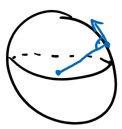
At any time there is a location on earth where the wind is not blowing.



Mathematical interpretation:

For any continuous vector field on S^2

$$\tau : S^2 \longrightarrow \mathbb{R}^3$$

$(\langle \tau(x), x \rangle = 0$ for $x \in S^2 \subset \mathbb{R}^3)$

there is a zero.

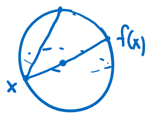
Lemma: Let $n \geqslant 2$ even. For each map $S^n \xrightarrow{f} S^n$ there is a $x \in S^n$ such that

$$f(x) \in \{x, -x\}.$$

Proof: Assume that $f(x) \notin \{x, -x\}$ for all $x \in S^n$.

Define

$$F(x,t) = \frac{(1-t)x + t \cdot f(x)}{\| (1-t)x + t \cdot f(x) \|}$$ is homotopy $\mathrm{id}_{S^n} \simeq f$



$$G(x,t) = \frac{(1-t)f(x) + t(-x)}{\| (1-t)f(x) + t(-x) \|}$$ is a homotopy $-\mathrm{id}_{S^n} \simeq f$.

$-\mathrm{id}_{S^n}$ is the composition of $n+1$ reflections.

Hence $H_n(-\mathrm{id}_{S^n}) : H_n(S^n) \longrightarrow H_n(S^n)$

is multiplication with $(-1)^{n+1} = -1$.

Hence $-\mathrm{id}_{S^n} \not\simeq \mathrm{id}_{S^n}$ ↯

□

Theorem Let $n \geqslant 2$ be even. Every continuous vector field on S^n vanishes at some point.

[every map $\tau : S^n \to \mathbb{R}^{n+1}$ with $\langle \tau(x), x \rangle = 0$ $\forall x \in S^n$ vanishes at some point]

Proof. Assume $\tau(x) \neq 0$ for all $x \in S^n$.

Define $f(x) = \dfrac{\tau(x)}{\|\tau(x)\|} \in S^n$.

By the lemma there is $x_0 \in S^n$ such that

$$f(x_0) \in \{x_0, -x_0\}.$$

This contradicts

$$0 = \langle \tau(x_0), x_0 \rangle = \|\tau(x_0)\| \cdot \langle f(x_0), x_0 \rangle$$

□

I hereby declare that this document has been composed by myself and describes my own work, unless otherwise acknowledged in the text.

Ich versichere wahrheitsgemäß, die Arbeit selbstständig verfasst, alle benutzten Hilfsmittel vollständig und genau angegeben und alles kenntlich gemacht zu haben, was aus Arbeiten anderer unverändert oder mit Abänderungen entnommen wurde sowie die Satzung des KIT zur Sicherung guter wissenschaftlicher Praxis in der jeweils gültigen Fassung beachtet zu haben.

Karlsruhe, 10.05.2022

Abstract

Some English abstract.

Zusammenfassung

Eine deutsche Zusammenfassung.

Contents

1 1 Background of the problem

some introduction.

Gromov was able to show that there is a Lipschitz null-homotopy with the tower of exponentials. He apparently was also able to show linear dependency for the case m=n. (reference Guth).

1.1 Some backgound assumptions

This paper deals with Lipschitz constants of maps between spheres. Most of the time we will only estimate the Lipschitz constants up to a constant $C(m,n)$, that only depends on the dimensions of the spheres. We denote equality/inequality up to a constant by $\sim, \lesssim, \gtrsim$ respectively. Throughout this paper let the unit spheres S^m, S^n be equipped with the length metric induced by the standard Riemannian metric (unless stated otherwise). That is, the distance between any two points is determined by the (Euclidean) length of the geodesics between them [1]. Note that while the topology is the same, the metric is different from the "default" metric inherited from the ambient Euclidean space. Occasionally we will consider objects that are homeomorphic to spheres when it is convinient (e.g. surface of a cube or of a simplex), but the conversion only changes things up to some constrant. On those objects we will still be using the length metric.

Statement of the problem.

[1] To be precise, the length of the geodesics is determined by the standard Riemmanian metric, where the metric is pulled back along the embedding of the spheres into their ambient Euclidean spaces ($\mathbb{R}^m$, $\mathbb{R}^n$, respectively). The lengths of geodesics are then precisely the respective Euclidean lengths of their embeddings. The reason to specify a metric so early on is that when we talk about Lipschitz continuity we are implicitly dealing with the metrics, not just with undelying topologies. However, since all of our results are up to a constant, suitable constant manipulation would show them to hold for the standard Euclidean metric as well. Nevertheless, we prefer to settle on a specific metric to avoid confusion or ambiguity.

1.2 Contracting the image of a lower dimensional sphere

1.2.1 Introduction to computations up to a constant

In this section we first consider Lipschitz maps from S^m to S^n when $m < n$. This case is fairly easy, as we know from topology that APPARENTLY THAT'S WRONG. we need piece-wise linearity or smoothness. We know that the image of S^m in S^n is not-surjective (citation). It is then contractible. In this section we want to show that the image of a lower dimensional sphere can be contracted in a Lipschitz way, and to provide a fairly tight Lipschitz constant.

