TUTORATO 23/11/2022

- es. 34 FOGLIO 6:

$$\lim_{n\to\infty}\left(1+\frac{1}{n!}\right)^n=\lim_{n\to\infty}\left[\left(1+\frac{1}{n!}\right)^{n!}\right]^{n/n!}=\lim_{n\to\infty}\left[\left(1+\frac{1}{n!}\right)^{n!}\right]^{\frac{\cancel{n}}{\cancel{n}(n-1)!}}=$$

$$=\lim_{n\to\infty}\underbrace{\left[\left(1+\frac{1}{n!}\right)^{n!}\right]}_{\to e}^{\underbrace{1/(n-1)!}_{\to 0}}=e^0=1$$

- es. 67 FOGLIO 6:

$$\lim_{n\to\infty}\left(1+\frac{7}{n}\right)^{2n}=\lim_{n\to\infty}\left[\left(1+\frac{1}{(n/7)}\right)^{n/7}\right]^{\frac{7}{\cancel{n}}\cdot 2\cancel{n}}=\lim_{n\to\infty}\underbrace{\left[\left(1+\frac{1}{(n/7)}\right)^{n/7}\right]}_{\to e}^{14}=e^{14}$$

- es. 54 FOGLIO 6:

$$\lim_{n\to\infty}\frac{n^2+n\sin n}{1+n^2+n}=\lim_{n\to\infty}\frac{n^2}{1+n^2+n}+\frac{n\sin n}{1+n^2+n}=\lim_{n\to\infty}\frac{\cancel{n^2}}{\cancel{n^2}\left(1+\frac{1}{n}+\frac{1}{n^2}\right)}+\frac{\cancel{n}\sin n}{n^{\cancel{2}}\left(1+\frac{1}{n}+\frac{1}{n^2}\right)}=$$

quantita' limitata per $n\to\infty$

$$=\lim_{n\to\infty}\underbrace{\frac{1}{1+\frac{1}{n}+\frac{1}{n^2}}}_{\to 1}+\underbrace{\frac{\sin n}{n\left(1+\frac{1}{n}+\frac{1}{n^2}\right)}}_{\to 0}=1$$

- es. 35 FOGLIO 6:

$$\lim_{n\to\infty}\sqrt[n]{a^{2n}+1}$$

$- \ a>1: \quad a^2=\sqrt[n]{a^{2n}}\leq\sqrt[n]{a^{2n}+1}\leq\sqrt[n]{a^{2n}+a^{2n}}=\sqrt[n]{2}\,a^2$

$\sqrt[n]{2} = 2^{1/n}\xrightarrow{n\to\infty}1$

$- \ 0\leq a\leq 1: \quad 1\leq\sqrt[n]{a^{2n}+1}\leq\sqrt[n]{1+1}=2^{1/n}\xrightarrow{n\to\infty}1$

$$\lim_{n\to\infty}\sqrt[n]{a^{2n}+1}=\begin{cases}a^2 & \text{se } a>1\\ 1 & \text{se } 0\leq a\leq 1\end{cases}$$

- es. 36 FOGLIO 6:

$$\lim_{n\to\infty} \frac{\log(1+n+n^3) - 3\log n}{n\left(1-\cos\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{\log\left(\frac{1+n+n^3}{n^3}\right)}{n\left(1-\cos\frac{1}{n^2}\right)} =$$

$$= \lim_{n\to\infty} \frac{\log\left(1+\frac{1}{n^2}+\frac{1}{n^3}\right)}{n\left(1-\cos\frac{1}{n^2}\right)} \cdot \frac{\frac{1}{n^2}+\frac{1}{n^3}}{\frac{1}{n^2}+\frac{1}{n^3}} = \lim_{n\to\infty} \frac{\log\left(1+\frac{1}{n^2}+\frac{1}{n^3}\right)}{\frac{1}{n^2}+\frac{1}{n^3}} \cdot \left(\frac{1}{n^2}+\frac{1}{n^3}\right) \cdot \frac{1}{n\left(1-\cos\frac{1}{n^2}\right)}$$

