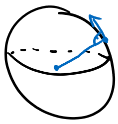
At any time there is a location on earth where the wind is not blowing.



Mathematical interpretation:

For any continuous vector field on $S^2$

$$\tau: S^2 \to \mathbb{R}^3$$

$(\langle \tau(x), x \rangle = 0$ for $x \in S^2 \subset \mathbb{R}^3)$

there is a zero.

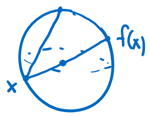
**Lemma:** Let $n \geq 2$ even. For each map $S^n \xrightarrow{f} S^n$ there is a $x \in S^n$ such that

$$f(x) \in \{x, -x\}.$$

**Proof:** Assume that $f(x) \notin \{x, -x\}$ for all $x \in S^n$.

Define

$$F(x,t) = \frac{(1-t)x + t \cdot f(x)}{\|(1-t)x + t \cdot f(x)\|}$$ is homotopy $\mathrm{id}_{S^n} \simeq f$




$$G(x,t) = \frac{(1-t)f(x) + t(-x)}{\|(1-t)f(x) + t(-x)\|}$$ is a homotopy $-\mathrm{id}_{S^n} \simeq f$.

$-\mathrm{id}_{S^n}$ is the composition of $n+1$ reflections.

Hence $H_n(-\mathrm{id}_{S^n}): H_n(S^n) \longrightarrow H_n(S^n)$

is multiplication with $(-1)^{n+1} = -1$.

Hence $-\mathrm{id}_{S^n} \not\simeq \mathrm{id}_{S^n}$ ↯

$\square$

**Theorem** Let $n \geq 2$ be even. Every continuous vector field on $S^n$ vanishes at some point.

[every map $\tau: S^n \to \mathbb{R}^{n+1}$ with $\langle \tau(x), x \rangle = 0$ $\forall x \in S^n$ vanishes at some point]

**Proof.** Assume $\tau(x) \neq 0$ for all $x \in S^n$.

Define $f(x) = \dfrac{\tau(x)}{\|\tau(x)\|} \in S^n$.

By the lemma there is $x_0 \in S^n$ such that

$$f(x_0) \in \{x_0, -x_0\}.$$

This contradicts

$$0 = \langle \tau(x_0), x_0 \rangle = \|\tau(x_0)\| \cdot \langle f(x_0), x_0 \rangle$$

$\square$

I hereby declare that this document has been composed by myself and describes my own work, unless otherwise acknowledged in the text.

Ich versichere wahrheitsgemäß, die Arbeit selbstständig verfasst, alle benutzten Hilfsmittel vollständig und genau angegeben und alles kenntlich gemacht zu haben, was aus Arbeiten anderer unverändert oder mit Abänderungen entnommen wurde sowie die Satzung des KIT zur Sicherung guter wissenschaftlicher Praxis in der jeweils gültigen Fassung beachtet zu haben.

Karlsruhe, 10.05.2022

# Abstract

Some English abstract.

# Zusammenfassung

Eine deutsche Zusammenfassung.

# Contents

# 1 1 Background of the problem

## 1.1 Covering with spherical caps.

The first case we consider are maps from $S^m$ to $S^n$ when $m < n$. Throughout this paper let the unit spheres $S^m$, $S^n$ be equipped with the length metric induced by the standard Riemannian metric. That is, the distance between any two points is determined by the (Euclidean) length of the geodesics between them [1]. Note that while the topology is the same, the metric is different from the "default" metric inherited from the ambient Eukledian space.

We first provide a quantitative argument for contractibility of a Lipschitz-map, namely that the image of any such map misses a ball in the target. We then produce a null-homotopy with a controlled Lipschitz constant.

**Claim** (Lemmas 1.2, 1.3)**.** *Let $f : S^m \to S^n$ be a Lipschitz conituous maps with a Lipschitz constant $L$. Then the image of $f$ misses a ball of radius $r$ for $r \gtrsim L^{\frac{-m}{n-m}}$*

*For each radius $r$ there is a Lipschitz-contraction $G : S^n \setminus B_r \times [0,1] \to S^n \setminus B_r$. $G$ has Lipschitz constant $\lesssim 1/r$ in the $S^n$ direction and $\lesssim 1$ in the $[0,1]$ direction.*

To prove the claim we will need a few tools. For the first part we will need an upper bound for the covering number of the sphere. We will start by recalling the necessary concepts, then proceed with a construction to procure the required bound. For the second part we will need the Riemannian metric on the sphere using polar coordinates. We will then demonstrate how to find a Lipschitz constant for Lipschitz maps between manifolds.