We start by showing that a Lipschitz map must in fact miss a whole open ball in the target:

Lemma 1.1. *Let $f : S^m \to S^n$ be a Lipschitz conituous maps with a Lipschitz constant L. Then the image of f misses a ball of radius r for $r \lesssim L^{\frac{-m}{n-m}}$*

Our strategy for proving this lemma will be to first cover the domain sphere by open balls of a set radius, then map that cover to the target sphere and show that the image of the cover cannot be surjective.

A ball with respect to the length-metric on the sphere is a spherical cap. The radius of the ball is the length of any geodesic from the center (the tip) of the cap to its edge. It is equal to the polar angle of the cap in radians.

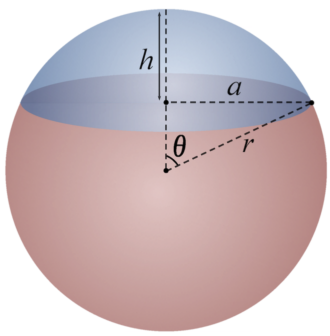


Figure 1.1: Spherical cap. Here $r = 1$, θ is the polar angle, $a = \sin\theta$, $h = 1 - \cos\theta$

For the proof of the lemma we will need to cover the sphere with spherical caps. To estimate the number of caps needed to cover the sphere we use a volume argument. A sloppy version of the argument would go as follows: we equip the sphere with a volume form that scales well with the polar angle and is equal to 1 on the whole sphere. Then the volume of the sphere is 1, the volume of each spherical cap is ρ^m. The cover should have area similar to that of the sphere (up to a constant). We then need $\sim 1/\rho^m$ spherical caps to cover the sphere.

You may object: why should the cover have volume similar to that of the sphere if there is an overlap? Why should the overlap scale well with the radius? Is it okay for us to change the metric if the result is stated w.r.t. to a different one (we could of course use the standard volume but then you would be right to point out that spherical cap volume might not scale well with the cover radius). However, this argument is only meant to provide us with an intuition, and we aim to show that this types of arguments can be formalized fairly easily:

Claim. *For any $\rho > 0$, the sphere S^m can be covered by $\sim 1/\rho^m$ balls of radius ρ.*

Proof. We want to estimate the number of $1/\rho$ balls needed to cover the sphere. Covering the sphere is up to a constant the same as covering the hemisphere. In fact, the the cardinality of the cover for $S^m \sim$ Hemisphere $S^m_+ \sim D^m$ (we can tranfer the cover back and forth by projecting the hemisphere onto the equator disk or wrapping a larger disk around the hemisphere [2] $= B^m \sim$ covering $\sqrt{2}B^m$ (scaling up) $\sim$ covering the m-box of side length two (it can be squeezed between the two balls, i.e. it contains the unit ball and it is contained in the $\sqrt{2}B^m$) $\sim$ covering the m-box of side length 1 (the unit m-box). It is easy to see why the volume argument should work now: the unit box can be clearly be covered by $\lceil 1/\rho \rceil^m$ boxes of side length ρ. Each ρ-box is contained in a ball of radius ρ and we are done. $\square$

Arguing up to a constant allows us great flexibility in choosing objects we are more comfortable working with. The constants we omitted can easily be traced back through the equivalence steps we took. However, if you are not yet comfortable working up to a constant there is a direct argument on the sphere without any equivalences or dropping constants that I provided in the appendix.

Lemma 1.2 (Image of misses a ball)**.** *Let $f : S^m \to S^n$ be a Lipschitz-conituous map with a Lipschitz constant L. Then the image of f misses a ball of radius r for $r \lesssim L^{-\frac{m}{n-m}}$*

Proof. For any $\rho > 0$, S^m can be covered by $\sim \rho^{-m}$ balls of radius ρ. The image of each such ball is contained in a ball of radius $L\rho$. Therefore, the image of f can be covered by $\lesssim \rho^{-m}$ balls of radius $L\rho$. We set $r := L\rho$. We now want to choose ρ small enough so that the cover misses a ball of radius r.

[2]Projecting the hemisphere S^m_+ down onto the unit disk D^m at the equator obviously only changes things up to a constant (depending only on m): we project down the cover centers and keep the radius as is. Showing that the cover can be transferred in the other direction is a little trickier: we start by scaling the cover up by a factor of $\pi/2$ to cover $\pi/2D^m$. We then wrap the larger disc around the hemisphere by taking (θ, r) to $(\theta, \rho) = (\theta, r), where,$ r is the radius and ρ is the polar angle. Distances can only reduce for the same reason that we can wrap a paper around a ball without tearing it and the paper will wrinkle: radial components of distances stay the same and angular components srink with a factor of sine as the radius increases. Of course if you are not convinced you can scale your disk up by another factor of two. Same as before, keeping the projected cover centers and the old cover radius yield a cover.

