

At any time there is a location on earth where the wind is not blowing.



Mathematical interpretation:

For any continuous vector field on  $S^2$

$$\tau: S^2 \rightarrow \mathbb{R}^3$$

$\langle \tau(x), x \rangle = 0$  for  $x \in S^2 \subset \mathbb{R}^3$ ,  
there is a zero.

Lemma: Let  $n \geq 2$  even. For each map  $S^n \xrightarrow{f} S^n$  there is a  $x \in S^n$  such that

$$f(x) \in \{x, -x\}.$$

Proof: Assume that  $f(x) \notin \{x, -x\}$  for all  $x \in S^n$ .

Define

$$\bar{F}(x, t) = \frac{(1-t)x + t \cdot f(x)}{\|(1-t)x + t \cdot f(x)\|} \quad \text{is homotopy} \\ \text{id}_{S^n} \simeq f$$



$$G(x, t) = \frac{(1-t)f(x) + t(-x)}{\|(1-t)f(x) + t(-x)\|} \quad \text{is a homotopy} \\ -\text{id}_{S^n} \simeq f.$$

$-\text{id}_{S^n}$  is the composition of  $n+1$  reflections.

$$\text{Hence } H_n(-\text{id}_{S^n}): H_n(S^n) \rightarrow H_n(S^n)$$

is multiplication with  $(-1)^{n+1} = -1$ .

$$\text{Hence } -\text{id}_{S^n} \neq \text{id}_{S^n} \quad \Rightarrow \quad \square$$

Theorem Let  $n \geq 2$  be even. Every continuous vector field on  $S^n$  vanishes at some point.

[every map  $\tau: S^n \rightarrow \mathbb{R}^{n+1}$  with  $\langle \tau(x), x \rangle = 0 \quad \forall x \in S^n$  vanishes at some point]

Proof. Assume  $\tau(x) \neq 0$  for all  $x \in S^n$ .

$$\text{Define } f(x) = \frac{\tau(x)}{\|\tau(x)\|} \in S^n.$$

By the lemma there is  $x_0 \in S^n$  such that

$$f(x_0) \in \{x_0, -x_0\}.$$

This contradicts

$$0 = \langle \tau(x_0), x_0 \rangle = \|\tau(x_0)\| \cdot \langle f(x_0), x_0 \rangle.$$

□

I hereby declare that this document has been composed by myself and describes my own work, unless otherwise acknowledged in the text.

Ich versichere wahrheitsgemäß, die Arbeit selbstständig verfasst, alle benutzten Hilfsmittel vollständig und genau angegeben und alles kenntlich gemacht zu haben, was aus Arbeiten anderer unverändert oder mit Abänderungen entnommen wurde sowie die Satzung des KIT zur Sicherung guter wissenschaftlicher Praxis in der jeweils gültigen Fassung beachtet zu haben.

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Karlsruhe, 10.05.2022

# **Abstract**

Some English abstract.

# **Zusammenfassung**

Eine deutsche Zusammenfassung.

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# 1 Background of the problem

some introduction.

## 1.1 Some background assumptions

This paper deals with Lipschitz constants of maps between spheres. Most of the time we will only estimate the Lipschitz constants up to a constant  $C(m, n)$ , that only depends on the dimensions of the spheres. We denote equality/inequality up to a constant by  $\sim, \lesssim, \gtrsim$  respectively. Throughout this paper let the unit spheres  $S^m, S^n$  be equipped with the length metric induced by the standard Riemannian metric (unless stated otherwise). That is, the distance between any two points is determined by the (Euclidean) length of the geodesics between them<sup>1</sup>. Note that while the topology is the same, the metric is different from the “default” metric inherited from the ambient Euclidean space. Occasionally we will consider objects that are homeomorphic to spheres when it is convenient (e.g. surface of a cube or of a simplex), but the conversion only changes things up to some constant. On those objects we will still be using the length metric.

Statement of the problem.

