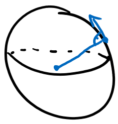
At any time there is a location on earth where the wind is not blowing.



Mathematical interpretation:

For any continuous vector field on S^2

$$\tau : S^2 \longrightarrow \mathbb{R}^3$$

$$(\langle \tau(x), x \rangle = 0 \text{ for } x \in S^2 \subset \mathbb{R}^3)$$

there is a zero.

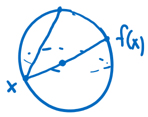
Lemma: Let $n \geq 2$ even. For each map $S^n \xrightarrow{f} S^n$ there is a $x \in S^n$ such that

$$f(x) \in \{x, -x\}.$$

Proof: Assume that $f(x) \notin \{x, -x\}$ for all $x \in S^n$.

Define

$$F(x,t) = \frac{(1-t)x + t \cdot f(x)}{\|(1-t)x + t \cdot f(x)\|}$$ is homotopy $\mathrm{id}_{S^n} \simeq f$



$$G(x,t) = \frac{(1-t)f(x) + t(-x)}{\|(1-t)f(x) + t(-x)\|}$$ is a homotopy $-\mathrm{id}_{S^n} \simeq f$.

$-\mathrm{id}_{S^n}$ is the composition of $n+1$ reflections.

Hence $$H_n(-\mathrm{id}_{S^n}) : H_n(S^n) \longrightarrow H_n(S^n)$$

is multiplication with $(-1)^{n+1} = -1$.

Hence $-\mathrm{id}_{S^n} \not\simeq \mathrm{id}_{S^n}$ ↯

$\square$

Theorem Let $n \geq 2$ be even. Every continuous vector field on S^n vanishes at some point.

[every map $\tau : S^n \to \mathbb{R}^{n+1}$ with $\langle \tau(x), x \rangle = 0 \; \forall x \in S^n$ vanishes at some point]

Proof. Assume $\tau(x) \neq 0$ for all $x \in S^n$.

Define $$f(x) = \frac{\tau(x)}{\|\tau(x)\|} \in S^n.$$

By the lemma there is $x_0 \in S^n$ such that

$$f(x_0) \in \{x_0, -x_0\}.$$

This contradicts

$$0 = \langle \tau(x_0), x_0 \rangle = \|\tau(x_0)\| \cdot \langle f(x_0), x_0 \rangle$$

$\square$

I hereby declare that this document has been composed by myself and describes my own work, unless otherwise acknowledged in the text.

Ich versichere wahrheitsgemäß, die Arbeit selbstständig verfasst, alle benutzten Hilfsmittel vollständig und genau angegeben und alles kenntlich gemacht zu haben, was aus Arbeiten anderer unverändert oder mit Abänderungen entnommen wurde sowie die Satzung des KIT zur Sicherung guter wissenschaftlicher Praxis in der jeweils gültigen Fassung beachtet zu haben.

Karlsruhe, 10.05.2022

Abstract

Some English abstract.

Zusammenfassung

Eine deutsche Zusammenfassung.

Contents

1 1 Background of the problem **5**

1.1 Some backgound assumptions . . . 5

1.2 Contracting the image of a lower dimensional sphere . . . 6

1.2.1 Introduction to computations up to a constant . . . 6

1.2.2 Detour: geometric suspension . . . 8

1.3 Simplicial approximation . . . 12

1.3.1 Approximating maps between spheres . . . 15

2 n is odd theorem **17**

1 1 Background of the problem

some introduction.

Gromov was able to show that there is a Lipschitz null-homotopy with the tower of exponentials. He apparently was also able to show linear dependency for the case m=n. (reference Guth).

1.1 Some backgound assumptions

This paper deals with Lipschitz constants of maps between spheres. Most of the time we will only estimate the Lipschitz constants up to a constant $C(m, n)$, that only depends on the dimensions of the spheres. We denote equality/inequality up to a constant by $\sim, \lesssim, \gtrsim$ respectively. Throughout this paper let the unit spheres S^m, S^n be equipped with the length metric induced by the standard Riemannian metric (unless stated otherwise). That is, the distance between any two points is determined by the (Euclidean) length of the geodesics between them [1]. Note that while the topology is the same, the metric is different from the "default" metric inherited from the ambient Euclidean space. Occasionally we will consider objects that are homeomorphic to spheres when it is convinient (e.g. surface of a cube or of a simplex), but the conversion only changes things up to some constrant. On those objects we will still be using the length metric.

Statement of the problem.

[1] To be precise, the length of the geodesics is determined by the standard Riemmanian metric, where the metric is pulled back along the embedding of the spheres into their ambient Euclidean spaces ($\mathbb{R}^m$, $\mathbb{R}^n$, respectively). The lengths of geodesics are then precisely the respective Euclidean lengths of their embeddings. The reason to specify a metric so early on is that when we talk about Lipschitz continuity we are implicitly dealing with the metrics, not just with undelying topologies. However, since all of our results are up to a constant, suitable constant manipulation would show them to hold for the standard Euclidean metric as well. Nevertheless, we prefer to settle on a specific metric to avoid confusion or ambiguity.