Expanding the radius of the cover to $2r$ yields a cover of the r-neighborhood of the image. We denote this $2r$-cover by C. If this larger cover does not cover the full sphere S^n, the image of f must miss a ball of radius r. The total volume of the cover C is at most the cardinality of C times the volume of a ball of radius $2r$ (which is a spherical cap of polar angle $2r$). We replace the cap volume by the larger volume of a disk $2r \cdot D^n$ by essentially the same argument as we used to transfer the disk cover from the disk to the hemisphere HEMISPHERE IS ESSENTIALLY A spherical cap. Can we use the "special" volume form argument here instead????? and [3]. The total cover volume is then at most $|C|\omega^n(\pi/2 \cdot B_2^n r)$, where ω^n denotes the Euclidean n-volume form?????. $\square$

We now set ρ so that this number is smaller than the volume of the sphere. So we get for $n > m$

$$|C|\omega^n(2r\text{-cap}) \lesssim \rho^{-m} r^n = L^n \rho^{n-m} \lesssim 1,$$

$$\rho \lesssim L^{-\frac{n}{n-m}},$$

$$r = L\rho \lesssim L^{-\frac{m}{n-m}}.$$

In particular, even if f is a constant map we can choose ρ small enough so that $r \leq \pi/2$

1.2.2 Detour: geometric suspension

If we equip the sphere S^2 with the usual pullback Riemannian metric, the resulting metric written in the matrix form is

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}.$$

Even more often in the literature one encounters the corresponding symmetric quadratic form - its first fundamental form - which can be written as:

$$\mathrm{d}s^2 = \mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2\theta \, \mathrm{d}\phi \otimes \mathrm{d}\phi,$$

or simply

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2.$$

[3]Projecting the hemisphere S^m_+ down onto the unit disk D^m at the equator obviously only changes things up to a constant (depending only on m): we project down the cover centers and keep the radius as is. Showing that the cover can be transferred in the other direction is a little trickier: we start by scaling the cover up by a factor of $\pi/2$ to cover $\pi/2D^m$. We then wrap the larger disc around the hemisphere by taking (θ, r) to $(\theta, \rho) = (\theta, r), where,$ r is the radius and ρ is the polar angle. Distances can only reduce for the same reason that we can wrap a paper around a ball without tearing it and the paper will wrinkle: radial components of distances stay the same and angular components srink with a factor of sine as the radius increases. Of course if you are not convinced you can scale your disk up by another factor of two. Same as before, keeping the projected cover centers and the old cover radius yield a cover.

We will now show that the metric is verbatim the same for S^m for $\forall m \geq 2$

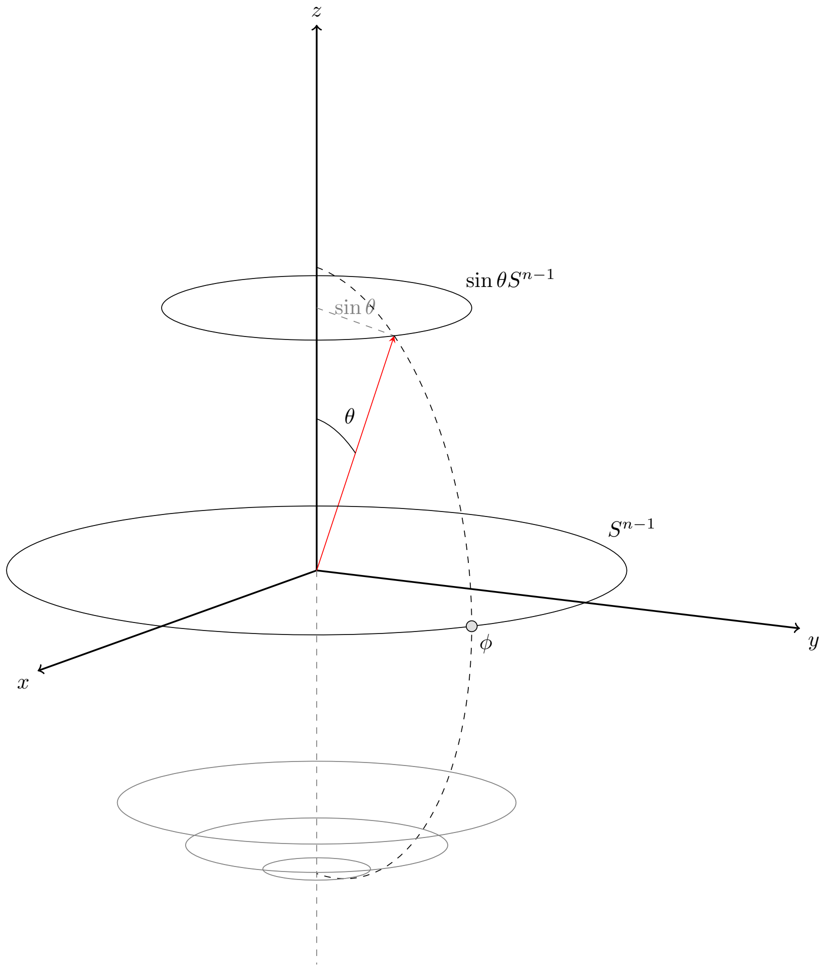


Figure 1.2: Polar coordinates

We can think of S^m as of several S^{m-1} stacked on top of each other (where S^{m-1} shrink to a single point at the poles). This is essentially the geometric version of suspension. Using the polar angle rather than height, we scale the equator S^{m-1} by $\sin\theta$.

