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Quench problems in Integrable Field Theories

Laureando
Alvise Bastianello

Relatore
Prof. Pasquale Calabrese

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Preface

In the world of many bodies quantum systems it seems natural questioning about possible equilibrium states and how the time evolution approaches these steady states. These models are usually very complex and not easy to solve, in most cases this task is impossible and some approximation are required.

A particular class of quantum systems, called integrable models, presents nice properties that provide simpler, but non trivial, probe for equilibrium and out equilibrium features in exactly solvable models; besides, recent developments in the field of ultracold atoms give access to test experimentally these theoretical models and increase the interest towards them.

In this thesis we approach integrable theories aiming to describe the equilibrium and the time evolution in the thermodynamical limit, that is to say when the size of the system is sent to infinity. In Chapters 1, 2 and 3 the necessary tools are presented.

Chapter 1 contains an overview of statistical physic that underlines the principal differences and analogies between the classical and quantum cases, in particular the role of integrals of motions is considered. At the quantum level the necessity to restrict us to some classes of observables to have relaxation is introduced, then the problem of quantum quenches is presented.

Chapter 2 introduces some particular integrable models suitable for our purpose, the construction of integrable quantum models proceeds trough a simple example, the Lieb-Liniger model, and uses it to recognize which are the basic features that the integrable models must have.

Chapter 3 considers the thermodynamic of integrable theories and introduce the basic tools to study equilibrium and out of equilibrium problems. The concept of local observable is also introduced and studied both in equilibrium and quenches situations.

The last chapters contain the original part of this thesis. Some results are already known from other works, but the approach we use seems suitable for generalizations and hopefully in future research could bring us to new non trivial results.

Chapter 4 is focused on local averages on squeezed states, a particular form of ini-

tial state. We establish a new graphical method that in the free bosonic and fermionic theories permits us to solve the entire dynamics, instead for non trivial integrable theories the expression for the steady state is obtained. The steady state was already known from others works, instead the time evolution in the free case is a new result: we hope to extend the dynamical calculation in the interacting case in future studies.

Chapter 5 is inspired by the previous one and by a recent article by Sotiriadis and Calabrese: they showed that in free bosonic systems initialized in a state that respects a property known as "cluster property", the averages of local observables proceed towards a steady value described by a GGE's density matrix. The approach they used seems not easy to extend to interacting integrable theories: guided by their results a large class of states that satisfy the cluster property has been individuated. These states appear as a natural generalization of the squeezed states analyzed in Chapter 4 and a similar graphical approach is attempted in the free bosonic case: with our method we can show that the steady state is described by a GGE as we already know, but this approach seems more suitable for a generalization to interacting theories. In particular, even if we perform the calculation only in the free bosonic case, at the end of this chapter some ideas to generalize our approach to the non trivial cases are discussed and it is our intention to proceed in this direction in future researches.

Contents

Chapter 1

Introduction

One of the most powerful tools for physicists is statistical physics: its most remarkable success is characterizing the relevant information of complex systems in terms of few meaningful parameters, even if they are governed by an enormous number of degrees of freedom.

Statistical classical physics is now a well known field, but in the quantum case we have still lots of interesting questions without a definitive answer. In particular, the problem of thermalisation attracted lots of theoretical and experimental work in the last years.

In this chapter we give a brief overview of statistical physics at classical and quantum level, in particular we stress the role of the integrals of motion, both in the classical and in the quantum case, in order to motivate the subsequent study of integrable theories. We also introduce quantum quenches.

1.1 Classical Statistical Physiscs

In this section we will give a brief overview of classical statistical mechanics: we will deal only with the equilibrium state, leaving to the references the difficult problem of thermalisation.

Imagine a system completely characterized by N coordinates $(q_1, q_2, .., q_N)$ and their conjugate momenta $(p_1, p_2, ..p_N)$; an Hamiltonian $H = H(\vec{q}, \vec{p})$ rules the dynamics. We can unambiguously identify the state of the system with a point in a $2N$ dimensional space formed by the couples (q_i, p_i): this space is called the Phase Space.

We can also give a probability distribution $\rho(\vec{q}, \vec{p})$ over the phase space and define the averages of observables through it:

$$\langle f \rangle_\rho = \int d^N q d^N p \, f(\vec{q}, \vec{p}) \rho(\vec{q}, \vec{p}), \qquad \int d^N q d^N p \, \rho(\vec{q}, \vec{p}) = 1 \tag{1.1}$$

Suppose we initialize the system in a particular state $q_i = q_i(0)$, $p_i = p_i(0)$ and let it evolve $q_i \to q_i(t)$, $p_i \to p_i(t)$; then consider $\rho_t = \rho(\vec{q}, \vec{p}, t)$ the evolution of the initial distribution and in particular its limit when $t \to \infty$.

An interesting question could be the following: does the infinite time limit exist? Is it independent from the initial conditions?

Suppose for now an affirmative answer, then for all the observables we would have $\lim_{t\to\infty} \langle f \rangle_{\rho_t} = \langle f \rangle_{\tilde{\rho}}$, with $\tilde{\rho}$ the equilibrium density of probability.

In this case we would describe the long time behavior of the system through only a distribution, independently of the initial conditions. Unfortunately this statement is not true, as a matter of fact we can state the following differential equation for ρ [1](chap. 6B):

$$\partial_t \rho = -\mathcal{H}\rho, \qquad \mathcal{H} = \sum_i \partial_{p_i} H(\vec{p},\vec{q})\partial_{q_i} - \partial_{q_i} H(\vec{p},\vec{q})\partial_{p_i} \tag{1.2}$$

Notice that the $\mathcal{H}$ operator is antihermitian: this implies that the evolution is unitary and in particular ρ will not have a well defined $t \to \infty$ limit because of the absence of dissipative effects.

Definition: A system is called Ergodic if the time average can be substituted with integration on the set $H(\vec{q},\vec{p}) = E$ constant:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T dt \, \langle f \rangle_{\rho_t} = \langle f \rangle_{\tilde{\rho}} \tag{1.3}$$

where $\tilde{\rho}$ is different from zero only on the manifold identified by $H(\vec{q},\vec{p}) = E$ and is flat over this subset. For a complete discussion of ergodic systems see [3] (chap. 15).

The idea of substituting temporal averages with averages on the phase space is the basic assumption of statistical physiscs, this approach is fully justified for ergodic systems, but it is not so easy to understand when a system is ergodic.

We can notice that if the system possesses an integral of motion I independent from the Hamiltonian, then the ergodicity is broken due to the constraint $I = constant$ during the motion.

This is not the entire story: when we use statistical physics we claim that the quantities we calculate are exactly the observed ones, without referring to any temporal average. We must consider that each measure is performed over a temporal interval, so we cannot pinpoint the exact value of the observable, but only temporal averages over a small time lapse τ: we can imagine that the time interval τ is large enough to have the time average equal to the statistical average.

We can outline the general approach that is used, some of these ideas have a counterpart in the quantum case.

Usually we consider systems coupled to a thermal bath, for this reason suppose a closed system divided in two subsystems, the first with $2n$ degrees of freedom $(q_1,p_1),..(q_n,p_n)$ and the second with $2N$ degrees of freedom $(q_{n+1},p_{n+1}),..(q_{n+N},p_{n+N})$, then we focus on observables that are functions only of the first $2n$ degrees of freedom.

We will suppose that we can implement in some meaningful way the limit $N \to \infty$: to implement this limit we need to imagine that the Hamiltonian is extensive over the two systems a part from some corrections decaying for $N \to \infty$. We will not be more precise.

Suppose $f(\vec{q}_n, \vec{p}_n)$ an observable for the reduced system, initialize the whole system in an initial state and let it evolve. Consider now the evolution of the observable $f(\vec{q}_n(t), \vec{p}_n(t))$, then we assume the following properties:

$$\begin{gathered} \exists\, \rho(\vec{q}_n, \vec{p}_n) \quad s.t.\ \forall f(\vec{q}_n, \vec{p}_n), \forall \tau > 0 \\ \lim_{t\to\infty} \lim_{N\to\infty} \frac{1}{\tau} \int_t^{t+\tau} dt' f(\vec{q}_n(t'), \vec{p}_n(t')) = \int d^n q\, d^n p\, \rho(\vec{q}_n, \vec{p}_n) f(\vec{q}_n, \vec{p}_n) \end{gathered} \tag{1.4}$$

When N is fixed the limit $t \to \infty$ is usually not well defined, but we hope to recast a sort of dissipative motion with a precise steady state through the thermodynamic limit $N \to \infty$.

If we suppose that the previous statement is true, that the whole system is ergodic and that the Hamiltonian is extensive over the two subsystems, then it can be shown that the density distribution must be of the Gibbs type:

$$\rho(\vec{q}_n, \vec{p}_n) \propto e^{-\beta H_n(\vec{q}_n, \vec{p}_n)} \tag{1.5}$$

where H_n is the Hamiltonian referred only to the small subsystem. The derivation can be found in all the basic texts of statistical mechanics, see for example [2] (chap. 7.1).

We will not comment further the problem of thermalisation, instead we will suppose the system reaches effectively a steady state and aim to describe immediately its asymptotic behavior.

We can argue in this way: suppose a system is described by a set of states, for simplicity imagine a discrete set labeled with integers i, then assign a probability distribution ρ_i. What is the best candidate to describe the equilibrium situation? We would like to select the "most probable distribution", but before we must give a quantitative definition of "most probable".

Definition[3]: Define the set $\Delta^n = \{(p_1, p_2, .., p_n) \in \mathbb{R}^n \ s.t.\ \sum_i p_i = 1\}$, then we will call a function $S : \Delta^n \to [0; +\infty), \quad S : (p_1, p_2..p_n) \to S^n(p_1, p_2, ..p_n) \in \mathbb{R}$ an entropy if it satisfies the following requirements [3]:

1. It is symmetric in its variables.
2. $S^n(1, 0, 0...0) = 0$.
3. $S^n(0, p_2, p_3, ..., p_n) = S^{n-1}(p_2, .., p_n)$.
4. $S^n(p_1, .., p_n) \leq S^n(\frac{1}{n}, .., \frac{1}{n})$ and $S^n(p_1, .., p_n) = S^n(\frac{1}{n}, .., \frac{1}{n}) \Leftrightarrow \forall i\, p_i = \frac{1}{n}$.

5. Suppose $i \in (1,..,m), j \in (1,..,n)$, $\pi_{i,j} \in \Delta^{mn}$, $p_j \in \Delta^n$, then:

$$S^{nm}(\pi_{1,1}, \pi_{1,2}.., \pi_{2,1}..\pi_{m,n}) = S^n(p_1,..,p_n) + \sum_j p_j S^m\left(\frac{\pi_{j,1}}{p_j}, \frac{\pi_{j,2}}{p_j}, .., \frac{\pi_{j,m}}{p_j}\right)$$

Given a probability distribution $\rho_i, i \in (0,..,n)$ we can consider $S^n(\rho_1, ..\rho_n)$.

We claim that the function S quantifies the "information" contained in the distribution ρ: the requirement (1.) is obvious, (2.) states that, if an event is certain (so there exists i *s.t.* $\rho_j = \delta_{i,j}$) S^n is zero. The entropy has its maximum for the most "uncertain" case, that is to say $\rho_i = \frac{1}{n}$ (4.).

(3.) states that an impossible event ($\rho_i = 0$) does not give further information, finally (5.) is motivated by the conditional probability: suppose that $\pi_{i,j}$ represents the probability of two events A_i and B_j, instead p_i is the probability that A_i is verified. Then $prob(B_j|A_i) = \frac{\pi_{i,j}}{p_i}$: with this observation (5.) becomes a natural requirement.

Theorem [3]: $S : \Delta^n \to [0; +\infty)$ $S : (p_1, p_2..p_n) \to S^n(p_1, p_2, ..p_n) \in \mathbb{R}$ is an entropy if and only if

$$S^n(p_1, ..p_n) \propto -\sum_i^n p_i \ln(p_i) \tag{1.6}$$

The function $S^n(p_1, ..p_n) = -\sum_i^n p_i \ln(p_i)$ is called Shannon Entropy [3].

The proof, with a complete discussion of the entropy, can be found in [3] (chap. 13.6). The Entropy is the best candidate to measure the information contained in a probability distribution, in particular the less is the information contained in p_i, the higher is the value of S. For a discussion of entropy from an information theory point of view see [4] (chap. 11.1, 11.2, 12.2.1).

We can define an analogous of the Shannon Entropy in the continuous case, that is to say suppose $\rho : \mathbb{R}^n \to \mathbb{R}^+$, $\int d\vec{x}\rho(\vec{x}) = 1$.

$$S[\rho] = -\int d\vec{x}\rho(\vec{x}) \log(\rho(\vec{x})) \tag{1.7}$$

Now we can be more quantitative when asking for the most probable density of probability: we evaluate the probability that contains the minimum amount of information maximizing S. It is obvious from point (4.) of the definition of S that the probability that maximizes S is the flat one.

This matches with the Ergodic Hypothesis: the most probable density distribution is flat on the allowed regions of the space of phases and is the density distribution we would use to perform time averages and asymptotic limits of observables, if the latter exists.

We can ask ourselves another question: suppose the system has its energy not exactly constrained, but its average is still fixed. We can look for the distribution that maximizes the entropy with this constraint through the functional equation:

$$\frac{\delta}{\delta\rho}\left(S[\rho] - \lambda\int d^n q\, d^n p\, \rho(\vec{q}_n,\vec{p}_n) - \beta\int d^n q\, d^n p\, \rho(\vec{q}_n,\vec{p}_n)H(\vec{q}_n,\vec{p}_n)\right) = 0 \qquad (1.8)$$

$$\Rightarrow \rho \propto e^{-\beta H}$$

here λ and β are two Lagrangian multipliers. We recognize the Maxwell Boltzmann distribution that rules the Canonical Ensemble and we can give a brief comment on what is happening.

The meaning of the hypothesis "the energy is not fixed, but only its average" can be read in this way: we are considering the system coupled with a larger one (a thermal bath) that makes the system proceeding towards a configuration with an almost definite energy. This makes sense if the Hamiltonian is extensive over the entire set of degrees of freedom, so a possible approach seems to be considering an effective Hamiltonian for the system in which the mean interaction with the bath is approximated with a fluctuating driving term and a dissipative contribution. For a useful discussion we could consider the Brownian Motion, for example see [1] (chap.5).

We can generalize what we have just said and suppose that, on average, a set of charges I_j is conserved. As before we can proceed maximizing the entropy and arrive at the result:

$$\frac{\delta}{\delta\rho}\left(S[\rho] - \lambda\int d^n q\, d^n p\, \rho(\vec{q}_n,\vec{p}_n) - \sum_j \beta_j\int d^n q\, d^n p\, \rho(\vec{q}_n,\vec{p}_n)I_j(\vec{q}_n,\vec{p}_n)\right) = 0 \quad (1.9)$$

$$\Rightarrow \rho \propto e^{-\sum_j \beta_j I_j}$$

This approach can lead immediately to the Grand Canonical Ensemble: since now we have used Phase Spaces with a fixed number of particles n, call it Γ_n. It is a trivial generalization to define a larger Phase Space through a direct product $\Gamma = \times_n \Gamma_n$: at this point we can consider distributions and entropies in this larger phase space and require that the number of particles is fixed on average. We arrive at the Grand Canonical distribution:

$$\rho \propto e^{-\beta H + \beta\mu n} \qquad (1.10)$$

At this point we need to think about some general physical requirements that the observables I_j must satisfy. If our hypothesis of considering the system as coupled to a thermodynamical bath is correct, we can suppose that the dynamics of the system around the equilibrium point is mostly governed by the Hamiltonian H. The objective of ρ is to describe a steady situation, so ρ must not evolve through the

time. This leads us to consider ρ as a function only of integrals of motion (in fact H and N are integrals of motion in the Grand Canonical case). This requirement is not enough: following this reasoning we could consider observables as H^2, H^3 and these are all integrals of motion.

If we work at fixed number of degrees of freedom and with the hypothesis that the Hamiltonian is the only integral of motion, we want to recast the Canonical Ensemble: we need to exclude terms such as H^n.

The fact is that only extensive integrals of motion must enter in the probability distribution, this permits us to satisfy the following physical requirement: consider a large system divided in two subsystems, suppose N_1 and N_2 are the numbers of degrees of freedom of the two subsystems, then call $N = N_1 + N_2$. Call ρ_N the probability distribution regarding the whole system and ρ_{N_1}, ρ_{N_2} the distributions restricted to the two subsystems. Then we expect that if we perform the limit $N_1, N_2 \to \infty$ the whole distribution factorizes $\lim_{N_1,N_2\to\infty} \rho_N = \rho_{N_1}\rho_{N_2}$. Since the Hamiltonian is supposed to be an extensive quantity, we have:

$$H = H_1 + H_2 + corrections \tag{1.11}$$

where H_1 and H_2 are the reduced Hamiltonians of the two subsystems, the corrections are suppressed for $N_1, N_2 \to \infty$.

In this case we have:

$$\rho_N \propto e^{-\beta H} \simeq e^{-\beta H_1} e^{-\beta H_2} \propto \rho_{N_1}\rho_{N_2} \tag{1.12}$$

This is not true if instead of H we use H^2 and so on, for this reason we decide to reject all the non-extensive integrals of motion.

So we expect that if a steady state is reached its probability distribution must be in the form:

$$\rho \propto e^{-\sum_j \beta_j I_j} \tag{1.13}$$

where I_j are all the extensive integrals of motion.

1.2 Quantum Statistical Physic

Extending the previous discussion to the quantum case is not an easy task. In the quantum case we do not have the phase space's concept and the equivalent of the ergodic theorem is not trivial. As in the classical case, a closed quantum system will never relax to a steady state because of the unitarity of the time evolution.

We still can say something on the time averages: suppose the system is initialized in the state $|\psi_0\rangle$, then

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T e^{-iHt} |\psi_0\rangle \langle\psi_0| e^{iHt} = \lim_{T\to\infty} \frac{1}{T} \int_0^T \sum_{n,m} e^{-i(E_n - E_m)t} \langle n|\psi_0\rangle \langle\psi_0|m\rangle |n\rangle \langle m| \tag{1.14}$$

If we suppose that is correct to exchange the summation and the average and that we do not have any degeneracy in the energy levels, we will end up with the diagonal ensemble ($c_n = \langle n|\psi_0\rangle$):

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T e^{-iHt} |\psi_0\rangle \langle\psi_0| e^{iHt} = \sum_n |c_n|^2 |n\rangle \langle n| \tag{1.15}$$

We can immediately observe that, if we look for an equivalent of the ergodic theorem over the density matrices we will surely fail, as a matter of fact the coefficients c_n strongly depend over the initial state $|\psi_0\rangle$: this is not the right approach.

In the classical case, when we looked for relaxation to a steady state, we restricted our observables to a subsystem: we can ask if there exists any restriction over the set of observables Θ that permits us to say that there is a density matrix ρ such that

$$\forall |\psi_0\rangle \quad \lim_{T\to\infty} \frac{1}{T} \int_0^T tr\left(\Theta e^{-iHt} |\psi_0\rangle \langle\psi_0| e^{iHt}\right) = tr(\Theta\rho) \tag{1.16}$$

This problem was first considered by Von Neumann [5], others techniques and results are instead available in [6] where systems with a random perturbation added to the Hamiltonian are considered. In [7] the Berry's conjecture is supposed to hold, that is to say a kind of average over the initial conditions is used.

We do not proceed any further in the general discussion of thermalisation, instead we will discuss the equilibrium situation through the quantum version of Entropy.

Given a density matrix ρ we define its Entropy (Von Neumann Entropy) [4] as:

$$S_N(\rho) = -tr(\rho \log(\rho)) \tag{1.17}$$

For the general discussion of this quantity see [4] (chap. 11.3) and for its interpretation in information theory [4] (chap 12.2.2). The only point we underline is that in the basis that diagonalizes $\rho = \sum_n \lambda_n |n\rangle \langle n|$ we find the classical entropy of the λ_n distribution, this stresses the link between the classical and quantum case:

$$S_N(\rho) = -tr(\rho \log(\rho)) = -\sum_n \lambda_n \log(\lambda_n) \tag{1.18}$$

As before we can ask ourselves which is the best candidate to describe the steady state situation, supposing that exists. We can interpret the Von Neumann entropy as a measure of the information contained in the density matrix and maximize S_N.

If we suppose to respect some constraints that fix the averages of a set of observables Q_i we have to impose:

$$\frac{\delta}{\delta\rho}\left(S_N(\rho) - \lambda tr(\rho) - \sum_i \beta_i tr(Q_i\rho)\right) = 0 \tag{1.19}$$

$$\Rightarrow \rho \propto e^{-\sum_i \beta_i Q_i}$$

If the system is closed and we do not fix the average of any quantity, we would obtain the microcanonical quantum ensemble: a flat distribution over the eigenstates of the Hamiltonian with fixed energy, let $|n\rangle$ be the eigenstates of H:

$$\rho_{micr} \propto \sum_{E-\frac{\Delta}{2}<E_n<E+\frac{\Delta}{2}} |n\rangle \langle n| \tag{1.20}$$

Instead if we consider an open quantum system the energy is not strictly fixed, but its average is still conserved and we obtain the canonical ensemble:

$$\rho_{can} \propto e^{-\beta H} \tag{1.21}$$

The correct way to obtain the canonical ensemble is to consider an open quantum system, but as in the classical case we could have that the mean of observables can be evaluated consistently with both density matrices, after that the infinite volume limit is performed.

As in the classical case, we have the problem of selecting which kind of observables can be inserted in the density matrix $\rho \propto e^{-\sum_j \beta_j Q_j}$. Surely, if we want that ρ describes a steady state, we need that all the Q_i are integrals of motion:

$$[H, Q_i] = 0 \qquad \forall i \tag{1.22}$$

We cannot proceed any further: in the quantum case we do not have the notion of "degrees of freedom" so the concept of extensive quantity is not so clear. We can still imagine that something could be said over the commutativity relations of the operators Q_i, as a matter of fact if we suppose $[Q_i, Q_j] = 0$ we would have $\rho \propto \prod_j e^{-\beta_j Q_j}$ regardless to the order of the Q_j. Remembering that in the classical case the commutation relations have to be replaced by the Poisson Brakets, it seems natural to study the integrable classical theories and their quantum counterpart. We will see them in Chapter 2.

1.3 Quantum systems out of equilibrium and quenches

In the previous section we discussed about equilibrium situations in quantum systems: at this point the next natural step is trying to understand if this equilibrium is reached and which conditions have to be fulfilled to have this behavior. To accomplish this task we can perturb a system and put it out of equilibrium, then we study its evolution and in particular look if a new equilibrium is reached and then analyze its characteristics.

A standard way to drive a system out of equilibrium is through global quantum quenches: the system is initialized in a known state, usually the ground state, then the coupling constants of the Hamiltonian are changed in time. In this way the initial ground state is no more an eigenstate of the Hamiltonian and the system has a non trivial evolution.

Quantum quenches stimulated lots of theoretical work, see for example [38], [39], [40], [41] and in recent years their predictions become experimentally testable in the field of ultracold atoms, for a rich overview of these topics see [23], [44].

In the following we will focus on global sudden quenches: the coupling constants are suddenly changed at $t = 0$ and then kept fixed.

We decide to introduce the problem through the simpler example we could imagine, that is to say a one dimensional free bosonic relativistic theory, since many key points are embedded in it.

Suppose to have a field Φ as follows:

$$\begin{aligned} &\Phi(x,t) = \int dk \frac{1}{\sqrt{4\pi E(k)}} \left(a^\dagger(k) e^{ikx - iE(k)t} + a(k) e^{-ikx + iE(k)t}\right) \\ &[a(k), a(q)] = 0, \qquad [a(k), a^\dagger(q)] = \delta(k-q); \qquad E(k) = \sqrt{m^2 + k^2} \end{aligned} \tag{1.23}$$

We will suppose that at the initial time $t = 0$ the system is in the ground state $|0_a\rangle$ of the theory, then the mass parameter is changed $m \to \mu$. After the quench the theory is still free and can be solved in terms of standard bosonic operators:

$$\begin{aligned} &\Phi(x,t) = \int dk \frac{1}{\sqrt{4\pi \xi(k)}} \left(b^\dagger(k) e^{ikx - i\xi(k)t} + b(k) e^{-ikx + i\xi(k)t}\right) \\ &[b(k), b(q)] = 0, \qquad [b(k), b^\dagger(q)] = \delta(k-q); \qquad \xi(k) = \sqrt{\mu^2 + k^2} \end{aligned} \tag{1.24}$$

The continuity condition for the field at $t = 0$ permits us to link $a(k)$ to $b(k)$ through a Bogoliubov transformation:

$$\begin{aligned} &\begin{pmatrix} b(k) \\ b^\dagger(-k) \end{pmatrix} = \begin{pmatrix} u(k) & v(k) \\ v(k) & u(k) \end{pmatrix} \begin{pmatrix} a(k) \\ a^\dagger(-k) \end{pmatrix} \\ &u(k) = \frac{1}{2}\left(\sqrt{\frac{\xi(k)}{E(k)}} + \sqrt{\frac{E(k)}{\xi(k)}}\right) \qquad v(k) = \frac{1}{2}\left(\sqrt{\frac{\xi(k)}{E(k)}} - \sqrt{\frac{E(k)}{\xi(k)}}\right) \end{aligned} \tag{1.25}$$

This relation permits us to write the initial state in terms of the post quench operators and vacuum $|0_b\rangle$:

$$a(k)\,|0_a\rangle = 0 \Rightarrow u(k)b(k) - v(k)b^\dagger(-k)\,|0_a\rangle = 0 \tag{1.26}$$

$$\Rightarrow |0_a\rangle \propto e^{\int dk \frac{v(k)}{u(k)} b^\dagger(k) b^\dagger(-k)}\,|0_b\rangle \tag{1.27}$$

States such as (1.27) are called squeezed states and we will study them in Chapter 4 in a more general contest. At this point the evolution of $|0_a\rangle$ can be easily computed, but since the evolution is unitary we cannot observe any relaxation directly on the state.

As we have already discussed in section 1.2, we could observe relaxation phenomena if we consider averages of some suitable observables, so the next natural step could be evaluating the averages of the field, for example the following quantities:

$$\langle 0_a| \left[\Phi(x,t)\right]^n |0_a\rangle \tag{1.28}$$

These quantities are easily computed in Heisenberg representation and we observe relaxation, for example we have:

$$\begin{aligned} \lim_{t\to\infty} \langle 0_a| \left[\Phi(x,t)\right]^2 |0_a\rangle &= \int dk\, \frac{1}{4\pi\xi(k)} \left(1+2v^2(k)\right) \\ \lim_{t\to\infty} \langle 0_a| \left[\Phi(x,t)\right]^4 |0_a\rangle &= 3\left[\int dk\, \frac{1}{4\pi\xi(k)} \left(1+2v^2(k)\right)\right]^2 \end{aligned} \tag{1.29}$$

With other checks we would find that these averages relax to steady quantities and the only information of the initial state we need to compute the long time averages is the coefficient $v^2(k)$. Notice that in this case $v^2(k) = \frac{1}{L}\langle 0_a| b^\dagger(k)b(k) |0_a\rangle$, where L is the length of the system introduced to regularize the factor $\delta(0)$.

It seems from our example that quenches could give steady averages for $[\Phi(x,t)]^n$ completely characterized by the post quench density of excitations.

We would have obtained different answers if we test other observables, for example the two particles correlator in the Fourier space $\langle b^\dagger(p)b^\dagger(k)b(p)b(k)\rangle$ does not show any relaxation: the fact is that the correlators of the field are local quantities, instead the correlators in momentum space are non local in the coordinate space. This it is not the rigth moment for a discussion on local operators and we postpone it to Chapter 3 in the more general framework of integrable theories, in this section we limit ourselves to few comments.

We already know that a closed quantum system cannot truly relax to a steady state because its time evolution is unitary, but local observables are quantities related only to a small portion of the system, so we can imagine to divide the whole system in two parts: the small part is probed with local observables, instead the other one behaves like a thermal bath for the first one and this can produce relaxation.

Things are different with correlators in the momentum space: these quantities are not localized in space and explore the whole system at once, so we can not separate the analogue of the thermal bath.

Since now we have found a class of observables whose averages proceed towards steady values, but we have not faced the problem of describing this equilibrium yet.

A first trial can be done with a Gibbs density matrix, but it does not work:

$$\begin{aligned} &\rho_{Th} \propto e^{-\beta H} \qquad\qquad \langle b^\dagger(k)b(k)\rangle_{Th} = \frac{1}{e^{\beta\xi(k)}-1} \\ &\langle[\Phi(x)]^2\rangle_{Th} = \int dk\, \frac{1}{4\pi\xi(k)} \left(1+2\langle b^\dagger(k)b(k)\rangle_{Th}\right) \\ &\langle[\Phi(x,t)]^4\rangle_{Th} = 3\left[\int dk\, \frac{1}{4\pi\xi(k)} \left(1+2\langle b^\dagger(k)b(k)\rangle_{Th}\right)\right]^2 \end{aligned} \tag{1.30}$$

These expressions are similar to eq. (1.29), if we substitute $\langle b^\dagger(k)b(k)\rangle_{Th} \to v^2(k)$. This also suggests the solution: consider the thermal density matrix

$$\rho_{Th} \propto e^{-\beta \int dk \xi(k) b^\dagger(k) b(k)} \tag{1.31}$$

and admit the possibility of substituting the energy with an arbitrary function $f(k)$, $\beta\xi(k) \to f(k)$. We obtain the following density matrix:

$$\rho_{Th} \to \rho_{GGE} \propto e^{-\int dk f(k) b^\dagger(k) b(k)} \tag{1.32}$$

this choice for the density matrix is called Generalized Gibbs Ensemble and will be extensively studied in Chapter 3. At this point we can choose the f function in this way:

$$\langle b^\dagger(k)b(k)\rangle_{GGE} = \frac{1}{e^{f(k)} - 1} = v^2(k) \tag{1.33}$$

The density matrix ρ_{GGE} seems to fulfill the task of describing the steady values of the averages of the field. In the passages above we constructed the density matrix ρ_{GGE} by hand, but at this point we would like to link it with the discussion in section 1.2, in particular with the role of integrals of motion.

In a free theory the operators $b^\dagger(k)b(k)$ are conserved, this permits us to determine an infinite set of conserved quantities:

$$Q_q = \int dk\, q(k) b^\dagger(k) b(k) \tag{1.34}$$

with $q(k)$ an arbitrary function. All these charges commute with each others: following the discussion of section 1.2, to describe the equilibrium situation we would have tried the following density matrix

$$\rho_{GGE} \propto e^{-\sum_q \beta_q Q_q} \Rightarrow \rho_{GGE} \propto e^{-\int dk f(k) b^\dagger(k) b(k)} \tag{1.35}$$

that we saw fulfill the task.

At the end of section 1.2 we faced the problem of choosing which conserved quantities have to be inserted in the density matrix, as a matter of fact a quantum system has always lots of conserved operators: for example the projectors on the eigenstates of the Hamiltonian. Observables such as (1.34) are somewhat special and we will have some comments.

Using the fact that the free theory can be constructed over the multiparticles states, we can introduce the concept of extensive integrals of motion. Set $|\vec{k}\rangle = b^\dagger(k_1)b^\dagger(k_2)..\,|0_b\rangle$:

$$Q_q\,|\vec{k}\rangle = \left(\sum_{k_i \in \vec{k}} q(k_i)\right) |\vec{k}\rangle \tag{1.36}$$

We notice that the action of Q_q on multiparticles states is simply the sum of the action on one particle states: since in the classical case the notion of extensive

integrals of motion has a special role, we could expect that also in the quantum case they are somewhat special.

The second comment regards the locality: since we aim to describe the equilibrium of local observables, we need to consider density matrices constructed with local quantities. It is possible to show that this kind of charges in the space representation are local quantities, but we postpone this discussion to Chapter 2 in a more general contest, in particular to section 2.3.1.

Even if this is only a trivial example, it embeds some of the basic features of a more general class of models, that is to say the integrable theories. We will introduce these models in Chapter 2, then equilibrium and quenches problems are extensively studied in Chapter 3.

1.4 Experiments with cold atoms

In recent years the progress in the field of ultra cold atomic gases has given access to test experimentally some quantum many-body systems. In particular the parameters that rule the dynamic of these systems are often well controlled, with the subsequent possibility of driving the systems out of equilibrium through quantum quenches. In particular, with a suitable external potential it is often possible to lower the effective dimension of the system, exploring the characteristic features of low dimensional systems.

Among all the experiments done in recent years, we select two of them: the first is an experimental realization of the Lieb-Liniger model that we will study in Chapter 2, the second one regards the behavior of correlation functions in the Bose-Hubbard model; other interesting experiments are [31], [32], but for a general and rich review of this topic we refer to [23], [44] and references therein.

1.4.1 A quantum Newton's cradle

The experiment that we are going to illustrate is almost a realization of the Lieb-Liniger model, for references see [21]. The authors used a blue detuned 2D optical lattice to create arrays of tightly confined ultracold ^{87}Rb atoms, the energy of the trapping potential was high compared to the energy of the atoms, in order to eliminate tunneling effects and construct a set of one-dimensional systems. An external harmonic trap was settled along the tubes in order to simulate the finite volume conditions, then the array of tubes was placed in a superposition of states of opposite momenta by the application of a transient optical phase grating. At this point the atoms divide themselves in two clouds of opposite momenta and start to oscillate around the center of the harmonic trap.

At regular time intervals the position of the atoms in some tubes was measured, in this way the authors were able to construct the evolution of the two clouds of atoms and they observe that these wavepackets conserved their identity with the same momentum after thousands of oscillations. This means, in particular, that

in the experiment's time scale the system does not reach a gaussian momenta's distribution, so the steady state is not described by a thermal distribution.

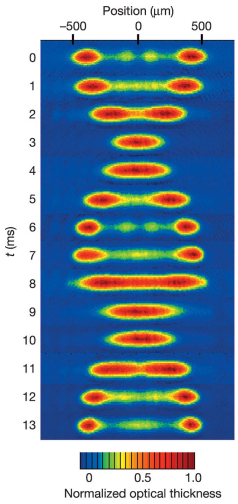


Figure 1.1: Position of the wavepackets, image taken from [21]

This behavior can be explained if we suppose that the atoms interact only through a two-body scattering due to a local potential, this in particular would imply that, except for the substitution of the finite volume conditions with an harmonic trap, the system is a good approximation of the Lieb-Liniger model: in particular the momenta are conserved trough the evolution.

For comparison, the same experiment has been repeated without the one dimensional confinement, so the system explored all the three spatial dimensions: in this case the authors observed that after few oscillations the density of momenta acquired a gaussian shape, that is to say the momentum distribution can be described through a thermal density matrix.

1.4.2 Light-cone-like spreading of correlations in a quantum many-body system

The authors of [22] trapped some cold atoms of ^{87}Rb in a set of equal 1D systems using optical lattice's techniques. In particular the 1D system was well approximated by a system of bosons on a lattice described by the Bose-Hubbard model:

$$H = \sum_j \left[-J(a_j^\dagger a_{j+1} + c.c.) + \frac{U}{2} a_j^\dagger a_j (a_j^\dagger a_j - 1) \right] \tag{1.37}$$

Basically it is a system of bosons trapped in an harmonic potential with some probability to move to the nearest sites plus a local two particles interaction. The ratio $\frac{U}{J}$ distinguishes two different phases: for $\frac{U}{J} \gg 1$ the eigenstates become well lo-

calized (this is the Mott-Insulator phase), instead when $\frac{U}{J} \ll 1$ the coupling between particles is almost removed and the states become delocalized.

The authors of the article prepared the system in the Mott-Insulator phase with an atom in each site, then they made a quench shifting $\frac{U}{J}$ near the phase transition's point and let the system evolve. They measured the observable $C_d(t) = \langle a_j(t) a_{j+d}(t) \rangle - \langle a_j(t) \rangle \langle a_{j+d}(t) \rangle$ at different times and what they found is quite interesting: the correlation function exhibits a finite propagation speed, as we can see in the picture.

