(i)

$$\int_0^{\pi/4} \tan x \, dx = \int_0^{\pi/4} \frac{\operatorname{sen} x}{\cos x} dx = -\int_0^{\pi/4} \frac{(-\operatorname{sen} x)}{\cos x} dx = \Big[-\log(\cos x)\Big]_{x=0}^{x=\pi/4} = -\log\frac{1}{\sqrt{2}} = \log\sqrt{2}$$

(ii)

$$\begin{cases} y' = \tan x \, y + 7 \\ y(0) = 1 \end{cases}$$

$y' = \tan(x) y + 7 \Rightarrow$ EQUAZIONE LINEARE DEL 1° ORDINE

$\left(a(x) = \tan(x), \; b(x) = 7\right)$

Risolvo l'equazione omogenea associata:

$y' = \tan(x) y$

Soluzione: $y = c e^{A(x)}$ ove $A(x) = \int a(x)\,dx = \int \tan(x)\,dx = -\log(\cos x)$

$\Rightarrow \; y = c e^{-\log(\cos x)}$

Ora cerco una soluzione particolare del tipo: $y_P(x) = c(x) e^{-\log(\cos x)}$

$$c'(x) e^{-\log(\cos x)} + \cancel{c(x) e^{-\log(\cos x)} \tan x} = \cancel{\tan(x)\, c(x) e^{-\log(\cos x)}} + 7$$

$c'(x) = 7 e^{\log(\cos x)} = 7 \cos x \Rightarrow c(x) = 7 \int \cos x \, dx = 7 \operatorname{sen} x$

Ne segue: $y_P(x) = 7 \operatorname{sen} x \, e^{-\log(\cos x)}$

Soluzione generale: $y(x) = c e^{-\log(\cos x)} + 7 \operatorname{sen} x \, e^{-\log(\cos x)}$

$= (7 \operatorname{sen} x + c) e^{-\log(\cos x)}$

$= \dfrac{7 \operatorname{sen} x + c}{\cos x}$

Condizione iniziale: $y(0) = 1 \Rightarrow 1 = c \Rightarrow \boxed{y(x) = \dfrac{7 \operatorname{sen} x + 1}{\cos x}}$

(Domanda BONUS)

$$\lim_{x \to \frac{\pi}{2}^-} \frac{\overbrace{7 \operatorname{sen} x + c}^{\to 7}}{\underbrace{\cos x}_{\to 0}} = \begin{cases} +\infty & \text{se } c > -7 \\ -\infty & \text{se } c < -7 \end{cases}$$

$\Rightarrow$ se $c = -7$:

$$\lim_{x \to \frac{\pi}{2}^-} \frac{7 \operatorname{sen} x + c}{\cos x} \overset{H}{=} \lim_{x \to \frac{\pi}{2}^-} \frac{7 \cos x}{-\operatorname{sen} x} = \lim_{x \to \frac{\pi}{2}^-} -\frac{7}{\underbrace{\tan x}_{\to +\infty}} = 0$$

(i)

$$\int \frac{e^x - 1}{e^x + 1}\, dx$$

$t = e^x \Rightarrow x = \log t \Rightarrow dx = \frac{dt}{t}$

Sostituisco t nell'integrale:

$$\int \frac{t-1}{t+1} \frac{dt}{t} = \int \frac{t-1}{t(t+1)}\, dt = -\int \frac{dt}{t} + \int \frac{2}{t+1}\, dt = -\log t + 2\log(t+1) + c$$

$$= 2\log\left(e^x + 1\right) - x + c$$

$$= \log\left(e^x + 1\right)^2 - x + c$$

$$\frac{t-1}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1} = \frac{At + Bt + A}{t(t+1)}$$

$$\begin{cases} A + B = 1 \Rightarrow -1 + B = 1 \Rightarrow B = 2 \\ A = -1 \end{cases}$$

(ii)

$y' + \frac{e^x - 1}{e^x + 1}\, y = 0 \Rightarrow$ EQUAZIONE LINEARE DEL 1° ORDINE (omogenea)