Point-wise this gives us that any point p of S^m can be parametrized in terms of the polar angle θ and the corresponding vector ϕ of the equator scaled down by $\sin\theta$ - polar coordinates with respect to S^{m-1} Figure 1.2. Fixing some direction z in $\mathbb{R}^{m+1}$ we can write out the parametrization:

$$\begin{aligned} \psi :& [0,\pi] \times S^{m-1} \longrightarrow S^m \\ &(\theta,\phi) \mapsto \sin\theta \cdot \phi + \cos\theta \cdot \vec{e_z},' \end{aligned} \tag{1.1}$$

where $\vec{e_z}$ denote the standard basis vector in the z direction. Computing partial derivatives yields

$$\frac{\partial\psi}{\partial\theta} = \cos\theta \cdot \phi - \sin\theta \cdot \vec{e_z},$$

$$\frac{\partial\psi}{\partial\phi} = \sin\theta \cdot \vec{e_z}.$$

Computing the spherical metric as a pullback of the $\mathbb{R}^{m+1}$ metric:

$$g_{\theta\theta} = \langle \cos\theta \cdot \phi - \sin\theta \cdot \vec{e_z}, \cos\theta \cdot \phi - \sin\theta \cdot \vec{e_z} \rangle = \cos^2\theta \cdot \langle \phi, \phi \rangle + \sin^2\theta \cdot \langle \vec{e_z}, \vec{e_z} \rangle = 1,$$

$$g_{\phi\theta} = g_{\theta\phi} = 0,$$

$$g_{\phi\phi} = \sin^2\theta$$

yielding the desired

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}.$$

Remark. Note that in this we could replace S^{m-1} with an arbitrary manifold M of non-zero dimension [4]. Remarkably, since we are not using any knowledge of the underlying manifold M to compute the suspension metric with respect to M, it is only the function that we use to shrink the manifold towards suspension poles that matters for this relative metric. Analogously, we could take an analytic version of any topological construction to obtain its geometric version.

The complement of a point in S^n is contractible. If we remove a ball from S^n, the leftover part can be contracted in a Lipschitz way.

Lemma 1.3. *For each radius r there is a Lipschitz-contraction $G : (S^n \setminus B_r) \times [0,1] \to S^n \setminus B_r$. G has Lipschitz constant $\lesssim 1/r$ in the S^n direction and $\lesssim 1$ in the $[0,1]$ direction.*

We choose the obvious contraction map:

$$G : (S^n \setminus B_r) \times [0,1] \to S^n \setminus B_r$$

$$G : (\rho, \theta, t) \to ((1-t)\rho, \theta)$$

Our goal is to compute its Lipschitz constants in both the sphere and the time direction. The strategy is to find the supremum of the differential applied to the appropriate tangent vectors and use it as an upper bound for the Lipschitz constants. The theoretical foundation for this approach is the mean value theorem for manifolds (REFERENCE).

State the mean value theorem, reference.

[4] For zero-dimensional manifolds $\mathrm{d}\phi^2$ vanishes, leaving $\mathrm{d}s^2 = \mathrm{d}\theta^2$ as the metric.

Proof. Let G be as above. Its differential is

$$\mathrm{d}G = \begin{pmatrix} 1-t & 0 & -\rho \\ 0 & 1 & 0 \end{pmatrix}$$

We start with the Lipschitz constant in the direction of the sphere by restricting to tangent vectors in the sphere direction, i.e. with the zero time component $(v_\rho, v_\theta, 0) \in T_p((S^n \setminus B_r) \times [0,1])$. It is of course the same as to fix t as a parameter and consider the family of maps G_t that are self-maps of the punctured sphere $S^n \setminus B_r$. We want compute the operator norm $\|\mathrm{d}G_t\|$ (REFERENCE):

$$\|\mathrm{d}G_t\| = \sup_{v\neq 0} \frac{\|\mathrm{d}G_t v\|_{G(p)}}{\|v\|_p} = \sup_{\|v\|_p=1} \|\mathrm{d}G_t v\|_{G(p)},$$

where $v = (v_\rho, v_\theta) \in T_p(S^n \setminus B_r), p = (\rho, \theta), G_t(p) = ((1-t)\rho, \theta)$ and we apply the sphere metric we computed in the section above. So for $\mathrm{d}G_t v$ we have:

$$\mathrm{d}G \begin{pmatrix} v_\rho \\ v_\theta \\ 0 \end{pmatrix} = \mathrm{d}G_t v = \begin{pmatrix} 1-t & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} v_\rho \\ v_\theta \end{pmatrix} = (1-t)^2 v_\rho^2 + v_\theta^2$$

$$\|v\|_p = 1 \Leftrightarrow v_\rho^2 + v_\theta^2 \sin^2\rho = 1$$

$$\|\mathrm{d}G_t v\|^2_{G(p)} = v_\rho^2(1-t)^2 + v_\theta^2 \sin^2((1-t)\rho) = v_\rho^2 \cdot (1-t)^2 + (1-v_\rho^2) \cdot \frac{\sin^2((1-t)\rho)}{\sin^2\rho}, \text{ where } 0 \leq v_\rho^2 \leq 1$$

