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Quantum Quench in one-dimensional gases

Candidato:
Paolo Pietro Mazza

Relatore:
Prof. Pasquale Calabrese

Correlatore:
Dott. Mario Collura

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PREFACE

In this work we discuss the relaxation properties of a quantum one dimensional gas. The interest in studying this kind of systems has recently grown due to the development of innovative experimental techniques that made possible the confinement of particles in quasi one-dimensional optical lattices weakly interacting with the environment. These properties can be explored using a quantum quench: we prepare the system in the ground state of a given hamiltonian, then we suddenly change a parameter and let it evolve unitarly. There are mainly two types of quantum quenches: local and global. We focus our attention on a global quantum quench and we see how the relaxation occurs as a many body effect.

The structure of the thesis is the following.

In the first chapter of this thesis there is a brief overview about the recent theoretical developments about thermalization of isolated quantum systems. In particular the attention is focused on the role of the spatial dimensionality and on the conservation laws. In fact it is widely believed that one dimensional integrable systems relax toward a non-thermal distribution that, in a certain sense, retains "more memory" of the initial conditions than a thermal one.

In the second chapter some notions about quantum integrability are given. We review the concept of classical integrability and we illustrate some criteria for the quantum case. This is useful in the classification of the models between integrable and non-integrable ones.

In the third chapter the model of interest is deeply analyzed. We deal with a many body hamiltonian with point-like interactions. This model, for general values of the coupling constant, is solvable via Bethe Ansatz techniques. However we are interested in two limiting cases: free bosons (no interaction) and Tonks-Girardeau limit, which is often referred as "hard-core" (HC) bosons gas. We show that the latter model is exactly mappable to a spinless free fermionic gas using a Jordan Wigner transformation. Furthermore we show some results from a problem in which the interaction parameter is quenched between $c = 0$ and $c = \infty$ (this corresponds to a quench from Bose-Einstein condensate to HC bosons), with periodic boundary conditions.

The fourth chapter is the core of this work, indeed it is the original contribution

to this thesis. We consider a global quantum quench in a *confined* one dimensional bosonic gas. This configuration is very interesting from an experimental point of view. We show that relaxation occurs in slightly different manner than in the periodic case: the stationary correlation function "feels" the boundaries also in the thermodynamic limit. Furthermore we find a compact expression for the time dependent density profile and for the fermionic correlation function. Both functions describe the non equilibrium behavior of the system. The solutions of the confined problem present difficulties which were absent in the periodic case. These have been overcome by some ingenious approximations which become exact in the thermodynamical limit, thus providing the analytical solution to the problem. In the course of the computation numerical analysis is often used as a support.
We found out that the long-time state of the confined system is translationally invariant (we demonstrated that non translationally invariant corrections are finite-size effects), in particular the stationary density profile is the same as in the homogeneous case, as naively expected. But the effects of the confinement are visible both in the stationary two point correlation function and in the non trivial evolution of the density profile.

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CHAPTER 1

INTRODUCTION

One of the hardest challenges in physics is the study of systems composed by an enormous number of particles. The difficulty in dealing with these kinds of systems is that it is practically impossible to get (and store...) informations about the positions and momenta of all the particles, and even if this were possible, it would be impossible (or very hard...) to solve the equations governing its dynamics.
These obstacles can be circumvented by introducing *macroscopical* quantities that describe the *collective* effect of the interactions between the microscopical constituents of the system. Examples of macroscopical quantities are: temperature, pressure, etc.... The words "collective" and "macroscopical" mean that it is possible to describe the state of the system in this simplified view only when we deal with a huge number of particles that together constitute a macroscopic system (i.e. it makes little sense to ask the temperature of a system with one particle). One fundamental assumption behind thermodynamics and classical statistical physics is that of *thermal equilibrium*: let us image two systems A and B with different temperatures T_A and T_B. If we put them in contact after a transient time the whole system $(A + B)$ will reach a stationary state characterized by an equilibrium intermediate temperature T_{A+B}; we say that the system *thermalized*. Statistical physics and thermodynamics do not tell us *how* the thermalization occurs but they fully describe the equilibrium state. The thermal state reached after the thermalization has a well defined temperature and then a well defined total energy, this steady state can be described and fully analyzed using the "Gibbs *ensembles*", based on the hypothesis that the time average coincides with the average over the *ensemble*. Let us briefly summarize some properties about Gibbs *ensembles*
The *microcanonical ensemble* [1] is a set of points in the phase space of a classi-

[1] Other *ensambles* can also be defined, i.e. the canonical and the gran-canonical. These are particularly useful when we want to analyze the thermal state of subsystems in a thermal bath. The energy and, in the grancanonical case the number of particles too, of the subsystem are not *fixed*, although the quantities related to the entire system are fixed. Differences between *ensembles* are present only for finite systems. When the thermodynamical limit is considered,

cal system such that each point is characterized by a fixed total energy(namely the energy belongs in a small interval [E and $E + \Delta E$]). Each point of this set defines a possible representative state for the system we are considering. From a macroscopical point of view all these states are indistinguishable, therefore all the points in the *ensemble* are equiprobable. When the system thermalizes, it can be found in any of those microscopical configurations. The ergodicity hypothesis states that, if the system is not integrable, it "explores" all the possible allowed micro-states in the phase-space and therefore the average over the the ensemble coincides with the time average. Integrable or quasi integrable systems present periodic or quasi periodic trajectories in the phase space and therefore the ergodic hypothesis is broken[2].
So far we have considered classical systems. In the quantum case the situation is more complicated. The most important difference in this case is the impossibility to define a phase-space and therefore the impossibility to translate the ergodic theorem from classical to quantum mechanics. Anyway this impossibility does not imply that a quantum statistical system has to be necessarily described by solving exactly its dynamics. It is possible to define the Gibbs *ensemble* also in this case introducing the density matrix representation. The density matrix representation is a formal procedure that it is particularly suitable when we deal with systems in which there is a lack of information that does not permit to construct a wave function for the entire system [1]. The averaging procedure using the density matrix has a twofold nature, it considers both the averaging over the intrinsic probabilities of the quantum mechanical description and the averaging due to the incompleteness of information of the whole system analyzed. Pure systems can also be described using density matrix representation, in this case only the probabilistic average over the quantum nature of the system is considered.
Before proceeding we must understand under which hypotheses the characterization of a thermal quantum state through the *ensembles* technique makes sense. Let us suppose to have an initial state described by the wave function:

$$|\Psi(0)\rangle = \sum_n c_n |\varphi_n\rangle, \tag{1.1}$$

where $|\varphi_n\rangle$ are stationary eigenfunctions of a certain Hamiltonan. The time evolved state is:

$$|\Psi(t)\rangle = \sum_n \mathrm{e}^{-i\frac{E_n}{\hbar}t} c_n |\varphi_n\rangle \tag{1.2}$$

If we consider a certain observable A, its expectation value at the generic time t is:

$$\langle\Psi(t)|A|\Psi(t)\rangle = \sum_{n,m} c_m(t)^* c_n(t) \langle\varphi_m|A|\varphi_n\rangle, \tag{1.3}$$

where $c_n(t) = \mathrm{e}^{-i\frac{E_n}{\hbar}t} c_n$. If a steady state exists, it must coincide with the time average over an infinite interval of time. The stationary expectation value of the

the characterization of the steady state is *equivalent* in the three pictures and the distinction becomes only formal.

[2] In the 2.1.1 this aspect will be discussed more in detail.

observable A is therefore:

$$\overline{\langle A\rangle} = \sum_{n,m} \overline{c_m(t)^* c_n(t)} \langle m|A|n\rangle \tag{1.4}$$

with $\overline{c_m(t)^* c_n(t)} = \overline{c_m^* c_n e^{i(E_m - E_n)t}} = |c_n|^2 \delta_{n,m}$.
Relation (1.4) holds only for an *isolated* system, if we consider a system in thermal contact with a bath the former relations are not valid. In order to assure the existence of a equilibrium state we must postulate that

$$\overline{c_m(t)^* c_n(t)} = |c_n|^2 \delta_{n,m}. \tag{1.5}$$

In statistical physics this postulate is known as *random phases postulate*, and states that a quantum subsystem in thermal equilibrium with a bath is an *incoherent superposition* of the eigenstates of a certain Hamiltonian. The physical meaning of this postulate is that the time average cancels all the quantum interferences among the states during the measure procedures. This is what is usually referred as a "statistical mixture". Under the assumption of the random phases postulate the value of a certain observable can be written as:

$$\langle A\rangle = \mathrm{Tr}(\rho A), \tag{1.6}$$

where $\rho = \sum_n |c_n|^2 |\varphi_n\rangle\langle\varphi_n|$ is the density matrix of the mixed state.
Let us consider an example that will clarify a bit why the random phases postulate is physically reasonable. Let us take a system S with two energy eigenstates ($|1\rangle_S$ and $|2\rangle_S$) in thermal equilibrium with the environment E that is in the state $|E\rangle_0$, the initial state of the composed system $S+E$ is described by the wave function:

$$|\phi_0\rangle_{S+E} = (\alpha|1\rangle + \beta|2\rangle) \otimes |E\rangle_0 = \alpha|1, E_0\rangle_{S+E} + \beta|2, E_0\rangle_{S+E}. \tag{1.7}$$

If we suppose to have a weak interaction between the environment and the system of the type:

$$H_{in} = C_1|1\rangle\langle 1| \otimes V_1 + C_2|2\rangle\langle 2| \otimes V_2$$

after a time interval the state will evolve in

$$|\phi_t\rangle_{S+E} = \alpha|1, E_t^{(1)}\rangle_{S+E} + \beta|2, E_t^{(2)}\rangle_{S+E}. \tag{1.8}$$

Let us note that the eigenstates of the system S are stationary instead the environment evolved in different way according to the different coupling with the eigenstates of the subsystem. The reduced density matrix of the system S is

$$\rho_S = \rho_{11}|1\rangle_S\langle 1| + \rho_{12}|1\rangle_S\langle 2| + \rho_{12}^*|2\rangle_S\langle 1| + \rho_{22}|2\rangle_S\langle 2|, \tag{1.9}$$

the matrix elements can be calculated using the definition of reduced density matrix $\mathrm{Tr}_E[|\phi_t\rangle_{S+E}\langle\phi_t|]$:

$$\begin{cases} \rho_{11} = |\alpha|^2 \\ \rho_{22} = |\beta|^2 \\ \rho_{12} = \alpha^*\beta\langle E_t^{(1)}|E_t^{(2)}\rangle \end{cases}. \tag{1.10}$$

At this point we can make some observations. The environment has a lot of degrees of freedom, by hypothesis it is much larger than the system S, therefore assuming casual time evolution the states are coupled randomly then it is highly probable that $\langle E_t^{(1)}|E_t^{(2)}\rangle \ll 1$, this means that the reduced density matrix is practically diagonal.
t For isolated quantum systems the situation is a bit different, in fact if the system is prepared in a pure state:

$$\rho_{in} = |\Psi\rangle\langle\Psi|, \tag{1.11}$$

the unitary time evolution due to the Schroedinger equation:

$$|\Psi(t)\rangle = \mathrm{e}^{-\frac{iHt}{\hbar}}|\Psi(0)\rangle, \tag{1.12}$$

will maintain it in a pure state all time. Therefore thermalization in this case can not be reached in the sense seen formerly because the behavior could display periodicity in time. For isolated quantum systems, then, the limit

$$\lim_{t\to\infty} |\Psi(t)\rangle \tag{1.13}$$

can be studied only passing through the exact non-equilibrium dynamics of $|\Psi(t)\rangle$. Actually, as we shall see later, with a suitably choose of the order of limits, an infinite subsystem of a certain isolated system can thermalize, or, said in another way, it acts as its *own thermal bath* [3].

1.1 Non equilibrium dynamics

In recent years a great theoretical effort has been devoted to the study of the unitary time evolution of isolated quantum systems. The main questions that we are trying to answer are:

- What are the characteristics of the steady state of an isolated quantum system?
- When this state is described by a thermal state with a effective temperature T_{eff}?
- What is the role played by the spatial dimensionality and the conservation laws? Do they affect the characterization of the steady state?

These questions arose for the first time in 1929 (Von Neumann et al. [4]) but before the last decades they were considered of purely academic interest. Recently the development of new experimental techniques have made possible the direct observation of non-equilibrium dynamics of isolated quantum systems. A fruitful comparison between theory and experiments has been possible thanks to the use of cold atomic gases and nanostructures. In particular it has been possible to construct quasi one-dimensional optical lattices that are so weakly interacting with the environment that their evolution can be considered unitary. Therefore it has been possible to see if, and eventually how, the spatial dimensionality affects the evolution of these systems. We refer to the section (1.2) for a more detailed

discussion of the experimental results.
The interest in understanding the out-of-equilibrium behavior of quantum interacting systems is not only important from a fundamental point of view, indeed there could be also technological implications. A quantum computer is the most remarkable field of application of these topics, in fact it will definitely require the capability of performing real time manipulations of interacting quantum systems. Therefore it is of crucial importance to understand the coherent dynamics of these systems since it could be one of the main points of various experimental setups and of future technologies.
There are several ways in which a system can be driven out of equilibrium. For example, it is possible to prepare a system in a equilibrium state and then apply a driving field, or pumping energy and particles. One of the most used technique both in theoretical and experimental field is that of the *quantum quench*: one prepares a quantum system in the ground state of a certain hamiltonian H_{bq} with a tunable parameter; at $t = t_0$ we change this parameter *suddenly*, the time evolution will be given therefore by another post-quench hamiltonian H_{pq}, this corresponds to an out-of-equilibrium evolution. It is important to remark that the changing in the parameter must be done instantaneously, in practice this means in a time interval $\Delta t \ll \Delta t_{evolution}$ where $\Delta t_{evolution}$ is the *minimum* time scale of the system.[3]In the opposite limit, we would explore an adiabatic regime.
There are mainly two types of quenches: global quenches and local quenches. In a local quench the change in the hamiltonian is localized, for example impurities on lattices, in the global quenches the hamiltonian is changed over the whole system. In this work we analyze a global quench; this means that we consider a certain hamiltonian (namely Lieb-Liniger model, see chapt. 3) with an interaction term that will be quenched between two different values of the coupling constant.
From a theoretical point of view the quench problems can be tackled in different ways:

- Numerically, using e.g. DMRG techniques or exact diagonalization;
- Analytically, exploiting various techniques (e.g. solving free theories after an exact mapping, or via Bethe Ansatz...)

In this work only analytical techniques are used. Numerical analysis has been only a support in order to understand the behavior of some quantities which were hard to treat analitically.
Before proceeding, it is of fundamental importance to point out some things about the infinite time limit for closed systems [5]. In fact for *finite* systems the $t \to \infty$ could not exists due to the effect of *quantum revivals* of matter waves. In order to bypass this obstacle it is sometimes convenient to consider time averaged quantities. This problem disappears when we work in the thermodynamic limit (TDL), that is when we consider a system composed by an infinite number of particles N lying on an infinite volume L such that the density $n = \frac{N}{L}$ is constant, the large-time limit must be taken *after* the TD one.
The revivals of matter waves have been observed in one of the first experiments

[3]It is usually related to the energy gap between the ground state and the first excited.

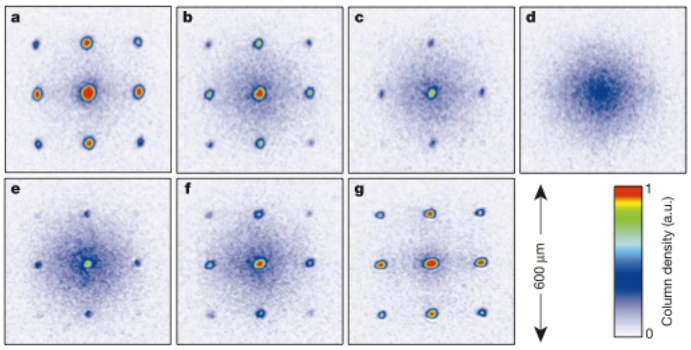

Figure 1.1: Figure from [6]. Interference patterns for different "hold times": (a) $\tau = 0$ μs, (b) $\tau = 100$ μs, (c) $\tau = 150$ μs, (d) $\tau = 250$ μs, (e) $\tau = 350$ μs, (f) $\tau = 400$ μs, (g) $\tau = 550$ μs. We note that after a hold time of 250μs the information about the initial pattern is completely lost, but it is restored after $\tau = 550\mu$s, this a classical example of *matter wave revival*.

about the out of equilibrium dynamics of closed systems performed by Greiner et al. in 2002 [6]. In this experiment it has been considered a Bose gas in a three dimensional cubic lattice, in this configuration the system is well approximated by a Bose-Hubbard model described by the hamiltonian

$$H = -J \sum_{\langle ij \rangle} (a_i^\dagger a_j + a_i a_j^\dagger) + \frac{U}{2} \sum_i n_i(n_i - 1), \tag{1.14}$$

where a_i are bosonic operators and $n_i = a_i^\dagger a_i$ are the number operators on site. The system is initially prepared in a "superfluid phase", that is, it is prepared in the ground state of hamiltonian (1.14) with $J \gg U$, in which the tunneling terms dominate. The on site interaction term U is suddenly quenched such that $U \gg J$, the time evolution is performed by the hamiltonian:

$$H_{pq} \approx \frac{U}{2} \sum_i n_i(n_i - 1). \tag{1.15}$$

The system is "hold" and left evolve for several time intervals and then is released from the trap. Absorption images of interference patterns after the release are shown in figure 1.1. As we can see after a time $\tau = 550$ μs the system returns in its initial configuration. This experiment shows that it is possible to maintain a system isolated for a time sufficient to show its coherent dynamics.

1.2 Experimental results

In this paragraph some of the most important experimental results, that encouraged theoretical interest in the arguments previously outlined, will be presented. The first one that we will discuss is *the quantum Newton's Cradle* [7] performed by T. Kinoshita, T. Wenger and D.S. Weiss. In this experiment it has been observed the non-equilibrium dynamics of a 1D Bose gas composed by ^{87}Rb atoms. To create a one dimensional system a Bose Einstein condensate is loaded in a

2D dimensional lattice, here, using a red-detuned crossed dipole trap, the bidimensional lattice is divided in arrays of 1D Bose gases weakly interacting. The dynamics in each tube is strictly one dimensional because the weak transverse excitation energy ($E_t = \hbar\omega_r$, $\frac{\omega_r}{2\pi} = 67$kHz) far exceeds the thermal energies of the trapped atoms and therefore the tunneling probability is very low.

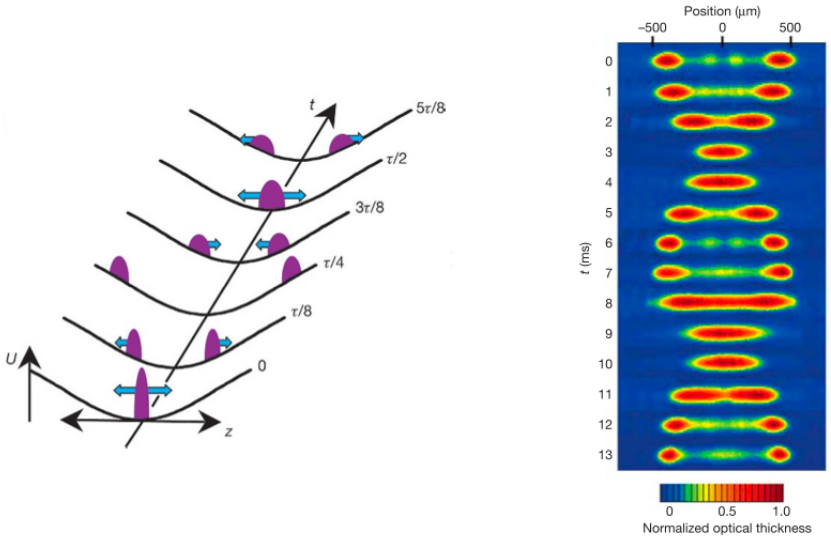

Figure 1.2: Figures from [7]. *Left*: Schematic representation of the atomic bundles in the 1D harmonic trap. *Right*: Absorption images in the first oscillation cycle. The dynamics is always 1D since the collision energy of the bundle is less than a quarter the transverse confinement energy.

Using two shortly separated pulses the atoms in every array are put in a superposition of momentum states with $p = \pm 2\hbar k$, in this way each bundle is split in two atomic clouds with opposite momenta. The system is left evolving, the bundles collide in the centre of the trap twice a cycle (Fig. 1.2), the collision energy is still less than a quarter of the minimum transverse excitation energy, ensuring that the dynamic remains 1D. There is not perfect recurrence due to dephasing effects. It is possible to extract also the momentum distribution. As we see in Fig 1.3, also after several periods of oscillations the distributions are far from being gaussian meaning that there is not thermalization.
These distributions, in a certain sense, carry more informations about the initial conditions. The same experiment has been done for 2D and 3D Bose systems (see Fig. 1.4) , in that case thermalization occurs after a few periods. Therefore the spatial dimensionality plays a crucial role in the thermalization of isolated quantum systems.

The next experiment we are going to describe is conceptually slightly different from the previous one, but is important to show how nowadays it is possible to manipulate microscopic systems in order to implement quantum quenches and then study the non-equilibrium dynamics [8]. In this experiment the expansion of a fermionic gas in a 2D lattice after a trap release is studied. Two cases are examined: interacting and non-interacting fermions (see Fig. 1.5)

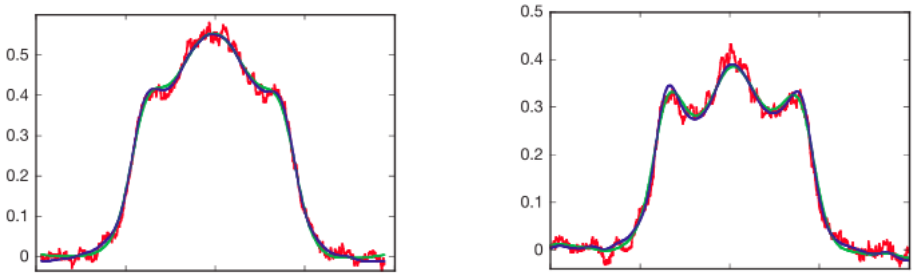

Figure 1.3: Figures from [7]. *Left*: Momentum distribution for $t > 1910\tau$, with τ oscillation period, and $\gamma = 3.2$. *Right*: Momentum distribution for $t > 390\tau$, with τ oscillation period, and $\gamma = 18$. $\gamma = |2/a_{1D}n_{1D}|$ where a_{1D} is the 1D scattering length and n_{1D} the average density. The red curves are real data taken in a certain t_{obs}, the blue and green curves are simulations from different models that take into account the losses during the measure. For details see [7].

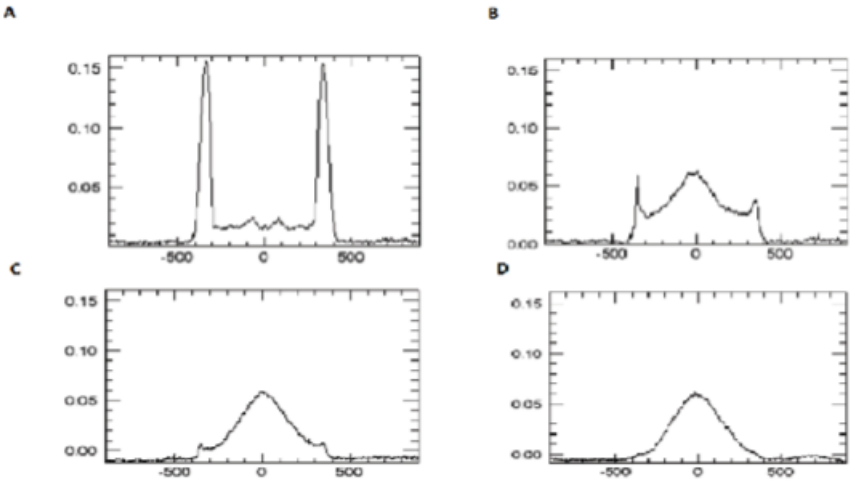

Figure 1.4: Momentum distribution of a 3D Bose gas in the same conditions described in [7]. $t = 0\tau$, $t = 2\tau$, $t = 4\tau$, $t = 9\tau$, with τ oscillation period, we can see that thermalization occurs after a few periods.