**Definition** (Covering, packing)**.** Let $(X, d)$ be a metric space, $K \subseteq X$.

[1] To be precise, the length of the geodesics is determined by the standard Riemmanian metric, where the metric is pulled back along the embedding of the spheres into their ambient Euclidean spaces ($\mathbb{R}^m$, $\mathbb{R}^n$, respectively). The lengths of geodesics are then precisely the respective Euclidean lengths of their embeddings. The reason to specify a metric so early on is that when we talk about Lipschitz continuity we are implicitly dealing with the metrics, not just with undelying topologies. However, since all of our results are up to a constant, suitable constant manipulation would show them to hold for the standard Eukledian metric as well. Nevertheless, we prefer to settle on a specific metric to avoid confusion or ambiguity.

A collection of points $C$ in $X$ is called an **$\rho$-covering** of $K$ if $K$ is contained in the union of $\rho$-balls around points in $C$, i.e. $K \subseteq \cup_{p \in C} B_\rho(p)$. In other words, for $\forall x \in K$ there is a $p$ in $C$ such that $d(p, x) \leq \rho$. Note that we do not require the centers of $\rho$-balls to lie in K. Such a covering is also called an **external $\rho$-covering**. The minimum $\rho$-covering cardinality is called the **(external) covering number** of $K$ denoted $N(K, d, \rho)$ or simply $N(\rho)$.

A collection $P$ of points in $K$ is called an **$\rho$-packing** if for $\forall p, q \in P$ $d(p, q) > \rho$. The maximum packing cardinality is called the **packing number** of $K$ and is denoted by $M(K, d, \rho)$ or simply $M(\rho)$.

**Observation.** Let P be an $\rho$-packing. Then the balls $B_{1/2\rho}(p)$ are pairwise disjoint (triangle inequality).
If P is maximal, then P is also an $\rho$-covering (by contraposition). In particular, this implies $N(\rho) \leq M(\rho)$

**Claim.** $M(2\rho) \leq N(\rho) \leq M(\rho)$

*Proof.* The second inequality follows from the observation above. To prove the first inequality, assume $M(2\rho) > N(\rho)$. Then by the pigeon-hole principle there are two points $x, y$ of the packaging contained in the same $\rho$-ball of the cover. By triangle inequality this yields a contradiction. □

We are now going to provide an upper bound for the covering number of a sphere. Geomtrically, we will be covering a sphere by spherical caps of equal size. We are interested in exloring the relationship between the size of the caps and the covering number.

**Definition** (spherical cap)**.** A closed **spherical cap** is the smaller portion of a unit sphere $S^m$ cut off by a plane (including the boundary). Formally, the spherical cap with angle $\rho \in (0, \pi/2]$ and center $x \in S^m$ is given by

$$cap(x, \rho) = \{y \in S^m : \langle x, y \rangle \geq cos\rho\}.$$

We will call a spherical cap with a polar angle $\rho$ a **$\rho$-cap**. Since we are dealing with a unit sphere, the polar angle in radians is precisely the length of any geodesic from the center (the tip) of the cap to its edge.

**Lemma 1.1.** *The covering number of a sphere $N(S^m, d, \rho) \lesssim \rho^{-m}$, where $d$ is the length-metric. That is, for any $\rho > 0$, the sphere $S^m$ can be covered by at most (up to a constant) $1/\rho^m$ $\rho$-caps.*

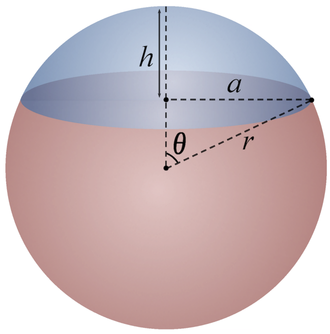



Figure 1.1: Spherical cap. Here $r = 1$, $\theta = \rho$, $a = \sin\theta$, $h = 1 - \cos\theta$

Remark: It is sufficient for us to show the upper bound up to a constant $c(m)$. The reason for that is that in later arguments we will be able to choose the radius of the cover small enough that any constant $c(m, n)$ can be "neutralized" for our purposes, so long as the quantities we omit do vary with $\rho$.

