(756) A Generalized Model - Life Contingencies

Ejercicios Resueltos

Federico von Brudersdorff, 892247

1. Consider the example in the text, with net single premium given by (16.6).

(a) What change should be made in the definition of ${}^a\bar{C}_x^g$ if the annuity benefit is to be payable monthly with first payment at the time of transfer to state g?

(b) Define a similar commutation function for the case where the annuity is to be payable quarterly with the first payment at the end of the policy year of transfer to state g.

(c) Write the net single premium for the benefit in (b) using a function of the type defined in (16.7).

(a) Cambiar la renta por su versión de pagos mensuales:

$$\boxed{v^{x+1/2}\, g_x\, \ddot{a}_{x+1/2}^{g(12)} = {}^a\bar{C}_x^g}$$

(b) Cambiar la renta por su versión de pagos trimestrales:

$$\boxed{v^{x+1/2}\, g_x\, \ddot{a}_{x+1/2}^{g(4)} = {}^a\bar{C}_x^g}$$

(c) Con ${}^a\bar{C}_x^g$ definida como en (b):

$$\frac{\sum_{z=x}^{y-1} {}^a\bar{C}_x^g}{D_x^T} = \boxed{\frac{{}^a_y\bar{M}_x^g}{D_x^T}}$$

2. The function ${}_yM_x^g$, where y is fixed, is tabulated for values of x. It is desired to obtain values of ${}_zM_x^g$, where z is a fixed age less than y. Show that

$${}_zM_x^g = {}_yM_x^g - {}_yM_z^g \qquad\qquad z < y$$

$$\begin{aligned}
{}_zM_x^g = M_x^g - M_z^g &= \sum_{s=x}^{\infty} {}^aC_s^g - \sum_{r=z}^{\infty} {}^aC_r^g \\
&= \sum_{s=x}^{\infty} {}^aC_s^g - \left(\sum_{r=y}^{\infty} {}^aC_r^g + \sum_{q=z}^{y-1} {}^aC_q^g \right) \\
&= \left(\sum_{s=x}^{\infty} {}^aC_s^g - \sum_{r=y}^{\infty} {}^aC_r^g \right) - \sum_{q=z}^{y-1} {}^aC_q^g \\
&= \sum_{s=x}^{y-1} {}^aC_s^g - \sum_{q=z}^{y-1} {}^aC_q^g \\
&\boxed{{}_zM_x^g = {}_yM_x^g - {}_yM_z^g}
\end{aligned}$$

3. Using ${}^{a}\bar{C}_x^g$ as defined in the text, obtain an approximating expression for

$$\int_{20}^{50} (x-20)\, v^x\, l_x^T\, \mu_x^g\, \bar{a}_x^g \,\mathrm{d}x$$

Usando una aproximación como la del texto:

$$\begin{aligned}
\int_{20}^{50} (x-20)\, v^x\, l_x^T\, \mu_x^g\, \bar{a}_x^g \,\mathrm{d}x &\doteqdot \sum_{x=20}^{49} (x-19-1/2)\, {}^{a}\bar{C}_x^g = \sum_{x=20}^{49} \left[(x-19)\, {}^{a}\bar{C}_x^g - \frac{1}{2}\, {}^{a}\bar{C}_x^g\right] \\
&\doteqdot \sum_{x=20}^{49} \left[(x-19)\, {}^{a}\bar{C}_x^g\right] - \sum_{x=20}^{49} \frac{1}{2}\, {}^{a}\bar{C}_x^g \\
&\doteqdot \sum_{x=20}^{49} \sum_{t=x}^{49} {}^{a}\bar{C}_t^g - \frac{1}{2}\, {}_{50}^{a}\bar{M}_{20}^g = \sum_{x=20}^{49} \left[\sum_{t=x}^{\infty} {}^{a}\bar{C}_t^g - \sum_{t=50}^{\infty} {}^{a}\bar{C}_t^g\right] - \frac{1}{2}\, {}_{50}^{a}\bar{M}_{20}^g \\
&\doteqdot \sum_{x=20}^{49} \left[{}^{a}\bar{M}_x^g - {}^{a}\bar{M}_{50}^g\right] - \frac{1}{2}\, {}_{50}^{a}\bar{M}_{20}^g = \sum_{x=20}^{49} {}^{a}\bar{M}_x^g - \sum_{x=20}^{49} {}^{a}\bar{M}_{50}^g - \frac{1}{2}\, {}_{50}^{a}\bar{M}_{20}^g \\
&\doteqdot {}_{50}^{a}\bar{R}_{20}^g - 30\, {}^{a}\bar{M}_{50}^g - \frac{1}{2}\, {}_{50}^{a}\bar{M}_{20}^g
\end{aligned}$$

$$\boxed{\int_{20}^{50} (x-20)\, v^x\, l_x^T\, \mu_x^g\, \bar{a}_x^g \,\mathrm{d}x \doteqdot {}_{50}^{a}\bar{R}_{20}^g - 30\, {}^{a}\bar{M}_{50}^g - \frac{1}{2}\, {}_{50}^{a}\bar{M}_{20}^g}$$