This system is well described by a Fermi-Hubbard model with an external harmonic potential that breaks the horizontal translational invariance:

$$H = -J \sum_{\langle ij \rangle, \sigma} (c^{\dagger}_{i,\sigma} c_{j,\sigma} + h.c.) + U \sum_{i} n_{i,\uparrow} n_{i,\downarrow} + V_{ext.}, \tag{1.16}$$

where the c_i are fermionic operators and σ are spins degrees of freedom. The system is initially prepared in a confined state, due to a tight harmonic trap, and then is released. This is an example of "inhomogeneus" quench: an external parameter of the hamiltonian is changed inhomogeneously along the whole system. Also these types of quenches have been studied from a theoretical point of view ([9]).

After the release the non-equilibrium dynamics is studied, in particular the *in-*

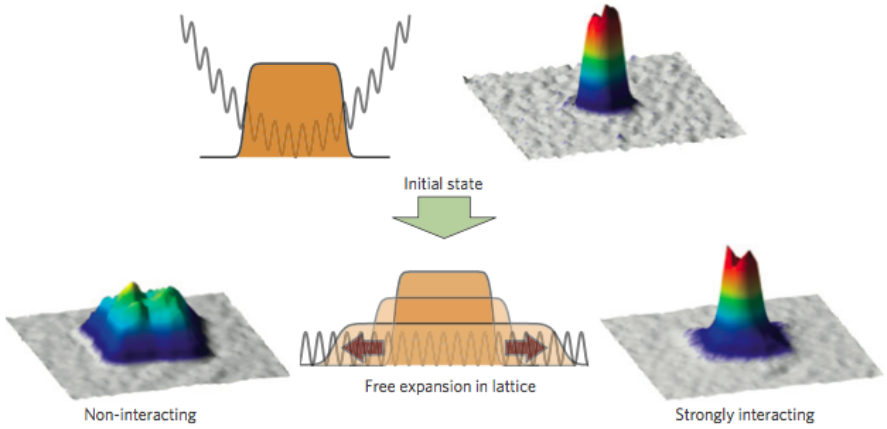

Figure 1.5: Figure from [8]. Schematic representation of the experiment. After the release of the harmonic trap, the *in-situ* density is studied both in the interacting case and in the non-interacting case, significative differences result.

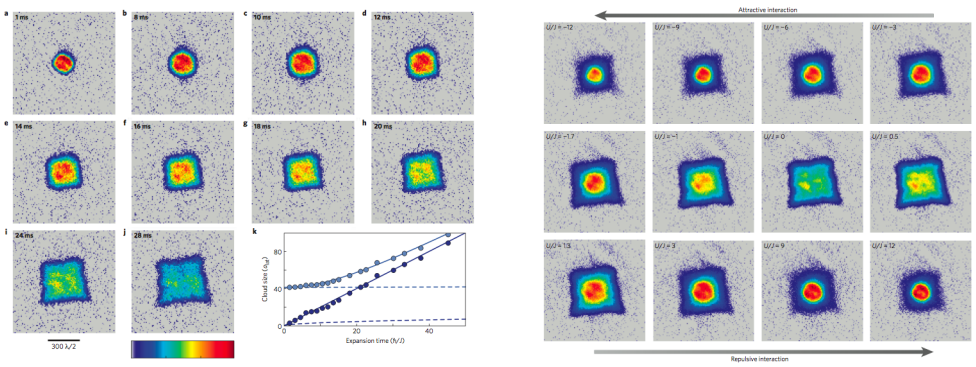

Figure 1.6: Figure from [8]. *Left*: time evolution of the density for non-interacting fermions. For large time (i.e. in the last panel) the system is homogeneous. *Right*: Density distributions 25 ms after the release for several values of the on-site interaction (both repulsive and attractive). The system is rotationally invariant especially for high values of the interaction (first and last panel).

situ density (see Fig.1.6). For non-interacting fermions the evolution is given only by the hopping term of (1.16), ($H_{pq} = -J \sum_{\langle ij \rangle,\sigma} (c^{\dagger}_{i,\sigma} c_{j,\sigma} + h.c.)$), the expansion is *ballistic*, that is, every excitation expands independently with constant quasi-momentum. This results in a square-shaped density distribution at the equilibrium. In the interacting case the situation is much more complicated, in this case the rotationally invariance of the density shape is preserved and the expansion is *diffusive*.

The next, and last, experiment that we are going to show was performed by Trotzky et al. in 2012 [10], they studied the relaxation properties of a one-dimensional Bose gas.

In this experiment a Bose Einstein condensate composed by ^{87}Rb atoms is loaded in a three dimensional lattice and using a crossed trap is divided in an array of 1D chains. According to the data, in the 3D lattice about $45 \cdot 10^3$ atoms are loaded. The trap is set in such a way that in every chain there are about 43 atoms. The

measured quantities have to be intended as an average over all the chains. Once that the 1D systems are created a "short lattice" with wave length $\lambda_{sl} = 765$nm is added along the direction of the chain with relative phase adjusted with the trapping lattice to load the sites of the short lattice alternately (for example only on "even" sites, see [10] for the technical details), as shown in the first picture of the Fig. 1.7.

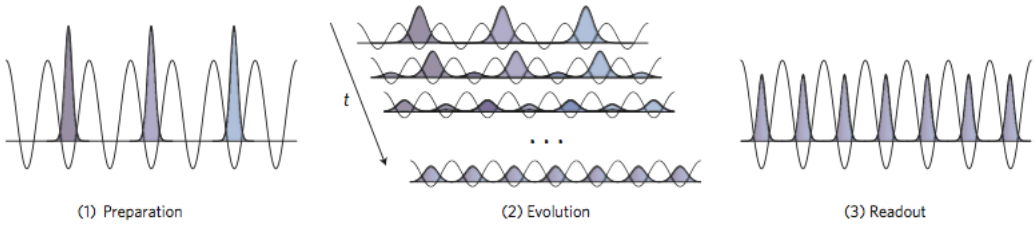

Figure 1.7: Figure taken from [10]. Schematic representation of the three steps of the experiment.

This system is well described by a Bose-Hubbard model with the same Hamiltonian (1.14) and everything is set in such a way that the tunneling probability is very low. Tuning the parameters, the potential is suddenly changed and the tunneling between adjacent sites is activated, the system is left evolving for a certain time t. After this time interval the potential in changed again and the tunneling is blocked, the average population on "odd" sites is then measured. This process is repeated for several values of t and four values of the ratio $\frac{J}{U}$.

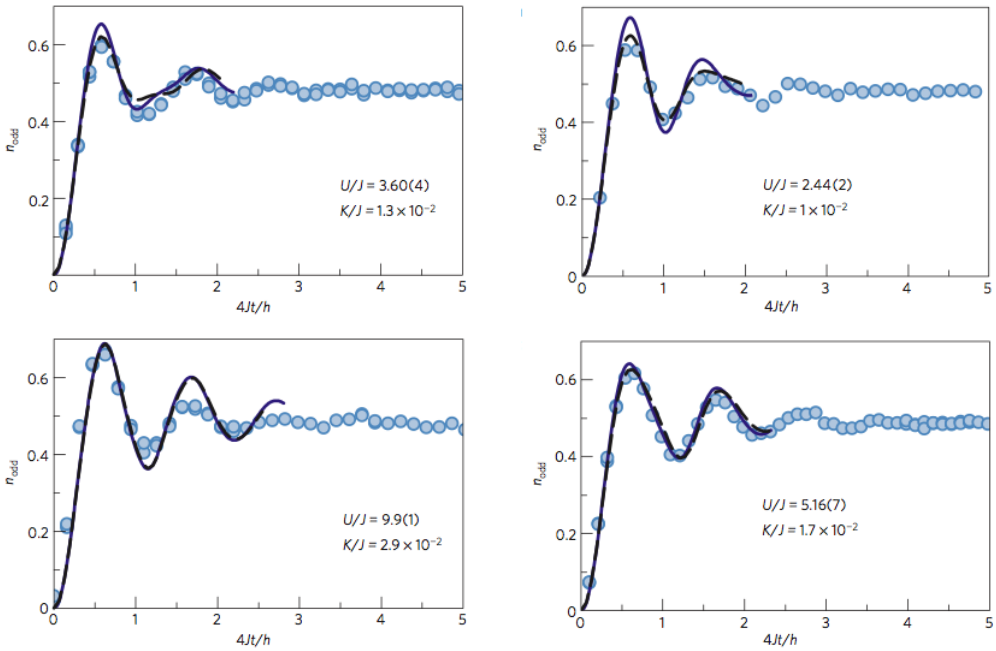

Figure 1.8: Figure taken from [10]. Evolution of the average population of the odd sites after the quench, the dots are the experimental data, the blue lines are results from numerical simulations.

In figure 1.8 the results of the measures are shown. The population in the odd

sites, that were empty in the initial state, after a transient period, approaches a constant stationary value. This fact agrees with numerical simulations performed using t-DMRG techniques.
The experiments discussed in this section are only a small sample of all the works done in the study of the non-equilibrium dynamics of isolated systems, other important examples, that we will not discuss here, can be found in references [14], [13], [12] and [11].

1.3 The Generalized Gibbs Ensemble

From the results of the experiments described in the former paragraph (especially [7] and [10]) it seems that isolated quantum systems prepared in a pure states, in the limit $t \to \infty$ after a quench, present characteristics typical of mixed states. As we have discussed in paragraph 1.1 systems prepared in a pure state evolve unitarily according to the Schroedinger equation, therefore they can never reach a steady state described by a density matrix. How can this apparent paradox be solved? The solution [15] is that in the thermodynamical limit *finite subsystems* reach an equilibrium steady state well described by a *reduced* density matrix that has all the characteristic of a mixed state. If we consider a system in a pure state $|\Psi(t)\rangle$, and a subsystem A, its reduced density matrix can be obtained tracing out the degrees of freedom of the complementary system $\overline{A}$:

$$\rho_A(t) = \mathrm{Tr}_{\overline{A}}(|\Psi(t)\rangle\langle\Psi(t)|). \tag{1.17}$$

The expectation value of an operator $\mathcal{O}_A$ which degrees of freedom are *within* the subsystem A reaches an equilibrium value that can be found from (1.17):

$$\lim_{t\to\infty} \langle\Psi(t)|\mathcal{O}_A|\Psi(t)\rangle = \mathrm{Tr}[\rho_A(\infty)\mathcal{O}_A]. \tag{1.18}$$

Therefore $\rho_A(\infty)$ is effectively a density matrix describing the equilibrium properties of a mixed state. If it happens that for *any* finite subsystem A of our system the limit:

$$\lim_{t\to\infty} \rho_A(t) = \rho_A(\infty) \tag{1.19}$$

exists we can make the following argument [15]: first of all, our entire system is decomposed in a subsystem A and its complementary $\overline{A}$, at this point we can take the thermodynamical limit of the system *keeping A finite.* Then, finally, we can make the system A very large[4] itself. Since relation (1.19) holds for *every* subsystem it will continue to be valid also for a system A built in this way. Then, if the order of the limits is respected, the subsystem A will reach an equilibrium state and all the operators within it will attain a value well described by a density matrix. Suggestively we can say that the system acts as its own bath.
The experiment previously described (Fig. 1.4) suggests that the reduced density matrix of the subsystem is thermal for 2D or 3D systems. In the configuration used in that case (a harmonic confining potential) a thermal density matrix is

[4]Strictly speaking the system A at this point can be considered practically infinite but still smaller than the entire system.

recognized by having a gaussian momentum distribution at the equilibrium. In this case we have only few conserved quantities (i.e. the energy and the number of particle) and the steady state can be described by a Gibbs ensemble with an effective temperature T_{eff}.
For one-dimensional systems the situation is a slightly different, as we have seen (fig. 1.3), and as conjectured by Rigol et al. [16] [17] [18] [19], for *integrable* quantum systems there is relaxation to a non-thermal distribution. Quantum integrable systems are characterized by an infinite set of higher conserved charges $\{Q_i\}$ that seem to play a key role in the description of the stationary state [20] [21] [22].
In [16] a model very similar to that considered in this thesis, but in the discrete case, it is studied: a one-dimensional lattice of length L loaded with *hard-core bosons.* This model is fully described by a Bose-Hubbard hamiltonian with no on-site interaction[5].

$$\hat{H} = -J \sum_{i=1}^{L} (\hat{b}_i^\dagger \hat{b}_{i+1} + h.c.), \tag{1.20}$$

where $[b_i, b_j^\dagger] = [b_i, b_j] = [b_i^\dagger, b_j^\dagger] = 0$ as long as $i \neq j$ and $\{b_i, b_i^\dagger\} = 1$, with $(b_i)^2 = (b_i^\dagger)^2 = 0$ for all i. This model can be mapped to a free fermion theory (see paragraph 3.2.1) and therefore it is exactly solvable. In [16] the authors questioned, assuming that after an out of equilibrium dynamics this system reaches a stationary state, what is the statistical many-body density matrix that would have described it. In order to answer this question it was conjectured that standard prescriptions from statistical mechanics apply, therefore it should be maximized the many-body entropy taking into account the constraints imposed by all the integrals of motion. This resulted in the Generalized Gibbs Ensemble (GGE):

$$\rho_{GGE} = \frac{\mathrm{e}^{-\sum_m \lambda_m \mathcal{I}_m}}{\mathrm{Tr}(\mathrm{e}^{-\sum_m \lambda_m \mathcal{I}_m})}, \tag{1.21}$$

where $\{\mathcal{I}_m\}$ is a *full* set of *local* (see paragraph 1.3.1) integrals of motion, and the λ_m are Lagrange multipliers fixed by *initial conditions* via:

$$\mathrm{Tr}(\mathcal{I}_m \rho_{GGE}) = \langle \mathcal{I}_m \rangle_{t=0}. \tag{1.22}$$

From (1.22) it is evident that the GGE carries an *infinite* amount of information about the initial conditions, this could explain why the equilibrium momentum distribution is not single peaked, as in the gaussian thermal case, but maintains a shape that in a certain sense "remembers" the initial distribution (this concept will be clearer soon). The integrals of motion $\{\mathcal{I}_m\}$ are defined as operators such that:

$$[\hat{H}, \mathcal{I}_m] = 0,$$

where $\hat{H}$ is the hamiltonian of our model. Therefore, being conserved quantities, they are the most intuitive choice to do if we want to catch the stationary state properties.

[5] This was the same system studied in experiment [7], it was this experiment that motivated Rigol et al. to investigate theoretically the thermalization properties of this system.

In order to understand why the generalized Gibbs ensemble has that form we can use the "maximum entropy argument" made by E.T. Jeynes [23]. In that article, information theory is exploited to define a rigorous criterion that could help in the construction of statistical ensembles when we have some constraints (e.g. many conserved quantities). Quoting from article [23]: "*Previously one constructed a theory based on equation of motion, supplemented by additional hypotheses of ergodicity [...] and the identification of the entropy was made at the end [...] Now, however, we can take entropy as our starting concept, and the fact that a probability distribution maximizes the entropy subject to certain constraints becomes the essential fact which justifies the use of that distribution for inference. [...] we make it possible to see statistical mechanics in a much more general light*".
In order to clarify slightly more this last concept let us consider a physical system with $n \gg 1$ states, to each state corresponds a certain probability p_i such that $\sum_i p_i = 1$. Let us suppose that we know $r \ll n$ expectation values of the operators $\mathcal{O}^{(l)}$:

$$E^{(l)} = \sum_i p_i \langle \phi_i | \mathcal{O}^{(l)} | \phi_i \rangle,$$

what is the most *probable* expectation value of another operator $\mathcal{F}(x)$ that cannot be decomposed in terms of $\mathcal{O}^{(l)}$? We would like to find it using constraints that implement the little informations we have. The only physical quantity that measures uncertainty, increasing with increasing uncertainty, that is positive and additive can be demonstrated to be the Shannon's entropy, defined as:

$$S(\{p_i\}) = -\sum_i p_i \ln(p_i). \tag{1.23}$$

Note that this quantity coincides with the Gibbs' entropy. If we know nothing about the expectation values, the entropy is maximized if $p_i = \frac{1}{n}\ \forall i$, this is exactly the microcanonical ensemble. If we, instead, have a constrain only on one expectation value (e.g. the energy) $E = \sum_i p_i \langle \phi_i | \mathcal{O} | \phi_i \rangle$, entropy can be maximized introducing a the Lagrange's multiplier β and this gives us the well-know canonical ensemble

$$p_i = Z^{-1} \mathrm{e}^{-\beta A_i}. \tag{1.24}$$

The same argument can be exploited when we have many constraints (as in the GGE ...). In this case we must use several Lagrange's multipliers β_l , one for every known expectation value, this, if the A_i commute leads to

$$p_i = \frac{\mathrm{e}^{-\sum_{l=1}^{r} \beta_l A_l}}{Z}, \tag{1.25}$$

where Z is a normalization factor (i.e. the partition function). This form is very similar to the (1.21). Let us return to the discussion of Ref. [16]: the predictive power of the generalized Gibbs ensemble is tested using numerical techniques. The system is prepared in the ground state of a spatially-periodic background-potential with period 4; $V_{ext.} = A \sum_i \cos\left(\frac{2\pi i}{T} b_i^\dagger b_i\right)$, then is released ($V_{ext.} = 0$) to a flat-bottom hard-wall box. In the paper, numerical simulations are used to study the *exact* out-of-equilibrium dynamics and the steady state attained.

The data are then compared with the results predicted by the Generalized Gibbs Ensemble.

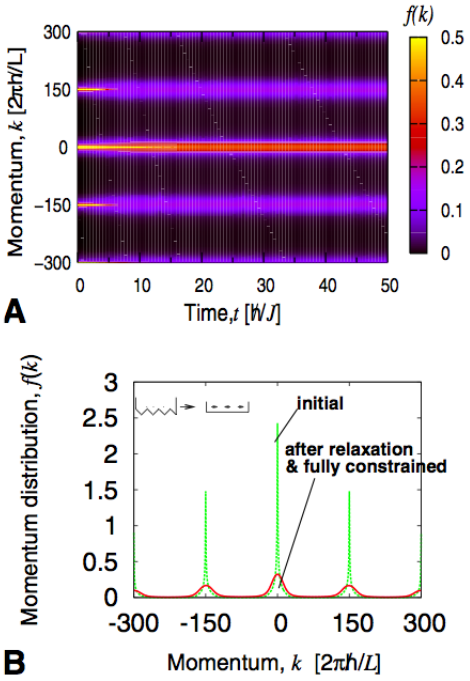

Figure 1.9: Figure taken from [16]. *Upper*: time evolution of the quasi-momentum distribution. *Lower*: distribution of the quasi-momentum after relaxation. The discrepancy between the results of the time evolution and the prediction of the GGE ensemble are less than the width of the line (see [16] for further details).

As it can be seen in figure 1.9 there is a perfect matching between the GGE predictions and the results from dynamical evolution, indeed the lines are completely overlapped. From Fig. 1.9 we can understand why the generalized stationary state carries "more memory" of the initial conditions, in fact also in the equilibrium state can be observed well-separated peaks in the momentum distribution similar to the initial one.

1.3.1 Locality

In the last paragraph it has been stated that the generalized Gibbs ensemble is built using a complete set of integral of motion $\mathcal{I}_m$. An important point to face concerns what kind of operators should be included in the definition (1.21). Any quantum system has too many integrals of motion; the easiest examples are the projectors onto the energy eigenstates $\mathcal{I}_n = |\psi_n\rangle\langle\psi_n|$ with $H|\psi_n\rangle = E_n|\psi_n\rangle$, since $[H, \mathcal{I}_n] = 0$. However the projectors can not have any role in describing the properties of a system after the thermalization. A possible solution to this dilemma has been conjectured by P. Calabrese, F.H.L. Essler and M. Fagotti in [15] and [24]. Since we focus on local properties of systems they proposed to include only *local integrals of motion*. These integrals can be written as sums (or, in the continuum case, as integral) of operators acting locally, $\mathcal{I}_n = \int dx J(x)$,

with a procedure that remembers the Noether's theorem. This conjecture has demonstrated to work in many cases , but the proposal is still under debate (see [25][26][27][28][29][30][31][32][33] for some criticism).

CHAPTER 2

INTEGRABILITY

In this chapter some notions of classical integrability following the Liouville's theorem are reviewed. We analyze the connection between the integrability of classical systems and the ergodic theorem which guarantees the validity of the *ensembles* technique. Furthermore the problems arising when a quantum system is considered are discussed. In particular, we follow some recent works ([37] or [42, chap. IV]) where it is shown a possible definition of quantum integrability. Although it is not intuitive, such a definition will be useful to give a first classification between integrable and non-integrable systems.

2.1 Classical Integrability

In order to show the precise mean of integrability in classical mechanics we will use a hamiltonian approach. Let us remind some basic facts that will be the background of our arguments[1]. Given a generic system described by the hamiltonian $H(p_j, q_j)$, where p_j and q_j is a set of conjugate variables, it is possible to characterize its dynamics by solving a system of coupled differential equations called the Hamilton's equations:

$$\begin{cases} \dot{q}_j = \frac{\partial H}{\partial p_j} \\ \dot{p}_j = -\frac{\partial H}{\partial q_j} \end{cases} \qquad \text{where} \qquad \dot{q} = \frac{dq}{dt} \qquad \text{and} \qquad \dot{p} = \frac{dq}{dt}. \tag{2.1}$$

For a system with n degrees of freedom we can introduce a $2n \times 2n$ matrix J:

$$J = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}, \tag{2.2}$$

such that the hamilton's equations can be written as:

$$\overrightarrow{\dot{x}} = J \cdot \nabla H,$$

[1] The notions exposed here can be found in all the classical books of mechanics. See for example [34] or [35]

where $\nabla H = (\frac{\partial H}{\partial p_j}, -\frac{\partial H}{\partial q_j})$ and $\overrightarrow{x} = (p, q)$.
The matrix J can be used to define the *Poisson brackets* on the space composed by differentiable functions defined on the $2n-$dimensional phase-space M:

$$\{F, G\} = (\nabla F, J \cdot \nabla G) = \sum_{i=1}^{N} \left(\frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} \right). \tag{2.3}$$

A hamiltonian system with n degrees of freedom is therefore characterized by a 2n-dimensional phase space, a Poisson structure and a Hamiltonian. For any function $G(p, q)$ defined on the phase space, it can be shown that the time evolution is given by the Poisson brackets with the Hamiltonian:

$$\frac{dG(p, q)}{dt} = \{H, G\}. \tag{2.4}$$

Since, trivially, $\{H, H\} = 0$, the hamiltonian is a conserved quantity and then the trajectories on the phase-space of the system live on a manifold with constant energy. The hamiltonian $H(\overrightarrow{x})$ can be expressed in terms of other variables $H(\overrightarrow{x}')$ provided that the $\overrightarrow{x}$ and the $\overrightarrow{x}'$ are related trough a *canonical* transformation, that is

$$\overrightarrow{x}' = A\overrightarrow{x} \qquad \text{with} \quad AJA^t = J \tag{2.5}$$

where A is $2n \times 2n$ matrix implementing the transformation and $\overrightarrow{x}$ and $\overrightarrow{x}'$ are $2n$-dimensional vectors of the type defined previously. Since the matrix J is non-degenerate the inverse, $J^{-1} = -J$ exists. Using this property it is possible to define a measure on the phase space by introducing the bilinear $\omega(\overrightarrow{x}, \overrightarrow{y})$:

$$\omega(\overrightarrow{x}, \overrightarrow{y}) = (\overrightarrow{x}, J^{-1}\overrightarrow{y}), \tag{2.6}$$

where $(,)$ denotes the scalar product. It is easy to show that the measure defined in this way is *invariant under canonical transformations*, indeed:

$$\omega(A\overrightarrow{x}, A\overrightarrow{y}) = (A\overrightarrow{x}, J^{-1}A\overrightarrow{y}) = -(\overrightarrow{x}, A^t J A\overrightarrow{y}) = (\overrightarrow{x}, J^{-1}\overrightarrow{y}) = \omega(\overrightarrow{x}, \overrightarrow{y}). \tag{2.7}$$

The Lioville's theorem states that a classical system with n degrees of freedom described by a Hamiltonian $H(\overrightarrow{x})$ is *integrable* if we can find n functions F_i defined smoothly over the phase-space M such that:

- the F_i are conserved charges, thus, for every $i = 1, \ldots, N$ one has

$$\{H, F_i\} = 0;$$

- the F_i are functionally independent of each other;
- all the F_i are in involution:

$$\{F_i, F_j\} = 0, \qquad \forall i, j.$$

This definition of integrability is sufficient for our purposes, anyway it can be improved. A more formal statement of the Liouville's theorem can be found for example in [34]. If a system is integrable, a particular type of conjugate variables can be introduced. These variables are called *action-angle variables* and the differential equations governing the dynamics can be integrated by quadratures [35]. The solution in terms of the action-angle variables displays a *periodic motion* on invariant tori in the phase-space. This periodicity is typical of integrable systems and it is one of the most important differences between classical integrable and non-integrable models. Indeed, if the evolution of the system in the phase-space is not chaotic, the ergodic theorem does not hold and the average values of the various quantities will depend on the particular orbit of the system in phase-space. Therefore, since the points of the sub-variety of the phase-space with constant energy are not equivalent, the *ensemble* technique fails.

