XII INCONTRO TUTORATO A.M.1

Ricordiamo che: • In generale vale $D(f(g(x))) = f'(g(x))\,g'(x)$

• INTEGRAZIONE PER PARTI: A partire da $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$

Date $f,g:[a,b] \to \mathbb{R}$ di classe $\mathcal{C}^1$, allora

$$\int_a^b f'(x)g(x)\,dx = \left[f(x)g(x)\right]_{x=a}^{x=b} - \int_a^b f(x)g'(x)\,dx$$

$\boxed{\text{ES. 1}}$

a) $\int \frac{x}{\cos^2(x)}dx = \int \underbrace{x}_{g(x)} \cdot \underbrace{\frac{1}{\cos^2(x)}}_{f'(x)} dx = \underbrace{x}_{g(x)}\underbrace{\tan(x)}_{f(x)} - \int 1 \cdot \tan(x)dx = x\tan(x) + \int \frac{-\sin(x)}{\cos(x)}dx$

$= x\tan(x) + \log|\cos(x)| + c$

b) $\int \frac{\ln(x)}{x^2}dx = \int \overbrace{\frac{1}{x^2}}^{f'(x)} \overbrace{\ln(x)}^{g(x)} dx = \frac{x^{-1}}{-1} \cdot \ln(x) - \int \frac{x^{-1}}{-1} \cdot \frac{1}{x} = -\frac{1}{x}\ln(x) - \int -\frac{1}{x} \cdot \frac{1}{x}dx$

$= -\frac{1}{x}\ln(x) + \int \frac{1}{x^2}dx = -\frac{1}{x}\ln(x) + \frac{x^{-1}}{-1} = -\frac{1}{x}\ln(x) - \frac{1}{x} + c$

c) $\int \overbrace{e^x}^{f(x)} \overbrace{\sin(x)}^{g'(x)} dx = e^x(-\cos(x)) - \int e^x(-\cos(x))\,dx = -e^x\cos(x) + \int e^x\cos(x)dx$

$= -e^x\cos(x) + e^x\sin(x) - \int e^x\sin(x)dx$

$\Rightarrow 2\int e^x\sin(x)dx = \frac{-e^x\cos(x) + e^x\sin(x)}{2}$

$\boxed{\text{ES. 2}}$

$$\int_0^{1/2} \frac{x^2+x+1}{(x+2)(x^2-1)}dx = \int_0^{1/2} \frac{x^2+x+1}{(x+2)(x+1)(x-1)} = \int_0^{1/2}\left(\frac{A}{x+2} + \frac{B}{x+1} + \frac{C}{x-1}\right)dx$$

$\Rightarrow A(x^2-1) + (Bx+2B)(x-1) + (Cx+C)(x+2)$

$= Ax^2 - A + Bx^2 - Bx + 2Bx - 2B + Cx^2 + 2Cx + Cx + 2C$

$= x^2(A+B+C) + x(-B+2B+2C+C) - A - 2B + 2C$

$$\begin{cases} A+B+C=1 \Rightarrow A = 1-B-C & (1)\\ B+3C=1 \Rightarrow B = 1-3C & (2)\\ -A-2B+2C=1 & (3)\end{cases}$$

Sostituisco (1) e (2) in (3)

$$-1+B+C-2B+2C=1 \rightarrow -1+1-3C+C-2+6C+2C=1$$

$$6C=3 \rightarrow C = 1/2$$

$$\rightarrow B = 1-\frac{3}{2} = -\frac{1}{2}$$

$$\rightarrow A = 1+\frac{1}{2}-\frac{1}{2} = 1$$

$$\Rightarrow \left[\log|x+2| - \frac{1}{2}\log|x+1| + \frac{1}{2}\log|x-1|\right]_0^{1/2}$$

$$= \log\left(\frac{5}{2}\right) - \frac{1}{2}\log\frac{3}{2} + \frac{1}{2}\log\left(\frac{1}{2}\right) - \log(2)$$

$$= \log(5) - \log(2) - \frac{1}{2}\log(3) + \cancel{\frac{1}{2}\log(2)} - \cancel{\frac{1}{2}\log(2)} - \log(2)$$

$$= \log\left(\frac{5}{4\sqrt{3}}\right)$$

$\boxed{\text{ES. 3}}$

$$\int_0^1 \frac{e^x+2}{e^x(e^x+1)}\,dx \Rightarrow \text{Metodo sostituzione: } e^x = t \Rightarrow \log t = x \Rightarrow dx = \frac{1}{t}\,dt$$