## 1.2 Contracting the image of a lower dimensional sphere

### 1.2.1 Introduction to computations up to a constant

For the first four lemmas we consider maps from  $S^m$  to  $S^n$  when  $m$  is less than  $n$ . We know that the image of  $S^m$  in  $S^n$  is not-surjective (citation). It is then contractible. In this section we want to show that the image of a lower dimensional sphere can be contracted in a Lipschitz way, and to provide a fairly tight Lipschitz constant. We start by showing that a Lipschitz map must in fact miss a whole open ball in the target:

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<sup>1</sup>To be precise, the length of the geodesics is determined by the standard Riemannian metric, where the metric is pulled back along the embedding of the spheres into their ambient Euclidean spaces ( $\mathbb{R}^m, \mathbb{R}^n$ , respectively). The lengths of geodesics are then precisely the respective Euclidean lengths of their embeddings. The reason to specify a metric so early on is that when we talk about Lipschitz continuity we are implicitly dealing with the metrics, not just with underlying topologies. However, since all of our results are up to a constant, suitable constant manipulation would show them to hold for the standard Euclidean metric as well. Nevertheless, we prefer to settle on a specific metric to avoid confusion or ambiguity.

**Lemma 1.1.** *Let  $f : S^m \rightarrow S^n$  be a Lipschitz continuous maps with a Lipschitz constant  $L$ . Then the image of  $f$  misses a ball of radius  $r$  for  $r \lesssim L^{\frac{-m}{n-m}}$*

A ball with respect to the length-metric on the sphere is a spherical cap. The radius of the ball is the length of any geodesic from the center (the tip) of the cap to its edge. It is equal to the polar angle of the cap in radians.

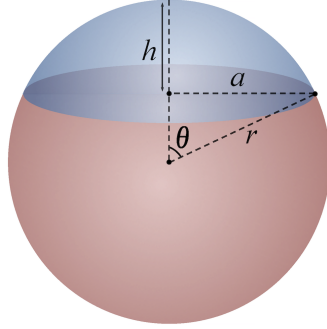


Figure 1.1: Spherical cap. Here  $r = 1$ ,  $\theta$  is the polar angle,  $a = \sin \theta$ ,  $h = 1 - \cos \theta$

For the proof of the lemma we will need to cover the sphere with spherical caps. To estimate the number of caps needed to cover the sphere we use a volume argument. A sloppy version of the argument would go as follows: We equip the sphere with a volume form that scales well with the polar angle and is equal to 1 on the whole sphere. Then the volume of the sphere is 1, the volume of each spherical cap is  $\rho^m$ . The cover should have area similar to that of the sphere (up to a constant). We then need  $\sim 1/\rho^m$  spherical caps to cover the sphere.

You may object: why should the cover have volume similar to that of the sphere if there is an overlap? Why should the overlap scale well with the radius? Is it okay for us to change the metric if the result is stated w.r.t. to a different one (we could of course use the standard volume but then you would be right to point out that spherical cap volume might not scale well with the cover radius). However, this argument is only meant to provide us with an intuition, and we aim to show that this types of arguments can be formalized fairly easily:

**Claim.** *For any  $\rho > 0$ , the sphere  $S^m$  can be covered by  $\sim 1/\rho^m$  balls of radius  $\rho$ .*

*Proof.* We want to estimate the number of  $1/\rho$  balls needed to cover the sphere. Covering the sphere is up to a constant the same as covering the hemisphere. In fact, the cardinality of the cover for  $S^m \sim \text{Hemisphere } S^m_+ \sim D^m$  by projecting down to the  $D^m$  at the equator<sup>2</sup> (which is just the unit ball  $B^m$ )  $\sim$  covering  $\sqrt{2}B^m$  (scaling up)  $\sim$  covering the m-box of side length two (it can be squeezed between the two balls, i.e. it contains

