(756) Population Theory - Life Contingencies

Ejercicios Resueltos

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1. Show that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{T_x}{l_x}\right) = \dfrac{\mu_x T_x}{l_x} - 1$.

Derivando por partes:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{T_x}{l_x}\right) = \frac{\mathrm{d}T_x}{\mathrm{d}x} * \frac{1}{l_x} + \frac{\mathrm{d}(l_x^{-1})}{\mathrm{d}x} * T_x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{T_x}{l_x}\right) = (-l_x) * \frac{1}{l_x} + (-1) * \frac{1}{l_x^2} * (-\mu_x) * l_x * T_x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{T_x}{l_x}\right) = (-1) + \frac{\mu_x * T_x}{l_x}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{T_x}{l_x}\right) = \frac{\mu_x * T_x}{l_x} - 1}$$

2. What mortality function is represented by each of the following when $i = 0$?

(a) $l_x \bar{A}_x$

(b) $\bar{D}_x$

(c) $\bar{N}_x$

(d) $l_x \bar{a}_x$

(e) $l_x (\bar{I}\bar{A})_x$

(f) $l_x (\bar{I}\bar{a})_x$

(a)

$$l_x \bar{A}_x\Big|_{i=0} = l_x * \int_0^\infty v^t \,{}_tp_x * \mu_{x+t}\ \mathrm{d}t\Big|_{i=0} = \int_0^\infty l_x * {}_tp_x * \mu_{x+t}\ \mathrm{d}t$$

$$l_x \bar{A}_x\Big|_{i=0} = \int_0^\infty l_{x+t} * \mu_{x+t}\ \mathrm{d}t$$

$$l_x \bar{A}_x\Big|_{i=0} = \int_0^\infty l_{x+t} * \mu_{x+t}\ \mathrm{d}t = (-l_{x+t})\Big|_{t=0}^{t=\infty}$$

$$\boxed{l_x \bar{A}_x\Big|_{i=0} = l_x}$$

(b)

$$\bar{D}_x\Big|_{i=0} = \int_0^1 D_{x+t}\ \mathrm{d}t\Big|_{i=0} = \int_0^1 v^t * l_{x+t}\ \mathrm{d}t\Big|_{i=0}$$

$$\bar{D}_x\Big|_{i=0} = \int_0^1 l_{x+t}\ \mathrm{d}t = L_x$$

$$\boxed{\bar{D}_x\Big|_{i=0} = L_x}$$

(c)

$$\bar{N}_x\Big|_{i=0} = \sum_{t=0}^{\infty} \bar{D}_{x+t}\ \mathrm{d}t\Big|_{i=0} = \int_0^\infty v^t * l_{x+t}\ \mathrm{d}t\Big|_{i=0}$$

$$\bar{N}_x\Big|_{i=0} = \int_0^\infty l_{x+t}\ \mathrm{d}t = (-T_{x+t})\Big|_{t=0}^{t=\infty}$$

$$\boxed{\bar{N}_x\Big|_{i=0} = T_x}$$

(d)

$$l_x \bar{a}_x\Big|_{i=0} = l_x * \int_0^\infty v^t * {}_tp_x\ \mathrm{d}t\Big|_{i=0}$$

$$l_x \bar{a}_x\Big|_{i=0} = l_x * \int_0^\infty {}_tp_x\ \mathrm{d}t = \int_0^\infty l_{x+t}\ \mathrm{d}t$$

$$\boxed{l_x \bar{a}_x\Big|_{i=0} = T_x}$$

(e)

$$l_x(\bar{I}\bar{A})_x\Big|_{i=0} = l_x * \int_0^{\infty} t * v^t * {}_tp_x * \mu_{x+t}\ \mathrm{d}t\Big|_{i=0} = l_x * \int_0^{\infty} t * {}_tp_x * \mu_{x+t}\ \mathrm{d}t$$

$$l_x(\bar{I}\bar{A})_x\Big|_{i=0} = \int_0^{\infty} t * l_{x+t} * \mu_{x+t}\ \mathrm{d}t = [-t * l_{x+t} - T_{x+t}]\Big|_{t=0}^{t=\infty}$$

$$\boxed{l_x(\bar{I}\bar{A})_x\Big|_{i=0} = T_x}$$

(f)

$$l_x(\bar{I}\bar{a})_x\Big|_{i=0} = l_x * \int_0^{\infty} t * v^t * {}_tp_x\ \mathrm{d}t\Big|_{i=0} = l_x * \int_0^{\infty} t * {}_tp_x\ \mathrm{d}t$$

$$l_x(\bar{I}\bar{a})_x\Big|_{i=0} = \int_0^{\infty} t * l_{x+t}\ \mathrm{d}t = [-t * T_{x+t} - Y_{x+t}]\Big|_{t=0}^{t=\infty}$$

$$l_x(\bar{I}\bar{a})_x\Big|_{i=0} = \int_0^{\infty} t * l_{x+t}\ \mathrm{d}t = [-t * T_{x+t} - Y_{x+t}]\Big|_{t=0}^{t=\infty}$$

$$\boxed{l_x(\bar{I}\bar{a})_x\Big|_{i=0} = Y_x}$$

3. Evaluate

(a) $\int_0^n t * l_{x+t} * \mu_{x+t}\ \mathrm{d}t$ (b) $\int_0^\infty t * l_{x+t}\ \mathrm{d}t$ (c) $\int_n^\infty l_{x+t}\ \mathrm{d}t$