1.2 Contracting the image of a lower dimensional sphere

1.2.1 Introduction to computations up to a constant

In this section we first consider Lipschitz maps from S^m to S^n when $m < n$. This case is fairly easy, as we know from topology that APPARENTLY THAT'S WRONG. we need piece-wise linearity or smoothness. We know that the image of S^m in S^n is not-surjective (citation). It is then contractible. In this section we want to show that the image of a lower dimensional sphere can be contracted in a Lipschitz way, and to provide a fairly tight Lipschitz constant.

We start by showing that a Lipschitz map must in fact miss a whole open ball in the target:

Lemma 1.1. *Let $f : S^m \to S^n$ be a Lipschitz conituous maps with a Lipschitz constant L. Then the image of f misses a ball of radius r for $r \lesssim L^{\frac{-m}{n-m}}$*

Our strategy for proving this lemma will be to first cover the domain sphere by open balls of a set radius, then map that cover to the target sphere and show that the image of the cover cannot be surjective.

A ball with respect to the length-metric on the sphere is a spherical cap. The radius of the ball is the length of any geodesic from the center (the tip) of the cap to its edge. It is equal to the polar angle of the cap in radians.

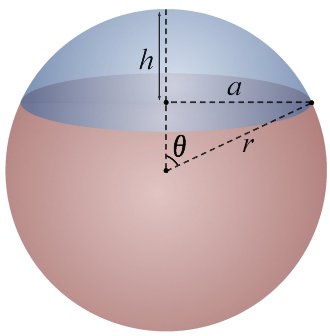


Figure 1.1: Spherical cap. Here $r = 1$, θ is the polar angle, $a = \sin\theta$, $h = 1 - \cos\theta$

For the proof of the lemma we will need to cover the sphere with spherical caps. To estimate the number of caps needed to cover the sphere we use a volume argument. A sloppy version of the argument would go as follows: we equip the sphere with a volume form that scales well with the polar angle and is equal to 1 on the whole sphere. Then the volume of the sphere is 1, the volume of each spherical cap is ρ^m. The cover should have area similar to that of the sphere (up to a constant). We then need $\sim 1/\rho^m$ spherical caps to cover the sphere.

You may object: why should the cover have volume similar to that of the sphere if there is an overlap? Why should the overlap scale well with the radius? Is it okay for us to change the metric if the result is stated w.r.t. to a different one (we could of course use the standard volume but then you would be right to point out that spherical cap volume might not scale well with the cover radius). However, this argument is only meant to provide us with an intuition, and we aim to show that this types of arguments can be formalized fairly easily:

Claim. *For any $\rho > 0$, the sphere S^m can be covered by $\sim 1/\rho^m$ balls of radius ρ.*

Proof. We want to estimate the number of $1/\rho$ balls needed to cover the sphere. Covering the sphere is up to a constant the same as covering the hemisphere. In fact, the the cardinality of the cover for $S^m \sim$ Hemisphere $S^m_+ \sim D^m$ (we can tranfer the cover back and forth by projecting the hemisphere onto the equator disk or wrapping a larger disk around the hemisphere [2] $= B^m \sim$ covering $\sqrt{2}B^m$ (scaling up) $\sim$ covering the m-box of side length two (it can be squeezed between the two balls, i.e. it contains the unit ball and it is contained in the $\sqrt{2}B^m$) $\sim$ covering the m-box of side length 1 (the unit m-box). It is easy to see why the volume argument should work now: the unit box can be clearly be covered by $\lceil 1/\rho \rceil^m$ boxes of side length ρ. Each ρ-box is contained in a ball of radius ρ and we are done. □

Arguing up to a constant allows us great flexibility in choosing objects we are more comfortable working with. The constants we omitted can easily be traced back through the equivalence steps we took. However, if you are not yet comfortable working up to a constant there is a direct argument on the sphere without any equivalences or dropping constants that I provided in the appendix.

Lemma 1.2 (Image of misses a ball)**.** *Let $f : S^m \to S^n$ be a Lipschitz-conituous map with a Lipschitz constant L. Then the image of f misses a ball of radius r for $r \lesssim L^{-\frac{m}{n-m}}$*

Proof. For any $\rho > 0$, S^m can be covered by $\sim \rho^{-m}$ balls of radius ρ. The image of each such ball is contained in a ball of radius $L\rho$. Therefore, the image of f can be covered by $\lesssim \rho^{-m}$ balls of radius $L\rho$. We set $r := L\rho$. We now want to choose ρ small enough so that the cover misses a ball of radius r.

[2] Projecting the hemisphere S^m_+ down onto the unit disk D^m at the equator obviously only changes things up to a constant (depending only on m): we project down the cover centers and keep the radius as is. Showing that the cover can be transferred in the other direction is a little trickier: we start by scaling the cover up by a factor of $\pi/2$ to cover $\pi/2D^m$. We then wrap the larger disc around the hemisphere by taking (θ, r) to $(\theta, \rho) = (\theta, r), where,$ r is the radius and ρ is the polar angle. Distances can only reduce for the same reason that we can wrap a paper around a ball without tearing it and the paper will wrinkle: radial components of distances stay the same and angular components srink with a factor of sine as the radius increases. Of course if you are not convinced you can scale your disk up by another factor of two. Same as before, keeping the projected cover centers and the old cover radius yield a cover.

