Karlsruhe Institute of Technology

On the topology of Lipschitz maps between spheres

Master Thesis of

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Karlsruhe, 18.08.2022

Abstract

We discuss the 2017 expository paper on quantitative topology by Larry Guth. The paper provides a more accessible take on two 2016 papers by Chambers, Dotterer, Ferry, Manin and Weinberger, which jointly gave an almost complete answer to Gromov's question: Given a Lipschitz continuous null-homotopic map between spheres $f : S^m \to S^n$, with Lipschitz constant L, what is the best Lipschitz constant over Lipschitz null-homotopies of f?

Zusammenfassung

Wir besprechen eine Übersichtsarbeit im Bereich der quantitativen Topologie von Larry Guth aus dem Jahr 2017. Seine Arbeit ermöglicht einen einfacheren Zugang zu zwei weiteren Arbeiten aus dem Jahr 2016, in denen es die Autoren Chambers, Dotterer, Ferry, Manin und Weinberger schaffen, eine fast vollständige Antwort auf die folgende Frage von Gromov zu geben: Gegeben sei eine nullhomotope Abbildung $f : S^m \to S^n$, Lipschitzstetig mit der Lipschitzkonstante L. Was ist die bestmögliche Lipschitzkonstante von allen Lipschitz-Nullhomotopien von f?

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1 Background of the problem

1.1 Introduction

This master thesis attempts to expand on and clarify some arguments from Larry Guth's expository article on quantitative topology[1]. Guth wrote an excellent introduction, and rather than attempt to add anything of substance to it I will merely summarize the main question and explain what I personally find appealing about it and about Guth's approach to the problem in particular.
In the mid-90s Gromov wrote an article on quantitative topology [2], where he posed a number of interesting problems. We will focus our attention on just one of them. Namely, he formulated and partially answered the following question:

Question. Given a Lipschitz continuous null-homotopic map between spheres $f : S^m \to S^n$, with Lipschitz constant L, what is the best Lipschitz constant over Lipschitz null-homotopies of f?

Gromov claimed that a Lipcshitz homotopy exists and bounded the Lipschitz constant by a tower of exponentials (see [1] for details). He conjectured however that for an optimal homotopy the Lipschitz constant should have polynomial dependency on L, and may even be linear. That is, the constant should be $C(m, n)L$ for some constant $C(m, n)$ that only depends on the dimensions of the spheres. Apparently, ha had also proved the conjecture for the case $m = n$. The result was is presented in [1] and was obtained by Guth from Gromov via personal communication.

In two 2016 papers Chambers, Dotterer, Ferry, Manin and Weinberger gave an almost complete answer to Gromov's question. Namely, for the case n is even they showed the Lipschitz constant of the homotopy to be up to $C(m, n)L^2$, and for the case n is odd they showed that the conjecture holds. I refer the reader to Guth's paper for an comlete and extensive introduction.

The question at hand is one of quantitative topology, which is interesting because it inherently deals with practical questions. In response to a statement of the form "we can null-homotope f to g" it asks "how fast can we do it?". Pretty much any practical application will introduce a metric and adding the quantiative aspect to topological questions allows us to take that metric into account. On the other hand, the quantitative aspect makes even the simpest of statements so much more difficult. But therein lies what makes it more interesting - to answer these types of questions we need to get creative. Quantitative topology relies on a variety of tools from algebraic topology, metric geometry and classical algebra, putting the area at the crossroads of several branches of mathematics. Specifically, Guth's expository paper captured my attention by bringing together a variety of topics such as simplicial complexes, higher homotopy groups, cohomology, a bit of differential geometry, degree theory, calculus on manifolds, Hopf fibrations etc. The style of the paper makes it feel more engaging and accessible, while the hands-on approach to proofs made the seemingly theoretical topics feel very practical.

1.2 Some underlying assumptions made explicit

The present work examines Lipschitz constants of maps between spheres. Most of the time we will only estimate the Lipschitz constants up to a constant $C(m, n)$, that only depends on the dimensions of the spheres. We denote equality/inequality up to a constant by $\sim, \lesssim, \gtrsim$ respectively. For this section let the unit spheres S^m, S^n be equipped with the length metric induced by the standard Riemannian metric (unless stated otherwise). That is, the distance between any two points is determined by the (Euclidean) length of the geodesics between them [1]. Note that while the topology is the same, the metric is different from the "default" metric inherited from the ambient Euclidean space. Later we will consider objects that are homeomorphic to spheres when it is convenient (e.g. surface of a cube or of a simplex), but the conversion only changes things up to some constant. On those objects we will still be using the length metric.