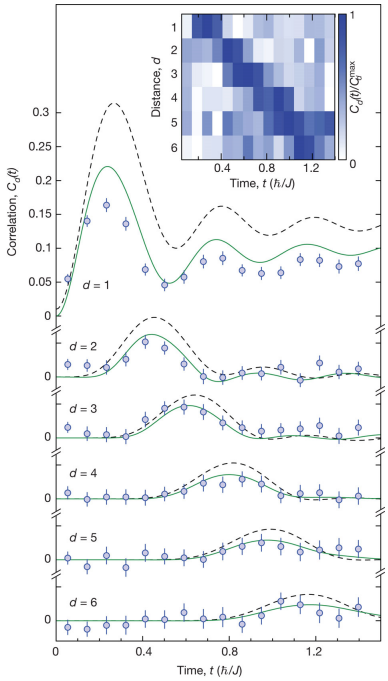


Figure 1.2: Evolution in time of correlator function, image taken from [22]

Chapter 2

Integrability

This section is devoted to present the concept of integrability: we will start with the notion of classical integrability to have some insight in the quantum case. The correct definition of quantum integrability is still debated [10], so we will proceed through an example to understand the main features that we require for a quantum integrable theory, then we will construct a definition suitable for our purpose.

As we already said, the presence of integrals of motion spoils the ergodic theorem and in general the thermalisation of the system, so it could be interesting to analyze what happens both in the classical case and in the quantum one.

2.1 Classical Integrability

We will start directly with the definition of integrability in classical systems, following [3] (chap. 11.4).

Definition[3]: A completely integrable system satisfies these requirements:

- The Hamiltonian of the system is time independent $H = H(\vec{p}, \vec{q})$.
- Suppose that the system has n coordinates $\vec{q} = (q_1, q_2, .., q_n)$ and there exist $I_1, I_2, ..I_n$ integrals of motion that are independent and in involution

$$\begin{cases} \sum_j \alpha_j \frac{\partial I_j}{\partial q_i} = 0 & \forall q_i \\ & \iff \alpha_j = 0 \quad \forall j \\ \sum_j \alpha_j \frac{\partial I_j}{\partial p_i} = 0 & \forall p_i \end{cases} \tag{2.1}$$

$$\{I_i, I_j\} = 0 \quad \forall i, j$$

where $\{\ ,\ \}$ are the Poissons brakets.

An integrable system has some nice properties, stated by two theorems due to Liouville and Arnol'd. The theorem due to Liouville states the possibility of finding a canonical trasformation such that the new conjugated momenta are the integrals of motion [3] (chap. 11), the theorem due to Arnol'd asserts that, if the manifold defined by the constraints $I_j(\vec{p},\vec{q}) = a_j$ is compact and connected, then is diffeomorphic to a n-dimensional thorus [3] (chap. 11).

Combining the two theorems we arrive at the conclusion that for completely integrable systems there exists a canonical transformation $(\vec{q},\vec{p}) \to (\vec{Q},\vec{P})$ such that the P_j are constants of motion and $Q_j \in S^1$ are periodic.
We have already said that when a system possesses an integral of motion independent from the Hamiltonian then the system cannot be ergodic, in particular the completely integrable systems are not ergodic.

It seems natural studying what happens to the ergodicity if we spoil the integrability.

Definition [3]: A system is said nearly integrable if its Hamiltonian can be split $H = H_0 + \epsilon h$ with H_0 a completely integrable Hamiltonian and ϵh a small perturbation.

The KAM theorem states that, if the perturbation is small enough, the torii of the completely integrable Hamiltonian are deformed by the perturbation, but quasi-periodic motions survive: in this case we have not ergodicity.

So even if the system does not possess integrals of motion different from the Hamiltonian, its evolution could be non ergodic.

2.2 Introduction to quantum integrability: the Lieb-Liniger model

We think that the easiest way to approach quantum integrability is through an example: we are going to consider a simple classical model and then define its quantum counterpart. Suppose a Newton's cradle, that is to say a one dimensional system of n identical particles interacting with hard core scattering.

We will suppose point-like particles on a circle interacting only via two bodies scattering. Under these hypothesis we have a number of conserved quantities equal to the number of particles, so equal to the number of degrees of freedom: in fact two identical bodies in one dimension can only exchange their momenta because of the conservation of total momentum and energy.

For this reason if we suppose a set of initial conditions $\{p_i'\}$ we will still have the same set of momenta during the entire evolution, this implies that the following quantities are conserved:

$$I_j = \sum_i^n p_i^j \qquad j \in \{1, 2.., n\} \tag{2.2}$$

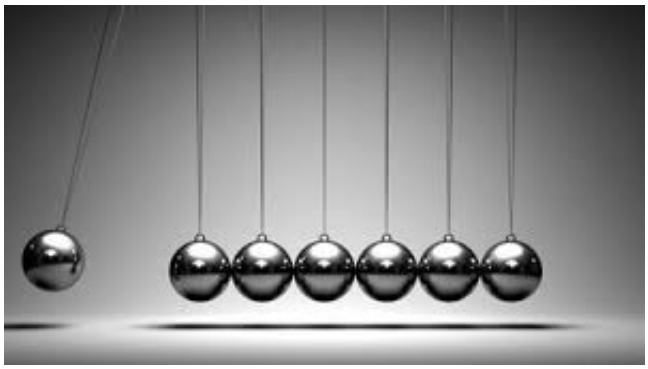


Figure 2.1: A Newton's cradle

These integrals are trivially independent and in involution. At this point we can look for a quantum analogue of the Newton's Cradle, what follows is mostly inspired by [8] (chap. 1).

This model is the obvious generalization of the classical Newton's Cradle: we will consider a system of bosons confined in a segment of length L with periodic boundary conditions, the bosons interact through a repulsive delta like potential.

In the second quantization's formalism the Hamiltonian can be written as:

$$\begin{aligned} H &= \int dx\, \partial_x\psi^\dagger(x)\partial_x\psi(x) + c\psi^\dagger(x)\psi^\dagger(x)\psi(x)\psi(x) \\ [\psi(x),\psi(y)] &= 0 \qquad [\psi(x),\psi^\dagger(y)] = \delta(x-y) \qquad c \geq 0 \end{aligned} \tag{2.3}$$

where for simplicity we measure time and distance in units of $\hbar$ and mass.

It is a trivial observation that the number of particles $\hat{N} = \int dx\, \psi^\dagger(x)\psi(x)$ is conserved, for this reason we can attack the problem using the first quantization's formalism:

$$|\phi\rangle \equiv \frac{1}{\sqrt{N!}} \int d^N x\, \phi(\vec{x}) \prod_j \psi^\dagger(x_j)\,|0\rangle \tag{2.4}$$

with the wavefunction $\phi(\vec{x})$ completely symmetric in its arguments. We can define the action of the Hamiltonian over the ϕ functions as:

$$H = \sum_j -\partial_{x_j}^2 + 2c\sum_{i<j}\delta(x_j - x_i) \tag{2.5}$$

The natural question is to evaluate the eigenvectors and eigenstates, this is done quite easily using the coordinate Bethe Ansatz.

Coordinate Bethe Ansatz

The coordinate Bethe Ansatz is a technique that permits to find the eigenvectors of the Lieb Liniger's Hamiltonian and other similar models. We can start with an observation: suppose that $\phi(\vec{x})$ is an eigenfunction of H and consider a region where $x_i \neq x_j$, then in this region H becomes trivially free and we expect that $\phi(\vec{x})$ can be written as a superposition of plane waves

$$\phi(\vec{x}) = \sum_{\sum_j k_j^2 = E} A^\rho(\vec{k}) e^{i\sum_j k_j x_j} \qquad \text{if } x_{\rho(1)} < x_{\rho(2)} < .. < x_{\rho(n)} \tag{2.6}$$

where ρ is a permutation of n indices. The Hamiltonian gives us the constraint that the wavefunction must obey when two coordinates coincide: suppose that $x_i = x_j = x$, call $\pi_{i,j}$ the permutation that exchanges i with j $(j > i)$. In what follows ρ is a permutation of indices such as $x_i < x_j$ and they are adjacent

continuity

$$\sum_{\sum_j k_j^2 = E} A^\rho(\vec{k}) e^{i\sum_{q\neq i,j} k_q x_q} e^{i(k_i+k_j)x} = \sum_{\sum_j k_j^2 = E} A^{\rho\pi_{i,j}}(\vec{k}) e^{i\sum_{q\neq i,j} k_q x_q} e^{i(k_i+k_j)x} \tag{2.7}$$

first derivative discontinuity

$$\begin{aligned} &\sum_{\sum_j k_j^2 = E} iA^\rho(\vec{k})(k_j - k_i) e^{i\sum_{q\neq i,j} k_q x_q} e^{i(k_i+k_j)x} + \\ &- \sum_{\sum_j k_j^2 = E} iA^{\rho\pi_{i,j}}(\vec{k})(k_i - k_j) e^{i\sum_{q\neq i,j} k_q x_q} e^{i(k_i+k_j)x} = \\ &= 2c \sum_{\sum_j k_j^2 = E} A^\rho(\vec{k}) e^{i\sum_{q\neq i,j} k_q x_q} e^{i(k_i+k_j)x} \end{aligned} \tag{2.8}$$

The information that ϕ must be symmetric in its arguments permits us to connect the constants A^ρ with A^{Id}. Given the vector $\vec{k} = (k_1, k_2, ..k_n)$ define the vector $\rho(\vec{k}) = (k_{\rho(1)}, k_{\rho(2)}, ..k_{\rho(n)})$, then the symmetry imposes $A^{\sigma\rho}(\rho(\vec{k})) = A^\sigma(\vec{k})$ where σ is a permutation of n elements: we can see $A^{Id}(\vec{k})$ as the free parameter of the equations and the other coefficients as their functions.

continuity

$$\sum_{\sum_j k_j^2 = E} A^\rho(\vec{k}) e^{i\sum_{q\neq i,j} k_q x_q} e^{i(k_i+k_j)x} = \sum_{\sum_j k_j^2 = E} A^\rho(\pi_{i,j}(\vec{k})) e^{i\sum_{q\neq i,j} k_q x_q} e^{i(k_i+k_j)x} \tag{2.9}$$

first derivative discontinuity

$$\begin{aligned}&\sum_{\sum_j k_j^2=E} iA^\rho(\vec{k})(k_j-k_i)e^{i\sum_{q\neq i,j}k_q x_q}e^{i(k_i+k_j)x}+\\&-\sum_{\sum_j k_j^2=E} iA^\rho(\pi_{i,j}(\vec{k}))(k_i-k_j)e^{i\sum_{q\neq i,j}k_q x_q}e^{i(k_i+k_j)x}=\\&=2c\sum_{\sum_j k_j^2=E} A^\rho(\vec{k})e^{i\sum_{q\neq i,j}k_q x_q}e^{i(k_i+k_j)x}\end{aligned} \tag{2.10}$$

These equations are a set of linear equations that decouples as follows: for each fixed $\pi_{i,j}$ each $A^\rho(\vec{k})$ is coupled with $A^\rho(\pi_{i,j}(\vec{k}))$ and not with other coefficients. But this equation must hold for any permutation ρ, so we can obtain a complete set of solution simply considering the equations for $\{A^{Id}(\rho(\vec{k}))\}_{\rho\in S^n}$, then the general solution can be obtained by superposition.

continuity

$$\sum_{\rho\in S^n} A^{Id}(\rho(\vec{k}))e^{i\sum_{q\neq i,j}k_{\rho(q)}x_q}e^{i(k_{\rho(i)}+k_{\rho(j)})x}=\sum_{\rho\in S^n} A^{Id}(\rho\pi_{i,j}(\vec{k}))e^{i\sum_{q\neq i,j}k_q x_q}e^{i(k_i+k_j)x} \tag{2.11}$$

first derivative discontinuity

$$\begin{aligned}&\sum_{\rho\in S^n} iA^{Id}(\rho(\vec{k}))(k_{\rho(j)}-k_{\rho(i)})e^{i\sum_{q\neq i,j}k_{\rho(q)}x_q}e^{i(k_{\rho(i)}+k_{\rho(j)})x}+\\&-\sum_{\rho\in S^n} iA^{Id}(\rho\pi_{i,j}(\vec{k}))(k_{\rho(i)}-k_{\rho(j)})e^{i\sum_{q\neq i,j}k_{\rho(q)}x_q}e^{i(k_{\rho(i)}+k_{\rho(j)})x}=\\&=2c\sum_{\rho\in S^n} A^{Id}(\rho(\vec{k}))e^{i\sum_{q\neq i,j}k_{\rho(q)}x_{\rho(q)}}e^{i(k_{\rho(i)}+k_{\rho(j)})x}\end{aligned} \tag{2.12}$$

The equation (2.11) is trivially satisfied, instead (2.12) requires:

$$\left(A^{Id}(\rho(\vec{k}))-A^{Id}(\rho\pi_{i,j}(\vec{k}))\right)i(k_{\rho(j)}-k_{\rho(i)})=c\left(A^{Id}(\rho(\vec{k}))+A^{Id}(\rho\pi_{i,j}(\vec{k}))\right) \tag{2.13}$$

$$\Rightarrow A^{Id}(\rho\pi_{i,j}(\vec{k}))=A^{id}(\rho(\vec{k}))\frac{i(k_{\rho(i)}-k_{\rho(j)})-c}{i(k_{\rho(i)}-k_{\rho(j)})+c} \tag{2.14}$$

The term in eq. (2.14) is a phase, so it is useful to define the following quantities:

$$S(k,p)\equiv e^{i\chi(k,q)}\equiv\frac{i(k-q)-c}{i(k-q)+c} \tag{2.15}$$

Notice that $S(k,p)S(p,k) = 1 \Rightarrow -\chi(k,q) = \chi(q,k) + 2\pi n \ n \in \mathcal{Z}$, with these definition we have simply

$$A^{Id}(\rho\pi_{i,j}(\vec{k})) = A^{id}(\rho(\vec{k}))S(k_{\rho(i)}, k_{\rho(j)}) = A^{id}(\rho(\vec{k}))e^{i\chi(k_{\rho(i)},k_{\rho(j)})} \tag{2.16}$$

At this point, through a repeated use of permutations $\pi_{i,j}$, we can connect the coefficient $A^{Id}(\rho(\vec{k}))$ with $A^{Id}(\vec{k})$.

The coefficients $A^{Id}(\rho(\vec{k}))$ satisfy $A^{Id}(\rho\pi_{i,j}(\vec{k})) = A^{Id}(\rho(\vec{k}))e^{i\chi(k_{\rho(i)},k_{\rho(j)})}$, $(j > i)$ if and only if:

$$A^{Id}(\rho(\vec{k})) = A^{Id}(\vec{k})e^{\frac{i}{2}\sum_{i<j}\chi(k_i,k_j) - \frac{i}{2}\sum_{i<j}\chi(k_{\rho(i)},k_{\rho(j)})} \qquad \forall\rho \in S^n$$

We can verify this statement by induction, for $\rho = Id$ is true.

Each permutation $\pi_{i,j}$ can be decomposed in a product of permutations that exchange two near elements, so it is sufficient to show that, for each permutation $\pi_{a,a+1}$, we have $A^{Id}(\rho\pi_{a,a+1}(\vec{k})) = A^{Id}(\rho(\vec{k}))e^{i\chi(k_{\rho(a)},k_{\rho(a+1)})}$. It is obvious that this solution is the most general. So we start with:

$$A^{Id}(\rho\pi_{a,a+1}(\vec{k})) = A^{Id}(\vec{k})e^{\frac{i}{2}\sum_{i<j}\chi(k_i,k_j) - \frac{i}{2}\sum_{i<j}\chi(k_{\rho\pi_{a,a+1}(i)},k_{\rho\pi_{a,a+1}(j)})} \tag{2.17}$$

We will examine only the phase factor:

$$\begin{aligned}
&\frac{i}{2}\sum_{i<j}\chi(k_i,k_j) - \frac{i}{2}\sum_{i<j}\chi(k_{\rho\pi_{a,b}(i)},k_{\rho\pi_{a,b}(j)}) = \\
=&\frac{i}{2}\sum_{i<j}\chi(k_i,k_j) - \frac{i}{2}\sum_{(\pi_{a,b})^{-1}(i)<(\pi_{a,b})^{-1}(j)}\chi(k_{\rho(i)},k_{\rho(j)}) = \\
&= \left[\frac{i}{2}\sum_{i<j}\chi(k_i,k_j) - \frac{i}{2}\sum_{i<j}\chi(k_{\rho(i)},k_{\rho(j)})\right] + \\
&\frac{i}{2}\left[\sum_{i<j}\chi(k_{\rho(i)},k_{\rho(j)}) - \sum_{(\pi_{a,b})^{-1}(i)<(\pi_{a,b})^{-1}(j)}\chi(k_{\rho(i)},k_{\rho(j)})\right]
\end{aligned} \tag{2.18}$$

With a little thought, it is quite easy to verify that the following equality is true:

$$\sum_{i<j}\chi(k_{\rho(i)},k_{\rho(j)}) - \sum_{(\pi_{a,b})^{-1}(i)<(\pi_{a,b})^{-1}(j)}\chi(k_{\rho(i)},k_{\rho(j)}) = 2\chi(k_{\rho(a)},k_{\rho(a+1)}) \tag{2.19}$$

So eq. (2.17) follows.

We found the eigenvectors of the Hamiltonian that are symmetric under particle's exchange, we will summarize their characteristics:

- Each eigenstate can be uniquely defined by a set of momenta $\{k_i\}$, the order does not matter.
- For each set of momenta $\{k_i\}$, we can write the eigenvector in coordinate representation

$$\sum_{\rho\in S^n} A^\sigma(\rho(\vec{k}))e^{i\sum_j k_{\rho(j)}x_j} \qquad x_{\sigma_1} < x_{\sigma_2} < .. < x_{\sigma_n}$$

$$A^{Id}(\rho\pi_{a,a+1}(\vec{k})) = A^{Id}(\vec{k})e^{\frac{i}{2}\sum_{i<j}\chi(k_i,k_j)-\frac{i}{2}\sum_{i<j}\chi(k_{\rho\pi_{a,a+1}(i)},k_{\rho\pi_{a,a+1}(j)})} \tag{2.20}$$

$$A^\sigma(\rho\sigma(\vec{k})) = A^{Id}(\rho(\vec{k})) \qquad S(k,q) = e^{i\chi(k,q)} = \frac{i(k-q)-c}{i(k-q)+c}$$

Given a set of momenta, the eigenvalue is simply $E_{\vec{k}} = \sum_i k_i^2$.

- It will be useful to define a convention to indicate the states: let $|\vec{k}\rangle = |k_1, k_2, .., k_n\rangle$ be the state that in coordinate representation has the following form

$$\langle\vec{x}|\vec{k}\rangle = \sum_{\rho\in S^n} \tilde{A}^\sigma(\rho(\vec{k}))e^{i\sum_j k_{\rho(j)}x_j} \qquad x_{\sigma_1} < x_{\sigma_2} < .. < x_{\sigma_n}$$

$$\tilde{A}^{Id}(\rho(\vec{k})) = e^{\frac{i}{2}\sum_{i<j}\chi(k_i,k_j)-\frac{i}{2}\sum_{i<j}\chi(k_{\rho(i)},k_{\rho(j)})} \tag{2.21}$$

$$\tilde{A}^\sigma(\rho\sigma(\vec{k})) = \tilde{A}^{Id}(\rho(\vec{k}))$$

Notice that the vectors are not normalized and that two vectors that differ only by a permutation of the elements of $\vec{k}$ are the same except a phase factor:

$$|k_1, k_2, ..., k_j, k_{j+1}, ..k_n\rangle = S(k_j, k_{j+1})\,|k_1, k_2, ..., k_{j+1}, k_j, ..k_n\rangle \tag{2.22}$$

Notice that, since $S(k,k) = -1$, if a vector $|\vec{k}\rangle$ contains two equal momenta it must be zero.

We have not required the periodicity of wavefunctions yet: this gives us a constraint over the sets of momenta allowed. Consider the set of coordinates $(x_1, x_2, .., x_n)$,

then the wavefunction at these points must be the same if evaluated at $(x_1 + L, x_2, ..x_n)$. Without loss of generality we can suppose $x_1 < x_2 < .. < x_n$.

$$A^{Id}(\rho(\vec{k})) = e^{iLk_{\rho(1)}} A^{\sigma}(\rho(\vec{k})) \tag{2.23}$$

where σ is the permutation that sends $(1, 2, 3..n) \to (2, 3, .., n, 1)$. Using $A^{\sigma}(\rho\sigma(\vec{k})) = A^{Id}(\rho(\vec{k}))$ we will end up with the following equation:

$$e^{i\sum_{j\neq 1}\chi(k_{\rho(1)},k_{\rho(j)})} e^{ik_{\rho(1)}L} = 1 \tag{2.24}$$

This must hold for any ρ, so we conclude:

$$\Rightarrow e^{i\sum_{j\neq i}\chi(k_i,k_j)} e^{ik_iL} = 1 \qquad \forall i \tag{2.25}$$

These are known as Bethe Equations.

If this system is the quantum analogue of a classical integrable system, we expect to find a complete set of first integrals of motion. We can define the charges Q_n through their action on the eigenstates of the Hamiltonian:

$$Q_n |\vec{k}\rangle = \sum_i k_i^n |\vec{k}\rangle \tag{2.26}$$

These charges commute with the Hamiltonian and with each other because they act diagonally over the same base, besides they are additive over the different particles. Observe that Q_0 is the number of particles, Q_1 the total momentum and Q_2 the energy. We can recast in some way the notion of "complete set" of first integrals: if we select a state and we know that is an eigenstate for Q_n for each n and we know each eigenvalue, then we can reconstruct the state $|\vec{k}\rangle$ a part from a phase factor.

2.3 Quantum integrable systems

In this section we will define and study a general integrable theory in one dimension. The correct definition is still debated [10], for this reason we will adopt an heuristic definition and then exhibit some examples that satisfy it. This approach is quite axiomatic, but permits us to include into the definition all the characteristics we want for an integrable model: we would recognize all the principal aspects of the Lieb-Liniger model, which inspired these requirements.

For simplicity we will consider systems with only one kind of particle in one spatial dimension. For a complete discussion that includes the multiparticle case we refer to [9] (chap. 16-17), which largely inspired this section.

Definition: We will say that a one-dimensional quantum theory with only one kind of particle and in a finite volume L is "integrable" if it satisfies the following requirements:

- A complete set of vectors of the Hilbert space can be identified by a set of quantum numbers $\{k_i\}$: they will be indicated as $|\vec{k}\rangle$.
- There are some conserved charges Q_n such that they commute with each other and they are additive over the quantum numbers k_i, that is to say:

$$Q_n |\vec{k}\rangle = \sum_{k_i \in \vec{k}} q_n(k_i) |\vec{k}\rangle \tag{2.27}$$

  We require that they are complete, that is to say that the knowledge of all the eigenvalues of Q_n over a state $|\vec{k}\rangle$ is enough to determine the set $\{k_i\}$.

  We require also that this set of charges contains the Hamiltonian and the Momentum operator.

- Not all the $|\vec{k}\rangle$ are independent. Consider two states $|\vec{k}\rangle$ and $|\vec{q}\rangle$, they are orthogonal if and only if the sets $\{k_i\}$ and $\{q_i\}$ are different.

  If the vectors $\vec{k}$ and $\vec{q}$ are equal a part from a permutation of their elements, then the two states $|\vec{k}\rangle$ and $|\vec{q}\rangle$ are equal a part from a phase factor. For later convenience we define the in-states $|\vec{k}\rangle_{in} = |k_1, k_2, .., k_n\rangle$ such that $k_1 \geq k_2 \geq .. \geq k_n$; an out-state is defined in the same manner but the quantum numbers are inversely ordered.

- There exists a function $S : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ called two-bodies scattering matrix such that:
  - $|S(k, p)| = 1$.
  - For each state $|\vec{k}\rangle$ the following relation holds

$$|k_1, k_2, ..k_j, k_{j+1}, ..k_n\rangle = S(k_j, k_{j+1}) |k_1, k_2, ..k_{j+1}, k_j, ..k_n\rangle \tag{2.28}$$

    For consistency of this definition, the S matrix must satisfy the unitarity condition:

$$S(p, q)S(q, p) = 1 \tag{2.29}$$

- The set of quantum numbers $\{k_i\}$ must satisfy the Bethe Equations, that is to say:

$$\prod_{j \neq i} S(k_i, k_j) e^{ip(k_i)L} = 1 \qquad \forall i \tag{2.30}$$

  where $p(k)$ is the one-particle momentum, L will be called the size of the system.

The definition we gave can be generalized to the case of different kinds of particles, this would imply the introduction of indexes to label each particle. Then the S matrix becomes a true matrix whose indices connect the different particles, but all the basic features remain the same, a part from the introduction of the Yang Baxter equation for the S matrix. For further reading see [9] (chap. 17).

Statistic of Bethe roots

The unitarity condition for the S matrix can give some constraints over the allowed quantum numbers: we can start with the following simple observation

$$S(p,q)S(q,p) = 1 \Rightarrow [S(p,p)]^2 = 1 \Rightarrow \quad S(p,p) = \pm 1 \tag{2.31}$$

Now consider a vector $|\vec{k}\rangle$ with $k_j = k_{j+1} = k$:

$$|k_1, k_2, ..k_j = k, k_{j+1} = k, ..k_n\rangle = S(k,k)\,|k_1, k_2, ..k_j = k, k_{j+1} = k, ..k_n\rangle \tag{2.32}$$

If we have $S(k,k) = -1$ we must conclude that this state is zero, if $S(k,k) = 1$ there is not any problem. The value of $S(k,k)$ establishes a Pauli's exclusion principle for the allowed vectors. We will say the model is Bose-like if $S(k,k) = 1\ \forall k$: in this case we do not have any restriction over the allowed values of $\vec{k}$. Instead is Fermi-like if $S(k,k) = -1\ \forall k$: in this case we must satisfy a Pauli's exclusion principle for the k_i contained in $\vec{k}$.

The Lieb-Liniger model, despite its bosonic feature in the standard field representation, is a fermi-like model from the point of view of its integrability for any non zero value of the coupling constant.

This axiomatic definition of integrable models is not very suitable for physical interpretation, we would like to find some criterion to determine integrable theories without explicitly solving all the dynamics: that is the task of the next section.

2.3.1 Relativistic integrable theories

In this section we consider relativistic integrable theories living in one dimension, most of the discussion is inspired by [11], but see also [14]. In the following we will suppose that the theory has no bound states, to avoid the case of different kinds of particles.

Consider a relativistic field theory with local interactions, for simplicity we will suppose a bosonic real field theory with action

$$\mathcal{S} = \int dx\, dt\ \left[\partial_\mu \Phi(x,t)\partial^\mu \Phi(x,t) + m^2 \Phi(x)\Phi(x) + \mathcal{L}_{int}(\Phi(x,t))\right] \tag{2.33}$$

where $\mathcal{L}_{int}$ is an analytic function of the field evaluated at the point (x,t); for clarity we switch to a different representation: we now that the field can be expanded in

terms of creation-annihilation operators

$$\Phi(x) = \int dk \frac{1}{\sqrt{4\pi E(k)}} \left(a^\dagger(k)e^{ikx} + a(k)e^{-ikx}\right)$$
$$[a(k), a(q)] = 0 \qquad [a(k), a^\dagger(q)] = \delta(k-q) \tag{2.34}$$

where $E(k) = \sqrt{m^2 + k^2}$, besides there exists a vacuum state $|0\rangle$ annihilated by all the $a(k)$. In terms of these operators the Hamiltonian has the following form

$$H = \int dk\, E(k) a^\dagger(k) a(k) - \int dx \mathcal{L}_{int}(\Phi(x)) \tag{2.35}$$

where the field in $\mathcal{L}_{int}$ has to be expressed in terms of the $a(k)$ operators. We will define now the creation-annihilation operators in the coordinate space as the Fourier transform of the $a(k)$ operators:

$$\psi(x) = \int \frac{dk}{\sqrt{2\pi}} e^{-ikx} a(k) \tag{2.36}$$

$$[\psi(x), \psi(y)] = 0, \;\; [\psi(x), \psi^\dagger(y)] = \delta(x-y), \;\; \psi(x)\,|0\rangle = 0 \tag{2.37}$$

We can interpret $\psi^\dagger(x)$ as the particle creation operator in the space representation, the field $\Phi(x)$ can be expressed as follows:

$$\Phi(x) = \int dy\, \psi^\dagger(y) \left[\int \frac{dk}{2\pi\sqrt{2E(k)}} e^{ik(x-y)}\right] + c.c. \tag{2.38}$$

The kernel of this integral is well localized:

$$\int \frac{dk}{2\pi\sqrt{2E(k)}} e^{ik(x-y)} \sim e^{-m|x-y|} \tag{2.39}$$

The field $\Phi(x)$ can be expanded in terms of the operators $\psi(y)$ with y in a neighbourhood of x of width $\sim m^{-1}$. This well justifies the definition and construction of asymptotic states. Consider a generic state $|\phi\rangle$, this can be expanded using the $\psi^\dagger$ operators:

$$|\phi\rangle = \sum_n \frac{1}{\sqrt{n!}} \int d^n x\, \phi_n(\vec{x}_n) \prod_j^n \psi^\dagger(x_j)\,|0\rangle \tag{2.40}$$

The functions $\phi_n(\vec{x}_n)$ are symmetric in their arguments, in particular the eigenstates can be so expanded.

The interaction term between the $\psi^\dagger(x)$ operators has a finite range, for example if $\mathcal{L}_{int}$ contains a term $\propto [\phi(x)]^n$ this has an interaction range roughly given by $\sim \frac{n}{m}$ with m the mass of the theory. This has the consequence that, if we are considering distant particles they are described by plane waves.

For these reasons we admit the following asymptotic behavior, faithful only in the case that the theory has no bound states:

$$\phi_n(\vec{x}_n) \simeq \frac{1}{\sqrt{\mathcal{N}}} \sum_{\vec{k}_n} A^{\sigma}(\vec{k}_n) e^{i \sum_j k_j x_j} \qquad x_{\sigma(1)} \ll x_{\sigma(2)} \ll .. \ll x_{\sigma(n)} \tag{2.41}$$

here $\mathcal{N}$ is a normalization factor introduced for later convenience, σ is a permutation. We also define the asymptotic in-states and out-states, more precisely we will define the "in" and "out" coefficients.

A coefficient $A^{Id}(\vec{k}_n)$ will be called in-coefficient and indicated as $A_{in}(\vec{k}_n)$ if and only if considering the vector of momenta $\vec{k}_n = (k_1, k_2, ..k_n)$ we have $k_1 \geq k_2 \geq .. \geq k_n$, an out-coefficient has the same definition, but with the elements of $\vec{k}$ reversely ordered.

If we relate somehow the "in" coefficients with the $A^{Id}(\vec{k}_n)$ coefficients and use the symmetrization requirement of the wavefunction, we can fix its asymptotic behavior; so the eigenstates could be identified by the "in" coefficients.

We define the S matrix elements from the following relation:

$$A^{Id}(\vec{q}_m) = \sum_{\vec{k}_n} S_{\vec{q}_m, \vec{k}_n} A_{in}(\vec{k}_n) \tag{2.42}$$

The S matrix has to be determined resolving the dynamic of the model. Since we are considering a relativistic invariant theory, it would be more convenient to express the S matrix and the A coefficients in terms of the rapidities:

$$\begin{gathered} k_i = m \sinh(\theta_i) \\ A^{Id}(\vec{q}_m) = \sum_{\vec{k}_n} S_{\vec{q}_m, \vec{k}_n} A_{in}(\vec{k}_n) \longrightarrow A^{Id}(\vec{\beta}_m) = \sum_{\vec{\theta}_n} S_{\vec{\beta}_m, \vec{\theta}_n} A_{in}(\vec{\theta}_n) \end{gathered} \tag{2.43}$$

Notice that since the theory is relativistic invariant, we expect that the S matrix will be invariant for a simultaneous shift of all the rapidities:

$$\beta_i \to \beta_i + \alpha \qquad \theta_i \to \theta_i + \alpha \tag{2.44}$$

At this point we have all the machinery to introduce the concept of local conserved charge.

A local conserved charge Q_s of spin s is an observable that commutes with the Hamiltonian and can be written as follows:

$$Q_s = \int dx \mathcal{Q}_s(x) \tag{2.45}$$

where $\mathcal{Q}_s(x)$ is a function of the fields $\Phi(x)$, $\Pi(x)$ and their derivatives.

We also suppose that if we perform a Lorentz transformation of rapidity β, then it transforms as:

$$Q_s \to e^{s\beta} Q_s \tag{2.46}$$

For references see [9].

Suppose that a relativistic theory with local interactions possess some local conserved charges Q_s for an infinite set of integers s and the charges are in involution. If the number of particles is conserved, then the theory is integrable. This is explained in [11] and we will sketch the proof.

These requirements impose some non trivial constraints on the action of Q_s on the eigenstates of the Hamiltonian: consider at first a one-particle state with rapidity θ, call it $|\theta\rangle$. This state must be an eigenstate for Q_s, suppose that $q_s(\theta)$ is its eigenvalue. We can use the Lorentz transformations to restrict the form of this eigenvalue, if we suppose a shift β in the rapidities we can say:

$$\begin{cases} |\theta\rangle \to |\theta + \beta\rangle \\ Q_s \to e^{s\beta} Q_s \end{cases} \quad \Rightarrow q_s(\theta + \beta) = q_s(\theta) e^{s\beta} \tag{2.47}$$

This fixes the behavior of the eigenvalue $q_s(\theta)$ as a function of θ: it must be in the form $q_s(\theta) = q_s e^{s\theta}$ where q_s is a real number.

Now consider n-particles states, they can be written as:

$$|\phi\rangle = \frac{1}{\sqrt{n!}} \int d^n x \, \phi_n(\vec{x}_n) \prod_j^n \psi^\dagger(x_j) \, |0\rangle \tag{2.48}$$

Since the number of particles commutes with Q_s, we can use a first quantization formalism and study the action of Q_s on the wavefunction. Call $\tilde{Q}_s$ the first quantization representation of Q_s, its action on the wavefunction is defined by the following equality:

$$Q_s \, |\phi\rangle = \frac{1}{\sqrt{n!}} \int d^n x \, \tilde{Q}_s(\phi_n)(\vec{x}_n) \prod_j^n \psi^\dagger(x_j) \, |0\rangle \tag{2.49}$$

The locality of Q_s tells us that $\tilde{Q}_s(\phi_n)(\vec{x}_n)$ is a function of $\phi_n(\vec{x}_n)$ and its derivatives at the same point, but does not depend from the wavefunction calculated in other points. Now remember we are looking for eigenvectors, so require:

$$\tilde{Q}_s(\phi_n)(\vec{x}_n) \propto \phi_n(\vec{x}_n) \tag{2.50}$$

We now combine the following two facts: this wavefunction must be also eigenvector of the Hamiltonian, so in the asymptotic region becomes a sum of plane waves; the operator $\tilde{Q}_s$ is local, so its action on a product of functions on distant points is additive. Combining this information with what we know about the one particle

states, in the asymptotic region ($x_{\sigma(1)} \ll x_{\sigma(2)} \ll .. \ll x_{\sigma(n)}$) we can say:

$$\tilde{Q}_s(\phi_n)(\vec{x}_n) = \sum_{\vec{\theta}_n} \frac{A^\sigma(\vec{\theta}_n)}{\sqrt{\mathcal{N}}} \tilde{Q}_s \left(e^{i\sum_j p(\theta_j)x_j}\right) = \sum_{\vec{\theta}_n} \frac{A^\sigma(\vec{\theta}_n)}{\sqrt{\mathcal{N}}} \left(q_s \sum_i e^{s\theta_i}\right) e^{i\sum_j p(\theta_j)x_j} \tag{2.51}$$

If we look for common eigenvectors for all the s, we obtain the consistency equation:

$$\sum_j e^{s\theta_j} = \sum_i e^{s\beta_i} \tag{2.52}$$

We have to satisfy this equation for all s and for each sets $\vec{\theta}$ and $\vec{\beta}$ of rapidities in the state. The only allowed solution is when the rapidities β_j and θ_i are the same a part from permutations of indices.

We can now look for eigenstates in the form:

$$\begin{aligned} |\phi\rangle &= \frac{1}{\sqrt{n!}} \int d^n x \phi_n(\vec{x}_n) \prod_j^n \psi^\dagger(x_j) |0\rangle \\ \phi_n(\vec{x}_n) &\simeq \frac{1}{\sqrt{\mathcal{N}}} \sum_{\rho \in S^n} A^\sigma(\rho(\vec{\theta}_n)) e^{i\sum_j k(\theta_{\rho(j)})x_j} \qquad x_{\sigma(1)} \ll x_{\sigma(2)} \ll .. \ll x_{\sigma(n)} \end{aligned} \tag{2.53}$$

with ρ a permutation of n indices whose action on the vector $\vec{\theta}$ is $\rho(\vec{\theta}) = \rho(\theta_1, \theta_2, ..) = (\theta_{\rho(1)}, \theta_{\rho(2)}, ..)$. In this representation the problem closely resembles the Lieb-Liniger model and the only difference is the connection between the coefficients $A^\sigma(\rho(\vec{\theta}_n))$: in the Lieb-Liniger model these are related with a S matrix that is factorized in two bodies processes.