So the value we are interested in maximizing is a convex combination of two terms, $(1-t)^2$ and $\frac{\sin^2((1-t)\rho)}{\sin^2\rho}$. We can find the supremum for each term, pick the larger one and be done. Instead let us first take a closer look at what is happening here. The two terms are just the operator norm in the directions of ρ and θ respectively. The reason why the norm is just a convex combination of the two is because the metric has no mixed terms, i.e. because the metric matrix dG_t is diagonal.

$$\|\mathrm{d}G_t v\|^2_{G(p)} = v_\rho^2 \cdot \frac{\|\mathrm{d}G \vec{v_\rho}\|^2}{\|\vec{v_\rho}\|^2} + (1-v_\rho^2) \cdot \frac{\|\mathrm{d}G\vec{v}_\theta\|^2}{\|\vec{v}_\theta\|^2}$$

$$\|\mathrm{d}G_t\| = \max\{\|\mathrm{d}G_{\theta,t}\|, \|\mathrm{d}G_{\rho,t}\|\}, \text{ where } \|\mathrm{d}G_{\theta,t}\| = \sup(1-t),$$

$$\|\mathrm{d}G_{\rho,t}\| = \sup_{v_\theta\neq 0} \frac{\|\mathrm{d}G\vec{v}_\theta\|_{G(p)}}{\|\vec{v}_\theta\|_p} = \sup_{v_\theta\neq 0} \frac{\|\vec{v}_\theta\|_{G(p)}}{\|\vec{v}_\theta\|_p} = \sup_{\substack{v_\theta\neq 0,\\ \rho\neq 0}} \frac{\sqrt{\sin^2((1-t)\rho)}}{\sqrt{\sin^2\rho}} = \sup_{\rho\neq 0} \frac{\sin((1-t)\rho)}{\sin\rho}$$

Direction ρ is the boring one, as $\sup(1-t) = 1$ is achieved at $t=0$, where the sine quotient also equals 1 for $t=0$. Thus, we can focus solely on the direction θ of the lateral spheres[5]. INSERT SPHERE CONTRACTION PICTURE HERE For large $r > \pi/2$ the

[5]We still have to address the case $\rho = 0$. This is the pole point where our metric repersentation is not

Lipschitz constant $L < 1$, as increasing t only reduces the fraction. Geometrically, for contraction then only shrinks the lateral spheres together with their tangent vectors. For $r < \pi/2$ we achieve the largest possible stretch of the tangent vectors when the latteral spheres S^{n-1} grow the most via G, that is, when p sits at the boundary of $S^n \setminus B_r$ and G(p) sits at the equator sphere. There

$$\rho = \pi - r;\ (1-t)\rho = \pi/2$$

$$\|\mathrm{d}G_t\| = \|\mathrm{d}G_{\rho,t}\| = \frac{\sin(\pi/2)}{\sin(\pi - r)} = \frac{1}{\sin r} \sim r^{-1}.$$

□

CONTINUE HERE!!!

The bound we proved is not particularly good. In the standard proof that the image of a lower dimensional sphere is not surjective one approximates the sphere by piece-wise linear maps. We can explore this idea further by introducing simplicial approximation.

1.3 Simplicial approximation

Simplicial complexes are often neglected in presentation, so it might be beneficial to agree on some basic definitions.

Definition (simplicial complex)**.** A simplicial complex K is a collection of simplices satisfying the following conditions: 1. Every face of a simplex in K also lies in K 2. A non-empty intersection of two simlices in K $\sigma_1 \cap \sigma_2 \neq \varnothing$ is a face of both σ_1 and σ_2.

Additionally we equip a simplicial complex K with coherent topology of its simplices: a subset U is open in K iff $U \cap \sigma$ is open for all $\sigma \in K$.

Observation. 1. A simplex σ is closed in K. 2. The interior of a single vertex is the vertex itself. The boundary of a vertex is empty. 3. A simplicial complex is a union of interiors of its simplices.

Thus by default a simplicial complex K has a topology but no metric. Let us denote a **geometric realization** of K by $|K|$. We can define a metric on K by choosing a geometric realization $|K|$ and on each simplex taking the subspace Euclidean metric it inherits from $|K|$ (essentially, the same procedure we used to define the topology). In other words, we choose a metric that when restricted to each simplex agrees with Clearly, such a metric agrees with the topology of K.

---

well defined. G fixes the pole and dG_t on the pole tangent space is identity. Hence at that point $\|dG_t|\rho = 0\| = 1$.

Definition (star)**.** Let K be a simplicial complex. The **closed star** of a simplex σ in K $St\sigma$ is the union of all simplices containing σ. The **open star** of a simplex $\sigma \in K$ $st\sigma$ is the union of interiors of all simplices containing σ.

Observation. Closed stars are closed. Open stars are open. $St\sigma$ is the closure of $st\sigma$.

Of a special interest to us are stars of vertices. A star of a vertex v is the combinatorial analog of a ball around v. A closed star of a vertex caputures all adjacent and incident edges, while open stars of vertices provide an open cover that is just shy of containing the adjacent vertices - this cover is expecially useful for simplicial approximation. Picture open/ closed STAR OF A VERTEX

A

Definition (triangulation)**.** Let K be a simplicial complex. X a topological space. A homeomorphism $\phi : K \to X$ is called a **triangulation** of X.

