TUTORATO 14/12/2022

Trovare lo sviluppo di Taylor all'ordine 2 (con resto $o(x^2)$) di

$$f(x) = \frac{1}{1-x-x^2}$$

in $x_0 = 0$.

Soluzione:

Uso la sostituzione $t = x + x^2$ (se $x \to 0$ anche $t \to 0$)

In questo modo: $f(t) = \frac{1}{1-t}$

e posso usare lo sviluppo di Taylor noto: $\frac{1}{1-t} = 1 + t + t^2 + o(t^2)$

Dunque avremo:

$(x+x^2)^2 = x^2 + o(x^2)$

$$f(x) = \frac{1}{1-x-x^2} = 1 + (x+x^2) + (x+x^2)^2 + o\left((x+x^2)^2\right) = 1 + x + x^2 + x^2 + o(x^2) + o(x^2)$$

$$= 1 + x + 2x^2 + o(x^2)$$

$$\lim_{x\to 0} \frac{2\arcsin x - \tan x - x}{x^5}$$

- $f(x) = \arcsin x \Rightarrow f(0) = 0$

  $f'(x) = \frac{1}{\sqrt{1-x^2}} \Rightarrow f'(0) = 1$

  $f''(x) = \frac{x}{\sqrt{(1-x^2)^3}} \Rightarrow f''(0) = 0$

  $f'''(x) = 3x^2(1-x^2)^{-\frac{5}{2}} + (1-x^2)^{-\frac{3}{2}} \Rightarrow f'''(0) = 1$

  $\Rightarrow f(x) = x + \frac{x^3}{3!} + o(x^3)$

- $f(x) = \tan x \Rightarrow f(0) = 0$

  $f'(x) = \frac{1}{\cos^2 x} \Rightarrow f'(0) = 1$

  $f''(x) = \frac{2\operatorname{sen} x}{\cos^3 x} \Rightarrow f''(0) = 0$

  $f'''(x) = \frac{2}{\cos^2 x}\left(-2 + \frac{3}{\cos^2 x}\right) \Rightarrow f'''(0) = 2$

  $\Rightarrow f(x) = x + \frac{x^3}{3} + o(x^3)$

Numeratore: $2\arcsin x - \tan x - x = \cancel{2x} + \cancel{\frac{x^3}{3}} + o(x^3) - \cancel{x} - \cancel{\frac{x^3}{3}} + o(x^3) - \cancel{x}$

Non posso concludere nulla perche' $\frac{o(x^3)}{x^5} \xrightarrow[x\to 0]{} ?$

Vado all'ordine 5 al numeratore:

$$\arcsin x = x + \frac{x^3}{6} + \frac{3x^5}{40} + o(x^5)$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + o(x^5)$$

Sostituisco: $2\arcsin x - \tan x - x = \cancel{2x} + \cancel{\frac{x^3}{3}} + \frac{3x^5}{20} + o(x^5) - \cancel{x} - \cancel{\frac{x^3}{3}} - \frac{2}{15}x^5 + o(x^5) - \cancel{x}$

$= \left(\frac{3}{20} - \frac{2}{15}\right)x^5 + o(x^5) = \frac{1}{60}x^5 + o(x^5)$

$$\lim_{x\to 0} \frac{2\arcsin x - \tan x - x}{x^5} = \lim_{x\to 0} \frac{\frac{1}{60}x^5 + o(x^5)}{x^5} = \frac{1}{60}$$

$$\lim_{x\to 0} \frac{e^{-x} + \log\left(\frac{1+x}{e}\right)}{3(\cosh x - 1)\sinh x}$$

Numeratore: $e^{-x} + \log\left(\frac{1+x}{e}\right) = e^{-x} + \log(1+x) - \log e = e^{-x} + \log(1+x) - 1$

- $e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + o(y^3)$

$\Rightarrow e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + o(x^3)$

- $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$

$\Rightarrow e^{-x} + \log(1+x) - 1 = \cancel{1} - \cancel{x} + \cancel{\frac{x^2}{2}} - \frac{x^3}{6} + o(x^3) + \cancel{x} - \cancel{\frac{x^2}{2}} + \frac{x^3}{3} + o(x^3) - \cancel{1}$