Expanding the radius of the cover to $2r$ yields a cover of the r-neighborhood of the image. We denote this $2r$-cover by C. If this larger cover does not cover the full sphere S^n, the image of f must miss a ball of radius r. The total volume of the cover C is at most the cardinality of C times the volume of a ball of radius $2r$ (which is a spherical cap of polar angle $2r$). We replace the cap volume by the larger volume of a disk $2r \cdot D^n$ by essentially the same argument as we used to transfer the disk cover from the disk to the hemisphere HEMISPHERE IS ESSENTIALLY A spherical cap. Can we use the "special" volume form argument here instead????? and [3]. The total cover volume is then at most $|C|\omega^n(\pi/2 \cdot B_2^n r)$, where ω^n denotes the Euclidean n-volume form?????. $\square$

We now set ρ so that this number is smaller than the volume of the sphere. So we get for $n > m$

$$|C|\omega^n(2r\text{-cap}) \lesssim \rho^{-m} r^n = L^n \rho^{n-m} \lesssim 1,$$

$$\rho \lesssim L^{-\frac{n}{n-m}},$$

$$r = L\rho \lesssim L^{-\frac{m}{n-m}}.$$

In particular, even if f is a constant map we can choose ρ small enough so that $r \leq \pi/2$

1.2.2 Detour: geometric suspension

If we equip the sphere S^2 with the usual pullback Riemannian metric, the resulting metric written in the matrix form is

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}.$$

Even more often in the literature one encounters the corresponding symmetric quadratic form - its first fundamental form - which can be written as:

$$\mathrm{d}s^2 = \mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2\theta \, \mathrm{d}\phi \otimes \mathrm{d}\phi,$$

or simply

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2.$$

[3]Projecting the hemisphere S^m_+ down onto the unit disk D^m at the equator obviously only changes things up to a constant (depending only on m): we project down the cover centers and keep the radius as is. Showing that the cover can be transferred in the other direction is a little trickier: we start by scaling the cover up by a factor of $\pi/2$ to cover $\pi/2D^m$. We then wrap the larger disc around the hemisphere by taking (θ, r) to $(\theta, \rho) = (\theta, r), where,$ r is the radius and ρ is the polar angle. Distances can only reduce for the same reason that we can wrap a paper around a ball without tearing it and the paper will wrinkle: radial components of distances stay the same and angular components srink with a factor of sine as the radius increases. Of course if you are not convinced you can scale your disk up by another factor of two. Same as before, keeping the projected cover centers and the old cover radius yield a cover.

We will now show that the metric is verbatim the same for S^m for $\forall m \geq 2$