<sup>2</sup>Projecting down onto the equator obviously only changes things up to a constant (depending only on  $m$ ): we project down the cover centers and keep the radius as is. Going the other direction is a little

the unit ball and it is contained in the  $\sqrt{2}B^m$ )  $\sim$  covering the  $m$ -box of side length 1 (the unit  $m$ -box). It is easy to see why the volume argument should work now: the unit box can be clearly be covered by  $\lceil 1/\rho \rceil^m$  boxes of side length  $\rho$ . Each  $\rho$ -box is contained in a ball of radius  $\rho$  and we are done.  $\square$

Arguing up to a constant allows us great flexibility in choosing objects we are more comfortable working with. The constants we omitted can easily be traced back through the equivalence steps we took. However, if you are not yet comfortable working up to a constant there is a direct argument on the sphere without any equivalences or constant dropping that I provided in the appendix.

**Lemma 1.2** (Image of misses a ball). *Let  $f: S^m \rightarrow S^n$  be a Lipschitz-continuous map with a Lipschitz constant  $L$ . Then the image of  $f$  misses a ball of radius  $r$  for  $r \lesssim L^{-\frac{m}{n-m}}$*

*Proof.* For any  $\rho > 0$ ,  $S^m$  can be covered by  $\sim \rho^{-m}$  balls of radius  $\rho$ . The image of each such ball is contained in a ball of radius  $L\rho$ . Therefore, the image of  $f$  can be covered by  $\lesssim \rho^{-m}$  balls of radius  $L\rho$ . We set  $r := L\rho$ . We now want to choose  $\rho$  small enough so that the cover misses a ball of radius  $r$ .

Expanding the radius of the cover to  $2r$  yields a cover of the  $r$ -neighborhood of the image. We denote this  $2r$ -cover by  $C$ . If this larger cover does not cover the full sphere  $S^n$ , the image of  $f$  must miss a ball of radius  $r$ . The total volume of the cover  $C$  is at most the cardinality of  $C$  times the volume of a ball of radius  $2r$ ,  $|C|\omega^n(B_2^n r)$ .  $\square$

We set  $\rho$  so that this number is smaller than the volume of the sphere. So we get for  $n > m$

$$\begin{aligned} |C|\omega^n(2r\text{-cap}) &\lesssim \rho^{-m}r^n = L^n \rho^{n-m} \lesssim 1, \\ \rho &\lesssim L^{-\frac{n}{n-m}}, \\ r = L\rho &\lesssim L^{-\frac{m}{n-m}}. \end{aligned}$$

In particular, even if  $f$  is a constant map we can choose  $\rho$  small enough so that  $r \leq \pi/2$   
 $\#\#$  Computing our first Lipschitz constant

### 1.2.2 Detour: geometric suspension

If we equip the sphere  $S^2$  with the usual pullback Riemannian metric, the resulting metric written in the matrix form is

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \theta \end{pmatrix}.$$

---

trickier: we start by scaling up the cover to  $\pi/2D^m$ . We then project from the larger disc onto the hemisphere by taking  $(\theta, r)$  to  $(\theta, \rho) = (\theta, r)$ , where,  $r$  is the radius and  $\rho$  is the polar angle. Same as before, distances can only reduce and thus keeping the radius yields a cover.

Even more often in the literature one encounters the corresponding symmetric quadratic form - its first fundamental form - which can be written as:

$$ds^2 = d\theta \otimes d\theta + \sin^2\theta \, d\phi \otimes d\phi,$$

or simply

$$ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2.$$

We will now show that the metric is verbatim the same for  $S^m$  for  $\forall m \geq 2$