A non trivial fact is that, if the interactions are local as in our assumption, then the presence of an infinite set of Q_s conserved charges and the relativistic invariance enforce the factorization of the S matrix. We would not prove this assertion referring to [11], or [9] (chap. 17.2.4).

At this point, with a different two bodies S matrix, the problem becomes analogous to Lieb-Liniger and the theory is clearly integrable. The Bethe Equations are obtained if we introduce a finite volume regularization putting the system in an interval of length L with periodic boundary conditions.

2.3.2 Zamolodchikov-Fadeev algebra

The Zamolodchikov-Fadeev algebra is an useful tool when we consider integrable theories directly in the infinite volume. Here we give only a brief introduction, for further readings see [12] (chap. 1) and references therein, [25] and for applications of this algebra to quenches problems see [35]. As starting point we summarize briefly the definition we gave for quantum integrable models in a finite volume.

1. The eigenstates of the Hamiltonian are identified by multi-particles states

$$|\vec{k}\rangle = |k_1, k_2, k_3..k_n\rangle \tag{2.54}$$

2. The Hamiltonian and Momentum act diagonally over these eigenstates

$$H\,|\vec{k}\rangle = \sum_{k_i \in \vec{k}} E(k_i)\,|\vec{k}\rangle\,, \qquad P\,|\vec{k}\rangle = \sum_{k_i \in \vec{k}} p(k_i)\,|\vec{k}\rangle \tag{2.55}$$

3. States that differ only by permutations of the quantum numbers k_i are related through the S matrix

$$|..k_j, k_{j+1}, ..\rangle = S(k_j, k_{j+1})\,|..k_{j+1}, k_j, ..\rangle \tag{2.56}$$

   If two states do not contain the same set of quantum numbers, they are orthogonal.

4. The Bethe Equations hold

$$\prod_{j \neq i} S(k_i, k_j) e^{ip(k_i)L} = 1 \qquad \forall i \tag{2.57}$$

We notice that in our definition the only information about the volume is contained in the Bethe Equations: they are obtained from boundary conditions on the segment L. For a definition of an integrable theory in infinite volume we would like to remove (4.), in this way the allowed quantum numbers k will not be discretized and they will assume continuous values.

Definition [9]: We will say that a model is an "quantum integrable theory in infinite volume " if (1.), (2.), (3.) are still true, instead (4.) is substituted with the following requirement.

The set of quantum numbers $\{k_i\}$ is a continuum and the in-states are so normalized:

$$\begin{aligned} |\vec{k}_n\rangle &= |k_1, k_2, ..k_n\rangle \qquad k_1 \geq k_2 \geq .. \geq k_n \\ |\vec{q}_m\rangle &= |q_1, q_2, ..q_m\rangle \qquad q_1 \geq q_2 \geq .. \geq q_m \\ \langle \vec{q}_m | \vec{k}_n \rangle &= \delta_{n,m} \prod_i (2\pi\delta(k_i - q_i)) \end{aligned} \tag{2.58}$$

in the Fermi case we must also impose that vectors with identical quantum numbers are zero. The normalization of the other states can be obtained through the S matrix.

We stress that the latter definition cannot be recast from the finite volume's one setting $L \to \infty$, we will have some comments at the beginning of Chapter 4. With this definition, we can introduce and motivate the Zamolodchikov-Fadeev algebra.

Definition [9]: A Zamolodchikov-Fadeev algebra is a set of operators $Z(k)$ such that there exists $|0\rangle_Z$ such that $Z(k)|0\rangle_Z = 0 \ \forall k$, besides we require:

$$\begin{aligned} Z(k)Z(q) &= S(k,q)Z(q)Z(k) \\ Z(k)Z^+(q) &= S(q,k)Z^+(q)Z(k) + 2\pi\delta(k-q) \end{aligned} \tag{2.59}$$

This algebra automatically describes the infinite volume integrable theories, as a matter of fact it is sufficient to define the states of the theory as:

$$|k_1, k_2..k_n\rangle = Z^+(k_1)Z^+(k_2)..Z^+(k_n)|0\rangle_Z \tag{2.60}$$

2.3.3 Examples of Integrable theories

This section is dedicated to recognize some theories as integrable models, we will not discuss the details leaving their complete description to the references.

Free Theories:

This is the most trivial example of integrable model: consider a set of creation and annihilation operators that satisfy standard commutation or anticommutation rules. The sign $\pm$ is $-$ in the Bose case, $+$ in the Fermi one:

$$\begin{aligned} &[a_k, a_q^\dagger]_\pm = a_k a_q^\dagger \pm a_q^\dagger a_k = \delta(k-q) \qquad [a_k, a_q]_\pm = 0 \\ &a_k|0\rangle = 0 \, \forall k \qquad\qquad H = \int dk E(k) a_k^\dagger a_k \end{aligned} \tag{2.61}$$

In a free theory, the standard creation and annihilation operators can be seen as the Zamolodchikov-Fadeev operators and the scattering matrix is $S = \pm 1$. We can define the states in this way:

$$|k_1, k_2, ..k_n\rangle = a^\dagger(k_1)a^\dagger(k_2)..a^\dagger(k_n)|0\rangle \tag{2.62}$$

If we want to consider a finite volume integrable theory we choose periodic boundary conditions, so the momenta become quantized and we also obtain the (trivial) Bethe Equations.

Lieb-Liniger:

This model does not need any comment, because it is the example we used to introduce integrable theories.

XXZ Model:

This model is constructed over a spin chain: suppose N lattice sites with $\frac{1}{2}-$ spin and the Hamiltonian

$$H = \sum_j \frac{1}{2}\left(S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+\right) + \Delta(S_j^z S_{j+1}^z - \frac{1}{4}) \tag{2.63}$$

Boundary periodic conditions are assumed. Remember the following commutators:

$$S_j^\pm = S_j^x \pm i S_j^y \quad [S_j^+, S_k^-] = 2S_j^z \delta_{j,k} \quad [S_j^z, S_k^\pm] = \pm\delta_{j,k} S_j^\pm \tag{2.64}$$

The construction of eigenstates is very similar to Lieb-Liniger, so we do not perform all the calculations leaving this task to [8] (chap. 2). As first step notice that the Hamiltonian commutes with the total spin in the z direction, so we can look for common eigenvectors:

$$\left[\sum_j S_j^z, H\right] = 0 \tag{2.65}$$

Define the vacuum as the state that has all the spin in the $+z$ direction:

$$|0\rangle \equiv \bigotimes_j |\uparrow_j\rangle \tag{2.66}$$

We can find all the eigenvectors of $\sum_j S_j^z$ acting with S^-:

$$|\vec{j}_n\rangle = |j_1, j_2..j_n\rangle \equiv \prod_i^n S_{j_i}^- |0\rangle \tag{2.67}$$

The most general eigenvector of H must be a superposition of $|\vec{j}_n\rangle$ at fixed n. We can evaluate the action of H over $|\vec{j}_n\rangle$:

$$H|\vec{j}_n\rangle = \sum_{i=1}^n \frac{1}{2}(1-\delta_{j_i+1,j_{i+1}})\left\{|..,j_i+1,j_{i+1}..\rangle + |..j_i,j_{i+1}-1..\rangle\right\} - \Delta(1-\delta_{j_i+1,j_{i+1}})|\vec{j}_n\rangle \tag{2.68}$$

The most general solution for $n=1$ is simply a plane wave:

$$|k\rangle = \sum_{m=1}^N e^{ikm}|m\rangle \qquad k = \frac{2\pi}{N}a,\ a \in \mathbb{Z} \tag{2.69}$$

This suggests us to look for solutions in the form:

$$|\vec{k}_n\rangle = \sum_{\rho\in S^n} A(\rho) \sum_{\vec{j}_n}^{n\ fixed} e^{i\sum_i k_{\rho(i)} j_i} |\vec{j}_n\rangle \tag{2.70}$$

It turns out that, solving the eigenvalues equation on these states, we find that the coefficients $A(\rho)$ have the following form

$$A(\rho) = constant \cdot (-1)^{sign(\rho)} e^{-\frac{i}{2}\sum_{1\leq a<b\leq N}\Phi(k_{\rho(a)},k_{\rho(b)})} \tag{2.71}$$

Where $sign(\rho)$ is the sign of the permutation and:

$$\Phi(k,q) = 2\arctan\frac{\Delta\sin\frac{k-q}{2}}{\cos\frac{k+q}{2} - \Delta\cos\frac{k-q}{2}} \tag{2.72}$$

As in Lieb-Liniger we can define a set of states $|\vec{k}\rangle$ eigenstates of H and the S matrix

$$S(k,q) \equiv -e^{i\Phi(k,q)} \Rightarrow \qquad S(k,q)S(q,k) = 1 \tag{2.73}$$

$$|..k_i k_{i+1}..\rangle = S(k_i,k_{i+1})\,|..k_{i+1},k_i..\rangle \tag{2.74}$$

These states are orthogonal unless they differ by a permutation of their momenta. The Bethe Equations are recovered imposing the periodic boundary conditions:

$$\prod_{j\neq i} S(k_i,k_j)e^{iNk_i} = 1 \tag{2.75}$$

Sinh-Gordon model:

This is the first integrable theory that we consider for which we cannot write down explicitly the eigenstates. It is a relativistic theory of a bosonic real field governed by the action $\mathcal{S}$

$$\mathcal{S} = \int dx\,dt\ \frac{1}{16\pi}\left(\partial_\mu\phi\right)^2 + \frac{\mu^2}{g^2}\cosh\left(\frac{g}{\sqrt{8\pi}}\phi(x)\right) \tag{2.76}$$

Here we will not report the details, but it can be shown that this model possesses an infinite set of local charges Q_s: so as we already said this is an integrable theory with the following S matrix see [9] (chap. 16.2):

$$S(\theta,\theta') = \frac{\sinh(\theta-\theta') - i\sin B}{\sinh(\theta-\theta') + i\sin B} \qquad B = \frac{g^2}{8}\frac{1}{1+\frac{g^2}{8\pi}} \tag{2.77}$$

An interesting link between the Sinh-Gordon model and the Lieb-Liniger model exists: the latter is a non relativistic limit of the first, for references see [36].

Chapter 3

Equilibrium and quenches in integrable theories

In this chapter we consider both equilibrium and quenches in the contest of integrable theories, in particular we introduce the concept of local operator and try to describe the expectation values of these observables as we briefly anticipated in Chapter 1.

In section 3.1 the necessary tools to describe equilibrium situations are introduced, then in section 3.2 quenches problems are considered: many of these concepts are fundamental for Chapters 4 and 5.

3.1 Equilibrium in integrable theories

This section is devoted to the study of equilibrium situations in integrable models, in particular we discuss the Generalized Gibbs Ensemble and the Thermodynamic Bethe Ansatz as a tecnique to deal with it. Then the concept of local observable is introduced and the GGE is applied to these operators through the Le Clair-Mussardo's conjecture.

3.1.1 Generalized Gibbs Ensemble

In the previous chapters we introduced the necessary tools to discuss some aspects of thermalisation in quantum integrable theories. Here we will introduce the GGE, that is to say a trial to describe the equilibrium state of a quantum integrable model. We refer directly to the original article [17].

We already know from Chapter 1 that the evolution of a closed quantum system is non dissipative and so the equilibrium cannot be reached in a strict sense, but it could happen that there exist some observables whose averages behave such as a global equilibrium is reached. For the time being we postpone the problem of understanding if such operators effectively exist and we try to describe the equilibrium, we still suppose that the system is initialized in some known state and then after a long time the steady situation is reached.

We have already posed this problem in Chapter 1 (section 1.2), we remind that the candidate equilibrium density matrix has the following form:

$$\rho \propto e^{-\sum_i \beta_i Q_i} \tag{3.1}$$

here Q_i are conserved charges: in quantum integrable theories we have an infinite number of conserved charges, this implies that if we consider a state $|\vec{k}\rangle$, then every function of $\vec{k}$ is conserved.

So the following form is well justified:

$$\rho \propto \sum_{\vec{k}}^{ordered} e^{F(\vec{k})} |\vec{k}\rangle \langle\vec{k}| \tag{3.2}$$

The ordered summation means that the vectors $\vec{k}$ obey the constraint $k_1 \geq k_2 \geq .. \geq k_j \geq ..$; this because a different order of k_j does not produce a new state. If $F(\vec{k})$ is an arbitrary function, this density matrix could describe every diagonal ensemble, but it is clear that we need a huge amount of information to construct it: we need some recipe to limit the freedom of choosing $F(\vec{k})$. For this purpose we reject integrals of motion that are not extensive over the particles, this implies that we require the additivity of the F function, that is to say:

$$F(\vec{k}) = \sum_{k_i \in \vec{k}} \epsilon(k_i) \tag{3.3}$$

$$\rho_{GGE} \propto \sum_{\vec{k}}^{ordered} e^{-\sum_{k_i \in \vec{k}} \epsilon(k_i)} |\vec{k}\rangle \langle\vec{k}| \tag{3.4}$$

This proposal for the density matrix with the normalization $\langle\vec{k}|\vec{k}\rangle = 1$, is called the Generalized Gibbs Ensemble. The function ϵ has to be chosen in such a way that the distribution of quantum numbers k calculated with this density matrix is the same as in the initial state.

The GGE density matrix appears as a generalization of the standard Gibbs Ensemble, where the function $\epsilon(k)$ is chosen to be the one-particle energy over the temperature. The next natural step is to study the initial conditions and the observables to measure: to have relaxation to the GGE we need to restrict both the sets of initial states and of the observables to consider.

About the initial state some negative examples can be shown: consider for example an initial state made of a finite superposition of eigenstates, then it is obvious that a generic observable evaluated on this state does not have any long time steady behavior, because the time evolution is driven by a finite linear combination of oscillating terms.

From now on we will restrict ourselves to the infinite size limit: in the next section we will show that in this limit the GGE is dominated by states with the number of particles of the same order of the size of the system, so we expect to verify the GGE

only if we excite states with an infinite number of particles. The analysis of the thermodynamic limit of the GGE is contained in 3.1.2 where the Thermodynamic Bethe Ansatz is explained.

We can also give some comments on the observables to consider: when we study statistical physics, usually the system is coupled with a thermal bath that provides the relaxation towards a steady state. We can mimic this bath in a closed quantum system in this way: suppose to consider averages that regards only a small portion of the system, if we perform the $L \to \infty$ limit we can suppose that the whole system acts as a thermal bath for the small one, in this case we could have relaxation towards steady situations. This motivates the introduction of local observables in 3.1.3.

3.1.2 Thermodynamic Bethe Ansatz

This section is devoted to a useful method for studying the Bethe Equations in the thermodynamical limit, in particular we will apply it to the evaluation of the partition function that arises from the GGE. Here we will present only a brief summary of this topic suitable for this thesis, the contents of this section and further readings are in [9] (chap. 19). We start considering the set of Bethe Equations:

$$\prod_{j\neq i}^{N} S(k_i, k_j) e^{iLp(k_i)} = 1 \tag{3.5}$$

Our first objective is studying the thermodynamical limit of these equations, that is to say $L \to \infty$, but $\frac{N}{L}$ constant. Motivated by our examples we suppose that the two body S matrix has the following form:

$$S(k, q) = \pm e^{i\chi(k,q)} \tag{3.6}$$

with $\chi(k, q)$ a well behaving function such that $\chi(k, k) = 0$; from the unitarity condition of S the χ function must be antisymmetric in its arguments $\chi(k, q) = -\chi(q, k)$. We can consider the logarithm of Bethe Equations as follows:

$$\begin{aligned} 1 = \prod_{j\neq i}^{N} S(k_i, k_j) e^{iLp(k_i)} &= (\pm)^{N-1} e^{i\sum_{j\neq i}\chi(k_i,k_j)+iLp(k_i)} \\ \Rightarrow \frac{1}{L}\sum_{j\neq i}^{N} \chi(k_i, k_j) + p(k_i) &= \frac{2\pi}{L} n_i + \frac{\pi}{L}\eta_N, \qquad n_i \in \mathbb{Z} \end{aligned} \tag{3.7}$$

where η_N is a parameter with value 0 or 1 in order to take care of the contribution of $(\pm)^{N-1}$.

The summation normalized on L has a very appealing aspect: we would like, in the thermodynamical limit, to exchange the summation with an integration up to $\mathcal{O}(L^{-1})$ corrections; we can do it thanks to the hypothesis $\frac{N}{L} = constant$.

Let $\rho(k)$ be the density of occupied quantum numbers k such that we can set:

$$\begin{aligned} &\frac{1}{L}\sum_{j\neq i}^{N}\chi(k_i,k_j) = \int dq\ \chi(k_i,q)\rho(q) + \mathcal{O}(L^{-1}) \\ &\Rightarrow \int dq\ \chi(k_i,q)\rho(q) + p(k_i) = \frac{2\pi}{L}n_i + \mathcal{O}(L^{-1}), \qquad n_i \in \mathbb{Z} \end{aligned} \tag{3.8}$$

It is convenient to define the following quantity:

$$\rho_t(k) = \frac{1}{2\pi}\partial_k\left[p(k) + \int dq\ \chi(k,q)\rho(q)\right] = \frac{1}{2\pi}\partial_k p(k) + \frac{1}{2\pi}\int dq\ \varphi(k,q)\rho(q) \tag{3.9}$$

Where we defined $\varphi(k,q) = \partial_k\chi(k,q)$. For now these equations are quite meaningless, but they will prove very useful in studying statistical problems and quench problems.

We can use this construction to evaluate the partition function of the GGE:

$$Z = \sum_{\vec{k}}^{ordered}\prod_{k_i\in\vec{k}} e^{-\beta\epsilon(k_i)} \tag{3.10}$$

where we introduced the β parameter in analogy with the Gibbs Ensemble.

The summation is over all the vectors $\vec{k}_N$ of N elements that satisfy the Bethe Equations. In the infinite volume limit we expect that $\log Z$ possesses an extensive part in the L parameter plus some corrections:

$$logZ = LA + corrections \tag{3.11}$$

The idea is to evaluate the summation through a functional integral approach: this is well justified only if $\langle N\rangle \sim L$, but the consistency of this assumption can be checked a posteriori.

$$Z = \sum_{\vec{k}_N}\prod_i e^{-\beta\epsilon(k_i)} = \sum_{\vec{k}_N} e^{-\beta\sum_i\epsilon(k_i)} \simeq \int D\rho e^{LS[\rho]-L\beta\int dk\,\rho(k)\epsilon(k)} \tag{3.12}$$

where $S[\rho]$ is a functional called entropy that counts the number of different configurations of the k_i that lead to the same distribution ρ. To evaluate $S[\rho]$ we can proceed as follows [9] (chap. 19.4).

Entropy in the Fermi case

Fix a distribution ρ and consider an interval around k of width Δ: in this interval there will be $N_k \simeq L\Delta\rho(k)$ particles. We must count the number of configurations

that give this result, with the constraint that the k numbers have to satisfy the Bethe Equations:

$$\int dq\ \chi(k_i,q)\rho(q) + p(k_i) = \frac{2\pi}{L} n_i + \mathcal{O}(L^{-1}), \qquad n_i \in \mathbb{Z} \tag{3.13}$$

Call $\tilde{\Delta}$ the width of the segment explored by n_i if we move k_i around k with width Δ:

$$\frac{1}{L}\tilde{\Delta} \simeq \frac{1}{2\pi}\partial_k \left[\int dq\ \chi(k,q)\rho(q) + p(k)\right] \Delta = \rho_t(k)\Delta \tag{3.14}$$

We can say that we have $\tilde{\Delta} \simeq L\rho_t(k)\Delta$ possible integers n_i such that the related k_i is in the desired interval. Counting the number of configurations such that a number of k_i equal to $L\rho(k)\Delta$ is in this interval can now be restated as: the number of possibilities of disposing $L\rho(k)\Delta$ identical objects in $\tilde{\Delta} \simeq L\rho_t(k)\Delta$ sites. Remember that two of these objects cannot occupy the same site due to the Fermi statistics of the quantum numbers. Let call $\#_k$ this number of configuration:

$$\#_k \simeq \binom{L\rho_t(k)\Delta}{L\rho(k)\Delta} \simeq e^{L\Delta(\rho_t(k)\log(\rho_t(k)) - \rho(k)\log(\rho(k)) - (\rho_t(k)-\rho(k))\log(\rho_t(k)-\rho(k)))} \tag{3.15}$$

In the last passage we use the Stirling formula assuming L large. If we indicate with $\#_\rho$ the number of configurations that lead to the same density we would have simply:

$$\begin{aligned} &\#_\rho = e^{LS[\rho]} \\ &S[\rho] = \int dk\ \rho_t(k)\log(\rho_t(k)) - \rho(k)\log(\rho(k)) - (\rho_t(k)-\rho(k))\log(\rho_t(k)-\rho(k)) \end{aligned} \tag{3.16}$$

Entropy in the Bose case

We can follow the same reasoning of the Fermi case: the only difference is that the quantum numbers do not have a Pauli exclusion principle to satisfy. With the same notation we would have:

$$\begin{aligned} &\#_k \simeq \binom{L\Delta(\rho_t(k)+\rho(k)) - 1}{L\Delta\rho(k)} \\ &\#_\rho = e^{LS[\rho]} \\ &S[\rho] = \int dk\,(\rho(k)+\rho_t(k))\log(\rho(k)+\rho_t(k)) - \rho\log\rho - \rho_t(k)\log\rho_t(k) \end{aligned} \tag{3.17}$$

Now we can return to our partition function Z: using a saddle point estimation we evaluate the leading term of $\log Z$:

$$\log Z = L\left(S[\rho] - \beta \int dk\ \rho(k)\epsilon(k)\right) + corrections \tag{3.18}$$

where ρ satisfies the saddle point equations

$$\frac{\delta}{\delta\rho}\left(S[\rho] - \beta \int dk\ \rho(k)\epsilon(k)\right) = 0 \tag{3.19}$$

$$\Rightarrow \begin{cases} \frac{\rho(q)}{\rho_t(q)-\rho(q)} \equiv e^{-\beta\tilde{\epsilon}(q)} \\ \beta\tilde{\epsilon}(q) = \beta\epsilon(q) - \frac{1}{2\pi}\int dk\,\varphi(k,q)\log\left(1+e^{-\beta\tilde{\epsilon}(k)}\right) \qquad Fermi\ \ case \\ \log Z = L\int dq \frac{\partial_q p(q)}{2\pi} log(1+e^{-\beta\tilde{\epsilon}(q)}) \end{cases} \tag{3.20}$$

$$\Rightarrow \begin{cases} \frac{\rho(p)}{\rho_t(p)+\rho(p)} \equiv e^{-\beta\tilde{\epsilon}(p)} \\ \beta\tilde{\epsilon}(p) = \beta\epsilon(p) + \frac{1}{2\pi}\int dk\,\varphi(k,p)\log\left(1-e^{-\beta\tilde{\epsilon}(k)}\right) \qquad Bose\ \ case \\ \log Z = -L\int dq \frac{\partial_q p(q)}{2\pi} log(1-e^{-\beta\tilde{\epsilon}(q)}) \end{cases} \tag{3.21}$$

$$\rho_t(k) = \frac{1}{2\pi}\partial_k p(k) + \frac{1}{2\pi}\int dq\ \varphi(k,q)\rho(q) \tag{3.22}$$

3.1.3 Local operators

This section is devoted to the definition and study of local operators in the limit $L \to \infty$.

Imagine a many body system with local interactions that is also integrable and the conserved charges are the momenta of the asymptotic states, see for example Lieb-Liniger or the models discussed in 2.3.1; the whole system is confined in a segment of length L.

As a starting point we will recall the properties of asymptotic states.

Asymptotic states

In integrable theories the number of particles is conserved, so we can define asymptotic states with a fixed number of particles. We will consider a bosonic field theory,

as we already said in 2.3.1 the eigenstates of the theory can be so expanded:

$$|\phi_{k_1,.,k_n}\rangle = \frac{1}{\sqrt{N(k)}} \int_0^L d^n x \; \phi_{k_1,.,k_n}(x_1,..x_n) \prod_j^n \psi^\dagger(x_j) \, |0\rangle \tag{3.23}$$

where $|0\rangle$ is the vacuum of the theory, ψ satisfies bosonic commutation rules $[\psi(x), \psi(y)] = 0$, $[\psi(x), \psi^\dagger(y)] = \delta(x-y)$. We will fix our notation requiring $k_1 \geq .. \geq k_n$ (in states), where k is the particle's momentum, $N(k)$ is a normalization we will fix later. We will indicate it in a more compact way as $\phi_{\vec{k}}(x_1,..x_n)$.

- $\phi_{\vec{k}}(x_1,..x_n)$ is symmetric in its variables.
- These states have a precise asymptotic limit: suppose that the variables can be divided into two groups $(x_1,..x_j)(x_{j+1}..x_n)$ such that $x_q << x_l \; \forall q \leq j, l \geq j+1$ where the length scale is the typical range of the interaction that we are considering. Then it is true that:

$$\phi_{\vec{k}}(x_1,..x_n) \simeq \sum_{\vec{k}=\vec{k}_1 \bigcup \vec{k}_2} S_{\vec{k}\vec{k}_1} \phi_{\vec{k}_1}(x_1,..x_j) \phi_{\vec{k}_2}(x_{j+1},..x_n) \tag{3.24}$$

  where $S_{\vec{k}\vec{k}_1}$ is the phase factor we need to order the momenta $(\vec{k}_1, \vec{k}_2) \to \vec{k}$ using the two body scattering matrix S. This expression is correct if we choose the following normalization for the one-particle wavefunction:

$$\phi_k(x) = e^{ikx} \tag{3.25}$$

  With this notation we fix the normalization $N(\vec{k}) = n! \int_0^L d^n x \; \phi^*_{\vec{k}}(\vec{x}) \phi_{\vec{k}}(\vec{x})$.

- The periodic boundary conditions will impose the set of Bethe equations on the physical states:

$$\phi_{\vec{k}}(x_1,..,x_j,..x_n) = \phi_{\vec{k}}(x_1,..,x_j+L,..x_n) \iff \prod_{j \neq i} S(k_i,k_j) e^{iLk_i} = 1 \tag{3.26}$$

  where $S(k,p)$ is the two bodies scattering matrix.

Local operators and their averages:

Here we will give the definition of local operators and study their expectation value on the eigenstates of the theory.

An operator Θ is local if there exists an interval I of finite length such that:

$$\forall x \notin I \qquad [\Theta, \psi(x)] = 0, \quad [\Theta, \psi^\dagger(x)] = 0 \tag{3.27}$$

This statement can be relaxed requiring that the commutator outside of the interval I goes to zero quickly enough, for example exponentially.

We also define the point operators as follows: an operator $O(x)$ is a point operator centered in x if it is an analytic function of the fields $\psi^\dagger(x)$ and $\psi(x)$ evaluated in x.

$$O(x) = \sum_{n,m} a_{n,m}[\psi^\dagger(x)]^n[\psi(x)]^m \tag{3.28}$$

where we admit that the coefficients $a_{n,m}$ can be also derivative operators acting on the fields.

Notice that a local operator Θ on an interval I can be expressed through multiplication and summation of point operators on a bounded interval:

$$\Theta = \int d\vec{x} f(\vec{x}) \prod_j O_j(x_j), \qquad f(\vec{x}) = 0 \; if \; \exists x_j \in \vec{x} \; s.t. \; x_j \notin I \tag{3.29}$$

This because each operator can be expressed as a function of the ψ, $\psi^\dagger$ operators and they are point operators.

Even if we are interested in general local operators, we start by considering point operators: later it will be clear the generalization to the local ones. So suppose:

$$\langle \phi_{\vec{k}}| \, O(q) \, |\phi_{\vec{p}}\rangle \, , \qquad O(q) = F(\psi^\dagger(q), \psi(q)) \tag{3.30}$$

with F an analytic function. Suppose that the vectors $\vec{k}$ and $\vec{p}$ have respectively N and M elements. From the definitions:

$$\begin{gathered} \langle \phi_{\vec{k}}| \, O(q) \, |\phi_{\vec{p}}\rangle = \\ = \frac{1}{\sqrt{N(\vec{k})}} \frac{1}{\sqrt{N(\vec{p})}} \int_0^L d^N x \, d^M y \, \phi^*_{\vec{k}}(\vec{x}) \phi_{\vec{p}}(\vec{y}) \, \langle 0| \prod_j \psi(x_j) O(q) \prod_i \psi^\dagger(y_i) \, |0\rangle \end{gathered} \tag{3.31}$$

We will now study $\langle 0| \prod_j \psi(x_j) O(q) \prod_i \psi^\dagger(y_i) \, |0\rangle$, it is quite easy to see that:

$$\begin{gathered} \langle 0| \prod_j \psi(x_j) O(q) \prod_i \psi^\dagger(y_i) \, |0\rangle = \\ = \sum_n {\sum_{\sigma,\rho}}' \prod_{j=1}^n \delta(x_{\sigma_j} - y_{\rho_j}) \prod_{j=n+1}^N \delta(x_{\sigma_j} - q) \prod_{j=n+1}^M \delta(y_{\rho_j} - q) \, \langle 0| \, O(q)^{N-n;M-n} \, |0\rangle \end{gathered} \tag{3.32}$$

where $O(q)^{N-n;M-n} = \left(\partial_a^{N-n} \partial_b^{M-n} F(a,b)\right)_{a=\phi^+(q),b=\phi(q)}$.

The sum over the permutations identifies the permutations that leave $\prod_{j=1}^n \delta(x_{\sigma_j} - y_{\rho_j})$ the same.

Expression (3.31) becomes:

$$\frac{1}{\sqrt{N(\vec{k})}}\frac{1}{\sqrt{N(\vec{p})}}\sum_n{}'\sum_{\sigma,\rho}\langle 0|\,O(q)^{N-n;M-n}\,|0\rangle\int_0^L d^Nx\,d^My\phi^*_{\vec{k}}(\vec{x})\phi_{\vec{p}}(\vec{y})\cdot$$
$$\cdot\prod_{j=1}^n\delta(x_{\sigma_j}-y_{\rho_j})\prod_{j=n+1}^N\delta(x_{\sigma_j}-q)\prod_{j=n+1}^M\delta(y_{\rho_j}-q) \tag{3.33}$$

Now we can use that the wavefunctions are symmetric in their arguments:

$$\sum_{\sigma,\rho}{}'\int_0^L d^Nx\,d^My\,\phi^*_{\vec{k}}(\vec{x})\phi_{\vec{p}}(\vec{y})\prod_{j=1}^n\delta(x_{\sigma_j}-y_{\rho_j})\prod_{j=n+1}^N\delta(x_{\sigma_j}-q)\prod_{j=n+1}^M\delta(y_{\rho_j}-q)=$$
$$=n!\binom{N}{n}\binom{M}{n}\int_0^L d^Nx\,d^My\,\phi^*_{\vec{k}}(\vec{x})\phi_{\vec{p}}(\vec{y})\prod_{j=1}^n\delta(x_j-y_j)\prod_{j=n+1}^N\delta(x_j-q)\prod_{j=n+1}^M\delta(y_j-q) \tag{3.34}$$

To simplify the notation set $\phi_{\vec{p}}(\vec{y}_n,q_{M-n})\equiv\phi_{\vec{p}}(y_1,...y_n,q,q,q..,q)$

$$\langle\phi_{\vec{k}}|\,O(q)\,|\phi_{\vec{p}}\rangle=\frac{1}{\sqrt{N(\vec{k})}}\frac{1}{\sqrt{N(\vec{p})}}\sum_n n!\binom{N}{n}\binom{M}{n}\cdot$$
$$\cdot\langle 0|\,O(q)^{N-n;M-n}\,|0\rangle\int_0^L d^nx\,\phi^*_{\vec{k}}(\vec{x}_n,q_{N-n})\phi_{\vec{p}}(\vec{x}_n,q_{M-n}) \tag{3.35}$$

Till now the formula is exact, in the next section we will attempt to evaluate the $L\to\infty$ limit.

Infinite length limit

Now we analyze the infinite length limit in the previous expression, in particular we need to estimate the expression below:

$$\int_0^L d^nx\,\phi^*_{\vec{k}}(\vec{x}_n,q_{N-n})\phi_{\vec{p}}(\vec{x}_n,q_{M-n}) \tag{3.36}$$

This expression is quite similar to the scalar product of two asymptotic states, but is not exactly the same. We expect some divergences due to the integration in the infinite volume.

We can proceed as follows: change the integration variables $Ly_i=x_i$

$$L^n\int_0^1 d^ny\,\phi^*_{\vec{k}}(L\vec{y}_n,q_{N-n})\phi_{\vec{p}}(L\vec{y}_n,q_{M-n}) \tag{3.37}$$

The argument is symmetric exchanging the variables of integration, so:

$$\begin{aligned} &L^n n! \int_{0<y_1<y_2<..<y_n<1} d^n y\, \phi^*_{\vec{k}}(L\vec{y}_n, q_{N-n})\phi_{\vec{p}}(L\vec{y}_n, q_{M-n}) = \\ &= L^n n! \sum_j \int_{0<y_1<..<y_j<\frac{q}{L}<y_{j+1}<..<y_n<1} d^n y\, \phi^*_{\vec{k}}(L\vec{y}_n, q_{N-n})\phi_{\vec{p}}(L\vec{y}_n, q_{M-n}) \end{aligned} \tag{3.38}$$

In the infinite length limit, there will be only one term that dominates the sum:

$$= L^n n! \left(\int_{0<\frac{q}{L}<y_1<..<y_j<y_{j+1}<..<y_n<1} d^n y\, \phi^*_{\vec{k}}(L\vec{y}_n, q_{N-n})\phi_{\vec{p}}(L\vec{y}_n, q_{M-n}) + \mathcal{O}(L^{-1}) \right) \tag{3.39}$$

We will study only the leading term, so we drop $\mathcal{O}(L^{-1})$ and check later the validity of this assumption. We can proceed further if we consider that, a part from a $\mathcal{O}(L^{-1})$ domain, we are in an asymptotic regime for the q variables with respect to the others, so we can use the asymptotic property of the wavefunction:

$$\phi_{\vec{k}}(\vec{x}, q_{N-n}) = \sum_{\vec{k}=\vec{k}_1 \bigcup \vec{k}_2} S_{\vec{k}\vec{k}_1} \phi_{\vec{k}_1}(\vec{x})\phi_{\vec{k}_2}(q_{N-n}) \tag{3.40}$$

So eq. (3.39) becomes:

$$\begin{aligned} &\sum_{\vec{k}=\vec{k}_1 \bigcup \vec{k}_2} \sum_{\vec{p}=\vec{p}_1 \bigcup \vec{p}_2} S^*_{\vec{k}\vec{k}_1} S_{\vec{p}\vec{p}_1} \phi^*_{\vec{k}_2}(q_{N-n})\phi_{\vec{p}_2}(q_{M-n}) \int_0^L d^n y\, \phi^*_{\vec{k}_1}(\vec{x})\phi_{\vec{p}_1}(\vec{x}) = \\ &= \sum_{\vec{k}=\vec{k}_1 \bigcup \vec{k}_2} \sum_{\vec{p}=\vec{p}_1 \bigcup \vec{p}_2} S^*_{\vec{k}\vec{k}_1} S_{\vec{p}\vec{p}_1} \phi^*_{\vec{k}_2}(q_{N-n})\phi_{\vec{p}_2}(q_{M-n}) \frac{1}{n!} \sqrt{N(\vec{k}_1)} \sqrt{N(\vec{p}_1)}\, \langle \phi_{\vec{k}_1} | \phi_{\vec{p}_1} \rangle \end{aligned} \tag{3.41}$$

We put this result in eq. (3.35) and obtain:

$$\begin{aligned} \Rightarrow \langle \phi_{\vec{k}} |\, O(q)\, | \phi_{\vec{p}} \rangle = \sum_n \sum_{\vec{k}=\vec{k}_1 \bigcup \vec{k}_2} \sum_{\vec{p}=\vec{p}_1 \bigcup \vec{p}_2} S^*_{\vec{k}\vec{k}_1} S_{\vec{p}\vec{p}_1} \binom{N}{n} \binom{M}{n} \frac{\sqrt{N(\vec{k}_1)N(\vec{p}_1)}}{\sqrt{N(\vec{k})N(\vec{p})}} \cdot \\ \cdot \langle 0 |\, O(q)^{N-n;M-n}\, | 0 \rangle\, \phi^*_{\vec{k}_2}(q_{N-n})\phi_{\vec{p}_2}(q_{M-n})\, \langle \phi_{\vec{k}_1} | \phi_{\vec{p}_1} \rangle \end{aligned} \tag{3.42}$$

At this point set $\mathcal{P}(\vec{k}) = \frac{1}{(N!)^2 L^N} N(\vec{k})$ and define the connected average:

$$\langle \vec{k}_2 |\, O(q)\, | \vec{p}_2 \rangle_{conn} \equiv \frac{1}{\sqrt{(N-n)!(M-m)!}} \langle 0 |\, O(q)^{N-n;M-n}\, | 0 \rangle\, \phi^*_{\vec{k}_2}(q_{N-n})\phi_{\vec{p}_2}(q_{M-n}) \tag{3.43}$$

We finally obtain:

$$\langle\phi_{\vec{k}}|\,O(q)\,|\phi_{\vec{p}}\rangle = \sum_{\vec{k}=\vec{k}_1\bigcup\vec{k}_2}\sum_{\vec{p}=\vec{p}_1\bigcup\vec{p}_2} S^*_{\vec{k}\vec{k}_1}S_{\vec{p}\vec{p}_1}\left(\frac{1}{L}\right)^{\frac{1}{2}(N+M)-n_1}\frac{\sqrt{\mathcal{P}(\vec{k}_1)\mathcal{P}(\vec{p}_1)}}{\sqrt{\mathcal{P}(\vec{k})\mathcal{P}(\vec{p})}}\cdot \\ \cdot\langle\vec{k}_2|\,O(q)\,|\vec{p}_2\rangle_{conn}\langle\phi_{\vec{k}_1}|\phi_{\vec{p}_1}\rangle \tag{3.44}$$

with n_1 the number of momenta in $\vec{k}_1$.