$$= \lim_{n\to\infty} \frac{\log\left(1+\frac{1}{n^2}+\frac{1}{n^3}\right)}{\frac{1}{n^2}+\frac{1}{n^3}} \cdot \left(\frac{1}{n^2}+\frac{1}{n^3}\right) \cdot \frac{1/n^4}{1-\cos\frac{1}{n^2}} \cdot \frac{n^{\cancel{4}\,3}}{\cancel{n}}$$

$$= \lim_{n\to\infty} \underbrace{\frac{\log\left(1+\frac{1}{n^2}+\frac{1}{n^3}\right)}{\frac{1}{n^2}+\frac{1}{n^3}}}_{\to 1} \cdot \underbrace{\frac{1/n^4}{1-\cos\frac{1}{n^2}}}_{\to 2} \cdot \underbrace{n^3\left(\frac{1}{n^2}+\frac{1}{n^3}\right)}_{=1+n \to +\infty} = +\infty$$

- es. 80 FOGLIO 6:

$$\lim_{n\to\infty} \sin\left(2\pi\sqrt{n^2+\sqrt{n}}\right) = \lim_{n\to\infty} \sin\left(2\pi\sqrt{n^2\left(1+n^{-3/2}\right)}\right) =$$

$$= \lim_{n\to\infty} \sin\left(2\pi n\sqrt{1+\frac{1}{\sqrt[3]{n}}}\right) = \lim_{n\to\infty} \sin\left[2\pi n\left(1+\frac{1}{\sqrt[3]{n}}+o\left(\frac{1}{\sqrt[3]{n}}\right)\right)\right] =$$

$$\sqrt{1+\frac{1}{\sqrt[3]{n}}} = 1+\frac{1}{2\sqrt[3]{n}}+o\left(\frac{1}{\sqrt[3]{n}}\right)$$

$$= \lim_{n\to\infty} \sin\Big(2\pi n + \underbrace{\frac{2\pi}{\sqrt{n}}+o\left(\frac{1}{\sqrt{n}}\right)}_{=:X_n}\Big) = \lim_{n\to\infty} \underbrace{\sin(2\pi n)}_{\to 0}\underbrace{\cos(X_n)}_{\to 1} + \underbrace{\cos(2\pi n)}_{\to 1}\underbrace{\sin(X_n)}_{\to 0} = 0$$

$$X_n \xrightarrow{n\to\infty} 0$$

- es. 100 FOGLIO 7:

$$a_n = n^{(-1)^n}$$

$$a_n = \begin{cases} n & \text{se } n \text{ e' pari} \\ n^{-1} = 1/n & \text{se } n \text{ e' dispari} \end{cases}$$

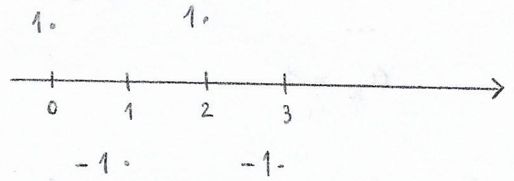
•) La sottosuccessione $a_{2n} \xrightarrow{n\to\infty} +\infty \Rightarrow \limsup\limits_{n\to\infty} a_n = +\infty$

•) La sottosuccessione $a_{2n+1} \xrightarrow{n\to\infty} 0 \Rightarrow \liminf\limits_{n\to\infty} a_n = 0$

• es. 101 FOGLIO 7 :

$$a_n = \frac{(-1)^n}{n} + \frac{1+(-1)^n}{2}$$

$$a_n = (-1)^n$$



•) $b_n = \sup\limits_{m\geq n} \{a_m\} = 1 \Rightarrow \limsup\limits_{n\to\infty} a_n = 1$

•) $c_n = \inf\limits_{m\geq n} \{a_m\} = -1 \Rightarrow \liminf\limits_{n\to\infty} a_n = -1$

•) $\limsup\limits_{n\to\infty} a_n = \limsup\limits_{n\to\infty} \frac{(-1)^n}{n} + \limsup\limits_{n\to\infty} \frac{1+(-1)^n}{2}$