(a)

$$\int_0^n t * l_{x+t} * \mu_{x+t}\ \mathrm{d}t = [-t * l_{x+t} - T_{x+t}]\Big|_{t=0}^{t=n} = [-n * l_{x+n} - T_{x+n}] - [-0 * l_{x+0} - T_{x+0}]$$

$$\boxed{\int_0^n t * l_{x+t} * \mu_{x+t}\ \mathrm{d}t = T_x - n * l_{x+n} - T_{x+n}}$$

(b)

$$\int_0^\infty t * l_{x+t}\ \mathrm{d}t = [-t * T_{x+t} - Y_{x+t}]\Big|_{t=0}^{t=\infty} = \lim_{t\to\infty} [-t * T_{x+t} - Y_{x+t}] - [-0 * T_{x+0} - Y_{x+0}]$$

$$\boxed{\int_0^\infty t * l_{x+t}\ \mathrm{d}t = Y_x}$$

(c)

$$\int_n^\infty l_{x+t}\ \mathrm{d}t = [-T_{x+t}]\Big|_{t=n}^{t=\infty} = \lim_{t\to\infty} (-T_{x+t}) - (-T_{x+n})$$

$$\boxed{\int_n^\infty l_{x+t}\ \mathrm{d}t = T_{x+n}}$$

4. Show that m_x is constant for all values of x if $l_x = k * \mathrm{e}^{-x}$

$$ {}_tp_x = \frac{l_{x+t}}{l_x} = \frac{k * \mathrm{e}^{-(x+t)}}{k * \mathrm{e}^{-x}} = \mathrm{e}^{-t} $$

$$ m_x = \frac{d_x}{L_x} = \frac{l_x * {}_tq_x}{\int_0^1 l_{x+t}} = \frac{l_x * (1 - {}_tp_x)}{l_x * \int_0^1 {}_tp_x} $$

$$ m_x = \frac{1 - \mathrm{e}^{-t}}{\int_0^1 \mathrm{e}^{-t}} $$

$$ \boxed{m_x = 1} $$

5. Estimate m_{25} if $l_{25} = 10075$ and $l_{26} = 9925$.

$$m_x = \frac{d_x}{L_x} = \frac{l_x - l_{x+1}}{L_x} \simeq \frac{l_x - l_{x+1}}{\frac{1}{2}\left(l_x + l_{x+1}\right)}$$

$$\Rightarrow m_{25} \simeq \frac{l_{25} - l_{26}}{\frac{1}{2}\left(l_{25} + l_{26}\right)}$$

$$\boxed{m_{25} \simeq 0,015}$$

6. Show that $L_{x+1} = L_x * \mathrm{e}^{\int_0^1 m_{x+t}\ \mathrm{d}t}$

$$\int_0^1 m_{x+t}\ \mathrm{d}t = \int_0^1 \left(-\frac{1}{L_{x+t}}\right) * \frac{\mathrm{d}L_{x+t}}{\mathrm{d}t}\ \mathrm{d}t = -\ln(L_{x+t})\Big|_{t=0}^{t=1}$$

$$\int_0^1 m_{x+t} = -\ln(\frac{L_{x+1}}{L_x}) \Rightarrow \mathrm{e}^{-\int_0^1 m_{x+t}\ \mathrm{d}t} = \mathrm{e}^{\ln\left(\frac{L_{x+1}}{L_x}\right)} = \frac{L_{x+1}}{L_x}$$

$$\frac{L_{x+1}}{L_x} * L_x = \boxed{L_{x+1} = L_x * \mathrm{e}^{\int_0^1 m_{x+t}\ \mathrm{d}t}}$$

7. Calculate the exact value of the central death rate at age x if mortality follows the de Moivre law.

Por la ley de de Moivre:

$$ {}_tp_x = \frac{w-(x+t)}{w-x} $$

$$ \Rightarrow m_x = \frac{d_x}{L_x} = \frac{l_x * (1 - {}_1p_x)}{l_x * \int_0^1 {}_tp_x \ \mathrm{d}t} = \frac{1 - \frac{w-(x+1)}{w-x}}{\int_0^1 \frac{w-(x+t)}{w-x} \ \mathrm{d}t} $$

$$ m_x = \frac{\frac{w-x-w+x+1}{w-x}}{\left. \frac{(w-x)*t-0,5*t^2}{w-x} \right|_{t=0}^{t=1}} = \frac{1}{w-x} * \frac{w-x}{(w-x)*1-0,5*1^2} $$