1.3 Contracting the image of a lower dimensional sphere

1.3.1 Introduction to computations up to a constant

In this section we consider Lipschitz maps from S^m to S^n when $m < n$. It is a classical result that such a map is null-homotopic - we first homotope the map f to a piece-wise linear or smooth map f_{approx}. Then we know that the image $f_{approx}(S^m)$ in S^n is not-surjective and therefore contractible. Our aim is to take these ideas and make them more quantitative. To start with, we isolate the part of contracting a non-surjective image in the target sphere. In this subsection we show that the image of a lower dimensional sphere can be contracted in a Lipschitz way and provide a tight Lipschitz constant for the contraction (that is, tight up to constant manipulations).

[1] To be precise, the length of the geodesics is determined by the standard Riemmanian metric, where the metric is pulled back along the embedding of the spheres into their ambient Euclidean spaces ($\mathbb{R}^m$, $\mathbb{R}^n$, respectively). The lengths of geodesics are then precisely the respective Euclidean lengths of their embeddings. The reason to specify a metric so early on is that when we talk about Lipschitz continuity we are implicitly dealing with the metrics, not just with underlying topologies. However, since all of our results are up to a constant, suitable constant manipulation would show them to hold for the standard Euclidean metric as well. Nevertheless, we prefer to settle on a specific metric to avoid confusion or ambiguity.

We start by showing that a Lipschitz map must in fact miss a whole open ball in the target:

Lemma 1.1 (Image of f misses a ball)**.** *Let $f : S^m \to S^n$ be a Lipschitz continuous maps with a Lipschitz constant L. Then the image of f misses a ball of radius r for $r \lesssim L^{\frac{-m}{n-m}}$*

Our strategy for proving this lemma will be to first cover the domain sphere by open balls of a set radius, then map that cover to the target sphere and show that the image of the cover cannot be surjective.

A ball with respect to the length-metric on the sphere is a spherical cap. The radius of the ball is the length of any geodesic from the center (the tip) of the cap to its edge. It is equal to the polar angle of the cap in radians.

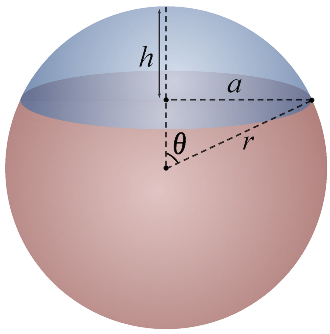

Figure 1.1: Spherical cap. Here $r = 1$, θ is the polar angle, $a = \sin\theta$, $h = 1 - \cos\theta$. Credit: Jhmadden, CC BY-SA 4.0 https://creativecommons.org/licenses/by-sa/4.0, via Wikimedia Commons

For the proof of the lemma we will need to cover the sphere with spherical caps. To estimate the number of caps needed to cover the sphere we use a volume argument. A sloppy version of the argument would go as follows: we equip the sphere with a volume form that scales well with the polar angle and is equal to 1 on the whole sphere. Then the volume of the sphere is 1, the volume of each spherical cap is ρ^m. The cover should have area similar to that of the sphere (up to a constant). We then need $\sim 1/\rho^m$ spherical caps to cover the sphere.

You may object: why should the cover have volume similar to that of the sphere if there is an overlap? Why should the overlap scale well with the radius? Is it okay for us to change the metric if the result is stated w.r.t. to a different one (we could of course use the standard volume but then you would be right to point out that spherical cap volume might not scale well with the cover radius). The argument above was only meant to provide us with an intuition. We aim to show that this kind of arguments can be formalized fairly easily:

Claim. *For any $\rho > 0$, the sphere S^m can be covered by $\sim 1/\rho^m$ balls of radius ρ.*

Proof. We want to estimate the number of $1/\rho$ balls needed to cover the sphere. Covering the sphere is up to a constant the same as covering the hemisphere. In fact, the the cardinality of the cover for $S^m \sim$ that of a hemisphere $S^m_+ \sim D^m$ (we can transfer the cover back and forth by projecting the hemisphere onto the equator disk or wrapping a larger disk around the hemisphere[2] $= B^m \sim$ covering $\sqrt{2}B^m$ (scaling up) $\sim$ covering the m-box of side length two (it can be squeezed between the two balls, i.e. it contains the unit ball and it is contained in the $\sqrt{2}B^m$) $\sim$ covering the m-box of side length 1 (the unit m-box). It is easy to see why the volume argument should work now: the unit box can clearly be covered by $\lceil 1/\rho \rceil^m$ boxes of side length ρ. Each ρ-box is contained in a ball of radius ρ and we are done. $\square$

Arguing up to a constant allows us great flexibility in choosing objects we are more comfortable working with. The constants we omitted can easily be traced back through the equivalence steps we took. However, if an argument up to a constant seems sketchy, there is a direct argument on the sphere without any equivalences or dropping constants that I provided in the appendix.