Infinite length limit with fixed number of particles, connected part:

In this section we will study the infinite length limit keeping fixed the number of particles, so we can suppose $N << \mathcal{O}(L)$. It can be seen immediately that with this normalization, supposing all the momenta different, in the infinite length limit $N(\vec{k}) = (d!)^2L^d\,(1+\mathcal{O}(L^{-1}))$ with d the number of momenta of the state.

Evaluating $\langle\phi_{\vec{k}_1}|\phi_{\vec{p}_1}\rangle$ in this limit is not too difficult. It is sufficient to remember that the momenta are ordered: this select the leading contribution with $\vec{k}_1\sim\vec{p}_1$. We suppose the momenta in $\vec{k}$ are all different.

$$\langle\phi_{\vec{k}_1}|\phi_{\vec{p}_1}\rangle = \prod_{k_j\in\vec{k}_1,p_j\in\vec{p}_1}^{n_1}\frac{e^{iL(p_j-k_j)}-1}{iL(p_j-k_j)}+\mathcal{O}(L^{-1}) \tag{3.45}$$

In this product the elements are ordered $p_1\geq p_2..\geq p_{n_1}$, $k_1\geq k_2..\geq k_{n_1}$.

$$\langle\phi_{\vec{k}}|\,O(q)\,|\phi_{\vec{p}}\rangle = \sum_{\vec{k}=\vec{k}_1\bigcup\vec{k}_2}\sum_{\vec{p}=\vec{p}_1\bigcup\vec{p}_2} S^*_{\vec{k}\vec{k}_1}S_{\vec{p}\vec{p}_1}\left(\frac{1}{L}\right)^{\frac{1}{2}(N+M)-n_1}\langle\vec{k}_2|\,O(q)\,|\vec{p}_2\rangle_{conn}\cdot \\ \cdot\prod_{k_j\in\vec{k}_1,p_j\in\vec{p}_1}^{n_1}\frac{e^{iL(p_j-k_j)}-1}{iL(p_j-k_j)} \tag{3.46}$$

In the infinite length limit the set of possible k numbers becomes a continuum, so it is possible to substitute summations with integrals. If we consider the limit $L\to\infty$ with a finite number of particles, the Bethe Equations decouple as if the system is free.

For this purpose consider the following expression

$$\sum_{\vec{k}} W(\vec{k})\,\langle\phi_{\vec{k}}|\,O(q)\,|\phi_{\vec{k}}\rangle \tag{3.47}$$

with $W(\vec{k})$ a regular function, then we use the previous expression for the average of

point operators and substitute the summations with integrals:

$$\sum_{\vec{k}} W(\vec{k}) \langle \phi_{\vec{k}} | O(q) | \phi_{\vec{k}} \rangle \to \int d^N k \left(\frac{L}{2\pi}\right)^N W(\vec{k}) \sum_{\vec{k}=\vec{k}_1 \bigcup \vec{k}_2} \sum_{\vec{k}=\vec{p}_1 \bigcup \vec{p}_2} S^*_{\vec{k}\vec{k}_1} S_{\vec{p}\vec{p}_1} \cdot \left(\frac{1}{L}\right)^{N-n_1} \langle \vec{k}_2 | O(q) | \vec{p}_2 \rangle_{conn} \prod_{k_j \in \vec{k}_1, p_j \in \vec{p}_1}^{n_1} \frac{e^{iL(p_j-k_j)}-1}{iL(p_j-k_j)} = \tag{3.48}$$

$$= \int d^N k \, W(\vec{k}) \sum_{\vec{k}=\vec{k}_1 \bigcup \vec{k}_2} \sum_{\vec{k}=\vec{p}_1 \bigcup \vec{p}_2} \frac{1}{(2\pi)^{N-n_1}} S^*_{\vec{k}\vec{k}_1} S_{\vec{p}\vec{p}_1} \langle \vec{k}_2 | O(q) | \vec{p}_2 \rangle_{conn} \prod_{k_j \in \vec{k}_1, p_j \in \vec{p}_1}^{n_1} \delta(p_j - k_j) \tag{3.49}$$

with the regularization $\delta(0) = \frac{L}{2\pi}$.

Set $\langle \vec{k}_\infty | O(q) | \vec{p}_\infty \rangle \equiv L^{\frac{1}{2}(N+M)} \langle \phi_{\vec{k}} | O(q) | \phi_{\vec{p}} \rangle$ in order to have the following correspondence:

$$\sum_{\vec{k}} W(\vec{k}) \langle \phi_{\vec{k}} | O(q) | \phi_{\vec{k}} \rangle \to \int \frac{d^N k}{(2\pi)^N} W(\vec{k}) \langle \vec{k}_\infty | O(q) | \vec{k}_\infty \rangle \tag{3.50}$$

Now we will give a new meaning to $\langle \vec{k} | O(q) | \vec{k} \rangle_{conn}$. Consider the expression for $\langle \vec{k}_\infty | O(q) | \vec{p}_\infty \rangle$ and slightly shift the bra's momenta in the complex plane $k_i \to k_i - i\epsilon_i$, $\epsilon_i > 0$.

ϵ will be small so in the following expression we will neglect it where possible: the non trivial behavior is only in $\frac{e^{iL(p_j-k_j)}-1}{iL(p_j-k_j)}$.

$$\langle \vec{k}^+_\infty | O(q) | \vec{p}_\infty \rangle = \sum_{\vec{k}=\vec{k}_1 \bigcup \vec{k}_2} \sum_{\vec{p}=\vec{p}_1 \bigcup \vec{p}_2} S^*_{\vec{k}\vec{k}_1} S_{\vec{p}\vec{p}_1} L^{n_1} \langle \vec{k}_2 | O(q) | \vec{p}_2 \rangle_{conn} \cdot \cdot \prod_{k_j \in \vec{k}_1, p_j \in \vec{p}_1}^{n_1} \frac{e^{-L\epsilon_j + iL(p_j-k_j)} - 1}{-\epsilon_j L + iL(p_j - k_j)} \tag{3.51}$$

In the infinite length limit we can safely neglect the exponential correction for each strictly positive value of ϵ_j

$$\langle \vec{k}^+_\infty | O(q) | \vec{p}_\infty \rangle = \sum_{\vec{k}=\vec{k}_1 \bigcup \vec{k}_2} \sum_{\vec{p}=\vec{p}_1 \bigcup \vec{p}_2} S^*_{\vec{k}\vec{k}_1} S_{\vec{p}\vec{p}_1} \langle \vec{k}_2 | O(q) | \vec{p}_2 \rangle_{conn} \prod_{k_j \in \vec{k}_1, p_j \in \vec{p}_1}^{n_1} \frac{1}{\epsilon_j - i(p_j - k_j)} \tag{3.52}$$

In order to select the connected part we must take the prescription of rejecting all the terms that behave like $\frac{1}{\epsilon}$ for some values of momenta.

It can be shown that also for local operators an analogous construction holds: it is sufficient to consider the point operators content of the local operator and proceed analogously

$$\langle\phi_{\vec{k}}|\,\Theta\,|\phi_{\vec{p}}\rangle = \sum_{\vec{k}=\vec{k}_1\bigcup\vec{k}_2}\sum_{\vec{p}=\vec{p}_1\bigcup\vec{p}_2} S^*_{\vec{k}\vec{k}_1}S_{\vec{p}\vec{p}_1}\left(\frac{1}{L}\right)^{\frac{1}{2}(N+M)-n_1}\frac{\sqrt{\mathcal{P}(\vec{k}_1)\mathcal{P}(\vec{p}_1)}}{\sqrt{\mathcal{P}(\vec{k})\mathcal{P}(\vec{p})}}\cdot \tag{3.53}$$
$$\cdot\,\langle\vec{k}_2|\,\Theta\,|\vec{p}_2\rangle_{conn}\,\langle\phi_{\vec{k}_1}|\phi_{\vec{p}_1}\rangle$$

where the connected part can be extracted from the $L\to\infty$ limit keeping finite the number of particles:

$$\langle\vec{k}^+_\infty|\,\Theta\,|\vec{p}_\infty\rangle = \sum_{\vec{k}=\vec{k}_1\bigcup\vec{k}_2}\sum_{\vec{p}=\vec{p}_1\bigcup\vec{p}_2} S^*_{\vec{k}\vec{k}_1}S_{\vec{p}\vec{p}_1}\,\langle\vec{k}_2|\,\Theta\,|\vec{p}_2\rangle_{conn}\prod_{k_j\in\vec{k}_1,p_j\in\vec{p}_1}^{n_1}\frac{1}{\epsilon_j - i(p_j-k_j)} \tag{3.54}$$

The pole structure of $\langle\vec{k}_\infty|\,\Theta\,|\vec{p}_\infty\rangle$, together with a series of symmetry hypothesis, leads to the Form Factors's Bootstrap program that permits us to reconstruct the connected terms (also called connected Form Factors) of lots of local operators.

We can derive some of the Bootstrap's equations from this approach, in particular the equations known as "kinematic singularities". Consider $\langle\vec{k}_\infty|\,O(q)\,|\vec{p}_\infty\rangle$, fix $\vec{k}=(k_1,k_2,..,k_m)$ and $\vec{p}=(p_1,p_2..,p_n)$; we consider now the case $p_n\sim k_m$. Let be $\vec{k}'$ the vector $\vec{k}$ without k_m and $\vec{p}'$ the same for $\vec{p}$.

As before, we would obtain this kind of decomposition:

$$\langle\vec{k}_\infty|\,O(q)\,|\vec{p}_\infty\rangle = \langle\vec{k}'_\infty|\,O(q)\,|\vec{p}'_\infty\rangle\,\frac{e^{iL(p_n-k_m)}-1}{i(p_n-k_m)} + terms \tag{3.55}$$

where "terms" does not contain any contribution like $\frac{1}{p_n-k_m}$. We want L to disappear completely from our equations, so we will use the Bethe Equations

$$e^{iLp_n}\prod_{j\neq n}S(p_n,p_j)=1; \qquad e^{iLk_m}\prod_{j\neq m}S(k_m,k_j)=1 \tag{3.56}$$

$$\Rightarrow\langle\vec{k}_\infty|\,O(q)\,|\vec{p}_\infty\rangle = \langle\vec{k}'_\infty|\,O(q)\,|\vec{p}'_\infty\rangle\,\frac{\prod_{j\neq m}S(k_m,k_j)\left(\prod_{j\neq n}S(p_n,p_j)\right)^{-1}-1}{i(p_n-k_m)} + terms \tag{3.57}$$

We can now take the residue in this expression and we arrive at the following relation:

$$\lim_{p_n\to k_m}(p_n-k_m)\,\langle\vec{k}_\infty|\,O(q)\,|\vec{p}_\infty\rangle = i\,\langle\vec{k}'_\infty|\,O(q)\,|\vec{p}'_\infty\rangle\left(1-\prod_{j\neq m}S(k_m,k_j)\prod_{j\neq n}S^{-1}(k_m,p_j)\right) \tag{3.58}$$

These equations, joint with other symmetries, permit us to write recursive relations for the averages of local operators.

Here we will not introduce the bootstrap approach, for references see [12] or also [9] (chap. 17) for the bootstrap approach to the S matrix. In [13] a practical bootstrap calculation for the Sinh-Gordon model is shown.

Even though we know the connected Form Factors we have not calculated the entire expression for the average yet: as a matter of fact to calculate $\mathcal{P}$ and the scalar product $\langle \phi_{\vec{k}_1} | \phi_{\vec{p}_1} \rangle$ we need the wavefunction and use it to perform a complicate multi-dimensional integration.

In the case of free theories, both in the Bose case ($S = 1$) and in the Fermi one ($S = -1$), the average of local operators can be evaluated quite easily on the Bethe States:

$$\langle \vec{k} | \Theta | \vec{p} \rangle = \sum_{\vec{k}=\vec{k}_1 \bigcup \vec{k}_2} \sum_{\vec{p}=\vec{p}_1 \bigcup \vec{p}_2} S^*_{\vec{k}\vec{k}_1} S_{\vec{p}\vec{p}_1} \left(\frac{1}{L}\right)^{\frac{1}{2}(N+M)-n_1} \langle \vec{k}_2 | \Theta | \vec{p}_2 \rangle_{conn} \delta_{\vec{k}_1,\vec{p}_1} \tag{3.59}$$

where $\delta_{\vec{k}_1,\vec{p}_1}$ is the multi-dimensional Kronecker delta and the exclusion principle in the Fermi case is at work. The summation is on all the distinct ways of splitting in two parts the sets $\vec{k}$ and $\vec{p}$. This derivation uses as quantum numbers directly the momenta, this is not restrictive because the relation between quantum numbers and momenta is supposed to be bijective. We can use different quantum numbers to label the particles, for example their rapidities: so suppose that the momentum associated to k is $p(k)$, then in eq. (3.48) appears a non trivial jacobian that we can insert in the definition of $\langle \vec{k}_\infty | O(q) | \vec{p}_\infty \rangle$ to fulfill eq. (3.50). We have this substitution:

$$\langle \vec{k}_\infty | O(q) | \vec{p}_\infty \rangle \to \frac{1}{\sqrt{\prod_{p_j \in \vec{p}_2} [2\pi\rho_t(p_j)] \prod_{k_j \in \vec{k}_2} [2\pi\rho_t(k_j)]}} \langle \vec{k}_\infty | O(q) | \vec{p}_\infty \rangle \tag{3.60}$$

where $\rho_t(k) = \frac{\partial_k p(k)}{2\pi}$. Also the connected part undergoes the same substitution, so we arrive at the final expression:

$$\langle \vec{k} | \Theta | \vec{p} \rangle = \sum_{\vec{k}=\vec{k}_1 \bigcup \vec{k}_2} \sum_{\vec{p}=\vec{p}_1 \bigcup \vec{p}_2} \frac{S^*_{\vec{k}\vec{k}_1} S_{\vec{p}\vec{p}_1}}{\sqrt{\prod_{p_j \in \vec{p}_2} [L2\pi\rho_t(p_j)] \prod_{k_j \in \vec{k}_2} [L2\pi\rho_t(k_j)]}} \langle \vec{k}_2 | \Theta | \vec{p}_2 \rangle_{conn} \delta_{\vec{k}_1,\vec{p}_1} \tag{3.61}$$

For non trivial integrable theories the diagonal terms in the Fermi case are known.

We give a result from [15], [16] that with our notation is written:

$$\langle\vec{p}|\,\Theta\,|\vec{p}\rangle = \sum_{\vec{p}=\vec{p}_1\bigcup\vec{p}_2}\frac{J(\vec{p}|\vec{p}_1)}{J(\vec{p})}\,\langle p_2|\,\Theta\,|p_2\rangle_{conn}\,;\qquad J(\vec{p})=\det(M)$$
$$M_{i,j}=2\pi L\left(\delta_{i,j}\left(\frac{\partial_{p_i}p(p_i)}{2\pi}+\frac{1}{L2\pi}\sum_{k\neq j}\varphi(p_j,p_k)\right)+(1-\delta_{i,j})\frac{1}{L2\pi}\varphi(p_j,p_i)\right) \tag{3.62}$$

here $J(\vec{p}|\vec{p}_1)$ is the determinant of the minor of M associated to the indices of $\vec{p}_1$, for example if $\vec{p}_1=(p_{j_1}.p_{j_2},p_{j_3}..)$ then $J(\vec{p}|\vec{p}_1)$ is constructed with the indices $j_1,j_2,j_3..$

For the off diagonal terms, the same articles provide a formulation based on the pole structure of the form factors program, but we will not use it in this thesis. For further reading see also [37].

3.1.4 Le Clair-Mussardo's conjecture

The Le Clair-Mussardo's conjecture regards the application of a GGE density matrix to averages of local operators. It was stated for the first time in [18] and then proven by Pozsgay in [19]. In this section we will give the results and describe the approach of the two articles.

The Le Clair-Mussardo's conjecture was originally stated for relativistc quantum integrable systems, so the Bethe States are labeled with the rapidities of the particles and the S matrix is supposed to be relativistic invariant: $S(\theta,\theta')=S(\theta-\theta')$.

Then let $\mathcal{O}$ be a local operator, the Le Clair-Mussardo's conjecture asserts:

$$\rho_{GGE}=\sum_{\vec{\theta}}^{ordered}e^{-\sum_{\theta_i\in\vec{\theta}}\epsilon(\theta_i)}\,|\vec{\theta}\rangle\,\langle\vec{\theta}| \tag{3.63}$$

$$\Rightarrow\frac{tr(\mathcal{O}\rho_{GGE})}{tr(\rho_{GGE})}=\sum_n\frac{1}{n!}\int\frac{d\theta^n}{(2\pi)^n}\prod_i\frac{e^{-\tilde{\epsilon}(\theta_i)}}{1-S(0)e^{-\tilde{\epsilon}(\theta_i)}}\,\langle\theta_1,\theta_2,..\theta_n|\,\mathcal{O}\,|\theta_n,\theta_2,..\theta_n\rangle_{conn} \tag{3.64}$$

where:

$$\begin{cases}\frac{\rho(p)}{\rho_t(p)-\rho(p)}=e^{-\tilde{\epsilon}(p)}\\ \tilde{\epsilon}(p)=\epsilon(p)-\frac{1}{2\pi}\int dk\,\varphi(k-p)\log\left(1+e^{-\tilde{\epsilon}(k)}\right)\end{cases}\qquad Fermi\ \ case\ S(0)=-1 \tag{3.65}$$

$$\begin{cases}\frac{\rho(p)}{\rho_t(p)+\rho(p)}=e^{-\tilde{\epsilon}(p)}\\ \tilde{\epsilon}(p)=\epsilon(p)+\frac{1}{2\pi}\int dk\,\varphi(k-p)\log\left(1-e^{-\tilde{\epsilon}(k)}\right)\end{cases}\qquad Bose\ \ case\ S(0)=1 \tag{3.66}$$

$$\rho_t(\theta) = \frac{m}{2\pi}\cosh(\theta) + \frac{1}{2\pi}\int dq\ \varphi(\theta - q)\rho(q), \qquad \varphi(\theta) = -i\partial_\theta \log S(\theta) \tag{3.67}$$

In the original article by Le Clair and Mussardo this formula was derived in the free case, then using some saddle point arguments on the partition function the general formula was argued. The conjecture, as we said, has been proven by Pozsgay: he used the expression for diagonal expectations values of local operators of eq (3.62) at the end of section 3.1.3 ([15],[16]), joined with some arguments by a method called Generalized Thermodynamic Bethe Ansatz [20] that we are going to explain in 3.2.

At this point we have the average of local observables given by a GGE density matrix, that is to say a candidate to describe the steady situation for these operators: the next problem is to understand which initial states give this kind of relaxation. This question does not have an exhaustive answer yet, especially in non trivial integrable theories which cannot be mapped in free theories. We will analyze this aspect in Chapter 4 and especially in Chapter 5.

3.2 Generalized Thermodynamic Bethe Ansatz

This approach was born to deal with the non equilibrium problem of quantum quenches. Here we restrict us to the case of a sudden quench: the Hamiltonian is suddenly changed and the state evolves with the post quench Hamiltonian, that will be supposed integrable.

Among all the different ways of dealing with quenches, we choose to describe the GTBA for some similarities with the techniques of Chapter 4 and for the role that this approach could have in future studies in quenches in interacting integrable theories, see [19], [43].

For our purpose it will be sufficient to consider the system initialized in a state $|\psi\rangle$ at $t = 0$; then we will suppose the time evolution $|\psi\rangle \to |\psi(t)\rangle = e^{-iHt}|\psi\rangle$ with H an integrable Hamiltonian. The problem is studying the time evolution of local operators in the infinite volume limit:

$$\langle \mathcal{O}(t)\rangle \equiv \lim_{L\to\infty} \frac{\langle\psi(t)|\,\mathcal{O}\,|\psi(t)\rangle}{\langle\psi|\psi\rangle} \tag{3.68}$$

In particular we are interested in the limit $t \to \infty$, what follows can be found in [20]. As a first step we expand $|\psi(t)\rangle$ in the eigenvectors representation of the integrable model:

$$|\psi(t)\rangle = \sum_{\vec{k}}^{ordered} e^{-\xi(\vec{k})} e^{-iE(\vec{k})t}\,|\vec{k}\rangle \tag{3.69}$$

We suppose the eigenstates $|\vec{k}\rangle$ normalized to unity, $E(\vec{k}) = \sum_{k\in\vec{k}} E(k)$ and ξ is defined as:

$$e^{-\xi(\vec{k})} = \langle\vec{k}|\psi\rangle \tag{3.70}$$

With this notation we have:

$$\begin{aligned}\langle\psi(t)|\,\mathcal{O}\,|\psi(t)\rangle &= \sum_{\vec{k},\vec{q}}^{ordered} e^{-(\xi^*(\vec{q})+\xi(\vec{k}))}e^{-it(E(\vec{k})-E(\vec{q}))}\,\langle\vec{q}|\,\mathcal{O}\,|\vec{k}\rangle \\ \langle\psi|\psi\rangle &= \sum_{\vec{k}}^{ordered} e^{-(\xi^*(\vec{k})+\xi(\vec{k}))}\end{aligned} \tag{3.71}$$

At this point the idea is implementing a functional-integral approach and using some characteristics of local operators. The authors assume that the initial state's norm is dominated by large numbers of particles $\sim L$ and show that in the infinite volume limit the evolution of the state can be computed with only a single summation instead of a double one in the following way

$$\langle\mathcal{O}(t)\rangle = \lim_{L\to\infty}\sum_{\vec{k}}^{ordered}\left[e^{-(\xi^*(\vec{p})+\xi(\vec{k}))}e^{-it(E(\vec{k})-E(\vec{p}))}\frac{\langle\vec{p}|\,\mathcal{O}\,|\vec{k}\rangle}{2}+c.c.\right] \tag{3.72}$$

where $|\vec{p}\rangle$ is a typical state chosen such that the distribution of its quantum numbers p_i follows a density driven by the saddle point equations determined by the norm:

$$\begin{aligned}&\xi^*(\vec{k})+\xi(\vec{k})\to LW[\rho] \\ &\langle\psi|\psi\rangle = \sum_{\vec{k}}^{ordered} e^{-(\xi^*(\vec{k})+\xi(\vec{k}))}\to\int D\rho e^{-LW[\rho]+LS[\rho]} \\ &\to\frac{\delta}{\delta\rho}\left(-W[\rho]+S[\rho]\right)=0\end{aligned} \tag{3.73}$$

where $S[\rho]$ is the entropy defined in section 3.1.2. We stress that to justify the functional-integral approach we need some regularity hypothesis for the functions $\xi(\vec{k})$ and write them as an extensive functional of the density ρ.

If this approach is justified, then the long time limit of this expression should be:

$$\lim_{t\to\infty}\langle O(t)\rangle = \lim_{L\to\infty}\langle\vec{p}|\,\mathcal{O}\,|\vec{p}\rangle \tag{3.74}$$

A basic hypothesis at the root of this formula is that the average $\langle p|\,\mathcal{O}\,|p\rangle$ is essentially insensitive to the microscopic difference among the states $|p\rangle$, if they respect the same global density distribution.

This approach suggests that the thermodynamic limit of time averages of local observables can be described through a typical state whose density of particles respects a set of saddle point equations. Besides the time evolution could be described by a finite number of excitations respect to this thermal state: these simplifications indicate that the GTBA could be used to study the dynamic of local observables in

interacting theories, for example a recent work [43] provides a long time asymptotic behavior with this technique. An useful discussion between quenches and GGE can be found in [30] and in [44].

Chapter 4

Local averages over squeezed states

In this chapter we deal with a specific problem: we want to study the averages of local observables initialized on a special class of states, called squeezed states. This kind of initial state is the natural byproduct of many quenches in free theories or in models that can be mapped to them, see for example [44] and reference therein.

The situation is more complex in truly interacting theories and simple quantum quench protocols can lead to more complex initial states, even if the model remains integrable; see for example the discussion in [35].

A part from these limitations, squeezed states remain a good starting point to study quenches in integrable theories and they are considered in [24], [19], [43].

This chapter is mostly an original contribution: the purpose is to study the time dependence of averages of local observables over squeezed states. The idea is a natural generalization of [24] where the time average of these observables has been calculated in the case of Infinite-Volume integrable theories, then a form in the interacting case was argued in close analogy with the Le Clair-Mussardo's conjecture. The formula in the interacting case has been proved in [19]. An interesting result of these articles is that the infinite time averages of local observables is well described by the GGE density matrix.

We summarize the approaches of these two articles and then describe the differences with respect to what we are going to do:

- In [24] an infinite volume approach is used: this gives problems of regularizing singularities such as $\delta(0)$, and only the time average in the free case has been calculated.
- In [19] the author considers only the time averages giving a proof for the interacting case. He used a finite volume regularization ([15], [16]) joined with arguments taken from the GTBA approach [20] that permits to calculate time

averages as expectation values over a special "thermal state". The GTBA approach indicates also that this time average equals the infinite time limit of these averages.

Now we outline the approach we are going to use and which results can be obtained:

- The system is put in a finite volume of length L and the results are obtained in the $L \to \infty$ limit. This approach avoids ill defined terms such as $\delta(0)$.
- A new graphical approach is constructed. With this new tool we can study the free case obtaining not only the result of [24], but the entire time dependence of the expectation value.
- Using the results of [15], [16] we managed to obtain the result by [19] without using GTBA arguments, even if functional integrals are still used.

Motivated by the known results that we can obtain and by the new result in the free case, we hope to use these techniques in future investigations to study the time dependence of these averages also in the interacting case.

Understanding the behavior of squeezed states is very important: besides giving us an example of steady behavior described by the GGE, they arise in many contexts a part from quenches, as boundary problems: see [26], [24], [34], [35] and references therein. Since most of the results of other works regarding these states are obtained directly in the infinite volume theory, the first necessary step will be to find their correct counterpart in the finite volume case.

4.1 Squeezed states

Here we define the squeezed states in the framework of integrable theories in infinite volume, then we find their natural counterpart in a finite volume theory. In [24] the authors use for a relativistic invariant theory the definition

$$|B\rangle = \exp\left(\int_0^\infty \frac{d\theta}{2\pi} K(\theta) Z^+(\theta) Z^+(-\theta)\right)|0\rangle \tag{4.1}$$

where $Z(\theta)$ satisfies the Zamolodchikov-Faddeev Algebra:

$$\begin{aligned} Z(\theta)Z(\beta) &= S(\theta,\beta)Z(\beta)Z(\theta) \\ Z(\theta)Z^+(\beta) &= S(\beta,\theta)Z^+(\beta)Z(\theta) + 2\pi\delta(\theta-\beta) \end{aligned} \tag{4.2}$$

Because of the relativistic invariance of the theory, the S matrix depends on the rapidities only through their difference $S(\theta,\beta) = S(\theta-\beta)$.

Even if we define the squeezed states in the finite volume case through this example, we will not restrict to relativistic invariant theories.

4.1.1 Infinite and finite volume theory

In this section we will link integrable theories in the infinite volume case with the finite volume theory. Let us recall the Bethe Equations:

$$Lp(\theta_i) + \sum_{j \neq i} \chi(\theta_i, \theta_j) = 2\pi n_i \tag{4.3}$$

where $\chi(\theta, \beta) = -i \log S(\theta, \beta)$, L is the length of the system and $(n_i)_N$ is an appropriate set of integers (or half integers) of N elements.

In the finite volume case we have a complete set of eigenstates $|\vec{\theta}\rangle$ whose rapidities satisfy the Bethe Equations:

$$\begin{aligned} &|..\theta_j, \theta_{j+1}..\rangle = S(\theta_j, \theta_{j+1}) \, |..\theta_{j+1}, \theta_j..\rangle \\ &\langle\vec{\theta}|\vec{\beta}\rangle = \prod_i \delta_{\theta_i, \beta_i} \qquad \text{if } \theta_i \geq \theta_2 \geq .. \geq \theta_j \geq ..; \quad \beta_i \geq \beta_2 \geq .. \geq \beta_j \geq .. \end{aligned} \tag{4.4}$$

Each state can be represented as an infinite superposition of in-states:

$$|\psi\rangle = \sum_{\vec{\theta}}^{ordered} f(\vec{\theta}) \, |\vec{\theta}\rangle \tag{4.5}$$

The summation is over all the rapidities allowed by Bethe equations; let us recall that the summation is restricted over the in-states, so the rapidities in $\vec{\theta}$ are ordered.

In the $L \to \infty$ limit the distribution of roots of Bethe equations becomes dense, so we can substitute the summations with integrals. This makes sense only if we consider the number of particles fixed, so $\frac{N}{L} \to 0$ and the weight function f is sufficiently smooth varying $\vec{\theta}$. The integration measure will be flat in the integers of the Bethe Equations, instead for the continuous limit in the momenta representation we must consider a jacobian.

With these observations, we define the truncated state $|\psi_N\rangle$ as

$$|\psi_N\rangle = \sum_{m=0}^{N} \sum_{\vec{\theta}_m}^{ordered} f(\vec{\theta}^m) \, |\vec{\theta}^m\rangle \tag{4.6}$$

here $\vec{\theta}^m$ is a vector of m elements. For fixed N we can implement the continuous limit as follows:

$$|\psi_N\rangle \to \sum_{m}^{N} \int_{\theta_1 \geq \theta_2 \geq .. \geq \theta_m} d^m\theta \left(\frac{\partial n}{\partial \theta}\right) f(\vec{\theta}^m) \, |\vec{\theta}^m\rangle \qquad \text{for } L \to \infty \tag{4.7}$$

where $\left(\frac{\partial n}{\partial \theta}\right)$ is the jacobian of the Bethe Equations:

$$\left(\frac{\partial n}{\partial \theta}\right) = \det |J|, \qquad J_{i,j} = \frac{\partial n_i}{\partial \theta_j} \tag{4.8}$$

Now we can use this expression to link the finite volume representation to the infinite volume one. Using the Zamolodchikov-Faddeev Algebra we would have constructed this type of asymptotic states:

$$|\vec{\theta}_\infty\rangle \equiv \prod_{\theta_1\geq\theta_2\geq..\geq\theta_n} Z^+(\theta_i)\,|0\rangle \Rightarrow \langle\vec{\beta}_\infty|\vec{\theta}_\infty\rangle = \prod_i (2\pi\delta(\theta_i-\beta_i)) \tag{4.9}$$

We will interpret the $|\vec{\theta}_\infty\rangle$ states as a continuous limit of the $|\vec{\theta}\rangle$ states. With this interpretation the $|\vec{\theta}_\infty\rangle$ states are proportional to $|\vec{\theta}\rangle$ states and we need to fix the normalization. Suppose $|\vec{\theta}_\infty\rangle \leftrightarrow F(\vec{\theta})\,|\vec{\theta}\rangle$ with F a real quantity, then in the infinite volume limit we can proceed as follows:

$$\begin{aligned} 1 &= \sum_{\vec{\theta}}^{ordered} \langle\vec{\theta}|\vec{\beta}\rangle = \int_{\theta_1\geq\theta_2..\geq\theta_N} d\theta^N \left(\frac{\partial n}{\partial\theta}\right) F(\vec{\theta})^{-1}F(\vec{\beta})^{-1}\,\langle\vec{\theta}_\infty|\vec{\beta}_\infty\rangle = \\ &= \int_{\theta_1\geq\theta_2..\geq\theta_N} d\theta^N \left(\frac{\partial n}{\partial\theta}\right) F(\vec{\theta})^{-1}F(\vec{\beta})^{-1}\prod_i (2\pi\delta(\theta_i-\beta_i)) = \left(\frac{\partial n}{\partial\beta}\right)(2\pi)^N F(\vec{\beta})^{-2} \end{aligned} \tag{4.10}$$

So we find the relation $|\vec{\theta}_\infty\rangle \leftrightarrow \sqrt{\left(\frac{\partial 2\pi n}{\partial\theta}\right)}\,|\vec{\theta}\rangle$.

$$|\psi_N\rangle \to \sum_m^N \int_{\theta_1\geq\theta_2\geq..\geq\theta_m} \frac{d^m\theta}{(2\pi)^{\frac{m}{2}}}\sqrt{\left(\frac{\partial n}{\partial\theta}\right)} f(\vec{\theta}^m)\,|\vec{\theta}^m_\infty\rangle \tag{4.11}$$

At this point it is natural to extend this correspondence to the entire vector and not only to the truncated one:

$$\sum_{\vec{\theta}}^{ordered} f(\vec{\theta})\,|\vec{\theta}\rangle \longleftrightarrow \sum_m \int_{\theta_1\geq\theta_2\geq..\geq\theta_m} \frac{d^m\theta}{(2\pi)^{\frac{m}{2}}}\sqrt{\left(\frac{\partial n}{\partial\theta}\right)} f(\vec{\theta}^m)\,|\vec{\theta}^m_\infty\rangle \tag{4.12}$$

In that way we can link the discrete theory with the continuous one, the recipe is simply to substitute summations with integrals using the appropriate measure and to substitute $|\vec{\theta}_\infty\rangle \leftrightarrow \sqrt{\left(\frac{\partial 2\pi n}{\partial\theta}\right)}\,|\vec{\theta}\rangle$. This is consistent with [29].

4.1.2 Cooper pair states

We study separately the case of states with quantum numbers organized in Cooper pairs, that is to say if in the set $(n_i)_N$ it appears the integer n, also $-n$ must be present. Because of the form of Bethe equations, a state made entirely of Cooper pairs in the n space means a state formed only by Cooper pairs in the rapidity space too. We will use the notation n^c to indicate the Cooper pair $(n,-n), n>0$, instead θ^c stands for $(\theta,-\theta), \theta>0$. Consider now a state made of Cooper pairs and its

truncated counterpart

$$\begin{aligned}
|\psi\rangle &= \sum_{\vec{\theta}}^{ordered} f(\vec{\theta}^m)\,|\vec{\theta^c}^{\,m}\rangle \\
|\psi_N\rangle &= \sum_{m}^{N}\sum_{\vec{\theta}}^{ordered} f(\vec{\theta}^m)\,|\vec{\theta^c}^{\,m}\rangle \\
|\vec{\theta^c}^{\,m}\rangle &= |\theta_1,-\theta_1,\theta_2,-\theta_2,..\theta_m,-\theta_m\rangle \qquad \theta_1\geq\theta_2\geq..
\end{aligned} \tag{4.13}$$

that is to say a state in which only Cooper pairs appear.