$= + \frac{x^3}{6} + o(x^3)$

Denominatore:

- $\cosh x = 1 + \frac{x^2}{2} + o(x^2) \Rightarrow \cosh x - 1 = \frac{x^2}{2} + o(x^2)$
- $\sinh x = x + o(x)$

$\Rightarrow 3(\cosh x - 1)\sinh x = 3\left(\frac{x^2}{2} + o(x^2)\right)(x + o(x)) = \frac{3x^3}{2} + o(x^3)$

$$\lim_{x\to 0} \frac{e^{-x} + \log\left(\frac{1+x}{e}\right)}{3(\cosh x - 1)\sinh x} = \lim_{x\to 0} \frac{\frac{x^3}{6} + o(x^3)}{\frac{3x^3}{2} + o(x^3)} = \frac{1}{\cancel{6}_{3}} \cdot \frac{\cancel{2}}{3} = \frac{1}{9}$$

$f(x) = 2e^x - \sqrt{1+4x} - \sqrt{1+6x^2}$

$f : [-\frac{1}{4}, +\infty) \longrightarrow \mathbb{R}$

$x_0 = 0$ e' punto di massimo/minimo (o nessuno dei due) per f?

Soluzione:

Scrivo lo sviluppo di Taylor per $f(x)$.

- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + o(x^3)$

$\Rightarrow 2e^x = 2 + 2x + x^2 + \frac{x^3}{3} + o(x^3)$

- $\sqrt{1+t} = 1 + \frac{t}{2} - \frac{t^2}{8} + \frac{t^3}{16} + o(t^3)$

$\Rightarrow \sqrt{1+4x} = 1 + 2x - \frac{16x^2}{8} + \frac{64x^3}{16} + o(x^3) = 1 + 2x - 2x^2 + 4x^3 + o(x^3)$

$\Rightarrow \sqrt{1+6x^2} = 1 + 3x^2 + o(x^3)$

$$f(x) = \cancel{2} + \cancel{2x} + \cancel{x^2} + \frac{x^3}{3} + o(x^3) - \cancel{1} - \cancel{2x} + \cancel{2x^2} - 4x^3 + o(x^3) - \cancel{1} - \cancel{3x^2} + o(x^3)$$

$$= \frac{1-12}{3} x^3 + o(x^3) = -\frac{11}{3} x^3 + o(x^3)$$ $\Rightarrow$ la prima derivata non nulla di $f(x)$ in x_0 e' la derivata terza $\Rightarrow$ x_0 nè max nè min

---

Studiare il grafico di:

$f(x) = xe^{-x}$

$g(x) = |x|e^{-x}$

Notiamo che $g(x) = |f(x)|$.

Soluzione:

$f(x) = xe^{-x}$

- Dominio: $f(x)$ e' definita per ogni $x \Rightarrow D : \mathbb{R}$
- Limiti/asintoti:

$$\lim_{x\to+\infty} xe^{-x} = \lim_{x\to+\infty} \frac{x}{e^x} \overset{H}{=} \lim_{x\to+\infty} \frac{1}{e^x} = 0 \Rightarrow y = 0 \text{ e' as. orizzontale}$$

$$\lim_{x\to-\infty} xe^{-x} = -\infty$$

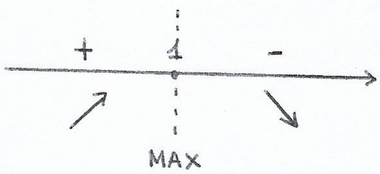
Asintoti obliqui per $x \to -\infty$? $m = \lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} e^{-x} = +\infty \Rightarrow$ No!