Now, let us restate the lemma before we prove it.

Lemma 1.1. *Let $f\colon S^m \to S^n$ be a Lipschitz-continuous map with a Lipschitz constant L. Then the image of f misses a ball of radius r for $r \lesssim L^{-\frac{m}{n-m}}$*

Proof. For any $\rho > 0$, S^m can be covered by $\sim \rho^{-m}$ balls of radius ρ. The image of each such ball is contained in a ball of radius $L\rho$. Therefore, the image of f can be covered by $\lesssim \rho^{-m}$ balls of radius $L\rho$. We set $r := L\rho$. Expanding the radius of the cover to $2r$ yields a cover of the r-neighborhood of the image. We denote this $2r$-cover by C. If this larger cover does not cover the full sphere S^n, the image of f must miss a ball of radius r. We now want to choose ρ small enough for that to be the case. We determine the desired ρ again by a volume argument.

The total volume of the cover C is at most the cardinality of C times the volume of a ball of radius $2r$ (which is a spherical cap of polar angle $2r$). That volume does not scale well with r, so we replace the cap volume by the larger nicely scaling volume of a disk $2r \cdot D^n$. We do this by essentially the same argument as we used to transfer the disk cover from the disk to the hemisphere (see footnote on previous page). The total cover volume is then at most $|C|\omega^n(\pi/2 \cdot B^n_2 r)$, where ω^n denotes the Euclidean n-volume form.

[2]Projecting the hemisphere S^m_+ down onto the unit disk D^m at the equator obviously only changes things up to a constant (depending only on m): we project down the cover centers and keep the radius as is. Showing that the cover can be transferred in the other direction is a little trickier: we start by scaling the cover up by a factor of $\pi/2$ to cover $\pi/2D^m$. We then wrap the larger disc around the hemisphere by taking (θ, r) to $(\theta, \rho) = (\theta, r)$, where $\theta \in S^{m-1}$, r is the radius and ρ is the polar angle. Distances can only reduce for the same reason that we can wrap a paper around a ball without tearing it and the paper will wrinkle: radial components of distances stay the same and angular components shrink with a factor of sine as the radius increases. Of course if you are not convinced you can scale your disk up by another factor of two. Same as before, keeping the projected cover centers and the old cover radius yield a cover.

We now set ρ so that this number is smaller than the volume of the sphere. So we get for $n > m$

$$\begin{gathered}\omega^n(C) < |C|\omega^n(\pi/2 \cdot B_2^n r) \lesssim \rho^{-m} r^n \overset{!}{\lesssim} \omega^n(S^n) \sim 1,\\ L^n \rho^{n-m} \lesssim 1,\\ \rho \lesssim L^{-\frac{n}{n-m}},\\ r = L\rho \lesssim L^{-\frac{m}{n-m}}.\end{gathered}$$

In particular, even if f is a constant map we can choose ρ small enough so that $r \leq \pi/2$. $\square$

1.3.2 Detour: geometric suspension

If we equip the sphere S^2 with the usual pullback Riemannian metric, the resulting metric written in the matrix form is

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}.$$

Even more often in the literature one encounters the corresponding symmetric quadratic form - its first fundamental form - which can be written as:

$$\mathrm{d}s^2 = \mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2\theta \, \mathrm{d}\phi \otimes \mathrm{d}\phi,$$

or simply

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2.$$

We will now show that the metric is verbatim the same for S^m for $\forall m \geq 2$

We can think of S^m as of several S^{m-1} stacked on top of each other (where S^{m-1} shrink to a single point at the poles). This is essentially the geometric version of suspension. Using the polar angle rather than height, we scale the equator S^{m-1} by $\sin\theta$.

Point-wise this gives us that any point p of S^m can be parametrized in terms of the polar angle θ and the corresponding vector ϕ of the equator scaled down by $\sin\theta$ - polar coordinates with respect to S^{m-1} Figure 1.2. Fixing some direction z in $\mathbb{R}^{m+1}$ we can write out the parametrization:

$$\begin{aligned}\psi :&[0,\pi] \times S^{m-1} \longrightarrow S^m \\ &(\theta, \phi) \mapsto \sin\theta \cdot \phi + \cos\theta \cdot \vec{e_z},\end{aligned}, \tag{1.1}$$

where $\vec{e_z}$ denote the standard basis vector in the z direction. Computing partial derivatives yields

$$\frac{\partial\psi}{\partial\theta} = \cos\theta \cdot \phi - \sin\theta \cdot \vec{e_z},$$