4. For a given age x (less than 65), the probability of retirement $q^r_{x+t} = 0,002(t+1)$ for all t. Show that:

$$ {}_{65}M^r_x = 0,002\, v^{\frac{1}{2}} * [S^T_x - S^T_{65} - (65-x)\, N^T_{65}]. $$

$$
\begin{aligned}
{}_{65}M^r_x &= \sum_{t=x}^{64} \bar{C}^r_t = \sum_{t=x}^{64} v^{t+\frac{1}{2}-x}\, l^T_t\, q^r_t = v^{\frac{1}{2}} \sum_{t=x}^{64} v^{t-x}\, l^T_t\, 0,002\,(t+1-x) \\
&= 0,002\, v^{\frac{1}{2}} \sum_{t=0}^{64-x} v^t\, l^T_{x+t}\,(t+1) = 0,002\, v^{\frac{1}{2}} \sum_{t=0}^{64-x} D(x+t)\,(t+1) \\
&\boxed{{}_{65}M^r_x = 0,002\, v^{\frac{1}{2}}\, \left[S^T_x - S^T_{65} - (65-x)N^T_{65}\right]}
\end{aligned}
$$

5. If an employee aged x retires or becomes permanently disables within the next year, he will receive an annual life income of \$4000. Find the net single premium for this one year's coverage, assuming that the income is payable on a continuous basis and given the following:

$$D_x^T = 1000 \qquad {}^a\bar{C}_x^r = 160 \qquad i_x = 1,5\, r_x \qquad \bar{a}^i_{x+\frac{1}{2}} = 0,08\, \bar{a}^r_{x+\frac{1}{2}}$$

$$\begin{aligned}
160 = {}^a\bar{C}_x^r = \bar{C}_x^r\, \bar{a}^r_{x+\frac{1}{2}} &= v^{1/2}\, r_x\, \bar{a}^r_{x+\frac{1}{2}} \\
&= v^{1/2}\, \frac{i_x}{1,5}\, \frac{\bar{a}^i_{x+\frac{1}{2}}}{0,08} = \frac{{}^a\bar{C}_x^i}{0,12} \\
19,20 &= {}^a\bar{C}_x^i
\end{aligned}$$

$$\Rightarrow \text{Prima} = 4000\, \frac{{}^a\bar{C}_x^i + {}^a\bar{C}_x^r}{D_x^T} = 4000\, \frac{19,20 + 160}{1000}$$

$$\boxed{\text{Prima} = 716,80}$$

6. How should (16.8) be modified if the plan provides that participants (entering the plan at age x) who are still in service at the end of $65 - x$ years must retire at that time?

(16.8)

$$PV_x^r = \frac{\bar{M}_x^r}{D_x^T} = \frac{\sum_{t=0}^{\infty} \bar{C}_{x+t}^r}{D_x^T} = \frac{\sum_{t=0}^{\infty} v^{t+1/2}\, r_{x+t}}{D_x^T}$$

Agregando los retiros forzados y eliminando los no retirados:

$$PV'^r_x = \frac{\sum_{t=0}^{\infty} v^{t+1/2}\, r_{x+t} - \sum_{t=65-x}^{\infty} v^{t+1/2}\, r_{x+t} + l_{65}^T\, v^{65-x}}{D_x^T}$$

$$PV'^r_x = \frac{\sum_{t=0}^{64-x} v^{t+1/2}\, r_{x+t} + l_{65}^T\, v^{65-x}}{D_x^T}$$

$$\Rightarrow \boxed{PV'^r_x = \frac{\bar{M}_x^r - \bar{M}_{65}^r + D_{65}^T}{D_x^T}}$$

7. A salary scale function has the property that $S_y = (1+i)^{y-x} S_x$. Find an expression for ${}_5Z_{65}$ in terms of S_x and functions involving interest only.

$$ {}_mZ_y \risingdotseq \frac{1}{2m}\left[S_y + 2(S_{y-1} + S_{y-2} ... + S_{y-(m-1)}) + S_{y-m}\right] $$

$$ \Rightarrow {}_5Z_{65} \risingdotseq \frac{1}{10}\{(1+i)^{65-x} + 2\left[(1+i)^{64-x} + (1+i)^{63-x} + (1+i)^{62-x} + (1+i)^{61-x}\right] + (1+i)^{60-x}\} * S_x $$

$$ \risingdotseq \frac{S_x}{10}(1+i)^{65-x}\{(1+i)^{0} + 2\left[(1+i)^{-1} + (1+i)^{-2} + (1+i)^{-3} + (1+i)^{-4}\right] + (1+i)^{-5}\} $$

$$ \risingdotseq \frac{S_x}{10}(1+i)^{65-x}\{\sum_{t=0}^{5}(1+i)^{-t} + \sum_{s=1}^{4}(1+i)^{-t}\} = \frac{S_x}{10}(1+i)^{65-x}\{\sum_{t=0}^{5} v^t + \sum_{s=1}^{4} v^t\} $$

$$ \boxed{{}_5Z_{65} \risingdotseq \frac{S_x}{10} * (1+i)^{65-x}\left(\ddot{a}_{\overline{6}|} + a_{\overline{4}|}\right)} $$

8. A pension plan provides a retirement income of %20 of the five-year final average salary. For an employee now aged x (where x is not within five years of the earliest retirement age) who is earning \$10000 per year, express the present value of this benefit in terms of commutation functions.