$$ m_x = \frac{1}{w-x-0,5} $$

$$ \boxed{m_x = \frac{2}{2(w-x)-1}} $$

8. From the following data, compute the curtate expectation of life at age 90 and estimate the complete expectation:

x	l_x
90	21
91	15
92	12
93	9
94	7

x	l_x
95	5
96	3
97	1
98	0

$$e_x = \sum_{t=1}^{\infty} {}_t p_x = \sum_{t=1}^{\infty} \frac{l_{x+t}}{l_x}$$

$$\Rightarrow e_{90} = \sum_{t=1}^{8} \frac{l_{90+t}}{l_{90}} = \frac{1}{l_{90}} \sum_{t=1}^{8} l_{90+t} = \frac{1}{21} * (15 + 12 + 9 + 7 + 5 + 3 + 1 + 0) = \frac{52}{21}$$

$$\boxed{e_x \simeq 2,47619}$$

$$\mathring{e}_x \simeq e_x + \frac{1}{2}$$

$$\boxed{\mathring{e}_x \simeq 2,97619}$$

9. Find $\mathring{e}_x$ assuming de Moivre's law.

Por de Moivre:

$$_tp_x = \frac{w-x-t}{w-x}$$

$$\Rightarrow \mathring{e}_x = \int_0^{\infty} {_tp_x}\ \mathrm{d}t = \int_0^{w-x} \frac{w-x-t}{w-x}\ \mathrm{d}t$$

$$\mathring{e}_x = \frac{(w-x)*t-0,5*t^2}{w-x}\Bigg|_{t=0}^{t=w-x} = \frac{(w-x)^2-0,5*(w-x)^2}{w-x} - 0$$

$$\boxed{\mathring{e}_x = \frac{w-x}{2}}$$

10.

(a) Show that $\dfrac{\mathrm{d}(\log T_x)}{\mathrm{d}x} = -\dfrac{1}{\mathring{e}_x}$.

(b) Show that if $\mu'_x = \mu_x + \dfrac{0,05}{\mathring{e}_x}$, then $p'_x = p_x * \left(\dfrac{T_{x+1}}{T_x}\right)^{0,05}$.

(a) Por regla de la cadena:

$$\frac{\mathrm{d}(\log T_x)}{\mathrm{d}x} = \frac{1}{T_x} * \frac{\mathrm{d}(T_x)}{\mathrm{d}x} = \frac{1}{T_x} * (-l_x)$$

$$\boxed{\frac{\mathrm{d}(\log T_x)}{\mathrm{d}x} = -\frac{1}{\mathring{e}_x}}$$

(b)

$$p'_x = \mathrm{e}^{-\int_0^1 \mu'_{x+t}\ \mathrm{d}t} = \mathrm{e}^{-\int_0^1 \left[\mu_{x+t} + \frac{0,05}{\mathring{e}_{x+t}}\right]\ \mathrm{d}t} = \mathrm{e}^{-\int_0^1 \mu_{x+t}\ \mathrm{d}t} * \mathrm{e}^{-\int_0^1 \frac{0,05}{\mathring{e}_{x+t}}\ \mathrm{d}t}$$

$$p'_x = p_x * \mathrm{e}^{0.05 * \log T_{x+t}|_{t=0}^{t=1}} = p_x * \mathrm{e}^{0,05 * \log \frac{T_{x+1}}{T_x}}$$

$$\boxed{p'_x = p_x * \left(\frac{T_{x+1}}{T_x}\right)^{0,05}}$$

11. Prove that $a_{\overline{e_x}|} > a_x$ for the case where $0 < e_x < 1$

$$a_{\overline{e_x}|} > v * e_x = v * \sum_{t=1}^{\infty} {}_tp_x \geqslant \sum_{t=1}^{\infty} v^t * {}_tp_x = a_x$$

$$\boxed{a_{\overline{e_x}|} > a_x}$$

12. For the l_x lives who survive to age x, write expressions for

(a) the total past lifetime subsequent to age y ($y < x$);

(b) the total future lifetime prior to age z ($z > x$);

(c) the average lifetime between ages y and z ($y < x < z$).

(a)

$$x * l_x - y * l_x = \boxed{(x - y) * l_x}$$

(b)

$$\int_0^{z-x} l_{x+t} \ \mathrm{d}t = \boxed{T_x - T_z}$$

(c)

$$\frac{\int_0^{z-x} l_{x+t} \ \mathrm{d}t + x * l_x - y * l_x}{l_x} = \boxed{x - y + \mathring{e}_{x:\overline{z-x}|}}$$

13. For those in a survivorship group who survive to age 30 but die before age 50, find

(a) the total past lifetime between ages 10 and 30;

(b) the average future lifetime subsequent to age 30.

