Karlsruhe Institute of Technology

On the topology of Lipschitz maps between spheres

Master Thesis of

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Karlsruhe, 18.08.2022

Abstract

We discuss the 2017 expository paper on quantitative topology by Larry Guth. The paper provides a more accessible take on two 2016 papers by Chambers, Dotterer, Ferry, Manin and Weinberger, which jointly gave an almost complete answer to Gromov's question: Given a Lipschitz continuous null-homotopic map between spheres $f : S^m \to S^n$, with Lipschitz constant L, what is the best Lipschitz constant over Lipschitz null-homotopies of f?

Zusammenfassung

Wir besprechen eine Übersichtsarbeit im Bereich der quantitativen Topologie von Larry Guth aus dem Jahr 2017. Seine Arbeit ermöglicht einen einfacheren Zugang zu zwei weiteren Arbeiten aus dem Jahr 2016, in denen es die Autoren Chambers, Dotterer, Ferry, Manin und Weinberger schaffen, eine fast vollständige Antwort auf die folgende Frage von Gromov zu geben: Gegeben sei eine nullhomotope Abbildung $f : S^m \to S^n$, Lipschitzstetig mit der Lipschitzkonstante L. Was ist die bestmögliche Lipschitzkonstante von allen Lipschitz-Nullhomotopien von f?

Contents

1 Background of the problem

This master thesis attempts to expand on and clarify some arguments from Larry Guth's expository paper on quantitative topology[1].

Guth wrote an excellent introduction, and rather than attempt to add anything of substance to it I will merely summarize the main question and explain what I personally find appealing about the topic and the exposition paper. Gromov stated and partially answered the following question:

Question. Given a Lipschitz continuous null-homotopic map between spheres $f : S^m \to S^n$, with Lipschitz constant L, what is the best Lipschitz constant over Lipschitz null-homotopies of f?

Gromov proved that a Lipcshitz homotopy exists and gave a proof for a huge Lipschitz bound with a tower of exponentials. He conjectured however that for an optimal homotopy the Lipschitz constant should have linear dependency on L, i.e. the constant should be $C(m,n)L$ for some constant $C(m,n)$ that only depends on the dimensions of the spheres. He proved the conjecture for the case $m = n$.

In two 2016 papers Chambers, Dotterer, Ferry, Manin and Weinberger gave an almost complete answer to Gromov's question. Namely, for the case n is even they showed the Lipschitz constant of the homotopy to be up to $C(m,n)L^2$, and for the case n is odd they showed that the conjecture holds. I refer you to Guth's paper for a more extensive introduction.

The question at hand is one of quantitative topology, which is interesting because the research area touches upon many different fields in mathematics. Specifically this expository paper captured my attention by bringing together a variety of topics such as simplicial complexes, higher homotopy groups, cohomology, a bit of differential geometry, degree theory, calculus on manifolds, Hopf fibrations etc. The style of the paper makes it feel more engaging and accessible, while the hands-on approach to proofs made the seemingly theoretical topics feel very practical.

1.1 Some background assumptions

This paper deals with Lipschitz constants of maps between spheres. Most of the time we will only estimate the Lipschitz constants up to a constant $C(m,n)$, that only depends on the dimensions of the spheres. We denote equality/inequality up to a constant by

$\sim, \lesssim, \gtrsim$ respectively. For this section let the unit spheres S^m, S^n be equipped with the length metric induced by the standard Riemannian metric (unless stated otherwise). That is, the distance between any two points is determined by the (Euclidean) length of the geodesics between them [1]. Note that while the topology is the same, the metric is different from the "default" metric inherited from the ambient Euclidean space. Later we will consider objects that are homeomorphic to spheres when it is convenient (e.g. surface of a cube or of a simplex), but the conversion only changes things up to some constraint. On those objects we will still be using the length metric.

1.2 Contracting the image of a lower dimensional sphere

1.2.1 Introduction to computations up to a constant

In this section we consider Lipschitz maps from S^m to S^n when $m < n$. It is a classical result that such a map is null-homotopic - we first homotope the map f to a piece-wise linear or smooth map f_{approx}. Then we know that the image $f_{approx}(S^m)$ in S^n is not-surjective and therefore contractible. Our aim is to take these ideas and make them more quantitative. To start with, we isolate the part of contracting a non-surjective image in the target sphere. In this subsection we show that the image of a lower dimensional sphere can be contracted in a Lipschitz way and provide a tight Lipschitz constant for the contraction (that is, tight up to constant manipulations).

We start by showing that a Lipschitz map must in fact miss a whole open ball in the target:

Lemma 1.1. *Let $f : S^m \to S^n$ be a Lipschitz continuous maps with a Lipschitz constant L. Then the image of f misses a ball of radius r for $r \lesssim L^{\frac{-m}{n-m}}$*

Our strategy for proving this lemma will be to first cover the domain sphere by open balls of a set radius, then map that cover to the target sphere and show that the image of the cover cannot be surjective.

A ball with respect to the length-metric on the sphere is a spherical cap. The radius of the ball is the length of any geodesic from the center (the tip) of the cap to its edge. It is equal to the polar angle of the cap in radians.

[1] To be precise, the length of the geodesics is determined by the standard Riemmanian metric, where the metric is pulled back along the embedding of the spheres into their ambient Euclidean spaces ($\mathbb{R}^m$, $\mathbb{R}^n$, respectively). The lengths of geodesics are then precisely the respective Euclidean lengths of their embeddings. The reason to specify a metric so early on is that when we talk about Lipschitz continuity we are implicitly dealing with the metrics, not just with underlying topologies. However, since all of our results are up to a constant, suitable constant manipulation would show them to hold for the standard Euclidean metric as well. Nevertheless, we prefer to settle on a specific metric to avoid confusion or ambiguity.

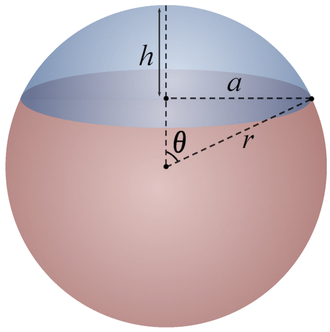


Figure 1.1: Spherical cap. Here $r = 1$, θ is the polar angle, $a = \sin\theta$, $h = 1 - \cos\theta$. Credit: Jhmadden, CC BY-SA 4.0 https://creativecommons.org/licenses/by-sa/4.0, via Wikimedia Commons

For the proof of the lemma we will need to cover the sphere with spherical caps. To estimate the number of caps needed to cover the sphere we use a volume argument. A sloppy version of the argument would go as follows: we equip the sphere with a volume form that scales well with the polar angle and is equal to 1 on the whole sphere. Then the volume of the sphere is 1, the volume of each spherical cap is ρ^m. The cover should have area similar to that of the sphere (up to a constant). We then need $\sim 1/\rho^m$ spherical caps to cover the sphere.