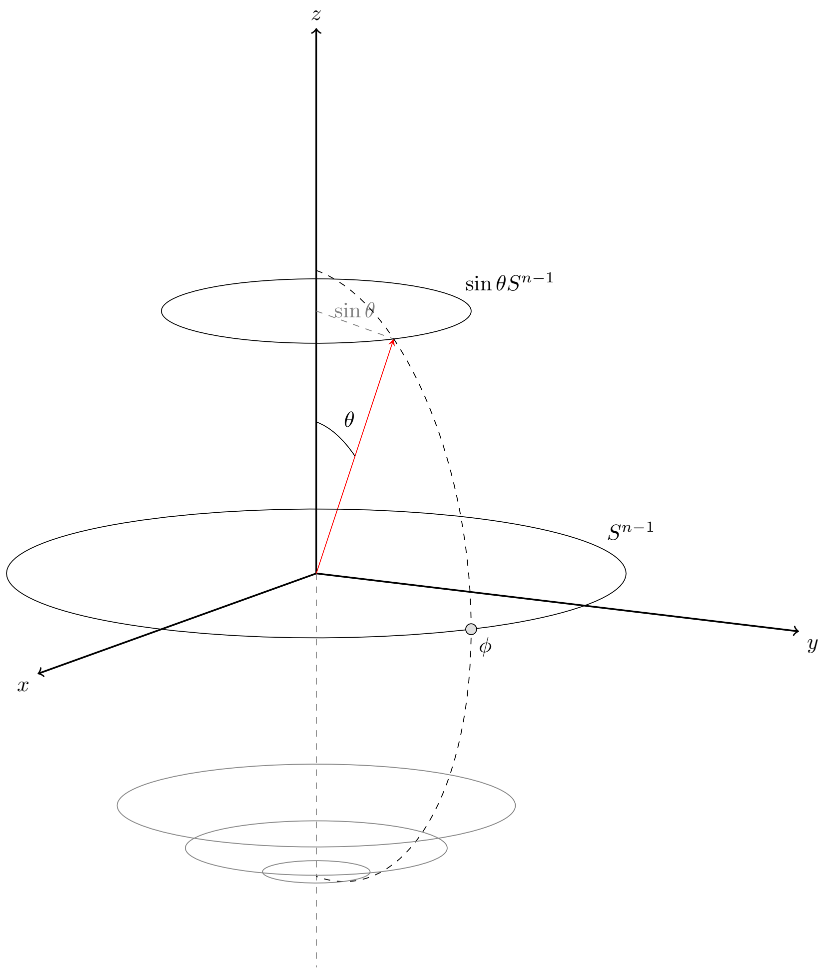

Figure 1.2: Polar coordinates

$$\frac{\partial\psi}{\partial\phi} = \sin\theta \cdot \vec{e_z}.$$

Computing the spherical metric as a pullback of the $\mathbb{R}^{m+1}$ metric:

$$g_{\theta\theta} = \langle \cos\theta \cdot \phi - \sin\theta \cdot \vec{e_z}, \cos\theta \cdot \phi - \sin\theta \cdot \vec{e_z} \rangle = \cos^2\theta \cdot \langle \phi, \phi \rangle + \sin^2\theta \cdot \langle \vec{e_z}, \vec{e_z} \rangle = 1,$$

$$g_{\phi\theta} = g_{\theta\phi} = 0,$$

$$g_{\phi\phi} = \sin^2\theta$$

yielding the desired

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}.$$

Remark. Note that in this we could replace S^{m-1} with an arbitrary manifold M of non-zero dimension [3]. Remarkably, since we are not using any knowledge of the underlying manifold M to compute the suspension metric with respect to M, it is only the function that we use to shrink the manifold towards suspension poles that matters for this relative metric. Analogously, we could take an analytic version of any topological construction to obtain its geometric version.

The complement of a point in S^n is contractible. If we remove a ball from S^n, the leftover part can be contracted in a Lipschitz way.

Lemma 1.2 (contraction lemma)**.** *For each radius r there is a Lipschitz-contraction $G : (S^n \setminus B_r) \times [0,1] \to S^n \setminus B_r$. G has Lipschitz constant $\lesssim 1/r$ in the S^n direction and $\lesssim 1$ in the $[0,1]$ direction.*

We choose the obvious contraction map:

$$G : (S^n \setminus B_r) \times [0,1] \to S^n \setminus B_r$$

$$G : (\rho, \theta, t) \to ((1-t)\rho, \theta)$$

Our goal is to compute its Lipschitz constants in both the sphere and the time direction. The strategy is to find the supremum of the differential applied to the appropriate tangent vectors and use it as an upper bound for the Lipschitz constants. The theoretical foundation for this approach is the mean value theorem for manifolds.