*Proof.* Let us first consider a maximal packing of our sphere with spherical caps. For any such packing the total volume of spherical caps cannot exceed the volume[2] of the sphere. As the caps in a packing are disjoint,

$$M(\rho) \leq \frac{\omega^m(S^m)}{\omega^m(\rho\text{-cap})}.$$

Now, $S^m$ can be covered by exactly two $\pi$-caps, so $\omega^m(S^m) = 2\omega^m(\pi\text{-cap})$. Rewriting the previous inequality we get:

$$M(\rho) \leq \frac{2\omega^m(\pi\text{-cap})}{\omega^m(\rho\text{-cap})}. \tag{1.1}$$

We would like to replace the $\rho$-caps in the inequality by $\rho$-disks, as they scale easier with $\rho$, and that would allow us to reduce the fraction. Projecting the cap down onto the disk at its base will reduce the volume[3], i.e. $\omega^m(\rho\text{-cap}) \geq \omega^m(\sin\rho D^m)$. Dividing both sides by the $m$-volume of a $\rho$-disk and simplifying we obtain the following inequality:

$$\frac{1}{\left(\frac{\pi}{2}\right)^m} \leq \frac{\sin^m\rho}{\rho^m} = \frac{\omega^m(\sin\rho D^m)}{\omega^m(\rho D^m)} \leq \frac{\omega^m(\rho\text{-cap})}{\omega^m(\rho D^m)},$$

where $\rho \in (0, \frac{\pi}{2}]$. Multiplying by $\left(\frac{\pi}{2}\right)^m$ we get:

$$1 \leq \left(\frac{\pi}{2}\right)^m \cdot \frac{\omega^m(\rho\text{-cap})}{\omega^m(\rho D^m)} \tag{1.2}$$

[2]We are referring to $m$-volumes. Think of surface areas in case $m = 2$.

[3]In fact it is easy to see that $\omega^m(\sin\rho D^m) \leq \omega^m(\rho\text{-cap}) \leq \omega^m(\rho D^m)$. We leave it for the appendix maybe.

Multiplying inequal (1.1) by a term (1.2) greater than 1 on the right yields:

$$N(\rho) \leq M(\rho) \lesssim \frac{\omega^m(\pi\text{-cap})}{\omega^m(\rho\text{-cap})} \lesssim \frac{\omega^m(\pi D^m)}{\omega^m(\rho D^m)} = \frac{\pi^m}{\rho^m} \sim \frac{1}{\rho^m}.$$

$\square$

**Lemma 1.2.** *Let $f\colon S^m \to S^n$ be a Lipschitz-conituous map with a Lipschitz constant $L$. Then the image of $f$ misses a ball of radius $r$ for $r \gtrsim L^{-\frac{m}{n-m}}$*

*Proof.* For any $\rho > 0$, $S^m$ can be covered by $\sim \rho^{-m}$ balls of radius $\rho$. The image of each such ball is contained in a ball of radius $L\rho$. Therefore, the image of $f$ is covered by $\lesssim \rho^{-m}$ balls of radius $L\rho$. We set $r := L\rho$. We now want to choose $\rho$ small enough so that the cover misses a ball of radius $r$.

Expanding the radius of the cover to $2r$ yields a cover of the $r$-neighborhood of the image. We denote this $2r$-cover by $C$. If this larger cover does not cover the full sphere $S^n$, the image of $f$ must miss a ball of radius $r$. The total volume of the cover $C$ is at most the cardinality of $C$ times the volume of a $2r$-cap of $S^n$, $|C|\omega^n(2r\text{-cap})$. We set $\rho$ so that this number is smaller than the volume of the sphere. So we get

$$|C|\omega^n(2r\text{-cap}) \lesssim \rho^{-m} r^n = L^n \rho^{n-m} \lesssim 1,$$

$$\rho \lesssim L^{-\frac{n}{n-m}},$$
$$r = L\rho \lesssim L^{-\frac{m}{n-m}}.$$

$\square$

# 2 Computing our first Lipschitz constant

## 2.1 Detour: geometric suspension

If we equip the sphere $S^2$ with the usual pullback Riemannian metric, the resulting metric written in the matrix form is

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}.$$

Even more often in the literature one encounters the corresponding symmetric quadratic form - its first fundamental form - which can be written as:

$$\mathrm{d}s^2 = \mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2\theta \; \mathrm{d}\phi \otimes \mathrm{d}\phi,$$

or simply

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta \; \mathrm{d}\phi^2.$$

We will now show that the metric is verbatim the same for $S^m$ for $\forall m \geq 2$

TIKZ PICTURE HERE

We can think of $S^m$ as of several $S^{m-1}$ stacked on top of each other (where $S^{m-1}$ shrink to a single point at the poles). This is essentially the geometric version of suspension. Using the polar angle rather than height, we scale the equator $S^{m-1}$ by $\sin\theta$.