You may object: why should the cover have volume similar to that of the sphere if there is an overlap? Why should the overlap scale well with the radius? Is it okay for us to change the metric if the result is stated w.r.t. to a different one (we could of course use the standard volume but then you would be right to point out that spherical cap volume might not scale well with the cover radius). However, this argument is only meant to provide us with an intuition. We aim to show that this types of arguments can be formalized fairly easily:

Claim. *For any $\rho > 0$, the sphere S^m can be covered by $\sim 1/\rho^m$ balls of radius ρ.*

Proof. We want to estimate the number of $1/\rho$ balls needed to cover the sphere. Covering the sphere is up to a constant the same as covering the hemisphere. In fact, the the cardinality of the cover for $S^m \sim$ Hemisphere $S^m_+ \sim D^m$ (we can transfer the cover back and forth by projecting the hemisphere onto the equator disk or wrapping a larger disk around the hemisphere [2] $= B^m \sim$ covering $\sqrt{2}B^m$ (scaling up) $\sim$ covering the m-box of

[2]Projecting the hemisphere S^m_+ down onto the unit disk D^m at the equator obviously only changes things up to a constant (depending only on m): we project down the cover centers and keep the radius as is. Showing that the cover can be transferred in the other direction is a little trickier: we start by scaling the cover up by a factor of $\pi/2$ to cover $\pi/2D^m$. We then wrap the larger disc around the hemisphere by taking (θ, r) to $(\theta, \rho) = (\theta, r)$, where $\theta \in S^{m-1}$, r is the radius and ρ is the polar angle. Distances can only reduce for the same reason that we can wrap a paper around a ball without tearing it and the paper will wrinkle: radial components of distances stay the same and angular components

side length two (it can be squeezed between the two balls, i.e. it contains the unit ball and it is contained in the $\sqrt{2}B^m$) ~ covering the m-box of side length 1 (the unit m-box). It is easy to see why the volume argument should work now: the unit box can be clearly be covered by $\lceil 1/\rho \rceil^m$ boxes of side length ρ. Each ρ-box is contained in a ball of radius ρ and we are done. $\square$

Arguing up to a constant allows us great flexibility in choosing objects we are more comfortable working with. The constants we omitted can easily be traced back through the equivalence steps we took. However, if an argument up to a constant seems sketchy, there is a direct argument on the sphere without any equivalences or dropping constants that I provided in the appendix.

Lemma 1.2 (Image of misses a ball)**.** *Let $f\colon S^m \to S^n$ be a Lipschitz-continuous map with a Lipschitz constant L. Then the image of f misses a ball of radius r for $r \lesssim L^{-\frac{m}{n-m}}$*

Proof. For any $\rho > 0$, S^m can be covered by $\sim \rho^{-m}$ balls of radius ρ. The image of each such ball is contained in a ball of radius $L\rho$. Therefore, the image of f can be covered by $\lesssim \rho^{-m}$ balls of radius $L\rho$. We set $r := L\rho$. We now want to choose ρ small enough so that the cover misses a ball of radius r.

Expanding the radius of the cover to $2r$ yields a cover of the r-neighborhood of the image. We denote this $2r$-cover by C. If this larger cover does not cover the full sphere S^n, the image of f must miss a ball of radius r. The total volume of the cover C is at most the cardinality of C times the volume of a ball of radius $2r$ (which is a spherical cap of polar angle $2r$). We replace the cap volume by the larger volume of a disk $2r \cdot D^n$ by essentially the same argument as we used to transfer the disk cover from the disk to the hemisphere (see footnote on previous page). The total cover volume is then at most $|C|\omega^n(\pi/2 \cdot B_2^n r)$, where ω^n denotes the Euclidean n-volume form. $\square$

We now set ρ so that this number is smaller than the volume of the sphere. So we get for $n > m$

$$|C|\omega^n(2r\text{-cap}) \lesssim \rho^{-m} r^n = L^n \rho^{n-m} \lesssim 1,$$
$$\rho \lesssim L^{-\frac{n}{n-m}},$$
$$r = L\rho \lesssim L^{-\frac{m}{n-m}}.$$

In particular, even if f is a constant map we can choose ρ small enough so that $r \leq \pi/2$

shrink with a factor of sine as the radius increases. Of course if you are not convinced you can scale your disk up by another factor of two. Same as before, keeping the projected cover centers and the old cover radius yield a cover.

1.2.2 Detour: geometric suspension

If we equip the sphere S^2 with the usual pullback Riemannian metric, the resulting metric written in the matrix form is

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}.$$

Even more often in the literature one encounters the corresponding symmetric quadratic form - its first fundamental form - which can be written as:

$$\mathrm{d}s^2 = \mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2\theta \, \mathrm{d}\phi \otimes \mathrm{d}\phi,$$

or simply

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2.$$

We will now show that the metric is verbatim the same for S^m for $\forall m \geq 2$

We can think of S^m as of several S^{m-1} stacked on top of each other (where S^{m-1} shrink to a single point at the poles). This is essentially the geometric version of suspension. Using the polar angle rather than height, we scale the equator S^{m-1} by $\sin\theta$.

Point-wise this gives us that any point p of S^m can be parametrized in terms of the polar angle θ and the corresponding vector ϕ of the equator scaled down by $\sin\theta$ - polar coordinates with respect to S^{m-1} Figure 1.2. Fixing some direction z in $\mathbb{R}^{m+1}$ we can write out the parametrization:

$$\begin{aligned} \psi : &[0, \pi] \times S^{m-1} \longrightarrow S^m \\ &(\theta, \phi) \mapsto \sin\theta \cdot \phi + \cos\theta \cdot \vec{e_z}, \end{aligned}, \tag{1.1}$$

where $\vec{e_z}$ denote the standard basis vector in the z direction. Computing partial derivatives yields

$$\frac{\partial \psi}{\partial \theta} = \cos\theta \cdot \phi - \sin\theta \cdot \vec{e_z},$$

$$\frac{\partial \psi}{\partial \phi} = \sin\theta \cdot \vec{e_z}.$$

Computing the spherical metric as a pullback of the $\mathbb{R}^{m+1}$ metric:

$$g_{\theta\theta} = \langle \cos\theta \cdot \phi - \sin\theta \cdot \vec{e_z}, \cos\theta \cdot \phi - \sin\theta \cdot \vec{e_z} \rangle = \cos^2\theta \cdot \langle \phi, \phi \rangle + \sin^2\theta \cdot \langle \vec{e_z}, \vec{e_z} \rangle = 1,$$

$$g_{\phi\theta} = g_{\theta\phi} = 0,$$

$$g_{\phi\phi} = \sin^2\theta$$

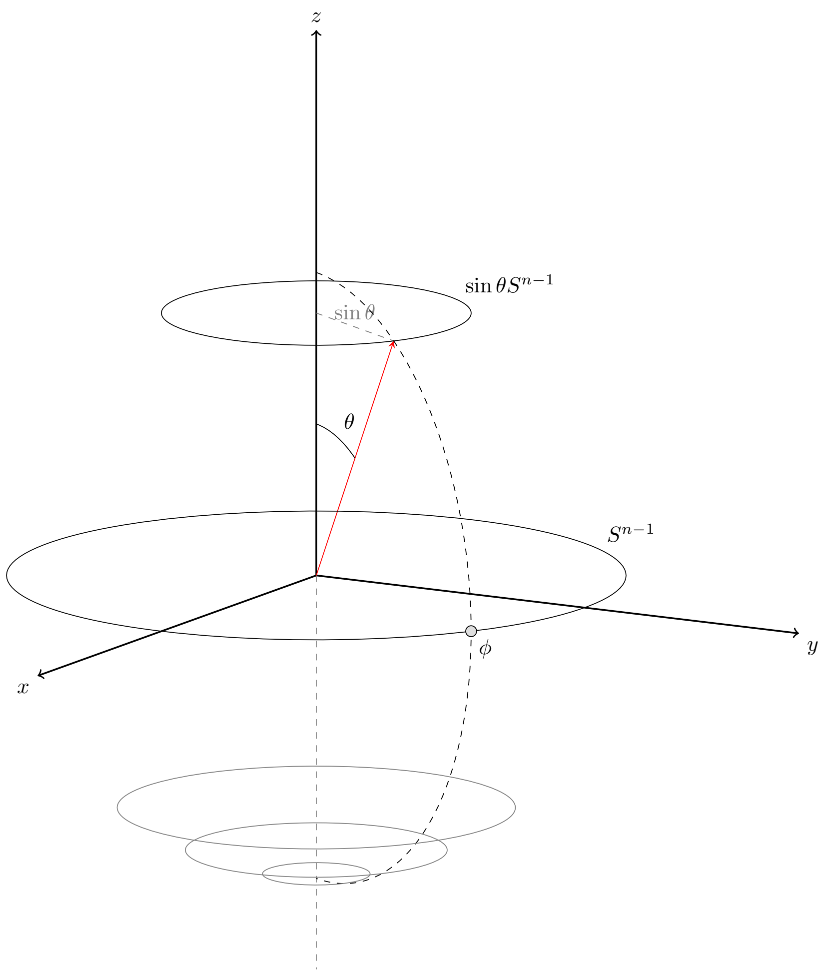


Figure 1.2: Polar coordinates

yielding the desired

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}.$$

Remark. Note that in this we could replace S^{m-1} with an arbitrary manifold M of non-zero dimension [3]. Remarkably, since we are not using any knowledge of the underlying manifold M to compute the suspension metric with respect to M, it is only the function that we use to shrink the manifold towards suspension poles that matters for this relative metric. Analogously, we could take an analytic version of any topological construction to obtain its geometric version.

The complement of a point in S^n is contractible. If we remove a ball from S^n, the leftover part can be contracted in a Lipschitz way.

Lemma 1.3 (contraction lemma)**.** *For each radius r there is a Lipschitz-contraction $G : (S^n \setminus B_r) \times [0,1] \to S^n \setminus B_r$. G has Lipschitz constant $\lesssim 1/r$ in the S^n direction and $\lesssim 1$ in the $[0,1]$ direction.*

We choose the obvious contraction map:

$$G : (S^n \setminus B_r) \times [0,1] \to S^n \setminus B_r$$

$$G : (\rho, \theta, t) \to ((1-t)\rho, \theta)$$

Our goal is to compute its Lipschitz constants in both the sphere and the time direction. The strategy is to find the supremum of the differential applied to the appropriate tangent vectors and use it as an upper bound for the Lipschitz constants. The theoretical foundation for this approach is the mean value theorem for manifolds.

Proof. Let G be as above. Its differential is

$$\mathrm{d}G = \begin{pmatrix} 1-t & 0 & -\rho \\ 0 & 1 & 0 \end{pmatrix}$$

We start with the Lipschitz constant in the direction of the sphere by restricting to tangent vectors in the sphere direction, i.e. with the zero time component $(v_\rho, v_\theta, 0) \in T_p((S^n \setminus B_r) \times [0,1])$. It is of course the same as to fix t as a parameter and consider the family of maps G_t that are self-maps of the punctured sphere $S^n \setminus B_r$. We want compute the operator norm $\|\mathrm{d}G_t\|$:

$$\|\mathrm{d}G_t\| = \sup_{v \neq 0} \frac{\|\mathrm{d}G_t v\|_{G(p)}}{\|v\|_p} = \sup_{\|v\|_p = 1} \|\mathrm{d}G_t v\|_{G(p)},$$

[3] For zero-dimensional manifolds $\mathrm{d}\phi^2$ vanishes, leaving $\mathrm{d}s^2 = \mathrm{d}\theta^2$ as the metric.

where $v = (v_\rho, v_\theta) \in T_p(S^n \setminus B_r), p = (\rho, \theta), G_t(p) = ((1-t)\rho, \theta)$ and we apply the sphere metric we computed in the section above. So for $\mathrm{d}G_t v$ we have:

$$\mathrm{d}G \begin{pmatrix} v_\rho \\ v_\theta \\ 0 \end{pmatrix} = \mathrm{d}G_t v = \begin{pmatrix} 1-t & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} v_\rho \\ v_\theta \end{pmatrix} = (1-t)^2 v_\rho^2 + v_\theta^2$$

$$\|v\|_p = 1 \Leftrightarrow v_\rho^2 + v_\theta^2 \sin^2 \rho = 1$$

$$\|\mathrm{d}G_t v\|_{G(p)}^2 = v_\rho^2 (1-t)^2 + v_\theta^2 \sin^2((1-t)\rho) = v_\rho^2 \cdot (1-t)^2 + (1-v_\rho^2) \cdot \frac{\sin^2((1-t)\rho)}{\sin^2 \rho}, \text{ where } 0 \leq v_\rho^2 \leq 1$$