Proof. Let G be as above. Its differential is

$$\mathrm{d}G = \begin{pmatrix} 1-t & 0 & -\rho \\ 0 & 1 & 0 \end{pmatrix}$$

[3] For zero-dimensional manifolds $\mathrm{d}\phi^2$ vanishes, leaving $\mathrm{d}s^2 = \mathrm{d}\theta^2$ as the metric.

We start with the Lipschitz constant in the direction of the sphere by restricting to tangent vectors in the sphere direction, i.e. with the zero time component $(v_\rho, v_\theta, 0) \in T_p((S^n \setminus B_r) \times [0,1])$. It is of course the same as to fix t as a parameter and consider the family of maps G_t that are self-maps of the punctured sphere $S^n \setminus B_r$. We want compute the operator norm $\|\mathrm{d}G_t\|$:

$$\|\mathrm{d}G_t\| = \sup_{v\neq 0} \frac{\|\mathrm{d}G_t v\|_{G(p)}}{\|v\|_p} = \sup_{\|v\|_p=1} \|\mathrm{d}G_t v\|_{G(p)},$$

where $v = (v_\rho, v_\theta) \in T_p(S^n \setminus B_r), p = (\rho, \theta), G_t(p) = ((1-t)\rho, \theta)$ and we apply the sphere metric we computed in the section above. So for $\mathrm{d}G_t v$ we have:

$$\mathrm{d}G \begin{pmatrix} v_\rho \\ v_\theta \\ 0 \end{pmatrix} = \mathrm{d}G_t v = \begin{pmatrix} 1-t & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} v_\rho \\ v_\theta \end{pmatrix} = (1-t)^2 v_\rho^2 + v_\theta^2$$

$$\|v\|_p = 1 \Leftrightarrow v_\rho^2 + v_\theta^2 \sin^2\rho = 1$$

$$\|\mathrm{d}G_t v\|^2_{G(p)} = v_\rho^2(1-t)^2 + v_\theta^2 \sin^2((1-t)\rho) = v_\rho^2 \cdot (1-t)^2 + (1-v_\rho^2) \cdot \frac{\sin^2((1-t)\rho)}{\sin^2\rho}, \text{ where } 0 \le v_\rho^2 \le 1$$

So the value we are interested in maximizing is a convex combination of two terms, $(1-t)^2$ and $\frac{\sin^2((1-t)\rho)}{\sin^2\rho}$. We can find the supremum for each term, pick the larger one and be done. Instead let us first take a closer look at what is happening here. The two terms are just the operator norm in the directions of ρ and θ respectively. The reason why the norm is just a convex combination of the two is because the metric has no mixed terms, i.e. because the metric matrix dG_t is diagonal.

$$\|\mathrm{d}G_t v\|^2_{G(p)} = v_\rho^2 \cdot \frac{\|\mathrm{d}G\vec{v}_\rho\|^2}{\|\vec{v}_\rho\|^2} + (1-v_\rho^2) \cdot \frac{\|\mathrm{d}G\vec{v}_\theta\|^2}{\|\vec{v}_\theta\|^2}$$

$$\|\mathrm{d}G_t\| = \max\{\|\mathrm{d}G_{\theta,t}\|, \|\mathrm{d}G_{\rho,t}\|\}, \text{ where } \|\mathrm{d}G_{\theta,t}\| = \sup(1-t),$$

$$\|\mathrm{d}G_{\rho,t}\| = \sup_{v_\theta \neq 0} \frac{\|\mathrm{d}G\vec{v}_\theta\|_{G(p)}}{\|\vec{v}_\theta\|_p} = \sup_{v_\theta \neq 0} \frac{\|\vec{v}_\theta\|_{G(p)}}{\|\vec{v}_\theta\|_p} = \sup_{\substack{v_\theta\neq 0, \\ \rho \neq 0}} \frac{\sqrt{\sin^2((1-t)\rho)}}{\sqrt{\sin^2\rho}} = \sup_{\rho\neq 0} \frac{\sin((1-t)\rho)}{\sin\rho}$$

Direction ρ is the boring one, as $\sup(1-t) = 1$ is achieved at $t = 0$, where the sine quotient also equals 1 for $t = 0$. Thus, we can focus solely on the direction θ of the lateral spheres[4].