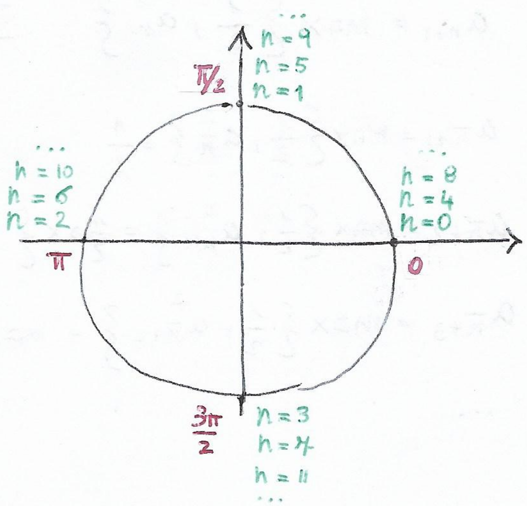
$$= \underbrace{\limsup_{n\to\infty} \frac{(-1)^n}{n}}_{\lim\limits_{n\to\infty} \frac{1}{n} = 0} + \underbrace{\limsup_{n\to\infty} \frac{1}{2} + \frac{(-1)^n}{2}}_{\frac{1}{2}+\frac{1}{2}\underbrace{\limsup\limits_{n\to\infty}(-1)^n}_{=1} \longrightarrow 1} = 1$$

•) $$\liminf_{n\to\infty} a_n = \underbrace{\liminf_{n\to\infty} \frac{(-1)^n}{n}}_{\lim\limits_{n\to\infty} -\frac{1}{n} = 0} + \underbrace{\liminf_{n\to\infty} \frac{1+(-1)^n}{2}}_{\frac{1}{2}+\frac{1}{2}\underbrace{\liminf\limits_{n\to\infty}(-1)^n}_{=-1/2} \longrightarrow 0} = 0$$

• es. 107 FOGLIO 7 :

$$a_n = \frac{n!}{2^n} \sin\left(n\frac{\pi}{2}\right)$$

$$\sin\left(n\frac{\pi}{2}\right) = \begin{cases} 0 & \text{se } n = 0,4,8\ldots \\ & \phantom{\text{se }} n = 2,6,10\ldots \\ 1 & \text{se } n = 1,5,9,\ldots \\ -1 & \text{se } n = 3,7,11,\ldots \end{cases}$$



$$\Rightarrow a_n = \begin{cases} 0 & \text{se } n = 0,2,4,6,\ldots \\ \frac{n!}{2^n} & \text{se } n = 1,5,9,\ldots \\ -\frac{n!}{2^n} & \text{se } n = 3,7,11,\ldots \end{cases}$$

$\lim\limits_{n\to\infty} \frac{n!}{2^n} = +\infty$

$a_n = \frac{n!}{2^n}$, $a_{n+1} = \frac{(n+1)!}{2^{n+1}} \Rightarrow \lim\limits_{n\to\infty} \frac{(n+1)!}{2^{n+1}} \cdot \frac{2^n}{n!} = \lim\limits_{n\to\infty} \frac{n+1}{2} = +\infty$

Per il criterio del rapporto vale il risultato sopra.

$\lim\limits_{n\to\infty} \frac{n!}{2^n} = +\infty \Rightarrow \limsup\limits_{n\to\infty} a_n = +\infty$

$\lim\limits_{n\to\infty} -\frac{n!}{2^n} = -\infty \Rightarrow \liminf\limits_{n\to\infty} a_n = -\infty$

∘ es. 118 FOGLIO 7:

$a_{n+1} = \max\left\{\frac{1}{2}, a_n^2\right\}$

∘) $|a_0| = 1$: $a_1 = \max\left\{\frac{1}{2}, a_0^2\right\} = \max\left\{\frac{1}{2}, 1\right\} = 1$

$a_2 = \max\left\{\frac{1}{2}, a_1^2\right\} = \max\left\{\frac{1}{2}, 1\right\} = 1$

...

$a_n = 1$

$\Rightarrow \lim\limits_{n\to\infty} a_n = 1$

∘) $|a_0| > 1$: a_n^2 e' sempre maggiore di $1/2$ per cui studiare il comportamento della successione equivale a studiare $b_{n+1} = b_n^2$.