So the value we are interested in maximizing is a convex combination of two terms, $(1-t)^2$ and $\frac{\sin^2((1-t)\rho)}{\sin^2 \rho}$. We can find the supremum for each term, pick the larger one and be done. Instead let us first take a closer look at what is happening here. The two terms are just the operator norm in the directions of ρ and θ respectively. The reason why the norm is just a convex combination of the two is because the metric has no mixed terms, i.e. because the metric matrix dG_t is diagonal.

$$\|\mathrm{d}G_t v\|_{G(p)}^2 = v_\rho^2 \cdot \frac{\|\mathrm{d}G \vec{v}_\rho\|^2}{\|\vec{v}_\rho\|^2} + (1 - v_\rho^2) \cdot \frac{\|\mathrm{d}G \vec{v}_\theta\|^2}{\|\vec{v}_\theta\|^2}$$

$$\|\mathrm{d}G_t\| = \max\{\|\mathrm{d}G_{\theta,t}\|, \|\mathrm{d}G_{\rho,t}\|\}, \text{ where } \|\mathrm{d}G_{\theta,t}\| = \sup(1-t),$$

$$\|\mathrm{d}G_{\rho,t}\| = \sup_{v_\theta \neq 0} \frac{\|\mathrm{d}G \vec{v}_\theta\|_{G(p)}}{\|\vec{v}_\theta\|_p} = \sup_{v_\theta \neq 0} \frac{\|\vec{v}_\theta\|_{G(p)}}{\|\vec{v}_\theta\|_p} = \sup_{\substack{v_\theta \neq 0, \\ \rho \neq 0}} \frac{\sqrt{\sin^2((1-t)\rho)}}{\sqrt{\sin^2 \rho}} = \sup_{\rho \neq 0} \frac{\sin((1-t)\rho)}{\sin \rho}$$

Direction ρ is the boring one, as $\sup(1-t) = 1$ is achieved at $t = 0$, where the sine quotient also equals 1 for $t = 0$. Thus, we can focus solely on the direction θ of the lateral spheres[4].
For large $r > \pi/2$ the Lipschitz constant $L < 1$, as increasing t only reduces the fraction. Geometrically, for $r > \pi/2$ contraction only shrinks the lateral spheres together with their tangent vectors. For $r < \pi/2$ we achieve the largest possible stretch of the tangent vectors when the lateral spheres S^{n-1} grow the most via G, that is, when p sits at the boundary of $S^n \setminus B_r$ and G(p) sits at the equator sphere. There

$$\rho = \pi - r; \ (1-t)\rho = \pi/2$$

$$\|\mathrm{d}G_t\| = \|\mathrm{d}G_{\rho,t}\| = \frac{\sin(\pi/2)}{\sin(\pi - r)} = \frac{1}{\sin r} \sim r^{-1}.$$

This shows the claim for the Lipschitz constant in the sphere direction.
Now for the Lipscthitz constant direction time,

[4] We still have to address the case $\rho = 0$. This is the pole point where our metric representation is not well defined. G fixes the pole and dG_t on the pole tangent space is identity. Hence at that point $\|dG_t|\rho = 0\| = 1$.

$$\mathrm{d}G\begin{pmatrix}0\\0\\v_t\end{pmatrix} = \mathrm{d}G_{\rho,\theta} = \frac{\partial G}{\partial t}\cdot v_t = -\rho\cdot v_t$$

$$\|\mathrm{d}G_{\rho,\theta}\| = sup\left|\frac{\partial G}{\partial t}\right| = \pi \sim 1.$$

$\square$

The bound we proved is not particularly good. In the standard proof that the image of a lower dimensional sphere is not surjective one approximates the sphere by piece-wise linear maps. We can explore this idea further by introducing simplicial approximation.

1.3 Simplicial approximation

Simplicial complexes are often neglected in presentation, so it might be beneficial to agree on some basic definitions.

Definition (simplicial complex)**.** A simplicial complex K is a collection of simplices satisfying the following conditions:

(1) Every face of a simplex in K also lies in K

(2) A non-empty intersection of two simlices in K $\sigma_1 \cap \sigma_2 \neq \varnothing$ is a face of both σ_1 and σ_2.

Additionally we equip a simplicial complex K with coherent topology of its simplices: a subset U is open in K iff $U \cap \sigma$ is open for all $\sigma \in K$.

Observation.

- A simplex σ is closed in K.
- The interior of a single vertex is the vertex itself. The boundary of a vertex is empty.
- A simplicial complex is a union of interiors of its simplices.

We will restrict our attention to finite simplicial complexes.
By default a simplicial complex K has a topology but no metric. A **geometric realization |K|** of K on the other hand carries the metric that restricts to the subspace Euclidean metric on each simplex. This metric thus obviously agrees with the topology of K (i.e. $K \cong |K|$). If K has $N+1$ vertices one can simply choose a realization as the subsimplex of the standard N simplex Δ^N.

Definition (star)**.** Let K be a simplicial complex. The **closed star** of a simplex σ in K **St**σ is the union of all simplices containing σ. The **open star** of a simplex $\sigma \in K$ stσ is the union of interiors of all simplices containing σ.

Observation: Closed stars are closed. Open stars are open. Stσ is the closure of stσ.

Of a special interest to us are stars of vertices. A star of a vertex v is the combinatorial analog of a ball around v. A closed star of a vertex captures all adjacent and incident edges, while open stars of vertices provide an open cover that is just shy of containing the adjacent vertices - this cover is expecially useful for simplicial approximation.

Claim (Lemma 2C.2 in Hatcher)**.** *Let* $v_1, v_2, \dots, v_k \in VertK$*. Then* $\mathrm{st}v_1 \cap \mathrm{st}v_2 \cap \dots \cap \mathrm{st}v_k$ *is either empty or* $\sigma = [v_1, v_2, \dots, v_k] \in K$ *and* $\mathrm{st}v_1 \cap \mathrm{st}v_2 \cap \dots \cap \mathrm{st}v_k = \mathrm{st}\sigma$*.*

Definition. Let K, J be simplicial complexes. We call a map $f_0 : VertK \to VertJ$ that takes the vertex set of K to the vertex set of J a **vertex map**.

A map $f : K \to J$ that is linear on each simplex of K w.r.t. the barycentric coordinates is called a **simplicial map**.

Observation: A simplicial map restricts to a vertex map. A vertex map that can be linearly extended to a simplicial map if for each simplex σ its vertices are mapped to vertices of some target simplex.

Now that we have collected all the necessary tools we proceed with simplicial approximation. We start with a classical result on simplicial approximation.