For large $r > \pi/2$ the Lipschitz constant $L < 1$, as increasing t only reduces the fraction. Geometrically, for $r > \pi/2$ contraction only shrinks the lateral spheres together with their tangent vectors. For $r < \pi/2$ we achieve the largest possible stretch of the tangent

[4] We still have to address the case $\rho = 0$. This is the pole point where our metric representation is not well defined. G fixes the pole and dG_t on the pole tangent space is identity. Hence at that point $\|dG_t|\rho = 0\| = 1$.

vectors when the lateral spheres S^{n-1} grow the most via G, that is, when p sits at the boundary of $S^n \setminus B_r$ and G(p) sits at the equator sphere. There

$$\rho = \pi - r;\ (1-t)\rho = \pi/2$$

$$\|\mathrm{d}G_t\| = \|\mathrm{d}G_{\rho,t}\| = \frac{\sin(\pi/2)}{\sin(\pi - r)} = \frac{1}{\sin r} \sim r^{-1}.$$

This shows the claim for the Lipschitz constant in the sphere direction.
Now for the Lipscthitz constant direction time,

$$\mathrm{d}G \begin{pmatrix} 0 \\ 0 \\ v_t \end{pmatrix} = \mathrm{d}G_{\rho,\theta} = \frac{\partial G}{\partial t} \cdot v_t = -\rho \cdot v_t$$

$$\|\mathrm{d}G_{\rho,\theta}\| = sup \left| \frac{\partial G}{\partial t} \right| = \pi \sim 1.$$

□

The bound we proved is not particularly good. In the standard proof that the image of a lower dimensional sphere is not surjective one approximates the sphere by piece-wise linear maps. We can explore this idea further by introducing simplicial approximation.

1.4 Simplicial approximation

Simplicial complexes are often neglected in presentation, so it might be beneficial to agree on some basic definitions.

Definition (simplicial complex)**.** A simplicial complex K is a collection of simplices satisfying the following conditions:

(1) Every face of a simplex in K also lies in K

(2) A non-empty intersection of two simlices in K $\sigma_1 \cap \sigma_2 \neq \varnothing$ is a face of both σ_1 and σ_2.

Additionally we equip a simplicial complex K with coherent topology of its simplices: a subset U is open in K iff $U \cap \sigma$ is open for all $\sigma \in K$.

Observation.

- A simplex σ is closed in K.
- The interior of a single vertex is the vertex itself. The boundary of a vertex is empty.
- A simplicial complex is a union of interiors of its simplices.

We will restrict our attention to finite simplicial complexes.
By default a simplicial complex K has a topology but no metric. A **geometric realization $|\mathbf{K}|$** of K on the other hand carries the metric that restricts to the subspace Euclidean metric on each simplex. This metric thus obviously agrees with the topology of K (i.e. $K \cong |K|$). If K has $N+1$ vertices one can simply choose a realization as the subsimplex of the standard N-simplex Δ^N.

Definition (star)**.** Let K be a simplicial complex. The **closed star** of a simplex σ in K $\mathbf{St}\sigma$ is the union of all simplices containing σ. The **open star** of a simplex $\sigma \in K$ $\mathrm{st}\sigma$ is the union of interiors of all simplices containing σ.

Observation: Closed stars are closed. Open stars are open. $\mathrm{St}\sigma$ is the closure of $\mathrm{st}\sigma$.

Of a special interest to us are stars of vertices. A star of a vertex v is the combinatorial analog of a ball around v. A closed star of a vertex captures all adjacent and incident edges, while open stars of vertices provide an open cover that is just shy of containing the adjacent vertices - this cover is expecially useful for simplicial approximation.

Claim (Lemma 2C.2 in Hatcher)**.** *Let $v_1, v_2, \ldots, v_k \in VertK$. Then $\mathrm{st}v_1 \cap \mathrm{st}v_2 \cap \cdots \cap \mathrm{st}v_k$ is either empty or $\sigma := [v_1, v_2, \ldots, v_k] \in K$ and $\mathrm{st}v_1 \cap \mathrm{st}v_2 \cap \cdots \cap \mathrm{st}v_k = \mathrm{st}\sigma$.*

Proof. Exercise (or see [3] for the solution). □

Definition. Let K, J be simplicial complexes. We call a map $f_0 : VertK \to VertJ$ that takes the vertex set of K to the vertex set of J a **vertex map**.

A map $f : K \to J$ that is linear on each simplex of K w.r.t. the barycentric coordinates is called a **simplicial map**.

Observation: A simplicial map restricts to a vertex map. A vertex map that can be linearly extended to a simplicial map if for each simplex σ its vertices are mapped to vertices of some target simplex.

Now that we have collected all the necessary tools we proceed with simplicial approximation. We start with a classical result on simplicial approximation.

Theorem 1.3 (2C.1 in Hatcher)**.** If K is a finite simplicial complex and J is an arbitrary simplicial complex, then any map $f : K \to J$ is homotopic to a map that is simplicial with respect to some iterated barycentric subdivision of K.