(a)

$$(l_{30} - l_{50}) * (30 - 10) = \boxed{20 * (l_{30} - l_{50})}$$

(b)

$$\frac{\int_0^{20} l_{30+t}\ \mathrm{d}t - 20 * l_{50}}{l_{30} - l_{50}} = \boxed{\frac{T_{30} - T_{50} - 20 * l_{50}}{l_{30} - l_{50}}}$$

14. A staff is mantained in a stationary condition by 300 annual entrants at exact age 20. Ten per cent leave at the end of five years, 5% of those remaining are promoted after ten years into jobs outside the staff, and at age 60 the balance retire on a pension. Express in terms of mortality table functions

(a) the size of the staff,

(b) the number of promoted each year,

(c) the number of pensioners on the book.

(a)

$$\frac{300}{l_{20}} * [T_{20} - 0,1T_{25} - 0,05 * (1 - 0,1)T_{30} - (1 - 0,1 - 0.05 * 0,9)T_{60}]$$

$$\boxed{\frac{300}{l_{20}} * [T_{20} - 0,1T_{25} - 0,045T_{30} - 0,855T_{60}]}$$

(b)

$$\boxed{\frac{300}{l_{20}} * 0,045l_{30}}$$

(c)

$$\boxed{\frac{300}{l_{20}} * 0,855 * T_{60}}$$

15. An organization of 1000 active members is kept in a stationary state by admission of a uniform number of entrants at exact age 30. There are no withdrawals other than by death except that $\frac{1}{4}$ of those who reach age 55 retire at that age, $\frac{1}{3}$ of those who reach age 60 in service retire at that age, and all those remaning in service are retired at 65. Express in terms of tabular functions

(a) the number of annual entrants at age 30,

(b) the total number of deaths in service each year.

(a)

$$\frac{x}{l_{30}} * \left(T_{30} - 0,25T_{55} - 0,75 * \frac{1}{3}T_{60} - 0,75 * \frac{2}{3}T_{65}\right) = 1000$$

$$\Rightarrow \boxed{x = \frac{1000 * l_{30}}{T_{30} - 0,25T_{55} - 0,25T_{60} - 0,5T_{65}}}$$

(b)

$$x = 1000 * \frac{(l_{30} - l_{55}) + 0,75\,(l_{55} - l_{60}) + 0,5\,(l_{60} - l_{65})}{T_{30} - 0,25T_{55} - 0,25T_{60} - 0,5T_{65}}$$

$$\boxed{x = \frac{l_{30} - 0,25l_{55} - 0,25l_{60} - 0,5l_{65}}{T_{30} - 0,25T_{55} - 0,25T_{60} - 0,5T_{65}}}$$

16. In a stationary community supported by 5000 annual births, each person contributes \$200 on attaining the age of 25 and \$10 on each succeeding birthday up to, and including, his 65th birthday. On each birthday thereafter, he receives an annuity payment of \$150. A payment of \$50 is made at the death of each contributor who dies before receiving the first annuity payment. Find expressions for

(a) the constant number of persons who have made their first contribution but not their last,

(b) the receipts in any year,

(c) the total annuity payments in any year,

(d) the death claims in any year.

(a)

$$\text{Población entre los 25 y los 65} * \text{Proporción grupo/población} = \boxed{(T_{25} - T_{65}) * \frac{5000}{l_0}}$$

(b)

$$\text{Proporción grupo/población} * \text{Población de 25 años} * \text{Pagos}$$

$$\boxed{\frac{5000}{l_0} * l_{25} * (200 + 10 * e_{25:\overline{40|}})}$$

(c)

$$\text{Proporción grupo/población} * \text{Población de 65 años} * \text{Pagos}$$

$$\boxed{\frac{5000}{l_0} * l_{65} * (150 * e_{65})}$$

(d)

$$\text{Proporción grupo/población} * \text{Muertes entre los 25 y los 66} * \text{Pago}$$

$$\boxed{\frac{5000}{l_0} * (l_{25} - l_{66}) * 50}$$

17.

(a) Show that $\frac{\mathrm{d}(l_x \bar{a}_x)}{\mathrm{d}x} = -l_x * \bar{A}_x$

(b) In a stationary community, all persons now aged x and over agree to contribute a single sum equally to a fund from which a unit will be paid at the death of each of them. Show that the payment to be made by each is $\frac{\bar{a}_x}{\mathring{e}_x}$