$\left(a(x) = -\frac{e^x - 1}{e^x + 1} \right)$

Soluzione: $y(x) = c\, e^{A(x)}$ ove $A(x) = \int a(x)\, dx = -\int \frac{e^x - 1}{e^x + 1}\, dx = -\log\left(e^x + 1\right)^2 + x$

$$\Rightarrow y(x) = c\left[e^{-\log(e^x+1)^2 + x}\right] = c\left[e^x \cdot e^{-\log(e^x+1)^2}\right]$$

$$= c\, \frac{e^x}{(e^x+1)^2}$$

(iii)

$$\begin{cases} y' + \frac{e^x - 1}{e^x + 1}\, y = \frac{1}{(e^x+1)^2} \\ y(0) = -1 \end{cases}$$

$\Rightarrow$ EQUAZIONE LINEARE DEL 1° ORDINE (non omogenea)

$\left(a(x) = \frac{-(e^x - 1)}{e^x + 1},\ b(x) = \frac{1}{(e^x+1)^2} \right)$

Soluzione eq. omogenea associata: $y(x) = c\, \frac{e^x}{(e^x+1)^2}$

Ora cerco una soluzione particolare del tipo: $y_P(x) = c(x)\, \frac{e^x}{(e^x+1)^2}$

$$c'(x)\frac{e^x}{(e^x+1)^2}-\frac{e^x-1}{e^x+1}\cancel{c(x)\frac{e^x}{(e^x+1)^2}}=\frac{1}{(e^x+1)^2}-\frac{e^x-1}{e^x+1}\cancel{c(x)\frac{e^x}{(e^x+1)^2}}$$

$$c'(x)\frac{e^x}{\cancel{(e^x+1)^2}}=\frac{1}{\cancel{(e^x+1)^2}} \Rightarrow c'(x)=e^{-x} \Rightarrow c(x)=\int e^{-x}dx=-\int -e^{-x}dx=-e^{-x}$$

Ne segue: $y_P(x)=-\cancel{e^{-x}}\frac{\cancel{e^x}}{(e^x+1)^2}=-\frac{1}{(e^x+1)^2}$

Soluzione generale: $y(x)=c\frac{e^x}{(e^x+1)^2}-\frac{1}{(e^x+1)^2}$

Condizione iniziale: $y(0)=-1 \Rightarrow -1=\frac{c}{4}-\frac{1}{4} \Rightarrow -4=c-1 \Rightarrow c=-3$

$$\Rightarrow \boxed{y(x)=\frac{1}{(e^x+1)^2}\left(-3e^x-1\right)}$$

(i)

$$\int\frac{dx}{x^2+4x+5}=\int\frac{dx}{\underbrace{x^2+4x+4}_{(x+2)^2}-4+5}=\int\frac{dx}{1+(x+2)^2}=\arctan(x+2)+c,\quad c\in\mathbb{R}$$

(ii)

$$\begin{cases} y'=y^2+4y+5 \\ y(0)=1 \end{cases}$$

$y'=y^2+4y+5 \Rightarrow$ EQUAZIONE A VARIABILI SEPARABILI

$\left(y'=f(y)\,g(x) \text{ ove } f(y)=y^2+4y+5 \text{ e } g(x)=1\right)$

$$\frac{dy}{dx}=y^2+4y+5 \Rightarrow \frac{dy}{y^2+4y+5}=dx \Rightarrow \int\frac{dy}{y^2+4y+5}=\int dx$$

$$\Rightarrow \arctan\left(y(x)+2\right)=x+c$$

Condizione iniziale: $y(0)=1 \Rightarrow \arctan(3)=c$

$$\Rightarrow y(x)=\tan\left(x+\arctan 3\right)-2$$

(i)

$\circ \int \log x \, dx = x \log x - \int x \frac{1}{x} \, dx = x(\log x - 1) + c \qquad , \; c \in \mathbb{R}$

Integrazione per parti:

$u(x) = 1 \Rightarrow U(x) = x$

$v(x) = \log x \Rightarrow v'(x) = \frac{1}{x}$

$\circ \int \log^2 x \, dx = x \log^2 x - \int \cancel{x} \frac{2 \log x}{\cancel{x}} \, dx = x \log^2 x - 2 \int \log x \, dx$