We do not actually intend to prove this result, but rather highlight some of the ideas that we want to translate to the Lipschitz setting:

1. We equip K with a metric as described above. In particular with this metric open stars are open and closed stars are closed in K. Distances within the simplex are also well behaved - all points are at most as far apart as the largest side length.
2. Observe that open stars form a covering of J. Taking pre-image of that cover yields an open cover of K. Since K is a finite simplicial complex it is in particular compact. We take the finite subcover and find its Lebesgue number (it exists by the Lebesgue number lemma). This gives us a way to determine the desired size of the simplices of K.
3. Now let us subdivide K until the simplices are small enough that closed star of a vertex v is contained in some cover element. This means we managed to contain the closed neighborhood of a vertex - adjacent edges and their vertices - fully in a reasonably small region of the the simplex. Edges can't wrap around our simplex multiple times and vertices cannot be too far apart. Meaning we have a chance of building a simplicial map.
   This is as much of the proof as we need for now - see [3] for more details and the contruction.

To translate this idea to a Lipschitz map setting we want to replace Lebesgue number using our Lipschitz constant. First, we need both our spaces to be metric. Now, we want distance between vertices in the target simplex J to be uniform (can normalize it to 1). To avoid counterexamples we restrict J to not have singletons.

The key insight of Guth's version as opposed to the classical approximation theorem in Hatcher is that in a Lipschitz setting we can avoid using the Lebesgue number of the open star cover altogether, relying on the Lipschitz constant instead:

If the image of a vertex maps close to some vertex we can just take it as our approximation. A bad case is if a vertex maps far from any vertex while still close to some edge. So let us consider what happens if some vertex v maps to the barycenter of an n-simplex Δ^n of side length 1. To contain $B(Im(v), c(n))$ in an open star of any vertex we would need to set the radius $c(n)$ to be less than the shortest distance from the barycenter to the face of the n-simplex, i.e. $c(n) := dist(barycenter, \partial\Delta^n)$. But that distance is determined by n and it grows smaller as n increases. Meaning this was indeed the worst case scenario we have determined the required constant without referring to the Lebesgue number!

Theorem 1.4 (Simplicial approximation of a Lipschitz map)**.** Let J be a finite simplicial complex of dimension n and let $|J|$ be an equilteral realization of J with edges of length 1. Let $f : |K| \to |J|$ be a Lipschitz map with Lipschitz constant L and let $c(n)$ be defined as in the discussion above. If $|K|$ has equilateral simplices of side length $c(n)/L$ then f can be approximated by a simplicial map with Lipschitz constant $L/c(n)$ and a homotopy H_{simp} with Lipschitz constants $L/c(n)$ and $c(n)$ in directions of the simplex K and in the time direction respectively.

More generally, if each simplex of $|K|$ is homeomorphic to some standard simplex of side-length $1/L$ with bi-lipschitz constant ~ 1 (i.e. homeomorphisms are Lipschitz in both directions with constants $\lambda(K), \gamma(K)$), then f_{simp} has Lipscthitz constant $C(K,n)L$ and H_{simp} has Lipschitz constant $C(K,n)L$ in the K direction and ~ 1 in the time direction.

Proof. By the discussion above we can guarantee that for each $v \in VertK$ there is a vertex $g(v) \in VertJ$ such that $\mathrm{St}v \subset \mathrm{st}g(v)$. Thus $g : VertK \to VertJ$ defines a vertex map. We want to show that it extends to a simplicial map. Let x be a point in the interior of $[v_1, v_2, \dots, v_k]$. Then $f(x)$ is contained in each of the stars $\mathrm{st}g(v_i)$. Thus, by the claim labelled "Lemma 2C.2" above, $\sigma = [g(v_1), g(v_2), \dots, g(v_k)]$ is a simplex in J and we can extend the vertex map g to a simplicial map f_{simp}. Again, by the claim above we conclude that $f(x) \in st(\sigma)$ and thus there is a simplex σ' that contains $f(x)$ in its interior and contains σ as a face (does not have to be a proper face, i.e. it is possible that $\sigma' = \sigma$). We conclude that $f(x), f_{simp}(x) \in \sigma'$. We can now simply take the straight line homotopy, i.e. (cf proof of 2C.1 in [3]).

$$H_{simp} = (1-t)f + tf_{simp}$$

It remains to verify that the Lispchitz constants hold. f_{simp} extends linearly on simplices, thus for equilateral K the constant multiple is determined entirely by the ratio of edge lengths, i.e. $1/c(n) > 1$. The Lipschitz constant o f_{simp} in this case is thus $L/c(n)$. For the more general version this is magnified by how much the shortest edge in K need to be stretched, which is at most the product $\lambda \cdot \gamma$ of the Lipschitz constants of the bi-lipschitz simplex homeomorphisms in both directions, which depend solely on K. The total constant thus amounts to $C(K,n) := \frac{\lambda\gamma}{c(n)} \cdot L \sim L$.