$$0,01k * (ES)_{y:\overline{m}|} * \frac{{}^{a}\bar{M}^{r}_{x}}{D^{T}_{x}} = 0,01k * (AS)_x * \frac{{}_{m}Z_y}{S_x} * \frac{{}^{a}\bar{M}^{r}_{x}}{D^{T}_{x}}$$

$$\Rightarrow 0,2 * (ES)_{y:\overline{5}|} * \frac{{}^{a}\bar{M}^{r}_{x}}{D^{T}_{x}} = 0,2 * 10000 * \frac{{}_{5}Z_y}{S_x} * \frac{{}^{a}\bar{M}^{r}_{x}}{D^{T}_{x}}$$

$$\boxed{\text{Valor actual del beneficio} = 2000 * \frac{{}^{Za}\bar{M}^{r}_{x}}{{}^{S}D^{T}_{x}}}$$

9. Express in terms of commutation functions the present value of the following benefits for an employee hired at age x:

(i) a retirement pension equal to 2% of annual salary at the rate of pay applicable in the year of retirement for each of the first 20 completed years of service, plus 1% of such salary for each of the following 10 completed years of service; and

(ii) a disability pension equal to \$100 a year for each completed year of service, such pension not to exceed \$3000 a year.

(i)

$$
\begin{aligned}
&2\%(ES)_x * \frac{{}^a\bar{R}^r_{x+1} - {}^a\bar{R}^r_{x+21}}{D^T_x} + 1\%(ES)_x * \frac{{}^a\bar{R}^r_{x+21} - {}^a\bar{R}^r_{x+31}}{D^T_x} \\
&= 2\%(AS)_x * \frac{S_y}{S_x} * \frac{{}^a\bar{R}^r_{x+1} - {}^a\bar{R}^r_{x+21}}{D^T_x} + 1\%(AS)_x * \frac{S_y}{S_x} \frac{{}^a\bar{R}^r_{x+21} - {}^a\bar{R}^r_{x+31}}{D^T_x} \\
&= 2\%(AS)_x * \frac{{}^{Sa}\bar{R}^r_{x+1} - {}^{Sa}\bar{R}^r_{x+21}}{{}^S D^T_x} + 1\%(AS)_x \frac{{}^{Sa}\bar{R}^r_{x+21} - {}^{Sa}\bar{R}^r_{x+31}}{{}^S D^T_x}
\end{aligned}
$$

$$
\Rightarrow \boxed{\text{Valor actual} = 1\%(AS)_x \frac{2\,{}^{Sa}\bar{R}^r_{x+1} - {}^{Sa}\bar{R}^r_{x+21} - {}^{Sa}\bar{R}^r_{x+31}}{{}^S D^T_x}}
$$

(ii) 30 años completos como máximo:

$$
{}^a\bar{R}^i_{x+1} - {}^a\bar{R}^i_{x+1+30}
$$

$$
\Rightarrow \boxed{\text{Valor actual} = 100 * \frac{{}^a\bar{R}^i_{x+1} - {}^a\bar{R}^i_{x+31}}{D^T_x}}
$$

10. To what does ${}^{S''i}\bar{R}^{w}_{x+1}$ reduce if $j = i$?

$$
\begin{aligned}
{}^{S''i}\bar{R}^{w}_{x+1}\Big|_{j=i} &= \sum_{t=0}^{\infty} \frac{S_{x+t}}{(1+j)^{x+t}} * {}^{i}\bar{M}^{w}_{x+1+t}\Big|_{j=i} \\
&= \sum_{t=0}^{\infty} \frac{S_{x+t}}{(1+j)^{x+t}} * \sum_{s=1}^{\infty} (1+i)^{x+t+s}\, \bar{C}^{w}_{x+t+s}\Big|_{j=i} \\
&= \sum_{t=0}^{\infty} \frac{S_{x+t}}{(1+j)^{x+t}} * \sum_{s=1}^{\infty} (1+i)^{x+t+s}\, v^{x+t+s}\, w_{x+t+s} *\Big|_{j=i} \\
&= \sum_{t=0}^{\infty} \frac{S_{x+t}}{(1+j)^{x+t}} * \sum_{s=1}^{\infty} (1+j)^{x+t+s+\frac{1}{2}}\, v^{x+t+s}\, w_{x+t+s} \\
&= \sum_{t=0}^{\infty} S_{x+t} * \sum_{s=1}^{\infty} (1+j)^{s}\, v^{x+t+s+\frac{1}{2}}\, w_{x+t+s} \\
&= \sum_{t=0}^{\infty} S_{x+t} * \sum_{s=1}^{\infty} v^{x+t+\frac{1}{2}}\, w_{x+t+s}
\end{aligned}
$$

$$
\boxed{{}^{S''i}\bar{R}^{w}_{x+1}\Big|_{j=i} = \sum_{t=0}^{\infty} \left(S_{x+t} * v^{x+t+\frac{1}{2}} \sum_{s=1}^{\infty} w_{x+t+s} \right)}
$$