(a)

$$\frac{\mathrm{d}(l_x \bar{a}_x)}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(l_x \int_0^\infty v^t * {}_tp_x \ \mathrm{d}t\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^\infty v^t * l_{x+t} \ \mathrm{d}t\right) = \int_0^\infty v^t * \frac{\mathrm{d}(l_{x+t})}{\mathrm{d}x} \ \mathrm{d}t$$

$$\frac{\mathrm{d}(l_x \bar{a}_x)}{\mathrm{d}x} = \int_0^\infty v^t * (-l_{x+t} * \mu_{x+t}) \ \mathrm{d}t = -l_x * \int_0^\infty v^t * {}_tp_x * \mu_{x+t} \ \mathrm{d}t$$

$$\boxed{\frac{\mathrm{d}(l_x \bar{a}_x)}{\mathrm{d}x} = -l_x * \bar{A}_x}$$

(b)

$$T_x * \text{Pago} = \int_0^\infty l_{x+t} * \bar{A}_{x+t} \ \mathrm{d}t$$

$$T_x * \text{Pago} = (-l_{x+t} * \bar{a}_{x+t})\Big|_{t=0}^{t=\infty} = l_x * \bar{a}_x$$

$$\text{Pago} = \frac{l_x * \bar{a}_x}{T_x} = \frac{l_x}{T_x} * \bar{a}_x$$

$$\boxed{\text{Pago} = \frac{\bar{a}_x}{\mathring{e}_x}}$$

18. An organization of 10000 members who enter at age 20 and retire at age 60 has reached a stationary condition. Each year there are 100 deaths, the average age at death being 40. What is the number entering and retiring each year?

$$\mathring{e}_{20:\overline{60}|} = 40 - 20 = 20$$
$$l_{20} - l_{60} = 100$$
$$T_{20} - T_{60} = 10000$$

$$x + \mathring{e}_{x:\overline{n}|} = x + \frac{(T_x - T_{x+n}) - n * l_{x+n}}{l_x - l_{x+n}} \Rightarrow 20 + 20 = 20 + \frac{(T_{20} - T_{60}) - 40 * l_{60}}{l_{20} - l_{60}}$$

$$40 = 20 + \frac{10000}{100} - \frac{40 * l_{60}}{100}$$

$$\Rightarrow l_{60} = \frac{(20 * 100 - 10000)}{40} = \boxed{200 = l_{60}}$$

$$\Rightarrow l_{20} = 200 + 100 = \boxed{300 = l_{20}}$$

19.

(a) Find the total number of future years that those in the stationary population now between ages x and $x+n$ will live before attaining age $x+n$.

(b) Find the total number of future years that those now between ages x and $x+n$ will live in the next m years.

(a)

$$\int_0^n \int_0^{n-t} l_{x+t+y}\ \mathrm{d}y\ \mathrm{d}t = \int_0^n \left(T_{x+t} - T_{x+n}\right)\ \mathrm{d}t = \left(-Y_{x+t} - T_n * t\right)\Big|_{t=0}^{t=n}$$

$$\boxed{\int_0^n \int_0^{n-t} l_{x+t+y}\ \mathrm{d}y\ \mathrm{d}t = Y_x - n * T_n - Y_n}$$

(b)

$$\int_0^m \int_0^n l_{x+t+y}\ \mathrm{d}y\ \mathrm{d}t = \int_0^m \left(T_{x+t} - T_{x+t+n}\right)\ \mathrm{d}t = \left(-Y_{x+t} + Y_{x+t+n}\right)\Big|_{t=0}^{t=n}$$

$$\boxed{\int_0^m \int_0^n l_{x+t+y}\ \mathrm{d}y\ \mathrm{d}t = Y_{x+n+m} + Y_x - Y_{x+n} - Y_{x+m}}$$

20.

(a) Find the average age at death of those in the stationary population who are now living between ages x and $x+n$.

(b) Find the average age at death of those in the stationary population who are now living between ages x and $x+n$ and who will die before attaining age $x+n$.

(a)

$$\begin{aligned}
\frac{\int_0^n l_{x+t} * [(x+t) + \mathring{e}_{x+t}]\ \mathrm{d}t}{\int_0^n l_{x+t}\ \mathrm{d}t} &= \frac{\int_0^n l_{x+t} * (x+t)\ \mathrm{d}t + \int_0^n l_{x+t} * \mathring{e}_{x+t}\ \mathrm{d}t}{T_x - T_{x+n}} \\
&= \frac{\int_0^n l_{x+t} * (x+t)\ \mathrm{d}t + \int_0^n T_{x+t}\ \mathrm{d}t}{T_x - T_{x+n}} \\
&= \frac{[-(x+t) * T_{x+t} - Y_{x+t}]\Big|_{t=0}^{t=n} + (-Y_{x+t})\Big|_{t=0}^{t=n}}{T_x - T_{x+n}} \\
&= \frac{x * T_x + Y_x - (x+n) * T_{x+n} - Y_{x+n} + Y_x - Y_{x+n}}{T_x - T_{x+n}}
\end{aligned}$$

$$\boxed{\frac{\int_0^n l_{x+t} * [(x+t) + \mathring{e}_{x+t}]\ \mathrm{d}t}{\int_0^n l_{x+t}\ \mathrm{d}t} = x + \frac{2(Y_x - Y_{x+n}) - n * T_{x+n}}{T_x - T_{x+n}}}$$