Theorem 1.4 (2C.1 in Hatcher)**.** If K is a finite simplicial complex and J is an arbitrary simplicial complex, then any map $f : K \to J$ is homotopic to a map that is simplicial with respect to some iterated barycentric subdivision of K.

We don't actually intend to prove this result, but rather highlight some of the ideas that we want to translate to the Lipschitz setting:

1. We equip K with a metric as described above. In particular with this metric open stars are open and closed stars are closed in K. Distances within the simplex are also well behaved - all points are at most as far apart as the largest side length.
2. Observe that open stars form a covering of J. Taking pre-image of that cover yields an open cover of K. Since K is a finite simplicial complex it is in particular compact. We take the finite subcover and find its Lebesgue number (it exists by the Lebesgue number lemma). This gives us a way to determine the desired size of the simplices of K.

3. Now let us subdivide K until the simplices are small enough that closed star of a vertex v is contained in some cover element. This means we managed to contain the closed neighborhood of a vertex - adjacent edges and their vertices - fully in a reasonably small region of the the simplex. Edges can't wrap around our simplex multiple times and vertices cannot be too far apart. Meaning we have a chance of building a simplicial map.
   This is as much of the proof as we need for now - see [2] for more details and the contruction.

To translate this idea to a Lipschitz map setting we want to replace Lebesgue number using our Lipschitz constant. First, we need both our spaces to be metric. Now, we want distance between vertices in the target simplex J to be uniform (can normalize it to 1). To avoid counterexamples we restrict J to not have singletons.

The key insight of Guth's version as opposed to the classical approximation theorem in Hatcher is that in a Lipschitz setting we can avoid using the Lebesgue number of the open star cover altogether, relying on the Lipschitz constant instead:

If the image of a vertex maps close to some vertex we can just take it as our approximation. A bad case is if a vertex maps far from any vertex while still close to some edge. So let us consider what happens if some vertex v maps to the barycenter of an n-simplex Δ^n of side length 1. To contain $B(Im(v), c(n))$ in an open star of any vertex we would need to set the radius $c(n)$ to be less than the shortest distance from the barycenter to the face of the n-simplex, i.e. $c(n) := dist(barycenter, \partial\Delta^n)$. But that distance is determined by n and it grows smaller as n increases. Meaning this was indeed the worst case scenario we have determined the required constant without referring to the Lebesgue number!

Theorem 1.5 (Simplicial approximation of a Lipschitz map)**.** Let J be a finite simplicial complex of dimension n and let $|J|$ be an equilteral realization of J with edges of length 1. Let $f : |K| \to |J|$ be a Lipschitz map with Lipschitz constant L and let $c(n)$ be defined as in the discussion above. If $|K|$ has equilateral simplices of side length $c(n)/L$ then f can be approximated by a simplicial map with Lipschitz constant $L/c(n)$ and a homotopy H_{simp} with Lipschitz constants $L/c(n)$ and $c(n)$ in directions of the simplex K and in the time direction respectively.

More generally, if each simplex of $|K|$ is homeomorphic to some standard simplex of side-length $1/L$ with bi-lipschitz constant ~ 1 (i.e. homeomorphisms are Lipschitz in both directions with constants $\lambda(K), \gamma(K)$), then f_{simp} has Lipscthitz constant $C(K,n)L$ and H_{simp} has Lipschitz constant $C(K,n)L$ in the K direction and ~ 1 in the time direction.

Proof. By the discussion above we can guarantee that for each $v \in VertK$ there is a vertex $g(v) \in VertJ$ such that $\mathrm{St}v \subset \mathrm{st}g(v)$. Thus $g : VertK \to VertJ$ defines a vertex map. We want to show that it extends to a simplicial map. Let x be a point in the interior of $[v_1, v_2, \ldots, v_k]$. Then $f(x)$ is contained in each of the stars $\mathrm{st}g(v_i)$. Thus by the claim 1.3 above $\sigma = [g(v_1), g(v_2), \ldots, g(v_k)]$ is a simplex in J and we can extend the vertex map g to a simplicial map f_{simp}. Again by claim 1.3 we conclude that $f(x) \in st(\sigma)$ and thus there is a simplex σ' that contains $f(x)$ in its interior and contains σ as a face (does not have to be a proper face, i.e. it is possible that $\sigma' = \sigma$). We conclude that $f(x), f_{simp}(x) \in \sigma'$. We can now simply take the straight line homotopy, i.e. (cf proof of 2C.1 in [2]).

$$H_{simp} = (1-t)f + tf_{simp}$$

It remains to verify that the Lispchitz constants hold. f_{simp} extends linearly on simplices, thus for equilateral K the constant multiple is determined entirely by the ratio of edge lengths, i.e. $1/c(n) > 1$. The Lipschitz constant o f_{simp} in this case is thus $L/c(n)$. For the more general version this is magnified by how much the shortest edge in K need to be stretched, which is at most the product $\lambda \cdot \gamma$ of the Lipschitz constants of the bi-lipschitz simplex homeomorphisms in both directions, which depend solely on K. The total constant thus amounts to $C(K,n) := \frac{\lambda\gamma}{c(n)} \cdot L \sim L$.

Finally, we have to determine the Lipschitz constants of the homotopy. Note that $C(K,n) > 1/c(n) > 1$, thus for a given t $H_{simp}(t)$ has Lipscihtz constant $(1-t)L + tC(K,n)L$. Meaning in the K direction the constant is at most $C(K,n)L$ and in the time direction the Lipschitz constant grows linearly with $C(K,n)$. This finishes the proof. □

1.3.1 Approximating maps between spheres

Definition (triangulation)**.** Let K be a simplicial complex, X a topological space. A homeomorphism $\phi : K \to X$ is called a **triangulation** of X.

Let $f : X \to Y$ be a map between metric spaces, $\phi : (K, d_k) \to X$, $\psi : (J, d_j) \to Y$ - bi-lipschitz triangulations. Then if there is a simplicial approximation g_{simp} of $g := \phi \circ f \circ \psi^{-1}$ we call $f_{simp} := \phi^-1 \circ g_{simp} \circ \psi$ the simplicial approximation of f.

$$\begin{array}{ccc} X & \xrightarrow{f_{simp}} & Y \\ \phi\uparrow & & \psi\uparrow \\ (K, d_k) & \xrightarrow{g_{simp}} & (J, d_j) \end{array}$$

We now want to find triangulations for out spheres so that we can apply the simplicial approximation to them. We will pick triangulations that suit our purposes for the main result.