11. To what does (16.34) reduce if $j = i$?

(16.34)

$$^{j}(TPC)_x\,(1+j)^{\frac{1}{2}}\,\frac{^{j}\bar{M}^{w}_x}{^{j}D^{T}_x}$$

$$\begin{aligned}
\Rightarrow {}^{j}(TPC)_x\,(1+j)^{\frac{1}{2}}\,\frac{^{j}\bar{M}^{w}_x}{^{j}D^{T}_x}\bigg|_{i=j} &= {}^{i}(TPC)_x\,(1+i)^{\frac{1}{2}}\,\frac{^{i}\bar{M}^{w}_x}{^{i}D^{T}_x}\\
&= {}^{i}(TPC)_x\,(1+i)^{\frac{1}{2}}\,\frac{\sum_{t=0}^{\infty}(1+i)^{x+t}\,v^{x+t+\frac{1}{2}} * w_{x+t}}{(1+i)^{x}\,l^{T}_x\,v^{x}}\\
&= {}^{i}(TPC)_x\,\frac{\sum_{t=0}^{\infty}(1+i)^{t+\frac{1}{2}}\,v^{t+\frac{1}{2}} * w_{x+t}}{l^{T}_x}
\end{aligned}$$

$$\Rightarrow \boxed{{}^{j}(TPC)_x\,(1+j)^{\frac{1}{2}}\,\frac{^{j}\bar{M}^{w}_x}{^{j}D^{T}_x}\bigg|_{i=j} = {}^{i}(TPC)_x\,\frac{\sum_{t=0}^{\infty} w_{x+t}}{l^{T}_x}}$$

12. A pension plan requires contributions by members of 3% of salary. It provides on withdrawal a return of contributions with interest at rate j. An employee now aged 40 has a monthly salary of \$500. Give an expression in terms of commutation functions for each of the following:

(a) the present value of the withdrawal benefit arising from contributions in the year of age 50 to 51;

(b) the total present value of the withdrawal benefit.

(a)

$$(ES)_{x+t} = (AS)_x \frac{S_{x+t}}{S_x}$$

$$\begin{aligned}
&\Rightarrow 3\%(AS)_x \frac{S_{x+t}}{S_x} * \frac{1}{(1+j)^{x+t}} * \frac{\frac{1}{2}\,{}^{j}\bar{C}^{w}_{x+t} + \sum_{n=1}^{\infty} {}^{j}\bar{C}^{w}_{x+t+n}}{D^{T}_{x}} \\
&= 3\%(AS)_x \frac{S_{x+t}}{(1+j)^{x+t}} * \frac{\frac{1}{2}\,{}^{j}\bar{C}^{w}_{x+t} + \sum_{n=1}^{\infty} {}^{j}\bar{C}^{w}_{x+t+n}}{{}^{S}D^{T}_{x}} \\
&= 3\%(AS)_x * \frac{\frac{1}{2}\,{}^{S}\bar{C}^{w}_{x+t} + {}^{S''j}\bar{M}^{w}_{x+t+1}}{{}^{S}D^{T}_{x}} \\
&= 3\% * (500 * 12) * \frac{\frac{1}{2}\,{}^{S}\bar{C}^{w}_{50} + {}^{S''j}\bar{M}^{w}_{51}}{{}^{S}D^{T}_{40}}
\end{aligned}$$

$$\boxed{\text{Valor presente} = 180 * \frac{\frac{1}{2}\,{}^{S}\bar{C}^{w}_{50} + {}^{S''j}\bar{M}^{w}_{51}}{{}^{S}D^{T}_{40}}}$$

(b)

$$\sum_{t=0}^{\infty} 180 * \frac{1/2\,{}^{S}\bar{C}^{w}_{40+t} + {}^{S''j}\bar{M}^{w}_{41+t}}{{}^{S}D^{T}_{40}} = \boxed{180 * \frac{1/2\,{}^{S}\bar{M}^{w}_{40} + {}^{S''j}\bar{R}^{w}_{41}}{{}^{S}D^{T}_{40}} = \text{Valor presente}}$$

13. For each of the following disability benefits, payable in the event of disability prior to age 60, find the net single premium at age 35 in terms of elementary functions and also in terms of commutation functions:

(a) a payment of \$100 at the end of a four months waiting period;

(b) a continuous disabled life annuity of \$1000 per year, commencing at the end of a 6 months waiting period;

(c) a continuous disabled life annuity of \$2000 per year, commencing at the end of a 3 months waiting period and payable up to age 65.

(a)

$$100 * \sum_{t=35}^{59} \frac{v^{\frac{4}{12}} i_t \, v^{t-35+\frac{1}{2}}}{l^{aa}_{35}} = 100 * \sum_{t=35}^{59} \frac{v^{\frac{1}{3}} i_t \, v^{t-35+\frac{1}{2}}}{l^{aa}_{35}}$$

$$\boxed{\frac{100}{l^{aa}_{35}} * \sum_{t=35}^{59} i_t \, v^{t-35+\frac{5}{6}} = 100 * v^{\frac{1}{3}} * \frac{{}_{60}\bar{M}^{i}_{35}}{D^{aa}_{35}}}$$

(b)

$$1000 * \sum_{x=35}^{59} \frac{v^{(x-35)+\frac{1}{2}+\frac{6}{12}} * i_x \; {}_{\frac{6}{12}|}\bar{a}^{i}_{(x+\frac{1}{2})}}{l^{aa}_{35}}$$

$$\boxed{\frac{1000}{l^{aa}_{35}} * \sum_{x=35}^{59} v^{x-34} * i_x \, \bar{a}^{i}_{x+1} = 1000 * \frac{{}^{\omega}_{60}\bar{M}^{i}_{35}}{D^{aa}_{35}}}$$