III Crescenza / Decrescenza. Max/Min :

Calcolo la derivata prima:

$$f'(x) = e^{-x} - x e^{-x} = (1-x)e^{-x}$$

$$f'(x) \geq 0 \iff 1 - x \geq 0 \iff x \leq 1$$



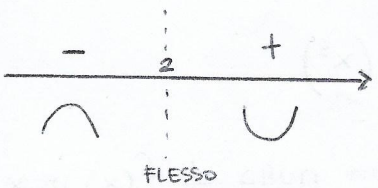
$$f(x) \begin{cases} \text{crescente in } (-\infty, 1) \\ \text{decrescente in } (1, +\infty) \end{cases}$$

IV Concavita', convessita', flessi :

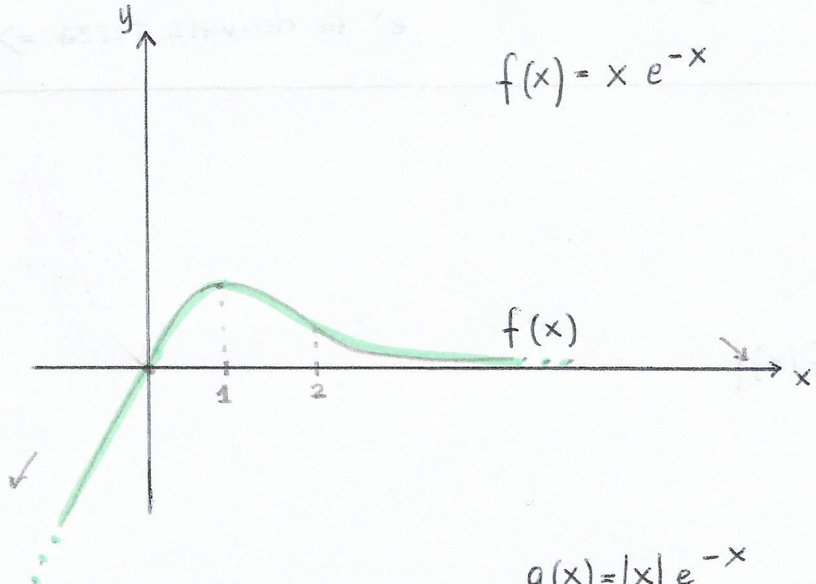
Calcolo la derivata seconda:

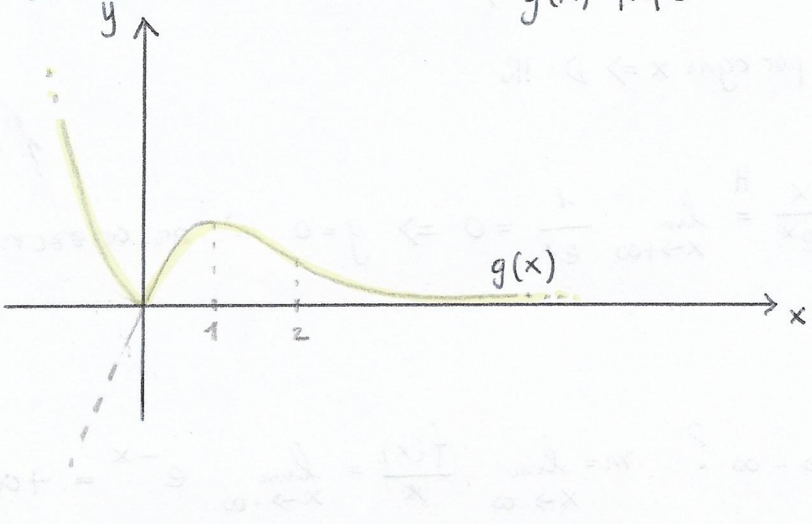
$$f''(x) = -e^{-x} - (1-x)e^{-x} = e^{-x}(-1-1+x) = (x-2)e^{-x}$$

$$f''(x) \geq 0 \iff x - 2 \geq 0 \iff x \geq 2$$



$$f(x) \begin{cases} \text{concava in } (-\infty, 2) \\ \text{convessa in } (2, +\infty) \end{cases}$$