ADD TIKZ-CD!! DEFINE SIMPLICIAL MAP SOMEWHERE!

We start with a classical result on simplicial approximation. Theorem. If K is a finite simplicial complex and J is an arbitrary simplicial complex, then any map $f : K \to J$ is homotopic to a map that is simplicial with respect to some iterated barycentric subdivision of K.

We don't actually intend to prove this result, but rather highlight some of the ideas that we want to translate to the Lipschitz setting: 1. We equip K with a metric as described above. In particular with this metric open stars are open and closed stars are closed in K. Distances within the simplex are also well behaved - all points are at most as far apart as the largest side length. 2. Observe that open stars form a covering of P. Taking pre-image of that cover yields an open cover of K. Since K is a finite simplicial complex it is in particular compact. We take the finite subcover and find its Lebesgue number (it exists by the Lebesgue number lemma). This gives us a way to determine the desired size of the simplices of K. 3. Now let us subdivide K until the simplices are small enough that closed star of a vertex v is contained in some cover element. This means we managed to contain the closed nighborhood of a vertex - adjacent edges and their vertices - fully in a reasonably small region of the the simplex. Edges can't wrap around our simplex multiple times and vertices cannot be too far apart. Meaning we have a chance of building a simplicial map.
This is as much of the proof as we need for now - see Hatcher for more details and the contruction.

To translate this idea to a Lipschitz map setting we want to replace Lebesgue number using our Lipschitz constant. We have to pay attention to several thnigs: both differentiable manifolds are equipped with length-metrics. Those metrics should agree with the simplicial structure - we would like edges of a simplex to be geodesic segments (and shortest paths), simplices should not budge out too much - e.g. there shouldn't be distances longer than the longest edge - to avoid weird counterexamples. We repeat step one. take the smallest diameter of the star and denote it by diam(J). diam(J) could be described purely in terms of the smallest side-length involved in the finite cover and a constant depending only on the size of the simplex.

We now want to equip the spheres with a simplicial structure and then proceed

Definition (star of a simplex). See Hatcher for definitions of basic concepts and proof. Some important notes: boundary of a single point. What an open star is intuitively, a picture? There are some nice pictures on wikipedia that I could borrow.

We require for the closed star to be contained in the open star so that we control for where vertices are approximated to.

Here is a quantitative

Theorem (Simplicial approximation). Let J be an arbitrary simplicial complex equipped with a metric such that preimage of each simplex under the exponential map is an equilateral simplex in the tangent space. L

2 n is odd theorem

Theorem ([CDMW]). Suppose that n is odd and $f : S^m \to S^n$ is a null-homotopic map with Lipschitz constant L. Then there is a null-homotopy $H : S^m \times [0,1] \to S^n$ with Lipschitz constant at most C(m,n)L.

The approach they take in this proof is as follows: Here are some preparatory steps for the proof:

1. We first take a simplicial approximation of the map f (ref Lemma 1.5). The simplicial approximation is still null-homotopic: cylinder composed with the cone picture here. Therefore in the proof we can directly start with a simplicial map of the sphere.
2. A null-homotopy from an arbitrary space X can be described as a map from the Cone of X to the target. Recall that the cone of X is simply the cyllinder of X with one end collapsed to a point: $X \times [0,1]/X \times \{0\}$. Cone of S^m is homeomorphic to the unit ball B^{m+1}. To see this, think of radius of the ball as the variable for the interval. (picture for S^1).
3. We endow B^{m+1} with a simplicial structure Tri_L using the procedure from blibla.

We now have the following setup: we are given a map $h : B^{m+1} \to S^n$ (the default null-homotopy for f) with $h_{|\partial B^{m+1}} \to S^n$ simplicial with Lipschitz constant L (our map f).

Our strategy for the proof is to "straighten out" h skeleton by skeleton: we iteratively homotope h relative to the boundary (so that the restriction to f stays intact) to maps $h_0, h_1, h_2, ..., h_m = H$, where each map is $\lesssim L-$Lipschitz when restricted to the j-th skeleton of (B^{m+1}, Tri_L) until we reach the m-th skeleton. The resulting map h_m is the desired homotopy H.

There are several more key ingredients to the proof: 1. At each skeleton we leverage knowledge about the corresponding homotopy group. We will quickly revise their definition and talk about homotopy groups of spheres. 2. CW-complexes, and, in particular, simplicial complexes satisfy are fibrations, i.e. they satisfy the homotopy extention property. We will briefly state what it means and prove this fact.

Before we proceed, let us adress some concerns one might have with this strategy:

Definition. Homotopy, relative homotopy. proof that lower homotopy groups are trivial.
Definition. Fibration?