In what follows we will indicate with $\left(\frac{\partial n}{\partial\theta}\right)$ the jacobian of the transformation between positive θ and positive n of the Cooper pairs, with $\left(\frac{\partial n^c}{\partial\theta^c}\right)$ the jacobian of the transformation between the entire set of $(n_i^c)_m$ and $(\theta_i^c)_m$, that to say we consider n and $-n$ as independent variables. We find the jacobian of the larger change of variables and only then we impose the Cooper pairs's structure. We can now implement the continuous limit on the truncated states as before

$$|\psi_N\rangle \to \sum_{m}^{N}\int_{\theta_1\geq\theta_2\geq..\geq\theta_m}\frac{d^m\theta}{(2\pi)^m}\left(\frac{\partial n^c}{\partial\theta^c}\right)^{-\frac{1}{2}}\left(\frac{\partial n}{\partial\theta}\right)f(\vec{\theta}^m)\,|\vec{\theta}^{\,c}_{\infty}{}^{m}\rangle \tag{4.14}$$

We can compute $\langle\vec{\beta^c}|\vec{\theta^c}\rangle$ considering the finite volume normalization: if we consider in states it is obvious that the correct regularization is $\langle\vec{\beta^c}_{\infty}|\vec{\theta^c}_{\infty}\rangle \to \alpha\prod_i\delta(\beta_i-\theta_i)$.

$$\begin{aligned}
1 &= \sum_{\vec{\beta}}^{ordered}\langle\vec{\beta^c}|\vec{\theta^c}\rangle \to \int_{\beta_1\geq\beta_2\geq..\geq\beta_N}\frac{d^N\beta}{(2\pi)^{2N}}\left(\frac{\partial n^c}{\partial\beta^c}\right)^{-\frac{1}{2}}\left(\frac{\partial n^c}{\partial\theta^c}\right)^{-\frac{1}{2}}\left(\frac{\partial n}{\partial\beta}\right)\langle\vec{\beta^c}_{\infty}|\vec{\theta^c}_{\infty}\rangle = \\
&= \alpha\frac{1}{(2\pi)^{2N}}\left(\frac{\partial n^c}{\partial\theta^c}\right)^{-1}\left(\frac{\partial n}{\partial\beta}\right)
\end{aligned} \tag{4.15}$$

$$\Rightarrow \alpha = (2\pi)^{2N}\left(\frac{\partial n^c}{\partial\theta^c}\right)\left(\frac{\partial n}{\partial\theta}\right)^{-1} \tag{4.16}$$

$$\Rightarrow \langle\vec{\beta^c}_{\infty}|\vec{\theta^c}_{\infty}\rangle = \left(\frac{\partial n^c}{\partial\theta^c}\right)\left(\frac{\partial n}{\partial\theta}\right)^{-1}\prod_i\left[(2\pi)^2\delta(\theta_i-\beta_i)\right] \tag{4.17}$$

So we establish the following link

$$\sum_{\vec{\theta}}^{ordered} f(\vec{\theta}^m)\,|\vec{\theta^c}^{\,m}\rangle \longleftrightarrow \sum_m\int_{\theta_1\geq\theta_2\geq..\geq\theta_m}\frac{d^m\theta}{(2\pi)^m}\left(\frac{\partial n^c}{\partial\theta^c}\right)^{-\frac{1}{2}}\left(\frac{\partial n}{\partial\theta}\right)f(\vec{\theta}^m)\,|\vec{\theta}^{\,c}_{\infty}{}^{m}\rangle \tag{4.18}$$

4.1.3 Squeezed states in finite volume

We can now construct the equivalent of squeezed states in the finite volume theory. Consider:

$$|B\rangle = \exp\left(\int_0^\infty \frac{d\theta}{2\pi} K(\theta) Z^+(\theta) Z^+(-\theta)\right)|0\rangle \tag{4.19}$$

Notice that $[Z^+(\theta)Z^+(-\theta), Z^+(\beta)Z^+(-\beta)] = 0$. This permits us to proceed as follows:

$$\begin{aligned}
&\exp\left(\int_0^\infty \frac{d\theta}{2\pi} K(\theta) Z^+(-\theta) Z(\theta)\right)|0\rangle = \sum_m \frac{1}{m!}\left(\int_0^\infty \frac{d\theta}{2\pi} K(\theta) Z^+(\theta) Z^+(-\theta)\right)^m = \\
&= \sum_m \int_{\theta_1\geq\theta_2\geq..\geq\theta_m} \frac{d^m\theta}{(2\pi)^m} \prod_{\theta_1\geq\theta_2\geq..\geq\theta_m} K(\theta_i) Z^+(\theta_i) Z^+(-\theta_i)\,|0\rangle \\
&\Rightarrow |B\rangle = \sum_m \int_{\theta_1\geq\theta_2\geq..\geq\theta_m} \frac{d^m\theta}{(2\pi)^m} \prod_i K(\theta_i)\,|\vec{\theta^c}_\infty{}^m\rangle
\end{aligned} \tag{4.20}$$

At this point we can use the result of the previous section and establish the finite volume analogue of the $|B\rangle$ state:

$$|B\rangle \longleftrightarrow |\phi\rangle = \sum_{\vec{\theta}}^{ordered} \left(\frac{\partial n^c}{\partial \theta^c}\right)^{\frac{1}{2}} \left(\frac{\partial n}{\partial \theta}\right)^{-1} \prod_i K(\theta_i)\,|\vec{\theta^c}\rangle \tag{4.21}$$

We will use this state as the correct regularization for $|B\rangle$ and we will attempt to evaluate the following kind of averages:

$$\lim_{L\to\infty} \frac{\langle\phi|\, O(x,t)\,|\phi\rangle}{\langle\phi|\phi\rangle} \tag{4.22}$$

where $O(x)$ is a local operator centered in x.

In the following we will not label the states with the rapidities, but with a more general quantum number $\theta_i \to k_i$. We still suppose that the momentum is an odd function of k, in this way the states $|\vec{k^c}\rangle$ are invariant under translations. We will make the assumption that the energy is even under the symmetry $k \to -k$, besides we will suppose:

$$S(k,q)S(-k,q)S(-k,-q)S(k,-q) = 1 \tag{4.23}$$

in order to preserve the condition:

$$[Z^+(k)Z^+(-k), Z^+(q)Z^+(-q)] = 0 \tag{4.24}$$

4.1.4 Normalization of the state in the infinite length limit

The goal of this section is to evaluate $\langle\phi|\phi\rangle$ in the infinite length limit: this will be a divergent quantity that can give us some insight in the calculation of the expectation values of local observables.

We need to distinguish two different cases: the fermion-like and the boson-like.

Boson-like norm

In this case we have not any constraint on the integers that enter in the Bethe equations. Consider this quantity:

$$\langle\phi|\phi\rangle = \sum_{\vec{k}} \left(\frac{\partial n^c}{\partial k^c}\right) \left(\frac{\partial n}{\partial k}\right)^{-2} \prod_i |K(k_i)|^2 \tag{4.25}$$

The first step is showing that in the infinite length limit only the contributions with $N \sim L$ particles will matter. Suppose that the terms we are summing give a cut off $\sim \Lambda$ in the space of quantum numbers k_i, so due to the Bethe equations we will have a cut off in the space of the integers n_i of order $\sim L\Lambda$. We can give a crude estimation of (4.25) if we approximate each term of the sum with a product of functions $\alpha\Phi[0, L\Lambda]$, where Φ is the characteristic function of the interval and α a constant:

$$\langle\phi|\phi\rangle \sim \sum_N \alpha^N \sum_{(n_i)_N}^{n_i<L\Lambda} = \sum_N \alpha^N \frac{(N+L\Lambda-1)!}{N!(L\Lambda-1)!} \simeq const \sum_N e^{-\frac{1}{2}M(1-\alpha)\left(\frac{N}{M}-1\right)^2} \tag{4.26}$$

where $M = L\Lambda\frac{\alpha}{\alpha-1}$.

When $L \to \infty$ the sum's terms are very peaked around $N \sim L$, this has two consequences: we can use directly the infinite length limit of $\left(\frac{\partial n^c}{\partial k^c}\right)\left(\frac{\partial n}{\partial k}\right)^{-2}$ with the number of particles of the same order of L and evaluate the summation through a functional integral.

We start with $\left(\frac{\partial n^c}{\partial k^c}\right)\left(\frac{\partial n}{\partial k}\right)^{-2}$. Consider the following kind of matrix:

$$J_{m,n} = L\left(\delta_{m,n}\left(\frac{\partial_k p(k_m)}{2\pi} + \frac{1}{L2\pi}\sum_{k_l\neq k_m}\varphi(k_m,k_l)\right) - (1-\delta_{m,n})\frac{1}{L2\pi}\varphi(k_n,k_m)\right) \tag{4.27}$$

Now we will show:

$$\det(J) = L^N\left(\prod_{k_m\in\vec{k}}\left(\frac{\partial_k p(k_m)}{2\pi} + \frac{1}{L2\pi}\sum_{k_l\neq k_m}\varphi(k_m,k_l)\right) + \mathcal{O}\left(\frac{1}{L}\right)\right) \tag{4.28}$$

This can be found in [19]. Define two matrices A and C as follows:

$$A_{m,n} = \delta_{m,n}\left(\frac{\partial_k p(k_m)}{2\pi} + \frac{1}{L2\pi}\sum_{k_l \neq k_m}\varphi(k_m,k_l)\right), \quad C_{m,n} = -\frac{1}{2\pi}(1-\delta_{m,n})\varphi(k_n,k_m)$$
$$det(J) = L^N \det(A + \frac{1}{L}C) = L^N \det(A) e^{tr\log(1+\frac{1}{L}A^{-1}C)} \tag{4.29}$$

We can estimate $tr\log(1+\frac{1}{L}A^{-1}C) = -\sum_{n=1}(-1)^n \frac{1}{L^n} tr(A^{-1}C)^n$ using the following relation

$$|tr(A^{-1}C)^n| \leq N\|A^{-1}C\|^n \tag{4.30}$$

where $\|\ \|$ is the matrix norm. If we suppose that $\|A^{-1}C\|$ remains bounded varying N (this would depend on the particular form of φ) and we perform $L \to \infty$ fixing $N \sim L$ we conclude:

$$tr\log(1+\frac{1}{L}A^{-1}C) = -\frac{N}{L}tr(A^{-1}C) + \mathcal{O}\left(\frac{1}{L}\right) \tag{4.31}$$

Since $tr(A^{-1}C) = 0$ we arrive at the following result:

$$\begin{aligned} &\det(J) = L^N\left(\det(A) + \mathcal{O}\left(\frac{1}{L}\right)\right) \Rightarrow \\ &\det|J| = L^N\left(\prod_{k_m\in\vec{k}}\left(\frac{\partial_k p(k_m)}{2\pi} + \frac{1}{L2\pi}\sum_{k_l\neq k_m}\varphi(k_m,k_l)\right) + \mathcal{O}\left(\frac{1}{L}\right)\right) \end{aligned} \tag{4.32}$$

Using this limit we can evaluate the leading term in $\left(\frac{\partial n^c}{\partial k^c}\right)\left(\frac{\partial n}{\partial k}\right)^{-2}$:

$$\begin{aligned} &\left(\frac{\partial n}{\partial k}\right) \simeq \prod_{k_i\in\vec{k}}\left[L\left(\frac{\partial_k p(k_i)}{2\pi} + \frac{1}{2\pi L}\varphi(-k_i,k_i) + \frac{1}{2\pi L}\sum_{k_m\neq k_i}\varphi(k_i,k_m)\right)\right] \\ &\left(\frac{\partial n^c}{\partial k^c}\right) \simeq \prod_{k_i\in\vec{k}}\left[L\left(\frac{\partial_k p(k_i)}{2\pi} + \frac{1}{2\pi L}\sum_{k_m\neq k_i}\varphi(k_i,k_m)\right)\right]^2 \end{aligned} \tag{4.33}$$

$$\begin{aligned} &\tilde{\rho}_t(k_i) = \frac{\partial_k p(k_i)}{2\pi} + \frac{1}{2\pi L}\sum_{k_j\neq k_i}\varphi(k_i,k_j) \\ &\Rightarrow \left(\frac{\partial n^c}{\partial k^c}\right)\left(\frac{\partial n}{\partial k}\right)^{-2} = e^{-\sum_{k_i>0}\frac{1}{\pi L\tilde{\rho}_t(k_i)}\varphi(-k_i,k_i)} + \mathcal{O}(L^{-1}) \end{aligned}$$

In what follows it will be clear that we can neglect the $\mathcal{O}(L^{-1})$ terms. Now we can evaluate the norm with the functional integral formalism.

Define for simplicity $\epsilon(k_i) = -\log\left(|K(k_i)|^2\right)$

$$\begin{aligned}
\langle\phi|\phi\rangle &= \int D\rho e^{LS_{Bose}[\rho] - L\int_0^{+\infty} dk\rho(k)\epsilon(k)} e^{-\int_{k>0} dk \frac{\rho(k)}{\pi\rho_t(k)}\varphi(-k,k)} \\
S_{Bose}[\rho,\rho_t] &= \int_0^{+\infty} dk(\rho+\rho_t)\log(\rho+\rho_t) - \rho\log\rho - \rho_t\log\rho_t \\
\rho_t(k) &= \frac{\partial_k p(k)}{2\pi} + \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\lambda\rho(\lambda)\varphi(k,\lambda)
\end{aligned} \tag{4.34}$$

with $\varphi(k,\lambda) = \partial_k\chi(k,\lambda)$. Remember that because of the Cooper pair's structure we must impose $\rho(k) = \rho(-k)$.

We can go further with a saddle point estimation and the result will be exact in the large length limit:

$$\delta\left(S[\rho,\rho_t] - \int_0^{+\infty} dk\rho(k)\epsilon(k)\right) = 0 \tag{4.35}$$

$$\frac{\rho_0(k)}{\rho_0(k)+\rho_t(k)} = |K(k)|^2 exp\left(\frac{1}{2\pi}\int d\lambda\varphi(\lambda,k)\log\left(\frac{\rho_t(\lambda)+\rho_0(\lambda)}{\rho_t(\lambda)}\right)\right) \tag{4.36}$$

$$\langle\phi|\phi\rangle \sim e^{\frac{L}{2\pi}\int_0^{+\infty} dk\partial_k p(k)\log(1+\frac{\rho_0}{\rho_t})} \tag{4.37}$$

Fermion-like norm

In this case the Pauli's exclusion is at work and we must impose that all the quantum numbers k_i are different. We can proceed as before and show that only terms with $N \sim L$ will contribute to the sum when $L \to \infty$.

We can still estimate each term of the sum with a product of characteristic functions normalized to some constant $\alpha\Phi[0, L\Lambda]$:

$$\langle\phi|\phi\rangle \simeq \sum_N^{L\Lambda} \alpha^N \sum_{(n_i)_N}^{n_i<L\Lambda} = \sum_N^{L\Lambda} \alpha^N \frac{(L\Lambda)!}{N!(L\Lambda-N)!} \simeq const \sum_N^{L\Lambda} e^{-\frac{1}{2}M(1+\alpha)\left(\frac{N}{M}-1\right)^2} \tag{4.38}$$

where $M = L\Lambda\frac{\alpha}{1+\alpha}$.

When $L \to \infty$ only terms with $N \sim L$ contribute and we can use a functional integral approach; beside we can approximate $\left(\frac{\partial n^c}{\partial k^c}\right)\left(\frac{\partial n}{\partial k}\right)^{-2} \simeq e^{-\sum_{k_i>0}\frac{1}{\pi L\rho_t(k_i)}\varphi(-k_i,k_i)}$.

Define $\epsilon(k_i) = -\log\left(|K(k_i)|^2\right)$

$$\begin{aligned}
\langle\phi|\phi\rangle &= \int D\rho e^{LS_{Fermi}[\rho] - L\int_0^{+\infty} dk\rho(k)\epsilon(k)} e^{-\int_{k>0} dk \frac{\rho(k)}{\pi\rho_t(k)}\varphi(-k,k)} \\
S_{Fermi}[\rho,\rho_t] &= \int_0^{+\infty} dk\rho_t\log\rho_t - \rho\log\rho - (\rho_t-\rho)\log(\rho_t-\rho) \\
\rho_t(k) &= \frac{\partial_k p(k)}{2\pi} + \frac{1}{2\pi}\int d\lambda\rho(\lambda)\varphi(k,\lambda)
\end{aligned} \tag{4.39}$$

We arrive at:

$$\frac{\rho_0(k)}{\rho_t(k)-\rho_0(k)} = |K(k)|^2 exp\left(\frac{1}{2\pi}\int \varphi(\lambda,\theta)\log\left(\frac{\rho_t(\lambda)}{\rho_t(\lambda)-\rho_0(\lambda)}\right)\right) \tag{4.40}$$

$$\langle\phi|\phi\rangle \sim e^{-\frac{L}{2\pi}\int_0^{+\infty} dk\partial_k p(k)\log(1-\frac{\rho_0}{\rho_t})} \tag{4.41}$$

In both cases we have this behavior: only terms with $N \sim L$ particles matter and their contribution diverges exponentially in L: we will use this information in what follows.

Remember that we are trying to evaluate $\lim_{L\to\infty}\frac{\langle\phi|O(x,t)|\phi\rangle}{\langle\phi|\phi\rangle}$.

We know that $\langle\phi|\phi\rangle$ diverges exponentially in L and only terms with the number of particles of the same order of L matter: we can suppose that only terms with $\mathcal{O}(L)$ particles will matter in the numerator too.

In a more satisfactory way we could recast the same estimation we used for $\langle\phi|\phi\rangle$ and show that we can consider only contributions with $\mathcal{O}(L)$ particles. In particular we can suppose $\left(\frac{\partial n^c}{\partial k^c}\right)\left(\frac{\partial n}{\partial k}\right)^{-2} \simeq e^{-\sum_{k_i>0}\frac{1}{\pi L\tilde{\rho_t}(k_i)}\varphi(-k_i,k_i)}$:

$$\langle\phi|\,O(x,t)\,|\phi\rangle \simeq \sum_{\vec{k},\vec{p}}^{\sim L}\prod_{k_i\in\vec{k}} K^*(k_i)e^{-\frac{\varphi(-k_i,k_i)}{2\pi L\tilde{\rho_t}(k_i)}}\prod_{p_j\in\vec{p}} K(p_j)e^{-\frac{\varphi(-p_j,p_j)}{2\pi L\tilde{\rho_t}(p_j)}}\langle\vec{k}^c|\,O(x,t)\,|\vec{p}^c\rangle =$$

$$= \sum_{\vec{k},\vec{p}}^{\sim L}\prod_{k_i\in\vec{k}}\left[K^*(k_i)e^{-\frac{\varphi(-k_i,k_i)}{2\pi L\tilde{\rho_t}(k_i)}}e^{it2E(k_i)}\right]\prod_{p_j\in\vec{p}}\left[K(p_j)e^{-\frac{\varphi(-p_j,p_j)}{2\pi L\tilde{\rho_t}(p_j)}}e^{-it2E(p_j)}\right]\langle\vec{k}^c|\,O(0,0)\,|\vec{p}^c\rangle \tag{4.42}$$

The sign $\sim L$ means that we are restricting our calculations to terms where the number of momenta in $\vec{k}$ and $\vec{p}$ is $\sim L$: other contributions do not matter in the normalized average when we perform $L\to\infty$.

In particular with GTBA arguments it can be shown that the long time limit is given by the diagonal ensemble:

$$\lim_{t\to\infty}\langle\phi|\,O(x,t)\,|\phi\rangle = \sum_{\vec{k}}^{\sim L}\prod_{k_i\in\vec{k}}\left[|K(k_i)|^2 e^{-\frac{\varphi(-k_i,k_i)}{\pi L\tilde{\rho_t}(k_i)}}\right]\langle\vec{k}^c|\,O(0,0)\,|\vec{k}^c\rangle \tag{4.43}$$

To proceed we expand $\langle\vec{k}^c|\,O(0,0)\,|\vec{p}^c\rangle$ as we did in section 3.1.3.

- In the free case:

$$\langle\vec{k}|\,O(0,0)\,|\vec{p}\rangle = \sum_{\vec{k}=\vec{k}_1\bigcup\vec{k}_2}\sum_{\vec{p}=\vec{p}_1\bigcup\vec{p}_2}\frac{S^*_{\vec{k}\vec{k}_1}S_{\vec{p}\vec{p}_1}\langle\vec{k}_2|\,O(0,0)\,|\vec{p}_2\rangle_{conn}}{\sqrt{\prod_{p_j\in\vec{p}_2}[L2\pi\rho_t(p_j)]\prod_{k_j\in\vec{k}_2}[L2\pi\rho_t(k_j)]}}\delta_{\vec{k}_1,\vec{p}_1} \tag{4.44}$$

where $\delta_{\vec{k}_1,\vec{p}_1}$ is the multi-dimensional kronecker delta and the exclusion principle in the Fermi case is at work, the summation is on all the distinct ways of splitting in two parts the sets $\vec{k}$ and $\vec{p}$.

- For a non trivial integrable theory the diagonal terms in the Fermi case are known, we give a result from [15],[16] that with our notation is written:

$$\langle\vec{p}|\, O(0,0)\, |\vec{p}\rangle = \sum_{\vec{p}=\vec{p}_1 \bigcup \vec{p}_2} \frac{J(\vec{p}|\vec{p}_1)}{J(\vec{p})} \langle p_2|\, O(0,0)\, |p_2\rangle_{conn} \tag{4.45}$$

$J(\vec{p})$ is defined as $J(\vec{p}) = \det(M)$, with

$$M_{i,j} = 2\pi L \left(\delta_{i,j} \left(\frac{\partial_{p_i} p(p_i)}{2\pi} + \frac{1}{L2\pi} \sum_{k\neq j} \varphi(p_i, p_k) \right) + (1-\delta_{i,j}) \frac{1}{L2\pi} \varphi(p_j, p_i) \right) \tag{4.46}$$

Instead $J(\vec{p}|\vec{p}_1)$ is the determinant of the minor of M associated to the indices of $\vec{p}_1$, for example if $\vec{p}_1 = (p_{j_1}.p_{j_2}, p_{j_3}..)$ then $J(\vec{p}|\vec{p}_1)$ is constructed with the indices $j_1, j_2, j_3..$

4.2 Free case

In this section we perform the calculation of the entire time dependence of local averages over squeezed states in the free case. We start by evaluating $\langle\phi|\, O(x,t)\, |\phi\rangle$:

$$\begin{aligned} \langle\phi|\, O(x,t)\, |\phi\rangle \simeq \sum_{\vec{k},\vec{p}}^{\sim L} \prod_{k_i\in\vec{k}} \left[K^*(k_i) e^{it2E(k_i)}\right] \prod_{p_j\in\vec{p}} \left[K(p_j) e^{-it2E(p_j)}\right] \cdot \\ \cdot \sum_{\vec{k}^c=\vec{k}_1\bigcup\vec{k}_2} \sum_{\vec{p}^c=\vec{p}_1\bigcup\vec{p}_2} \frac{S^*_{\vec{k}\vec{k}_1} S_{\vec{p}\vec{p}_1} \langle\vec{k}_2|\, O(0,0)\, |\vec{p}_2\rangle_{conn}}{\sqrt{\prod_{p_j\in\vec{p}_2}[L2\pi\rho_t(p_j)] \prod_{k_j\in\vec{k}_2}[L2\pi\rho_t(k_j)]}} \delta_{\vec{k}_1,\vec{p}_1} \end{aligned} \tag{4.47}$$

The next section will be devoted to find a convenient graphical way to handle this expression.

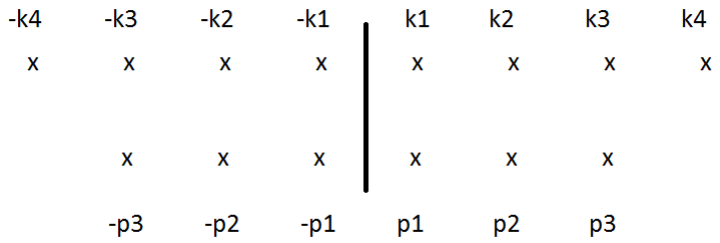
4.2.1 Graphical representation

We start noticing that each term in the summation is identified by the factor $\langle\vec{k}_2|\, O(0,0)\, |\vec{p}_2\rangle_{conn} \delta_{\vec{k}_1,\vec{p}_1}$: if we know it, the entire term inside the sum can be reconstructed, so we can look for a graphical representation for this term.

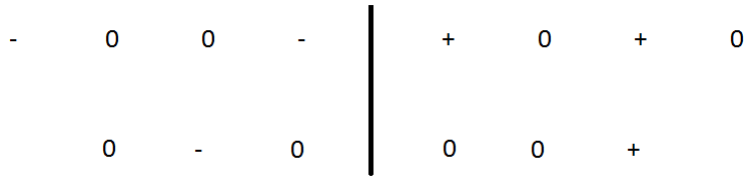
We will construct the graphical rules step by step: begin drawing two lines of points, then the upper one will represent the numbers in $\vec{k}$ and the lower one will

represent $\vec{p}$. The quantum numbers are ordered from left to right in an increasing way.

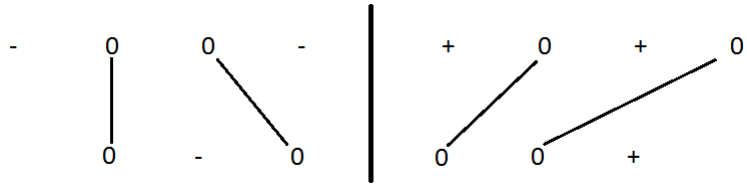
To remind ourselves of of the Cooper pairs structure of the sets we will draw a vertical line: the quantum numbers on the left are obtained from the quantum numbers on the right reflecting the picture through this line and exchanging their signs.



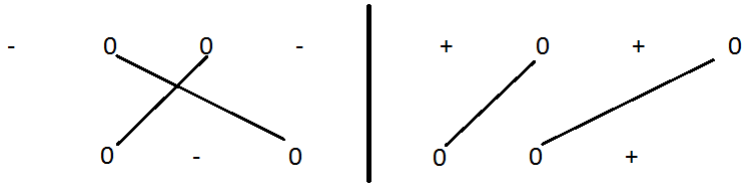
From now on we will not indicate explicitly the k_i or p_i. The next step is representing the variables inside $\langle \vec{k}_2 | O(0,0) | \vec{p}_2 \rangle_{conn}$: instead of "X" we put a "-" sign for the quantum numbers at the left of the line that enter in the connected average, "+" if they are located on the right. Others will be indicate with "0" and they will give the "delta" structure. For example we can imagine:



Now we draw the constraints due to the "delta" factors. The set of quantum numbers in the upper row represented by "0" must be equal to the set of "0" in the lower row. Since the sets are ordered, the quantum numbers are equal in pairs from the left to the right. We will join these couples with a continuous line.



Instead the following graph would not be correct:



The graph above is not correct because the equality of quantum numbers imposed by the delta structure must respect their order: we have the graphical recipe that two continuous lines cannot cross each others.

To make the situation simpler, we will adopt the following rule: a continuous line cannot cross the vertical line.

We can give a partial justification of this fact: if a line crosses the vertical one, we are imposing the equality between a positive number and a negative one. This is possible only if they are both zero: in the Fermi case this situation is not allowed and in the Bose case we will see this case contributes $\sim L^{-1}$ to the final result and so it can be neglected.

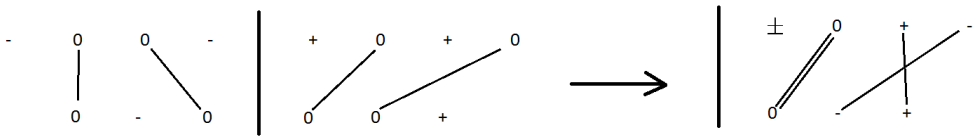
We will truly understand this fact only at the end of the calculation: we choose to anticipate this result in order to keep the notation simpler and we will have some comments when we will able to prove this statement.

In this representation it is not clear which quantum numbers are equal: we must consider at the same time the delta structure and the symmetry through the vertical line. To take care automatically of the last requirement we fold the graph along the vertical line: imagine of drawing this graph on a piece of paper, then fold it along the symmetry line and put together the two halves.

We will use the following notation:

- Two "0" coincide, then write "0".
- "0" and "-" coincide, write "-".
- "0" and "+" coincide, write "+".
- "+" and "-"coincide, write "$\pm$".

For example consider the graph below:



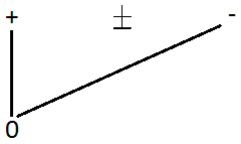
In the last kind of graph it is clear which quantum numbers are equal.

Here we will state other two recipes: for now we can give only a partial justification, the true proof of the following facts can be given only later. We anticipate these facts to keep the notation simpler.

In the last kind of representation two continuous lines cannot cross each other.

The reason is quite the same as the one at the root of the recipe that continuous lines cannot cross the vertical line: suppose $p \geq p'$ and $k \geq k'$ are joined $p \leftrightarrow k'$ and $p' \leftrightarrow k$. This means that $p = k'$ and $k = p'$, but then their order imposes $p = p'$: in the fermionic case this situation is impossible, in the bosonic case this contribution will be suppressed $\mathcal{O}(L^{-1})$. For the bosonic case we will give later a proof of this assertion.

The following graph is not allowed:

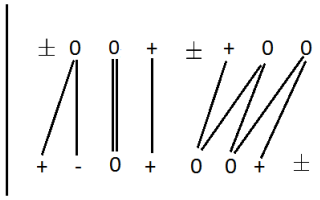


The motivation is similar to the previous one: in the fermionic case this graph cannot exist because it has two quantum numbers equal in the same row (in this case the "bra" row), in the bosonic case this configuration will not be zero only if all the quantum numbers are equals, but this will contribute $\mathcal{O}(L^{-1})$. We will see this later, now take it as a rule.

It will be much more comfortable if we could draw directly the last kind of graph, without the "bending" procedure. Here we will give the recipe, which can be obtained if we consider the graphs in the previous form and then bend them.

- Draw a vertical line, than dispose on two rows the symbols 0, +, -, ± only on the right side of that vertical line.
- Draw the connections in this way: the "0" signs can have two links, the "+" and "-" signs have only one link and the symbol ± cannot have any connection. Each connection can link only opposite rows and has a "charge" : if the link has on one side a + sign the connection will be "positive", if it has a "-" sign the connection is "negative". Positive connections can link only "+" signs and "0", negative connections can link only "-" and "0". The links departing from "0" must be one positive and the other negative.
- The connections cannot cross each other and we cannot insert a "±" symbol between two connected momenta: in this way we will automatically drop terms that will give zero contributions in the $L \to \infty$ limit.

With these recipes we can determine a "block structure" in the graphs, for example consider the following picture:



We will call a block "connected" if all of its momenta are linked through continuous lines: each graph will be trivially composed by connected blocks.

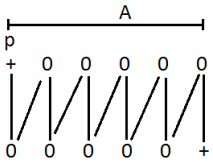
We give a special name to the following type of block, that to say "bubble":

Now we will give the rules for the graphs:

- Draw a graph, then each connected block will give the following contribution:
  - The bubbles:

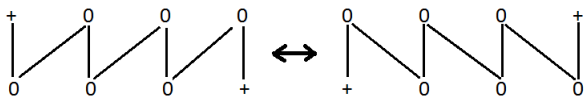
$$\rightarrow |K(p)|^2$$

  - Observe that the following blocks are "time independent". The blocks which exchange "+" with "-" have an identical contribution, apart from the quantum numbers that have to be inserted in the connected average.

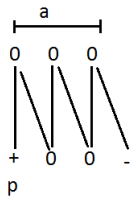


$$\rightarrow \frac{1}{L2\pi\rho_t(p)}|K(p)|^A$$

Besides, the following kinds of block must be counted only once

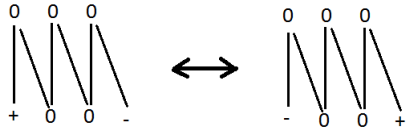


– The blocks:

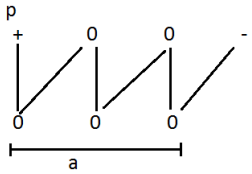


$$\to \frac{1}{L2\pi\rho_t(p)} K(p) e^{-2itE(p)} |K(p)|^a$$

Besides the two graphs count as the same:

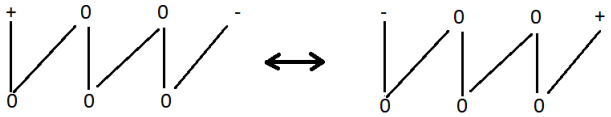


– The blocks:



$$\to \frac{1}{L2\pi\rho_t(p)} K^*(p) e^{2itE(p)} |K(p)|^a$$

And the following must be considered the same:



– The $\pm$ blocks give $K(p)e^{-2iE(p)t}\frac{1}{L2\pi\rho_t(p)}$ if posed in the lower row, instead if they are in the upper row contribute as $K^*(p)e^{2iE(p)t}\frac{1}{L2\pi\rho_t(p)}$.

- Insert the quantum numbers in the connected average: the sign "+" related to the quantum number "p" means a "p" in the connected average, "-" means "$-p$" in the connected average. The sign "$\pm$" means a Cooper pair $(p, -p)$ in the connected average. We can take care of the phases organizing the quantum numbers in this way (this will be true also for the interacting case): indicate with $\vec{\alpha}$ the quantum numbers of the blocks that have no time dependence, $\vec{\gamma}^c$ and $\vec{\eta}^c$ will be Cooper pairs vectors such that in $\vec{\eta}^c$ there are $(p, -p)$ Cooper

pairs, in $\vec{\gamma}^c$ the Cooper pairs are reversely ordered $(-p, p)$. Finally we will call $\vec{\alpha}^+$ the vector $\vec{\alpha}$ reversely ordered. Put the momenta in the connected average in the following way:

$$\langle \vec{\alpha}^+ \vec{\gamma}^c | \, O(0,0) \, | \vec{\eta}^c \vec{\alpha} \rangle_{conn} \tag{4.48}$$

At this point we are ready to evaluate $\lim_{L\to\infty} \frac{\langle \phi | O(x,t) | \phi \rangle}{\langle \phi | \phi \rangle}$: this will be done in the next section through a partial summation over "bubble" graphs.

4.2.2 Eliminating the divergences

In the previous section we saw that the quantum numbers in bubbles do not contribute to the connected averages: we will see soon that their contributions will be canceled by the term $\langle \phi | \phi \rangle$. We can immediately solve the free bosonic case, this gives us some insight in the free fermionic and in the interacting cases.

Free Bosonic case

The free bosonic case in this formalism is very simple.

In the free bosonic case we do not have Bethe Equations and a Pauli's exclusion principle: this means that when we sum over all the possible graphs the contribution due to bubbles simply factorizes.

The term $\langle \phi | \, O(x,t) \, | \phi \rangle$ can be evaluated summing over all the possible graphs, each graph is made of connected blocks and bubbles. Then the sum over all the possible graphs is equal to the summation over all the possible connected blocks times the summation over bubbles:

$$\langle \phi | \, O(x,t) \, | \phi \rangle = \sum_{connected, bubbles} [connected][bubbles] = \sum_{connected} [connected] \sum_{bubbles} [bubbles] \tag{4.49}$$

We can apply the same graphical machinery to evaluate the norm $\langle \phi | \phi \rangle$ and we obtain that the graphs are made only of bubbles:

$$\langle \phi | \phi \rangle = \sum_{bubbles} [bubbles] \tag{4.50}$$

With these results we obtain that the average can be evaluated simply summing over the connected graphs without bubbles:

$$\frac{\langle \phi | \, O(x,t) \, | \phi \rangle}{\langle \phi | \phi \rangle} = \sum_{connected} [connected] \tag{4.51}$$

We will proceed further in the calculations after we have studied the Fermi case. The free Fermi case is more difficult because the Pauli principle forbids this simple factorization and the interacting case is far more difficult due to the presence of non trivial Bethe Equations.

Fermi case, Bose case derived again

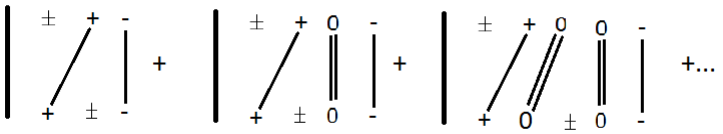
As we just said the Fermi case is not straightforward as the Bose case because we need to take care of the Pauli's principle, but we still have almost a factorization.