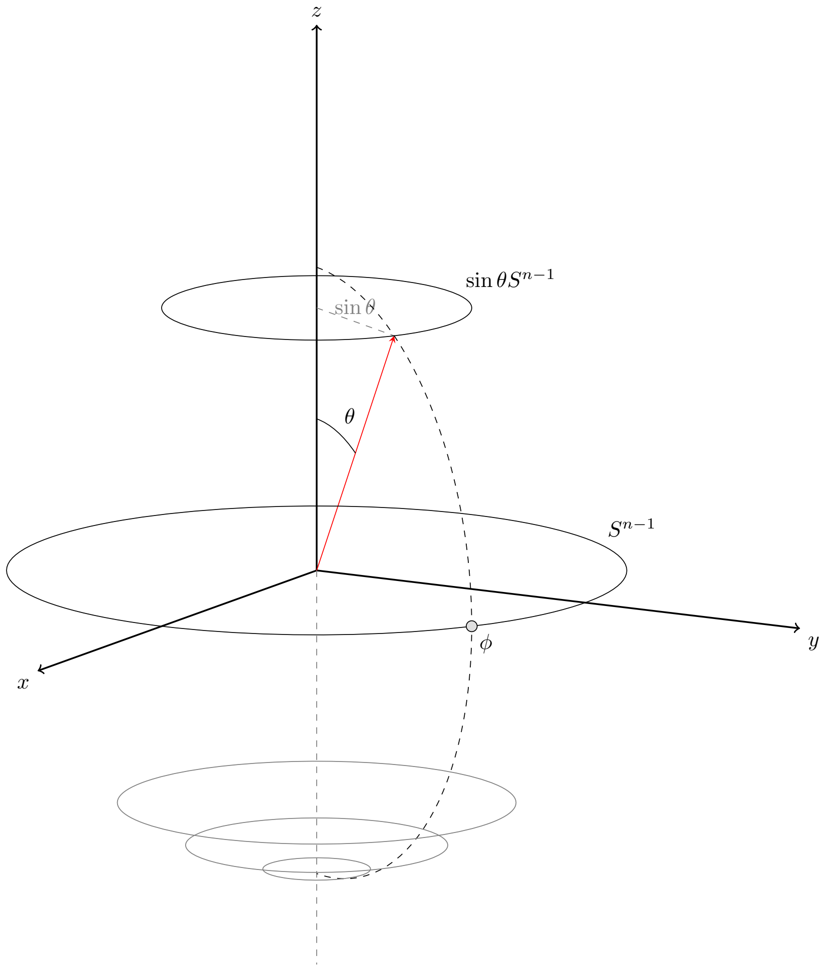


Figure 1.2: Polar coordinates

We can think of S^m as of several S^{m-1} stacked on top of each other (where S^{m-1} shrink to a single point at the poles). This is essentially the geometric version of suspension. Using the polar angle rather than height, we scale the equator S^{m-1} by $\sin\theta$.

Point-wise this gives us that any point p of S^m can be parametrized in terms of the polar angle θ and the corresponding vector ϕ of the equator scaled down by $\sin\theta$ - polar coordinates with respect to S^{m-1} Figure 1.2. Fixing some direction z in $\mathbb{R}^{m+1}$ we can write out the parametrization:

$$\begin{aligned} \psi :& [0,\pi] \times S^{m-1} \longrightarrow S^m \\ &(\theta,\phi) \mapsto \sin\theta \cdot \phi + \cos\theta \cdot \vec{e_z}, \end{aligned} \tag{1.1}$$

where $\vec{e_z}$ denote the standard basis vector in the z direction. Computing partial derivatives yields

$$\frac{\partial \psi}{\partial \theta} = \cos\theta \cdot \phi - \sin\theta \cdot \vec{e_z},$$

$$\frac{\partial \psi}{\partial \phi} = \sin\theta \cdot \vec{e_z}.$$

Computing the spherical metric as a pullback of the $\mathbb{R}^{m+1}$ metric:

$$g_{\theta\theta} = \langle \cos\theta \cdot \phi - \sin\theta \cdot \vec{e_z}, \cos\theta \cdot \phi - \sin\theta \cdot \vec{e_z} \rangle = \cos^2\theta \cdot \langle \phi, \phi \rangle + \sin^2\theta \cdot \langle \vec{e_z}, \vec{e_z} \rangle = 1,$$

$$g_{\phi\theta} = g_{\theta\phi} = 0,$$

$$g_{\phi\phi} = \sin^2\theta$$

yielding the desired

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}.$$

Remark. Note that in this we could replace S^{m-1} with an arbitrary manifold M of non-zero dimension [4]. Remarkably, since we are not using any knowledge of the underlying manifold M to compute the suspension metric with respect to M, it is only the function that we use to shrink the manifold towards suspension poles that matters for this relative metric. Analogously, we could take an analytic version of any topological construction to obtain its geometric version.

The complement of a point in S^n is contractible. If we remove a ball from S^n, the leftover part can be contracted in a Lipschitz way.

Lemma 1.3 (contraction lemma)**.** *For each radius r there is a Lipschitz-contraction $G : (S^n \setminus B_r) \times [0, 1] \to S^n \setminus B_r$. G has Lipschitz constant $\lesssim 1/r$ in the S^n direction and $\lesssim 1$ in the $[0, 1]$ direction.*

We choose the obvious contraction map:

$$G : (S^n \setminus B_r) \times [0, 1] \to S^n \setminus B_r$$

$$G : (\rho, \theta, t) \to ((1 - t)\rho, \theta)$$

Our goal is to compute its Lipschitz constants in both the sphere and the time direction. The strategy is to find the supremum of the differential applied to the appropriate tangent vectors and use it as an upper bound for the Lipschitz constants. The theoretical foundation for this approach is the mean value theorem for manifolds (REFERENCE).

State the mean value theorem, reference.

[4]For zero-dimensional manifolds $\mathrm{d}\phi^2$ vanishes, leaving $\mathrm{d}s^2 = \mathrm{d}\theta^2$ as the metric.

Proof. Let G be as above. Its differential is

$$\mathrm{d}G = \begin{pmatrix} 1-t & 0 & -\rho \\ 0 & 1 & 0 \end{pmatrix}$$

We start with the Lipschitz constant in the direction of the sphere by restricting to tangent vectors in the sphere direction, i.e. with the zero time component $(v_\rho, v_\theta, 0) \in T_p((S^n \setminus B_r) \times [0,1])$. It is of course the same as to fix t as a parameter and consider the family of maps G_t that are self-maps of the punctured sphere $S^n \setminus B_r$. We want compute the operator norm $\|\mathrm{d}G_t\|$ (REFERENCE):

$$\|\mathrm{d}G_t\| = \sup_{v \neq 0} \frac{\|\mathrm{d}G_t v\|_{G(p)}}{\|v\|_p} = \sup_{\|v\|_p = 1} \|\mathrm{d}G_t v\|_{G(p)},$$