2.1.1 Integrability and ergodicity

Let us consider a system of N particles described by a certain hamiltonian $H(p,q)$ that defines a sub-variety of constant energy in the phase-space on which the orbit of the temporal evolution lives. The ergodic hypothesis states that, for every observable $F(p,q)$ defined on the subvariety $H(p,q) = E$, the late-time average can be calculated averaging over an infinite number of copies of the system satisfying the only requirement of fixed energy. This picture coincides with the Gibbsian ensemble technique [2] discussed formerly and that it is formalized by the introduction of a density distribution defined in the phase-space $\rho(p,q,t)$ such that

$$\rho(p,q)d^3qd^3p = \text{number of copies contained in the volume } d^3qd^3p. \tag{2.8}$$

The *ensemble average* of the quantity $F(p,q)$ is defined as

$$\langle F(p,q)\rangle = \frac{\int d^{3N}pd^{3N}qF(p,q)\rho(p,q)}{\int d^{3N}pd^{3N}q\rho(p,q)}. \tag{2.9}$$

The ergodic condition can be thus rewritten as

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T dtF(q(t),p(t)) = \frac{\int d^{3N}pd^{3N}qF(p,q)\rho(p,q)}{\int d^{3N}pd^{3N}q\rho(p,q)}, \tag{2.10}$$

where $q(t), p(t) \in \Gamma(p,q)$ and $\Gamma(p,q,)$ is the trajectory of the system in the phase-space. One fundamental assumption behind the ergodic condition and the validity of the Gibbs ensemble technique, as sketched in the last paragraph, is that all the copies of the system must be *equiprobable*. This statement can be rephrased as: the points satisfying the condition of constant energy must be "touched" at least once by the path representing the time evolution of the system in the phase-space. In fact the averaging procedure over the ensemble takes in account *all* the points compatible with the requirement of fixed energy, therefore if some of them do not concur at the time evolution Eq. (2.10) does not hold. Those systems whose trajectories display a certain regularity can not be treated as ergodic ones. In this category both integrable and *quasi-integrable* systems fall.

Quasi integrable systems are distinguished by the property that are very "close" to be integrable, examples of this kind are integrable systems subjected to weak non linear perturbations. These systems, although strictly speaking are non-integrable, display an absence of ergodicity because the trajectories in the phase-space are quasi periodic and, in the infinite time limit, do not cover all the allowed sub-variety in the phase-space[2]. Non-integrable systems are instead qualitatively different. Indeed in this case the evolution is "chaotic" and as time passes the phase-space on which the system lives is completely explored and thus the relation (2.10) is valid.This means that for "random" time evolution all the copies of the system that preserves the condition of energy conservation are *de-facto* equivalent and then we have ergodicity.
What has been reported here about the ergodic problem is only partial, indeed we avoid to mention a lot of details and problematics that are behind the concept of ergodicity, but what has been discussed should be sufficent to give an overview and to remind some important concepts on which all the classical statistical physics (and not only) is based.

2.2 Quantum Integrability

The aim of this paragraph is to introduce the reader to some concepts relating the question of quantum integrability and quantum integrable systems. Our discussion will follow what reported in the works [42, chap. IV] or [37] and [38]. First of all, we must point out that in this thesis, as reported in chapt. 4, we consider a post-quench hamiltonian which can be mapped to a free fermionic one, and the initial state was prepared in the ground state a free bosonic hamiltonian. Therefore we are dealing with integrable models since, no matters what kind of definition of quantum integrability we adopt, free theories should fall into the integrable class. The first naive tentative that one can do in order to give a proper definition of quantum integrability is translating in "quantum language" the Liouville's theorem that holds for classical systems. Then, if we replace the Poisson brackets with the commutators under the prescription

$$\{,\} \to \frac{i}{\hbar}[,], \tag{2.11}$$

we can say that:

Definition 1. *A system is quantum integrable if we can find a complete set of independent commuting operators (i.e.* $[\mathcal{I}_m, \mathcal{I}_n] = 0 \ \forall m, n = 1, \dots, dim(\mathcal{H})$*).*

This definition, although seems to be quite reasonable, shows some pitfalls. The first question is that a classical definition transported in quantum mechanics could be ill-conditioned by the fact that the degrees of freedom are counted in a total different way in the two cases. In classical mechanics the number of degrees of freedom is just the number of couples of conjugate variables that one needs

[2]The first and perhaps most famous demonstration of quasi periodic motion for quasi integrable systems is the so called FPU (Fermi, Pasta, Ulam) problem, all the details can be found in reference [36]

to specify the dynamics, instead in quantum mechanics the number of degrees of freedom of a system is just the dimensionality of its Hilbert space. For this reason the requirement of completeness it has been added in the definition, in other words: the cardinality of the conserved charges has to be the same as the dimensionality of the Hilbert space $\mathcal{H}$. The second, and more severe, problem is that according to the definition 1 every quantum system with finite-dimensional Hilbert space would be integrable. In fact by the spectral theorem every hermitian Hamiltonian is diagonalizable, then one can define $dim(\mathcal{H})$ commuting operators, $\mathcal{I}_m = |\psi_m\rangle\langle\psi_m|$. This is not acceptable if we want a classical correspondence, therefore we must abandon the idea of translating "word by word" the integrability condition from classical to quantum language.
Another commonly definition used in literature is:

Definition 2. *A quantum system is integrable if its full set of eigenstates can be constructed, i.e. if it is exactly solvable.*

Also this definition could be satisfactory at first sight, especially because reminds the classical feature of angle-action variables. But this sounds as a consequence of integrability rather than as a definition. Furthermore it sounds too simplistic because it is not specified if models solvable with different methods should display different physical behavior.
Before proceeding, we follow what done in [37] and itemize some inescapable requirements that we want to be fulfilled by a proper definition:

- it should be unambiguous;
- it should define different classes to which different models belong;
- all the classes should display different physical behavior.

We realize that without a general and rigorous definition it is very hard to fulfill all these criteria. In particular let us note that the second definition we gave (that at the moment is the most reasonable one) fails in the third criterion. Another subtle point concerns the number (i.e. the cardinality) of the conserved charges. In the first definition, we required the completeness in the sense that the cardinality of the conserved charges should be the same of the dimensionality of the Hilbert space. A theorem by Von Neumann [38] states that: given a set of commuting operators $\{\mathcal{I}_m\}$, it is possible to construct another operator I such that every $\mathcal{I}_m$ can be viewed as a functions of I, $\mathcal{I}_m = f_m(I)$, so the *number* of the conserved charges can be ill-defined. Furthermore the request of completeness is often accompanied by the request of locality (see last chapter 1.3.1), but it is not clear what should mean: "*a complete set of local charges*", since the "locality" requirement restricts the number of possible operators. The confusion probably arises from the misunderstanding between the concept of *infinite* and the concept of *complete* set of charges [37].
A possible escape from all these problematics has been proposed by J.S. Caux and J.J. Mossel and is reported in [37] and [42]. The definition they gave is a very formal one but fulfills all the criteria and, until now, seems to work, contrarily to the previous ones we considered. It is worth to introduce some preliminary

concepts. Let us define a *size sequence* as a strictly increasing string of integer numbers, $(N_1, \dots, N_m, \dots)$ with $N_1 < N_2 < N_3 \dots etc.$ To each number in the string it is associated a Hilbert space $\mathcal{H}^{(N_i)}$ obtained by tensoring a finite dimension "basic" Hilbert space (with $dim\mathcal{H}_i = d_i$) $\mathcal{H}_i$ N_i times, so that $dim\left(\mathcal{H}^{(N_i)}\right) = \prod_{j=1}^{N_i} d_j \equiv d^{N_i}$ is also finite. Higher dimensional Hilbert spaces in the sequence are built following the simple rule: $\mathcal{H}^{N_{i+1}} = \mathcal{H}^{N_i} \otimes \mathcal{H}_j$. In the basic Hilbert spaces $\mathcal{H}_j$ self adjoint operators can be represented by $d_j \times d_j$ hermitian matrices that can be decomposed in a chosen basis, therefore all the operators acting on the tensored higher dimensional Hilbert space $\mathcal{H}^{N_i}$ can be decomposed as:

$$\mathcal{O}^{(N_i)} = \sum_{i_1,\dots,i_{N_i}} \mathcal{O}^{(N_i)}_{i_1,\dots,i_{N_i}} \mathbf{e}^{i_1 \dots i_{N_i}}, \tag{2.12}$$

where $\mathbf{e}^{i_1 \cdots_{N_i}}$ is the basis obtained tensoring N_i times the chosen basis of the fundamental Hilbert space $\mathcal{H}_i$. The number of non-zero entries in (2.12) is denoted as $\mathbb{N}_e\left(\mathcal{O}^{(N_i)}\right)$. Considering a size sequence of operators $(\mathcal{O}^{(N_1)}, \mathcal{O}^{(N_2)} \dots)$ it is defined the *density character* of $\mathcal{O}^{(N_i)}$ as the "nature" (i.e. linear, polynomial, exponential, etc...) of the *minimal* function $f(N_i)$ such that $\mathbb{N}_e\left(\mathcal{O}^{(N_i)}\right) < f(N_i)$. At this point it is possible to introduce a size sequence of *Hermitian* operators $(H^{(N_1)}, H^{(N_2)}, H^{(N_3)} \dots)$ that can be interpreted as Hamiltonias acting on the respective Hilbert spaces. Since these operators are hermitian and finite-dimensional, automatically possess complete sets of conserved charges $\mathcal{Q}^{(N_i)}_m$ which can be arranged in such a way that several size sequences can be defined: $(Q^{(N_i)}_m, Q^{(N_{i+1})}_m, Q^{(N_{i+2})}_m \dots)$ for all $m = 1, \dots, m_{max} \leq d^{(N_i)}$.
After these preliminary concepts we are ready to report the definition.

Definition 3 (Caux & Mossel)**.** A Hamiltonian of density character $O(f(N))$ is *quantum integrable* if it is a member of a sequence $(H^{(N_1)}, H^{(N_2)}, H^{(N_3)} \dots)$ of operators, having $O(f(N))$ density character in a certain basis too, for which it is possible to define a sequence of sets of operators $(\{\mathcal{Q}^{(N_1)}\}, \{\mathcal{Q}^{(N_2)}\}, \{\mathcal{Q}^{(N_3)}\}, \dots)$ such that:

- all operators $\mathcal{Q}^{N_i}_m$ in $\{\mathcal{Q}^{(N_i)}\}$ commute with each other and with the relative hamiltonian $H^{(N_i)}$;
- the operators in $\{\mathcal{Q}^{(N_i)}\}$ are algebraically independent;
- the cardinality of the set $\{\mathcal{Q}^{(N_i)}\}$ becomes unbounded in the infinite size limit;
- each member of a set $\{\mathcal{Q}^{(N_i)}\}$ can be embedded within a sequence of operators $(Q^{(N_i)}_m, Q^{(N_{i+1})}_m, Q^{(N_{i+2})}_m \dots)$ with $O(f(N))$ in a given basis.

This definition is very formal and at first sight hard, anyway it fulfills all the required criteria; in particular it can be done a partition between different classes of integrable models according to their density character $\mathcal{O}(f(N))$.For all the details see [37]. Let us note in particular that, by definition 3, free discrete

models are *linear* quantum integrable in the sites basis (i.e. of density character $O(N)$ where N is the number of sites)[3].

[3]Let us note that in the continuum case this means that we have an infinite number of conserved charges, indeed any continuum theory can be obtained from a lattice regularization in which subset of conserved charges can be chosen $O(N)$.

CHAPTER 3

THE LIEB-LINIGER MODEL

In this chapter the model of our interest is introduced. We will show that it is solvable using the coordinate Bethe ansatz and we will outline how to find the general solution in the periodic case. Furthermore, the limit of "hard-core" bosons is discussed in detail and we show, in two different ways, that in this limit the model is solvable, since it can be mapped to a free fermion theory. Finally, the last part of the chapter is devoted to the discussion of some important results for a quench from non-interacting to strongly interacting bosons with PBC, without confining potential following [47].

3.1 The model

The Lieb-Liniger model is an example of interacting model and describes one dimensional bosons with a contact interaction. It was introduced and solved in 1963 [39] and it gives a good description for bosons in effective 1D lattices, especially when the limit of strong interaction is considered [40, sec. 5]. Exact solutions can be found both for attractive and repulsive interaction. The hamiltonian in first quantized notation for a system composed by N particles is:

$$\mathcal{H}_N = \sum_{i=1}^{N} -\frac{\partial^2}{\partial x_i^2} + 2c \sum_{j>i} \delta(x_i - x_j), \tag{3.1}$$

where we set $\hbar = 2m = 1$, the constant c represents the interaction strength and we will always consider the repulsive case (i.e. $c > 0$). The aim in this paragraph is to find the eigenfunctions $\chi_N(x_1 \dots x_N)$ of (3.1):

$$\mathcal{H}_N \chi_N(x_1 \dots x_N) = E_N(k_1 \dots k_N) \chi_N(x_1 \dots x_N). \tag{3.2}$$

Since we are dealing with a bosonic system we must impose the symmetry of the functions $\chi_N(x_1 \dots x_N)$ under the exchange of the coordinates, that is:

$$\chi_N(x_1 \dots x_i \dots x_j \dots x_N) = \chi_N(x_1 \dots x_j \dots x_i \dots x_N),$$

for every couple (i, j).
So far we have not discussed about boundary conditions, these are relevant since they can drastically change the wave function especially when working with finite N. Different boundary conditions can be implemented: we can study a one dimensional bosonic gas lying on a ring of circumference L, (this coincides with the imposition of periodic boundary conditions) or we can consider a box of length L, meaning that we are fixing the boundaries. The model can also be solved in the most general case of twisted boundary conditions. The effects of the boundaries are expected to vanish when the limit $L \to \infty$ is considered. Since our purpose is to give only an outline of the methods used in finding the exact solutions of (3.1), we will consider the easiest case of periodic boundary conditions.
The wave function has to be symmetric in the spatial coordinates, therefore we can consider only the following domain

$$\mathcal{D} : x_1 < x_2 < \cdots < x_N, \tag{3.3}$$

in $\mathcal{D}$ the (3.1) reduces to a free Hamiltonian

$$\mathcal{H}_N^0 = \sum_{i=1}^{N} -\frac{\partial^2}{\partial x_i^2}, \tag{3.4}$$

with the following constraint:

$$\left(\frac{\partial}{\partial x_{i+1}} - \frac{\partial}{\partial x_i} - c\right)\chi_N = 0, \qquad x_{i+1} = x_i, \tag{3.5}$$

imposed by the δ interaction. In order to understand why the constraint (3.5) works, let us show how it arises when we have only two particles.Tthe generalization for N bosons is straightforward. Introducing the variables

$$\begin{cases} z = x_2 - x_1 \\ Z = \frac{x_2 + x_1}{2} \end{cases}$$

the Hamiltonian (3.1) can be written as:

$$\mathcal{H}_2 = -2\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{2\partial Z^2} + 2c\delta(z), \tag{3.6}$$

if we now integrate equation (3.2) for the Hamiltonian (3.6) in the variable z over an interval of measure $\epsilon \ll 1$ (i.e. $-\frac{\epsilon}{2} < z < \frac{\epsilon}{2}$), we obtain:

$$\left[-2\frac{\partial}{\partial z}\chi_2\right]_{-\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} + \left(\frac{\partial^2}{\partial Z^2}\chi_2\right)\epsilon + 2c\chi_2 = O(\epsilon). \tag{3.7}$$

In the limit $\epsilon \to 0$ (3.7) becomes exactly (3.5):

$$\left(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1} - c\right)\chi_2 = 0, \qquad x_2 = x_1, \tag{3.8}$$

this means that, if the wave function $\chi_2(x_1, x_2)$ is an eigenfunction of the hamiltonian, the condition (3.8) has to be fullfiled.

Before considering the general solution of (3.1) with the boundary conditions (3.5), let us see what happens in the simplest possible case: $N = 2$. The wave function in this case can be written as:

$$\chi_2(x_1, x_2) = C_1 \mathrm{e}^{ik_1x_1+ik_2x_2} + C_2 \mathrm{e}^{ik_2x_1+ik_1x_2}, \tag{3.9}$$

where the coefficients C_1 and C_2 are determined by the constraint (3.8):

$$(ik_1 - ik_2 - c)C_1 = (ik_2 - ik_1 - c)C_2, \tag{3.10}$$

$$\frac{C_2}{C_1} = -\mathrm{e}^{i\phi(k_1-k_2)} \qquad \phi(k_1 - k_2) = -i \ln \frac{c + i(k_1 - k_2)}{c - i(k_1 - k_2)}, \tag{3.11}$$

the function $\phi(k_1 - k_2)$ can be thought as a *phase-shift* due to the contact interaction. The general symmetric solution is:

$$\begin{aligned} \chi_2(x_1, x_2, k_1, k_2) = \mathrm{sgn}(x_2 - x_1)[&\mathrm{e}^{ik_1x_1+ik_2x_2-i\mathrm{sgn}(x_2-x_1)\phi(k_1,k_2)/2} \\ &- \mathrm{e}^{ik_2x_1+ik_1x_2+i\mathrm{sgn}(x_2-x_1)\phi(k_1,k_2)/2}]. \end{aligned} \tag{3.12}$$

For a generic number of particles N we can proceed in the following way [1]. We make the *ansatz* that the solution can be written as a superposition of plane waves

$$\psi(x_1 \dots x_N) = \sum_{\mathcal{P}} A_{\mathcal{P}} \mathrm{e}^{ik_{\mathcal{P}_1}x_1+\dots+ik_{\mathcal{P}_N}x_N}, \tag{3.13}$$

where k_i are the (quasi)momenta, $\mathcal{P}$ is a permutation of the k_i and the sum is extended over all the permutations. All the possible permutations are done by exchanging two possible indices at a time, therefore the coefficients $A_{\mathcal{P}}$ can be found exploiting the results from the two particles case

$$\frac{A_{\mathcal{P}}}{A_{\mathcal{P}'}} = \frac{i(k_i - k_j) + c}{i(k_i - k_j) - c} = -\mathrm{e}^{i\phi(k_i-k_j)}, \tag{3.14}$$

where k_i and k_j are the momenta exchanged by the permutation. Therefore we obtain [39]:

$$A_{\mathcal{P}} = C(-1)^{\mathcal{P}} \prod_{j<i} (k_{\mathcal{P}_j} - k_{\mathcal{P}_i} + ic) \tag{3.15}$$

where C is a normalization constant. The wave functions that we found in this way are eigenfunctions of the Hamiltonian (3.4) with eigenvalue

$$E = \sum_{j=1}^{N} k_j^2. \tag{3.16}$$

They are also eigenfunctions of the momentum operator, with total momentum

$$P = \sum_{i=1}^{N} k_i. \tag{3.17}$$

[1] A part from the original reference [39], there are several reviews about the coordinate Bethe Ansatz techniques used in the solution of the Lieb-Liniger model, e.g. [41] or [42, chapt. 2]. The reference [41] is more complete.

So far we said nothing about the permitted values of the quasi-momenta k_i; these values must be found imposing the boundary conditions. As discussed formerly we will consider periodic boundary conditions (PBC). If the particles lying in a length L, PBC are imposed if for every $1 \leq j \leq N$ we have:

$$\psi(x_1 \dots x_j \dots x_N) = \psi(x_1 \dots x_j + L \dots x_N), \tag{3.18}$$

this effectively means that our system has the geometry of a ring. If a particle moves around the circle it will acquire a phase due to two different contributions. The first contribution will be the dynamical phase (i.e. e^{ik_jL}). The second contribution will come from the scattering with all the others particles around the ring, when PBC are considered. The sum of these contributions is a integer multiple of 2π [41]. More formally we can say that if we take two permutation of the momenta[2] such that the difference between the two($\mathcal{P}$ and $\mathcal{P}'$) is the shift of one index (i.e. $\mathcal{P}'_N = \mathcal{P}_{N-1}$ and $\mathcal{P}'_1 = \mathcal{P}_N \dots$), the following condition must hold

$$\frac{A_{\mathcal{P}'}}{A_{\mathcal{P}}} = \mathrm{e}^{ik_{\mathcal{P}_N}L}, \tag{3.19}$$

this is equivalent to:

$$\mathrm{e}^{ik_jL} = \prod_{j \neq i} \left(\frac{k_j - k_i + ic}{k_j - k_i - ic} \right), \qquad j = 1, \dots, N. \tag{3.20}$$

The (3.20) is a set of N equations for N variables (the quasi-momenta k_i) called *Bethe equations*. Although the number of equations is the same of the number of variables the solution is not unique since different solutions can be shifted by integer multiples of 2π. This fact is evident is we take the logarithm of (3.20):

$$k_jL = -i \sum_{i \neq j} i\pi + \sum_{i \neq j} -i \ln \left(\frac{c + i(k_i - k_j)}{c - i(k_i - k_j)} \right) + 2\pi \overline{I}_j, \tag{3.21}$$

that is :

$$k_jL = (N-1)\pi + \sum_{i \neq j} \phi(k_i - k_j) + 2\pi \overline{I}_j. \tag{3.22}$$

The numbers $\overline{I}_j$ are integers if N is odd, half-integers otherwise. From equation (3.22) it is clear that the sum of the dynamical phase and the phase acquired due to the interaction has to be a integer multiple of 2π. Let us note that the set of integers $\overline{I_j}$ are quantum numbers that identify the state of the system. It is not hard to convince ourselves that the ground state is the one in which the integers $\overline{I}_j$ are symmetrically distributed around the zero

$$\overline{I}_j = -\frac{N+1}{2} + j \qquad j = 1 \dots N. \tag{3.23}$$

This fact can be proved in different ways but we will omit the demonstration. It is important to remark that the Bethe solution of the Lieb Liniger model provides all the quasi-momenta to be distinct, indeed if we have $k_i = k_j$ for a given (i, j) the wave function vanishes. In particular, this last point shows that, also if we are considering bosons in real space, the contact interaction reveals a "fermionic nature" in momentum space.

[2]Different orderings of the particle can be expressed as different permutations of the momenta.

3.2 Tonks-Girardeau limit

The limit for $c \to \infty$ in the Hamiltonian (3.1) is usually referred as "Tonks-Girardeau" limit since it was solved for the first time by Girardeau [43] and, in its classical version by Tonks [44]. In this regime the Hamiltonian (3.1) describes a one-dimensional system of *strongly* interacting bosons. What we calculated so far for the generic Lieb-Liniger gas is still valid and the solution for the Girardeau gas could be found considering, with a certain care, the limit for $c \to \infty$ of the previous results. However we will derive the correct solution focusing on the physical meaning of the limit considered.
Imposing a strong repulsive contact interaction means that the probability for two particles to be "close" to each other is very low. For $c \to \infty$, when dealing with point like particles[3], this is equivalent to say that, given a many-body eigenfunction of the hamiltonian $\chi_N(x_1 \dots x_N)$, the following condition must be respected:

$$\chi_N(x_1 \dots x_i \dots x_j \dots x_N) = 0 \qquad \text{if} \qquad x_i = x_j, \tag{3.24}$$

for every couple (i, j). Equation (3.24) sounds like a *fermionic* requirement. Effectively in this regime, since it is not allowed multiple spatial occupation, the bosonic gas is physically *undistinguishable* from a free fermionic one, as will be clearer soon. The Hamiltonian for a Girardeau gas can thus be written as:

$$\mathcal{H}_{TG} = \sum_{i=1}^{N} -\frac{\partial^2}{\partial x_i^2}, \tag{3.25a}$$

$$\chi_N(x_1 \dots x_i \dots x_j \dots x_N) = 0 \qquad \text{if} \qquad x_i = x_j, \tag{3.25b}$$

let us note that relations (3.25) are, as expected, exactly the same as (3.4) with the constraint (3.5) in the limit $c \to \infty$. A possible solution to (3.25) is a determinant of a $N \times N$ matrix with entries $\mathrm{e}^{ik_i x_l}$ (i.e. a Slater determinant). However we must take into account the fact the we are still dealing with a *bosonic* system, therefore the wave function should be symmetric under the exchange of two spatial coordinates. Following [43] the wave function is:

$$\chi_N(x_1 \dots x_N | k_1 \dots k_N) = \frac{C}{\sqrt{N!}} \det[\exp\{ik_l x_j\}] \prod_{i<j} \mathrm{sgn}(x_l - x_j), \tag{3.26}$$

that is:

$$\chi_{N,G}(x_1, x_2, \dots, x_N) = \prod_{i<j} \mathrm{sgn}(x_i - x_j) \chi_{N,F}(x_1, x_2 \dots, x_N), \tag{3.27}$$

where $\chi_{N,F}(x_1, x_2 \dots, x_N)$ is a many-body eigenstate of a free fermionic hamiltonian. The allowed values for the momenta are found, as usual, imposing the boundary condition. Following what has been done for the general Lieb-Liniger model, (i.e. imposing PBC) we find

$$\mathrm{e}^{ik_j L} = (-1)^{N-1} \qquad \text{for} \qquad j = 1, \dots, N. \tag{3.28}$$

[3] A more realistic physical situation would prescribe to consider the particles as spheres of finite ray, this has been done classically and is fully described in [44].