$u(x) = 1 \Rightarrow U(x) = x$

$= x(\log^2 x - 2 \log x + 2) + c$

$v(x) = \log^2 x \Rightarrow v'(x) = \frac{2 \log x}{x}$

(ii)

$$\begin{cases} y' = e^y \log x \\ y(1) = 0 \end{cases}$$

$y' = e^y \log x \Rightarrow$ EQUAZIONE A VARIABILI SEPARABILI

$\left(y' = f(y) g(x) \text{ ove } f(y) = e^y \text{ e } g(x) = \log x \right)$

$\frac{dy}{dx} = e^y \log x \Rightarrow \frac{dy}{e^y} = \log x \, dx \Rightarrow \int \frac{dy}{e^y} = \int \log x \, dx$

$-\int -e^{-y} \, dy = x(\log x - 1) + c$

$e^{-y} = -x(\log x - 1) - c$

$\Rightarrow y(x) = -\log\left[-x(\log x - 1) - c\right]$

Pongo la condizione del problema di Cauchy:

$y(1) = 0 \Rightarrow 0 = -\log(1 - c) \Rightarrow c = 0$

$\Rightarrow \boxed{y(x) = -\log\left[-x(\log x - 1)\right]}$

(i)

$y'' + 4y = \cos(2x) \Rightarrow$ EQUAZIONE DEL SECONDO ORDINE (non omogenea)

L'equazione omogenea associata è:

$$y'' + 4y = 0$$

Equazione caratteristica: $z^2 + 4 = 0$

$$\Delta = -16 = 16i^2$$

$$z = \frac{\pm\sqrt{16i^2}}{2} = \pm\frac{\overset{2}{\cancel{4}}i}{\cancel{8}} = \pm 2i$$

Soluzione dell'omogenea associata:

$$y(x) = a\cos(2x) + b\,\text{sen}(2x)$$

$y'' + 4y = \cos(2x) \Rightarrow g(x) = H(x)\cos(2x)$ ove $H(x) = 1 \Rightarrow \deg H(x) = 0$

$\beta = 2 \Rightarrow 2i$ è soluzione dell'equazione caratteristica

Soluzione particolare: $y_P(x) = x\left[c_1\cos(2x) + c_2\,\text{sen}(2x)\right]$ ove c_1, c_2 hanno lo stesso grado di $H(x)$

$\Rightarrow y'_P(x) = c_1\cos(2x) + c_2\,\text{sen}(2x) + 2x\left(-c_1\,\text{sen}(2x) + c_2\cos(2x)\right)$

$\Rightarrow y''_P(x) = -2c_1\,\text{sen}(2x) + 2c_2\cos(2x) + 2\left(-c_1\,\text{sen}(2x) + c_2\cos(2x)\right) + 4x\left(-c_1\cos(2x) - c_2\,\text{sen}(2x)\right)$

$= 4\left(-c_1\,\text{sen}(2x) + c_2\cos(2x)\right) - 4y_P$

$$4\left(-c_1\,\text{sen}(2x) + c_2\cos(2x)\right) - \cancel{4y_P} + \cancel{4y_P} = \cos(2x)$$

$$-4c_1\,\text{sen}(2x) + 4c_2\cos(2x) = \cos(2x)$$

$\Rightarrow \begin{cases} -4c_1 = 0 \Rightarrow c_1 = 0 \\ 4c_2 = 1 \Rightarrow c_2 = 1/4 \end{cases} \Rightarrow y_P(x) = \frac{1}{4}x\,\text{sen}(2x)$

Soluzione: $\boxed{y(x) = a\cos(2x) + b\,\text{sen}(2x) + \frac{1}{4}x\,\text{sen}(2x)}$

(ii)

Problema di Cauchy: $\begin{cases} y'' + 4y = \cos(2x) \\ y(0) = 1,\ y'(0) = 2 \end{cases}$

$y(0)=1 \Rightarrow 1=a$

$y'(x)=-2a\,\text{sen}(2x)+2b\cos(2x)+\frac{1}{4}\text{sen}(2x)+\frac{1}{2}x\cos(2x)$