where $v = (v_\rho, v_\theta) \in T_p(S^n \setminus B_r), p = (\rho, \theta), G_t(p) = ((1-t)\rho, \theta)$ and we apply the sphere metric we computed in the section above. So for $\mathrm{d}G_t v$ we have:

$$\mathrm{d}G \begin{pmatrix} v_\rho \\ v_\theta \\ 0 \end{pmatrix} = \mathrm{d}G_t v = \begin{pmatrix} 1-t & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} v_\rho \\ v_\theta \end{pmatrix} = (1-t)^2 v_\rho^2 + v_\theta^2$$

$$\|v\|_p = 1 \Leftrightarrow v_\rho^2 + v_\theta^2 \sin^2\rho = 1$$

$$\|\mathrm{d}G_t v\|_{G(p)}^2 = v_\rho^2 (1-t)^2 + v_\theta^2 \sin^2((1-t)\rho) = v_\rho^2 \cdot (1-t)^2 + (1-v_\rho^2) \cdot \frac{\sin^2((1-t)\rho)}{\sin^2\rho}, \text{ where } 0 \leq v_\rho^2 \leq 1$$

So the value we are interested in maximizing is a convex combination of two terms, $(1-t)^2$ and $\frac{\sin^2((1-t)\rho)}{\sin^2\rho}$. We can find the supremum for each term, pick the larger one and be done. Instead let us first take a closer look at what is happening here. The two terms are just the operator norm in the directions of ρ and θ respectively. The reason why the norm is just a convex combination of the two is because the metric has no mixed terms, i.e. because the metric matrix dG_t is diagonal.

$$\|\mathrm{d}G_t v\|_{G(p)}^2 = v_\rho^2 \cdot \frac{\|\mathrm{d}G \vec{v_\rho}\|^2}{\|\vec{v_\rho}\|^2} + (1-v_\rho^2) \cdot \frac{\|\mathrm{d}G \vec{v}_\theta\|^2}{\|\vec{v}_\theta\|^2}$$

$$\|\mathrm{d}G_t\| = \max\{\|\mathrm{d}G_{\theta,t}\|, \|\mathrm{d}G_{\rho,t}\|\}, \text{ where } \|\mathrm{d}G_{\theta,t}\| = \sup(1-t),$$

$$\|\mathrm{d}G_{\rho,t}\| = \sup_{v_\theta \neq 0} \frac{\|\mathrm{d}G\vec{v}_\theta\|_{G(p)}}{\|\vec{v}_\theta\|_p} = \sup_{v_\theta \neq 0} \frac{\|\vec{v}_\theta\|_{G(p)}}{\|\vec{v}_\theta\|_p} = \sup_{\substack{v_\theta \neq 0, \\ \rho \neq 0}} \frac{\sqrt{\sin^2((1-t)\rho)}}{\sqrt{\sin^2\rho}} = \sup_{\rho \neq 0} \frac{\sin((1-t)\rho)}{\sin\rho}$$

Direction ρ is the boring one, as $\sup(1-t) = 1$ is achieved at $t = 0$, where the sine quotient also equals 1 for $t = 0$. Thus, we can focus solely on the direction θ of the lateral spheres[5]. INSERT SPHERE CONTRACTION PICTURE HERE For large $r > \pi/2$ the

[5]We still have to address the case $\rho = 0$. This is the pole point where our metric repersentation is not

Lipschitz constant $L < 1$, as increasing t only reduces the fraction. Geometrically, for contraction then only shrinks the lateral spheres together with their tangent vectors. For $r < \pi/2$ we achieve the largest possible stretch of the tangent vectors when the latteral spheres S^{n-1} grow the most via G, that is, when p sits at the boundary of $S^n \setminus B_r$ and G(p) sits at the equator sphere. There

$$\rho = \pi - r; \ (1-t)\rho = \pi/2$$

$$\|\mathrm{d}G_t\| = \|\mathrm{d}G_{\rho,t}\| = \frac{\sin(\pi/2)}{\sin(\pi - r)} = \frac{1}{\sin r} \sim r^{-1}.$$

□

CONTINUE HERE!!!

The bound we proved is not particularly good. In the standard proof that the image of a lower dimensional sphere is not surjective one approximates the sphere by piece-wise linear maps. We can explore this idea further by introducing simplicial approximation.

1.3 Simplicial approximation

Simplicial complexes are often neglected in presentation, so it might be beneficial to agree on some basic definitions.

Definition (simplicial complex)**.** A simplicial complex K is a collection of simplices satisfying the following conditions:

(1) Every face of a simplex in K also lies in K

(2) A non-empty intersection of two simlices in K $\sigma_1 \cap \sigma_2 \neq \varnothing$ is a face of both σ_1 and σ_2.

Additionally we equip a simplicial complex K with coherent topology of its simplices: a subset U is open in K iff $U \cap \sigma$ is open for all $\sigma \in K$.

Observation.

- A simplex σ is closed in K.
- The interior of a single vertex is the vertex itself. The boundary of a vertex is empty.
- A simplicial complex is a union of interiors of its simplices.

well defined. G fixes the pole and dG_t on the pole tangent space is identity. Hence at that point $\|dG_t|\rho = 0\| = 1$.