These equations coincide with the limit $c \to \infty$ of the Bethe equations. Let us note that the energy eigenvalues are identical to the fermionic ones, indeed in this singular limit, all the possible observables are genuinely *fermionic*. Actually, in a more realistic situation, c is set such that it is larger than a certain energy scale $c^2 \gg E$, in this context the model would be purely bosonic if the momentum of the particles is $p \gg c$ (i.e. $c \approx 0$), purely fermionic otherwise.
Lastly, we must point out that in the Lieb-Liniger model the coupling constant has the dimensions of an inverse length therefore, except in the limits $c = 0$ and $c \to \infty$, there is a natural energy scale. Following [39] an adimensional non trivial parameter, $\gamma = c\rho^{-1}$ where $\rho = \frac{N}{L}$, can be introduced. The Tonks-Girardeau limit is recovered when

$$\gamma \gg 1, \tag{3.29}$$

that is for high values of the coupling constant or for extreme diluted systems.

3.2.1 Jordan-Wigner transformations

As we have seen in the last paragraph, there is a *one-to-one correspondence* between the Hilbert space of a free fermionic system and the Hilbert space of a strongly interacting bosonic one. This mapping can be implemented by mean of a Jordan-Wigner transformation. The procedure that will be showed for the Lieb-Liniger is valid quite in general; indeed this mapping to free-fermion theories is often used for the solution of several one-dimensional lattice models (e.g. strongly interacting Bose-Hubbard model or spin chains). In order to solve the Lieb-Liniger model we have to rewrite the Hamiltonian (3.1) in a "second quantization" notation

$$H = \int_0^L dx[\partial_x \hat{\phi}^\dagger(x)\partial_x \hat{\phi}(x) + c\hat{\phi}^\dagger(x)\hat{\phi}^\dagger(x)\hat{\phi}(x)\hat{\phi}(x)], \tag{3.30}$$

where, if we use periodic bound conditions, $\hat{\phi}(x) = \frac{1}{\sqrt{L}}\sum_p \mathrm{e}^{ipx}\hat{a}_p$. The $\hat{a}_p$ are bosonic operators, therefore their algebra is:

$$[a_p, a_q^\dagger] = \delta_{pq}, \tag{3.31}$$

where $a_p^\dagger$ creates a particle of momentum p and a_q annihilates a particle of momentum q. The allowed values for the momenta p are $2\pi k/L$ with k integer.
In the limit of strong interaction ($c \to \infty$) it is convenient to rewrite the hamiltonian in terms of the "hard-core" bosons operators $\hat{\Phi}(x)$, $\hat{\Phi}(x)^\dagger$. They are related to the free bosonic fields by a *non-linear* transformation:

$$\hat{\Phi}^\dagger(x) = P_x \hat{\phi}^\dagger(x) P_x, \tag{3.32}$$

in which P_x is the projector on the truncated Hilbert space with at the most one boson in x, $P = |0_x\rangle\langle 0_x| + |1_x\rangle\langle 1_x|$. The Tonks-Girardeu Hamiltonian can thus be written as a free hamiltonian in the hard-core fields

$$H = \int_0^L dx[\partial_x \hat{\Phi}^\dagger(x)\partial_x \hat{\Phi}(x)], \tag{3.33}$$

provided that they obey an "effective" Pauli principle implemented by an unusual algebra:

$$[\hat{\Phi}(x)]^2 = [\hat{\Phi}^\dagger(x)]^2 = 0, \qquad \{\hat{\Phi}(x), \hat{\Phi}^\dagger(x)\} = 1, \tag{3.34a}$$

$$[\hat{\Phi}(x), \hat{\Phi}(y)] = [\hat{\Phi}(x), \hat{\Phi}^\dagger(y)] = 0 \qquad x \neq y. \tag{3.34b}$$

Although (3.33) seems quite simple to solve, this is not the case due to the non trivial relations (3.34). This problem can be circumvented by introducing the Jordan-Wigner mapping

$$\hat{\Psi}(x) = \mathrm{e}^{i\pi \int_0^x dz \hat{\Phi}(x)^\dagger \hat{\Phi}(x)} \hat{\Phi}(x). \tag{3.35}$$

It is straightforward to show that (3.35) is a purely anti-commutating field:

$$\{\hat{\Psi}(y), \hat{\Psi}^\dagger(x)\} = \delta(x - y). \tag{3.36}$$

The fields defined through the Jordan-Wigner transformation are genuinely fermionic and the Hamiltonian (3.33) becomes therefore:

$$H = \int_0^L dx [\partial_x \hat{\Psi}^\dagger(x) \partial_x \hat{\Psi}(x)]. \tag{3.37}$$

If we use PBC (3.35) can be decomposed as

$$\hat{\Psi}(x) = \sum_k \mathrm{e}^{ikx} \eta_k \qquad \hat{\eta}_k = \int dx \ \mathrm{e}^{-ikx} \hat{\Psi}(x). \tag{3.38}$$

Plugging 3.38 in 3.37 we obtain:

$$H = \sum_{k=-\infty}^{\infty} k^2 \hat{\eta}_k^\dagger \hat{\eta}_k, \qquad \{\hat{\eta}_q^\dagger, \hat{\eta}_k\} = \delta_{q,k}. \tag{3.39}$$

We note that the Hamiltonian (3.39) is diagonal in the proper modes, confirming that the Tonks-Girardeau model has purely fermionic properties.
As we have already seen formerly the Lieb-Liniger is not the only model that (in a particular limit) can be mapped to a free fermionic theory by mean of a Jordan-Wigner transformation, another important example is the *Bose-Hubbard* model (1.14) in the limit of strong on-site interaction. When the limit $U \to \infty$ is considered, the only permitted values of the lattice site occupation are: $n_i = 0, 1$ (actually this exactly the discrete analogous of a Girardeau gas). The hamiltonian of the hard-core Bose-Hubbard model has been introduced formerly (1.20) together with the commutation relations for the fields b_i and $b_i^\dagger$. In this case the Jordan-Wigner transformations are defined as:

$$c_i = e^{i\pi \sum_{j<i} b_j^\dagger b_j} b_i = \prod_{j<i} (1 - 2 b_j^\dagger b_j) b_i, \tag{3.40a}$$

$$c_i^\dagger = b_i^\dagger e^{-i\pi \sum_{j<i} b_j^\dagger b_j} = b_i^\dagger \prod_{j<i} (1 - 2 b_j^\dagger b_j). \tag{3.40b}$$

Also in this case it is straightforward to show that c_i and $c_i^\dagger$ obey anti-commutation relations

$$\{c_i, c_j^\dagger\} = \delta_{i,j}. \tag{3.41}$$

Using (3.40) the Hamiltonian (1.20) becomes:

$$H = -J \sum_i (c_i^\dagger c_{i+i} + h.c.). \tag{3.42}$$

This expression is very similar to (1.16), indeed it is exactly the Hamiltonian of a system of non-relativistic *spinless* fermions hopping on a lattice without on-site interaction.
Another important example of discrete hamiltonian that can be diagonalized using Jordan-Wigner transformation is the *XX-model*

$$H_{XX} = -\frac{1}{2} \sum_{j=1}^{L} \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y - 2h\sigma_j^z \tag{3.43}$$

where $\sigma_j^{x,y,z}$ are the Pauli matrices on lattice site j and h is a magnetic field. In this case the JW is defined as

$$c_i = \left(\prod_{j<i} \sigma_j^z \right) \frac{\sigma_i^x + i\sigma_i^y}{2}. \tag{3.44}$$

After the transformation the hamiltonian can be written as:

$$H_{XX} = -\sum_{i=1}^{L} (c_i^\dagger c_{i+i} + h.c.) - 2h \left(c_j^\dagger c_j - \frac{1}{2} \right). \tag{3.45}$$

Let us note that if we set $h = 0$ (3.45) is equal to (3.42). This demonstrates that also very different models can be resolved using the same physical picture of *free fermionic* theories thanks to the Jordan-Wigner transformation.

3.3 Quantum quench from free to hard-core boson in a periodic 1-D gas

In this paragraph some results from [47] are reported. In the paper a global quantum quench, with the same protocol that we used in this thesis, is studied. It is considered a 1D system composed by N particles lying over a ring of length L prepared in the ground state of the Lieb-Liniger Hamiltonian with $c = 0$. The interaction parameter is suddenly quenched in such a way that the time evolution is governed by a Tonks-Girardeau Hamiltonian ((3.1) with $c \to \infty$). All the analytical results are given in the thermodynamical limit with $n = \frac{N}{L}$ constant. The particularity of this quench protocol is that there is a *non-linear* relation (Eq. (3.32)) between the pre- and post-quench operators. In [47] the attention is focused both on the dynamical and on the stationary properties of the two-point and of the density-density correlation function. One of the most important thing to remark is that in the hardcore limit there is a *linear* relation

between the local integrals of motion $\mathcal{I}_m$ and the fermionic occupation modes $\hat{n}_j$, although these are typically non-local. This means that all the information about the stationary state is encoded in the fermionic two point correlation function. The existence of such linear relation is important because it makes possible the construction of the GGE avoiding the divergences that arise in the local integrals of motion when the continuum limit is considered, making necessary a proper *regularization* (for the details [45]). In order to demonstrate this last point let us follow what done in [9]. For free fermionic fields the local charges can be written in the continuum case as:

$$\hat{I}_j = \int dx\ \hat{\Psi}^\dagger(x)(-i)^j \frac{\partial^j}{\partial x^j}\Psi\hat{(}x). \tag{3.46}$$

It is not hard to show that (3.46) is equivalent to:

$$\hat{I}_j = \sum_k k^j \hat{n}_k, \tag{3.47}$$

Indeed the relation (3.46) can be written as:

$$\begin{aligned}
\hat{I}_j = &\int\int dxdy(-i)^j \hat{\Psi}^\dagger(x)\delta^{(j)}(x-y)\hat{\Psi}(y) = \\
&\int\int dxdy\hat{\Psi}^\dagger(x)\left[\sum_k \frac{1}{L}k^j \mathrm{e}^{ik(x-y)}\right]\hat{\Psi}(y) = \\
&\frac{1}{L}\sum_k \int\int dxdy\hat{\Psi}^\dagger(x)k^j \mathrm{e}^{ik(x-y)}\hat{\Psi}(y) = \\
&\sum_k k^j \hat{n}_k,
\end{aligned} \tag{3.48}$$

where we used the decomposition (3.38) and fact that for finite values of L the momentum is discrete. Therefore, there some coefficients γ_k exist such that:

$$\sum_k \gamma_k \hat{n}_k = \sum_m \lambda_m \hat{I}_m. \tag{3.49}$$

The fact that the GGE properties of the system are fully described only by the occupation fermionic number is a clear indication that in the infinite time limit the Wick's theorem holds in its usual form and therefore all the multipoint function can be expressed as combination of the two-point fermionic function. The fermionic correlation $\langle\hat{\Psi}^\dagger(x)\hat{\Psi}(y)\rangle$ is found to be time-independent and its expression in the thermodynamical limit is

$$C_F(x,y) = n\mathrm{e}^{-2n|x-y|}. \tag{3.50}$$

If we set $x = y$ we find the spatial density $n(x) = n$. Therefore if we prepare the system in a translationally invariant state the two-point function does not evolve, this is one of the most remarkable differences with the confined case.
The calculation of the density-density correlation function is less trivial. One astonishing result in this case is that for finite times the Wick's theorem does not apply and therefore the dynamical multipoint functions have to be calculated

esplicitly.
The density operator is defined as

$$\hat{\rho}(x,t) = \hat{\Psi}^\dagger(x,t)\hat{\Psi}(x,t), \tag{3.51}$$

let us note that the fermionic density coincides with the hard-core bosonic one and for this reason the density-density correlation function is experimentally relevant. The density-density function is defined in general as:

$$\begin{aligned}\langle\hat{\rho}(x_1,t_1),\hat{\rho}(x_2,t_2)\rangle = \frac{1}{L^2}\sum_{k_1,k_2,k_3,k_4} & \mathrm{e}^{-i(k_1-k_2)x_1-i(k_3-k_4)x_2}\mathrm{e}^{i(k_1^2-k_2^2)t_1}\mathrm{e}^{i(k_3^2-k_4^2)t_2} \\ & \times\langle\psi_0|\hat{\eta}^\dagger_{k_1}\hat{\eta}_{k_2}\hat{\eta}^\dagger_{k_3}\hat{\eta}_{k_4}|\psi_0\rangle,\end{aligned} \tag{3.52}$$

where, again, the decomposition (3.38) has been used for the fermionic field. The four points correlation in the proper modes can not be found using the standard prescriptions of the Wick's theorem because the initial state is definided as

$$|\psi_0\rangle = \frac{(a_0^\dagger)^N}{\sqrt{N!}}|0\rangle \qquad \text{with} \qquad [a_p^\dagger, a_q] = \delta_{p,q}, \tag{3.53}$$

and the relation between the fermionic operators η_k and the pure bosonic ones a_p is non trivial.[4] The only way to proceed is going back to the real space

$$\begin{aligned}\langle\psi_0|\hat{\eta}^\dagger_{k_1}\hat{\eta}_{k_2}\hat{\eta}^\dagger_{k_3}\hat{\eta}_{k_4}|\psi_0\rangle = \frac{1}{L^2}\int_0^L & dz_1dz_2dz_3dz_4\mathrm{e}^{i(k_1z_1-k_2z_2+k_3z_3-k_4z_4)} \\ & \cdot\langle\psi_0|\hat{\Psi}^\dagger(z_1)\hat{\Psi}(z_2)\hat{\Psi}^\dagger(z_3)\hat{\Psi}(z_4)|\psi_0\rangle.\end{aligned} \tag{3.54}$$

In order to calculate the four-points function (3.54) all the possible permutations of the spatial indices must be taken in account. Moreover one has to consider also the fact that two or more fermionic operators can act in the same spatial point. Furthermore, there are minus signs arising from the ordering of the fermionic operators and extra minus signs coming from the commutation rules of the Jordan-Wigner string, see e.g. appendix of [46]. Then the four-points function is given by:

$$\begin{aligned}\langle\psi_0|\hat{\Psi}^\dagger(z_1)\hat{\Psi}(z_2)\hat{\Psi}^\dagger(z_3)\hat{\Psi}(z_4)|\psi_0\rangle = & \,\delta(z_2-z_3)n\mathrm{e}^{-2n|z_4-z_1|} \\ & + \sum_{\mathcal{P}}\theta(z_{\mathcal{P}})\sigma_{\mathcal{P}}n^2\mathrm{e}^{-2n(z_{\mathcal{P}_4}-z_{\mathcal{P}_3}+z_{\mathcal{P}_2}-z_{\mathcal{P}_1})},\end{aligned} \tag{3.55}$$

where $\theta(z_{\mathcal{P}}) = \theta(z_{\mathcal{P}_4}-z_{\mathcal{P}_3})\theta(z_{\mathcal{P}_3}-z_{\mathcal{P}_2})\theta(z_{\mathcal{P}_2}-z_{\mathcal{P}_1})$. The sum is intended to be done over all possible permutations and $\sigma_{\mathcal{P}}$ is the sign of the permutation. Plugging this expression in (3.52) one obtains

$$\begin{aligned}\langle\psi_0|\hat{\eta}^\dagger_{k_1}\hat{\eta}_{k_2}\hat{\eta}^\dagger_{k_3}\hat{\eta}_{k_4}|\psi_0\rangle = & \,n(k_1)\delta_{k_2,k_3}\delta_{k_1,k_2} + (\delta_{k_1,k_2}\delta_{k_3,k_4}-\delta_{k_2,k_3}\delta_{k_1,k_4})n(k_1)n(k_3) \\ & + \delta_{k_1,-k_3}\delta_{k_2,-k_4}\frac{k_1k_2}{4n^2}n(k_1)n(k_2).\end{aligned} \tag{3.56}$$

[4] Namely, it is exactly the same that holds for the fields (3.32), i.e. the multi-occupied spatial states have to be drop out.

Notice that, if the Wick's theorem held, only the *first* line of (3.56) would have been obtained. By using (3.56) the density-density correlation can be explicitly found. Let us set $\Delta x = x_2 - x_1$ and $\Delta t = t_2 - t_1$ then

$$\begin{aligned}\langle\hat{\rho}(x,t),\hat{\rho}(x+\Delta x,t+\Delta t)\rangle = \frac{1+i\mathrm{sgn}(\Delta t)}{2\sqrt{2\pi|\Delta t|}}\mathrm{e}^{-i\frac{\Delta x^2}{4\Delta t}}\int\frac{dk}{2\pi}\mathrm{e}^{ik\Delta x-ik^2\Delta t}n(k)\\ + n^2 - \left|\int\frac{dk}{2\pi}\mathrm{e}^{ik\Delta x-ik^2\Delta t}n(k)\right|^2\\ + \left|\frac{1}{2n}\int\frac{dk}{2\pi}\mathrm{e}^{ik\Delta x-ik^2(\Delta t+2t)}kn(k)\right|.\end{aligned} \tag{3.57}$$

The anomalous term in the second line of (3.56) originates the last term of (3.57), which is the only one that depends on the time after the quench. It easy to convince that when $t \to \infty$ it vanishes because the phase of the integrand is highly oscillating, therefore, in the stationary state, all the contributions come from the terms in the first line of (3.56). This means that in the GGE the four-point functions can be obtained using the Wick's theorem.

CHAPTER 4

QUANTUM QUENCH IN A CONFINED 1-D GAS

In this chapter we analyze the stationary properties and the non-equilibrium dynamics of a confined strongly interacting one dimensional Bose gas. The situation in this case is slightly different from the one studied in [47]. Indeed we consider a confined system and see how the trap influences the dynamics and the stationary state reached after the quench. This configuration is potentially interesting from an experimental point of view (see paragraph 1.2). We study the simplest confined situation that can be implemented: the potential well trap described by the Hamiltonian

$$\hat{H}(x) = \sum_{i=1}^{N} -\frac{\partial^2}{\partial x_i^2} + V, \tag{4.1}$$

where

$$\begin{cases} V = 0 & \text{if } 0 < x < L \\ V \to \infty, & \text{otherwise} \end{cases}. \tag{4.2}$$

We set $\hbar = 2m = 1$ and L represents the extension of the trap. At the time $t = 0^-$ our system is composed by N non-interacting spin-less bosons. In order to simplify the problem we will consider the thermodynamical limit (TDL), that is $N \to \infty$ and $L \to \infty$ maintaining $n = \frac{N}{L}$ fixed. The n in our case has to be intended as an average density, indeed the initial state is inhomogeneous and, in order to preserve this inhomogeneity, we must pay attention in performing the TDL.
Let us find the single-particle wave functions of the bosons, that, since there is no interaction, are the fundamental bricks whit whom we will build everything. We must solve:

$$\begin{cases} \frac{\partial^2}{\partial x^2}\varphi(x) = -E\varphi(x) \\ \varphi(0) = \varphi(L) = 0. \end{cases} \tag{4.3}$$

The solution of (4.3) is of the form

$$\varphi(x) = A \sin(\sqrt{E}x + \delta), \tag{4.4}$$

the constant A is fixed requiring the normalization of the wave function in the interval $[0, L]$ and the constant δ is given by the boundary conditions. The correct normalized wave functions are

$$\begin{cases} \varphi_j(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{j\pi}{L}x\right) \\ j \in \mathbb{N} \end{cases} . \tag{4.5}$$

We prepare our system in the ground state, that is the one with $j = 1$. The initial density is given by

$$n_0(x) = \frac{2N}{L} \sin^2\left(\frac{\pi}{L}x\right). \tag{4.6}$$

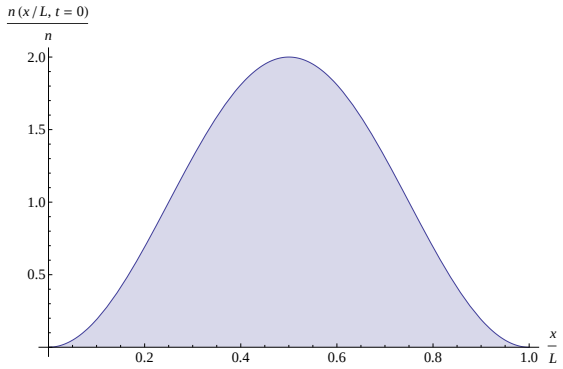

Figure 4.1: Initial density

4.1 The quench protocol

Following what done in chapter 3, it is convenient to introduce fields and describe everything in second quantization, this a typical approach in many-body problems. Let us define

$$\hat{\phi}(x) = \sum_{j=1}^{\infty} \varphi_j(x) a_j \qquad \hat{\phi}^\dagger(x) = \sum_{j=1}^{\infty} \varphi_j(x) a_j^\dagger, \tag{4.7}$$

where the $\varphi_j(x)$ are the same of (4.5) and the $a_j^\dagger$ and a_j are commutating raising and lowering operators, so their algebra is

$$[a_j, a_k^\dagger] = \delta_{j,k}. \tag{4.8}$$

The initial hamiltonian can thus be written as

$$H = \int_0^L dx (\partial_x \hat{\phi}^\dagger(x) \partial_x \hat{\phi}(x) + V \hat{\phi}^\dagger(x) \hat{\phi}(x) + c \hat{\phi}^\dagger(x) \hat{\phi}^\dagger(x) \hat{\phi}(x) \hat{\phi}(x)), \tag{4.9}$$

with $c = 0$.
As in the [47], at time $t = 0$ we suddenly turn on an infinitely strong contact interaction and let the system evolve, in other words we consider as evolving hamiltonian the one with $c \to \infty$. In this limit (Tonks Girardeau limit paragraph 3.2)

bosons behave as they were impenetrable. Since there is an interaction we cannot quantize the fields naively as done in (4.7), again, we can define "hard-core" bosonic fields that obey a "hybrid" algebra reflecting their fermionic behavior

$$[\hat{\Phi}(x), \hat{\Phi}^\dagger(y)] = 0, \qquad \{\hat{\Phi}(x), \hat{\Phi}^\dagger(x)\} = 1, \qquad [\hat{\Phi}^\dagger(x)]^2 = [\hat{\Phi}(x)]^2 = 0. \tag{4.10}$$

The relation between these fields and the free bosonic ones is non-trivial and can be written as:

$$\hat{\Phi}^\dagger(x) = P_x \hat{\phi}^\dagger(x) P_x, \tag{4.11}$$

where $P_x = |0_x\rangle\langle 0_x| + |1_x\rangle\langle 1_x|$ is a local projector.
It is possible to rewrite the post quench hamiltonian in terms of the hard-core bosons fields:

$$H = \int_0^L dx (\partial_x \hat{\Phi}^\dagger(x) \partial_x \hat{\Phi}(x)), \tag{4.12}$$

The crucial passage at this point is to use the Jordan-Wigner transformations, introduced in the last chapter, that permit to rewrite the strong interacting bosonic Hamiltonian as a free fermionic one, in the same spirit of what done in [47]

$$\hat{\Psi}(x) = \exp\left\{ i\pi \int_0^x dz \hat{\Phi}^\dagger(z)\hat{\Phi}(z) \right\} \hat{\Phi}(x), \tag{4.13a}$$

$$\hat{\Psi}^\dagger(x) = \hat{\Phi}^\dagger(x) \exp\left\{ -i\pi \int_0^x dz \hat{\Phi}^\dagger(z)\hat{\Phi}(z) \right\}. \tag{4.13b}$$

The fermionic fields are defined as

$$\hat{\Psi}(x) = \sum_{p=1}^{\infty} \varphi_p(x) \xi_p \qquad \xi_p = \int_0^L dx \varphi_p(x) \hat{\Psi}(x) \tag{4.14}$$

where ξ_p are anti-commutating fermionic operators (i.e. $\{\xi_p, \xi_q^\dagger\} = \delta_{p,q}$) and the $\varphi_p(x)$ are the single-particle solutions of the Schroedinger equation found in (4.5). In terms of these fermionic fields the hamiltonian can be written as

$$H = \int_0^L dx (\partial_x \hat{\Psi}^\dagger(x) \partial_x \hat{\Psi}(x)) \qquad \text{or} \qquad H = \sum_k k^2 \hat{\xi}_k^\dagger \hat{\xi}_k. \tag{4.15}$$

Therefore the evolution of our strong interacting bosonic system can be studied from that of a free theory.