We start with a triangulation of S^n by the boundary of the unilateral $n+1$-simplex $\partial Delta^{n+1}$. This obviously uses very few vertices, thereby limiting the quality of our approximation, so let me try to motivate this choice of triangulation (the motivation will become apparent in the upcoming proofs): for the main result in case $m \geq n$ we need all vertices to be pairwise incident (i.e. any two vertices to share an edge). This will allow us to "approximate" the null-homotopy to some extent without any further geometric information about it. Furthermore, we would like simplices to be both equilateral and to have equal area. This already determines our triangulation uniquely (up to rotations). Additionally, $\partial Delta^{n+1}$ is defined for all dimensions (as opposed to, say, a triangulation of S^2 by the surface of icosahedron, that does not generalize well to other dimensions). $\partial Delta^{n+1}$ is bi-lipschitz homeomorphic to S^n with bi-lipschitz constants ~ 1 only depending on n.

It is notably more difficult to triangulate S^m so that its triangulation fits the theorem. In fact, the proof of the main theorem requires us to be able to triangulate not only S^m but the whole unit ball, B^{m+1}. One difficulty with it is that the Lipschitz constant of our approximation is determined entirely by the shortest side length(s) in the metric simplicial complex: $(length(s) \cdot L \cdot 1/c(n))^{-1}$. At the same time the side-length needs to be strictly less than $c(n)/L$ We formulate this as an exercise. See appendix for discussion.

Exercise. Find a family of geometric simplicial complexes $(K, |K|)$ together with a bi-lipschitz triangulation of the unit ball $Tri_L : |K| \to B^{m+1}$ such that the Lipschitz constant in the direction of the ball is less than 1. We require furthermore that each simplex in $(K, |K|)$ is bilipschitz homeomorphic to the unilateral simplex Δ^{m+1}. We require the maximum over Lipschitz constants of maps $\Delta \to B^{m+1}$ to be bounded by $c(n)/L$.

Theorem 1.6 (1.4 in)**.** If $m < n$ and $f : S^m \to S^n$ has Lipschizt constant L, then there is a null-homotopy with Lipschitz constant $\lesssim L$. In fact the null-homotopy has Lipschitz constant $\lesssim L$ in the S^m directions and $\lesssim 1$ in the [0,1] direction.

Proof. Consider the map between simplices instead. Approximate g using Theorem 1.5. g_{simp} is piecewise linear hence not surjective. Thus g_{simp} misses a whole simplex! Now back on the sphere simplicial approximation of f f_{simp} misses a ball of radius 1 in S^n. Applying Lemma 1.3 finishes the proof. □

Remark. Lemmas 1.2 and Theorem 1.5 show that Lipschitz maps for dim $m < n$ are null-homotopic independently. Both is stronger than what we need for the main proof. Note that we did not use Lemmas 1.1-1.2.

Remark. This bound is tight (up to constant manipulations). I did not verify this, Guth recommends it as an exercise.

2 Main theorem

Theorem ([3]). Suppose that n is odd[1] and $f : S^m \to S^n$ is a null-homotopic map with Lipschitz constant L. Then there is a null-homotopy $H : S^m \times [0,1] \to S^n$ with Lipschitz constant at most $C(m,n)L$.

2.1 Proof angle and outline

Here are some preparatory steps for the proof:

1. A null-homotopy from an arbitrary space X can be described as a map from the cone of X to the target. Recall that the **cone** of X **CX** is simply the cylinder of X with one end collapsed to a point: $X \times [0,1]/X \times \{0\}$. Cone of S^m is homeomorphic to the unit ball B^{m+1}. To see this, think of radius of the ball as the variable for the interval. Thus, a null-homotopy of f can be viewed as a map from B^{m+1}.
2. We first take a simplicial approximation of the map f (ref Theorem 1.5) using triangulations Tri_L and Tri_{S^n} as defined in the pre-discussion to Theorem 1.6. The simplicial approximation is still null-homotopic: glue the two homotopies together at f. Therefore it suffices to prove the claim for $f : S^m, Tri_L \to S^n, Tri_{S^n}$.
3. We endow B^{m+1} with a simplicial structure Tri_L using the procedure from the pre-discussion to Theorem 1.6. We fix an orientation on both simplicial complexes by ordering all the vertices.

We now have the following setup: we are given a map $h : (B^{m+1}, Tri_L) \to (S^n, Tri_{S^n})$ (the default null-homotopy for f) with $h_{|\partial B^{m+1}} \to (S^n, Tri_{S^n})$ simplicial with Lipschitz constant L (our map f).

Our strategy for the proof is to "straighten out" h skeleton by skeleton: we iteratively homotope h relative to the boundary (so that the restriction to f stays intact) to maps $h^0, h^1, h^2, ..., h^m = H$, where each map is $\lesssim L-$Lipschitz when restricted to the j-th skeleton of (B^{m+1}, Tri_L) until we reach the m-th skeleton. The resulting map h_m is the desired null-homotopy H.

There are several more key ingredients to the proof:

[1] We only need this assumption in the last part of the main proof, namely in the "Higher dimensional skeleta" subsection. The rest of the discussion applies for arbitrary n. In particular we are allowed to use S^2 for examples.

1. At each skeleton we leverage knowledge about the corresponding homotopy group.
2. CW-complexes, and, in particular, simplicial complexes are fibrations, i.e. they satisfy the homotopy extension property.

We treat the case $m < n$ separately to avoid the following problem: for $m = n - 1$ this strategy might fail for a constant map f with some complicated null-homotopy. See [1] for a more detailed discussion.

2.2 Homotoping skeleta up to n-1.

To recap, here is our setting: we are given a map h between metric simplicial complexes $h : (B^{m+1}, Tri_L) \to (S^n, Tri_{S^n})$ (a null-homotopy for f), which is simplicial on the boundary, i.e. $h|_{\partial B^{m+1}} = f : (S^m, Tri_L) \to (S^n, Tri_{S^n})$ simplicial. We denote the skeleta of (B^{m+1}, Tri_L) by $X^0 \subset X^1 \subset X^2 \cdots \subset X^m \subset X^{m+1} = (B^{m+1}, Tri_L)$.
Our goal for this section is to homotope h to a map h^{n-1} that agrees with h on the boundary and is simplicial on the $n-1$-skeleton, $h^{n-1}_{|X^{n-1}}$ simplicial. Then h^{n-1} is Lipschitz on the $n-1$-skeleton with the Lipschitz constant $L/c(n)$.