(c)

$$2000 * \sum_{x=35}^{59} \frac{v^{(x-35)+\frac{1}{2}+\frac{3}{12}} * i_x \; {}_{\frac{3}{12}|}\bar{a}^{i}_{(x+\frac{1}{2}):\overline{65-(x+\frac{1}{2})}|}}{l^{aa}_{35}}$$

$$\boxed{\frac{2000}{l^{aa}_{35}} * \sum_{x=35}^{59} v^{x-34,25} * i_x \, \bar{a}^{i}_{(x+\frac{3}{4}):\overline{64,25-x}|} = 2000 * \frac{{}^{65}_{60}\bar{M}^{i}_{35}}{D^{aa}_{35}}}$$

14. In deriving formulas (16.46a) and (16.46b), the following approximations is used:

$$\ddot{a}^{i(12)}_{[x+\frac{1}{2}]+\frac{m}{12}:\overline{x+n-z-\frac{6+m}{12}}|} \doteqdot \bar{a}^{i}_{[x+\frac{1}{2}]+\frac{m}{12}:\overline{x+n-z-\frac{6+m}{12}}|} + k, \text{ where } k = \frac{1}{24}$$

What would be the expression for k on the basis of tha standard approximations?

$$\alpha = [x + \frac{1}{2}] + \frac{m}{12} \qquad\qquad \beta = x + n - z - \frac{6+m}{12}$$

$$\begin{aligned}
\ddot{a}^{i(12)}_{\alpha:\overline{\beta}|} &\simeq \ddot{a}^{i}_{\alpha:\overline{\beta}|} - \frac{12-1}{24} * (1 - {}_{\beta}E_{\alpha}) \\
&\simeq \ddot{a}^{i}_{\alpha:\overline{\beta}|} - \frac{11}{24} * (1 - {}_{\beta}E_{\alpha}) \\
&\simeq \ddot{a}^{i}_{\alpha:\overline{\beta}|} - \frac{11}{24} * (1 - {}_{\beta}E_{\alpha}) \\
&\simeq \bar{a}^{i}_{\alpha:\overline{\beta}|} + \frac{\left(1 - {}_{\beta}E_{\alpha}\right)}{2} - \frac{11}{24} * (1 - {}_{\beta}E_{\alpha})
\end{aligned}$$

$$\boxed{\ddot{a}^{i(12)}_{[x+\frac{1}{2}]+\frac{m}{12}:\overline{x+n-z-\frac{6+m}{12}}|} \doteqdot \bar{a}^{i}_{[x+\frac{1}{2}]+\frac{m}{12}:\overline{x+n-z-\frac{6+m}{12}}|} + \frac{1}{24} * \left(1 - {}_{x+n-z-\frac{6+m}{12}}E_{[x+\frac{1}{2}]+\frac{m}{12}}\right)}$$

$$\boxed{k = \frac{1}{24} * \left(1 - {}_{x+n-z-\frac{6+m}{12}}E_{[x+\frac{1}{2}]+\frac{m}{12}}\right)}$$

15. Find, in terms of commutation functions, the net annual premium at age 30 for a disability income benefit of \$10 per month issued as a rider with a 20-year endowment policy. The waiting period is 6 months, and the disability income ceases upon maturity of the endowment.

$$I * \frac{{}^{x+n}\bar{M}_x^i + {}^1/{}_{24}\, v^{m/12}\; {}_{x+n}\bar{M}_x^i}{N_x^{aa} - N_{x+n}^{aa}}$$

$$\boxed{120 * \frac{{}^{50}\bar{M}_{30}^i + {}^1/{}_{24}\, v^{1/2}\; {}_{50}\bar{M}_{30}^i}{N_{30}^{aa} - N_{50}^{aa}}}$$

16. For the benefit described in exercise 15, find the net annual premium at age 45 if disability must occur before age 60.

$$I * \frac{{}^{x+n}_{\;\;y}\bar{M}^i_x + {}^1\!/_{24}\, v^{m/12}\; {}_y\bar{M}^i_x}{N^{aa}_x - N^{aa}_y}$$

$$\boxed{120 * \frac{{}^{65}_{60}\bar{M}^i_{45} + {}^1\!/_{24}\, v^{1/2}\; {}_{60}\bar{M}^i_{45}}{N^{aa}_{45} - N^{aa}_{60}}}$$

17. A disability waiver of premium benefit included in an ordinary life policy issued at age 30 provides for waiver of premium falling due after a six month waiting period and during continued disability, but only if disability occurs before age 55. Premiums falling due during the waiting period are waived retroactively. If the gross annual premium to be waived is \$100, find the net annual premium for the waiver of premium benefit.

Waiver original:

$$\frac{{}^{x+n}_{\quad y}\bar{M}^i_x}{N^{aa}_x - N^{aa}_y} = \frac{{}^{\omega}_{55}\bar{M}^i_{30}}{N^{aa}_{30} - N^{aa}_{55}}$$

Beneficio retroactivo:

$$\frac{\frac{m}{12}\, v^{m/12}\, {}_{y}\bar{M}^i_x}{N^{aa}_x - N^{aa}_y} = \frac{\frac{6}{12}\, v^{6/12}\, {}_{55}\bar{M}^i_{30}}{N^{aa}_{30} - N^{aa}_{55}}$$

$$\Rightarrow 100 * \left(\frac{{}^{\omega}_{55}\bar{M}^i_{30}}{N^{aa}_{30} - N^{aa}_{55}} + \frac{\frac{6}{12}\, v^{6/12}\, {}_{55}\bar{M}^i_{30}}{N^{aa}_{30} - N^{aa}_{55}} \right) = \boxed{50 * \left(\frac{2 * {}^{\omega}_{55}\bar{M}^i_{30} + v^{1/2}\, {}_{55}\bar{M}^i_{30}}{N^{aa}_{30} - N^{aa}_{55}} \right)}$$

18. For the benefit described in exercise 17, find the net annual premium if the benefit is included:

(a) in a 15-payment 25-year endowment issued at age 35;

(b) in a 30-payment life policy issued at age 45.