The approach we are going to use can also be applied to interacting theories as we will do in section 4.3: here we consider only the free cases and we derive again the free Bose case in this way as a check of correctness.

In the free boson case we saw that the contribution of bubbles is canceled by $\langle\phi|\phi\rangle$, it seems plausible that a similar cancellation still survives also in the Fermi case (and also in the interacting one). The value of $\langle\phi|\phi\rangle$ is dominated by the contributions where the number of particles is of the same order of L, so it is natural to evaluate $\langle\phi|\, O(x,t)\,|\phi\rangle$ supposing that only the set of quantum numbers in the bubbles has $\sim L$ elements; instead the number of the ones that enter in the connected average will be kept finite for $L \to \infty$.

With these assumptions we will obtain an expression for the average and then we will check its validity trough the convergence of the final result.

The idea is still to sum over all the possible ways of inserting the bubbles between the connected blocks, this summation can be evaluated through a functional integral approximation. Graphically we would like to sum this type of graphs:



In a more formal way the contribution of each graph can be decomposed in some connected pieces and bubbles: indicate it as $[connected][bubbles]$

$$\langle O(x,t)\rangle = \lim_{L\to\infty} \frac{\sum_{connected,bubbles}[connected][bubbles]}{\langle\phi|\phi\rangle} \tag{4.52}$$

Since the number of bubbles is $\sim L$, we can evaluate their contribution through a functional integral approach similarly to what we did in section 4.1.4. for the norm of the state.

Set $\epsilon(p) = -\log(|K(p)|^2)$:

$$\begin{aligned}
\langle O(x,t)\rangle &= \lim_{L\to\infty} \frac{\sum_{connected,bubbles}[connected][bubbles]}{\langle\phi|\phi\rangle} = \\
&= \lim_{L\to\infty} \frac{\sum_{bubbles}\sum_{connected}[connected][bubbles]}{\langle\phi|\phi\rangle} = \\
&= \lim_{L\to\infty} \frac{\int D\rho \sum_{connected}[connected] e^{L\int'_{k\geq 0} dk\; s(\rho(k),\rho_t(k))-\rho(k)\epsilon(k)}}{\int D\rho e^{L\int dk\; s(\rho(k),\rho_t(k))-\rho(k)\epsilon(k)}}
\end{aligned} \tag{4.53}$$

In the free case we have $\rho_t = \frac{\partial_k p(k)}{2\pi}$, the density of entropy "s" has to be chosen between:

$$\begin{aligned} s_{Bose}(\rho, \rho_t) &= (\rho + \rho_t)\log(\rho + \rho_t) - \rho \log \rho - \rho_t \log \rho_t \\ s_{Fermi}(\rho, \rho_t) &= \rho_t \log \rho_t - \rho \log \rho - (\rho_t - \rho)\log(\rho_t - \rho) \end{aligned} \tag{4.54}$$

The integral $\int'$ is defined as follows:

- In the Bose case it is a simple integral.
- In the Fermi case the integral excludes a tiny interval around the connected blocks. Suppose p is the quantum number of the connected block, then the domain to exclude is an interval around p of length $\Delta p = \frac{1}{L\rho_t(p)}$. We can explain the reason for this modification considering this graph:

Where $p_1, ..$ are the quantum numbers and $n_1..$ the relative integers that enter in the Bethe Equations. If we analyze the situation without the connected block (so the same case but without the block associate to p_2) we would have, due to the exclusion principle, the constraint $n_1 - n_3 \geq 1$. Instead if we consider the presence of the connected block we have the double constraint $n_1 - n_2 \geq 1$, $n_2 - n_3 \geq 1 \Rightarrow n_1 - n_3 \geq 2$. For this reason the presence of p_2 leads to the exclusion of a segment of length 1 in the domain of the n integers. Our formalism is implemented in the space of quantum numbers so we must consider the length of the segment excluded in this domain:

$$\Delta n = \rho_t(p) L \Delta p \tag{4.55}$$

Then an excluded segment of width "1" in the "n" space means an interval of length $\Delta p = \frac{1}{L\rho_t(p)}$ in the quantum numbers' space.

Suppose p_m are the quantum numbers of the connected blocks, because of what we have just said we define $\int'$ over a test function $f(p)$ as follows

$$\int' dp\, f(p) = \int dp\, f(p) - \frac{1}{L}\sum_m \frac{1}{\rho_t(p_m)} f(p_m) \tag{4.56}$$

Notice that the Cooper pair's structure implies that if we have p_m in a connected block, then we have also $-p_m$.

Consider now the expression:

$$\langle O(x,t)\rangle = \lim_{L\to\infty} \frac{\int D\rho \sum_{connected}[connected] e^{L\int'_{k\geq 0} dk\, s(\rho(k),\rho_t(k))-\rho(k)\epsilon(k)}}{\int D\rho e^{L\int_{k\geq 0} dk\, s(\rho(k),\rho_t(k))-\rho(k)\epsilon(k)}} \tag{4.57}$$

There is a slight difference between the fermionic and bosonic case due to $\int'$. We recall the results of Appendix 4.5 that permits us to calculate this ratio:

$$\lim_{L\to\infty} \frac{\int D\rho A[\rho,\frac{1}{L}]e^{LF[\rho]}}{\int D\rho B[\rho,\frac{1}{L}]e^{LF[\rho]}} = \frac{A[\rho_0]}{B[\rho_0]} \tag{4.58}$$

where $F[\rho]$ is a functional with only one maximum in ρ_0 such that its second derivative is negative defined; $A[\rho,\frac{1}{L}]$ and $B[\rho,\frac{1}{L}]$ are well behaved functionals of ρ in order to guarantee the convergence of the integrals and with a well defined $L\to\infty$ limit.

In order to apply the results of Appendix 4.5, we need to put the limit in the form above, so we can proceed as follows:

$$\begin{aligned} &\frac{\int D\rho \sum_{connected}[connected] e^{L\int'_{k\geq 0} dk\, s(\rho(k),\rho_t(k))-\beta\rho(k)\epsilon(k)}}{\int D\rho e^{L\int_{k\geq 0} dk\, s(\rho(k),\rho_t(k))-\beta\rho(k)\epsilon(k)}} = \\ &= \frac{\int D\rho \sum_{connected}[connected] e^{W[\rho]} e^{L\int_{k\geq 0} dk\, s(\rho(k),\rho_t(k))-\beta\rho(k)\epsilon(k)}}{\int D\rho e^{L\int_{k\geq 0} dk\, s(\rho(k),\rho_t(k))-\beta\rho(k)\epsilon(k)}} \end{aligned} \tag{4.59}$$

where $W[\rho]$ is defined as:

$$W[\rho] = L\int'_{k\geq 0} dk\, s(\rho(k),\rho_t(k)) - \beta\rho(k)\epsilon(k) - L\int_{k\geq 0} dk\, s(\rho(k),\rho_t(k)) - \beta\rho(k)\epsilon(k) \tag{4.60}$$

$$\begin{aligned} &\Rightarrow \lim_{L\to\infty} \frac{\int D\rho \sum_{connected}[connected] e^{L\int'_{k\geq 0} dk\, s(\rho(k),\rho_t(k))-\beta\rho(k)\epsilon(k)}}{\int D\rho e^{L\int_{k\geq 0} dk\, s(\rho(k),\rho_t(k))-\beta\rho(k)\epsilon(k)}} = \\ &= \lim_{L\to\infty} \sum_{connected} [connected] e^{\lim_{L\to\infty} W[\rho]} \end{aligned} \tag{4.61}$$

Where the distribution in the last expression must be taken at the saddle point.
Now we will study separately the Bose and Fermi cases.
In the Bose case we have simply $W[\rho]=0$, so we conclude:

$$\lim_{L\to\infty} \frac{\int D\rho \sum_{connected}[connected] e^{L\int'_{k\geq 0} dk\, s(\rho(k),\rho_t(k))-\beta\rho(k)\epsilon(k)}}{\int D\rho e^{L\int_{k\geq 0} dk\, s(\rho(k),\rho_t(k))-\beta\rho(k)\epsilon(k)}} = \lim_{L\to\infty} \sum_{connected} [connected] \tag{4.62}$$

We recall for completeness the saddle-point equations in the free Bose case:

$$\frac{\rho(k)}{\rho(k)+\rho_t(k)} = |K(k)|^2 \qquad \rho_t(k) = \frac{\partial_k p(k)}{2\pi} \tag{4.63}$$

Observe that we find again the result for the free Bose case of the beginning of section 4.2.2.

Instead in the Fermi case $W[\rho]$ is not zero because of the presence of $\int'$ instead of $\int$:

$$\begin{aligned} W[\rho] &= L\int'_{\geq 0} dk\, s(\rho(k),\rho_t(k)) - \beta\rho(k)\epsilon(k) - L\int_{\geq 0} dk\, s(\rho(k),\rho_t(k)) - \beta\rho(k)\epsilon(k) = \\ &= -\sum_{p>0\in connected} \frac{1}{\rho_t(p)}\left(s(\rho(p),\rho_t(p)) - \rho(p)\epsilon(p)\right) \end{aligned} \tag{4.64}$$

where the last summation is restricted to the positive quantum numbers in the connected blocks. We recall now the saddle point equation and the expression for the entropy in the free Fermi case

$$\begin{aligned} &\frac{\rho(k)}{\rho_t(k)-\rho(k)} = |K(k)|^2 \qquad \rho_t(k) = \frac{\partial_k p(k)}{2\pi} \\ &s(\rho,\rho_t) = \rho_t \log\rho_t - \rho\log\rho - (\rho_t-\rho)\log(\rho_t-\rho) \end{aligned} \tag{4.65}$$

At the saddle point the following relation holds:

$$s(\rho(p),\rho_t(p)) - \rho(p)\epsilon(p) = \rho_t(p)\log\left(\frac{\rho_t(p)}{\rho_t(p)-\rho(p)}\right) \tag{4.66}$$

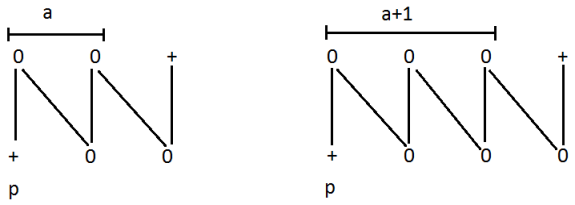
$$\Rightarrow W[\rho] = -\sum_{p>0\in connected} \log\left(\frac{\rho_t(p)}{\rho_t(p)-\rho(p)}\right) = -\sum_{p>0\in connected} \log\left(1+|K(p)|^2\right) \tag{4.67}$$

At the end we obtain:

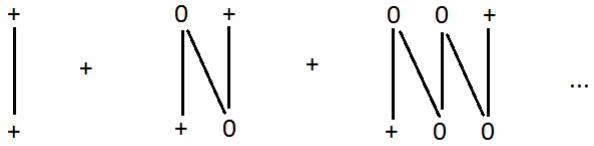
$$\begin{aligned} &\frac{\int D\rho \sum_{connected}[connected] e^{L\int'_{k\geq 0} dk\, s(\rho(k),\rho_t(k)) - \beta\rho(k)\epsilon(k)}}{\int D\rho e^{L\int_{k\geq 0} dk\, s(\rho(k),\rho_t(k)) - \beta\rho(k)\epsilon(k)}} = \\ &= \lim_{L\to\infty} \sum_{connected}[connected] \prod_{p>0\in connected}\left(\frac{1}{1+|K(p)|^2}\right) \end{aligned} \tag{4.68}$$

We have eliminated the contribution of bubbles and we have obtained a simpler summation over graphs made only by connected blocks: these blocks have to be "renormalized" in the Fermi case.

We can proceed further and analyze the relation between the following two blocks:

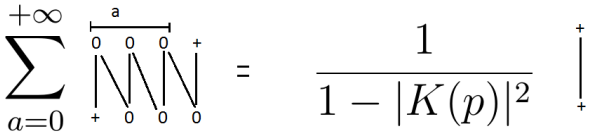


that is to say we look for the relation between two equal blocks, but with different "length". In the fermionic case we cannot have blocks whose length is bigger than zero (as length we mean the number of "0" in a row) because of the exclusion principle: no equal quantum numbers in the same row. Instead in the Bose case incrementing by one unit the length of the block means simply an extra $|K(p)|^2$ factor. The natural idea in the boson case is "summing" over the possible lengths of each block:

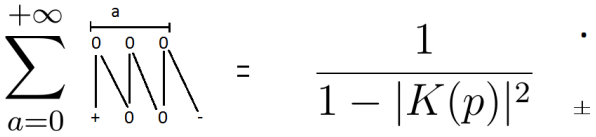


$$\frac{1}{L2\pi\rho_t(p)}\left(|K(p)|^2+|K(p)|^4+..\right)=\frac{1}{L2\pi\rho_t(p)}\sum_{n=1}^{+\infty}|K(p)|^{2n}=\frac{1}{L2\pi\rho_t(p)}\frac{|K(p)|^2}{1-|K(p)|^2}$$

We can write in a graphical representation:



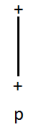
With the same reasoning we can say:



With these results we can introduce a new graphical representation in which no "0" appears: in fact we have already summed over all bubbles (so their "0" disappear), then in the Fermi case we cannot have any "0" left, instead in the Bose case we sum over the different lengths of graphs as above.

New graphical rules:

- Draw graphs as before, but not any "0" symbol must be drawn.
- We give the contribution of each block.

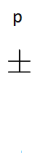


$$\frac{1}{L2\pi\rho_t(p)}\frac{|K(p)|^2}{1-|K(p)|^2} \quad Bose\ case, \qquad \frac{1}{L2\pi\rho_t(p)}\frac{|K(p)|^2}{1+|K(p)|^2} \quad Fermi\ case$$

The contribution of the block with "-" is analogous, except for the presence in the connected average of $-p$ instead of p. A "$\pm$" in the lower row means the following contribution:

$$\frac{1}{L2\pi\rho_t(p)}\frac{e^{-2itE(p)}K(p)}{1-|K(p)|^2} \quad Bose\ case, \qquad \frac{1}{L2\pi\rho_t(p)}\frac{e^{-2itE(p)}K(p)}{1+|K(p)|^2} \quad Fermi\ case$$

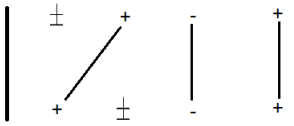
Instead a "$\pm$" in the upper row has this contribution:



$$\frac{1}{L2\pi\rho_t(p)}\frac{e^{2itE(p)}K^*(p)}{1-|K(p)|^2} \quad Bose\ case, \qquad \frac{1}{L2\pi\rho_t(p)}\frac{e^{2itE(p)}K^*(p)}{1+|K(p)|^2} \quad Fermi\ case$$

- Enter the corresponding quantum numbers in the connected average as we have already stated.
- Sum over the quantum numbers respecting their order.
- Sum over all possible graphs to obtain $\frac{\langle\phi|O(x,t)|\phi\rangle}{\langle\phi|\phi\rangle}$.

At this point, a typical graph will have this aspect:



We can proceed further: consider the difference between the two following graphs

The only difference is that the last two blocks are exchanged. The contribution to the terms to sum is the same, the only difference is the domain of summation: we have exchanged the order of the last quantum numbers.

Consider a graph with some connected blocks and then all the graphs obtained from it exchanging the positions of these blocks. The contribution to the sum of all these graphs is the same, the difference is only in the different domains of the summation over the quantum numbers. Summing over the quantum numbers in a graph and then summing over all the graphs obtained permuting the blocks will be equivalent (a part from some symmetry factors we are going to explain) to the contribution of only a graph, but summing its quantum numbers without restrictions.

We state this fact more precisely: consider a graph such that there are n_+ blocks with "+" at the end of the link, n_- blocks with "-" signs at the end of the connections, $n_\pm^{up}$ "$\pm$" symbols in the upper row and "$n_\pm^{dw}$" in the lower row. We indicate "$[n_+, n_-, n_\pm^{up}, n_\pm^{dw}]$" the equivalence class of the graphs with the previous characteristics.

The following relation is true:

$$\sum_{[graph]\in[n_+,n_-,n_\pm^{up},n_\pm^{dw}]} \sum^{ordered} [graph] = \sum^{free} \frac{1}{n_+!n_-!n_\pm^{up}!n_\pm^{dw}!}[n_+, n_-, n_\pm^{up}, n_\pm^{dw}] \qquad (4.69)$$

In the last summation we have to consider only a representative graph of the equivalence class, the "free" summation means that we do not have to restrict the quantum numbers to a precise order, but their domain is still $[0; +\infty)$. The factorial terms are motivated by the fact that exchanging identical blocks we do not have any new graph, so we have:

$$\lim_{L\to\infty} \frac{\langle\phi|\, O(x,t)\,|\phi\rangle}{\langle\phi|\phi\rangle} = \sum_{n_+,n_-,n_\pm^{up},n_\pm^{dw}} \sum_{free} \frac{1}{n_+!n_-!n_\pm^{up}!n_\pm^{dw}!}[n_+, n_-, n_\pm^{up}, n_\pm^{dw}] \qquad (4.70)$$

We can perform one last step, then write the final formula: suppose to sum the contributions as follows

$$\lim_{L\to\infty}\frac{\langle\phi|\,O(x,t)\,|\phi\rangle}{\langle\phi|\phi\rangle}=\sum_{n,n_\pm^{up},n_\pm^{dw}}\sum_{n_++n_-=n}\sum_{free}\frac{1}{n_+!n_-!n_\pm^{up}!n_\pm^{dw}!}[n_+,n_-,n_\pm^{up},n_\pm^{dw}] \quad (4.71)$$

If we sum over n_+, n_- with the constrain $n = n_+ + n_-$ we are summing identical contributions, a part from the quantum numbers that enter in the connected average which "change sign" when we change n_+ and n_-. An automatic way to take care of this sum is summing over all the possible quantum numbers in $(-\infty;+\infty)$ and substitute $\frac{1}{n_+!n_-!}\to\frac{1}{n!}$. With this last observation we arrive at the following expression, where the $\pm$ sign in the weight functions is "+" for fermions and "-" for bosons.

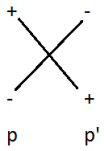
$$\lim_{L\to\infty}\frac{\langle\phi|\,O(x,t)\,|\phi\rangle}{\langle\phi|\phi\rangle}=\sum_{n,n_\pm^{up},n_\pm^{dw}}\frac{1}{n!n_\pm^{up}!n_\pm^{dw}!}\sum_{\vec\alpha,-\infty}^{+\infty}\sum_{\vec\gamma,0}^{\infty}\sum_{\vec\eta,0}^{+\infty}\langle\vec\alpha^{+}\vec\gamma^{c}|\,O(0,0)\,|\vec\eta^{c}\vec\alpha\rangle_{conn}\cdot$$
$$\cdot\prod_{p\in\vec\alpha}\left[\frac{1}{L2\pi\rho_t(p)}\frac{|K(p)|^2}{1\pm|K(p)|^2}\right]\prod_{p\in\vec\eta}\left[\frac{1}{L2\pi\rho_t(p)}\frac{K(p)e^{-i2tE(p)}}{1\pm|K(p)|^2}\right]\prod_{p\in\vec\gamma}\left[\frac{1}{L2\pi\rho_t(p)}\frac{K^*(p)e^{i2tE(p)}}{1\pm|K(p)|^2}\right] \quad (4.72)$$

where the vectors $\vec\alpha,\vec\gamma,\vec\eta$ have $n, n_\pm^{up}, n_\pm^{dw}$ elements. The summation over the elements of $\vec\alpha$ runs from $-\infty$ to $+\infty$, the other two sums are constrained to positive numbers. There are no more constraints over the summations.

Now we can safely perform the continuous approximation and evaluate the summation with integrals. The measure is simply $\sum_p\to L\int dp\rho_t(p)$. We observe that all the explicit L dependence is canceled as it should:

$$\lim_{L\to\infty}\frac{\langle\phi|\,O(x,t)\,|\phi\rangle}{\langle\phi|\phi\rangle}=\sum_{n,n_\pm^{up},n_\pm^{dw}}\frac{1}{n!n_\pm^{up}!n_\pm^{dw}!}\int_{-\infty}^{+\infty}\frac{d^n\alpha}{(2\pi)^n}\int_0^{+\infty}\frac{d^{n_\pm^{up}}\gamma}{(2\pi)^{n_\pm^{up}}}\frac{d^{n_\pm^{dw}}\eta}{(2\pi)^{n_\pm^{dw}}}\cdot$$
$$\cdot\langle\vec\alpha^{+}\vec\gamma^{c}|\,O(0,0)\,|\vec\eta^{c}\vec\alpha\rangle_{conn}\prod_{p\in\vec\alpha}\frac{|K(p)|^2}{1\pm|K(p)|^2}\prod_{p\in\vec\eta}\frac{K(p)e^{-i2tE(p)}}{1\pm|K(p)|^2}\prod_{p\in\vec\gamma}\frac{K^*(p)e^{i2tE(p)}}{1\pm|K(p)|^2} \quad (4.73)$$

At this point we can return to the problem of the graphs we dropped: we consider a simple case but the discussion is identical for the others. Consider the following excluded block:



The only non zero contribution of this block is when $p = p'$, so we can perform the same summation over the bubbles as the ordinary connected block. In the fermion case this graph will be trivially 0, instead in the Bose case it will give:

$$\frac{1}{(L2\pi\rho_t(p))^2}|K(p)|^4 \tag{4.74}$$

This contribution has to be summed over the possible momenta p, but $\sum_p \sim L$ so this term after the summation is still $\mathcal{O}(L^{-1})$ and so gives zero contribution in the $L \to \infty$ limit.

4.2.3 Conclusions for the free case

At the end of our calculation we obtained this result:

$$\begin{aligned}\lim_{L\to\infty} \frac{\langle\phi|\, O(x,t)\,|\phi\rangle}{\langle\phi|\phi\rangle} = \sum_{n,n_\pm^{up},n_\pm^{dw}} \frac{1}{n!n_\pm^{up}!n_\pm^{dw}!}\int_{-\infty}^{+\infty}\frac{d^n\alpha}{(2\pi)^n}\int_0^{+\infty}\frac{d^{n_\pm^{up}}\gamma}{(2\pi)^{n_\pm^{up}}}\frac{d^{n_\pm^{dw}}\eta}{(2\pi)^{n_\pm^{dw}}}\cdot \\ \cdot\langle\vec{\alpha}^+\vec{\gamma}^c|\, O(0,0)\,|\vec{\eta}^c\vec{\alpha}\rangle_{conn}\prod_{p\in\vec{\alpha}}\frac{|K(p)|^2}{1\pm|K(p)|^2}\prod_{p\in\vec{\eta}}\frac{K(p)e^{-i2tE(p)}}{1\pm|K(p)|^2}\prod_{p\in\vec{\gamma}}\frac{K^*(p)e^{i2tE(p)}}{1\pm|K(p)|^2}\end{aligned} \tag{4.75}$$

This expression permits us to evaluate all the time dependence of local observables on squeezed states. Its convergence depends on the connected averages and on the function K which often have to be regularized, for example see [27] and references therein.

Usually, a part from the regularization of the K functions, this summation is fast convergent because of the presence of the factorial terms: the leading terms for the long time behavior are easy extracted. In the long time limit the presence of fast oscillating terms $e^{i2E(k)t}$ immediately tells us that the steady state can be obtained summing only the terms such that $n_\pm^{up} = n_\pm^{dw} = 0$ and this result coincides with [24]. In particular, if we compare the long time limit with the Le Clair-Mussardo's conjecture, we see that the steady state is described by a GGE with pseudo energy $\epsilon(k) = -\log|K(k)|^2$.

The corrections to the steady state are easily computed keeping the lower orders in $n_\pm^{dw} + n_\pm^{up}$, in particular if we have a gapped dispersion law, such as for massive relativistic theories, we have that the leading contributions in the long time limit must have $n_\pm^{dw} = n_\pm^{up}$.

4.3 Long time averages in the interacting case

In this section we implement our graphical formalism to obtain the time average of local observables in the interacting fermionic theory, in this way we obtain the result of [19]. We are going to use results from [15] and [16]. Even if the final result is

already known, this approach could give us some useful insight into the interacting time-dependent case.

To calculate the long time averages we can restrict ourselves to the diagonal ensemble:

$$\lim_{t\to\infty}\langle\phi|\,O(x,t)\,|\phi\rangle=\sum_{\vec{k}}^{\sim L}\prod_{k_i\in\vec{k}}\left[|K(k_i)|^2e^{-\frac{\varphi(-k_i,k_i)}{\pi L\tilde{\rho}_t(k_i)}}\right]\langle\vec{k}^c|\,O(0,0)\,|\vec{k}^c\rangle \tag{4.76}$$

We need the expression of the expectation value in terms of connected averages from [16]:

$$\begin{aligned}
&\langle\vec{p}|\,O(x)\,|\vec{p}\rangle=\sum_{\vec{p}=\vec{p}_1\bigcup\vec{p}_2}\frac{J(\vec{p}|\vec{p}_1)}{J(\vec{p})}\langle p_2|\,O(x)\,|p_2\rangle_{conn}\\
&J(\vec{p})=\det(M)\\
&M_{i,j}=2\pi L\left(\delta_{i,j}\left(\frac{\partial_k p(k_i)}{2\pi}+\frac{1}{L2\pi}\sum_{l\neq j}\varphi(k_i,k_l)\right)+(1-\delta_{i,j})\frac{1}{L2\pi}\varphi(k_j,k_i)\right)
\end{aligned} \tag{4.77}$$

where $J(\vec{p}|\vec{p}_1)$ is the determinant of the minor of M associated to the indices of $\vec{p}_1$: if $\vec{p}_1=(p_{j_1}.p_{j_2},p_{j_3}..)$ then $J(\vec{p}|\vec{p}_1)$ is constructed with the indices $j_1,j_2,j_3..$

Motivated by the results of the free cases, we can evaluate $\frac{J(\vec{p}|\vec{p}_1)}{J(\vec{p})}$ when the number of particles is $\sim L$. We can use the same passages we used to evaluate $\left(\frac{\partial n}{\partial k}\right)$ in section 4.1.4:

$$\begin{aligned}
&J(\vec{k}^c)=(2\pi L)^{2N}\left(\prod_{k_j\in\vec{k}^c}\left(\frac{\partial_k p(k_j)}{2\pi}+\frac{1}{L2\pi}\sum_{k_l\neq k_j}\varphi(k_j,k_l)\right)+\mathcal{O}\left(\frac{1}{L}\right)\right)\\
&J(\vec{k}^c|\vec{p}_1)=(2\pi L)^{n_1}\left(\prod_{k_j\in\vec{p}_1}\left(\frac{\partial_k p(k_j)}{2\pi}+\frac{1}{L2\pi}\sum_{k_l\neq k_j}\varphi(k_j,k_l)\right)+\mathcal{O}\left(\frac{1}{L}\right)\right)
\end{aligned} \tag{4.78}$$

with N the number of Cooper pairs, n_1 the number of momenta in $\vec{p}_1$.

Call n_2 the number of quantum numbers in $\vec{p}_2$:

$$\frac{J(\vec{p}|\vec{p}_1)}{J(\vec{p})}=\frac{1}{(2\pi L)^{n_2}}\prod_{k_j\in\vec{p}_2}\left(\frac{\partial_k p(k_j)}{2\pi}+\frac{1}{L2\pi}\sum_{k_l\neq k_j}\varphi(k_j,k_l)\right)^{-1} \tag{4.79}$$

We omit some $\mathcal{O}(L^{-1})$ terms.

With the definition $\tilde{\rho}_t(k_j)=\frac{\partial_k p(k_j)}{2\pi}+\frac{1}{L2\pi}\sum_{k_l\neq k_j}\varphi(k_j,k_l)$ we have:

$$\langle\vec{k}^c|\,O(x)\,|\vec{k}^c\rangle=\sum_{\vec{p}=\vec{p}_1\bigcup\vec{p}_2}\prod_{p_k\in\vec{p}_2}\left(\frac{1}{2\pi L\tilde{\rho}_t(p_k)}\right)\langle p_2|\,O(x)\,|p_2\rangle_{conn} \tag{4.80}$$

In order to have a simpler notation, we anticipate a result: when we will calculate the normalized average, the contribution of "bubbles" will be canceled similarly to the free cases. We will see that this process leads us to the following expression:

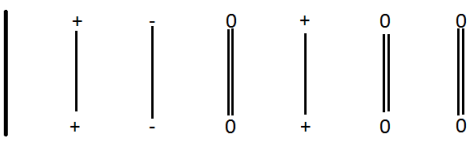
$$\lim_{t\to\infty} \langle\phi| O(x,t) |\phi\rangle \sim \sum_{free\, p\in\vec{p}_2} \prod_{p_k\in\vec{p}_2} \left(\frac{1}{2\pi L\tilde{\rho}_t(p_k)}\right) \langle p_2| O(x) |p_2\rangle_{conn} [regular\, function] \tag{4.81}$$

The "free summation" is over the free quantum numbers in $\vec{p}_2$: since it could contain Cooper pairs, some quantum numbers in $\vec{p}_2$ could be constrained by the Cooper pair structure.

At this point we need to estimate the leading term, we can see that whenever a Cooper pair appears in $\vec{p}_2$ the contribution is suppressed for $L \to \infty$. We can say that $\sum_{free\, p\in\vec{p}_2} \sim L^{n_{free}}$ with n_{free} the number of free quantum numbers in $\vec{p}_2$, instead $\prod_{p_k\in\vec{p}_2} \left(\frac{1}{2\pi L\tilde{\rho}_t(p_k)}\right) \sim L^{-n_2}$ with n_2 the number of elements in $\vec{p}_2$, so we can conclude that the only terms that matter at the end of the calculation are the ones that $n_{free} = n_2$, so without any Cooper pair in $\vec{p}_2$.

If there are no Cooper pairs in $\vec{p}_2$ we can use the same graphical machinery as the free case and exclude blocks with time dependence: we are allowed to use only the following blocks

Instead we cannot use $\pm$. The terms associated to each block are the same as before, apart from the slight modification of the contribution $2\pi\tilde{\rho}_t(p)$ and the small exponential factor $exp\left(-\frac{\varphi(-k,k)}{\pi L\tilde{\rho}_t(k)}\right)$ that is trivial in the free case. We report a typical graph:



We still try to sum over the "bubbles": now the theory is interacting and we have

to introduce some modifications in the previous steps. Set $\epsilon = -\log|K|^2$:

$$\langle O(x, t\to\infty)\rangle = \\ = \lim_{L\to\infty} \frac{\int D\rho \sum_{connected}[connected] e^{L\int'_{k\geq 0} dk\, s(\rho(k),\rho'_t(k))-\rho(k)\epsilon(k)} e^{-\int_{k>0} dk \frac{\rho(k)}{\pi\rho_t(k)}\varphi(-k,k)}}{\int D\rho e^{L\int_{k\geq 0} dk\, s(\rho(k),\rho_t(k))-\rho(k)\epsilon(k)} e^{-\int_{k>0} dk \frac{\rho(k)}{\pi\rho_t(k)}\varphi(-k,k)}} \tag{4.82}$$

On a test function $f(p)$ we still have:

$$\int' dp\, f(p) = \int dp\, f(p) - \frac{1}{L}\sum_m \frac{1}{\rho_t(p_m)} f(p_m) \tag{4.83}$$

where p_m are the quantum numbers of the connected blocks.

The distribution ρ_t is now defined through an integral equation:

$$\rho_t(k) = \frac{\partial_k p(k)}{2\pi} + \frac{1}{2\pi}\int d\lambda \rho(\lambda)\varphi(k,\lambda) \tag{4.84}$$

The definition of ρ'_t considers also the presence of the quantum numbers of the connected blocks in the Bethe Equations:

$$\rho'_t(k) = \frac{\partial_k p(k)}{2\pi} + \frac{1}{2\pi}\int' d\lambda \rho(\lambda)\varphi(k,\lambda) + \frac{1}{2\pi L}\sum_m \varphi(k,p_m) + \frac{1}{2\pi L}\sum_m \varphi(k,-p_m) \tag{4.85}$$

here the quantum numbers p_m are the positive ones in the connected blocks and each negative quantum number in each Cooper pairs is explicitly written. In the following we will continue with this convention.

We report the saddle point equation in the interacting Fermi case:

$$\frac{\rho_{Fermi}(k)}{\rho_t(k)-\rho_{Fermi}(k)} = e^{-\epsilon(k)} exp\left(\frac{1}{2\pi}\int \varphi(\lambda,k)\log\left(\frac{\rho_t(\lambda)}{\rho_t(\lambda)-\rho_{Fermi}(\lambda)}\right)\right) \tag{4.86}$$

Similarly to the free case we arrive at the following expression:

$$\langle O(x,t\to\infty)\rangle = \lim_{L\to\infty}\sum_{connected}[connected] e^{\lim_{L\to\infty} W[\rho]} \tag{4.87}$$

where $W[\rho]$ is defined as follows:

$$W[\rho] = L\int'_{k\geq 0} dk\, s(\rho(k),\rho'_t(k)) - \rho(k)\epsilon(k) - L\int_{k\geq 0} dk\, s(\rho(k),\rho_t(k)) - \rho(k)\epsilon(k) \tag{4.88}$$

here ρ satisfies the saddle point equations.

We calculate $\lim_{L\to\infty} W[\rho]$, omitting $\mathcal{O}(L^{-1})$ terms we have:

$$\begin{aligned} W[\rho] &= L\int_{k\geq 0}' dk\, s(\rho(k),\rho_t'(k)) - \rho(k)\epsilon(k) - L\int_{k\geq 0} dk\, s(\rho(k),\rho_t(k)) - \rho(k)\epsilon(k) = \\ &= L\int_{k\geq 0} dk s(\rho(k),\rho_t'(k)) - s(\rho(k),\rho_t(k)) - \sum_m \frac{s(\rho(p_m),\rho_t(p_m)) - \rho(p_m)\epsilon(p_m)}{\rho_t(p_m)} = \\ &= L\int_{k\geq 0} dk(\rho_t' - \rho_t)\partial_{\rho_t} s(\rho(k),\rho_t(k)) - \sum_m \frac{s(\rho(p_m),\rho_t(p_m)) - \rho(p_m)\epsilon(p_m)}{\rho_t(p_m)} \end{aligned} \tag{4.89}$$

as in the free case, the summation above is restricted to the positive quantum numbers in the connected blocks. In the following we use the symmetry of the density distribution $\rho(k) = \rho(-k)$ to extend the domains of the integrals from $[0;+\infty)$ to $(-\infty;+\infty)$ and take care of the negative quantum numbers in the connected blocks: the integrals below are performed on the whole real axis and the summations are restricted to positive p_m numbers.

$$\begin{aligned} &\lim_{L\to\infty} W[\rho] = \sum_m \frac{1}{2\pi}\int dk \partial_{\rho_t} s(\rho,\rho_t)\varphi(k,p_m) + \\ &-\sum_m \frac{1}{\rho_t(p_m)}\left(\frac{\rho(p_m)}{2\pi}\int dk\varphi(k,p_m)\partial_{\rho_t} s(\rho(k),\rho_t(k)) + s(\rho(p_m),\rho_t(p_m)) - \rho(p_m)\epsilon(p_m)\right) \end{aligned} \tag{4.90}$$

We can impose the saddle point equations in the expression above

$$\partial_{\rho_t} s(\rho,\rho_t) = \log\left(\frac{\rho_t}{\rho_t - \rho}\right) \tag{4.91}$$

$$\begin{aligned} &\Rightarrow \frac{1}{2\pi}\int dk\varphi(k,p_m)\partial_{\rho_t} s(\rho(k),\rho_t(k)) = \frac{1}{2\pi}\int d\lambda\varphi(\lambda,p_m)\log\left(\frac{\rho_t(\lambda)}{\rho_t(\lambda) - \rho_{Fermi}(\lambda)}\right) = \\ &= \epsilon(p_m) + \log\left(\frac{\rho(p_m)}{\rho_t(p_m) - \rho(p_m)}\right) \end{aligned} \tag{4.92}$$

$$\begin{aligned} &\Rightarrow \frac{\rho(p_m)}{2\pi}\int dk\varphi(k,p_m)\partial_{\rho_t} s(\rho(k),\rho_t(k)) + s(\rho(p_m),\rho_t(p_m)) - \rho(p_m)\epsilon(p_m) = \\ &\quad = \rho_t(p_m)\log\left(\frac{\rho_t(p_m)}{\rho_t(p_m) - \rho(p_m)}\right) \end{aligned} \tag{4.93}$$

$$\Rightarrow \lim_{L\to\infty} W[\rho] = \sum_m \frac{1}{2\pi}\int dk\varphi(k,p_m)\log\left(\frac{\rho_t}{\rho_t - \rho}\right)_{(k)} - \sum_m \log\left(\frac{\rho_t(p_m)}{\rho_t(p_m) - \rho(p_m)}\right) \tag{4.94}$$

The contribution we obtain is similar to the free case plus an extra factor. Notice that the expression is factorized over the blocks.