(b)

$$\begin{aligned}
\frac{\int_0^n \int_0^{n-t} (x+t+y) * l_{x+t+y} * \mu_{x+t+y}\ \mathrm{d}y\ \mathrm{d}t}{\int_0^n \int_0^{n-t} l_{x+y+t} * \mu_{x+y+t}} &= \frac{\int_0^n [-(x+t+y) l_{x+t+y} - T_{x+t+y}]\Big|_{y=0}^{y=n-t}\ \mathrm{d}t}{\int_0^n (-l_{x+t+y})\Big|_{y=0}^{y=n-t}\ \mathrm{d}t} \\
&= \frac{\int_0^n\ [(x+t) l_{x+t} + T_{x+t} - (x+n) l_{x+n} - T_{x+n}]\ \mathrm{d}t}{\int_0^n (l_{x+t} - l_{x+n})\ \mathrm{d}t}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\{[-(x+t) T_{x+t} - Y_{x+t}] + (-Y_{x+t}) - t * (x+n) l_{x+n} - t * T_{x+n}\}\Big|_{t=0}^{t=n}}{(-T_{x+t} - t * l_{x+n})\Big|_{t=0}^{t=n}} \\
&= \frac{x * T_x + Y_x + Y_x - (x+n) T_{x+n} - Y_{x+n} - Y_{x+n} - n * (x+n) l_{x+n} - n * T_{x+n}}{T_x - n * l_{x+n} - T_{x+n}}
\end{aligned}$$

$$\boxed{\frac{\int_0^n l_{x+t} * [(x+t) + \mathring{e}_{x+t}]\ \mathrm{d}t}{\int_0^n l_{x+t}\ \mathrm{d}t} = x + \frac{2 * (Y_x - n * T_{x+n} - Y_{x+n}) - n^2 * l_{x+n}}{T_x - n * l_{x+n} - T_{x+n}}}$$

21. Find the average attained age of those in the stationary population now living at ages under 30.

$$\frac{\int_0^{30} x * l_x \ \mathrm{d}x}{\int_0^{30} l_x \ \mathrm{d}x} = \frac{\left[-xT_x - Y_x\right]\Big|_{x_0}^{x=30}}{\left[-T_x\right]\Big|_{x=0}^{x=30}}$$

$$\boxed{\frac{\int_0^{30} x * l_x \ \mathrm{d}x}{\int_0^{30} l_x \ \mathrm{d}x} = \frac{Y_0 - 30T_{30} - Y_{30}}{T_0 - T_{30}}}$$

22. Find the average age at death of those in the stationary population who are now living between ages 20 and 30 and who will die between ages 30 and 50.

$$\frac{\int_0^{10}\int_{10-t}^{30-t}(20+t+y)l_{20+t+y} * \mu_{20+t+y}\ \mathrm{d}y\ \mathrm{d}t}{\int_0^{10}\int_{10-t}^{30-t} l_{20+t+y} * \mu_{20+t+y}\ \mathrm{d}y\ \mathrm{d}t} = \frac{\int_0^{10}\left[-(20+t+y)l_{20+t+y} - T_{20+t+y}\right]\Big|_{y=10-t}^{y=30-t}\ \mathrm{d}t}{\int_0^{10}(-l_{20+t+y})\Big|_{y=10-t}^{y=30-t}\ \mathrm{d}t}$$

$$= \frac{\int_0^{10}\left(30l_{30} + T_{30} - 50l_{50} - T_{50}\right)\ \mathrm{d}t}{\int_0^{10}(l_{30} - l_{50})\ \mathrm{d}t}$$

$$= \frac{10 * (30l_{30} + T_{30} - 50l_{50} - T_{50})}{10 * (l_{30} - l_{50})}$$

$$\boxed{\frac{\int_0^{10}\int_{10-t}^{30-t}(20+t+y)l_{20+t+y} * \mu_{20+t+y}\ \mathrm{d}y\ \mathrm{d}t}{\int_0^{10}\int_{10-t}^{30-t} l_{20+t+y} * \mu_{20+t+y}\ \mathrm{d}y\ \mathrm{d}t} = 30 + \frac{T_{30} - 20l_{50} - T_{50}}{l_{30} - l_{50}}}$$

23.

(a) Given $e_{25:\overline{n}|} = n\left(1 - \dfrac{n+1}{150}\right)$ for all n, derive an expression for ${}_np_{25}$.