(a) Waiver original:

$$\frac{{}^{x+n}\bar{M}^i_x}{N^{aa}_x - N^{aa}_{x+n}} = \frac{{}^{50}\bar{M}^i_{35}}{N^{aa}_{35} - N^{aa}_{50}}$$

Beneficio retroactivo:

$$\frac{\frac{m}{12}\, v^{m/12}\; {}_{x+n}\bar{M}^i_x}{N^{aa}_x - N^{aa}_{x+n}} = \frac{\frac{6}{12}\, v^{6/12}\; {}_{50}\bar{M}^i_{35}}{N^{aa}_{35} - N^{aa}_{50}}$$

$$\Rightarrow 100 * \left(\frac{{}^{50}\bar{M}^i_{35}}{N^{aa}_{35} - N^{aa}_{50}} + \frac{\frac{6}{12}\, v^{6/12}\; {}_{50}\bar{M}^i_{35}}{N^{aa}_{35} - N^{aa}_{50}} \right) = \boxed{50 * \left(\frac{2 * {}^{50}\bar{M}^i_{35} + v^{1/2}\; {}_{50}\bar{M}^i_{35}}{N^{aa}_{35} - N^{aa}_{50}} \right)}$$

(b) Waiver original:

$$\frac{{}^{x+n}_{\;y}\bar{M}^i_x}{N^{aa}_x - N^{aa}_y} = \frac{{}^{75}_{55}\bar{M}^i_{45}}{N^{aa}_{45} - N^{aa}_{55}}$$

Beneficio retroactivo:

$$\frac{\frac{m}{12}\, v^{m/12}\; {}_{y}\bar{M}^i_x}{N^{aa}_x - N^{aa}_y} = \frac{\frac{6}{12}\, v^{6/12}\; {}_{55}\bar{M}^i_{45}}{N^{aa}_{45} - N^{aa}_{55}}$$

$$\Rightarrow 100 * \left(\frac{{}^{75}_{55}\bar{M}^i_{45}}{N^{aa}_{45} - N^{aa}_{55}} + \frac{\frac{6}{12}\, v^{6/12}\; {}_{55}\bar{M}^i_{45}}{N^{aa}_{45} - N^{aa}_{55}} \right) = \boxed{50 * \left(\frac{2 * {}^{75}_{55}\bar{M}^i_{45} + v^{1/2}\; {}_{55}\bar{M}^i_{45}}{N^{aa}_{45} - N^{aa}_{55}} \right)}$$

19. Write formulas for the active life reserves corresponding to (16.49a) and (16.49b) for the case where $x+n<y$ and $t<n$.

(16.49a)

$$120\left[\frac{{}_{y}^{\omega}\bar{M}_{x+t}^{i}+\frac{1}{24}*v^{m/12}*{}_{y}\bar{M}_{x+t}^{i}}{D_{x+t}^{aa}}-P_{x}^{I}*\ddot{a}_{x+t:\overline{y-x-t|}}^{aa}\right]$$

Con $x+n<y$ y $t<n$, el único cambio es en el plazo de pagos de prima:

$$\boxed{120\left(\frac{{}_{y}^{\omega}\bar{M}_{x+t}^{i}+\frac{1}{24}*v^{m/12}*{}_{y}\bar{M}_{x+t}^{i}}{D_{x+t}^{aa}}-P_{x}^{I}*\ddot{a}_{x+t:\overline{n-t|}}^{aa}\right)}$$

(16.49b)

$$P\left[\frac{{}_{y}^{x+n}\bar{M}_{x+t}^{i}+\frac{1}{24}*v^{m/12}*{}_{y}\bar{M}_{x+t}^{i}}{D_{x+t}^{aa}}-P_{x}^{W}*\ddot{a}_{x+t:\overline{y-x-t|}}^{aa}\right]$$

Con $x+n<y$ y $t<n$, cambio el plazo de pagos y de prestación:

$$\boxed{P\left(\frac{{}^{x+n}\bar{M}_{x+t}^{i}+\frac{1}{24}*v^{m/12}*{}_{x+n}\bar{M}_{x+t}^{i}}{D_{x+t}^{aa}}-P_{x}^{W}*\ddot{a}_{x+t:\overline{n-t|}}^{aa}\right)}$$

20. If the reserve given by (16.49a) is denoted by 120 ${}_tV_x^I$, show that

$$ {}_{t+1}V_x^I = \left({}_tV_x^I + P_x^I\right) u_{x+t}^{aa} - \left({}^{\omega}\bar{k}_{x+t}^i + {}^1\!/_{24}\, v^{m/12}\, \bar{k}_{x+t}^i\right), $$

where

$$ u_x^{aa} = \frac{D_x^{aa}}{D_{x+1}^{aa}} \qquad \bar{k}_x^i = \frac{\bar{C}_x^i}{D_{x+1}^{aa}} \qquad {}^{\omega}\bar{k}_x^i = \frac{{}^{\omega}\bar{C}_x^i}{D_{x+1}^{aa}} $$