Studiare il grafico di:

$$f(x) = (4x+3)\,e^{1/x}$$

- Dominio: $f(x)$ e' definita per $x \neq 0$. $D: (-\infty, 0) \cup (0, +\infty)$
- Limiti/asintoti:

$$\lim_{x\to 0^+} \underbrace{(4x+3)}_{\to 3}\,\underbrace{e^{1/x}}_{\to +\infty} = +\infty \Rightarrow x = 0 \text{ asintoto verticale a dx}$$

$$\lim_{x\to 0^-} \underbrace{(4x+3)}_{\to 3}\,\underbrace{e^{1/x}}_{\to 0} = 0$$

$$\lim_{x\to +\infty} \underbrace{(4x+3)}_{\to +\infty}\,\underbrace{e^{1/x}}_{\to 1} = +\infty$$

$$\lim_{x\to -\infty} \underbrace{(4x+3)}_{\to -\infty}\,\underbrace{e^{1/x}}_{\to 1} = -\infty$$

No as. orizzontali $\Rightarrow$ As. obliquo?

$$m = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{4x+3}{x} e^{1/x} = \lim_{x\to+\infty} \underbrace{\frac{\cancel{x}(4+3/x)}{\cancel{x}}}_{\to 4}\,\underbrace{e^{1/x}}_{\to 1} = 4$$

$$q = \lim_{x\to+\infty} (f(x) - mx) = \lim_{x\to+\infty} (4x+3)e^{1/x} - 4x$$

$$= \lim_{x\to+\infty} 4x(e^{1/x} - 1) + 3\underbrace{e^{1/x}}_{\to 3} = 7$$

$$\lim_{x\to+\infty} 4\,\frac{e^{1/x}-1}{1/x} \underset{t=1/x}{=} \lim_{t\to 0^+} 4\,\underbrace{\frac{e^t - 1}{t}}_{\to 1} = 4$$

$y = 4x + 7$ e' asintoto obliquo per $x \to +\infty$ (anche per $x \to -\infty$)

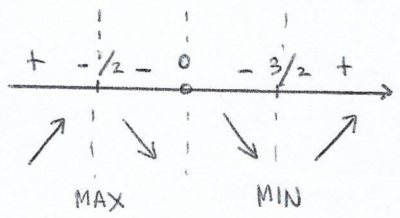
- Crescenza/decrescenza. Max/Min

$$f'(x) = 4e^{1/x} + (4x+3)e^{1/x}\left(-\frac{1}{x^2}\right) = e^{1/x}\,\frac{4x^2-4x-3}{x^2}$$

$f'(x) \geq 0 \iff 4x^2 - 4x - 3 \geq 0 \iff x \leq -\frac{1}{2}$ e $x \geq 3/2$

$\Delta = 16 + 48 = 64$

$x = \frac{4 \pm 8}{8} = \left\langle \begin{array}{l} 3/2 \\ -1/2 \end{array} \right.$



$f(x)$ crescente in $\left(-\infty, -\frac{1}{2}\right]$ e $\left[\frac{3}{2}, +\infty\right)$

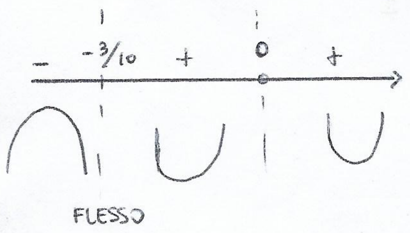
$f(x)$ decrescente in $\left(-\frac{1}{2}, 0\right)$ e $\left(0, \frac{3}{2}\right)$

- Concavita'/Convessita'. Flessi

$$f''(x) = \left(4 - \frac{4}{x} - \frac{3}{x^2}\right) D e^{1/x} + e^{1/x} D\left(4 - \frac{4}{x} - \frac{3}{x^2}\right)$$