(b) Find the derivative of $\mathring{e}_x$ with respect to l_x

(c) Show that $\int_0^\infty v^t * \mathring{e}_{x:\overline{t}|}\ \mathrm{d}t = \dfrac{\bar{a}_x}{\delta}$

(a)

$$
\begin{aligned}
e_{25:\overline{n}|} &= \sum_{t=1}^{n} {}_tp_{25} = n\left(1 - \frac{n+1}{150}\right) \\
&= \sum_{t=1}^{n-1} {}_tp_{25} + {}_np_{25} = e_{25:\overline{n-1}|} + {}_np_{25}(n-1)\left(1 - \frac{(n-1)+1}{150}\right) + {}_np_{25} = \\
&= (n-1)\left(1 - \frac{n}{150}\right) + {}_np_{25} \\
&\Rightarrow {}_np_{25} = n\left(1 - \frac{n+1}{150}\right) - (n-1)\left(1 - \frac{n}{150}\right) = 1 - \frac{n}{150} - \frac{n}{150}
\end{aligned}
$$

$$\boxed{{}_np_{25} = 1 - \frac{n}{75}}$$

(b) Derivando por partes, y suponiendo μ_x constante:

$$
\begin{aligned}
\frac{\mathrm{d}\mathring{e}_x}{\mathrm{d}l_x} &= \frac{\mathrm{d}}{\mathrm{d}l_x}\left(\frac{T_x}{l_x}\right) = \frac{\mathrm{d}T_x}{\mathrm{d}l_x} * \frac{1}{l_x} + T_x * \frac{\mathrm{d}}{\mathrm{d}l_x}\left(\frac{1}{l_x}\right) = \frac{\mathrm{d}}{\mathrm{d}l_x}(l_x * \int_0^\infty {}_tp_x\ \mathrm{d}t) * \frac{1}{l_x} - T_x * \frac{1}{l_x^2} \\
&= \frac{\mathrm{d}}{\mathrm{d}l_x}(l_x * \frac{1}{\mu_x}) * \frac{1}{l_x} - \mathring{e}_x * \frac{1}{l_x} = \frac{1}{\mu_x l_x} - \mathring{e}_x * \frac{1}{l_x}
\end{aligned}
$$

$$\boxed{\frac{\mathrm{d}\mathring{e}_x}{\mathrm{d}l_x} = \frac{1 - \mu_x \mathring{e}_x}{\mu_x l_x}}$$

(c)

$$
\begin{aligned}
\int_0^\infty v^t * \mathring{e}_{x:\overline{t}|}\ \mathrm{d}t &= \int_0^\infty v^t * \int_0^t \frac{l_{x+s}}{l_x}\ \mathrm{d}s\ \mathrm{d}t = \int_0^\infty \frac{v^t}{l_x} * \int_0^t l_{x+s}\ \mathrm{d}s\ \mathrm{d}t = \int_0^\infty \frac{v^t}{l_x} * [T_x - T_{x+t}]\ \mathrm{d}t \\
&= \mathring{e}_x * \int_0^\infty v^t\ \mathrm{d}t - \frac{1}{l_x}\int_0^\infty v^t * T_{x+t}\ \mathrm{d}t = \mathring{e}_x * \frac{(-1)}{\ln(v)} - \frac{1}{l_x} * \left[0 - \frac{T_x}{\ln(v)} + \frac{l_x \bar{a}_x}{\ln(v)}\right] \\
&= \frac{-\mathring{e}_x}{-\delta} + \frac{\mathring{e}_x}{-\delta} - \frac{\bar{a}_x}{-\delta} = \frac{\mathring{e}_x}{\delta} - \frac{\mathring{e}_x}{\delta} + \frac{\bar{a}_x}{\delta}
\end{aligned}
$$

$$\boxed{\int_0^\infty v^t * \mathring{e}_{x:\overline{t}|}\ \mathrm{d}t = \frac{\bar{a}_x}{\delta}}$$

24. In a stationary community of 600000 lives, the number of deaths is 10000 annually. The complete expectation of life on attaining majority at age 21 is 50 years. If one-third of the population is under age 21, how many lives attain majority each year, and what is the average age at death of those who die under age 21?

$$\text{Ingresos} = \text{Egresos} = l_0 = 10000 \qquad T_0 = 600000 \qquad T_{21} = 400000$$

$$\begin{aligned}
\frac{\int_0^\infty (21+t) * l_{21+t} * \mu_{21+t}\ \mathrm{d}t}{l_{21}} &= \frac{\left[-(21+t) * l_{21+t} - T_{21+t}\right]\Big|_{t=0}^{t=\infty}}{l_{21}} \\
&= \frac{21 * l_{21} + T_{21}}{l_{21}} = 21 + \frac{T_{21}}{l_{21}} = 21 + \mathring{e}_{21} \\
\Rightarrow l_{21} &= \frac{T_{21}}{\mathring{e}_{21}} = \frac{400000}{50} \\
&\boxed{l_{21} = 8000}
\end{aligned}$$

$$\begin{aligned}
\frac{\int_0^{21} t * l_t * \mu_t\ \mathrm{d}t}{l_0 - l_{21}} &= \frac{\left[-t * l_{21} - T_{21}\right]\Big|_{t=0}^{t=21}}{l_0 - l_{21}} \\
&= \frac{T_0 - 21 l_{21} - T_{21}}{l_0 l_{21}} \\
&= \frac{600000 - 21 * 8000 - 400000}{10000 - 8000} \\
&\boxed{\text{Edad media de muerte} = 16}
\end{aligned}$$

25. Find the total lifetime of those in the stationary population now living between ages 20 and 70 who will die between ages 60 and 80 within 50 years from now.