.

$$ \begin{aligned}
{}_{t+1}V_x^I &= \left[\frac{{}^{\omega}_{y}\bar{M}_{x+t+1}^i + \frac{1}{24} * v^{m/12} * {}_y\bar{M}_{x+t+1}^i}{D_{x+t+1}^{aa}} - P_x^I * \ddot{a}_{x+t+1:\overline{y-x-t-1|}}^{aa}\right] \\
&= \frac{{}^{\omega}_{y}\bar{M}_{x+t}^i - {}^{\omega}\bar{C}_{x+t}^i + \frac{1}{24} * v^{m/12} * \left({}_y\bar{M}_{x+t}^i - \bar{C}_{x+t}^i\right)}{D_{x+t+1}^{aa}} - P_x^I * \ddot{a}_{x+t+1:\overline{y-x-t-1|}}^{aa} \\
&= \frac{{}^{\omega}_{y}\bar{M}_{x+t}^i - {}^{\omega}\bar{C}_{x+t}^i + \frac{1}{24} * v^{m/12} * \left({}_y\bar{M}_{x+t}^i - \bar{C}_{x+t}^i\right)}{D_{x+t+1}^{aa}} - P_x^I * \left(\ddot{a}_{x+t:\overline{y-x-t|}}^{aa} - 1\right) * \frac{D_{x+t}^{aa}}{D_{x+t+1}^{aa}} \\
&= \frac{{}^{\omega}_{y}\bar{M}_{x+t}^i + \frac{1}{24} * v^{m/12}\ {}_y\bar{M}_{x+t}^i - \frac{1}{24} * v^{m/12}\, \bar{C}_{x+t}^i - {}^{\omega}\bar{C}_{x+t}^i}{D_{x+t+1}^{aa} * \frac{D_{x+t}^{aa}}{D_{x+t}^{aa}}} - P_x^I * \left(\ddot{a}_{x+t:\overline{y-x-t|}}^{aa} - 1\right) * u_{x+t}^{aa} \\
&= \frac{{}^{\omega}_{y}\bar{M}_{x+t}^i + \frac{1}{24} * v^{m/12}\ {}_y\bar{M}_{x+t}^i}{D_{x+t}^{aa}} * u_{x+t}^{aa} - \frac{\frac{1}{24} * v^{m/12}\, \bar{C}_{x+t}^i - {}^{\omega}\bar{C}_{x+t}^i}{D_{x+t+1}^{aa}} - P_x^I * \left(\ddot{a}_{x+t:\overline{y-x-t|}}^{aa} - 1\right) * u_{x+t}^{aa} \\
&= \frac{{}^{\omega}_{y}\bar{M}_{x+t}^i + \frac{1}{24} * v^{m/12}\ {}_y\bar{M}_{x+t}^i}{D_{x+t}^{aa}} * u_{x+t}^{aa} - \frac{1}{24} * v^{m/12}\, \bar{k}_{x+t}^i - {}^{\omega}\bar{k}_{x+t}^i - P_x^I * \left(\ddot{a}_{x+t:\overline{y-x-t|}}^{aa} - 1\right) * u_{x+t}^{aa}
\end{aligned} $$

$$ \boxed{{}_{t+1}V_x^I = \left({}_tV_x^I + P_x^I\right) u_{x+t}^{aa} - \left({}^{\omega}\bar{k}_{x+t}^i + {}^1\!/_{24}\, v^{m/12}\, \bar{k}_{x+t}^i\right)} $$

21. A pension plan provides at retirement an annual pension of 20% of one year's salary at the rate of pay in the year of retirement plus, for each completed year of service at retirement, 1% of m-year final average salary. The pension is paid on a five years certain and life thereafter basis. The employee contributes 1/2% of each year's salary. Employee contributions cease after 30 years even if the employee has not retired. If the employee terminates or dies prior to retirement, the accumuluation of his contributions at rate of interest j is returned. What is the present value of all future benefits for an employee hired at age x at an annueal salary of \$5000?

Beneficio por ingresos:

$$
\begin{aligned}
20\%\,(AS)_x \sum_{y=x}^{\infty} \frac{S_y}{S_x} * \frac{\bar{C}^r_y\,(\bar{a}_{\overline{5|}} + {}_{5|}\bar{a}^r_{y+1/2})}{D^T_x} &= 20\%\,5000 \sum_{y=x}^{\infty} \frac{{}^{Sa}\bar{C}^r_y}{{}^{S}D^T_x} \\
&= 1000 * \frac{{}^{Sa}\bar{M}^r_x}{{}^{S}D^T_x}
\end{aligned}
$$

Beneficio por años de servicio:

$$
\begin{aligned}
\sum_{y=x+1}^{\infty} (y-x)1\%(AS)_x * \frac{{}_mZ_y}{S_x} * \frac{\bar{C}^r_y\,(\bar{a}_{\overline{5|}} + {}_{5|}\bar{a}^r_{y+1/2})}{D^T_x} &= 1\%\,5000 \sum_{y=x+1}^{\infty} (y-x) * \frac{{}^{Za}\bar{C}^r_y}{{}^{S}D^T_x} \\
&= 50 * \frac{{}^{Za}\bar{R}^r_{x+1}}{{}^{S}D^T_x}
\end{aligned}
$$