$b_1 = b_0^2$

$b_2 = b_1^2 = (b_0^2)^2 = b_0^4$

$b_3 = b_2^2 = (b_0^4)^2 = b_0^8$

...

$b_n = b_0^{2^n} \xrightarrow[b_0>1]{n\to\infty} +\infty$

$\Rightarrow \lim\limits_{n\to\infty} a_n = +\infty$

∘) $|a_0| < 1$: $b_n = b_0^{2^n} \xrightarrow[b_0<1]{n\to\infty} 0$

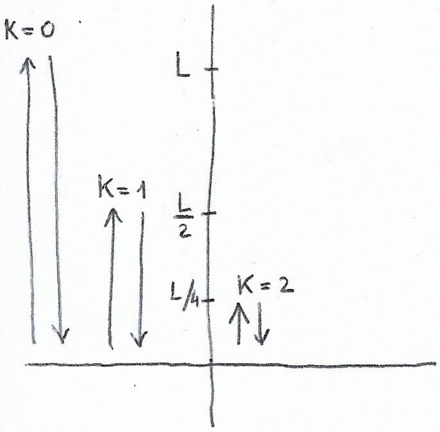
$a_{n+1} = \max\left\{\frac{1}{2}, a_n^2\right\}$ $\exists\, \bar{n}$ tale che $\max\left\{\frac{1}{2}, a_{\bar{n}}^2\right\} = 1/2$ perche' $b_n \to 0$

$$\left.\begin{aligned} a_{\bar{n}+1} &= \max\left\{\tfrac{1}{2}, a_{\bar{n}}^2\right\} = \tfrac{1}{2} \\ a_{\bar{n}+2} &= \max\left\{\tfrac{1}{2}, a_{\bar{n}+1}^2\right\} = \max\left\{\tfrac{1}{2}, \tfrac{1}{4}\right\} = \tfrac{1}{2} \\ a_{\bar{n}+3} &= \max\left\{\tfrac{1}{2}, a_{\bar{n}+2}^2\right\} = \max\left\{\tfrac{1}{2}, \tfrac{1}{4}\right\} = \tfrac{1}{2} \\ &\dots \end{aligned}\right\} a_n = \tfrac{1}{2} \ \forall n \geq \bar{n}.$$

$\Rightarrow \lim\limits_{n\to\infty} a_n = \frac{1}{2}$

Riassumendo: $\lim\limits_{n\to\infty} a_n = \begin{cases} +\infty & \text{se } |a_0| > 1 \\ 1 & \text{se } |a_0| = 1 \\ 1/2 & \text{se } |a_0| < 1 \end{cases}$

∘ es. 4.20 FOGLIO 6



$K=0 \Rightarrow S_0 = 2L$

$K=1 \Rightarrow S_1 = L$

$K=2 \Rightarrow S_2 = L/2$

...

Al rimbalzo K-esimo la pallina percorre:

$$S_K = \frac{2L}{2^K} = \frac{L}{2^{K-1}}$$

Quando tocca terra per l'n-esima volta la pallina ha percorso lo spazio:

$$S_n = \sum_{K=0}^{n} \frac{L}{2^{K-1}} = L \sum_{K=0}^{n} \frac{1}{2^{K-1}} = 4L\left(1 - 2^{-1-n}\right)$$

$$\sum_{K=0}^{n} \frac{1}{2^{K-1}} = \left(2 + 1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{n-1}}\right) \cdot \frac{1 - 1/2}{1 - 1/2}$$

$$= \frac{1}{1 - 1/2} \cdot \left(2 + \cancel{1} + \cancel{\frac{1}{2}} + \dots + \cancel{\frac{1}{2^{n-1}}} - \cancel{1} - \cancel{\frac{1}{2}} - \dots - \cancel{\frac{1}{2^{n-1}}} - \frac{1}{2^n}\right)$$

$$= \frac{1}{1 - 1/2} \cdot \left(2 - \frac{1}{2^n}\right) = 2\left(2 - 2^{-n}\right) = 4 - 2^{1-n} = 4\left(1 - 2^{-1-n}\right)$$