4.2 Generalized Gibbs Ensemble results

In this section we want to study the stationary properties of the density and of the fermionic correlation function. As stated in paragraph 1.3.1, the Generalized Gibbs Ensemble is constructed considering only the *local* integrals of motion. However, as we have seen, for free fermionic theories a linear relation between the local conserved charges and the momentum occupation number exists (see paragraph 3.3). Therefore the GGE can be written as:

$$\rho_{GGE} = Z^{-1} \exp\left(-\sum_k \lambda_k \hat{n}_k \right), \tag{4.16}$$

where $Z = \prod_k \left[1 + \mathrm{e}^{-\lambda_k}\right]$ is the partition function. The coefficients λ_k are Lagrange multipliers that must be fixed imposing:

$$\mathrm{Tr}\left(\rho_{GGE}\hat{n}_k\right) = n_k. \tag{4.17}$$

Since, by definition, $Z = \mathrm{Tr}\left[\exp\left(-\sum_k \lambda_k \hat{n}_k\right)\right]$, equation (4.17) can be rewritten as

$$-\frac{\partial}{\partial\lambda_k}\ln(Z_{GGE}) = \frac{1}{\mathrm{e}^{\lambda_k}+1} = n_k. \tag{4.18}$$

Equation (4.18) suggests that all the information about the equilibrium state is in the fermionic occupation number. Then the characterization of the system in the infinite time limit can be done by considering only the diagonal terms in the time evolution of operators. In this sense it is clear that the GGE describes time averaged quantities that must coincide with the stationary ones.
In order to understand this last statement, let us consider the formal expression for the time evolved fermionic correlation function:

$$\left\langle \Psi^\dagger(y)\Psi(x)\right\rangle_t = \left\langle \mathrm{e}^{i\hat{H}t}\Psi^\dagger(y)\Psi(x)\mathrm{e}^{-i\hat{H}t}\right\rangle. \tag{4.19}$$

This expression can be rewritten in a clearer way, using the definition of fields given in (4.14), as

$$\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\varphi_q(y)\varphi_p(x)\left\langle\ \mathrm{e}^{i\hat{H}t}\hat{\xi}_q^\dagger\hat{\xi}_p\mathrm{e}^{-i\hat{H}t}\right\rangle, \tag{4.20}$$

and, since the fermionic hamiltonian is diagonal in the proper modes ξ_p, equation (4.20) becomes:

$$\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\varphi_q(y)\varphi_p(x)\mathrm{e}^{i(\epsilon_q-\epsilon_p)t}\left\langle\ \hat{\xi_q}^\dagger\hat{\xi_p}\right\rangle, \tag{4.21}$$

where, $\epsilon_p = \left(\frac{p\pi}{L}\right)^2$.
Let us now calculate the time average of the expression (4.21) over an interval $[0,T]$

$$\begin{aligned}&\frac{1}{2T}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\varphi_q(y)\varphi_p(x)\int_0^{+T}\ dt\ \mathrm{e}^{i(\epsilon_q-\epsilon_p)t}\left\langle\ \hat{\xi_q}^\dagger\hat{\xi_p}\right\rangle = \\ &\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\varphi_q(y)\varphi_p(x)\frac{\mathrm{e}^{i(\epsilon_q-\epsilon_p)T}-1}{i(\epsilon_q-\epsilon_p)T}\left\langle\hat{\xi_q}^\dagger\hat{\xi_p}\right\rangle,\end{aligned} \tag{4.22}$$

if we now consider the limit for $T\to\infty$ we obtain

$$\lim_{T\to\infty}\frac{\mathrm{e}^{i(\epsilon_q-\epsilon_p)T}-1}{i(\epsilon_q-\epsilon_p)T} = \delta_{pq}. \tag{4.23}$$

Therefore, plugging in (4.22) we have

$$\overline{\langle\Psi^\dagger(y)\Psi(x)\rangle} \equiv \left\langle\Psi^\dagger(y)\Psi(x)\right\rangle_{GGE} = \sum_{q=1}^{\infty}\varphi_q(y)\varphi_q(x)\left\langle\ \hat{\xi_q}^\dagger\hat{\xi_q}\right\rangle. \tag{4.24}$$

4.2.1 Initial fermionic correlation function

The first quantity we calculate is the initial correlation function. This is a fundamental step that will enable us in finding both the dynamical and the stationary properties. Using Jordan Wigner transformations defined in (4.13) and the anti-commutation relations, it is straightforward to show that [47]

$$\left\langle \Psi^\dagger(x)\Psi(y)\right\rangle_{t=0} = \sum_{j=0}^{\infty}\frac{(-2)^j}{j!}\int_x^y dz_1\ldots\int_x^y dz_j\left\langle\, \Phi^\dagger(x)\Phi^\dagger(z_1)\ldots\Phi^\dagger(z_j)\Phi(z_j)\ldots\Phi(z_1)\Phi(y)\right\rangle. \tag{4.25}$$

At this point we make a fundamental hypothesis. We assume that the expectation value of the multi-point function of hard-core and *real* bosonic fields is the same, this means

$$\left\langle\, \phi^\dagger(x)\phi^\dagger(z_1)\ldots\phi(z_1)\phi(y)\right\rangle = \left\langle\, \Phi^\dagger(x)\Phi^\dagger(z_1)\ldots\Phi(z_1)\Phi(y)\right\rangle. \tag{4.26}$$

As will be showed in the appendix A.1 using a rigorous lattice regularization this approximation makes completely sense. In fact this assumption is physically reasonable, because as long as we consider a continuum of sites we cannot (more correctly it is highly improbable...) have multiple occupied sites especially with finite N.
By hypothesis the initial state $|N\rangle$ is the ground state, therefore it can be written in terms of the raising operators as

$$|N\rangle = \frac{a_1^{\dagger N}}{\sqrt{N!}}|0\rangle. \tag{4.27}$$

The $\phi_q(x)$ are real bosonic fields, then from (4.7) and (4.27) we have

$$\begin{aligned}
&\langle\phi^\dagger(x)\phi^\dagger(z_1)\ldots\phi^\dagger(z_j)\phi(z_j)\ldots\phi(z_1)\phi(y)\rangle = \\
&\sum_{\{p\}}\varphi^*_{p_1}(x)\varphi^*_{p_2}(z_1)\ldots\varphi_{p_{2j+1}}(z_1)\varphi_{p_{2j+2}}(y)\langle N|a^\dagger_{p_1}a^\dagger_{p_2}\ldots a_{p_{2j+1}}a_{p_{2j+2}}|N\rangle = \\
&\sum_{\{p\}}\left(\prod_{i=1}^{2j+1}\delta_{p_i,1}\right)\varphi^*_{p_1}(x)\varphi^*_{p_2}(z_1)\ldots\varphi_{p_{2j+1}}(z_1)\varphi_{p_{2j+2}}(y)\langle N|a^\dagger_{p_1}\ldots a_{p_{2j+2}}|N\rangle = \\
&\prod_{i=1}^{j}|\varphi_1(z_i)|^2\varphi_1(x)^*\varphi_1(y)\,\langle N|\,a_1^{\dagger j+1}a_1^{j+1}\,|N\rangle.
\end{aligned} \tag{4.28}$$

Between the second and the third line of (4.28) we used that

$$a_q\,|N\rangle = \delta_{q1}\sqrt{N}\,|N-1\rangle. \tag{4.29}$$

Using (4.29) it is also straightforward to see that

$$\langle N|\,a_1^{\dagger j+1}a_1^{j+1}\,|N\rangle = \frac{N!}{(N-j-1)!}. \tag{4.30}$$

Equation (4.25) thus can be rewritten in terms of the wave function of the ground state

$$\left\langle \Psi^\dagger(x)\Psi(y)\right\rangle_{t=0} = \sum_{j=0}^{\infty} \frac{(-2)^j}{j!}\varphi_1(x)\varphi_1(y)\frac{N!}{(N-j-1)!}\left(\int_x^y dz \frac{2}{L}\sin^2\left(\frac{\pi}{L}z\right)\right)^j, \tag{4.31}$$

which is valid only if $y > x$, otherwise we must flip the integration extremes. This point can be understood observing how is defined the Jordan-Wigner transformation (4.13). Taking into account this fact the result of the integral in (4.31) is:

$$\frac{|y-x|}{L} + \frac{1}{2\pi}\text{sign}(y-x)\left(\sin\left(\frac{2\pi x}{L}\right) - \sin\left(\frac{2\pi y}{L}\right)\right). \tag{4.32}$$

Equation (4.31) can be rewritten in a more compact way, using the binomial formula and the prosthaphaeresis relations, as

$$\begin{aligned}\left\langle \Psi^\dagger(x)\Psi(y)\right\rangle_{t=0} = &\frac{2N}{L}\sin\left(\frac{\pi}{L}x\right)\sin\left(\frac{\pi}{L}y\right) \\ &\times\left[1 - 2\left(\frac{|y-x|}{L} + \frac{1}{\pi}\text{sign}(y-x)\cos\left(\frac{\pi}{L}(x+y)\right)\sin\left(\frac{\pi}{L}(x-y)\right)\right)\right]^{N-1}.\end{aligned} \tag{4.33}$$

4.2.2 Some general features about quantum-quenches in trap

The relation obtained formerly is valid for finite values of N and L. As already stated, we would like to find an intelligent (and of course correct...) way to consider the thermodynamic limit of (4.33). In the periodic case analyzed in [47] this passage was quite straightforward. The importance of considering the TDL relied on the fact that the initial fermionic correlation function simplified considerably and this fact permitted to calculate both stationary and dynamic quantities. In the confined case the situation is quite different and unfortunately much more complicated. The complication arises when we deal with the typical wave-functions of a confined problem. Let us consider a normalized single-particle wave function of a confined system. In order to fix the ideas we can initially take a function defined over a symmetric compact domain $\mathcal{D} = \left[-\frac{L}{2}, \frac{L}{2}\right]$, where L is the length of the trap[1]

$$\lambda_p(x) \qquad \text{with} \qquad \int_{-\frac{1}{2}}^{+\frac{1}{2}} dx\ ||\lambda_p(x)||^2 = 1. \tag{4.34}$$

In the previous relation we reabsorbed a factor L such that the integration domain becomes $\mathcal{D}' = \left[-\frac{1}{2}, \frac{1}{2}\right]$. Using $\lambda_p(x)$ we can define a *monotonically increasing* function

$$\Lambda_p(x) = \int_0^x dz\ ||\lambda_p(z)||^2 \qquad x \in \mathcal{D}'. \tag{4.35}$$

[1] Although we are dealing with a compact domain, this argument is general and is valid for problems defined in on the whole real axis as, for example, the harmonic trap, where $\frac{1}{\sqrt{\omega}} = L$ is the typical decay length.

The initial fermionic correlation function has the same structure of (4.33):

$$\left\langle \Psi^{\dagger}(x)\Psi(y)\right\rangle_{t=0} = N\lambda_0^*(x)\,\lambda_0(y)\,[1-2\,|\Lambda_0(y)-\Lambda_0(x)|]^{N-1}\,. \tag{4.36}$$

At this point we should take the limit $N\to\infty$, $L\to\infty$ Keeping $\frac{N}{L}=n$ constant. In the limit $N\to\infty$:

$$\begin{cases} N\lambda_0^*(x)\,\lambda_0(y)\,[1-2\,|\Lambda_0(y)-\Lambda_0(x)|]^{N-1}\to 0 & \text{if} \quad x\neq y \\ N\lambda_0^*(x)\,\lambda_0(y)\,[1-2\,|\Lambda_0(y)-\Lambda_0(x)|]^{N-1}\to\infty & \text{otherwise} \end{cases}. \tag{4.37}$$

Let us set:

$$F(x,y,N) = N\lambda_0^*(x)\,\lambda_0(y)\,[1-2\,|\Lambda_0(y)-\Lambda_0(x)|]^{N-1}$$

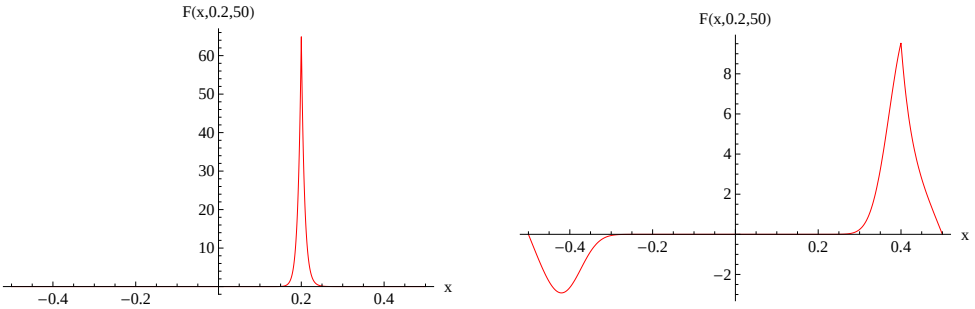

Figure 4.2: $F(x,y,N)$ in the case of a symmetric trap (i.e. $\lambda_0(x)=\sqrt{2}\cos\pi x$) *Left*: In this figure we set $y=0.2$, we are quite far from the boundary and the function is "well-behaved", in the limit $N\to\infty$ it is a Dirac delta. *Right*: In this case $y=0.4$ we are close to the point in which $\partial_x\Lambda_p(x)=0$. As we can see near the boundaries the delta approximation is not so good, in order to "see" a delta function for points close but still different from the nodes of the wave function, we have to consider very high values of N.

It easy to test numerically that, fixed the value of a variable, for large N, we have:

$$I(y,N) = \int_{x\in\mathcal{D}'} dx\; F(x,y,N) = 1. \tag{4.38}$$

Therefore we can say that:

$$\lim_{N\to\infty} F(x,y,N) = \delta(x-y). \tag{4.39}$$

Actually the situation is not so smooth. Relation (4.39) is valid only for that points in which $\partial_x\Lambda_p(x)\neq 0$ (that is $\lambda_p(x)\neq 0\ldots$), otherwise the limit is identically zero. We can say, therefore, that $F(x,y,N)$ in the limit $N\to\infty$ is pointwise convergent to a Dirac delta function, except in the null-measure set represented by the points in which the wave function is zero (i.e. in the "nodes" of the wave function). Note that, as long as our initial state is the ground state of a certain system, there are no nodes in the wave function (except at the boundaries...). If the limit (4.39) holds we have then

$$\lim_{N\to\infty}\left\langle \Psi^{\dagger}(x)\Psi(y)\right\rangle_{t=0} = \delta(x-y). \tag{4.40}$$

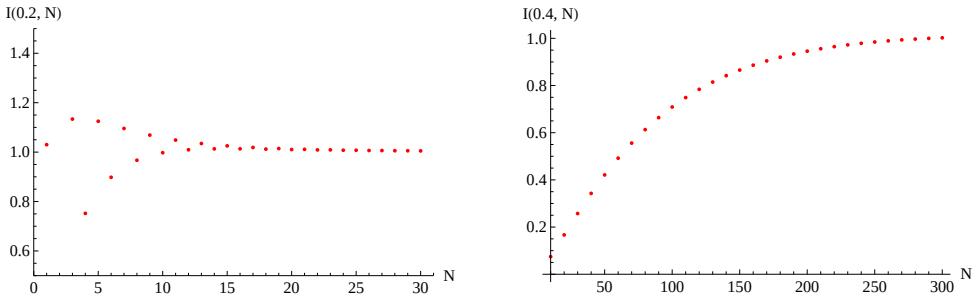

Figure 4.3: Integral of the same $F(x, y, N)$ of figure 4.2 as a function of N. *Left*: we set $y = 0.2$, being far from the boundaries the convergence of the integral is fast. *Right*: in this case we set $y = 0.4$, the convergence is much slower than the former case.

This expression is very simple but it is incorrect. In fact if we calculated the fermionic occupation number (see next paragraph...) using (4.40) as initial correlation function, we would have a constant occupation, this does not make sense.
The thermodynamic limit of (4.36) can also be performed in another way following what done in [47]. Let us preliminary point out that the wave functions describing a confined system have all the same structure:

$$\phi_p(x) = \frac{1}{\sqrt{L}}\psi_p\left(\frac{x}{L}\right), \tag{4.41}$$

where L is the typical confining length. Using this property we can rewrite the initial correlation as:

$$\left\langle \Psi^\dagger(x)\Psi(y)\right\rangle_{t=0} = n\psi_0^*\left(\frac{x}{L}\right)\psi_0\left(\frac{y}{L}\right)\left[1 - \frac{2n}{N}\left|\int_x^y dz\ ||\psi(z/L)||^2\right|\right]^{N-1}, \tag{4.42}$$

and since:

$$\lim_{N\to\infty}\left(1 + \frac{\alpha}{N}\right)^N = \mathrm{e}^\alpha,$$

equation (4.42) becomes:

$$\left\langle \Psi^\dagger(x)\Psi(y)\right\rangle_{t=0} = n\psi_0^*\left(\frac{x}{L}\right)\psi_0\left(\frac{y}{L}\right)\mathrm{e}^{-2n\left|\int_x^y dz\ ||\psi(z/L)||^2\right|}. \tag{4.43}$$

Also this expression gives an unphysical result for the fermionic occupation number, unless we expand the wave functions near the center of the distribution. In that case we would obtain

$$\left\langle \Psi^\dagger(x)\Psi(y)\right\rangle_{t=0} = Cn\mathrm{e}^{-2Cn|x-y|}, \tag{4.44}$$

with C constant depending on the initial distribution (for our problem $C = 2$). This expression is identical to that obtained in the homogeneous case meaning that we are simply expanding on the top of the initial distribution losing completely the effect of the trap. The only way in which we can proceed is to not consider the thermodynamic limit soon but maintain N and L finite as much as possible, but this will complicate all the calculations.

4.2.3 Fermionic mode occupation

In order to obtain the GGE results for the spatial density and for the correlation function we must calculate the fermionic mode occupation number that, since the hamiltonian is diagonal in the fermionic modes, is a conserved quantity. From equation (4.14) we see that it is possible to obtain an expression for the occupation number n_j using the initial fermionic correlator. In fact

$$n_j = \int_0^L dx \int_0^L dy \varphi_j(x)\varphi_j(y) \left\langle \Psi^\dagger(x)\Psi(y) \right\rangle_{t=0}, \tag{4.45}$$

that is

$$n_j = \frac{4N}{L^2} \int_0^L dx \int_0^L dy \, \sin\left(\frac{j\pi}{L}x\right) \sin\left(\frac{j\pi}{L}y\right) \sin\left(\frac{\pi}{L}x\right) \sin\left(\frac{\pi}{L}y\right) [1-2\beta(x,y)]^{N-1}, \tag{4.46}$$

where,

$$\beta(x,y) = \left(\frac{|y-x|}{L} + \frac{1}{\pi} \text{sign}(y-x) \cos\left(\frac{\pi}{L}(x+y)\right) \sin\left(\frac{\pi}{L}(x-y)\right) \right). \tag{4.47}$$

We can slightly simplify this expression using the Werner formulas and, since the integrand function depends only on $(x-y)$ and $(x+y)$, we can change variables obtaining:

$$\begin{aligned} n_j = \frac{N}{2\pi^2} \int_0^{2\pi} dt \int_{|t-\pi|-\pi}^{\pi-|t-\pi|} dz \, [\cos(jz) - \cos(jt)][\cos(z) - \cos(t)] \\ \times \left[1 - \frac{2}{\pi}(|z| - \text{sign}(z)\cos(t)\sin(z))\right]^{N-1}, \end{aligned} \tag{4.48}$$

in which $t = \frac{L(x+y)}{\pi}$ and $z = \frac{L(x-y)}{\pi}$.
This is a highly non trivial integral and in order to perform it we need a preliminary numerical analysis that will show its behavior in the thermodynamic limit. Numerical results of the integration of (4.48) are in figure 4.4. As we can see, apart for small values of j ($j \leq 10$), this function is well behaved. The disordered cloud of values for small j is caused by numerical errors that occur in the integration of (4.48) due to the oscillating behavior of the integrand. We can ignore this for two reasons: the function is limited for all j, so we can drop a finite number of points changing nothing, but, especially, because $n_j = f\left(\frac{j}{N}\right)$, then in the thermodynamic limit small values of j do not contribute.
After these considerations we can further simplify, in the TDL, equation (4.48)[2]

$$n_j = \frac{1}{\pi} \int_0^\pi dt \frac{1}{\left(1 + \frac{j^2\pi^2}{4N^2A(t)^2}\right)} = 1 - \sqrt{\frac{\tilde{j}(\tilde{j} + \sqrt{\tilde{j}^2+4})}{2(4+\tilde{j}^2)}}, \tag{4.49}$$

where $A(t) = 1 + \cos(t)$ and $\tilde{j} = \frac{j\pi}{2N}$.
This integral can be performed exactly, however we are not interested in its

[2]In appendix B.1 formal passages that led from (4.48) to (4.49) are reported.

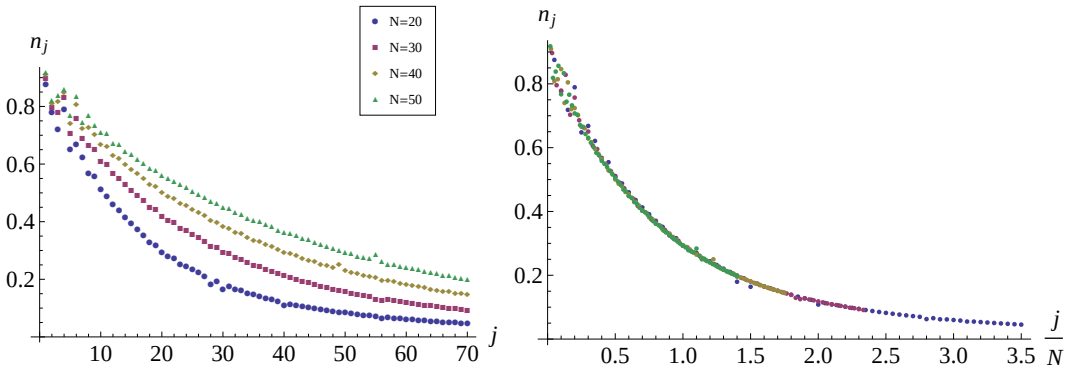

Figure 4.4: *Left*: numerical results for n_j as a function of j. *Right*: numerical results for n_j as a function of the rescaled variable $\frac{j}{N}$ for many values of N.

exact expression, we need only to compare this approximation with the previous numerical results in order to see if what done makes any sense. This has been reported in figure 4.5, as shown, a part the initial disorder, there is an almost perfect matching, considering the approximations done both in the numerical analysis and in the manipulation of (4.48), this is a surprising result.

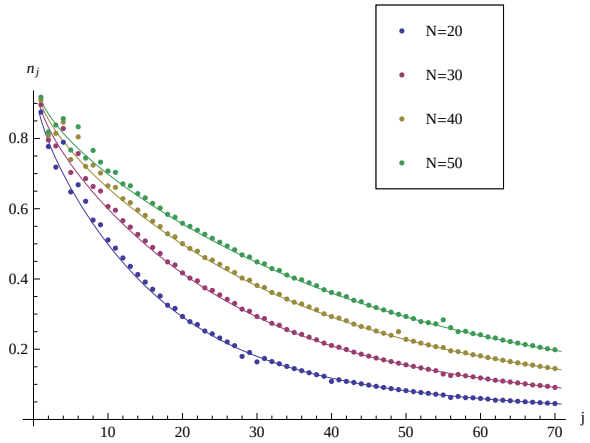

Figure 4.5: Comparison between numerical and analytical results for n_j for the same values of N of the fig. 4.4.

