VI Incontro Tutorato A.M. 1

$\boxed{\text{ES. 10}}$

Per prima cosa dimostriamo che $\lim_{x\to 0} \frac{\arctan(x)}{x} = 1$

<u>DIM</u>: Pongo $t = \arctan x \Rightarrow x = \tan(t)$ per $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\lim_{t\to 0} \frac{t}{\tan t} = \lim_{t\to 0}\left(\underbrace{\frac{t}{\sin t}}_{\downarrow 1} \cdot \underbrace{\cos t}_{\downarrow 1}\right) = 1$$

Dunque $\lim_{x\to 0}\left(\frac{\arctan(x) - x}{x}\right) = 0 \Rightarrow \arctan(x) = x + o(x)$

Andiamo allora a riscrivere ciascuna componente:

- $\arctan(x) = x + o(x)$

$\Rightarrow o(\alpha \cdot f(x)) = o(f(x))$

- $\cos(2x) = 1 - \frac{(2x)^2}{2} + o\left((2x)^2\right) = 1 - 2x^2 + o(4x^2) = 1 - 2x^2 + o(x^2)$

$\Rightarrow 1 - \cos(2x) = 1 - 1 + 2x^2 + o(x^2) = 2x^2 + o(x^2)$

- $\sin(x) = x + o(x) \Rightarrow \sin^2(x) = (x + o(x))^2 = x^2 + o(x)^2 + 2x\,o(x) = x^2 + o(x^2)$

$o(x^m)^n = o(x^{m\cdot n})$

$f(x)\cdot o(g(x)) = o(f(x)\cdot g(x))$

Sostituisco nel limite iniziale, ottenendo:

$$\lim_{x\to 0} \frac{3(x+o(x)) + (2x^2 + o(x^2))\cdot(x^2 + o(x^2))}{27x^4 + 5(x + o(x))} = \lim_{x\to 0} \frac{3x + \underbrace{3o(x)}_{o(x)} + 2x^4 + \underbrace{2x^2 o(x^2) + o(x^2)x^2 + o(x^2)o(x^2)}_{o(x^4)}}{27x^4 + 5(x + o(x))}$$

$$= \lim_{x\to 0} \frac{3x + o(x) + 2x^4 + o(x^4)}{27x^4 + 5x + o(x)} = \lim_{x\to 0} \frac{3x + \overbrace{o(x) + 2o(x) + o(o(x))}^{= o(x)}}{27 o(x) + 5x + o(x)} = \lim_{x\to 0} \frac{3x + o(x)}{5x + o(x)} = \frac{3}{5}$$

$x^4 \underset{x\to 0}{=} o(x)$ poiché $\lim_{x\to 0} \frac{x^4}{x} = x^3 = 0$

Volendo è possibile procedere anche nella seguente maniera:

$$\lim_{x\to 0} \frac{3\frac{\arctan(x)}{x}\cdot x + \frac{1-\cos(2x)}{4x^2}\cdot\frac{\sin^2(x)}{x^2}\cdot 4x^4}{27x^4 + 5\frac{\sin(x)}{x}\cdot x} = \frac{3}{5}$$

Nel limite gli unici termini a sopravvivere sono quelli cerchiati in verde

$\boxed{\text{ES - 16}}$

Riscrivendo ciascuno dei termini:

- $\cos(5x) = 1 - \frac{(5x)^2}{2} + o\left((5x)^2\right) = 1 - \frac{25x^2}{2} + o(x^2)$

- $\tan(3x) = 3x + o(3x) = 3x + o(x)$

- $\sin(x) = x + o(x)$

$$\Rightarrow \lim_{x\to 0} \frac{\left(\frac{25}{2}x^2 + o(x^2)\right)\left(3x + o(x)\right)}{\left(x + o(x) - x^3\right)^3} = \circledast$$

(numeratore $\to N$, denominatore $\to D$)

$$N: \left(\frac{25}{2}x^2 + o(x^2)\right)\left(3x + o(x)\right) = \frac{75}{2}x^3 + \underbrace{\frac{25}{2}x^2 o(x)}_{1} + \underbrace{o(x^2)\,3x}_{2} + \underbrace{o(x^2)\,o(x)}_{3} = \frac{75}{2}x^2 + o(x^3)$$