2.2.1 The zero-skeleton

We first aim to find a map h^0 that is simplicial on the 0-th skeleton. In order to do that we need to figure out where to map the vertices, i.e. to define a vertex map on X^0. This is a good moment to adress the choice of triangulation for S^n. If we could use some procedure similar to that of Theorem 1.5, we could ensure that the neighboring vertices are mapped to the same simplex. But we triangulate B^{m+1} without any knowledge of h, so h could map vertices of a simplex anywhere in the target, no matter how fine of a triangulation we prescribe to B^{m+1} at the beginning. Since we later want to build a simplicial map on the 1-skeleton we cannot have the vertex map taking incident vertices of Tri_L (i.e. vertices that share an edge) to non-incident vertices in the triangulated S^n. This forces us to choose a model complex where any pair of vertices share an edge. We also need the model simplicial complex to have the same dimension as the sphere. This makes the boundary of an $n+1$-simplex the unique choice of the model simplicial complex for the triangulation of the target sphere. The equilateral {n+1}-simplex has the most symmetry[2] and was convenient for proof of 1.5. From now on we will allow ourselves to write B^{m+1}, S^n meaning the metric simplicial complexes (B^{m+1}, Tri_L) and $(S^n, Tri_{S^n}) = \partial\Delta^{n+1}$ respectively.
$h(X^0)$ is a disjoint set of points in S^n. For each of the points we choose a vertex of S^n that is closest according to the metric it inherits from $\partial\Delta^{n+1}$ (the piece-wise straight

[2] As mentioned before, symmetry of the triangulation will be important to us later on. We will address it explicitly later in the proof why we would like the target to have equilateral simplices of equal area.

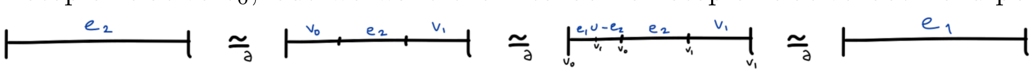
line Euclidean length that restricts to Euclidean metric on each simplex). S^n is path-connected, which gives a homotopy relative to the target vertex. To fit the more general framework, we could say that $\pi_0(S^n) = 0$. We could even pick a straight-line homotopy to keep this procedure as deterministic as possible. Having fixed the vertex homotopies, we denote the union of the vertex homotopies together with the "constant" homotopy[3] on the boundary sphere by $g^0 : S^m \cup X^0 \times [0,1] \to \partial\Delta^{n+1}$. The starting map of the homotopy g^0_0 agrees with h, i.e. $h_{|\partial B^{m+1} \cup X^0} = g^0_0$. We extend g^0 to the whole ball using the homotopy extension property to obtain $\bar{g}^0$:

$$\begin{tikzcd}
(S^m \cup X^0) \times [0,1] \bigcup_{\{} 0\} B^{m+1} \times \{0\} \arrow[r, "g^0 \cup h"] \arrow[d, hook] & \partial\Delta^{n+1} \\
B^{m+1} \times [0,1] \arrow[ur, dashed, "\bar{g}^0"'] &
\end{tikzcd}$$

$\bar{g}^0$ takes h to $h^0 := \bar{g}^0_1$.

2.2.2 The 1-skeleton

Let us explicitly do one more skeleton before stating the general version for the k-th skeleton. If $n = k = 1$ we are done and we move on to the next section of the proof. Else $m \geq n > 1$. We start with the map $h^0 : B^{m+1} \to \partial\Delta^{n+1}$, $h^0_{|X^0}$ is simplicial. Now, h^0 might take edges anywhere, only the end-points are prescribed. An edge image could wrap around S^n multiple times, making the Lipschitz constant huge. We want to homotope all edges to straight edges relative endpoints (that is, using homotopies that fix endpoints for $\forall t$). We want to show that such homotopies exist using the fact that the fundamental group of the sphere S^n is trivial, i.e. $\pi_1(S^n) = 1$ for $n > 1$. Let $\sigma = [v0, v1]$ be a 1-simplex in B^{m+1}, $e_1, e_2 : \sigma \cong [0,1] \to S^n$ be two edges in S^n that agree on the end-points (the images of vertices are determined by $h^0_{|X^0}$ and are not necessarily distinct in S^n). The edges we are interested in are $e_1 := h_0(\sigma)$ and e_2 - the simplicial image of σ, but we prove the claim for any two edges in S^n that agree on the boundary. We give the simplex σ an orientation from v_0 to v_1 (from 0 to 1 when viewed as the interval) and we denote by $-e_2$ the edge in the opposite direction. Then $e_1 \cup -e_2$ is a map from an oriented circle S^1 to S^n. Fixing v_0 as a base point makes $e_1 \cup -e_2$ into a representative of an element of the fundamental group $\pi_1(S^n)$, which is trivial because $n > 1$. Now, this already shows us that the edges are homotopic relative v_0, but we want them to be homotopic relative both end-points.



To fix this we homotope e_2 relative to boundary to a directed edge constant on the first

[3] By "constant" we mean the homotopy that does not change over time, $g^0_{0|S^m} = g^0_{t|S^m} = f$ for $\forall t \in [0,1]$. It is also sometimes called the "trivial" homotopy.

and last third of the interval, and run through the entire e_2 in the middle. We can now homotope the point corresponding to $t = 1/4$ to $e_1 \cup -e_2$ in the first third of the interval by what we have shown earlier. We now collapse $-e_2$ and e_2 and expand e_1 to the whole edge. All of these homotopies were done relative boundary.

We define the homotopy g^1 on $X^1 \cup S^m$ - the union of edge homotopies relative the 0-skeleton and the boundary sphere S^m and extend it to the ball using the homotopy extension property:

$$\begin{tikzcd}
(S^m \cup X^1) \times [0,1] \bigcup_{\{} 0\} B^{m+1} \times \{0\} \arrow[r, "g^1 \cup h^0"] \arrow[d, hook] & \partial\Delta^{n+1} \\
B^{m+1} \times [0,1] \arrow[ur, dashed, "\bar{g}^1"'] &
\end{tikzcd}$$

The 1-end of the obtained homotopy $barg^1_1$ is the desired map h^1. Its restriction to the 1-skeleton, $h^1_{|X^1}$, is simplicial and therefore Lipschitz. The Lipschitz constant is $L/c(n) > L$ (strictly greater but up to a constant depending on n). It is worth noting that we have no interest in the geometry of the homotopy $\bar{g}^1$ between h^0 and h^1. In particular, we do not know if it is Lipschitz continuous, and it is completely irrelevant. We only construct that homotopy in order to construct the map h^1.

This finishes the discussion of the 1-skeleton. Let us now repeat the argument in full generality.