We should show that

$$\sum_{j=1}^{\infty} n_j = N, \tag{4.50}$$

but it is hard to perform this sum exactly. In the next paragraph we will show some results that are coherent with the condition (4.50). Actually, using relation (4.49), we should test that

$$\int_0^{\infty} dj\ n_j = N, \tag{4.51}$$

since we obtained it in the thermodynamic limit. From (4.50) we have that

$$\frac{1}{\pi}\int_0^{\pi} dt \int_0^{\infty} dj \frac{1}{\left(1+\frac{j^2\pi^2}{4N^2A(t)^2}\right)} = \frac{1}{\pi}\int_0^{\pi} dt\ NA(t) = N. \tag{4.52}$$

4.2.4 Density profile

Using relation (4.49) we can find the shape of the density profile in the GGE, that is the time evolved spatial density. The definition of $n(x)$ is:

$$n(x) = \sum_{j=1}^{\infty} n_j |\varphi_j(x)|^2. \tag{4.53}$$

We can evaluate equation (4.53) at the point $x = \frac{L}{2}$

$$n\left(\frac{L}{2}\right) = \sum_{j=1}^{\infty} n_j \left|\varphi_j\left(\frac{L}{2}\right)\right|^2 = \frac{2}{L}\sum_{j=1,\ j=odd}^{\infty} n_j. \tag{4.54}$$

If equation (4.50) holds, then

$$\sum_{j=1,\ j=odd}^{\infty} n_j = \frac{1}{2}\sum_{j=1}^{\infty} n_j = \frac{N}{2}, \tag{4.55}$$

therefore, we should find that $n\left(\frac{L}{2}\right) = \frac{N}{L} = n$.
Plugging (4.49) and (4.5) in equation (4.53) we find that

$$n(x) = \frac{2}{\pi L}\sum_{j=1}^{\infty}\int_0^{\pi} dt \frac{1}{\left(1+\frac{j^2\pi^2}{4N^2A(t)^2}\right)} \sin^2\left(\frac{j\pi}{L}x\right). \tag{4.56}$$

This expression can be studied numerically, the result is in figure 4.6.

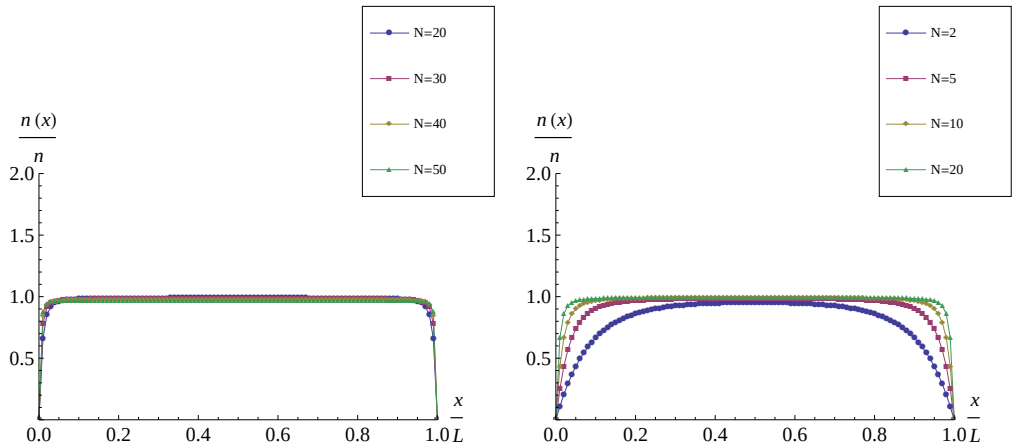

Figure 4.6: Numerical results of the density profile.

In the second picture of the figure 4.6 it is possible to see that $n(x)$ approaches to

a constant profile as N grows. The behavior of $n(x)$ close to the boundaries can be calculated analytically. Since both n_j and $\varphi_j(x)$ are functions of $\frac{j}{L}$ in (4.56) we can change variable. In the thermodynamic limit $L \to \infty$, so $\frac{j}{L}$ becomes a continuos variable and this permits us to replace the sum with an integral. Then we can write $n(x)$ as:

$$n(x) = \frac{2}{\pi}\int_0^\infty dq \int_0^\pi dt \frac{\sin^2(qx)}{\pi\left(1+\frac{q^2}{4n^2A(t)^2}\right)}. \tag{4.57}$$

At this point the trick is to exchange the order of integration doing the integral in q before that in t. Then

$$n(x) = \frac{2}{\pi}\int_0^\pi dt \frac{1}{\pi}\int_0^\infty dq \frac{\sin^2(qx)}{\left(1+\frac{q^2}{4n^2A(t)^2}\right)}. \tag{4.58}$$

The integral in q is essentially a Fourier transform, giving:

$$n(x) = n - n\mathrm{e}^{-4n|x|}(\mathrm{I}_0(4n|x|) - \mathrm{I}_1(4n|x|)), \tag{4.59}$$

where $\mathrm{I}_0(x)$ and $\mathrm{I}_1(x)$ are modified Bessel's functions.
Let us observe that, due to the presence of the exponential cut in (4.59), this result is valid only near the boundary (that is near the origin in the figure 4.6). For symmetry reasons the the shape of the distribution near the point $x = L$ is the same.

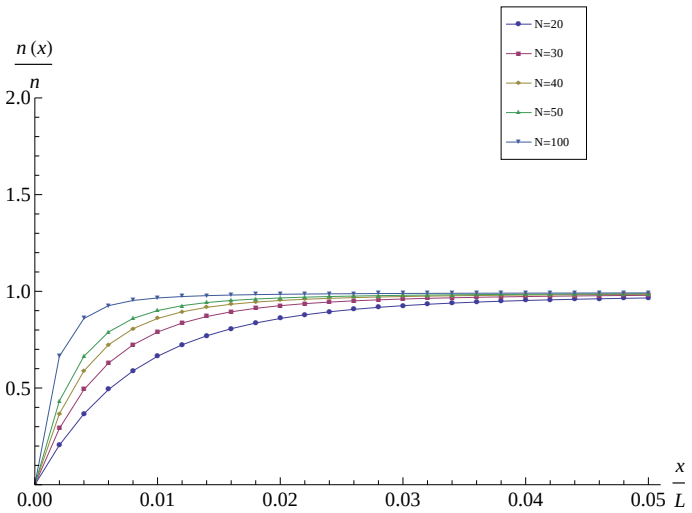

Figure 4.7: Detail of the edge. As N grows the exponential cut disappears.

It is important to remark that the exponential "Bessel modulated" cut on the edge is only a *finite-size* effect. In fact, in the thermodynamic limit (i.e. $L \to \infty$), it is clear that we lose the information about the boundaries, but this is not a problem as long as we preserve the shape of the distribution inducted by the trap. For this reason the correct rescaling of the spatial variable is

$$\tilde{x} = \frac{x}{L}, \tag{4.60}$$

Otherwise the risk is to expand near a boundary (as for example in (4.59)...) or near the center of the distribution. In the latter case we would flatten the shape recovering to the homogeneous case. In other words, we would lose totally the information of the effects inducted by the trap.
In order to show that the exponential cut at the boundaries is a finite-size effect, let us rewrite equation (4.59) as a function of $\tilde{x}$

$$n(\tilde{x}) = n - n\mathrm{e}^{-4nL\tilde{x}}(\mathrm{I}_0(4nL\tilde{x}) - \mathrm{I}_1(4nL\tilde{x})). \tag{4.61}$$

We note that the correction to the constant density is exponentially suppressed with L, therefore it vanishes in the TDL. The value of $n(x = \frac{L}{2})$ can be easily computed either from the TDL of (4.61) or considering the limit for $x \to \infty$ in (4.59)

$$n\left(\frac{L}{2}\right) = n(\tilde{x})_{TDL} = n(x \to \infty) = n. \tag{4.62}$$

Then from both numerical and analytical evidences we can say that the condition (4.50) holds.

4.2.5 Two-points correlation function

The fermionic correlation function in the GGE can be found following the same steps done previously. In the case of the correlation function it is senseless to rescale the spatial variable as done in (4.60), because we are interested in studying also the correlation of points that can be arbitrarily close to each other. Therefore, while the rescaling of the spatial variable is necessary in the analysis of the density profile, it is harmful in the study of the correlation because it would mean to consider only points infinitely separated.
The fermionic correlation function $C_F(x, y)$ is defined as:

$$C_F(x, y) = \sum_{j=1}^{\infty} n_j \varphi_j(x)\varphi_j(y), \tag{4.63}$$

using equation (4.49) this expression can be rewritten as:

$$C_F(x, y) = \frac{2}{\pi L}\sum_{j=1}^{\infty}\int_0^{\pi} dt \frac{1}{\left(1 + \frac{j^2\pi^2}{4N^2A(t)^2}\right)} \sin\left(\frac{j\pi}{L}x\right)\sin\left(\frac{j\pi}{L}y\right), \tag{4.64}$$

In the first picture of figure 4.8 we note that near the boundaries the translational invariance is broken, but this, as for the density, is a finite size effect.
Using the same trick of (4.58) we can rewrite equation (4.64) as:

$$C_F(x, y) = \frac{2}{\pi}\int_0^{\pi} dt \int_0^{\infty} dq \frac{\sin(qx)\sin(qy)}{\pi\left(1 + \frac{q^2}{4n^2A(t)^2}\right)}, \tag{4.65}$$

With some algebra we can solve this integral and its result is

$$\begin{aligned} C_F(x, y) = {} & n\mathrm{e}^{-2n|x-y|}(\mathrm{I}_0(2n|x - y|) - \mathrm{I}_1(2n|x - y|)) \\ & - n\mathrm{e}^{-2n|x+y|}(\mathrm{I}_0(2n|x + y|) - \mathrm{I}_1(2n|x + y|)). \end{aligned} \tag{4.66}$$

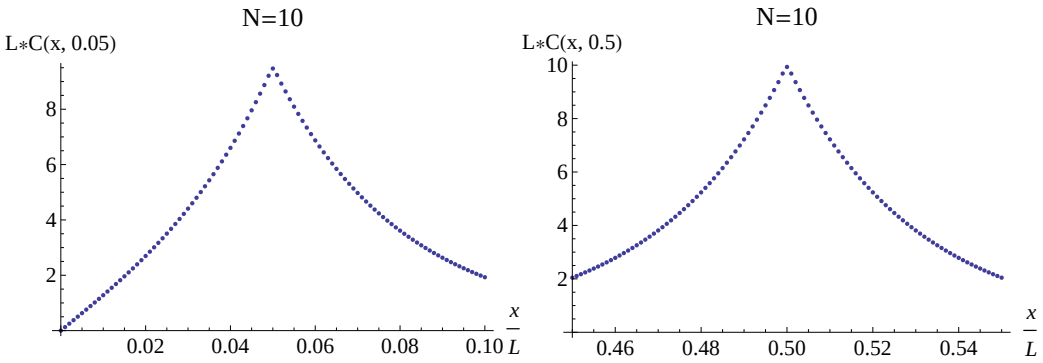

Figure 4.8: *Left*: Numerical result for the correlation near the boundary (in this case we set $y = (0.05)L$). *Right*: numerical results for the correlation near the center of the distribution (i.e. $y = (0.5)L$).

Before proceeding it is proper to make some clarifications. Formerly we said that for the correlation function the rescaling of the spatial variable was harmful, this is not properly correct. In fact our goal is to find the behavior of the stationary correlation in the thermodynamic limit. Therefore we must focus our attention far from the boundaries of the system, in order to avoid the same finite size effects that arise in the study of the density. For this purpose let us consider the following change of variables:

$$\begin{cases} x + y = u \\ x - y = v \end{cases} . \tag{4.67}$$

Focusing "far from the boundaries" means that $\frac{u}{L} \sim O(1)$. Then the correct rescaling to be considered in this case is

$$\begin{cases} \tilde{u} = \frac{u}{L} \\ v = v \end{cases} . \tag{4.68}$$

In terms of $\tilde{u}$ and v equation (4.66) becomes

$$C_F(x,y) = n\mathrm{e}^{-2n|v|}(\mathrm{I}_0(2n|v|) - \mathrm{I}_1(2n|v|)) - n\mathrm{e}^{-2Ln\tilde{u}}(\mathrm{I}_0(2Ln\tilde{u}) - \mathrm{I}_1(2Ln\tilde{u})). \tag{4.69}$$

It is clear that relation (4.69) gives the correlation for points x and y such that $|x - y| \ll L$, but this is not a problem because infinitely distant points are uncorrelated.
Therefore the stationary correlation function in the thermodynamic limit is, as expected, translationally invariant

$$C_F(x,y)_{x,y\gg 1} = n\mathrm{e}^{-2n|x-y|}(\mathrm{I}_0(2n|x-y|) - \mathrm{I}_1(2n|x-y|)). \tag{4.70}$$

In [47] the same calculation is done without trap, and in that case the result is

$$C_F(x,y)_{free} = n\mathrm{e}^{-2n|x-y|}. \tag{4.71}$$

Expressions (4.70) and (4.71) are different, meaning that, also where the density is constant and we have translational invariance, there is a non-trivial effect of the boundary conditions in the correlation function (see fig. 4.10). It is important to

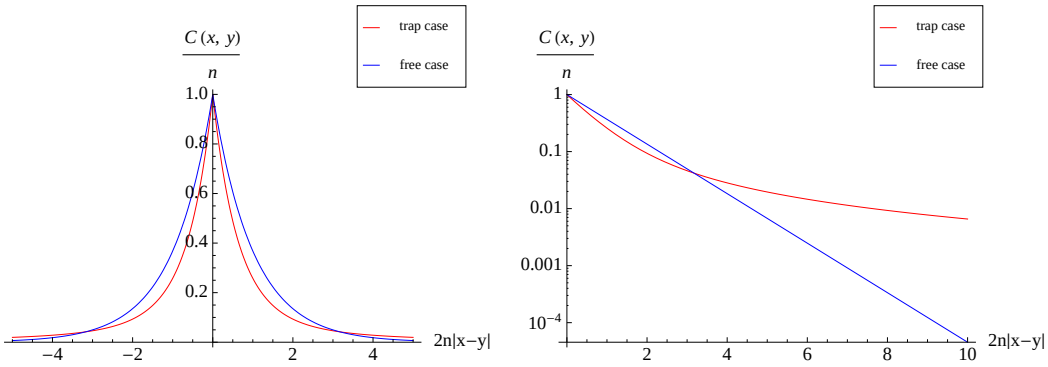

Figure 4.9: Comparison between the correlation in the free case and in the trap case both in natural (*left*) and in Log scale (*right*).

remark that the trap suppresses the *exponential decay* of the correlation function. Indeed for $|v| \gg 1$ we have

$$C(v) \simeq v^{-\frac{3}{2}}. \tag{4.72}$$

This behavior is due to the fact that the fermionic occupation number $n(j)$, if $\frac{j}{N} \ll 1$, can be written as

$$n(j) = 1 - C\sqrt{\frac{j}{N}} + O\left(\frac{j}{N}\right)^{\frac{3}{2}}, \tag{4.73}$$

where C is a constant. Therefore, in the Fourier transform of (4.73), a term proportional to $v^{-\frac{3}{2}}$ arises. This term dominates at long distances.

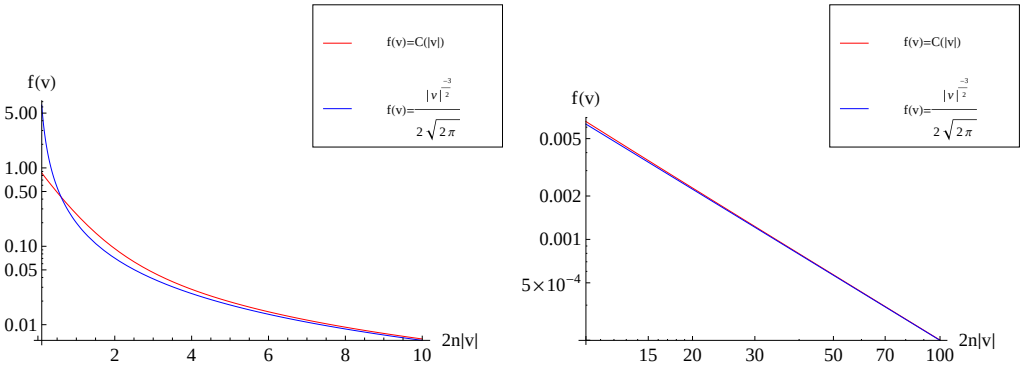

Figure 4.10: Comparison between the correlation and its asymptotic expansion. *Left*: for small values of v the behavior of the two functions if quite different (log scale). *Right*: as v grows the two functions almost overlap (log-log scale). This confirms that for large distances the decay of the correlation is algebraic.

4.3 Temporal evolution of the density profile

In this section we discuss the dynamical evolution of the density profile. Before starting we have to understand if this calculation makes sense, in other words we should wonder whether the density profile evolve. In the case treated in [47], the spatial density and the two-points function were constant in time. In our problem we start with a non-translationally invariant shape given by (4.6) and, as seen in the last paragraph, the stationary density is constant. Therefore it is interesting to understand how this happens.
The time evolution of the density profile is harder to find because we have to consider also the off-diagonal elements in (4.21). The formal definition of the time dependent density is

$$n(x,t) = \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} \varphi_q(x)\varphi_p(x) \mathrm{e}^{i(\epsilon_q-\epsilon_p)t} \left\langle \hat{\xi_q}^{\dagger}\hat{\xi_p} \right\rangle, \tag{4.74}$$

where $\epsilon_p = \left(\frac{p\pi}{L}\right)^2$. Then, in order to find the time-dependent behavior, we must calculate

$$\left\langle \hat{\xi_q}^{\dagger}\hat{\xi_p} \right\rangle = \frac{4N}{L^2}\int_0^L dx \int_0^L dy \ \sin\left(\frac{p\pi}{L}x\right)\sin\left(\frac{q\pi}{L}y\right)\sin\left(\frac{\pi}{L}x\right)\sin\left(\frac{\pi}{L}y\right) \times [1-2\beta(x,y)]^{N-1}, \tag{4.75}$$

where $\beta(x,y)$ is the same defined in (4.47). Also in this case we can change variables as done in (4.46):

$$\begin{cases} \frac{L(x+y)}{\pi} = u \\ \frac{L(x-y)}{\pi} = v \end{cases} \rightarrow \begin{cases} x = \frac{\pi(u+v)}{2L} \\ y = \frac{\pi(u-v)}{2L} \end{cases} . \tag{4.76}$$

The expression (4.75) is then

$$\frac{N}{\pi^2}\int_0^{2\pi} du \int_{|u-\pi|-\pi}^{\pi-|u-\pi|} dv \sin\left[\frac{p}{2}(u+v)\right]\sin\left[\frac{q}{2}(u-v)\right](\cos(v)-\cos(u)) \times \left[1-\frac{2}{\pi}(|v|-\mathrm{sign}(v)\sin(v)\cos(u))\right]^{N-1}. \tag{4.77}$$

This expression is more complicated than (4.48) because in this case $p \neq q$. In order to simplify it we can do an ansatz. We suppose that the only relevant contributions to (4.77) come, in the TDL, from high values of p and q. This means considering $p+q \gg 1$, nothing can be said, instead, about $p-q$. This hypothesis, as we have seen formerly, works very well for the diagonal term. Let us expand the term "$\sin\left[\frac{p}{2}(u+v)\right]\sin\left[\frac{q}{2}(u-v)\right]$" and see how can be simplified under this assumption

$$\sin[\dots]\sin[\dots] = \left(\cos\left[\frac{pv}{2}\right]\sin\left[\frac{pu}{2}\right]+\cos\left[\frac{pu}{2}\right]\sin\left[\frac{pv}{2}\right]\right) \times \left(\cos\left[\frac{qv}{2}\right]\sin\left[\frac{qu}{2}\right]-\cos\left[\frac{qu}{2}\right]\sin\left[\frac{qv}{2}\right]\right). \tag{4.78}$$

If we expand the product, the first term is

$$\cos\left(\frac{p}{2}v\right)\cos\left(\frac{q}{2}v\right)\sin\left(\frac{p}{2}u\right)\sin\left(\frac{q}{2}u\right) = \frac{1}{2}\cos\left(\frac{p}{2}v\right)\cos\left(\frac{q}{2}v\right)\left(\cos\left(\frac{p-q}{2}u\right)-\cos\left(\frac{p+q}{2}u\right)\right). \quad (4.79)$$

Since $p+q \gg 1$ and the integration interval in the u contains always an integer number of periods of the cosine, we can ignore "$\cos\left(\frac{p+q}{2}u\right)$". If we do the same thing with all the terms of the previous product and then sum up we obtain

$$\sin[\dots]\sin[\dots] \simeq \frac{1}{2}\left[\sin\left(\frac{p+q}{2}v\right)\sin\left(\frac{p-q}{2}u\right)+\cos\left(\frac{p+q}{2}v\right)\cos\left(\frac{p-q}{2}u\right)\right]. \quad (4.80)$$

The integral (4.77) becomes then

$$\frac{N}{2\pi^2}\int_0^{2\pi} du \int_{|u-\pi|-\pi}^{\pi-|u-\pi|} dv \left[\sin\left(\frac{p+q}{2}v\right)\sin\left(\frac{p-q}{2}u\right) + \cos\left(\frac{p+q}{2}v\right)\cos\left(\frac{p-q}{2}u\right)\right](\dots). \quad (4.81)$$

Since the integration interval of the v is symmetric, there is no contribution from the term "$\sin\left(\frac{p+q}{2}v\right)\sin\left(\frac{p-q}{2}u\right)$", therefore equation (4.81) is:

$$\frac{N}{2\pi^2}\int_{-\pi}^{\pi} du \cos\left(\frac{p-q}{2}(u+\pi)\right)\int_{|u|-\pi}^{\pi-|u|} dv \cos\left(\frac{p+q}{2}v\right)(\cos(v)+\cos(u)) \times\left[1-\frac{2}{\pi}(\beta(u,v))\right]^{N-1}. \quad (4.82)$$

At this point we note that if $p-q = odd$, "$\cos\left(\frac{p-q}{2}(u+\pi)\right)$" is an odd function integrated over a symmetric interval, then is zero. Following the same steps used to obtain (4.49) from equation (4.48) (these passages are explained in the appendix B.1...) we have

$$\begin{cases} \left\langle \hat{\xi_q}^{\dagger}\hat{\xi_p}\right\rangle = \frac{1}{\pi}\int_0^{\pi} d\tau \frac{\cos\left[\frac{p-q}{2}(\tau+\pi)\right]}{\left(1+\frac{(p+q)^2\pi^2}{16N^2A(\tau)^2}\right)} & \text{if } p-q = \text{even} \\ 0 & \text{if } p-q = \text{odd} \end{cases}, \quad (4.83)$$

where $A(\tau) = 1+\cos(\tau)$ as usual.
It is important to remember that relatoin (4.83) has been obtained assuming that $p+q \gg 1$. In order to understand if this makes any sense let us compare numerical results from the exact expression (4.77) and from (4.83). In this case we have two free indices (that are p and q) then the numerical analysis is a bit tricky. In order to perform it we can fix the value of a variable (e.g. q) and study the behavior of the off-diagonal expectation value as a function of the free index p. This has been done for several values of q and for $N = 50$. The results of this analysis are in figure 4.11.

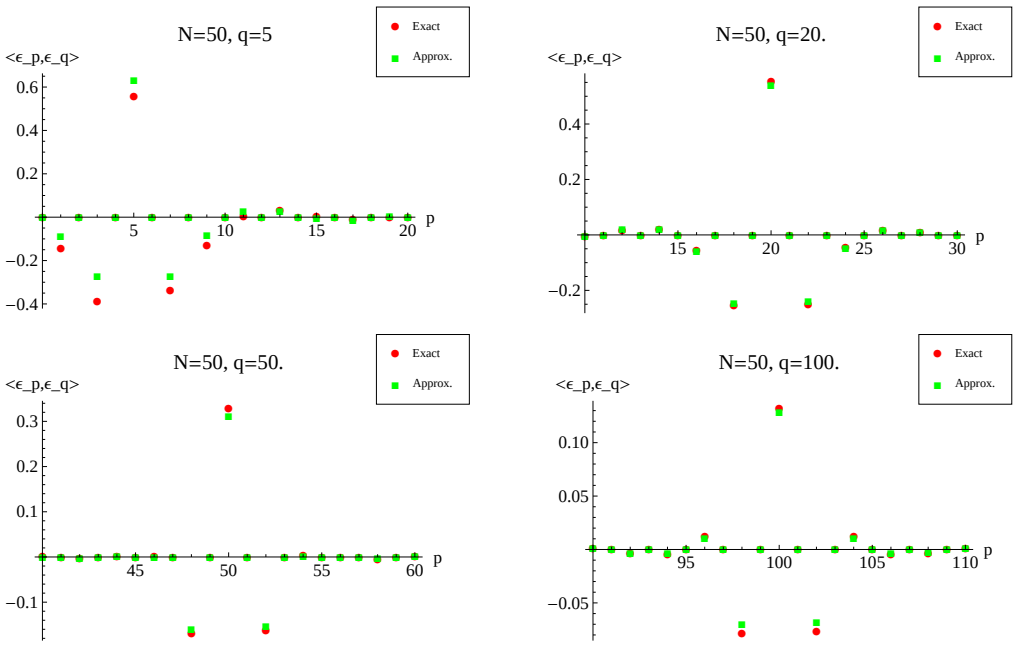

Figure 4.11: Numerical results for the off-diagonal expectation value as a function of the free index.