① : $\frac{25}{2}x^2 o(x) = o\left(\frac{25}{2}x^3\right) = o(x^3)$ $\quad \to o(\alpha f) = o(f)$

$\hookrightarrow g\cdot o(f) = o(f\cdot g)$

② : $o(x^2)\,3x = o(3x^3) = o(x^3)$

③ : $o(x^2)\,o(x) = o(x^3)$

$\hookrightarrow o(f)\cdot o(g) = o(fg)$

$$\Rightarrow ① + ② + ③ = o(x^3) + o(x^3) + o(x^3) = o(x^3)$$

$$\partial: \left(x+o(x)-x^3\right)^3 = \left(x+o(x)-o(x)\right)^3 = \left(x+o(x)\right)^3$$

$x^3 = o(x)$

$$= x^3 + 3x^2 o(x) + 3x\, o(x^2) + o(x^3)$$

$$= x^3 + o(x^3)$$

$$* = \lim_{x\to 0} \frac{\frac{75}{2}x^3 + o(x^3)}{x^3 + o(x^3)} = \frac{75}{2}$$

Procedendo in un'altra maniera

$$\lim_{x\to 0} \frac{\overbrace{\frac{1-\cos(5x)}{25x^2}}^{\to 1/2} \cdot \overbrace{\frac{\tan(3x)}{3x}}^{\to 1} \cdot 75x^3}{\underbrace{\left(\frac{\sin x}{x}\cdot x - x^3\right)^3}_{x^3\left(\frac{\sin x}{x} - 1\right)^3}} = \frac{75}{2}$$

$$\lim_{x\to 0} \frac{\frac{1-\cos(5x)}{25x^2} \cdot \frac{\tan(3x)}{3x} \cdot 75x^3}{\underbrace{\left(\frac{\sin(x)}{x}\cdot x - x^3\right)^3}_{x^3\left(\frac{\sin(x)}{x} - x^2\right)^3}} = \frac{75}{2}$$

$\boxed{\text{ES. 1c}}$

Consideriamo una radice alla volta

- $\sqrt[3]{2+x^3} = \sqrt[3]{x^3\left(1+\frac{2}{x^3}\right)} = x\sqrt[3]{1+\frac{2}{x^3}}$ $\quad \frac{2}{x^3} \to 0$ per $x\to\infty$

$$\Rightarrow \sqrt[3]{1+\frac{2}{x^3}} = \left(1+\frac{2}{x^3}\right)^{1/3} = 1 + \frac{1}{3}\cdot\frac{2}{x^3} + o\left(\frac{2}{x^3}\right) = 1 + \frac{2}{3x^3} + o\left(\frac{1}{x^3}\right)$$

$$\Rightarrow \sqrt[3]{2+x^3} = x\left(1+\frac{2}{3x^3}+o\left(\frac{1}{x^3}\right)\right) = x + \frac{2}{3x^2} + o\left(\frac{1}{x^2}\right)$$

$$\circ\ \sqrt[3]{1+2x^2+x^3} = \sqrt[3]{x^3\left(1+\frac{2}{x}+\frac{1}{x^3}\right)} = x\sqrt[3]{1+\frac{2}{x}+\frac{1}{x^3}}$$

$$\Rightarrow \sqrt[3]{1+\frac{2}{x}+\frac{1}{x^3}} = \left[1+\left(\frac{2}{x}+\frac{1}{x^3}\right)\right]^{1/3} = 1+\frac{1}{3}\left(\frac{2}{x}+\frac{1}{x^3}\right)+o\left(\frac{2}{x}+\frac{1}{x^3}\right)$$

$$= 1+\frac{2}{3x}+\frac{1}{3x^2}+o\left(\frac{1}{x}+\frac{1}{x^3}\right)$$

$$\Rightarrow \sqrt[3]{1+2x^2+x^3} = x\left(1+\frac{2}{3x}+\frac{1}{3x^3}+o\left(\frac{1}{x}+\frac{1}{x^3}\right)\right) = x+\frac{2}{3}+\frac{1}{3x^2}+o\left(1+\frac{1}{x^2}\right)$$