Proof. To recap, here is our setting: we are given a map $h : B^{m+1} \to S^n$ (a null-homotopy for f), which is simplicial, when restricted to the boundary, i.e. $h|_{\partial B^{m+1}} = f : (S^m, Tri_L) \to (S^n, Tri_{S^n})$ simplicial. We denote the skeleta of the simplicial complex B^{m+1} by $X^0 \subset X^1 \subset X^2 \cdots \subset X^m \subset X^{m+1} = B^{m+1}$.
We first aim to find a map h_0 that is simplicial on the 0-th skeleton. For that we need to figure out where to map the vertices of Tri_L. If we could use some procedure similar to that of Theorem REFERENCE simplicial] we could ensure that the neighboring vertices are mapped to the same simplex. But we triangulate B^{m+1} without any knowledge of h, so h could map vertices of a simplex anywhere in the target, no matter how fine of a triangulation we prescribe to B^{m+1} at the beginning. Since we later want to build a simplicial map on the 1-skeleton we need edges to map to edges, which forces us to choose the boundary of a standard simplex as triangulation of the target.[1]
h(X^0) is a disjoint set of points in S^n. For each of the points we choose a vertex of Tri_{S^n} that is closest according to the surface metric. S^n is path-connected, which gives a homotopy relative to the vertex. To fit the more general framework, we could say that $\pi_0(S^n) = 0$. We could even pick a straight-line homotopy by taking geodesics to keep this procedure as deterministic as possible. Having fixed the homotopies we use the homotopy extension property to obtain the homotopy g^0:

$$\begin{tikzcd} X^0 \times [0,1] \cup B^{m+1} \times \{0\} \arrow[r, "label"] \arrow[d, hook] & S^n \\ B^{m+1} \times [0,1] \arrow[ur, dashed, "g"'] & \end{tikzcd}$$

[1]There are other reasons which make this choice of triangulation of the target especially convinient. Namely, later in the proof we would like the target to have equilateral simplices of equal area.

3 Appendix

3.1 Covering a sphere by spherical caps.

Here we provide a direct and more thourough argument to cover the sphere S^m by spherical caps. In this argument we do not drop constants. We start with a quick introduction to the topic of covering and packing:

Definition (Covering, packing)**.** Let (X, d) be a metric space, $K \subseteq X$.

A collection C of points in X is called an $\boldsymbol{\rho}$**-covering** of K if K is contained in the union of ρ-balls around points in C, i.e. $K \subseteq \cup_{p \in C} B_\rho(p)$. In other words, for $\forall x \in K$ there is a p in C such that $d(p, x) \leq \rho$. Note that we do not require the centers of ρ-balls to lie in K. Such a covering is also called an **external** $\boldsymbol{\rho}$**-covering**. The minimum ρ-covering cardinality is called the **(external) covering number** of K denoted $N(K, d, \rho)$ or simply $N(\rho)$.

A collection P of points in K is called an $\boldsymbol{\rho}$**-packing** if for $\forall p, q \in P$ $d(p, q) > \rho$. The maximum packing cardinality is called the **packing number** of K and is denoted by $M(K, d, \rho)$ or simply $M(\rho)$.

Observation. Let P be a ρ-packing. Then the balls $B_{1/2\rho}(p)$ are pairwise disjoint (triangle inequality).
If P is maximal, then P is also an ρ-covering (by contraposition). In particular, this implies $N(\rho) \leq M(\rho)$

Claim. $M(2\rho) \leq N(\rho) \leq M(\rho)$

Proof. The second inequality follows from the observation above. To prove the first inequality, assume $M(2\rho) > N(\rho)$. Then by the pigeon-hole principle there are two points x, y of the packaging contained in the same ρ-ball of the cover. By triangle inequality this yields a contradiction. $\square$

We are now going to provide an upper bound for the covering number of a sphere. Geomtrically, we will be covering a sphere by spherical caps of equal size. We are interested in exloring the relationship between the size of the caps and the covering number.

Definition (spherical cap)**.** A closed **spherical cap** is the smaller portion of a unit sphere S^m cut off by a plane (including the boundary). Formally, the spherical cap with angle $\rho \in (0, \pi/2]$ and center $x \in S^m$ is given by

$$cap(x, \rho) = \{y \in S^m : \langle x, y\rangle \geq cos\rho\}.$$

We will call a spherical cap with a polar angle ρ a $\boldsymbol{\rho}$**-cap**. Since we are dealing with a unit sphere, the polar angle in radians is precisely the length of any geodesic from the center (the tip) of the cap to its edge.

Figure 1.1

Lemma 3.1. *The covering number of a sphere $N(S^m, d, \rho) \lesssim \rho^{-m}$, where d is the length-metric. That is, for any $\rho > 0$, the sphere S^m can be covered by at most (up to a constant) $1/\rho^m$ ρ-caps.*

Remark: It is sufficient for us to show the upper bound up to a constant $c(m)$. The reason for that is that in later arguments we will be able to choose the radius of the cover small enough that any constant $c(m, n)$ can be "neutralized" for our purposes, so long as the quantities we omit do not vary with ρ.