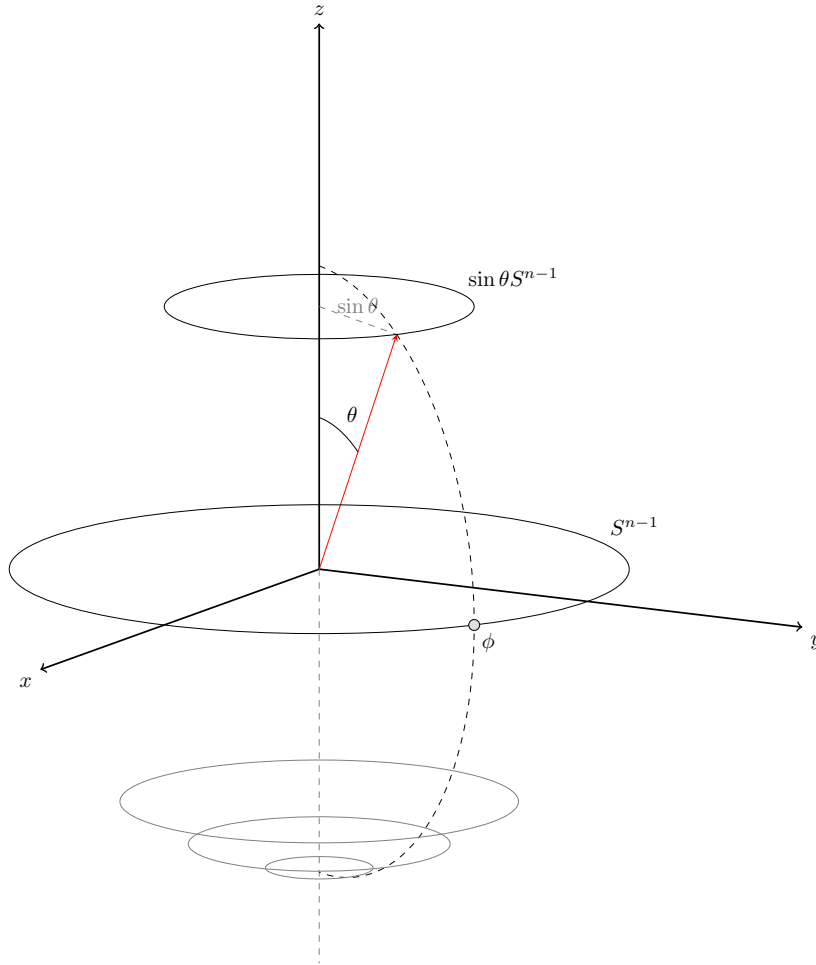


Figure 1.2: Polar coordinates

We can think of  $S^m$  as of several  $S^{m-1}$  stacked on top of each other (where  $S^{m-1}$  shrink to a single point at the poles). This is essentially the geometric version of suspension. Using the polar angle rather than height, we scale the equator  $S^{m-1}$  by  $\sin \theta$ .

Point-wise this gives us that any point  $p$  of  $S^m$  can be parametrized in terms of the



polar angle  $\theta$  and the corresponding vector  $\phi$  of the equator scaled down by  $\sin \theta$  - polar coordinates with respect to  $S^{m-1}$  Figure 1.2. Fixing some direction  $z$  in  $\mathbb{R}^{m+1}$  we can write out the parametrization:

$$\begin{aligned} \psi : [0, \pi] \times S^{m-1} &\longrightarrow S^m \\ (\theta, \phi) &\mapsto \sin \theta \cdot \phi + \cos \theta \cdot \vec{e}_z, \end{aligned} \tag{1.1}$$

where  $\vec{e}_z$  denote the standard basis vector in the  $z$  direction. Computing partial derivatives yields

$$\begin{aligned} \frac{\partial \psi}{\partial \theta} &= \cos \theta \cdot \phi - \sin \theta \cdot \vec{e}_z, \\ \frac{\partial \psi}{\partial \phi} &= \sin \theta \cdot \vec{e}_z. \end{aligned}$$

THIS DOES NOT EXPLAIN THE WHY. also missing all the stuff about taking the product metric (maybe more relevant below). Would be nice to provide some reference to some theory (like, what we did with Max, perhaps less in-depth than what we did with Leutzing).