We will restrict our attention to finite simplicial complexes.
By default a simplicial complex K has a topology but no metric. A **geometric realization |K|** of K on the other hand carries the metric that restricts to the subspace Euclidean metric on each simplex. This metric thus obviously agrees with the topology of K (i.e. $K \cong |K|$). If K has $N+1$ vertices one can simply choose a realization as the subsimplex of the standard N simplex Δ^N.

Definition (star)**.** Let K be a simplicial complex. The **closed star** of a simplex σ in K **St**σ is the union of all simplices containing σ. The **open star** of a simplex $\sigma \in K$ stσ is the union of interiors of all simplices containing σ.

Observation: Closed stars are closed. Open stars are open. Stσ is the closure of stσ.

Of a special interest to us are stars of vertices. A star of a vertex v is the combinatorial analog of a ball around v. A closed star of a vertex caputures all adjacent and incident edges, while open stars of vertices provide an open cover that is just shy of containing the adjacent vertices - this cover is expecially useful for simplicial approximation.

Claim (Lemma 2C.2 in Hatcher)**.** *Let $v_1, v_2, \dots, v_k \in VertK$. Then $stv_1 \cap stv_2 \cap \dots \cap stv_k$ is either empty or $\sigma = [v_1, v_2, \dots, v_k] \in K$ and $stv_1 \cap stv_2 \cap \dots \cap stv_k = st\sigma$.*

Definition. Let K, J be simplicial complexes. We call a map $f_0 : VertK \to VertJ$ that takes the vertex set of K to the vertex set of J a vertex map.
A map $f : K \to J$ that is linear on each simplex of K w.r.t. the barycentric coordinates is called a simplicial map.

Observation: A simplicial map restricts to a vertex map. A vertex map that can be linearly extended to a simplicial map if for each simplex σ its vertices are mapped to vertices of some target simplex.

Now that we have collected all the necessary tools we proceed with simplicial approximation. We start with a classical result on simplicial approximation.

Theorem 1.4 (cf. Hatcher)**.** If K is a finite simplicial complex and J is an arbitrary simplicial complex, then any map $f : K \to J$ is homotopic to a map that is simplicial with respect to some iterated barycentric subdivision of K.

We don't actually intend to prove this result, but rather highlight some of the ideas that we want to translate to the Lipschitz setting:

1. We equip K with a metric as described above. In particular with this metric open stars are open and closed stars are closed in K. Distances within the simplex are also well behaved - all points are at most as far apart as the largest side length.

2. Observe that open stars form a covering of J. Taking pre-image of that cover yields an open cover of K. Since K is a finite simplicial complex it is in particular compact. We take the finite subcover and find its Lebesgue number (it exists by the Lebesgue number lemma). This gives us a way to determine the desired size of the simplices of K.
3. Now let us subdivide K until the simplices are small enough that closed star of a vertex v is contained in some cover element. This means we managed to contain the closed nighborhood of a vertex - adjacent edges and their vertices - fully in a reasonably small region of the the simplex. Edges can't wrap around our simplex multiple times and vertices cannot be too far apart. Meaning we have a chance of building a simplicial map.
   This is as much of the proof as we need for now - see Hatcher for more details and the contruction.

To translate this idea to a Lipschitz map setting we want to replace Lebesgue number using our Lipschitz constant. First we need both our spaces to be metric. We have to pay attention to several thnigs:

FRame THE DISCUSSION!

We want distance between vertices in J to be uniform (can normalize it to 1). J cannot have singletons.

The key observation here is that in a Lipschitz setting we can avoid using the Lebesgue number of the open star cover altogether. If the image of a vertex maps close to some vertex we can just take it as our approximation. A bad case is if a vertex maps far from any vertex while still close to some edge. So let us consider what happens if some vertex v maps to the barycenter of an n-simplex Δ^n of side length 1. To contain $B(Im(v), c(n))$ in an open star of any vertex we would need to set the radius $c(n)$ to be less than the shortest distance from the barycenter to the face of the n-simplex, i.e. $c(n) := dist(barycenter, \partial\Delta^n) - \epsilon$. But that distance is determined by n and it grows smaller as n increases. Meaning this was indeed the worst case scenario we have determined the required constant without referring to the Lebesgue number!

Theorem (Simplicial approximation of a Lipschitz map) . Let J be a finite simplicial complex of dimension n and let $|J|$ be an equilteral realization of J with edges of length 1. Let $f : |K| \to |J|$ be a Lipschitz map with Lipschitz constant L and let $c(n)$ be defined as in the discussion above. If $|K|$ has equilateral simplices of side length $c(n)/L$ then f can be approximated by a simplicial map with Lipschitz constant $L/c(n)$ and a homotopy H_simp with Lipschitz constant .... More generally, if each simplex of $|K|$ is homeomorphic to some standard simplex of side-length $1/L$ with bi-lipschitz constant ~ 1 (i.e. homeomorphisms are Lipschitz in both directions with constants $\lambda(K), \gamma(K)$) then the constants are $C(K, n)L$ for f_simp and blah in direction and ~ 1 in the time direction respectively.