Point-wise this gives us that any point $p$ of $S^m$ can be parametrized in terms of the polar angle $\theta$ and the corresponding vector $\phi$ of the equator scaled down by $\sin\theta$ - polar coordinates with respect to $S^{m-1}$. Fixing some direction $z$ in $\mathbb{R}^{m+1}$ we can write out the parametrization:

$$\begin{aligned} \psi :& [0,\pi] \times S^{m-1} \longrightarrow S^m \\ & (\theta, \phi) \mapsto \sin\theta \cdot \phi + \cos\theta \cdot \vec{e_z}, \end{aligned} \tag{2.1}$$

where $\vec{e_z}$ denote the standard basis vector in the $z$ direction. Computing partial derivatives yields

$$\frac{\partial \psi}{\partial \theta} = \cos\theta \cdot \phi - \sin\theta \cdot \vec{e_z},$$

$$\frac{\partial\psi}{\partial\phi} = \sin\theta \cdot \vec{e_z}.$$

Computing the spherical metric as a pullback of the $\mathbb{R}^{m+1}$ metric:

$$g_{\theta\theta} = \langle \cos\theta \cdot \phi - \sin\theta \cdot \vec{e_z}, \cos\theta \cdot \phi - \sin\theta \cdot \vec{e_z} \rangle = \cos^2\theta \cdot \langle \phi, \phi \rangle + \sin^2\theta \cdot \langle \vec{e_z}, \vec{e_z} \rangle = 1,$$

$$g_{\phi\theta} = g_{\theta\phi} = 0,$$

$$g_{\phi\phi} = \sin^2\theta$$

yielding the desired

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}.$$

Remark. Note that in this we could replace $S^{m-1}$ with an arbitrary manifold $M$ of non-zero dimension FOOTNoTE(for zero-dimensional manifolds $\mathrm{d}\phi^2$ vanishes, leaving $\mathrm{d}s^2 = \mathrm{d}\theta^2$ as the metric). Remarkably, since we are not using any knowledge of the underlying manifold $M$ to compute the suspension metric with respect to $M$, it is only the function that we use to shrink the manifold towards suspension poles that matters for this relative metric. Analogously, we could take an analytic version of any topological construction to obtain its geometric version.

The complement of a point in $S^n$ is contractible. If we remove a ball from $S^n$, the leftover part can be contracted in a Lipschitz way.

**Lemma 2.1.** *For each radius $r$ there is a Lipschitz-contraction $G : S^n \setminus B_r \times [0,1] \to S^n \setminus B_r$. $G$ has Lipschitz constant $\lesssim 1/r$ in the $S^n$ direction and $\lesssim 1$ in the $[0,1]$ direction.*

## 2.2 Detour: manifolds with boundaries

We want to show that we can contract the target sphere $S^n$ in a Lipschitz way. For that we need to contruct a differentiable map between the cylinder of $S^n$ and $S^n$. Reminder: the (topological) cylinder is the cartesian product with the interval. So we want a map between manifolds, both equipped with a metric. For the sake of consistency, we would prefer to equip both with the length metric. Naturally, we could take the product Riemannian metric. But the inerval is not a manifold, nor is the (topological) cylinder! For it is strictly speaking not Euclidean at the points on the boundary - in the interval dimension we can only move in one direction from the boundary M kreuz $\{0\}$. At those boundary points we do, however, have homeomorphism to the Euclidean half-space $\mathbb{R}^{m+1}$. We would like to relax the usual definition of a manifold to include manifolds with boundary:

**Definition** (manifold with boundary)**.** definition here

Thus, the old manifolds are just manifolds with an empty boundary. Notably, the relaxed definition encompasses basic topological objects, such as the (closed) unit disk, the Moebius strip and topological cylinders as manifolds, the latter allowing us to consider differentiable homotopies.

All the usual definitions of dimension, tangent spaces etc apply to manifolds with boundaries. A manifold with a boundary also always admits a Riemannian metric:

**Definition** (Double)**.** A double a manifold with a boundary is bla glued along their boundaries. A double is a manifold without a boundary.

**Observation.** A double of a manifold $M$ admits a Reimannian metric. Selecting a metric and restricting to $M$ yields a Riemmanian metric on $M$.

## Derivative of a differentiable map w.r.t. the metric

In this section we want to learn how to find Lipschitz constants for a given differentiable map between manifolds. We want learn how to compute the differential directly using the corresponding metrics, with respect to a given parametrization As usual we will equip our spaces with the length metric

Equipping our spaces with a specific metric allows for explicit computations of c'(t) for a given curve c(t), explicit computations of lengths of tangent vectors etc. In particular it allows us to compute partial derivatives w.r.t. to our chosen parametrization as local dilation and to give an upper bound to dilation of a given map between manifolds.

THIS IS STILL HORRIBLE - REWRITE!

**Definition** (Dilation, local dilation)**.** definition here

Zusammenhang mit L, mit c'(t), mit Ableitung und Differential.

*Proof.* □