Proof. Let us first consider a maximal packing of our sphere with spherical caps. For any such packing the total volume of spherical caps cannot exceed the volume[1] of the sphere. As the caps in a packing are disjoint,

$$M(\rho) \leq \frac{\omega^m(S^m)}{\omega^m(\rho\text{-cap})}.$$

Now, S^m can be covered by exactly two $\frac{\pi}{2}$-caps, so $\omega^m(S^m) = 2\omega^m(\frac{\pi}{2}\text{-cap})$. Rewriting the inequality above we get:

$$M(\rho) \leq \frac{2\omega^m(\frac{\pi}{2}\text{-cap})}{\omega^m(\rho\text{-cap})}. \tag{3.1}$$

We would like to replace the ρ-caps in the inequality by ρ-disks, as they scale easier with ρ, and that would allow us to reduce the fraction. Projecting the cap down onto the disk at its base will reduce the volume[ˆcap-size], i.e. $\omega^m(\rho\text{-cap}) \geq \omega^m(\sin\rho D^m)$. Dividing both sides by the m-volume of a ρ-disk and simplifying we obtain the following inequality:

$$\frac{1}{(\frac{\pi}{2})^m} \leq \frac{\sin^m\rho}{\rho^m} = \frac{\omega^m(\sin\rho D^m)}{\omega^m(\rho D^m)} \leq \frac{\omega^m(\rho\text{-cap})}{\omega^m(\rho D^m)},$$

where $\rho \in (0, \frac{\pi}{2}]$. Multiplying by $\left(\frac{\pi}{2}\right)^m$ we get:

$$1 \leq \left(\frac{\pi}{2}\right)^m \cdot \frac{\omega^m(\rho\text{-cap})}{\omega^m(\rho D^m)} \tag{3.2}$$

[1]We are referring to m-volumes. Think of surface areas in case $m = 2$.

Multiplying inequal (3.1) by a term (3.2) greater than 1 on the right yields:

$$N(\rho) \leq M(\rho) \leq \frac{2\omega^m(\frac{\pi}{2}\text{-cap})}{\omega^m(\rho\text{-cap})} \leq \left(\frac{\pi}{2}\right)^m \cdot \frac{2\omega^m(\frac{\pi}{2}D^m)}{\omega^m(\rho D^m)} = \left(\frac{\pi}{2}\right)^{2m} \cdot \frac{2}{\rho^m} \sim \frac{1}{\rho^m}.$$

□

UP UNTIL HERE SHOULD BE FINE

For proof of next Lemma 1.2 (explicit constants):

$$|C|\omega^n(2r\text{-cap}) \leq \left(\frac{\pi}{2}\right)^{2m-n} \cdot \frac{2}{\rho^m} \cdot (2r)^n \cdot \omega^n(\tfrac{\pi}{2}\text{-cap}) \leq \omega^n(S^n) = 2\omega^n(\tfrac{\pi}{2}\text{-cap}),$$

simplified, this becomes

$$\left(\frac{\pi}{2}\right)^{2m-n} \cdot \frac{(2r)^n}{\rho^m} \leq 1.$$

Using $r = L\rho$ and $m < n$ we choose $\rho > 0$ small enough to obey

$$\rho \leq \left(\frac{L}{\pi}\right)^{-\frac{n}{n-m}} \cdot \left(\frac{\pi}{2}\right)^{-\frac{2m}{n-m}}.$$

it then follows for r

$$r = L\rho \leq \left(\frac{\pi^2 L}{4}\right)^{-\frac{m}{n-m}} \cdot \pi^{\frac{n}{n-m}}.$$

3.1.1 Detour: manifolds with boundaries

We want to show that we can contract the target sphere S^n in a Lipschitz way. For that we need to construct a differentiable map between the cylinder of S^n and S^n. Reminder: the (topological) cylinder is the cartesian product with the interval. So we want a map between manifolds, both equipped with a metric. For the sake of consistency, we would prefer to equip both with the length metric. Naturally, we could take the product Riemannian metric. But the inerval is not a manifold, nor is the (topological) cylinder! For it is strictly speaking not Euclidean at the points on the boundary - in the interval dimension we can only move in one direction from the boundary $M \times \{0\}$. At those boundary points we do, however, have homeomorphism to the Euclidean half-space $\mathbb{R}^{m+1}$. We would like to relax the usual definition of a manifold to include manifolds with boundary:

Definition (manifold with boundary)**.** definition here

Thus, the old manifolds are just manifolds with an empty boundary. Notably, the relaxed definition encompasses basic topological objects, such as the (closed) unit disk, the Moebius strip and topological cylinders as manifolds, the latter allowing us to consider differentiable homotopies.

All the usual definitions of dimension, tangent spaces etc apply to manifolds with boundaries. A manifold with a boundary also always admits a Riemannian metric:

Definition (Double)**.** A double a manifold with a boundary is bla glued along their boundaries. A double is a manifold without a boundary.

Observation. A double of a manifold M admits a Reimannian metric. Selecting a metric and restricting to M yields a Riemmanian metric on M. cite stackexchange because credit should be given where credit is due.

3.1.2 Derivative of a differentiable map w.r.t. the metric

In this section we want to learn how to find Lipschitz constants for a given differentiable map between manifolds. We want learn how to compute the differential directly using the corresponding metrics, with respect to a given parametrization As usual we will equip our spaces with the length metric

Equipping our spaces with a specific metric allows for explicit computations of c'(t) for a given curve c(t), explicit computations of lengths of tangent vectors etc. In particular it allows us to compute partial derivatives w.r.t. to our chosen parametrization as local dilation and to give an upper bound for dilation of a given map between manifolds.

THIS IS STILL HORRIBLE - REWRITE!