Proof. By the discussion above we can guarantee that for each $v \in VertK$ there is a vertex $g(v) \in VertJ$ such that $\mathrm{St}v \subset \mathrm{st}g(v)$. Thus $g : VertK \to VertJ$ defines a vertex map. We want to show that it extends to a simplicial map. Let x be a point in the interior of $[v_1, v_2, \dots, v_k]$. Then $f(x)$ is contained in each of the stars stg(v_i)\$. Thus by the claim 1.3 above $\sigma = [g(v_1), g(v_2), \dots, g(v_k)]$ is a simplex in J and we can extend the vertex map g to a simplicial map f_{simp}. Again by claim 1.3 we conclude that $f(x) \in st(\sigma)$ and thus there is a simplex σ' that contains $f(x)$ in its interior and contains σ as a face (does not have to be a proper face, i.e. it is possible that $\sigma' = \sigma$). We conclude that $f(x), f_{simp}(x) \in \sigma'$. We can now simply take the straight line homotopy. (cf HATCHER). It remains to verify that the Lispchitz constants hold. f_{simp} extends linearly on simplices, thus for equilateral K the constant is deterined entirely by the ratio of edge lengths, i.e. 1/c(n). For the more general version this is magnified by how much the shortest edge in K need to be stretched, which is at most the product $\lambda \cdot \gamma$ of the Lipschitz constants of the bi-lipschitz simplex homeomorphisms in both directions, which depend solely on K. The total constant thus amounts to $C(K, n) := frac\lambda\gamma c(n) \cdot L \sim L$.
Finally, we have to determine the Lipschitz constants of the homotopy. I'll think about it tomorrow.

1.3.1 Approximating maps between spheres

Definition (triangulation)**.** Let K be a simplicial complex, X a topological space. A homeomorphism $\phi : K \to X$ is called a **triangulation** of X.

Let $f : X \to Y$ be a map between mertic spaces, $\phi : (K, d_k) \to X$, $\psi : (J, d_j) \to Y$ - bi-lipschitz triangulations. Then if there is a simplicial approximation g_{simp} of $g := \phi \circ f \circ \psi^{-1}$ we call $f_{simp} := \phi^{-}1 \circ g_{simp} \circ \psi$ the simplicial approximation of f. \begin{center}

$$\begin{tikzcd} X \arrow[r, "f_{simp}"] & Y \\ (K, d_k) \arrow[u, "\phi"] \arrow[r, "g_{simp}"] & (J, d_j) \arrow[u, "\psi"] \end{tikzcd}$$

end{center} We now want to find triangulations for out spheres so that we can apply the simplicial approximation to them. We will pick triangulations that suit our purposes for the main result.

We start with a triangulation of S^n by the boundary of the unilateral $n+1$-simplex $\partial Delta^{n+1}$. This obviously uses very few vertices, thereby limiting the quality of our approximation, so let me try to motivate this choice of triangulation (the motivation will become apparent in the upcoming proofs): for the main result in case $m \geq n$ we need all vertices to be incident. This will allow us to "approximate" the null-homotopy to some extent witout any further geometric information about it. Furthermore, we would like simplices to be both equilateral and to have equal area. This already determines our triangulation uniquely (up to rotations). Additionally, $\partial Delta^{n+1}$ is defined for all dimensions (as opposed to, say, a triangulation of Sˆ2 by the surface of icosahedron, that does not gereralize well to other dimensions). $\partial Delta^{n+1}$ is bi-lipschitz homeomorphic to Sˆn with bi-lipschitz constants ~ 1 only depending on n.

It is notably more difficult to triangulize S^m so that its triangulation fits the theorem. In fact, the proof of the main theorem requires us to be able to triangulate not only Sˆm but the whole unit ball, B^{m+1}. One difficulty with it is that the Lipschitz constant of our approximation is determined entirely by the shortest side length(s) in the metric simplicial complex: $(length(s) \cdot L \cdot 1/c(n))^{-1}$. At the same time the side-length needs to be stricty less than $c(n)/L$ We formulate this as an exercise and leave the solution for the appendix.

Exercise. Find a family of geometric simplicial complexes $(K, |K|)$ together with a bi-lipschitz triangulation of the unit ball $Tri_L : |K| \to B^{m+1}$ such that the Lipschitz constant in the direction of the ball is less than 1. We require furthermore that each simplex in $(K, |K|)$ is bilipschitz homeomorphic to the unilateral simplex Δ^{m+1}. We require the maximum over Lipschitz constants of maps $\Delta \to B^{m+1}$ to be bounded by $c(n)/L$.

Theorem 1.5 (cf 1.4 Guth)**.** If $m < n$ and $f : S^m \to S^n$ has Lipschizt consntat L, then tehre is a null-homotopy with Lipschitz constant $\lesssim L$. In fact the null-homotopy has Lipschitz constant $\lesssim L$ in the S^m directions and $\lesssim 1$ in the [0,1] direction.