Finally, we have to determine the Lipschitz constants of the homotopy. Note that $C(K,n) > 1/c(n) > 1$, thus for a given t $H_{simp}(t)$ has Lipscihtz constant $(1-t)L + tC(K,n)L$. Meaning in the K direction the constant is at most $C(K,n)L$ and in the time direction the Lipschitz constant grows linearly with $C(K,n)$. This finishes the proof. $\square$

1.4.1 Approximating maps between spheres

Definition (triangulation)**.** Let K be a simplicial complex, X a topological space. A homeomorphism $\phi : K \to X$ is called a **triangulation** of X.

Let $f : X \to Y$ be a map between metric spaces, $\phi : (K, d_k) \to X$, $\psi : (J, d_j) \to Y$ - bi-lipschitz triangulations. Then if there is a simplicial approximation g_{simp} of $g := \phi \circ f \circ \psi^{-1}$ we call $f_{simp} := \phi^{-}1 \circ g_{simp} \circ \psi$ the simplicial approximation of f.

$$\begin{tikzcd}
X \arrow[r, "f_{simp}"] & Y \\
(K, d_k) \arrow[u, "\phi"] \arrow[r, "g_{simp}"] & (J, d_j) \arrow[u, "\psi"]
\end{tikzcd}$$

We now want to find triangulations for our spheres so that we can apply the simplicial approximation to them. We will pick triangulations that suit our purposes for the main result.

We start with a triangulation of S^n by the boundary of the unilateral $n+1$ - simplex $\partial\Delta^{n+1}$. This obviously uses very few vertices, thereby limiting the quality of our approximation, so let me try to motivate this choice of triangulation (the motivation will become apparent in the upcoming proofs): for the main result in case $m \geq n$ we need all vertices to be pairwise incident (i.e. any two vertices to share an edge). This will allow us to "approximate" the null-homotopy to some extent without any further geometric information about it. Furthermore, we would like simplices to be both equilateral and to have equal area. This already determines our triangulation uniquely (up to rotations). Additionally, $\partial\Delta^{n+1}$ is defined for all dimensions (as opposed to, say, a triangulation of S^2 by the surface of icosahedron, that does not generalize well to other dimensions). $\partial\Delta^{n+1}$ is bi-lipschitz homeomorphic to S^n with bi-lipschitz constants ~ 1 only depending on n.

It is notably more difficult to triangulate S^m so that its triangulation fits the theorem. In fact, the proof of the main theorem requires us to be able to triangulate not only S^m but the whole unit ball, B^{m+1}. One difficulty with it is that the Lipschitz constant of our approximation is determined entirely by the shortest side $length(s)$ in the metric simplicial complex: $(length(s) \cdot L \cdot 1/c(n))^{-1}$. At the same time the side-length needs to be strictly less than $c(n)/L$ We formulate this as an exercise. See appendix for discussion.

Exercise. Find a family of geometric simplicial complexes $(K, |K|)$ together with a bi-lipschitz triangulation of the unit ball $Tri_L : |K| \to B^{m+1}$ such that the Lipschitz constant in the direction of the ball is less than 1. We require furthermore that each simplex in $(K, |K|)$ is bilipschitz homeomorphic to the unilateral simplex Δ^{m+1}. We require the maximum over Lipschitz constants of maps $\Delta \to B^{m+1}$ to be bounded by $c(n)/L$.

Theorem 1.5 (1.4 in Guth)**.** If $m < n$ and $f : S^m \to S^n$ has Lipschizt constant L, then there is a null-homotopy with Lipschitz constant $\lesssim L$. In fact the null-homotopy has Lipschitz constant $\lesssim L$ in the S^m directions and $\lesssim 1$ in the [0,1] direction.

Proof. Consider the map between simplices instead. Approximate g using Theorem 1.4. g_{simp} is piecewise linear hence not surjective. Thus g_{simp} misses a whole simplex! Now back on the sphere simplicial approximation of f f_{simp} misses a ball of radius ~ 1 in S^n. Applying Lemma 1.2 finishes the proof. □

Remark. Lemmas 1.1 and Theorem 1.4 show that Lipschitz maps for dim $m < n$ are null-homotopic independently. Both are stronger than what we need for the main proof. Note that we did not use Lemmas 1.1-1.2.

Remark. This bound is tight (up to constant manipulations). I did not verify this, Guth recommends it as an exercise (see discussion after Lemma 1.3 in [1]).

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