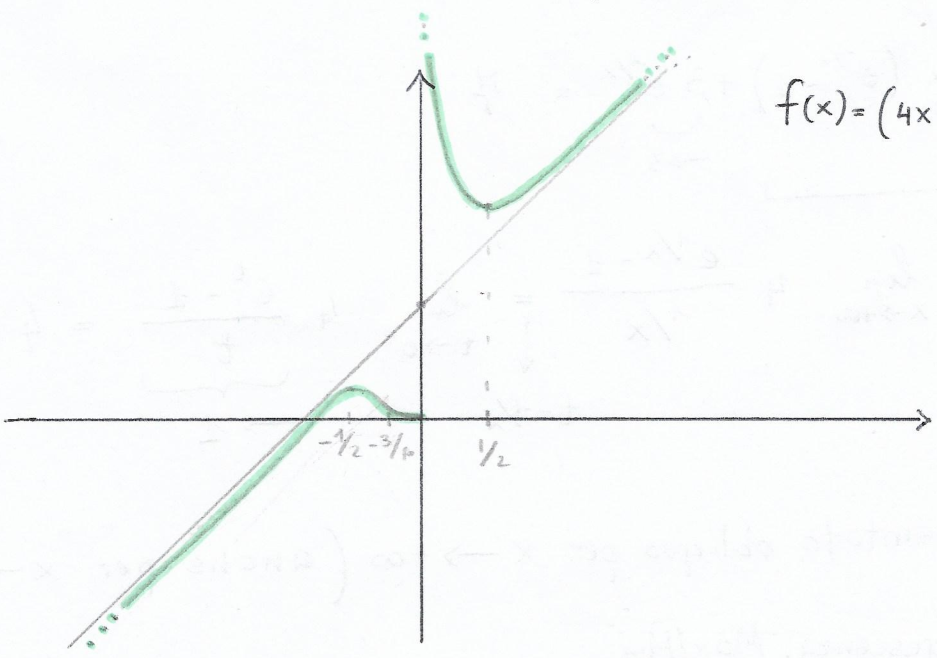
$$= -\left(4 - \frac{4}{x} - \frac{3}{x^2}\right) \frac{e^{1/x}}{x^2} + e^{1/x}\left(\frac{4}{x^2} + \frac{6}{x^3}\right) = \frac{10x+3}{x^4} e^{1/x}$$

$f''(x) \geq 0 \iff 10x + 3 \geq 0 \iff x \geq -3/10$



$f(x)$ concava in $\left(-\infty, -\frac{3}{10}\right)$

$f(x)$ convessa in $\left(-\frac{3}{10}, 0\right)$ e $(0, +\infty)$



$f(x) = (4x+3)\, e^{1/x}$

Consideriamo $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \cos x - \frac{1+ax^2}{1+bx^2} \qquad (a \in \mathbb{R},\ b>0)$$

Trovare a, b t.c. $f(x) = cx^n + o(x^n)$ con più grande possibile.

Soluzione:

Scrivo lo sviluppo di Taylor per $f(x)$

- $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{240} + o(x^6)$
- $\frac{1}{1+t} = 1 - t + t^2 - t^3 + o(t^3) \Rightarrow \frac{1}{1+bx^2} = 1 - bx^2 + b^2x^4 - b^3x^6 + o(x^6)$

$$\Rightarrow f(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + o(x^6) - (1+ax^2)\left(1 - bx^2 + b^2x^4 - b^3x^6 + o(x^6)\right)$$

$$= \cancel{1} - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + o(x^6) - \cancel{1} + bx^2 - b^2x^4 + b^3x^6 - ax^2 + abx^4 - ab^2x^6 + o(x^6)$$

$$= x^2\left(b - a - \frac{1}{2}\right) + x^4\left(ab - b^2 + \frac{1}{24}\right) + x^6\left(b^3 - ab^2 - \frac{1}{720}\right) + o(x^6)$$

Cerco a, b t.c. i primi due termini si annullino:

$$\begin{cases} b - a - \frac{1}{2} = 0 \Rightarrow b = a + \frac{1}{2} \Rightarrow b = -\frac{5}{12} + \frac{1}{2} \Rightarrow \boxed{b = \frac{1}{12}} \\ ab - b^2 + \frac{1}{24} = 0 \Rightarrow -\left(a + \frac{1}{2}\right)^2 + \left(a + \frac{1}{2}\right)a + \frac{1}{24} = 0 \end{cases}$$