Computing the spherical metric as a pullback of the  $\mathbb{R}^{m+1}$  metric:

$$g_{\theta\theta} = \langle \cos \theta \cdot \phi - \sin \theta \cdot \vec{e}_z, \cos \theta \cdot \phi - \sin \theta \cdot \vec{e}_z \rangle = \cos^2 \theta \cdot \langle \phi, \phi \rangle + \sin^2 \theta \cdot \langle \vec{e}_z, \vec{e}_z \rangle = 1,$$

$$g_{\phi\theta} = g_{\theta\phi} = 0,$$

$$g_{\phi\phi} = \sin^2 \theta$$

yielding the desired

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \theta \end{pmatrix}.$$

Remark. Note that in this we could replace  $S^{m-1}$  with an arbitrary manifold  $M$  of non-zero dimension <sup>3</sup>. Remarkably, since we are not using any knowledge of the underlying manifold  $M$  to compute the suspension metric with respect to  $M$ , it is only the function that we use to shrink the manifold towards suspension poles that matters for this relative metric. Analogously, we could take an analytic version of any topological construction to obtain its geometric version.

The complement of a point in  $S^n$  is contractible. If we remove a ball from  $S^n$ , the leftover part can be contracted in a Lipschitz way.

**Lemma 1.3.** *For each radius  $r$  there is a Lipschitz-contraction  $G : S^n \setminus B_r \times [0, 1] \rightarrow S^n \setminus B_r$ .  $G$  has Lipschitz constant  $\lesssim 1/r$  in the  $S^n$  direction and  $\lesssim 1$  in the  $[0, 1]$  direction.*

---

<sup>3</sup>For zero-dimensional manifolds  $d\phi^2$  vanishes, leaving  $ds^2 = d\theta^2$  as the metric.

*Proof.* There is a very obvious contraction map: □

$$\begin{aligned} G : S^n \setminus B_r \times [0, 1] &\rightarrow S^n \setminus B_r \\ G : (\rho, \theta, t) &\rightarrow ((1 - t)\rho, \theta) \end{aligned}$$

We want to compute its Lipschitz constants in both the sphere and the time direction. In order to do that we first compute the differential and apply it to the appropriate tangent vectors. The theoretical foundation for that is the mean value theorem for manifolds (REFERENCE)

$$dG = \begin{pmatrix} 1 - t & 0 & -\rho \\ 0 & 1 & 0 \end{pmatrix}$$

## 2 Appendix

### 2.1 Covering a sphere by spherical caps.

Here we provide a direct and more thorough argument to cover the sphere  $S^m$  by spherical caps. In this argument we do not drop constants. We start with a quick introduction to the topic of covering and packing:

**Definition** (Covering, packing). Let  $(X, d)$  be a metric space,  $K \subseteq X$ .

A collection  $C$  of points in  $X$  is called an  **$\rho$ -covering** of  $K$  if  $K$  is contained in the union of  $\rho$ -balls around points in  $C$ , i.e.  $K \subseteq \cup_{p \in C} B_\rho(p)$ . In other words, for  $\forall x \in K$  there is a  $p$  in  $C$  such that  $d(p, x) \leq \rho$ . Note that we do not require the centers of  $\rho$ -balls to lie in  $K$ . Such a covering is also called an **external  $\rho$ -covering**. The minimum  $\rho$ -covering cardinality is called the **(external) covering number** of  $K$  denoted  $N(K, d, \rho)$  or simply  $N(\rho)$ .

A collection  $P$  of points in  $K$  is called an  **$\rho$ -packing** if for  $\forall p, q \in P$   $d(p, q) > \rho$ . The maximum packing cardinality is called the **packing number** of  $K$  and is denoted by  $M(K, d, \rho)$  or simply  $M(\rho)$ .

**Observation.** Let  $P$  be a  $\rho$ -packing. Then the balls  $B_{1/2\rho}(p)$  are pairwise disjoint (triangle inequality).