Sostituisco tutto nel limite:

$$\lim_{x\to\infty} x+\frac{2}{3x^2}+o\left(\frac{1}{x^2}\right)-x-\frac{2}{3}-\frac{1}{3x^2}-o\left(1+\frac{1}{x^2}\right) =$$

$$= \lim_{x\to\infty} -\frac{2}{3}+\underbrace{\frac{1}{3x^2}+o\left(\frac{1}{x^2}\right)-o\left(1+\frac{1}{x^2}\right)}_{o(1)\ \Rightarrow\ \lim_{x\to\infty}\frac{1/x^2}{1}=\lim_{x\to\infty}\frac{1}{x^2}=0} = -\frac{2}{3}$$

Procedendo in altra maniera:

OBIETTIVO È RIPORTARCI A ESPRESSIONE DEL TIPO $(A-B)(A^2+AB+B^2)=A^3-B^3$

$$\lim_{x\to\infty}\left[(2+x^3)^{1/3}-(1+2x^2+x^3)^{1/3}\right]\cdot\frac{\left[(2+x^3)^{2/3}+\left[(2+x^3)(1+2x^2+x^3)\right]^{1/3}+(1+2x^2+x^3)^{2/3}\right]}{\left[(2+x^3)^{2/3}+\left[(2+x^3)(1+2x^2+x^3)\right]^{1/3}+(1+2x^2+x^3)^{2/3}\right]}$$

$$= \lim_{x\to\infty}\frac{2+\cancel{x^3}-1-2x^2-\cancel{x^3}}{\left[(2+x^3)^{2/3}+\underbrace{\left[(2+x^3)(1+2x^2+x^3)\right]^{1/3}}_{(2+4x^2+2x^3+x^3+2x^5+x^6)^{1/3}}+(1+2x^2+x^3)^{2/3}\right]}$$

$$= \lim_{x\to\infty}\frac{\cancel{x^2}\left(-2+\frac{1}{x^2}\right)}{\cancel{x^2}\left\{\left(1+\frac{2}{x^3}\right)^{2/3}+(1+\dots)^{1/3}+(1+\dots)^{2/3}\right\}} = -\frac{2}{3}$$

ES. 1d

$$\lim_{x\to 0} \frac{\tan(x) - \sin(x)}{x} = *$$

Sfruttando gli o-piccolo si é visto che: $\sin(x) = x + o(x)$
$\tan(x) = x + o(x)$

$$\Rightarrow \lim_{x\to 0} \frac{x + o(x) - x - o(x)}{x^3} = \lim_{x\to 0} \frac{o(x)}{x^3} \rightarrow \text{NON VA BENE}$$

Proviamo con un altro metodo:

$$* = \lim_{x\to 0} \frac{\sin(x)}{x^3}\left(\frac{1}{\cos(x)} - 1\right) = \lim_{x\to 0} \frac{\sin(x)}{x} \cdot \frac{\left(\frac{1-\cos(x)}{\cos(x)}\right)}{x^2}$$

$$= \lim_{x\to 0} \underbrace{\frac{\sin(x)}{x}}_{1} \cdot \underbrace{\frac{1-\cos(x)}{x^2}}_{1/2} \cdot \underbrace{\frac{1}{\cos(x)}}_{1} = \frac{1}{2}$$

ES. 2

$$\lim_{x\to 0^+} \frac{e^{-1/x}}{x^4} \Rightarrow \text{CAMBIO DI VARIABILE} \quad y = \frac{1}{x}$$

$$\Rightarrow \lim_{y\to +\infty} \frac{e^{-y}}{y^{-4}} = \lim_{y\to +\infty} \frac{y^4}{e^y} = 0$$

Dunque $e^{-1/x}$ é $o(x^4)$ per $x \to 0^+$