Para el grupo que hoy tiene entre 20 y 30:

$$\begin{aligned}
\int_0^{10}\int_0^{20-t}(60+y)l_{60+y}*\mu_{60+y}\ \mathrm{d}y\ \mathrm{d}t &= \int_0^{10}\left[-(60+y)*l_{60+y}-T_{60+y}\right]\Big|_{y=0}^{y=20-t}\ \mathrm{d}t\\
&= \int_0^{10}\left[60*l_{60}+T_{60}-(80-t)l_{80-t}-T_{80-t}\right]\ \mathrm{d}t\\
&= \left[t*60*l_{60}+t*T_{60}-(80-t)*T_{80-t}-Y_{80-t}-Y_{80-t}\right]\Big|_{t=0}^{t=10}\\
&= 600l_{60}+10T_{60}-70T_{70}-Y_{70}-Y_{70}+80T_{80}+Y_{80}+Y_{80}\\
&= 600*l_{60}+10*T_{60}-70*T_{70}-2*Y_{70}+80*T_{80}+2*Y_{80}
\end{aligned}$$

Para el grupo que hoy tiene entre 30 y 60:

$$\begin{aligned}
\int_0^{30}\int_0^{20}(60+y)*l_{60+y}*\mu_{60+y}\ \mathrm{d}y\ \mathrm{d}t &= \int_0^{30}\left[-(60+y)*l_{60+y}-T_{60+y}\right]\Big|_{y=0}^{y=20}\ \mathrm{d}t\\
&= \int_0^{30}\left[60*l_{60}+T_{60}-80*l_{80}-T_{80}\right]\ \mathrm{d}t\\
&= 30*\left[60*l_{60}+T_{60}-80*l_{80}-T_{80}\right]\\
&= 1800*l_{60}+30*T_{60}-2400*l_{80}-30*T_{80}
\end{aligned}$$

Para el grupo que hoy tiene entre 60 y 70:

$$\begin{aligned}
\int_0^{10}\int_0^{20-t}(60+t+y)*l_{60+t+y}*\mu_{60+t+y}\ \mathrm{d}y\ \mathrm{d}t &= \int_0^{10}\left[-(60+t+y)*l_{60+t+y}-T_{60+t+y}\right]\Big|_{y=0}^{y=20-t}\ \mathrm{d}t\\
&= \int_0^{10}\left[-80l_{80}-T_{80}+(60+t)l_{60+t}+T_{60+t}\right]\ \mathrm{d}t\\
&= \left[-t*80l_{80}-t*T_{80}-(60+t)*T_{60+t}-2Y_{60+t}\right]\Big|_{t=0}^{t=10}\\
&= 60T_{60}+2Y_{60}-70T_{70}-2Y_{70}-800*l_{80}-10T_{80}
\end{aligned}$$

Sumando los 3 grupos:

$$\boxed{\text{Años vividos} = 2400l_{60}+100T_{60}+2Y_{60}-140T_{70}-4Y_{70}-3200l_{80}+40T_{80}+2Y_{80}}$$

26.

(a) Show that $\frac{\mathrm{d}}{\mathrm{d}t}({}_tp_x\mathring{e}_{x+t}) = -{}_tp_x$.

(b) Show that $\int_0^\infty v^t\,{}_tp_x\mathring{e}_{x+t}\ \mathrm{d}t = \frac{1}{\delta}(\mathring{e}_x - \bar{a}_x)$.

(a)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}({}_tp_x\mathring{e}_{x+t}) &= \frac{\mathrm{d}}{\mathrm{d}t}\left({}_tp_x\frac{T_{x+t}}{l_{x+t}}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left({}_tp_x\frac{T_{x+t}}{l_x * {}_tp_x}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{T_{x+t}}{l_x}\right) = \frac{1}{l_x} * \frac{\mathrm{d}T_{x+t}}{\mathrm{d}t} \\
&= \frac{-l_{x+t}}{l_x}
\end{aligned}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}({}_tp_x\mathring{e}_{x+t}) = -{}_tp_x}$$

(b) Integrando por partes, usando el resultado anterior:

$$\begin{aligned}
\int_0^\infty v^t\,{}_tp_x\mathring{e}_{x+t}\ \mathrm{d}t &= \left[{}_tp_x * \frac{v^t}{\ln(v)} * \mathring{e}_{x+t}\right]\Bigg|_{t=0}^{t=\infty} - \frac{1}{\ln(v)} * \int_0^\infty v^t(-{}_tp_x)\ \mathrm{d}t \\
&= -\frac{\mathring{e}_x}{-\delta} + \frac{\bar{a}_x}{-\delta}
\end{aligned}$$

$$\boxed{\int_0^\infty v^t\,{}_tp_x\mathring{e}_{x+t}\ \mathrm{d}t = \frac{\mathring{e}_x - \bar{a}_x}{\delta}}$$