We note for small values of $p+q$ the approximation, as expected, is not perfect, but as the indices grow the matching becomes better. Indeed for high values of the $p+q$ the markers in the figure 4.11 are almost perfectly overlapped[3] [4].
Using equation (4.83) we can rewrite (4.74) as

$$n(x,t) = \frac{2}{L\pi}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\int_0^{\pi} d\tau \frac{\sin\left(\frac{p\pi}{L}x\right)\sin\left(\frac{q\pi}{L}x\right)}{\left(1+\frac{(p+q)^2\pi^2}{16N^2A^2(\tau)}\right)}\cdot\left(\cos\left[\frac{p-q}{2}(\tau+\pi)\right]\right)\mathrm{e}^{i\frac{\pi^2}{L^2}(p+q)(p-q)t}. \tag{4.84}$$

The domain of the double summation is in figure 4.12. We must remember that the only contribution comes from the cuples (p,q)=(odd, odd) or (p,q)=(even, even). In order to consider this fact we can change variables in the double summation setting

$$\begin{cases} p+q = 2r \\ p-q = 2l \end{cases}. \tag{4.85}$$

All the dots in the figure 4.12 are covered using the following summation extremes:

$$\begin{cases} 1 \le r < \infty \\ -r+1 \le l \le r-1 \end{cases}. \tag{4.86}$$

[3] One can object, observing Fig.4.11, that the approximation worsens for high values of p and q (compare, for example, the second picture and the last two ones). At first sight this is true, but we must take into account the fact that (4.77) for high values of q and p is highly oscillating and therefore it is harder to catch its correct behavior. However, the approximation is still better in the last panel, than in the first one.

[4] If we set $p=q$ we obtain exactly relation (4.49).

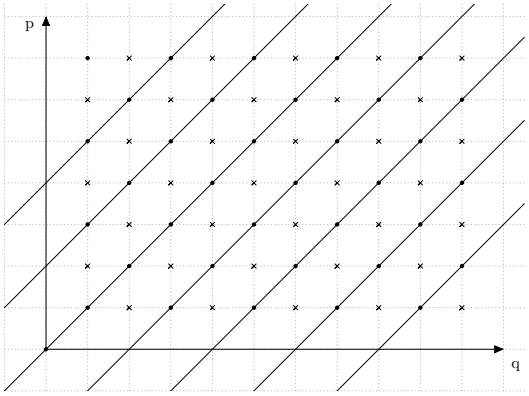

Figure 4.12: domain of the double sommation

The equation (4.84) becomes then:

$$n(x,t) = \frac{1}{L\pi}\sum_{r=1}^{\infty}\sum_{l=-r+1}^{r-1}\int_0^{\pi} d\tau\ \frac{\cos\left(\frac{2l\pi}{L}x\right) - \cos\left(\frac{2r\pi}{L}x\right)}{\left(1+\frac{r^2\pi^2}{4N^2A^2(\tau)}\right)}\cdot(\cos[l(\tau+\pi)])\mathrm{e}^{i\frac{4\pi^2}{L^2}r\cdot l\cdot t}. \tag{4.87}$$

In order to simplify this relation we can consider the following rescaling of the variables:

$$\begin{cases}\tilde{r} = \frac{r}{L}\\ \tilde{t} = \frac{t}{L}\\ \tilde{x} = \frac{x}{L}\end{cases}, \tag{4.88}$$

this permits us to substitute the sum over r with an integral and to extend the sum over l in the domain ($-\infty$ to $+\infty$). Equation (4.87), in the limit $L \to \infty$, becomes

$$n(\tilde{x},\tilde{t}) = \frac{1}{\pi}\int_0^{\infty} d\tilde{r}\sum_{l=-\infty}^{\infty}\int_0^{\pi} d\tau\ \frac{\cos(2l\pi\tilde{x}) - \cos(2L\tilde{r}\pi\tilde{x})}{\left(1+\frac{\tilde{r}^2\pi^2}{4n^2A^2(\tau)}\right)}\cdot(\cos[l(\tau+\pi)])\cos(4\pi^2\tilde{r}\cdot l\cdot\tilde{t}), \tag{4.89}$$

where we used the symmetry of the summation domain. In (4.89) the term "$\cos(2L\tilde{r}\pi\tilde{x})$" has a highly oscillating phase then we can drop it out. This will simplify the integral but will make us lose information about the behavior around the boundaries. However the "lost of the boundaries" is not dramatic, in fact, as we observed for the stationary density, the exponential cut at the edges is only a finite-size effect.
We can further simplify relation (4.89) using the parity of the integrand in the variable r

$$n(\tilde{x},\tilde{t}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tilde{r}\sum_{l=-\infty}^{\infty}\int_0^{\pi} d\tau\ \frac{\cos(2l\pi\tilde{x})}{\left(1+\frac{\tilde{r}^2\pi^2}{4n^2A^2(\tau)}\right)}(\cos[l(\tau+\pi)])\mathrm{e}^{i4\pi^2\tilde{r}\cdot l\cdot\tilde{t}}. \tag{4.90}$$

The integral in r is simply a Fourier transform:

$$\int_{-\infty}^{+\infty} dr \ \frac{\mathrm{e}^{i4\pi^2 r\cdot l\cdot t}}{1+\frac{r^2\pi^2}{4n^2A(\tau)^2}} = 2nA(\tau)\mathrm{e}^{-8n\pi|A(\tau)l|t}. \tag{4.91}$$

Then equation (4.89) becomes

$$n(\tilde{x},\tilde{t}) = \frac{n}{\pi}\sum_{l=-\infty}^{\infty}\cos(2l\pi\tilde{x})\int_0^{\pi} d\tau \ \cos[l(\tau+\pi)]A(\tau)\mathrm{e}^{-8n\pi|A(\tau)l|\tilde{t}}, \tag{4.92}$$

let us note that this expression depends neither on N nor on L and then is well defined in the thermodynamic limit. In order to understand whether equation (4.92) is correct, let us explore two limiting cases: $t=0$ and $t\to\infty$.
If we set $t=0$ the integral in τ can be performed exactly

$$n(\tilde{x},0) = \frac{n}{\pi}\sum_{l=-\infty}^{\infty}\cos(2l\pi\tilde{x})\frac{1-2l^2-2\cos(l\pi)}{l(l^2-1)}\sin(l\pi), \tag{4.93}$$

the only non-zero contributions to the sum come from $l=0$ and $|l|=1$, substituting, we have

$$n(\tilde{x},0) = 2n\sin^2(\pi\tilde{x}). \tag{4.94}$$

that is exactly the same as (4.6).
If we consider $t\to\infty$ we must set $l=0$ and the result is:

$$n(x,t\to\infty) = n. \tag{4.95}$$

This value is the same of (4.59) but, as already said, with a cut on the edges.
Equation (4.92) can be rewritten as:

$$n(\tilde{x},\tilde{t}) = n + \frac{2n}{\pi}\sum_{l=1}^{\infty}\cos(2l\pi\tilde{x})\int_0^{\pi} d\tau \ \cos[l(\tau+\pi)]A(\tau)\mathrm{e}^{-8n\pi|A(\tau)l|\tilde{t}}, \tag{4.96}$$

in this form it is clear why after a transient time the system reaches a stationary state. Indeed, the time-dependent part of (4.96) is exponentially suppressed. The transient part of the previous formula is negligible if

$$n\cdot\tilde{t} \gg \frac{1}{8\pi|A(\tau)l|}.$$

Let us note, in particular, that this time is inversely proportional to the average density.
Using that, if $l\in\mathbb{Z}$,

$$\frac{1}{\pi}\int_0^{\pi} du \ \cos(lu)\mathrm{e}^{z\cos(u)} = (-1)^m\mathrm{I}_m(z), \tag{4.97}$$

the integral in (4.96) can be performed exactly. Then, the time-dependent density can be written as

$$n(\tilde{x},\tilde{t}) = n\left(1+\sum_{l=1}^{\infty}\cos(2l\pi\tilde{x})(2\mathrm{I}_l(8n\pi l\tilde{t})-\mathrm{I}_{l+1}(8n\pi l\tilde{t})-\mathrm{I}_{l-1}(8n\pi l\tilde{t}))\mathrm{e}^{-8n\pi l\tilde{t}}\right). \tag{4.98}$$

Since for the modified Bessel's functions holds that

$$\frac{\partial}{\partial x}\mathrm{I}_m(x) = \frac{1}{2}(\mathrm{I}_{m-1}(x) + \mathrm{I}_{m+1}(x)), \tag{4.99}$$

equation (4.98) can expressed as

$$n(\tilde{x},\tilde{t}) = n\left(1 - 2\sum_{l=1}^{\infty}\cos(2l\pi\tilde{x})(\partial_z \mathrm{e}^{-z}\mathrm{I}_m(z))\right), \tag{4.100}$$

where $z = 8n\pi l\tilde{t}$.

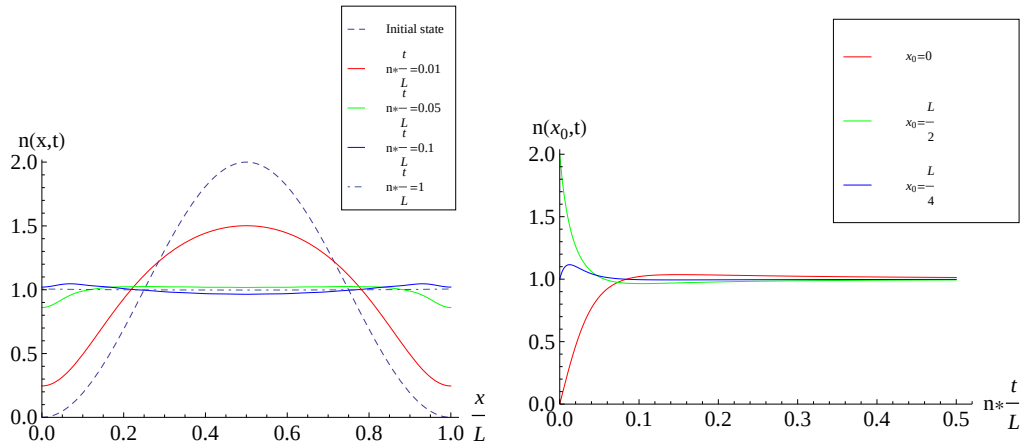

Figure 4.13: Numerical results for $n(x,t)$. *Left*: Density profile for some values of $n\tilde{t}$. *Right*: Value of the density for three different spatial points, as a function of time.

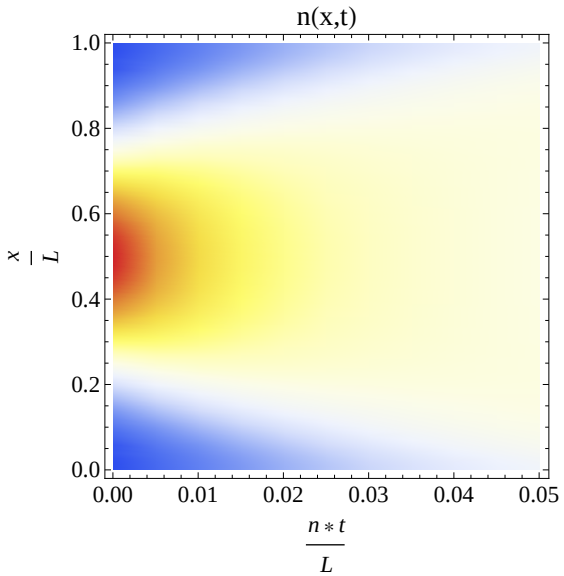

Figure 4.14: Evolution of the density profile.

The former results make sense, however we would like to test numerically the

approximations we have used to transform (4.87) in (4.96). For this purpose we can try to work with the exact relation of the time-dependent density (i.e. (4.87)). It is important to point out that there are some difficulties in the numerical analysis in this case. Since the sum in r should be performed from 0 to ∞ we truncated it, but the effect of a cut-off could be non trivial. Moreover it is very hard to set high values of the cut-off because the computation time grows very rapidly.
Let us define the function:

$$\Delta(x,t,L,N)_\Lambda = -\frac{1}{L\pi}\sum_{r=1}^{\Lambda}\sum_{l=-r+1}^{r-1}\int_0^\pi d\tau\ \frac{\cos\left(\frac{2r\pi}{L}x\right)}{\left(1+\frac{r^2\pi^2}{4N^2A^2(\tau)}\right)}(\cos[l(\tau+\pi)])\mathrm{e}^{i\frac{4\pi^2}{L^2}r\cdot l\cdot t}, \tag{4.101}$$

this is exactly the part that we ignored in relation (4.90). Equation (4.101) contributes to nullify the density at the edges, however it is interesting to see whether there are contributions from (4.101) also in the middle of the distribution, that is for $\frac{x}{L}=\frac{1}{2}$.
In figure 4.15 there are the results of our analysis. As it is possible to see the con-

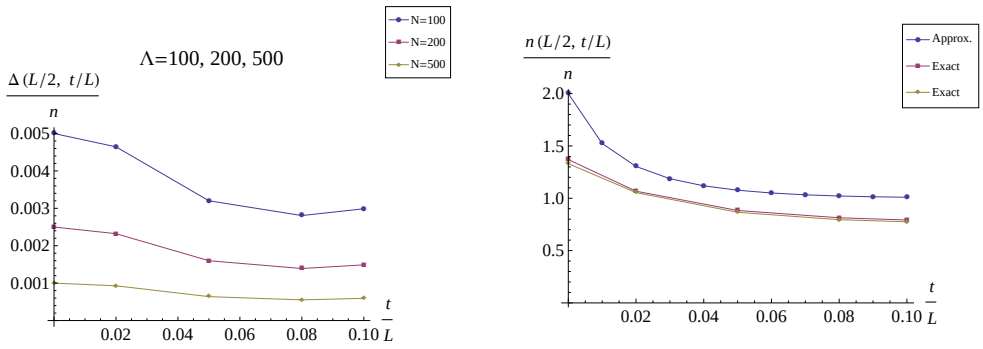

Figure 4.15: *Left*: numerical study of (4.101) for $N = 100, 200, 500$ and $\Lambda = 100, 200, 500$ respectively, $x = \frac{L}{2}$. *Right*: comparison between the approximated value of the time-dependent spatial density previously found and the correct truncated value 4.102 in $x = \frac{L}{2}$.

tribution of (4.101) is *compatible* with zero. It significative to see that, although Λ increases, as N grows Δ becomes smaller.
Let us now compare the results of the approximated time-dependent density and the exact one.
As we have seen formerly, $n(\tilde{x},\tilde{t})$ around $\tilde{x}=\frac{1}{2}$ can be written as:

$$n(x,t,L,N)_\Lambda = \frac{1}{L\pi}\sum_{r=1}^{\Lambda}\sum_{l=-r+1}^{r-1}\int_0^\pi d\tau\frac{\cos\left(\frac{2l\pi}{L}x\right)}{\left(1+\frac{r^2\pi^2}{4N^2A^2(\tau)}\right)}(\cos[l(\tau+\pi)])\mathrm{e}^{i\frac{4\pi^2}{L^2}r\cdot l\cdot t}. \tag{4.102}$$

In this case too we have to pay attention to the value of Λ.
The results from the right panel of figure 4.15 are "unsatisfactory" because the "exact" curves are always below the approximated one[5], even if they behave in a

[5]However, the correct values are the "approximated" ones.

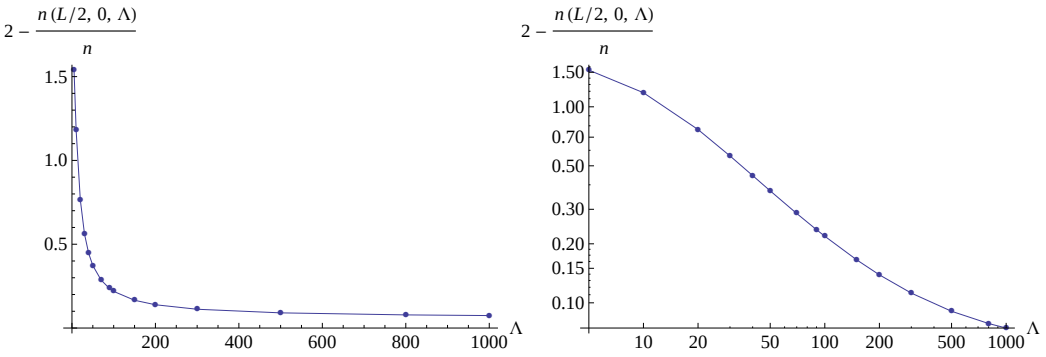

Figure 4.16: Truncated density as a function of Λ in linear and log-log scale.

similar way. This is the greatest difficulty in performing a correct numerical analysis of equation (4.102), indeed, unfortunately, the convergence of $n(x,t,L,N)_\Lambda$ is extremely slow with Λ. Therefore, in order to catch the correct behavior of (4.102), we would need extremely high values of the cut-off, but this is very hard from a computational point of view.
Let us see how the truncated density defined in (4.102) approaches to the correct value as the cut-off grows. For this purpose we consider the value of the density for $x = \frac{L}{2}$ and $t = 0$, that, from previous results, we know to be $2n$.
Observing figure 4.16 we can understand how the convergence of the truncated density is slow. For this reason numerical analysis in this sense is incomplete and unsatisfactory.

4.4 Time-dependent two-points correlation function

Using same techniques of section 4.3 it is possible to calculate the dynamical fermionic two-points correlation function. This will complete our insight into the non-equilibrium behavior of the system.
The formal definition of the dynamic correlator is

$$C(x,y,t) = \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} \varphi_p(x)\varphi_q(y) \mathrm{e}^{i(\epsilon_q-\epsilon_p)t} \left\langle \hat{\xi_q}^\dagger \hat{\xi_p} \right\rangle . \tag{4.103}$$

For the off-diagonal expectation value we have seen that the approximation (4.83) holds, therefore

$$C(x,y,t) = \frac{2}{L\pi}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\int_0^\pi d\tau \, \frac{\sin\left(\frac{p\pi}{L}x\right)\sin\left(\frac{q\pi}{L}y\right)}{\left(1+\frac{(p+q)^2\pi^2}{16N^2A^2(\tau)}\right)} \left(\cos\left[\frac{p-q}{2}(\tau+\pi)\right]\right) \mathrm{e}^{i\frac{\pi^2}{L^2}(p+q)(p-q)t}. \tag{4.104}$$

We see that this expression is a bit more complicated than (4.84). Let us consider the following variables:

$$\begin{cases} x+y=u \\ x-y=v \end{cases} . \tag{4.105}$$

The correlation function can be written as

$$C(u,v,t) = \frac{2}{L\pi}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\int_0^{\pi} d\tau \frac{\sin\left(\frac{p\pi}{2L}(u+v)\right)\sin\left(\frac{q\pi}{2L}(u-v)\right)}{\left(1+\frac{(p+q)^2\pi^2}{16N^2A^2(\tau)}\right)} \times\left(\cos\left[\frac{p-q}{2}(\tau+\pi)\right]\right)\mathrm{e}^{i\frac{\pi^2}{L^2}(p+q)(p-q)t}. \quad (4.106)$$

In relation (4.106) the term "$\sin\left(\frac{p\pi}{2L}(u+v)\right)\sin\left(\frac{q\pi}{2L}(u-v)\right)$" can be approximated as:

$$\frac{1}{2}\cos\left(\frac{\pi(p+q)v}{2L}\right)\cos\left(\frac{\pi(p-q)u}{2L}\right)$$

(see appendix B.2 for all the details).
Using the same spatial variables defined in (4.68) for the stationary correlator and considering again

$$\begin{cases} p+q=2r \\ p-q=2l \end{cases}, \quad (4.107)$$

with

$$\begin{cases} 1\le r<\infty \\ -r+1\le l\le r-1 \end{cases}, \quad (4.108)$$

expression (4.106) becomes:

$$C(u,v,t) = \frac{1}{L\pi}\sum_{r=1}^{\infty}\sum_{l=-r+1}^{r-1}\int_0^{\pi} d\tau\ \frac{\cos\left(\frac{r\pi v}{L}\right)\cos\left(\frac{l\pi u}{L}\right)}{\left(1+\frac{r\pi^2}{4N^2A^2(\tau)}\right)}\left(\cos\left[l(\tau+\pi)\right]\right)\mathrm{e}^{i\frac{4\pi^2}{L^2}l\cdot r\cdot t}. \quad (4.109)$$

Rescaling the variables in the usual way:

$$\begin{cases} \tilde{r}=\frac{r}{L} \\ \tilde{u}=\frac{u}{L} \\ \tilde{t}=\frac{t}{L} \end{cases}, \quad (4.110)$$

and considering the limit $L\to\infty$, equation (4.109) becomes

$$C(v,\tilde{u},\tilde{t}) = \frac{1}{4\pi}\int_{-\infty}^{+\infty} d\tilde{r}\sum_{l=-\infty}^{+\infty}\int_0^{\pi} d\tau\ \frac{\cos[\tilde{r}(\pi v+4\pi^2 l\tilde{t})]+\cos[\tilde{r}(\pi v-4\pi^2 l\tilde{t})]}{\left(1+\frac{\tilde{r}\pi^2}{4n^2A^2(\tau)}\right)}\left(\cos\left[l(\tau+\pi)\right]\right). \quad (4.111)$$

The integral in $\tilde{r}$ can be simply done, and the final result is:

$$C(u,v,\tilde{t}) = C(v,\tilde{t}\to\infty)+\frac{n}{\pi}\sum_{l=1}^{\infty}(-1)^l\cos(\pi l\tilde{u})\Upsilon(n,l,\tilde{t}), \quad (4.112)$$

where $C(v,\tilde{t}\to\infty)$ is the stationary correlation found in (4.70) and

$$\Upsilon(n,l,\tilde{t}) = \int_0^{\pi} d\tau\ \cos(l\tau)A(\tau)(\mathrm{e}^{-2n|A(\tau)(v+4\pi^2\cdot l\cdot\tilde{t})|}+\mathrm{e}^{-2n|A(\tau)(v-4\pi^2\cdot l\cdot\tilde{t})|}). \quad (4.113)$$

It is clear that as $\tilde{t} \to \infty$ we find again the stationary part calculated previously. In particular let us note that this part depends only on $v = x - y$, then it is correctly translationally invariant. In the paragraph 4.2.1 we found an expression for the initial correlation for finite values of N and L because, as stated later, it was not trivial to find a correct way to perform the thermodynamic limit. In order to have an expression for the initial correlation function one may wonder whether considering the limit for $\tilde{t} \to 0$ of expression (4.112) could be useful. Unfortunately this could not give the correct result, for mainly one reason. Indeed it is not said that $\tilde{t} \to 0$ corresponds to $t \to 0$, because, since $\tilde{t} = t/L$, also for finite values of t, as $L \to \infty$, we find $\tilde{t} = 0$. Then it is not clear what we would calculate performing the limit $\tilde{t} \to \infty$, especially because we have nothing to compare to. Equation (4.112) depends on v and $\tilde{u}$, these two variables work on different scales. In fact the correlation is a *local* property of the system and the "local variable" is $v = x - y$, indeed it is the only that "survives" when the system reaches a stationary state. The role of the variable $\tilde{u}$ is quite different, indeed it implements the anisotropy inducted by the trap. This anisotropy is a *global* property of the system, in fact we have to consider $u \approx L$ or $\tilde{u} \approx 1$. Therefore, in order to study equation (4.112), we can fix the value of $\tilde{u}$ and see the behavior of the correlation as a function of the local variable v. Numerical analysis of (4.112) is in figure 4.17.