If  $P$  is maximal, then  $P$  is also an  $\rho$ -covering (by contraposition). In particular, this implies  $N(\rho) \leq M(\rho)$

**Claim.**  $M(2\rho) \leq N(\rho) \leq M(\rho)$

*Proof.* The second inequality follows from the observation above. To prove the first inequality, assume  $M(2\rho) > N(\rho)$ . Then by the pigeon-hole principle there are two points  $x, y$  of the packaging contained in the same  $\rho$ -ball of the cover. By triangle inequality this yields a contradiction.  $\square$

We are now going to provide an upper bound for the covering number of a sphere. Geometrically, we will be covering a sphere by spherical caps of equal size. We are interested in exploring the relationship between the size of the caps and the covering number.

**Definition** (spherical cap). A closed **spherical cap** is the smaller portion of a unit sphere  $S^m$  cut off by a plane (including the boundary). Formally, the spherical cap with angle  $\rho \in (0, \pi/2]$  and center  $x \in S^m$  is given by

$$\text{cap}(x, \rho) = \{y \in S^m : \langle x, y \rangle \geq \cos \rho\}.$$

We will call a spherical cap with a polar angle  $\rho$  a  **$\rho$ -cap**. Since we are dealing with a unit sphere, the polar angle in radians is precisely the length of any geodesic from the center (the tip) of the cap to its edge.

Figure 1.1

**Lemma 2.1.** *The covering number of a sphere  $N(S^m, d, \rho) \lesssim \rho^{-m}$ , where  $d$  is the length-metric. That is, for any  $\rho > 0$ , the sphere  $S^m$  can be covered by at most (up to a constant)  $1/\rho^m$   $\rho$ -caps.*

Remark: It is sufficient for us to show the upper bound up to a constant  $c(m)$ . The reason for that is that in later arguments we will be able to choose the radius of the cover small enough that any constant  $c(m, n)$  can be “neutralized” for our purposes, so long as the quantities we omit do not vary with  $\rho$ .

*Proof.* Let us first consider a maximal packing of our sphere with spherical caps. For any such packing the total volume of spherical caps cannot exceed the volume<sup>1</sup> of the sphere. As the caps in a packing are disjoint,

$$M(\rho) \leq \frac{\omega^m(S^m)}{\omega^m(\rho\text{-cap})}.$$

Now,  $S^m$  can be covered by exactly two  $\frac{\pi}{2}$ -caps, so  $\omega^m(S^m) = 2\omega^m(\frac{\pi}{2}\text{-cap})$ . Rewriting the inequality above we get:

$$M(\rho) \leq \frac{2\omega^m(\frac{\pi}{2}\text{-cap})}{\omega^m(\rho\text{-cap})}. \quad (2.1)$$

We would like to replace the  $\rho$ -caps in the inequality by  $\rho$ -disks, as they scale easier with  $\rho$ , and that would allow us to reduce the fraction. Projecting the cap down onto the disk at its base will reduce the volume<sup>2</sup>, i.e.  $\omega^m(\rho\text{-cap}) \geq \omega^m(\sin \rho D^m)$ . Dividing both sides by the  $m$ -volume of a  $\rho$ -disk and simplifying we obtain the following inequality:

$$\frac{1}{(\frac{\pi}{2})^m} \leq \frac{\sin^m \rho}{\rho^m} = \frac{\omega^m(\sin \rho D^m)}{\omega^m(\rho D^m)} \leq \frac{\omega^m(\rho\text{-cap})}{\omega^m(\rho D^m)},$$

---

<sup>1</sup>We are referring to  $m$ -volumes. Think of surface areas in case  $m = 2$ .