We can give the new graphical rules:

The only allowed blocks are the following:

Both give the same contribution, a part from the quantum numbers in the connected average:

$$\frac{1}{2\pi L\rho_t(p)}|K(p)|^2\left(1-\frac{\rho(p)}{\rho_t(p)}\right)e^{\frac{1}{2\pi}\int dk\varphi(k,p)\log\left(\frac{\rho_t(k)}{\rho_t(k)-\rho(k)}\right)}$$

Adopt the following definition:

$$|\tilde{K}(p)|^2\equiv|K(p)|^2e^{\frac{1}{2\pi}\int dk\varphi(k,p)\log\left(\frac{\rho_t(k)}{\rho_t(k)-\rho(k)}\right)} \tag{4.95}$$

At the saddle point we have the following relations:

$$\begin{aligned}&\frac{\rho(p)}{\rho_t(p)-\rho(p)}=|\tilde{K}(p)|^2\\&|\tilde{K}(p)|^2=|K(p)|^2e^{\frac{1}{2\pi}\int dk\varphi(k,p)\log\left(1+|\tilde{K}(p)|^2\right)}\\&1-\frac{\rho(p)}{\rho_t(p)}=\frac{1}{1+|\tilde{K}(p)|^2}\end{aligned} \tag{4.96}$$

$$\Rightarrow\frac{1}{2\pi L\rho_t(p)}|K(p)|^2\left(1-\frac{\rho(p)}{\rho_t(p)}\right)e^{\frac{1}{2\pi}\int dk\varphi(k,p)\log\left(\frac{\rho_t(k)}{\rho_t(k)-\rho(k)}\right)}=\frac{1}{2\pi L\rho_t(p)}\frac{|\tilde{K}(p)|^2}{1+|\tilde{K}(p)|^2} \tag{4.97}$$

From now on the calculation is identical to the free case, a part from the substitution

$$\frac{1}{2\pi\rho_t(p)}\frac{|K(p)|^2}{1+|K(p)|^2}\to\frac{1}{2\pi\rho_t(p)}\frac{|\tilde{K}(p)|^2}{1+|\tilde{K}(p)|^2} \tag{4.98}$$

We obtain the final result:

$$\langle O(x, t \to \infty) \rangle = \sum_n \frac{1}{n!} \int_{-\infty}^{+\infty} \frac{d^n \alpha}{(2\pi)^n} \langle \vec{\alpha} | O(0,0) | \vec{\alpha} \rangle_{conn} \prod_{p \in \vec{\alpha}} \frac{|\tilde{K}(p)|^2}{1 + |\tilde{K}(p)|^2} \tag{4.99}$$

This is exactly the result given in [19]: if we compare this expression with the Le Clair-Mussardo's conjecture, we see that the interacting steady state is described by a GGE where $\epsilon(k) = -\log |K(k)|^2$.

4.4 Conclusions to Chapter 4

We want to summarize what has been done in this chapter and what we hope to obtain with future investigations.

We developed a new graphical approach to handle averages of local observables over squeezed states, we recast in a different way results from [24] and [19] and obtained as new result the time evolution for the free fermionic and bosonic cases. Since the time average in the interacting case has been computed using this method, we hope to generalize this approach and calculate time dependence also in interacting theories.

To accomplish this task would be an highly non trivial result: usually two times intervals are explored, that is to say immediately after a sudden quench and the long time behavior as we studied along this chapter. The squeezed states are usually obtained changing a parameter in the Hamiltonian, so a possible technique to explore the time behavior of the system is via a perturbative approach in the shift of the coupling constant: this method usually provides good results in a small time interval after the quench and is not suitable to study the entire time evolution. On the other hand, GTBA arguments have been used recently in [43] to study the long time behavior of local averages on squeezed states, but the entire time evolution is still unexplored.

Expressions such as the ones we obtained in this chapter, even if they are expressed as an infinite series, are not perturbative series and can be used to explore the entire time evolution; besides the factorial terms provide a fast convergence of the series that can be safely truncated after few terms and numerically studied.

4.5 Appendix

In this Appendix we study the following limit:

$$\lim_{L \to \infty} \frac{\int D\rho A[\rho, \frac{1}{L}] e^{LF[\rho]}}{\int D\rho B[\rho, \frac{1}{L}] e^{LF[\rho]}} \tag{4.100}$$

where $F[\rho]$ is a functional with only one maximum such that its second derivative is negative defined, $A[\rho, \frac{1}{L}]$ and $B[\rho, \frac{1}{L}]$ are well behaved functionals of ρ in order to

guarantee the convergence of the integrals and with a well defined $L \to \infty$ limit. Suppose ρ_0 is the maximum, then we can expand $F[\rho]$ around that point:

$$\frac{\int D\rho A[\rho, \frac{1}{L}] e^{LF[\rho_0]+L\Phi^2[\rho-\rho_0]+L\mathcal{O}[\rho-\rho_0]^3}}{\int D\rho B[\rho, \frac{1}{L}] e^{LF[\rho_0]+L\Phi^2[\rho-\rho_0]+L\mathcal{O}[\rho-\rho_0]^3}} \tag{4.101}$$

where Φ^2 is a quadratic operator applied to $\rho - \rho_0$. Change integration variable $\sigma = \sqrt{L}(\rho - \rho_0)$. The translation does not affect the measure of the functional integral, instead the dilatation gives a multiplicative contribution that is canceled by the ratio of the integrals:

$$\frac{\int D\sigma A[\rho_0 + \frac{\sigma}{\sqrt{L}}, \frac{1}{L}] e^{\Phi^2[\sigma]+\frac{1}{\sqrt{L}}\mathcal{O}[\sigma]^3}}{\int D\sigma B[\rho_0 + \frac{\sigma}{\sqrt{L}}, \frac{1}{L}] e^{\Phi^2[\sigma]+\frac{1}{\sqrt{L}}\mathcal{O}[\sigma]^3}} \tag{4.102}$$

Now we can perform the $L \to \infty$ limit

$$\lim_{L\to\infty} \frac{\int D\rho A[\rho, \frac{1}{L}] e^{LF[\rho]}}{\int D\rho B[\rho, \frac{1}{L}] e^{LF[\rho]}} = \frac{A[\rho_0]}{B[\rho_0]} \tag{4.103}$$

Chapter 5

Local averages and cluster property

In Chapter 4 we dealt with quenches protocols that give squeezed states as initial condition for the post-quench evolution and we saw that these states lead to GGE predictions. We already said, in the introduction to Chapter 4, that many quench protocols in interacting theories produce more complicated initial states [35], so we would like to consider a larger class of states.

This chapter is mostly motivated by [28]: in this article the authors analyze the long time behavior of local operators in a free bosonic theory after generic quenches. In particular they showed that if the initial state satisfies a requirement called "cluster property" then the long time behavior of the observable is described by a GGE. All the formalism is in Heisenberg's representation and the approach uses strongly the possibility of computing the time evolution for the bosonic field: this fact is highly non trivial in interacting theories and does not appear of easy generalization.

Since studing directly the effect of reasonably physical quenches in interacting theories is not so simple (see for example [35]), we choose to proceed backward: the idea is to use the results of [28] to individuate a class of initial states larger than the squeezed states that leads to GGE predictions for local observables. We will express these states in terms of the creation operators of the field in the Fourier space: this way of proceed seems generalizable also to interacting theories, as a matter of fact the creation operators of the field in the momentum representation are the Zamolodchikov-Fadeev operators of the free theory, so a formalism that uses this language could be useful to have some insight in the interacting case.

The particular form of cluster states, even if more complicated than the squeezed states, suggests that a graphical calculation similar to the one in Chapter 4 can be attempted: in the case of free bosonic theories we managed to perform the calculation of the time averages of local observables, which is also believed to be the long time limit. Even if we cannot find a compact formula for these averages we have two main results: a graphical approach that can be used to perform perturbative calculations and we also recast the non perturbative information that the steady state of local operators is described by a GGE density matrix.

The novelty of this approach is that it appears more appealing for a generalization to interacting theories, as a matter of fact at the end of the calculation it will be clear what are the basic features of the states that lead to the GGE in the long time limit: even if we do not perform any explicit calculation in interacting theories, we can guess a natural generalization of cluster states in the interacting case and the squeezed states are still a particular case of this guess. Motivated by these results we hope to find in future studies a more general set of states than the squeezed states that leads to the GGE.

5.1 Cluster states

In this chapter we deal with a free bosonic theory described by a field $\Phi(x)$ and its conjugated momentum $\Pi(x)$ satisfying the canonical commutation relation:

$$[\Phi(x), \Pi(y)] = i\delta(x-y) \tag{5.1}$$

The goal of this section is to characterize some states such that the cluster property holds, that is to say a state $|\phi\rangle$ such that if we evaluate the averages of the field on distant points we have the following behavior, see references in [28]:

$$\langle \prod_i \Phi(x_i) \prod_j \Phi(y_j) \rangle \to \langle \prod_i \Phi(x_i) \rangle \langle \prod_j \Phi(y_j) \rangle \quad \text{if } |x_i - y_j| \to +\infty \ \forall i,j \tag{5.2}$$

This property is not suitable for what follows, for this reason we are going to determine a stronger requirement that implies the cluster property. The field of a free theory can be decomposed in terms of creation-annihilation operators as follows:

$$\begin{aligned} &\Phi(x) = \int dk \frac{1}{\sqrt{4\pi E(k)}} \left(a^\dagger(k) e^{ikx} + a(k) e^{-ikx} \right) \\ &[a(k), a(q)] = 0, \qquad [a(k), a^\dagger(q)] = \delta(k-q) \end{aligned} \tag{5.3}$$

Such that the evolution of the field assumes this expression:

$$e^{iHt}\Phi(x)e^{-iHt} = \int dk \frac{1}{\sqrt{4\pi E(k)}} \left(a^\dagger(k) e^{ikx} e^{-itE(k)} + a(k) e^{-ikx} e^{itE(k)} \right) \tag{5.4}$$

For simplicity we will suppose a gapped dispersion law to avoid singularities in the previous equations. We can define the Fourier transform of the $a^\dagger$:

$$\psi^\dagger(x) = \int \frac{dk}{\sqrt{2\pi}} e^{ikx} a^\dagger(k) \tag{5.5}$$

These operators satisfy the standard commutation rules:

$$[\psi(x), \psi(y)] = 0 \qquad [\psi(x), \psi^\dagger(y)] = \delta(x-y) \tag{5.6}$$

We can interpret them as creation-annihilation operators in the position representation and construct states from the vacuum $|0\rangle$ with them. Notice that $\Phi(x)$ cannot be expressed as a function of $\psi^\dagger(x)$ and $\psi(x)$ evaluated at the same point x:

$$\begin{aligned}\Phi(x) &= \int dk \frac{1}{\sqrt{4\pi E(k)}} \left(a^\dagger(k)e^{ikx} + a(k)e^{-ikx}\right) = \\ &= \int dk \int dy \frac{1}{2\pi\sqrt{2E(k)}} \left(\psi^\dagger(y)e^{-iky}e^{ikx} + \psi(y)e^{iky}e^{-ikx}\right) = \\ &= \int dy \psi^\dagger(y) \left[\int \frac{dk}{2\pi\sqrt{2E(k)}} e^{ik(x-y)}\right] + c.c.\end{aligned} \tag{5.7}$$

The distribution $\int \frac{dk}{2\pi\sqrt{2E(k)}} e^{ik(x-y)}$ is not a delta function, but usually the dispersion law localizes this function: for example with a relativistic energy $E(k) = \sqrt{k^2+m^2}$ it can be shown that the integral decays as $\sim e^{-m|x-y|}$. So to evaluate $\Phi(x)$ we need the field $\psi(y)$ only in a neghbourhood of x of extension $\sim \frac{1}{m}$. Using this observation, we define another property for states.

Suppose a state $|\phi\rangle$ satisfies the following requirement, that we will call Modified Cluster Property (MCP):

$$\begin{aligned}&\langle \prod_i^{n+1} \psi^\dagger(x_i) \prod_{i=n+1}^{N} \psi(x_i) \prod_j^{m+1} \psi^\dagger(y_j) \prod_{j=m+1}^{M} \psi(y_j)\rangle \simeq \\ &\simeq \left(\langle \prod_i^{n+1} \psi^\dagger(x_i) \prod_{i=n+1}^{N} \psi(x_i)\rangle\right)\left(\langle \prod_j^{m+1} \psi^\dagger(y_j) \prod_{j=m+1}^{M} \psi(y_j)\rangle\right)\end{aligned} \tag{5.8}$$

$$\text{if } |x_i - y_j| \to +\infty \ \forall i,j$$

It is easy to prove that with the dispersion relation $E(k) = \sqrt{k^2+m^2}$ the MCP implies the cluster property due to the fact that the field $\Phi(x)$ can be expressed through the fields $\psi^\dagger(y)$, $\psi(y)$ with y confined in a neighbourhood of x of length $\sim m^{-1}$.

In the following sections we will look for states that satisfy the MCP: this will imply automatically that they satisfy the cluster property.

5.1.1 General observations and notations, introduction of cluster states

In this section we will construct the cluster states: they are a large class of states that satisfy the MCP and so the cluster property. Consider a generic state $|\phi\rangle$, we

know that can be represented in terms of $\psi^\dagger$ operators:

$$|\phi\rangle = \sum_n \frac{1}{n!} \int d^n x \, \phi_n(x_1, x_2, ..x_n) \prod_{i=1}^{n} \psi^\dagger(x_i) \, |0\rangle \tag{5.9}$$

where because of the commutation relations of $\psi^\dagger$ we can choose $\phi_n(\vec{x}_n)$ as a symmetric function.

We now will start to study which characteristics $|\phi\rangle$ must have to guarantee the MCP.

Suppose we consider a delocalized state, then if it satisfies the MCP its decomposition cannot contain a finite number of particles. In other words, delocalized states as the following one cannot satisfy the MCP:

$$|\phi^m\rangle = \sum_{n=0}^{m} \frac{1}{n!} \int d^n x \, \phi_n(x_1, x_2, ..x_n) \prod_{i=1}^{n} \psi^\dagger(x_i) \, |0\rangle \tag{5.10}$$

As a matter of facts consider $|\phi^m\rangle$: it contains states with at most m particles.

Then define $n(x) = \psi^\dagger(x)\psi(x)$ and $n(\vec{x}_{m+1}) = \prod_{j=1}^{m+1} n(x_j)$: we have automatically $\langle n(\vec{x}_{m+1})\rangle = 0$, but on the other hand the MCP imposes $\lim_{|x_j - x_i| \to \infty} \langle n(\vec{x}_{m+1})\rangle = \prod_{j=1}^{m+1} \langle n(x_j)\rangle$.

Due to the delocalization of $|\phi^m\rangle$ we can always find a set of distant points such that $\langle n(x_j)\rangle \neq 0 \, \forall j$, so we conclude that the MCP does not hold.

In the following we will restrict ourselves to delocalized states, as a matter of fact in the end we will be interested in translational invariant states.

We can immediately determine a set of states such that the MCP holds: states such that the wavefunction is factorized over the variables as the following:

$$|\tilde{\phi}\rangle = \sum_n \frac{1}{n!} \int d^n x \prod_j W(x_j) \prod_j \psi^\dagger(x_j) \, |0\rangle = e^{\int dx W(x) \psi^\dagger(x)} \, |0\rangle \tag{5.11}$$

Using these states we have simply ($x \neq y$):

$$\langle \psi(x)\psi(y)\rangle = \langle \psi(x)\rangle \, \langle \psi(y)\rangle \qquad \langle \psi^\dagger(x)\psi(y)\rangle = \langle \psi^\dagger(x)\rangle \, \langle \psi(y)\rangle \tag{5.12}$$

This is a trivial consequence of the factorization of the wavefunction, so in general the MCP holds for these states. This constraint on the wavefunction is too strong, but can guide us: we can observe that the MCP contains a limit procedure, so a natural trial is requiring that the wavefunction mostly "factorizes" if evaluated on distant sets of points.

Definition: We will say that a wavefunction $\phi_n(\vec{x}_n)$ clusterizes if for each set A, B such that $\exists D > 0\, s.t. \forall x_j \in A, \forall x_i \in B\ |x_j - x_i| > D$ the following relation is true

$$\lim_{D\to\infty} \phi_n = \phi'_{|A|}(x_j \in A)\phi''_{|B|}(x_j \in B) \tag{5.13}$$

that is to say the wavefunction becomes the product of two functions of the two set of variables. Here $|A|$ and $|B|$ are respectively the number of elements in A and in B.

We observe that the "factorized wavefunction" trivially clusterizes.

This property alone does not guarantee the MCP. For example consider factorized states as follows

$$|\bar{\phi}\rangle = \sum_n \frac{1}{n!} \int d^n x \prod_j W_n(x_j) \prod_j \psi^\dagger(x_j)\,|0\rangle \tag{5.14}$$

with W_n different functions for each n. The wavefunction of this state still clusterizes (it is trivially factorized), but this state will not in general satisfy the MCP: as an extreme example we could imagine there exists m such that $W_{n>m} = 0$, in this case we will have a finite number of particles and we already know this case does not work.

Consider again the following trivial state:

$$|\tilde{\phi}\rangle = \sum_n \frac{1}{n!} \int d^n x \prod_j W(x_j) \prod_j \psi^\dagger(x_j)\,|0\rangle \tag{5.15}$$

Define $\tilde{\phi}_n(\vec{x}_n) = \prod_j^n W(x_j)$, now we will analyze the (trivial) properties of cluster decomposition of these wavefunctions. Suppose two sets of variables A, B such that $\exists D > 0 s.t. \forall x_j \in A, x_i \in B\ |x_j - x_i| > D$, let $|A|, |B|$ be the number of elements in each set:

$$\begin{aligned} &\tilde{\phi}_n(\vec{x}_n) = \tilde{\phi}_{|A|}(x_j \in A)\tilde{\phi}_{|B|}(x_i \in B) \\ &\Rightarrow \lim_{D\to\infty} \tilde{\phi}_n(\vec{x}_n) = \tilde{\phi}_{|A|}(x_j \in A)\tilde{\phi}_{|B|}(x_i \in B) \end{aligned} \tag{5.16}$$

The last equality can be imposed on more general functions that the factorized ones.

Definition: We will call $|\phi\rangle$ a cluster state if satisfies the following properties:

$$|\phi\rangle = \sum_n \frac{1}{n!} \int d^n x\, \phi_n(\vec{x}_n) \prod_{i=1}^n \psi^\dagger(x_i)\,|0\rangle \tag{5.17}$$

where $\phi_n(\vec{x}_n)$ are completely symmetric functions that clusterize in this way: suppose two sets of variables A, B such that $\exists D > 0 s.t. \forall x_j \in A, x_i \in B |x_j - x_i| > D$, let be

$|A|, |B|$ the number of elements in each set, then we have

$$\lim_{D\to\infty} \phi_n(\vec{x}_n) = \phi_{|A|}(x_j \in A)\phi_{|B|}(x_i \in B) \tag{5.18}$$

This definition gives us a natural nested structure: we will define the clusters of n variables $C_n(x_1, x_2..x_n)$ in the following recursive way

$$\begin{aligned}
&\phi_1(x) = C_1(x) \\
&\phi_2(x_1, x_2) = C_1(x_1)C_1(x_2) + C_2(x_1, x_2) \\
&\phi_3(x_1, x_2, x_3) = C_1(x_1)C_1(x_2)C_1(x_3) + C_1(x_1)C_2(x_2, x_3) + C_1(x_2)C_2(x_1, x_3) + \\
&+ C_1(x_3)C_2(x_2, x_1) + C_3(x_1, x_2, x_3)
\end{aligned} \tag{5.19}$$

Each cluster $C_n(x_1, ..x_n)$ is a symmetric function with this property:

$$\forall j, i \in (1, 2..n), i \neq j \qquad \lim_{|x_j - x_i|\to\infty} C_n(x_1, ..x_n) = 0 \tag{5.20}$$

At this point it will be useful to represent these states in a different form. A generic state can be put in the exponential form:

$$|\phi\rangle = exp\left(\sum_n \left[\int d^n x \, \frac{C_n(x_1, ..x_n)}{n!} \prod_{j=1}^{n} \psi^\dagger(x_j)\right]\right)|0\rangle \tag{5.21}$$

We obtain the cluster states if and only if the C_n go to zero for large distances. To verify this assert we must compare this expression with the form below:

$$|\phi\rangle = \sum_n \frac{1}{n!} \int d^n x \, \phi_n(\vec{x}_n) \prod_{i=1}^{n} \psi^\dagger(x_i)\,|0\rangle \tag{5.22}$$

We have to expand the exponential form: for now concentrate on the contribution that has j_1 powers on C_1, j_2 powers of C_2 and so on; call it $[\vec{j}]$

$$[\vec{j}] = \prod_n \frac{1}{j_n!} \left(\int d^n x \, \frac{C_n(x_1, ..x_n)}{n!} \prod_{j=1}^{n} \psi^\dagger(x_j)\right)^{j_n} |0\rangle \tag{5.23}$$

The product is ordered in an increasing way in n. The next step is to expand the operator content, we will label the variables $x^n_{q_n,k}$

$$[\vec{j}] = \int \left(\prod_n \left[\frac{1}{j_n!} \prod_{q_n=1}^{j_n} \frac{C_n(x^n_{q_n,1}, ..x^n_{q_n,n})}{n!}\right]\right) \left[\prod_n \left(\prod_{q_n}^{j_n} \left(\prod_{k=1}^{n} \psi^\dagger(x^n_{q_n,k})\right)\right)\right] |0\rangle \tag{5.24}$$

It is useful to define a new set of variables, so consider the following map of indices:

$$(n, q_n, k) \to n(q_n - 1) + k + \sum_{m=1}^{n-1} m j_m \equiv \alpha \tag{5.25}$$

The map is bijective, note that $\alpha \in (1, 2...N)$ where $N = \sum_n n j_n$.

We can define a new set of variables as $x^n_{q_n,k} = y_\alpha$. We represent the function in the new notation as:

$$\begin{aligned} &\left(\prod_n \left[\frac{1}{j_n!}\prod_{q_n=1}^{j_n}\frac{C_n(x^n_{q_n,1},..x^n_{q_n,n})}{n!}\right]\right) = \left(\prod_n \left[\frac{1}{j_n!}\prod_{q_n=1}^{j_n}\frac{C_n}{n!}\right]\right)_{(y_1,..,y_N)} \\ &\Rightarrow [\vec{j}] = \int d^N y \left(\prod_n \left[\frac{1}{j_n!}\prod_{q_n=1}^{j_n}\frac{C_n}{n!}\right]\right)_{(y_1,..,y_N)} \prod_{m=1}^{N}\psi^\dagger(y_m)\,|0\rangle \end{aligned} \tag{5.26}$$

If we sum the other contributions with N particles we obtain this final form:

$$|\phi\rangle = \sum_N \int d^N y \sum_{N=\sum_n n j_n} \left(\prod_n \left[\frac{1}{j_n!}\prod_{q_n=1}^{j_n}\frac{C_n}{n!}\right]\right)_{(y_1,..,y_N)} \prod_{m=1}^{N}\psi^\dagger(y_m)\,|0\rangle \tag{5.27}$$

where $\sum_{N=\sum_n n j_n}$ is the summation over all the possible indices j_n such that $\sum_n n j_n = N$. At this point the wavefunction is not clearly symmetric, but we can proceed in this way: the $\psi^\dagger$ commute and the domain is symmetric in the variables y_m, this means that the integral is equivalent (a part from a $N!$ normalization) to the integral of this symmetrized function

$$\begin{aligned} &\left[\sum_{N=\sum_n n j_n}\left(\prod_n \left[\frac{1}{j_n!}\prod_{q_n=1}^{j_n}\frac{C_n}{n!}\right]\right)\right]^{symmetric}_{(y_1,..,y_N)} = \\ &\sum_\sigma \left[\sum_{N=\sum_n n j_n}\left(\prod_n \left[\frac{1}{j_n!}\prod_{q_n=1}^{j_n}\frac{C_n}{n!}\right]\right)_{(y_{\sigma(1)},..,y_{\sigma(N)})}\right] \end{aligned} \tag{5.28}$$

where σ are permutations of N indexes. Then:

$$|\phi\rangle = \sum_N \frac{1}{N!}\int d^N y \left[\sum_{N=\sum_n n j_n}\left(\prod_n \left[\frac{1}{j_n!}\prod_{q_n=1}^{j_n}\frac{C_n}{n!}\right]\right)\right]^{symmetric}_{(y_1,..,y_N)} \prod_{m=1}^{N}\psi^\dagger(y_m)\,|0\rangle \tag{5.29}$$

At this point we give the following definition:

$$\begin{aligned}\phi_N(y_1,..,y_N) &\equiv \left[\sum_{N=\sum_n n j_n}\left(\prod_n\left[\frac{1}{j_n!}\prod_{q_n=1}^{j_n}\frac{C_n}{n!}\right]\right)\right]^{symmetric}_{(y_1,..,y_N)} \\ \Rightarrow |\phi\rangle &= \sum_N \frac{1}{N!}\int d^N y\,\phi_N(\vec{y}_N)\prod_{m=1}^{N}\psi^\dagger(y_m)\,|0\rangle\end{aligned} \tag{5.30}$$

The function $\phi_N(\vec{y}_N)$ is clearly symmetric.

Here we have an implicit relation between ϕ_N and C_n, but it is easy to notice that

$$\phi_N = C_N + F(C_1, C_2, ..C_{N-1}) \tag{5.31}$$

with F a function of C_1, C_2 and so on, in particular F is a sum of products of C_n $n < N$. If we know ϕ_N we can use this relation as iterative definition for C_N

$$C_N = \phi_N - F(C_1, C_2, ..C_{N-1}) \tag{5.32}$$

so we conclude that each state can be put in the exponential form.

It is quite easy to convince ourselves that this recursive relation matches with the one for the wavefunction for the cluster states (5.19) and the requirement to impose to have cluster states is that the $C_n(x_1, x_2, ..x_n)$ must decay to zero when a variable is sent to infinity.

5.1.2 Cluster states and cluster property

As the name suggests, the cluster states satisfy (under some hypothesis of regularity) the cluster property. In this section we will show that the cluster states satisfy MCP and this implies, as we have already said, that they also satisfy the cluster property.

We start studying one point averages, that is to say $\langle[\psi^\dagger(x)]^a[\psi(x)]^b\rangle$. We use the notation of Wick contractions and the exponential form of the states, for simplicity define the following operators:

$$\begin{aligned}&\hat{C}_n^\dagger \equiv \int d^n x \frac{C_n(\vec{x}_n)}{n!}\prod_j \psi^\dagger(x_j); \qquad \hat{C}_n \equiv \int d^n x \frac{C_n^*(\vec{x}_n)}{n!}\prod_j \psi(x_j) \\ &\Rightarrow exp\left(\sum_n\left[\int d^n x\,\frac{C_n(x_1,..x_n)}{n!}\prod_{j=1}^{n}\psi^\dagger(x_j)\right]\right) = e^{\sum_n \hat{C}_n^\dagger}\end{aligned} \tag{5.33}$$

We can start to consider the effect of contractions on the exponential operator $e^{\sum_n \hat{C}_n^\dagger}$, let Θ be a generic operator:

$$\Theta\underbrace{[e^{\hat{C}_n^\dagger}]} = \Theta\underbrace{\sum_{q=0}\frac{1}{q!}[\hat{C}_n^\dagger]^q} = \Theta\underbrace{\hat{C}_n^\dagger}\sum_{q=1}\frac{1}{(q-1)!}[\hat{C}_m^\dagger]^{q-1} = \Theta\underbrace{\hat{C}_n^\dagger e^{\hat{C}_n^\dagger}} \tag{5.34}$$

We see that the contraction recasts the exponential operator. In the same way we can study the effect of two contractions:

$$\Theta[e^{\hat{C}_n^\dagger}] = \left(\Theta\hat{C}_n^\dagger + \Theta\hat{C}_n^\dagger\hat{C}_n^\dagger\right) e^{\hat{C}_n^\dagger} \tag{5.35}$$

Higher order contractions behave in the same way: the contractions of an operator with $e^{\hat{C}_n^\dagger}$ give another operator times $e^{\hat{C}_n^\dagger}$. A trivial generalization show the same behavior for $e^{\sum_n \hat{C}_n^\dagger}$.

With this observation we can study this kind of averages:

$$\frac{1}{a!b!}\frac{\langle 0|\, e^{\sum_n \hat{C}_n}(\psi^\dagger(x))^a(\psi(x))^b e^{\sum_n \hat{C}_n^\dagger}\,|0\rangle}{\langle 0|\, e^{\sum_n \hat{C}_n} e^{\sum_n \hat{C}_n^\dagger}\,|0\rangle} \tag{5.36}$$

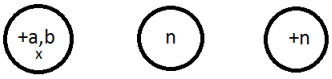
The factors $a!$ and $b!$ are inserted for later convenience. Performing this average is quite complicated and for this purpose we will construct a graphical representation.

Graphical representation

We are looking for a convenient graphical machinery to handle the following expression:

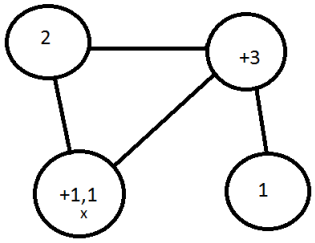
$$\frac{1}{a!b!}\,\langle 0|\, e^{\sum_n \hat{C}_n}(\psi^\dagger(x))^a(\psi(x))^b e^{\sum_n \hat{C}_n^\dagger}\,|0\rangle \tag{5.37}$$

We represent the operators $(\psi^\dagger(x))^a(\psi(x))^b$, $\hat{C}_n$ and $\hat{C}_n^\dagger$ with circles

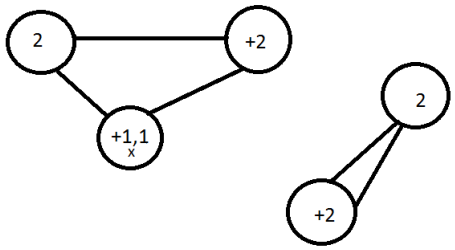


We represent the Wick's contractions as lines connecting different circles: observe that $\hat{C}_n^\dagger$ can be joined only with $\hat{C}_m$ or $(\psi^\dagger(x))^a(\psi(x))^b$. In order to perform all the possible contractions, from $(\psi^\dagger(x))^a(\psi(x))^b$ must depart "a" lines toward $\hat{C}_n$ and "b" lines toward $\hat{C}_n^\dagger$, from each $\hat{C}_n$ must depart "n" lines.

For example typical graphs could be the following:



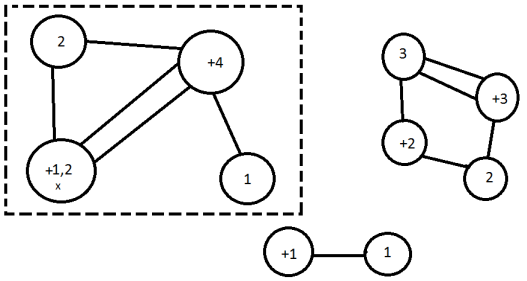
$$\rightarrow \int dydz\; C_2^*(x,y)C_3(x,y,z)C_1^*(z)$$



$$\rightarrow \frac{1}{2}\int dzdy\, C_2^*(x,y)C_2(y,x)C_2^*(z,z)C_2(z,z)$$

In this second graph a symmetry factor arises, the recipe to take care of these symmetry factors directly from the graphical machinery is quite complicated and we do not describe it because we do not need to perform any explicit calculation with these graphs.

Consider now a general graph, inside it we can recognize a part connected to $(\psi^\dagger(x))^a(\psi(x))^b$: we will call it the CO ("Connected to the Operator") part, in the next picture the CO part is contained in the dotted line



It is quite easy to see that the part disconnected from the operator resum to $\langle\phi|\phi\rangle$, this because the symmetry factors due to CO are factorized from the others.

For this reason we can directly evaluate the average as the summation over all the possible CO contributions:

$$\frac{1}{a!b!}\frac{\langle 0|\, e^{\sum_n \hat{C}_n}(\psi^\dagger(x))^a(\psi(x))^b e^{\sum_n \hat{C}_n^+}\,|0\rangle}{\langle\phi|\phi\rangle} = \sum_{CO}[CO] \tag{5.38}$$

Two points averages

The calculation for two (or more) points averages is quite the same, we will show the MCP in the restricted case of two distant points, but the general proof will appear as a trivial generalization. For this reason we will consider the following average:

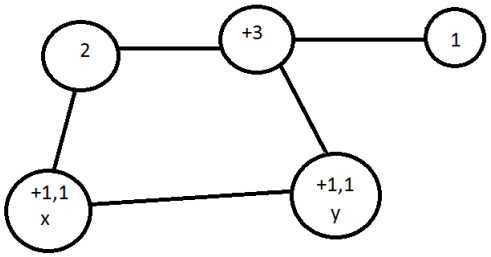
$$\frac{1}{a!b!c!d!}\,\langle(\psi^\dagger(x))^a(\psi(x))^b(\psi^\dagger(y))^c(\psi(y))^d\rangle \tag{5.39}$$

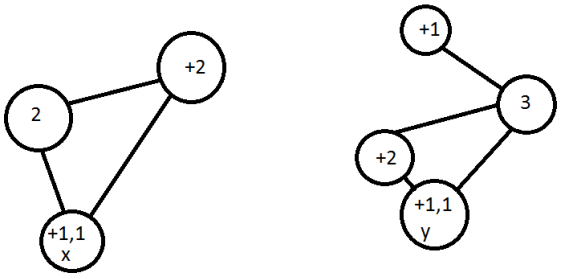
We can generalize the previous graphical machinery to this case: we represent $(\psi^\dagger(x))^a(\psi(x))^b$ and $(\psi^\dagger(y))^c(\psi(y))^d$ as two circles

A part from the slight modification of considering "x" and "y" labels, the notation remains the same. Also in this case we have the cancellation of $\langle\phi|\phi\rangle$ as in the one point case, so we can consider only the CO graphs:

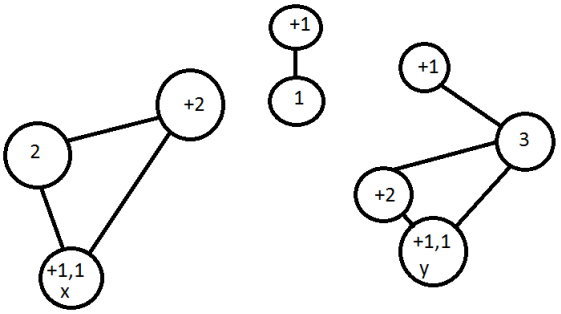
$$\frac{1}{a!b!c!d!}\langle(\psi^\dagger(x))^a(\psi(x))^b(\psi^\dagger(y))^c(\psi(y))^d\rangle = \sum_{CO}[CO] \tag{5.40}$$

where the "CO" graphs are such that each $\hat{C}_n^\dagger$ and $\hat{C}_n$ is linked with $(\psi^\dagger(x))^a(\psi(x))^b$ and(or) $(\psi^\dagger(y))^c(\psi(y))^d$. For example the following two graphs are CO:





Instead the following graph is not CO:



It is useful to define the connected and disconnected CO graphs: we will call a graph disconnected if $(\psi^\dagger(x))^a(\psi(x))^b$ and $(\psi^\dagger(y))^c(\psi(y))^d$ are not linked each others.