Beneficio por terminación (w) o muerte (d):

$$
\sum_{t=0}^{29} 0,5\%(AS)_x \, \frac{\frac{1}{2}\,{}^{S}\bar{C}^?_{x+t} + {}^{S''k}\bar{M}^?_{x+t+1}}{D^T_x} = 25 * \frac{\frac{1}{2}\,{}^{S}_{x+30}\bar{M}^?_x + {}^{S''j}\bar{R}^?_{x+1}}{{}^{S}_{x+31}D^T_x}
$$

$$
\boxed{\text{Valor presente} = 1000\,\frac{{}^{Sa}\bar{M}^r_x}{{}^{S}D^T_x} + 50\,\frac{{}^{Za}\bar{R}^r_{x+1}}{{}^{S}D^T_x} + 25\,\frac{\frac{1}{2}\,{}^{S}_{x+30}\bar{M}^w_x + {}^{S''j}_{x+31}\bar{R}^w_{x+1}}{{}^{S}D^T_x} + 25\,\frac{\frac{1}{2}\,{}^{S}_{x+30}\bar{M}^d_x + {}^{S''j}_{x+31}\bar{R}^d_{x+1}}{{}^{S}D^T_x}}
$$

22. A disability rider, attached at issue to an ordinary life policy on (x), provides the following benefits if (x) becomes totally disabled before age y and reamins disabled during a waiting period of 6 months:

(i) a monthly income commencing at the end of the waiting period and payable for life during continued disability;

(ii) waiver of the life insurance premiums falling due on or after the initial date of disability and during continued disability.

Find the amount of the monthly income and the amount of the annual premium to be waived (assumed payable continuously) if the net annual premium for the disability ride is

$$\frac{1720\ {}^{\omega}_{y}\bar{M}^{i}_{x} + 200\, v^{1/2}\ {}_{y}\bar{M}^{i}_{x}}{N^{aa}_{x} - N^{aa}_{y}}$$

$$\frac{1720\ {}^{\omega}_{y}\bar{M}^{i}_{x} + 200\, v^{1/2}\ {}_{y}\bar{M}^{i}_{x}}{N^{aa}_{x} - N^{aa}_{y}} = (12A_i)\,\frac{{}^{\omega}_{y}\bar{M}^{i}_{x} + 1/24\, v^{m/12}\ {}_{y}\bar{M}^{i}_{x}}{N^{aa}_{x} - N^{aa}_{y}} + A_{ii}\,\frac{\frac{m}{12}\, v^{m/12}\ {}_{y}\bar{M}^{i}_{x}}{N^{aa}_{x} - N^{aa}_{y}} + A_{ii}\,\frac{{}^{\omega}_{y}M^{i}_{x}}{N^{aa}_{x} - N^{aa}_{y}}$$

$$\Rightarrow 1720\ {}^{\omega}_{y}\bar{M}^{i}_{x} + 200\, v^{1/2}\ {}_{y}\bar{M}^{i}_{x} = (12A_i)\left({}^{\omega}_{y}\bar{M}^{i}_{x} + 1/24\, v^{m/12}\ {}_{y}\bar{M}^{i}_{x}\right) + A_{ii}\,\frac{m}{12}\, v^{m/12}\ {}_{y}\bar{M}^{i}_{x} + A_{ii}\ {}^{\omega}_{y}M^{i}_{x}$$

$$\Rightarrow \begin{cases} 1720\ {}^{\omega}_{y}\bar{M}^{i}_{x} = (12A_i)\ {}^{\omega}_{y}\bar{M}^{i}_{x} + A_{ii}\ {}^{\omega}_{y}M^{i}_{x} \\ 200\, v^{1/2}\ {}_{y}\bar{M}^{i}_{x} = (12A_i)1/24\, v^{m/12}\ {}_{y}\bar{M}^{i}_{x} + A_{ii}\,\frac{m}{12}\, v^{m/12}\ {}_{y}\bar{M}^{i}_{x} \end{cases}$$

Por lo tanto:

$$\begin{aligned} 1720\ {}^{\omega}_{y}\bar{M}^{i}_{x} &= (12A_i)\ {}^{\omega}_{y}\bar{M}^{i}_{x} + A_{ii}\ {}^{\omega}_{y}M^{i}_{x} \\ 1720 &= (12A_i) + A_{ii} \\ 1720 - (12A_i) &= A_{ii} \end{aligned}$$

y, con $m = 6$,

$$\begin{aligned} 200\, v^{1/2}\ {}_{y}\bar{M}^{i}_{x} &= (12A_i)\,\frac{1}{24}\, v^{6/12}\ {}_{y}\bar{M}^{i}_{x} + A_{ii}\,\frac{6}{12}\, v^{6/12}\ {}_{y}\bar{M}^{i}_{x} \\ 200 &= \frac{A_i}{2} + \frac{A_{ii}}{2} \\ \Rightarrow 400 &= A_i + A_{ii} = A_i + (1720 - 12A_i) \end{aligned}$$

$\boxed{A_i = 120}$ $\boxed{A_{ii} = 280}$