$$-\cancel{a^2} - \frac{1}{4} - a + \cancel{a^2} + \frac{1}{2}a + \frac{1}{24} = 0$$

$$-\frac{a}{2} - \frac{5}{24} = 0 \Rightarrow \boxed{a = -\frac{5}{12}}$$

Ora verifico che gli a, b trovati annullino anche il termine di ordine 3:

$$\begin{cases} a = -\frac{5}{12},\ b = \frac{1}{12} \\ b^3 - ab^2 - \frac{1}{720} = 0 \Rightarrow \left(\frac{1}{12}\right)^3 + \frac{5}{12}\left(\frac{1}{12}\right)^2 - \frac{1}{720} = 0 \end{cases}$$

$$\left(\frac{1}{12}\right)^3(1+5) - \frac{1}{720} = 0 \Rightarrow \frac{6}{12^3} - \frac{1}{720} = 0 \Rightarrow \frac{1}{2\cancel{4}\cdot 12^2} - \frac{1}{720} = 0$$

$$\Rightarrow \frac{1}{288} - \frac{1}{720} = 0 \quad \text{impossibile}$$

Per $a = -\frac{5}{12}$ e $b = \frac{1}{12}$ $\Rightarrow f(x) = \left(\frac{1}{288} - \frac{1}{720}\right)x^6 + o(x^6)$

$$= \frac{1}{144}\left(\frac{1}{2} - \frac{1}{5}\right)x^6 + o(x^6)$$

$$= \frac{1}{\cancel{144}_{48}} \cdot \frac{\cancel{3}}{10} x^6 + o(x^6) = \frac{1}{480} x^6 + o(x^6)$$

Studiare il grafico di:

$$f(x) = e^{-|x|}\sqrt{x^2 - 5x + 6}$$

Soluzione:

Dominio di definizione: $x^2 - 5x + 6 \geq 0 \iff x \leq 2$ e $x \geq 3$

$\Delta = 25 - 24 = 1$

$x = \frac{5 \pm 1}{2} = \begin{cases} 3 \\ 2 \end{cases}$

$\Rightarrow D: (-\infty, 2] \cup [3, +\infty)$

Possiamo scrivere $f(x)$ come: $f(x) = e^{-|x|}\sqrt{(x-2)(x-3)}$

$\boxed{x \to 3^+}$

$f(3) = 0$

Quando $x \to 3^+$: $\left.\begin{array}{l} e^{-|x|} \sim e^{-3} \\ \sqrt{(x-2)(x-3)} \sim \sqrt{x-3} \end{array}\right\} f(x) \sim e^{-3}\sqrt{x-3}$

$\boxed{x \to 2^-}$

$f(2) = 0$

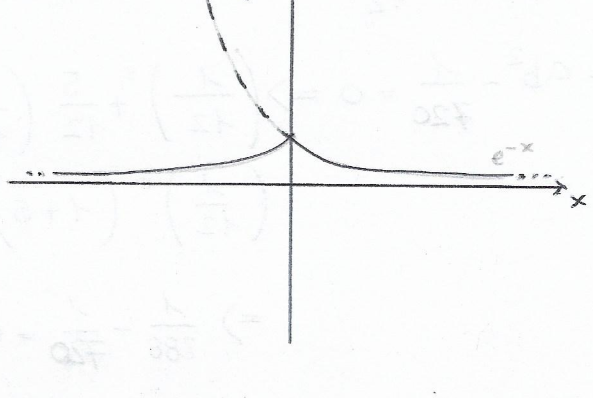
Quando $x \to 2^-$: $\left.\begin{array}{l} e^{-|x|} \sim e^{-2} \\ \sqrt{(x-2)(x-3)} \sim \sqrt{2-x} \end{array}\right\} f(x) \sim e^{-2}\sqrt{2-x}$

$\boxed{x \to 0}$

$f(x) \sim \sqrt{6}\, e^{-|x|}$

Grafico di $e^{-|x|}$

↳ funzione pari $(\varphi(x) = \varphi(-x))$



$$\lim_{x \to \pm\infty} e^{-|x|}\sqrt{x^2-5x+6} = 0$$