With these notations we have the tools to study the MCP on these states, notice that we have the following relation:

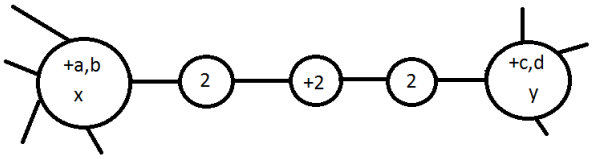
$$\frac{1}{a!b!c!d!}\langle(\psi^\dagger(x))^a(\psi(x))^b\rangle\langle(\psi^\dagger(y))^c(\psi(y))^d\rangle = \sum_{disconnected\,graphs}[disconnected\,graphs] \tag{5.41}$$

For this reason to show that the MCP holds we must verify that:

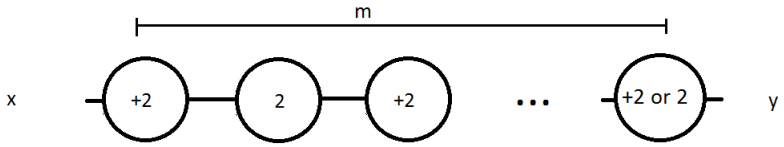
$$\lim_{|x-y|\to\infty}\sum_{connected\,graphs}[connected\,graphs] = 0 \tag{5.42}$$

We will study a simple case, after it will be clear which are the basic ideas and that their implementation is only a matter of notation: suppose the presence only of $\hat{C}_n$ with $n = 1, 2$, instead $\hat{C}_n = 0\,\forall n > 2$. We will suppose that $C_2(x_1, x_2)$ is non zero only in a finite range, that is to say there exists $\xi > 0$ such that $|x_1 - x_2| > \xi \Rightarrow C_n(x_1, x_2) = 0$.

Consider a generic connected graph: to connect the two points x, y we are forced to consider this situation (we draw only the part of the graph that matters)



It is useful to define a "chain" of m elements as the following graph:



We will call $\mathcal{C}(x, y, m)$ the contribution of a chain of m elements when its extrema are contracted with creation-annihilation operators in x and y:

$$\mathcal{C}(x, y, m) = \int d^m z\; C_2(x, z_1)C_2^*(z_1, z_2)C_2(z_2, z_3)... \tag{5.43}$$

The two points x and y, if connected, must be linked with chains. Observe that a chain has a finite range, that is to say a chain of m elements can connect points such that $|x-y| < m\xi$, this implies that if the two points are sent to infinity we need chains of increasing length to connect them. If the summation over graphs makes

sense, we expect that $\sum_{m=0}^{\infty} \mathcal{C}(x,y,m)$ converges. In particular we will suppose that, uniformly over the values of x, y:

$$\lim_{k\to\infty} \sum_{m=k}^{\infty} \mathcal{C}(x,y,m) = 0 \tag{5.44}$$

This is true for a large class of functions, for example we can imagine $C_2(x,y) = \alpha\varphi(x-y)\Theta(\xi - |x-y|)$ where Θ is the Heaviside function, φ is a bounded function $|\varphi| < 1$ and $0 < \alpha < 1$.

If this type of convergence is true we can proceed as follows:

$$\begin{aligned} \left[\sum_{m=0}^{\infty} \mathcal{C}(x,y,m)\right]_{|x-y|>D} &= \sum_{m>\frac{D}{\xi}} \mathcal{C}(x,y,m) \\ \Rightarrow \lim_{D\to\infty} \left[\sum_{m=0}^{\infty} \mathcal{C}(x,y,m)\right]_{|x-y|>D} &= \lim_{D\to\infty} \left[\sum_{m>\frac{D}{\xi}} \mathcal{C}(x,y,m)\right]_{|x-y|>D} = 0 \end{aligned} \tag{5.45}$$

So the connected graphs will disappear in the $|x-y| \to \infty$ limit. On the contrary, if C_2 does not go to zero when a variable is sent to infinity we would have not any problem in connecting the two points and so we would expect the MCP not to be true.

The basic ingredients of this proof are some convergences properties of the summation of graphs and the finite range of the functions C_2. The general case can be studied in the same way, supposing that each C_n is non zero only over a typical length ξ and that we have a uniform convergence of the summations of graphs. The basic ingredients, a part from a cumbersome notation, are already contained in the special case above.

5.1.3 Cluster states in momentum representation

For successive use of cluster states, it will be useful studying them in the momentum representation

$$\begin{aligned} a^\dagger(p) = \int \frac{dx}{\sqrt{2\pi}} e^{-ipx}\psi^\dagger(x), \qquad \psi^\dagger(x) = \int \frac{dp}{\sqrt{2\pi}} e^{ipx} a^\dagger(p) \\ \Rightarrow [a(p), a(q)] = 0, \qquad [a(p), a^\dagger(q)] = \delta(p-q) \end{aligned} \tag{5.46}$$

Each $\hat{C}_n^\dagger$ operator can be expressed in the momenta space as follows:

$$\begin{aligned}\hat{C}_n^\dagger &= \int d^n x \frac{C_n(\vec{x}_n)}{n!} \prod_j \psi^\dagger(x_j) = \int d^n x \frac{C_n(\vec{x}_n)}{n!} \prod_j \left[\int \frac{dp_j}{\sqrt{2\pi}} e^{ip_j x_j} a^\dagger(p_j) \right] = \\ &= \int d^n p \frac{1}{n!} \left(\int \frac{d^n x}{(2\pi)^{\frac{n}{2}}} e^{i\sum_j p_j x_j} C_n(\vec{x}_n) \right) \prod_j a^\dagger(p_j)\end{aligned} \tag{5.47}$$

We define the following quantity:

$$K_n(\vec{p}_n) \equiv \int \frac{d^n x}{(2\pi)^{\frac{n}{2}}} e^{i\sum_j p_j x_j} C_n(\vec{x}_n) \tag{5.48}$$

In this way we arrive at the following expression:

$$\begin{aligned}&\hat{C}_n^\dagger = \int d^n k \frac{K_n(\vec{k}_n)}{n!} \prod_j a^\dagger(k_j) \\ &\Rightarrow |\phi\rangle = e^{\sum_n \int d^n x \frac{C_n(\vec{x}_n)}{n!} \prod_j \psi^\dagger(x_j)} |0\rangle = e^{\sum_n \int d^n p \frac{K_n(\vec{p}_n)}{n!} \prod_j a^\dagger(p_j)} |0\rangle\end{aligned} \tag{5.49}$$

here the K_n functions are clearly completely symmetric in their arguments.

We could have also proceeded in an other way:

$$\begin{aligned}|\phi\rangle &= \sum_n \frac{1}{n!} \int d^n x \phi_n(\vec{x}_n) \prod_j \psi^\dagger(x_j) |0\rangle = \\ &= \sum_n \frac{1}{n!} \int d^n x \phi_n(\vec{x}_n) \prod_j \left[\int \frac{dp_j}{\sqrt{2\pi}} e^{ip_j x_j} a^\dagger(p_j) \right] |0\rangle = \\ &= \sum_n \frac{1}{n!} \int d^n p \tilde{\phi}_n(\vec{p}_n) \prod_j a^\dagger(p_j) |0\rangle\end{aligned} \tag{5.50}$$

$$\tilde{\phi}_n(\vec{p}_n) \equiv \int \frac{d^n x}{\left(\sqrt{2\pi}\right)^n} e^{i\sum_j p_j x_j} \phi_n(\vec{x}_n)$$

As we are able to extract the C_n from ϕ_n, we would like to extract the K_n from $\tilde{\phi}_n$ without using the Fourier transform. We will restrict ourselves to consider translationally invariant states.

Translationally invariant states

Here we characterize the cluster states directly in the momentum representation under the hypothesis of translational symmetry:

$$\begin{aligned}&\forall \epsilon, \forall n \quad \phi_n(x_1 + \epsilon, ..x_n + \epsilon) = \phi_n(x_1, .., x_n) \\ &\Leftrightarrow C_n(x_1 + \epsilon, ..x_n + \epsilon) = C_n(x_1, .., x_n)\end{aligned} \tag{5.51}$$

If the C_n are translational invariant we expect a term $\delta(\sum_j p_j)$ in the expression for $K_n(\vec{p}_n)$, so we define $\tilde{K}_n$ as:

$$K_n(\vec{p}_n) \equiv \delta\left(\sum_j p_j\right) \tilde{K}_n(\vec{p}_n) \tag{5.52}$$

The function $\tilde{K}_n$ can be explicitly written as:

$$\tilde{K}_n(\vec{p}_n) = \frac{2\pi}{n!} \sum_\sigma \int \frac{d^{n-1}x}{(2\pi)^{\frac{n}{2}}} e^{i\sum_j p_{\sigma(j)} x_j} C_n(0, x_1, .., x_{n-1}) \tag{5.53}$$

where σ are permutations of n indices: with this definition $\tilde{K}_n$ is still symmetric. We have no others "δ" singularity in $\tilde{K}_n$, this because $\lim_{|x_j|\to\infty} C_n(0, x_1, x_2.., x_{n-1}) = 0$.

$$|\phi\rangle = \sum_n \frac{1}{n!} \int d^n p \tilde{\phi}_n(\vec{p}_n) \prod_j a^\dagger(p_j) |0\rangle = e^{\sum_n \int d^n p \frac{\delta\left(\sum_j p_j\right) \tilde{K}_n(\vec{p}_n)}{n!}} \Pi_j a^\dagger(p_j) |0\rangle \tag{5.54}$$

We can proceed as in section 5.1.1 when we expressed the ϕ_n functions in terms of C_n, the passages are identical. We find a cluster structure of the delta singularities of $\tilde{\phi}_n$:

$$\begin{aligned}
&\tilde{\phi}_1(p) = \delta(p)\tilde{K}_1(p) \\
&\tilde{\phi}_2(p_1, p_2) = \delta(p_1)\delta(p_2)\tilde{K}_1(p_1)\tilde{K}_1(p_2) + \delta(p_1 + p_2)\tilde{K}_2(p_1, p_2) \\
&\tilde{\phi}_3(p_1, p_2, p_3) = \delta(p_1)\delta(p_2)\delta(p_3)\tilde{K}_1(p_1)\tilde{K}_1(p_2)\tilde{K}_1(p_3) + \\
&+ \delta(p_1 + p_2)\delta(p_3)\tilde{K}_2(p_1, p_2)\tilde{K}_1(p_3) + \delta(p_1 + p_3)\delta(p_2)\tilde{K}_2(p_1, p_3)\tilde{K}_1(p_2) + \\
&+ \delta(p_3 + p_2)\delta(p_1)\tilde{K}_2(p_3, p_2)\tilde{K}_1(p_1) + \delta(p_1 + p_2 + p_3)\tilde{K}_3(p_1, p_2, p_3)
\end{aligned} \tag{5.55}$$

With this observation we can in principle extract the $\tilde{K}_n$ factors from $\tilde{\phi}_n$ studying the delta structure of this function.

5.2 Time averages of local operators on cluster states

In this section we study the time average of local observables in a free bosonic theory. In this approach we do not use the explicit form of the dispersion relation: this implies that what follows is true for any free bosonic theory and not only if it is relativistically invariant. The considerations that follow seem that could have natural generalization to interacting integrable theories.

Introductory remarks

For later convenience, we need to introduce a finite volume regularization: we require that the field $\psi(x)$ lives on a finite volume of length L with periodic bonduary conditions. This discretizes the momenta admitted.

$$\begin{aligned} a_k^\dagger &= \int_0^L \frac{dx}{\sqrt{L}} e^{-ikx}\psi^\dagger(x), \qquad k = \frac{2\pi}{L}n,\ n \in \mathbb{Z} \\ \psi^\dagger(x) &= \frac{1}{\sqrt{L}}\sum_k e^{ikx}a_k^\dagger \qquad [a(k), a(q)] = 0, \qquad [a(k), a^\dagger(q)] = \delta_{k,q} \end{aligned} \tag{5.56}$$

The cluster states in the momentum representation will now assume this form:

$$\begin{aligned} |\phi\rangle &= e^{\sum_n \hat{C}_n^\dagger}|0\rangle, \qquad \hat{C}_n^\dagger = \frac{1}{n!}\sum_{p_1,..p_n} K_n(\vec{p}_n)\prod_j a_{p_j}^\dagger \\ K_n(\vec{p}_n) &\equiv \int_0^L \frac{d^n x}{L^{\frac{n}{2}}} e^{i\sum_j p_j x_j} C_n(\vec{x}_n) \end{aligned} \tag{5.57}$$

If we suppose that the C_n functions are translationally invariant we can extract a global delta contribution from $K_n(\vec{p}_n)$:

$$K_n(\vec{p}_n) = \delta_{(\sum_j p_j, 0)} \frac{1}{L^{\frac{n}{2}-1}} \tilde{K}_n(\vec{p}_n) \tag{5.58}$$

with $\tilde{K}_n(\vec{p}_n)$ a regular smooth function with no L dependence: in the following we will use states that are invariant under translations. As in 5.1.3 we can expand the exponential state in this form

$$|\phi\rangle = \sum_n \frac{1}{n!}\sum_{p_1,p_2,..p_n} \phi_n(\vec{p}_n)|\vec{p}_n\rangle \tag{5.59}$$

where we defined $|\vec{p}_n\rangle = \prod_j a_{p_j}^\dagger |0\rangle$. The ϕ_n function clusterizes as in section 5.1.3, with the modification of substituting the delta functions with the Kronecker delta normalized over the appropriate power of L.

Time averages

In this section we evaluate the time average of local observables on cluster states:

$$\langle O(x)\rangle_{Time} \equiv \lim_{L\to\infty}\lim_{T\to\infty}\frac{1}{T}\int_0^T dt \frac{\langle\phi| O(x,t)|\phi\rangle}{\langle\phi|\phi\rangle} \tag{5.60}$$

where $O(x)$ is a local operator centered in x and $|\phi\rangle$ is a cluster state:

$$|\phi\rangle = \sum_n \frac{1}{n!}\sum_{p_1,p_2..p_n} \phi_n(\vec{p}_n)|\vec{p}_n\rangle \tag{5.61}$$

To take care of the factors $n!$ it is more convenient to have ordered summations, then the expectation value take the following form:

$$\langle\phi|\, O(x,t)\,|\phi\rangle = \sum_{n,m} \sum_{\vec{p}_n,\vec{k}_n}^{ordered} \phi_n^*(\vec{k}_n)\phi_m(\vec{p}_m)\,\langle\vec{k}_n|\, O(0)\,|\vec{p}_m\rangle\, e^{it(E(\vec{k}_n)-E(\vec{p}_m))} \tag{5.62}$$

where $E(\vec{p}_n) = \sum_j E(p_j)$, the summation is ordered such as $k_1 \geq k_2 \geq ..$ and $p_1 \geq p_2 \geq ...$ We recall from 3.1.3 that for the free bosonic theory we can write:

$$\langle\vec{k}|\, O(0)\,|\vec{p}\rangle = \sum_{\vec{k}=\vec{k}_1\bigcup\vec{k}_2} \sum_{\vec{p}=\vec{p}_1\bigcup\vec{p}_2} \frac{\langle\vec{k}_2|\, O(0)\,|\vec{p}_2\rangle_{conn}}{\sqrt{\prod_{k_j\in\vec{k}_2}[L2\pi\rho_t(k_j)]\prod_{p_j\in\vec{p}_2}[L2\pi\rho_t(p_j)]}}\delta_{\vec{k}_1,\vec{p}_1} \tag{5.63}$$

Notice that if we choose the momenta as the quantum numbers to represent the states we have $\rho_t(k) = \frac{1}{2\pi}$.

If we are interested in time averages we can restrict ourselves to evaluate only the diagonal ensemble:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \langle\phi|\, O(x,t)\,|\phi\rangle = \sum_n \sum_{\vec{p}_n}^{ordered} \sum_{\vec{p}_n=\vec{p}_1\bigcup\vec{p}_2} |\phi_n(\vec{p}_n)|^2 L^{-n_2}\,\langle\vec{p}_2|\, O(0)\,|\vec{p}_2\rangle_{conn} \tag{5.64}$$

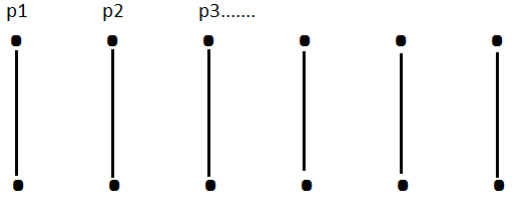
where n_2 is the number of momenta in $\vec{p}_2$.

The next step is finding a convenient graphical way to handle this expression: we will use the fact that the ϕ_n function clusterizes.

Graphical representation for time averages

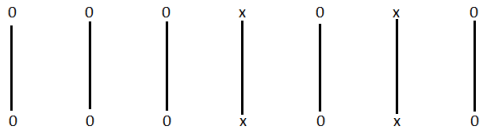
The graphical machinery we are going to present is mostly inspired by the one used for squeezed states.

We represent the momenta as an ordered set of points: suppose $p_1 \geq p_2 \geq ..$, then we draw two rows of points; the upper one represents the momenta in the bra, the lower one the momenta in the ket

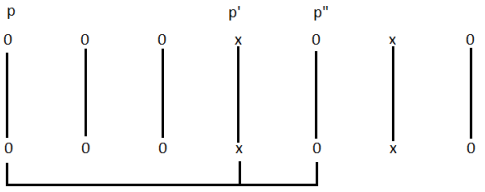


We join the momenta in the upper and lower row to remind us they are the same.

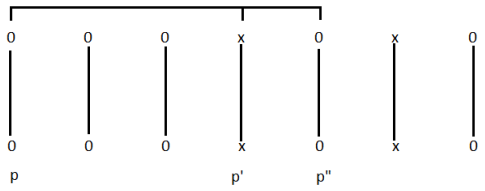
As second step, we must distinguish which momenta are in $\vec{p}_1$ and which are in $\vec{p}_2$: instead of points we will draw a "X" if the momentum is in $\vec{p}_1$, a "0" if it is in $\vec{p}_2$.



Now we need to represent the wavefunctions: we know they can be decomposed in terms of $K_n(\vec{p}_n)$, so we will connect externally the momenta that enter in each K_n. For example consider the connection in the following graph:



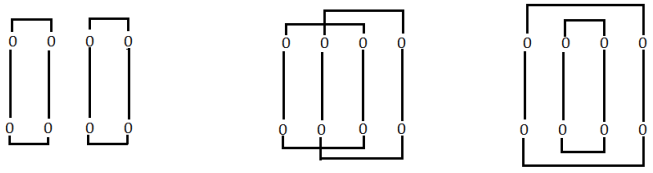
This connection represents $K_3(p, p', p")$, instead the connections in the upper row will represent terms of ϕ_n^*:



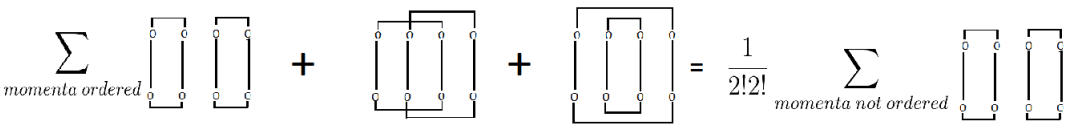
This graph will mean $K_3^*(p, p', p")$. Each factor "X" means a momentum in the connected average and a factor $\frac{1}{L}$: with these rules we can represent each factor in the summation.

Performing the summation over all the momenta is equivalent to sum over all the ways of connecting the momenta and then sum over all the ordered momenta.

The next step will be to combine the summation over the graphs with the summation over the momenta and express the result as a new summation in which we have removed the order constraint over the set of the momenta. We show an example to make things easier, so consider the following graphs:



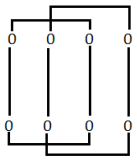
Summing these graphs and then summing over all the possible momenta (but still ordered) is equivalent to consider only the first graph and sum over all the possible momenta without any order, then renormalize the contribution with a symmetry factor equal to the number of permutations of the momenta that leave the graph the same. Pictorially:



From now on we will sum in this way: draw a graph, sum its momenta in all the possible ways without any order in the momenta and divide the contribution for the number of permutations that leave the graph invariant. Then sum over all the possible inequivalent graphs: two graphs are inequivalent if we cannot pass from one to the other simply permuting the momenta. Now we have removed the "ordering constraint" and we can study the graphs in an easier way.

We will say that two momenta are connected if we can pass from one to the other along the links of the graph. It is obvious that if p is connected to p' and p' is connected to p", then p is connected to p": we can define a connected block as the maximum set of momenta such that all the momenta are connected to each other.

It is obvious that each graph is made of many connected blocks, for example the following one is made of two connected blocks:



This definition is useful because in the summation, a part from the symmetry factor, the contributions of the connected blocks are factorized.

Some connected blocks contain "X" symbols: we will call them Blocks Connected to the Operator (BCO). Instead the blocks that do not contain any "X" are called "norm blocks"(NB).

The symmetry factor of any graph factorizes in the symmetry factor regarding the NB and the symmetry factor regarding the BCO. This simply because we cannot pass from a NB to a BCO only with permutations of the momenta. For this reason the entire summation factorizes:

$$\sum_{momenta} [graph] = \left(\sum_{momenta} [BCO] \right) \left(\sum_{momenta} [NB] \right)$$

The same graphical machinery can be constructed to evaluate $\langle\phi|\phi\rangle$: we would have obtained trivial graphs made only of NB.

For this reason the following passages hold:

$$\langle O(x)\rangle_{Time} = \frac{\left(\sum_{momenta}[BCO]\right)\left(\sum_{momenta}[NB]\right)}{\sum_{momenta}[NB]} = \sum_{momenta}[BCO] \tag{5.65}$$

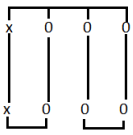
With these passages we reduce ourselves to consider only a restricted class of blocks.

At this point it is useful to write clearly the graphical rules to evaluate $\langle O(x)\rangle_{Time}$:

- Draw graphs that contain only BCO.
- Each connection in the block and the "X" symbol count as we already said, then in each block sum over all the possible momenta. Divide the contribution of the graphs for its symmetry factor and sum all the contributions of inequivalent graphs.

Since now we have not used any information over the $\tilde{K}_n$ functions (a part from the fact that they are symmetric in their variables).

Now we will try to implement our informations on the delta structure of $\tilde{K}_n$ functions and the $L \to \infty$ limit. We start from an example: consider the following graph



Its contribution will be:

$$\begin{aligned}&\frac{1}{2}\sum_{p_1,p_2,p_3,p_4}\frac{1}{L}\langle p_1|\,O\,|p_1\rangle_{conn} K_4^*(p_1,p_2,p_3,p_4)K_2(p_1,p_2)K_2(p_3,p_4) = \\ &= \frac{1}{2}\sum_{p_1,p_4}\frac{1}{L^2}\langle p_1|\,O\,|p_1\rangle_{conn}\tilde{K}_4^*(p_1,-p_1,-p_4,p_4)\tilde{K}_2(p_1,-p_1)\tilde{K}_2(-p_3,p_4)\end{aligned} \tag{5.66}$$

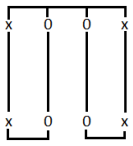
If we consider the $L \to \infty$ limit we can switch from summations to integrals with the recipe:

$$\sum_p \to \int L\rho_t(p)dp = \int \frac{L}{2\pi}dp \tag{5.67}$$

So we obtain the following expression:

$$= \frac{1}{2}\int dp_1 dp_4 \frac{1}{(2\pi)^2} \langle p_1| O |p_1\rangle_{conn} \tilde{K}_4^*(p_1, -p_1, -p_4, p_4)\tilde{K}_2(p_1, -p_1)\tilde{K}_2(-p_4, p_4) \tag{5.68}$$

We obtained a finite contribution in the $L \to \infty$ limit.
It will be instructive to consider this second graph:



With the same passages we obtain the following contribution:

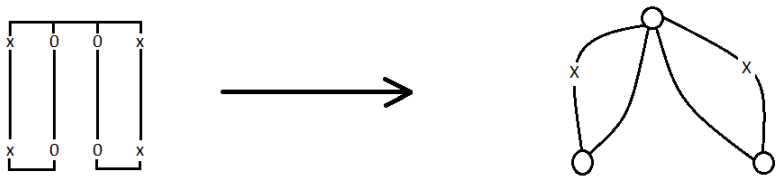
$$\frac{1}{2L}\int dp_1 dp_4 \frac{1}{(2\pi)^2} \langle p_4, p_1| O |p_1, p_4\rangle_{conn} \tilde{K}_4^*(p_1, -p_1, -p_4, p_4)\tilde{K}_2(p_1, -p_1)\tilde{K}_2(-p_4, p_4) \tag{5.69}$$

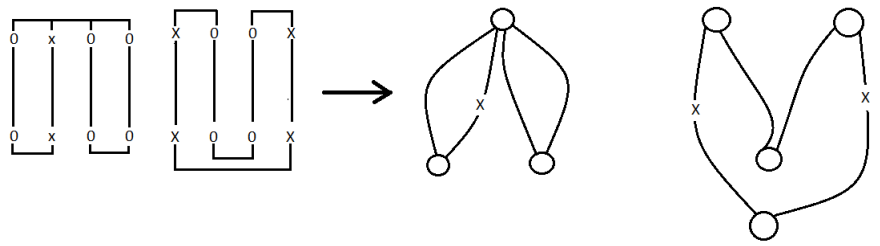
This term will be zero in the $L \to \infty$ limit. It will be very useful if we can classify the graphs considering the L powers in their final contribution.

In general we have that the L powers of each block come from two contributions: the L powers of the terms represented by the blocks and the contribution of the summation over each degree of freedom of the graph, that is to say the number of momenta that are not fixed by the delta structure inside the summation.

The graphical representation we used since now is not suitable for an "L" counting, for this reason we define the dual graph in this way:

- A dual graph is divided into blocks: each block is the dual counterpart of a direct connected block.
- In a dual connected block the connections of the momenta are replaced by the new vertices. We can proceed in this way: put a circle over each link that joins different momenta and eliminate the 0 and X symbols continuing the lines. Put an "X" over the links that come from "X" momenta.





It is quite easy to show that the L powers dependence of the graph can be seen in the dual blocks in this way: each continuous line without "X" in the dual block gives a contribution $\frac{1}{L}$, the continuous line with "X" gives $\frac{1}{L^2}$ contributions. Each vertex gives an L factor.

We have now the problem of counting the degrees of freedom of the graphs. In the connected dual blocks we can identify loops determined by the continuous lines: it is a trivial observation that the number of degrees of freedom of the block is equal to the number of its loops. Indicate with V the number of vertices in a connected block, C is the number of links, D is the number of loops and finally $|X|$ is the number of "X" we draw in the block. The L dependence of the contribution of such a block is simply $\sim L^{D+V-C-|X|}$.

Consider a general graph made of some connected blocks, the L's powers contribution will be simply the sum of the power contributions of each block.

$$[contribution\, of\, the\, graph] \to L^{\sum_j (D_j+V_j-C_j-|X_j|)} \tag{5.70}$$

Understanding the degree of divergence of the graphs is now a topological problem.

It is quite easy to prove that, inside a connected block, we have the topological conservation law $D + V - C = 1$.

With this observation we arrive at the following relation for the "L" counting:

$$[contribution\, of\, the\, graph] \to L^{\sum_j (1-|X_j|)} \tag{5.71}$$

Remember that all the blocks we are considering are blocks connected to the operator: so we have the constraint $|X_j| \geq 1$. With this last observation we arrive at the following important result: there are no divergences in the infinite length limit, besides only the graphs formed by BCO with only one "X" symbol give non zero contribution.

Using the fact that we are interested only in the $L \to \infty$ limit, we can restrict ourselves to calculate only the connected blocks with only a "X". We can proceed further rearranging the summation over graphs in a meaningful way.

Consider at first the summation over all the graphs made of only one connected block. The overall contribution, with the symmetry factors, can be recast in the

following way:

$$\sum_{graphs\ only\ one\ block} [graph] = \int \frac{dp}{2\pi} \langle p| O |p\rangle_{conn} h(p) \tag{5.72}$$

where this expression define the function $h(p)$. Consider now the contribution from graphs made of n BCO. It is very easy to convince ourselves than the weight function is simply a product of h functions, one for each variable, a part from a symmetry factor:

$$\sum_{graphs\ only\ n\ blocks} [graph] = \frac{1}{n!} \int \frac{d^n p}{(2\pi)^n} \langle \vec{p}_n| O |\vec{p}_n\rangle_{conn} \prod_j h(p_j) \tag{5.73}$$

With this observation we arrive at the following final conclusion:

$$\begin{aligned} &\lim_{t\to\infty} \frac{\langle\phi| O(x,t) |\phi\rangle}{\langle\phi|\phi\rangle} = \sum_{momenta} [BCO] = \sum_n \sum_{graphs\ with\ n\ blocks} [graph] = \\ &= \sum_n \frac{1}{n!} \int \frac{d^n p}{(2\pi)^n} \langle \vec{p}_n| O |\vec{p}_n\rangle_{conn} \prod_j h(p_j) \end{aligned} \tag{5.74}$$

$$\lim_{t\to\infty} \frac{\langle\phi| O(x,t) |\phi\rangle}{\langle\phi|\phi\rangle} = \sum_n \frac{1}{n!} \int \frac{d^n p}{(2\pi)^n} \langle \vec{p}_n| O |\vec{p}_n\rangle_{conn} \prod_j h(p_j) \tag{5.75}$$

We can calculate the long time limit in the infinite length limit evaluating the function $h(p)$. If the form of the state is given, we can perform this task perturbatively on the $\tilde{K}_n$ functions: evaluate the defining expression for $h(p)$ and truncate the summation over the graphs to some order.

Instead of this approach, we can arrive at an interesting non-perturbative result. Define $n(k) = \lim_{L\to\infty} \frac{2\pi}{L} \frac{\langle\phi|a^\dagger(k)a(k)|\phi\rangle}{\langle\phi|\phi\rangle}$, we can evaluate this quantity simply replacing $\langle \vec{p}_n| O |\vec{p}_n\rangle_{conn} \to \delta_{n,1}\delta_{k,p}\frac{2\pi}{L}$ before exchanging summations with integrals. With identical counting of L divergences we will end simply with:

$$n(p) = h(p) \tag{5.76}$$

So we arrive at this final expression

$$\langle O(x)\rangle_{Time} = \sum_n \frac{1}{n!} \int \frac{d^n p}{(2\pi)^n} \langle \vec{p}_n| O |\vec{p}_n\rangle_{conn} \prod_j n(p_j) \tag{5.77}$$

here $n(p)$ is the density of momenta in p: to perform time averages we need only this quantity.

If we compare this expression wit Le Clair-Mussardo's conjecture we have just showed that the time average of local observables evaluated over cluster states is described by the GGE.

This was already known because of the results of [28] joint with the fact that cluster states satisfy the cluster property, but this approach could be hopefully generalized to interacting theories.

5.3 Conclusions to Chapter 5 and ideas for future studies

In this section we want to summarize the main results acquired in this chapter and propose some ideas for further investigation.

- Guided by the results of [28], we constructed in section 5.1 a special class of states that we call cluster states.
- Even if we already know that the time average of local operators on these states is described by the GGE by construction, we give a different proof of this fact in section 5.2. The passages of this section do not use the explicit dispersion relation of the theory, so they hold for all free bosonic theories.

The content of section 5.2 is quite suitable to be generalized to interacting integrable theories, we summarize the basic hypothesis that bring to GGE in this insection:

- The wavefunction in the momentum space clusterizes defining a precise delta structure.
- The state can be written in an exponential form: this is important because if we contract an exponential state with an operator we have again an exponential state. This is the basic motivation that, joined with the delta structure of the wavefunction, causes the weight function of the average to factorize over the different variables in the infinite length limit.
- The averages of local operators over two states constructed with the creation-annihilation operators in the momentum space can be decomposed in a sum of connected averages: this leads to the elimination of divergences due to the norm of the state.

Using the information of the free bosons case as a guideline we could try to construct an analogue of cluster states in a general integrable theory. For free bosons a cluster state in the Bose case can be written as:

$$|\phi\rangle = \prod_n e^{\hat{C}_n^\dagger} |0\rangle\,; \quad \hat{C}_n^\dagger = \int d^n k \frac{1}{n!} K_n(\vec{k}_n) \prod_j a^\dagger(k_j); \quad K_n(\vec{p}_n) = \delta\left(\sum_j p_j\right) \tilde{K}_n(\vec{p}_n) \tag{5.78}$$

The first trial could be substituting the $a^\dagger$ operators with the Zamolodchikov-Faddeev operators and $\tilde{K}_n$ with a function that follows these commutation rules, that is to say we could consider states of this form:

$$|\phi\rangle = \prod_n e^{\hat{C}_n^\dagger} |0\rangle\,; \quad \hat{C}_n^\dagger = \int d^n k \frac{1}{n!} K_n(\vec{k}_n) \prod_j Z^+(k_j); \quad K_n(\vec{p}_n) = \delta\left(\sum_j p_j\right) \tilde{K}_n(\vec{p}_n) \tag{5.79}$$

$$\begin{aligned} &Z(p)Z(q) = S(p,q)Z(q)Z(p), \qquad Z(p)Z^+(q) = S(q,p)Z^+(q)Z(p) + \delta(p-q) \\ &\tilde{K}_n(p_1, ..p_j, p_{j+1}...p_n) = S(p_{j+1}, p_j)\tilde{K}_n(p_1, ..p_{j+1}, p_j...p_n) \end{aligned} \tag{5.80}$$

Even if this expression satisfies the first requirement, we have a problem with the second: if we want that the contraction of a local operators with a cluster state is again in the form "local operator multiplied cluster state", we have a problem with the non trivial commutative relations of the $\hat{C}_n^\dagger$ operators. This problem is avoided if $[\hat{C}_n^\dagger, \hat{C}_m^\dagger] = 0\ \forall n, m$, so we impose this requirement: this is a constraint over the $\tilde{K}_n$ functions and depends on the specific form of the S matrix.

$$[\hat{C}_n^\dagger, \hat{C}_m^\dagger] = 0 \qquad \forall n, m \tag{5.81}$$

This new definition does not restrict the previous one in the free-bosons case ($S(p,q) = 1$ and $[\hat{C}_n^\dagger, \hat{C}_m^\dagger] = 0 \quad \forall n, m$). In the free fermion case we have $S(p,q) = -1$: if we impose $\tilde{K}_{2n+1} = 0\ \forall n \in \mathbb{N}$ we can satisfy the requirement $[\hat{C}_n^\dagger, \hat{C}_m^\dagger] = 0$ $\forall n, m$, so we are still considering a quite general case.

If we consider an interacting theory with a non trivial S matrix that is galilean or relativistic invariant, then the squeezed states (that is to say $\tilde{K}_n \neq 0$ only if $n = 2$) will satisfy the requirement.

We hope that this approach with this kind of generalization could give some insight in an interacting integrable theory providing a larger class of states than the squeezed states that exploits GGE features on local observables.

Conclusions

This last section is a small overview of this thesis and in particular takes the results from Chapters 4 and 5.

In this work we considered the typical out of equilibrium problem of quantum quenches and we focused on a precise class of quantum theories, known as integrable models.

Integrable models are an interesting framework to study equilibration in quantum systems for many reasons: they are many body systems whose dynamic, at least in principle, is solvable and so we have the possibility of computing the entire time evolution of observables.

Although the integrable models have some nice properties, the presence of an infinite set of local conserved charges means an increased complexity of the stationary density matrix: the usual Gibbs Ensemble does not work and we must introduce the Generalized Gibbs Ensemble.

The notion of local operator permits us to decouple the system in two subsystems in the coordinate space: the first one is probed through measures of local observables while the second acts as a thermal bath for the previous one. This permits to reach local equilibrium.

Even if we introduced the notion of locality in the coordinate space, we shifted to another representation of local operators: in Chapter 3 we introduced the decomposition of local observables in their connected parts which usually have negligible contributions when we increase the number of particles. This has the following suggestive interpretation: if multiparticles states are excited, then the local averages depend directly from few particles through the connected averages while the other particles have the role of thermal bath. All of these considerations support the original contents of Chapter 4 and Chapter 5.

In Chapter 4 we studied the evolution of local observables supposing the quench protocol initializes the system in a particular class of states, called squeezed states: we constructed a new approach that permits us to show the equilibration to a GGE. Even if this result was already known from other works, the same approach provides the entire time evolution in the simpler case of free integrable theories. We believe that the same method could be also generalized to the interacting case and we will investigate it in future works.

Chapter 5 has another objective: squeezed states are a natural product of many quench protocols, but we could imagine more complicated operations that initialize the systems in other states. In particular, in the case of interacting theories is not

easy to understand which is the initial state given by a quench protocol, even in simple cases. These facts stimulate the interest in finding more general initial states that make local observables to relax to a GGE: in the framework of a free bosonic theory we found a large class of initial states that provide relaxation to GGE. The method we used in this chapter uses the typical formalism of integrable theories and we aim also to generalize it to interacting theories. With this method we will look for states different than squeezed states that lead to GGE: this will be done in future investigations.

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