$$\Rightarrow \int_1^e \underbrace{\frac{t+2}{t(t+1)}\cdot\frac{1}{t}}\,dt$$

$$\hookrightarrow \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t+1}$$

Da cui otteniamo $At(t+1)+Bt+B+Ct^2 = At^2+At+Bt+B+Ct^2$

$$\rightarrow t^2(A+C)+t(A+B)+B$$

$$\begin{cases} A+C=0 \Rightarrow C=1 \\ A+B=1 \Rightarrow A=-1 \\ B=2 \end{cases}$$

$$\rightarrow \int_1^e \left(-\frac{1}{t} + \frac{2}{t^2} + \frac{1}{t+1}\right) dt = \left[-\log|t| - \frac{2}{t} + \log|t+1|\right]_1^e$$

$$= \left[-\log(e^x) - \frac{2}{e^x} + \log(e^x+1)\right]_0^1$$

$$= \left[-\log(e^1) - \frac{2}{e} + \log(e+1)\right] - \left[0 - 2 + \log(2)\right]$$

$$= -1 - \frac{2}{e} + \log(e+1) + 2 - \log(2) = 1 - \frac{2}{e} + \log\left(\frac{e+1}{2}\right)$$

$\boxed{\text{ES. 4}}$

$$\int_0^1 \arctan\left(\frac{u-3}{u}\right) du$$

Procediamo per sostituzione $x = \frac{u-3}{2} \rightarrow dx = \frac{du}{2} \rightarrow du = 2dx$

$$\int_{-3/2}^{-1} \arctan(x)\, 2dx = ⊛$$

Possiamo integrare per parti, visto che

$$\int \arctan(x)dx = x\arctan(x) - \frac{1}{2}\int 2x \frac{1}{1+x^2} dx = x\arctan(x) - \frac{1}{2}\log(1+x^2) + c$$

$$\rightarrow ⊛ = 2\left[x\arctan(x) - \frac{1}{2}\log(1+x^2)\right]_{x=-3/2}^{x=-1}$$

$$= 2\left[-\arctan(-1) - \frac{1}{2}\log(2) + \frac{3}{2}\arctan\left(-\frac{3}{2}\right) + \frac{1}{2}\log\left(1+\frac{9}{4}\right)\right]$$

$$= \frac{\pi}{2} - 3\arctan\left(\frac{3}{2}\right) + \log\left(\frac{13}{8}\right)$$

ES. 5

a) Proviamo con una integrazione per parti

$$\int \underbrace{e^{-x}}_{f(x)} \underbrace{\cos(x)}_{g'(x)}\,dx = e^{-x}\sin(x) + \underbrace{\int \sin(x)\left(+e^{-x}\right)dx}$$

$$\int \underbrace{e^{-x}}_{h(x)} \underbrace{\sin(x)}_{k'(x)}\,dx = e^{-x}\left(-\cos(x)\right) - \int \left(+\cos(x)\right)\left(+e^{-x}\right)dx$$

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$$\rightarrow \int e^{-x}\cos(x)\,dx = e^{-x}\sin(x) - e^{-x}\cos(x) - \int \cos(x)\,e^{-x}\,dx$$

$$\cancel{2}\int e^{-x}\cos(x)\,dx = \frac{e^{-x}}{2}\left(\sin(x)-\cos(x)\right) + c = F(x)$$

Se imponiamo $F(\pi)=1 \Rightarrow \dfrac{e^{-\pi}}{2}\left(-\cos(\pi)+\sin(\pi)\right) + c = 1$

$$\rightarrow c = 1 - \frac{e^{-\pi}}{2}$$

b) Abbiamo che $a_n = \displaystyle\int_0^n e^{-x}\cos(x)\,dx = \left[\frac{e^{-x}}{2}\left(\sin(x)-\cos(x)\right)\right]_0^n =$

$$= \frac{e^{-n}}{2}\left(\sin(n)-\cos(n)\right) - \frac{1}{2}(0-1) = \underbrace{\frac{e^{-n}}{2}\left(\sin(n)-\cos(n)\right)}_{\to 0 \text{ per } n\to\infty} + \frac{1}{2}$$

$$\Rightarrow \lim_{n\to\infty} a_n = \frac{1}{2}$$

$\boxed{\text{ES. 6}}$

Abbiamo $\displaystyle\int \frac{x}{\underbrace{x^2+2x+2}} dx$

$\to \Delta = 4-8 = -4 < 0 \to (x+\alpha)^2+\gamma^2$

$$\to \frac{1}{2}\int \frac{2x+2-2}{x^2+2x+2}dx = \underbrace{\frac{1}{2}\int\frac{2x+2}{x^2+2x+2}dx}_{\frac{1}{2}\log(x^2+2x+2)} - \int\frac{1}{\underbrace{x^2+2x+2}_{\arctan(x+1)}}dx = ⊛$$

$\to x^2+2x+2 = (x+1)^2+1$

$$⊛ = \frac{1}{2}\log(x^2+2x+2) - \arctan(x+1) + c$$