$y'(0)=2 \Rightarrow 2=2b \Rightarrow b=1$

Soluzione p. di Cauchy: $\boxed{y(x)=\cos(2x)+\text{sen}(2x)+\frac{1}{4}x\,\text{sen}(2x)}$

$$\begin{cases} y'(x)\,y(x)=(1+y(x)^2)(x\,\text{sen}x) \\ y(\pi)=1 \end{cases}$$

$y'(x)\,y(x)=(1+y(x)^2)(x\,\text{sen}x)$

$\Rightarrow y'(x)=\frac{1+y(x)^2}{y(x)}(x\,\text{sen}x)$ EQ. A VARIABILI SEPARABILI

$\left(y'=f(y)g(x) \text{ ove } f(y)=\frac{1+y^2}{y},\ g(x)=x\,\text{sen}x\right)$

$\frac{dy}{dx}=\frac{1+y^2}{y}(x\,\text{sen}x) \Rightarrow \frac{y}{1+y^2}dy=(x\,\text{sen}x)\,dx \Rightarrow \int\frac{y}{1+y^2}dy=\int x\,\text{sen}x\,dx$

- $\int\frac{y}{1+y^2}dy=\frac{1}{2}\int\frac{2y}{1+y^2}dy=\frac{1}{2}\log(1+y^2)+c$
- $\int x\,\text{sen}x\,dx=-x\cos x+\int\cos x\,dx=-x\cos x+\text{sen}x+c$

Integrazione per parti:

$u(x)=\text{sen}x \Rightarrow U(x)=-\cos x$

$v(x)=x \Rightarrow v'(x)=1$

$\Rightarrow \frac{1}{2}\log(1+y^2)=-x\cos x+\text{sen}x+c$

$\log(1+y^2)=2(-x\cos x+\text{sen}x+c)$

$1+y^2=e^{2(-x\cos x+\text{sen}x+c)} \Rightarrow y(x)=\sqrt{e^{2(-x\cos x+\text{sen}x+c)}-1}$

$y(\pi)=1: \quad 1=\sqrt{e^{2(\pi+c)}-1} \Rightarrow 2=e^{2(\pi+c)}$

$\Rightarrow \log 2=2\pi+2c \Rightarrow c=\frac{\log 2-2\pi}{2}$

Soluzione p. di Cauchy: $y(x)=\sqrt{2e^{-2x\cos x+2\text{sen}x-2\pi}-1}$

$$\begin{cases} y'(x) = \left(\frac{1}{1+x^2} + e^{2x}\right) y(x)^2 \\ y(0) = 1 \end{cases}$$

$y'(x) = \left(\frac{1}{1+x^2} + e^{2x}\right) y(x)^2 \Rightarrow$ EQ. A VARIABILI SEPARABILI

$\left(y' = f(y)g(x) \text{ ove } f(y) = y^2 \text{ e } g(x) = \frac{1}{1+x^2} + e^{2x}\right)$

$$\frac{dy}{dx} = \left(\frac{1}{1+x^2} + e^{2x}\right) y(x)^2 \Rightarrow \frac{dy}{y^2} = \left(\frac{1}{1+x^2} + e^{2x}\right) dx \Rightarrow \int \frac{dy}{y^2} = \int \left(\frac{1}{1+x^2} + e^{2x}\right) dx$$

- $\int y^{-2}\, dy = -\frac{1}{y} + c$
- $\int \left(\frac{1}{1+x^2} + e^{2x}\right) dx = \arctan x + \frac{1}{2} e^{2x} + c$

$$\Rightarrow -\frac{1}{y} = \arctan x + \frac{1}{2} e^{2x} + c \Rightarrow y(x) = -\frac{1}{\arctan x + \frac{1}{2} e^{2x} + c}$$

$$y(0) = 1 \Rightarrow 1 = -\frac{1}{1/2 + c} \Rightarrow 1 = -c - 1/2 \Rightarrow c = -3/2$$

Soluzione p. di Cauchy: $\boxed{y(x) = -\frac{1}{\arctan x + \frac{1}{2} e^{2x} - 3/2}}$