<sup>2</sup>In fact it is easy to see that  $\omega^m(\sin \rho D^m) \leq \omega^m(\rho\text{-cap}) \leq \omega^m(\rho D^m)$ . We leave it for the appendix maybe.

where  $\rho \in (0, \frac{\pi}{2}]$ . Multiplying by  $(\frac{\pi}{2})^m$  we get:

$$1 \leq \left(\frac{\pi}{2}\right)^m \cdot \frac{\omega^m(\rho\text{-cap})}{\omega^m(\rho D^m)} \quad (2.2)$$

Multiplying inequal (2.1) by a term (2.2) greater than 1 on the right yields:

$$N(\rho) \leq M(\rho) \leq \frac{2\omega^m(\frac{\pi}{2}\text{-cap})}{\omega^m(\rho\text{-cap})} \leq \left(\frac{\pi}{2}\right)^m \cdot \frac{2\omega^m(\frac{\pi}{2}D^m)}{\omega^m(\rho D^m)} = \left(\frac{\pi}{2}\right)^{2m} \cdot \frac{2}{\rho^m} \sim \frac{1}{\rho^m}.$$

□

UP UNTIL HERE SHOULD BE FINE

For proof of next Lemma 1.2 (explicit constants):

$$|C|\omega^n(2r\text{-cap}) \leq \left(\frac{\pi}{2}\right)^{2m-n} \cdot \frac{2}{\rho^m} \cdot (2r)^n \cdot \omega^n(\frac{\pi}{2}\text{-cap}) \leq \omega^n(S^n) = 2\omega^n(\frac{\pi}{2}\text{-cap}),$$

simplified, this becomes

$$\left(\frac{\pi}{2}\right)^{2m-n} \cdot \frac{(2r)^n}{\rho^m} \leq 1.$$

Using  $r = L\rho$  and  $m < n$  we choose  $\rho > 0$  small enough to obey

$$\rho \leq \left(\frac{L}{\pi}\right)^{-\frac{n}{n-m}} \cdot \left(\frac{\pi}{2}\right)^{-\frac{2m}{n-m}}.$$

it then follows for r

$$r = L\rho \leq \left(\frac{\pi^2 L}{4}\right)^{-\frac{m}{n-m}} \cdot \pi^{\frac{n}{n-m}}.$$

### 2.1.1 Detour: manifolds with boundaries

We want to show that we can contract the target sphere  $S^n$  in a Lipschitz way. For that we need to construct a differentiable map between the cylinder of  $S^n$  and  $S^n$ . Reminder: the (topological) cylinder is the cartesian product with the interval. So we want a map between manifolds, both equipped with a metric. For the sake of consistency, we would prefer to equip both with the length metric. Naturally, we could take the product Riemannian metric. But the interval is not a manifold, nor is the (topological) cylinder! For it is strictly speaking not Euclidean at the points on the boundary - in the interval dimension we can only move in one direction from the boundary  $M \times \{0\}$ . At those boundary points we do, however, have homeomorphism to the Euclidean half-space  $\mathbb{R}^{m+1}$ . We would like to relax the usual definition of a manifold to include manifolds with boundary:

**Definition** (manifold with boundary). definition here

Thus, the old manifolds are just manifolds with an empty boundary. Notably, the relaxed definition encompasses basic topological objects, such as the (closed) unit disk, the Moebius strip and topological cylinders as manifolds, the latter allowing us to consider differentiable homotopies.

All the usual definitions of dimension, tangent spaces etc apply to manifolds with boundaries. A manifold with a boundary also always admits a Riemannian metric:

**Definition** (Double). A double a manifold with a boundary is bla glued along their boundaries. A double is a manifold without a boundary.

**Observation.** A double of a manifold  $M$  admits a Riemannian metric. Selecting a metric and restricting to  $M$  yields a Riemannian metric on  $M$ . cite stackexchange because credit should be given where credit is due.

### 2.1.2 Derivative of a differentiable map w.r.t. the metric

In this section we want to learn how to find Lipschitz constants for a given differentiable map between manifolds. We want learn how to compute the differential directly using the corresponding metrics, with respect to a given parametrization As usual we will equip our spaces with the length metric

Equipping our spaces with a specific metric allows for explicit computations of  $c'(t)$  for a given curve  $c(t)$ , explicit computations of lengths of tangent vectors etc. In particular it allows us to compute partial derivatives w.r.t. to our chosen parametrization as local dilation and to give an upper bound for dilation of a given map between manifolds.

THIS IS STILL HORRIBLE - REWRITE!