Proof. Consider the map between simplices instead. Approximate g using Theorem 1.3. g_{simp} is piecewise linear hence not surjective (REFERENCE ARGUMENT either above or in the appendix!). Thus g_simp misses a whole simplex! Now back on the sphere simplicial approximation of f \$f_{simp} misses a ball of radius 1 in S^n. Applying Lemma 1.3 **??** □

Remark. Lemmas 1.2 and Propositoin bla shows that Lipschitz maps for dim $m < n$ are null-homotopic inedpendently. Both is stronger than what we need for the main proof. Note that we did not use Lemmas 1.1-1.2. ADD A REMARK AT THE BEGINNING THAT THESE CAN BE SKIPPED.

Remark. This bound is basically tight. I did not verify this, Guth recommends it as an exercise.

2 n is odd theorem

Theorem ([CDMW]). Suppose that n is odd and $f : S^m \to S^n$ is a null-homotopic map with Lipschitz constant L. Then there is a null-homotopy $H : S^m \times [0,1] \to S^n$ with Lipschitz constant at most $C(m,n)L$.

The approach they take in this proof is as follows: Here are some preparatory steps for the proof:

1. We first take a simplicial approximation of the map f (ref Lemma 1.5). The simplicial approximation is still null-homotopic: cylinder composed with the cone picture here. Therefore in the proof we can directly start with a simplicial map of the sphere.
2. A null-homotopy from an arbitrary space X can be described as a map from the Cone of X to the target. Recall that the cone of X is simply the cyllinder of X with one end collapsed to a point: $X \times [0,1]/X \times \{0\}$. Cone of S^m is homeomorphic to the unit ball B^{m+1}. To see this, think of radius of the ball as the variable for the interval. (picture for S^1).
3. We endow B^{m+1} with a simplicial structure Tri_L using the procedure from blibla.

We now have the following setup: we are given a map $h : B^{m+1} \to S^n$ (the default null-homotopy for f) with $h_{|\partial B^{m+1}} \to S^n$ simplicial with Lipschitz constant L (our map f).

Our strategy for the proof is to "straighten out" h skeleton by skeleton: we iteratively homotope h relative to the boundary (so that the restriction to f stays intact) to maps $h_0, h_1, h_2, ..., h_m = H$, where each map is $\lesssim L-$Lipschitz when restricted to the j-th skeleton of (B^{m+1}, Tri_L) until we reach the m-th skeleton. The resulting map h_m is the desired homotopy H.

There are several more key ingredients to the proof: 1. At each skeleton we leverage knowledge about the corresponding homotopy group. We will quickly revise their definition and talk about homotopy groups of spheres. 2. CW-complexes, and, in particular, simplicial complexes satisfy are fibrations, i.e. they satisfy the homotopy extention property. We will briefly state what it means and prove this fact.

Before we proceed, let us adress some concerns one might have with this strategy:

Definition. Homotopy, relative homotopy. proof that lower homotopy groups are trivial.
Definition. Fibration?

Proof. To recap, here is our setting: we are given a map $h : B^{m+1} \to S^n$ (a null-homotopy for f), which is simplicial, when restricted to the boundary, i.e. $h|_{\partial B^{m+1}} = f : (S^m, Tri_L) \to (S^n, Tri_{S^n})$ simplicial. We denote the skeleta of the simplicial complex B^{m+1} by $X^0 \subset X^1 \subset X^2 \cdots \subset X^m \subset X^{m+1} = B^{m+1}$.
We first aim to find a map h_0 that is simplicial on the 0-th skeleton. For that we need to figure out where to map the vertices of Tri_L. If we could use some procedure similar to that of Theorem REFERENCE simplicial] we could ensure that the neighboring vertices are mapped to the same simplex. But we triangulate B^{m+1} without any knowledge of h, so h could map vertices of a simplex anywhere in the target, no matter how fine of a triangulation we prescribe to B^{m+1} at the beginning. Since we later want to build a simplicial map on the 1-skeleton we need edges to map to edges, which forces us to choose the boundary of a standard simplex as triangulation of the target.[1]
h(X^0) is a disjoint set of points in S^n. For each of the points we choose a vertex of Tri_{S^n} that is closest according to the surface metric. S^n is path-connected, which gives a homotopy relative to the vertex. To fit the more general framework, we could say that $\pi_0(S^n) = 0$. We could even pick a straight-line homotopy by taking geodesics to keep this procedure as deterministic as possible. Having fixed the homotopies we use the homotopy extension property to obtain the homotopy g^0:

$$\begin{tikzcd} X^0 \times [0,1] \cup B^{m+1} \times \{0\} \arrow[r, "label"] \arrow[d, hook] & S^n \\ B^{m+1} \times [0,1] \arrow[ur, dashed, "g"'] & \end{tikzcd}$$

[1]There are other reasons which make this choice of triangulation of the target especially convinient. Namely, later in the proof we would like the target to have equilateral simplices of equal area.