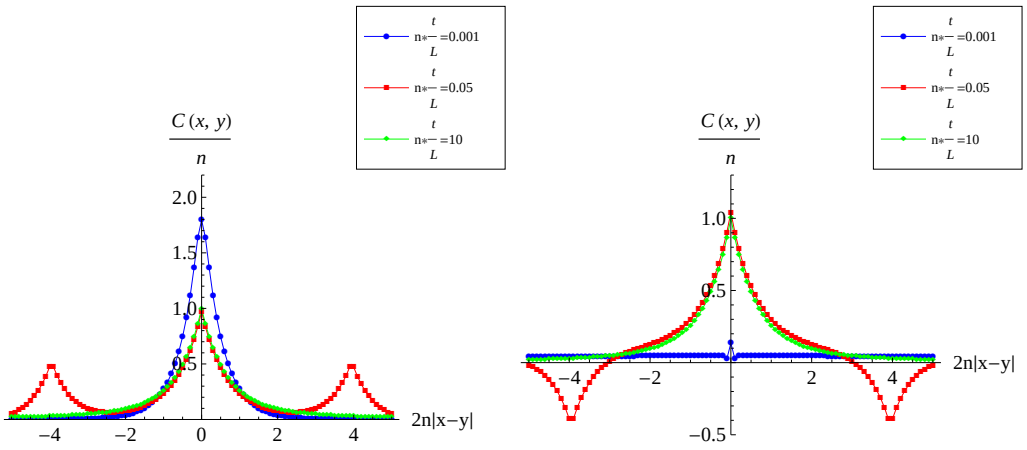

Figure 4.17: Dynamical behavior of the fermionic correlation function. *Left*: we set $\tilde{u} = 1$, this means that we are studying the correlation in a neighborhood of the centre of the distribution (i.e. $x \approx y \approx \frac{L}{2}$). *Right*: in this case $\tilde{u} = 0.1$, then we study the correlation near the edge of the system. We note that the behavior is qualitatively different in the two cases, anyway, for large values of $\tilde{t}$, the correlation is the same (green curve).

CONCLUSIONS AND OUTLOOKS

In this work we have studied the dynamical and stationary properties of a confined one dimensional Bose gas after a global quantum quench. The quench protocol considered was the easiest one that could be implemented in the Lieb-Liniger model. The difficulties in our cases arose from the fact that we were dealing with a confined gas. The hamiltonian describing our system could be written as

$$H = \int_0^L dx(\partial_x\hat{\phi}^\dagger(x)\partial_x\hat{\phi}(x) + c\hat{\phi}^\dagger(x)\hat{\phi}^\dagger(x)\hat{\phi}(x)\hat{\phi}(x) + \phi(x)^\dagger\phi(x)V(x)), \quad (4.114)$$

in which $V(x)$ played the role of trapping external potential.
We studied the simplest trap that could be considered: the potential well, that corresponds to an external potential

$$\begin{cases} V = 0 & \text{if } 0 < x < L \\ V \to \infty, & \text{otherwise} \end{cases}. \quad (4.115)$$

The particularity of the trapped systems is that in the initial state there is not *translational invariance*. In order to preserve this inhomogeneity, particular care has been devoted in performing the thermodynamical limit. In fact, as we have stated in paragraph 4.2.2, if one does naively the TDL the risk is to obtain *unphysical* results. This particular feature makes trapped systems harder to handle from a computational point of view.
One of the most remarkable result obtained is that, contrarily to the case described in [47], the two-point functions (i.e. both the fermionic correlation and the spatial density) *evolve*. This property has to be attributed to the initial non-trivial state. Therefore we could test the validity of the Generalized Gibbs Ensemble predictions by the analysis of the stationary and the dynamic correlation. Since we used a free fermionic theory the *local* integrals of motion could be expressed as linear combination of the momentum occupation number (as demonstrated in 3.3) simplifying considerably the calculations. Then the GGE has been constructed as:

$$\rho_{GGE} = Z^{-1}\exp\left(-\sum_k \lambda_k\hat{n}_k\right), \quad (4.116)$$

where $Z = \prod_k \left[1 + \mathrm{e}^{-\lambda_k}\right]$ and the Lagrange multipliers λ_k were fixed by the relation

$$\langle \hat{n}_k \rangle = \frac{1}{\mathrm{e}^{\lambda_k} + 1}.$$

The stationary value of the two-points correlation function in the thermodynamical limit was found to be:

$$C_F(x, y) = n\mathrm{e}^{-2n|x-y|}(\mathrm{I}_0(2n|x-y|) - \mathrm{I}_1(2n|x-y|)). \tag{4.117}$$

The effect of the trap in this case can be recognized in *correction* implemented by the Bessel functions. This correction for $C_F(x, y)$ "survives" also in thermodynamical limit, that is when we focus the attention on the centre of the trap and the boundaries are set at infinity. This correlation has been calculated using the *diagonal* GGE ensemble. The time-dependent correlation, instead, was harder to compute but with a bit of work its correct expression could be found too,

$$C(u, v, \tilde{t}) = C(v, \tilde{t} \to \infty) + \frac{n}{\pi} \sum_{l=1}^{\infty} (-1)^l \cos(\pi l \tilde{u}) \Upsilon(n, l, \tilde{t}), \tag{4.118}$$

where $\Upsilon(n, l, \tilde{t}) = \int_0^\pi d\tau \ \cos(l\tau) A(\tau) (\mathrm{e}^{-2n|A(\tau)(v+4\pi^2 \cdot l \cdot \tilde{t})|} + \mathrm{e}^{-2n|A(\tau)(v-4\pi^2 \cdot l \cdot \tilde{t})|})$. As we noted, when the infinite time limit $\tilde{t} \to \infty$ was considered the dynamical correlation approaches exactly the value calculated using the diagonal GGE, and this aspect is true also for the spatial density.

There are many unsolved features that can led to future developments of this type of problems. The first issue concerns the study of the same quench protocol in another confining potential, better if more realistic from an expermental point of view. It could be worth to consider a harmonic trap (i.e. $V(x) = \frac{\omega^2}{2} x^2$). In this case, since the functions involved are less elementary than sines and cosines, it is harder to achieve the same analytical results we found in this work, however, adapting the techniques used here, it does not seem to be impossible.

Another possible further improvement that can be done is the computation of the dynamical density-density correlation function in the same spirit of what done in [47]. Indeed it may be interesting to see whether, as expected, the Wick's theorem also in this case holds only in the stationary state. Anyway this challenge could be too hard from a computational point of view and we are pessimistic about the possibility of finding good analytical results. In conclusion we have to point out that, although in this work we used only free theories, we expect that *qualitatively* these results should be similar for a generic quench $c = 0 \to c = c_0$. However in this case the problem should be treated with numerical techniques or with Thermodynamical Bethe Ansatz [48] [49].

APPENDIX A

A.1 Setup on lattice

In this section we want to demonstrate that the approximation (4.26) makes sense. For this purpose we use a rigorous lattice regularization following the steps of what done in the homogeneuos case [47].
Let us consider a system of N boson hopping on a one dimensional lattice composed by M sites with lattice spacing δ, the length of the lattice is $L = M\delta$. The wave function of the ground state is

$$\varphi_1(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi}{L}x\right). \tag{A.1}$$

Then the ground state for a single particle on lattice is given by:

$$|GS\rangle_{1-P} = \sqrt{\frac{2}{M}} \sum_{i=1}^{M} \sin\left(\frac{\pi i}{M}\right) b_i^\dagger \prod_{\bigotimes i} |0\rangle_i, \tag{A.2}$$

where $b_i^\dagger$ is the construction operator on the i^{th} site. In order to have a well-defined continuum limit, in (A.2), we used that, the relation between the lattice operators and the continuum ones is

$$b_m = \sqrt{\delta}\hat{\phi}(\delta m) \qquad c_m = \sqrt{\delta}\hat{\Psi}(\delta m) \qquad a_m = \sqrt{\delta}\hat{\Phi}(\delta m). \tag{A.3}$$

Since we are dealing with non-interacting bosons the initial state of our system is a BEC:

$$|BEC\rangle_N = \sqrt{\frac{2^N}{M^N N!}} \left(\sum_{i=1}^{M} \sin\left(\frac{\pi i}{M}\right) b_i^\dagger\right)^N \prod_{\bigotimes i} |0\rangle_i. \tag{A.4}$$

We are interested in the continuum limit that is defined as $M \to \infty$, $\delta \to 0$ with $L = M \cdot \delta$ constant. The hard-core bosons operators are defined as in the continuum case:

$$a_i = P_i b_i P_i, \tag{A.5}$$

with $P_i = |0_i\rangle\langle 0_i| + |1_i\rangle\langle 1_i|$. In discrete case too, it is possible to define a Jordan-Wigner mapping from hard-core bosons to free fermions

$$a_i = e^{-i\pi \sum_{j<i} c_j^\dagger c_j} c_i = \prod_{j<i}(1 - 2c_j^\dagger c_j)c_i, \tag{A.6}$$

$$c_i = e^{i\pi \sum_{j<i} a_j^\dagger a_j} a_i = \prod_{j<i}(1 - 2a_j^\dagger a_j)a_i, \tag{A.7}$$

with $\{c_j^\dagger, c_i\} = \delta_{ij}$.
The initial fermionic correlation function for $k < l$ can be written as

$$\langle BEC|c_k^\dagger c_l|BEC\rangle = \langle BEC|a_k^\dagger \prod_{j=k}^{l-1}(1 - 2a_j^\dagger a_j)a_l|BEC\rangle, \tag{A.8}$$

that is

$$\langle BEC|c_k^\dagger c_l|BEC\rangle = \langle BEC|a_k^\dagger \sum_r (-2)^r \sum_{k<n_1\ldots<n_r<l} a_{n_1}^\dagger a_{n_1} \ldots a_{n_r}^\dagger a_{n_r} a_l|BEC\rangle. \tag{A.9}$$

Therefore, in order to find the initial correlation function, we have to deal with terms of type

$$\langle BEC|a_k^\dagger a_{n_1}^\dagger a_{n_1} \ldots a_{n_r}^\dagger a_{n_r} a_l|BEC\rangle. \tag{A.10}$$

In order to calculate terms like (A.10) we must expand the multinomial of equation (A.4)

$$|BEC\rangle_N = \sqrt{\frac{2^N}{M^N N!}} \sum_{i_1 \ldots i_M} \binom{N}{i_1, \ldots, i_M} (p_1 b_1^\dagger)^{i_1} \ldots (p_M b_M^\dagger)^{i_M}|0\rangle, \tag{A.11}$$

where $p_i = \sin\left(\frac{\pi i}{M}\right)$. Let us consider the ket of expression (A.10)

$$a_k^\dagger a_{n_1}^\dagger a_{n_1} \ldots a_{n_r}^\dagger a_{n_r} a_l|BEC\rangle. \tag{A.12}$$

If we want a non-zero result we must fix the value of some indeces, i.e. $i_l = i_{n_r} = \ldots i_{n_1} = 1$ e $i_k = 0$. This because when we rewrite the hard-core operators in terms of the real ones we have some projectors that annihilate all the multi-occupied sites. Equation (A.12) becomes therefore,

$$\sqrt{\frac{2^N}{M^N N!}} \sum_{\{i'\}} \binom{N}{i_1, \ldots, i_k = 0, \ldots, i_{n_1} = 1, \ldots, i_{n_r} = 1, i_l = 1, i_M} \times (p_1 b_1^\dagger)^{i_1} \ldots b_k^\dagger \ldots (p_M b_M^\dagger)^{i_M}|0\rangle \tag{A.13}$$

in which $\{i'\} = \{i_1 \ldots i_M\}/\{i_k, i_{n_r} \ldots i_{n_1}, \ldots i_l\}$. Let us note that in (A.13) there is only one $b_k^\dagger$ and there are not $b_l^\dagger$. When we consider the scalar product between the ket defined in (A.13) and a bra defined in (A.4) the only non-zero contributions will come from terms in which there is a perfect matching between

the powers of all operators. Therefore the "non-zero" bra in the scalar product (4.28) is that with $i_k = i_{n_1} = \dots i_{n_r} = 1$ e $i_l = 0$. Using that:

$$\langle 0|p_i^n b_i^n p_i^n b_i^{\dagger n}|0\rangle = p_i^{2n} n!, \tag{A.14}$$

we obtain

$$\langle BEC|\dots|BEC\rangle = \frac{2^N p_k p_l}{M^N N!} \prod_{s=n_1\dots n_r} p_s^2 \sum_{\{i'\}} \binom{N}{i_1,\dots,i_M}^2 p_1^{2i_1} i_1! \dots p_M^{2i_M} i_M!. \tag{A.15}$$

Since

$$\sum_{j\in\{i'\}} i_j = N - r - 1, \tag{A.16}$$

equation (A.15) can be rewritten as:

$$\langle \dots \rangle = \frac{2^N p_k p_l}{M^N} p_{n_1}^2 \dots p_{n_r}^2 N(N-1)\dots(N-r) \left(\sum_{j\in\{i'\}} p_j^2 \right)^{N-r-1}, \tag{A.17}$$

where we used again the multinomial expansion. The indices $\{i^{'}\}$ are $M - r - 2$ and their distribution depends on how are chosen the other $r - 2$ indices on lattice. In the continuum limit between the site l and k there is an infinite number of operators, however $r \leq N - 1$ because the operators $a_{n_1}^\dagger a_{n_1} \dots a_{n_r}^\dagger a_{n_r}$ act a $N - 1$ particles state and N is finite.

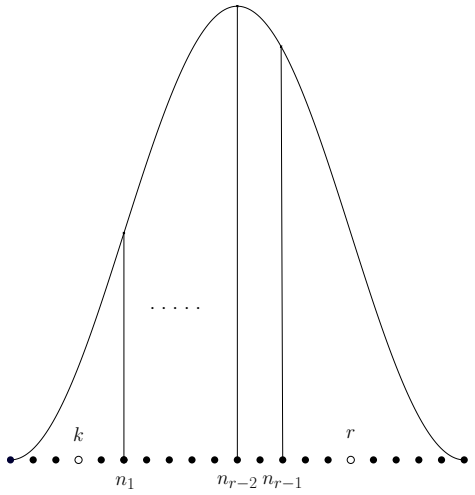

Figure A.1: Schematic representation of the lattice.

Therefore in the continuum limit $\sum_{j\in\{i'\}} p_j^2$ is a sum of infinite terms distributed practically uniformly over the lattice, then we can approximate it as:

$$p_j^2 = \tfrac{1}{2} \quad \forall j \in \{i^{'}\} \tag{A.18}$$

(we are flattening the distribution with its average value over the lattice). Equation (A.10) is then:

$$\langle BEC|a_k^\dagger a_{n_1}^\dagger a_{n_1}...a_{n_r}^\dagger a_{n_r} a_l|BEC\rangle = \\ 2^{r+1} p_k p_{n_1}^2 \dots p_{n_r}^2 p_l \frac{N}{M}\left(\frac{N-1}{M}\right)\dots\left(\frac{N-r}{M}\right)\left(1-\frac{r+2}{M}\right)^{N-r-1}. \quad (A.19)$$

Let us note that, a part for a normalization factor, if we set $p_k = \dots p_l = 1$ we obtain exactly the free expression described in [47].
At this point we have to sum this expression over the indices n_i. In order to do this non trivial passage we can consider the following approximation:

$$\sum_{n_1=k}^{l}\sum_{n_2=n_1}^{l}\cdots\sum_{n_r=n_{r-1}}^{l} p_{n_1}^2 p_{n_2}^2\dots p_{n_r}^2 = \frac{1}{r!}\left(\sum_{n=k+1}^{l} p_n^2\right)^r, \quad (A.20)$$

that is formally valid only when deal with continuos indices (i.e. with integrals...). In the continuum limit holds also that

$$2^{r+1}\frac{N}{M}\left(\frac{N-1}{M}\right)\dots\left(\frac{N-r}{M}\right)\left(1-\frac{r+2}{M}\right)^{N-r-1} \simeq \\ \left(\frac{2}{M}\right)^{r+1} N(N-1)\dots(N-r). \quad (A.21)$$

Then plugging (A.19) in (A.9), using (A.20) and (A.21), we find:

$$\langle\dots\rangle = \frac{2N}{M} p_k p_l \sum_{r=0}^{N-1}(-2)^r \frac{(N-1)\dots(N-r)}{r!}\left(\frac{2}{M}\sum_{n=k+1}^{l} p_n^2\right)^r, \quad (A.22)$$

that is,

$$\langle\dots\rangle = \frac{2N}{M} p_k p_l \left[1-2\left(\frac{2}{M}\sum_{n=k+1}^{l} p_n^2\right)\right]^{N-1}. \quad (A.23)$$

The goal now is to rewrite (A.23) in the continuum limit. This can be done remembering the correct power counting of the δ. From relation (A.3), we have that:

$$\langle c_k^\dagger c_l\rangle = \delta\langle\Psi^\dagger(x)\Psi(y)\rangle, \quad (A.24)$$

and then

$$\langle\Psi^\dagger(x)\Psi(y)\rangle = \frac{2N}{M\delta} p_k p_l \left[1-2\left(\frac{2}{M\delta}\sum_{n=k+1}^{l}\delta p_n^2\right)\right]^{N-1}. \quad (A.25)$$

Since $\delta\sum = \int$ equation (A.25) becomes:

$$\langle\Psi^\dagger(x)\Psi(y)\rangle = \frac{2N}{L}\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{L}\right)\left[1-2\left(\frac{2}{L}\int_x^y dz\ \sin^2\left(\frac{\pi z}{L}\right)\right)\right]^{N-1}, \quad (A.26)$$

this is the same expression we found in (4.33).

APPENDIX B

B.1 Demonstration of equation (4.49)

In this section we obtain relation (4.49) for the n_j starting from the definition (4.48). Firstly we set

$$\tau = t - \pi,$$

the integral (4.48), then, becomes:

$$n_j = \frac{N}{2\pi^2}\int_{-\pi}^{\pi} d\tau \int_{|\tau|-\pi}^{\pi-|\tau|} dz(\cos(jz) - (-1)^j\cos(j\tau))(\cos(z)+\cos(\tau)) \times \left[1 - \frac{2}{\pi}(|z| + \text{sign}(z)\cos(\tau)\sin(z))\right]^{N-1}. \quad \text{(B.1)}$$

At this point we note that the integration domain of the τ contains always an integer number of periods of the function $\cos(j\tau)$, so we expect that for large j the contribution from $\cos(j\tau)$ is suppressed with respect to the contribution from $\cos(jz)$, then we can ignore it. Equation (B.1) can be rewritten as:

$$n_j = \frac{2N}{\pi^2}\int_0^{\pi} d\tau \int_0^{\pi-|\tau|} dz(\cos(jz))(\cos(z)+\cos(\tau)) \left[1 - \frac{2}{\pi}(|z| + \text{sign}(z)\cos(\tau)\sin(z))\right]^{N-1}, \quad \text{(B.2)}$$

where we used the parity of the integrand function. The function:

$$\left|\left[1 - \frac{2}{\pi}(|z| + \text{sign}(z)\cos(\tau)\sin(z))\right]\right| = \beta(\tau, z, N), \quad \text{(B.3)}$$

in the integration domain of the z is always $\beta(\tau, z, N) \leq 1$. For large N the only relevant contribution will come from the values of $z \ll 1$. This fact permits us to Taylor expand the integrand function to the first useful order, that is $O(z)$, we must pay attention to the integration extremes. Considered $\epsilon \ll 1$ and, since $\text{sign}(z)z = |z|$, equation (B.2) becomes:

$$n_j = \frac{2N}{\pi^2}\int_0^{\pi} d\tau \int_0^{\epsilon} dz(\cos(jz))(1+\cos(\tau)) \left[1 - \frac{2z}{\pi}(1+\cos(\tau))\right]^{N-1}. \quad \text{(B.4)}$$

We cannot expand $\cos(jz)$ because for large j the argument is not small. It is possible to rewrite equation (B.4) as

$$n_j = \frac{2N}{\pi^2}\int_0^\pi d\tau \int_0^\epsilon dz(\cos(jz))(1+\cos(\tau))\mathrm{e}^{N\ln\left[1-\frac{2z}{\pi}(1+\cos(\tau))\right]}, \tag{B.5}$$

and, since $\log(1+x) \simeq x$ if $x \ll 1$,

$$n_j = \frac{2N}{\pi^2}\int_0^\pi d\tau \int_0^\epsilon dz(\cos(jz))(1+\cos(\tau))\mathrm{e}^{-N\frac{2z}{\pi}(1+\cos(\tau))}. \tag{B.6}$$

The integrand (B.6) is exponentially suppressed then, although we obtained it in the limit $z \ll 1$, it possible to consider $\epsilon \to \infty$ without changing significantly its value. Let us note that this is true especially for large values of N. Therefore n_j can be expressed as:

$$n_j = \frac{2N}{\pi^2}\Re\left(\int_0^\pi d\tau \int_0^\infty dz\ \mathrm{e}^{ijz}(1+\cos(\tau))\mathrm{e}^{-\frac{2N}{\pi}(1+\cos(\tau))z}\right). \tag{B.7}$$

Using that:

$$\Re\left(\int_0^\infty dz\ \mathrm{e}^{ikz}\mathrm{e}^{-\alpha z}\right) = \frac{1}{\alpha\left(1+\frac{k^2}{\alpha^2}\right)}, \tag{B.8}$$

the result of integral (B.7) is:

$$n_j = \frac{1}{\pi}\int_0^\pi d\tau \frac{1}{\left(1+\frac{j^2\pi^2}{4N^2(1+\cos(\tau))^2}\right)}, \tag{B.9}$$

that is exactly the same as (4.49).

B.2 Approximation used in the dynamical correlation function

In this paragraph it is explained the approximation that permitted us to simplify expression (4.106).
Using well known trigonometrical relations the term:

$$\sin\left(\frac{p\pi}{2L}(u+v)\right)\sin\left(\frac{q\pi}{2L}(u-v)\right)$$

can be written as:

$$\begin{aligned}&\frac{1}{2}\cos\left(\frac{\pi(p+q)v}{2L}\right)\cos\left(\frac{\pi(p-q)u}{2L}\right) - \frac{1}{2}\cos\left(\frac{\pi(p-q)v}{2L}\right)\cos\left(\frac{\pi(p+q)u}{2L}\right) + \\ &\frac{1}{2}\sin\left(\frac{\pi(p+q)v}{2L}\right)\sin\left(\frac{\pi(p-q)u}{2L}\right) - \frac{1}{2}\sin\left(\frac{\pi(p+q)u}{2L}\right)\sin\left(\frac{\pi(p-q)v}{2L}\right).\end{aligned} \tag{B.10}$$

It is not hard to show that in the thermodynamic limit the only relevant contributions to the dynamical correlation function will come from the first term of (B.10), the others are or identically zero or, simply, finite size corrections. For this

purpose it is important to remember that we are working under the assumption that $p+q \gg 1$ and $\frac{u}{L} \sim O(1)$, therefore we can rescale as usual

$$\begin{cases} \tilde{u} = \frac{u}{L} \\ v = v \\ \tilde{r} = \frac{p+q}{2L} \\ l = \frac{p-q}{2} \end{cases} . \tag{B.11}$$

In terms of the new variables (B.10) is

$$\begin{aligned} &\frac{1}{2}\cos(\pi\tilde{r}v)\cos(\pi l\tilde{u}) - \frac{1}{2}\cos\left(\frac{\pi l v}{L}\right)\cos(\pi\tilde{u}\tilde{r}L) + \\ &\frac{1}{2}\sin(\pi\tilde{r}v)\sin(\pi l\tilde{u}) - \frac{1}{2}\sin(\pi\tilde{u}\tilde{r}L)\sin\left(\frac{\pi l v}{L}\right). \end{aligned} \tag{B.12}$$

We can see immediately that as $L \to \infty$ the last term is zero. The third term does not contribute because it is odd in the variable l and in (4.109) the summation domain of l is symmetric. At this point only two terms remain:

$$\frac{1}{2}\cos(\pi\tilde{r}v)\cos(\pi l\tilde{u}) - \frac{1}{2}\cos(\pi\tilde{u}\tilde{r}L). \tag{B.13}$$

The first one gives the time-dependent correlation formerly calculated, the second one instead gives only a finite size correction. In fact, following the steps of what done in paragraph 4.4, it is easy to show that:

$$\frac{1}{L\pi}\sum_{r=1}^{\infty}\sum_{l=-r+1}^{r-1}\int_0^{\pi} d\tau\ \frac{\cos(\pi\tilde{u}\tilde{r}L)}{\left(1+\frac{r\pi^2}{4N^2A^2(\tau)}\right)}\left(\cos\left[l(\tau+\pi)\right]\right)\mathrm{e}^{i\frac{4\pi^2}{L^2}l\cdot r\cdot t} \propto \mathrm{e}^{-|L|}, \tag{B.14}$$

therefore it vanishes in the thermodynamic limit.

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