2.2.3 The k-th skeleton (for k up to $n-1$)

We have constructed $h^{k-1} : B^{m+1} \to S^n$, which is the null-homotopy $S^m \times [0,1] \to S^n$, where $h^{k-1}_0 = f$ and t is the radius of the unit ball. h^{k-1} is simplicial on the $k-1$-skeleton X^{k-1}. We aim to construct h^k, which is simplicial on the k-th skeleton, $k < n$. This requires us to straighten out each k-simplex σ^k of B^{m+1} relative its boundary, i.e. to homotope $h^{k-1}_{|\sigma}$ to a simplicial map relative boundary. As before, we can do so because the k-th homotopy group of the target sphere, $\pi_k(S^n)$, is trivial and a pair of simplices that represent the same homotopy group element are homotopic relative boundary. The next lemma states this fact in full generality.

Observation. Let $\lambda : \Delta^k \to S^n$, then its restriction to the boundary is null-homotopic. Fix a null-homotopy H. Quotienting out the boundary and its image descends to a map between spheres, $\bar{\lambda}$. Equipped with an orientation it represents an element of the homotopy group $\pi_k(S^n)$.

$$
\begin{tikzcd}
D^k \arrow[r, "\lambda"] \arrow[d] & S^n \\
D^k/\partial D^k \arrow[ur, "\lambda \cup H"'] &
\end{tikzcd}
$$

Lemma 2.1 (homotoping simplices relative boundary)**.** *Let $\lambda_1, \lambda_2 : D^k \to S^n$ be two maps that agree on the boundary S^{k-1}. Then the following are equivalent:*

(i). The disk maps are homotopic relative boundary sphere, $\lambda_1 \simeq_\partial \lambda_2$

(ii). Quotient maps over the boundary $\bar{\lambda_1}$, $\bar{\lambda_1}$ are in the same homotopy class of $\pi_k(S^n)$

(iii). The difference $\lambda_1 \cup_\partial -\lambda_2$ is null-homotopic.

Proof. Clearly, $(i) \Rightarrow (ii)$ (collapse the boundary); $(i) \Rightarrow (iii)$ (homotope λ_1 to λ_2, then collapse $\lambda_2 \cup_\partial -\lambda_2$); $(ii) \Rightarrow (i)$:
Fix a null-homotopy H of $\partial\Delta^k$ ($H_0 = Id$). Parametrize the disk D^k by radius as cone of S^{k-1}. Shrink λ to radius $1/2$, extend the rest to be the same as boundary, i.e. set $\lambda_{1|\partial} \forall r \in [1/2, 1]$. Collapse at $r = 3/4$ and replace boundary mapping with H for $r \in [1/2, 3/4]$, $-H$ for $r \in [3/4, 1]$. For $r \in [0, 3/4]$ the map is now $\lambda_2 \cup H$, homotope that part to $\lambda_2 \cup H$. Now collapse $H \cup -H$ to obtain λ_2. All homotopies were constructed relative boundary. $(iii) \Rightarrow (i)$:
Shrink λ_1 and collapse at $r = 3/4$ as above, pasting in $-\lambda_2 \cup \lambda_2$ for $r \in [1/2, 1]$. For $r \in [0, 3/4]$ the map collapses, leaving λ_2. ☐

We have shown existence of homotopies of k-simplices to simplicial maps relative boundaries. We now construct the homotopy g^k on $X^k \cup S^m$ as the union of those homotopies, relative $X^k \cup S^m$. We can extend it to the unit ball by the H.E.P.

$$
\begin{tikzcd}
(S^m \cup X^k) \times [0,1] \bigcup_{\{0\}} B^{m+1} \times \{0\} \arrow[r, "g^k \cup h^{k-1}"] \arrow[d, hook] & \partial\Delta^{n+1} \\
B^{m+1} \times [0,1] \arrow[ur, dashed, "\bar{g}^k"'] &
\end{tikzcd}
$$

We proceed this way until the $n-1$-st skeleton, constructing the desired map h^{n-1}. Its restriction to the $n-1$-st skeleton X^{n-1} is simplicial and thus Lipschitz continuous with Lipschitz constant L.

2.3 The n-skeleton

At the n-th skeleton we run into a problem. Namely, for two maps λ_1, λ_2 from an n-simplex Δ of B^{m+1} that we glue at the boundary, $\lambda_1 \cup -\lambda_2 : S^n \to S^n$ is a self-map of the $n-sphere$ and has degree defined for it. If this map has non-zero degree then the maps are no longer homotopic (not even relative some base point), so there is a good chance that we cannot homotope $h^{n-1}_{|\Delta}$ to a simplicial map on Δ. To investigate this matter further we introduce a new notion of degree, with the initial motivation of considering maps from Δ individually (and without having to fix a boundary homotopy as in Lemma 2.1).

Definition (relative degree)**.** We define **relative degree** of a map $\lambda : \Delta \to S^n$ as

$$\omega_\lambda(\Delta) := \text{relative degree } (\lambda) := \int_\Delta g^* dvol_{S^n},$$

where $dvol^n_S$ denotes the volume form on $(S^n, Tri_{S^n}) = \partial\Delta^{n+1}$ with integral 1, i.e. $\int_{\partial\Delta^{n+1}} dvol_{S^n} = 1$. $g^*dvol_{S^n}$ denotes the pullback n-form on B^{m+1}

For a map λ that quotients over the n-sphere, or in other words, that maps the boundary $\partial\Delta$ to a single vertex of S^n, the usual degree $\deg(\lambda)$ is defined for it. We choose the volume form $dvol_{S^n}$ so that Relative degree (λ) counts the number of times λ wraps around the target sphere, i.e. so that the degrees match: Relative degree $(\lambda) = \deg(\lambda)$.

For a λ simplicial on the boundary the relative degree counts the number of simpices λ covers divided by the number of n-simplices in the target sphere[4]. That is, for $n = 2$ S^2 has four 2-faces, then each covered face adds 1/4-th to the relative degree. Relative degree is invariant up to homotopies relative boundary, in fact the following claim holds:

Claim. *The difference of relative degrees is an integer. Two maps from n-simplices that agree on the boundary are homotopic relative boundary if and only if they have the same relative degree.*

Proof. Glue the simplex maps in question along the boundary.
$\omega_{\lambda_1}(\Delta) - \omega_{\lambda_2}(\Delta)$
$= \#$ simplices traversed by $\lambda_1 - \#$ simplices traversed by λ_2
$= \#$ simplices traversed by $\lambda_1 \cup -\lambda_2 = deg(\lambda_1 \cup -\lambda_2)$.
The second part of the claim follows by Lemma 2.1. □

Thus, relative degrees determine homotopy classes of simplex maps relative boundary. For maps simplicial on the boundary the homotopy classes rel boundary are then precisely

[4]Equal simplex area allows us to use relative degree to "count" simplices (by adding an equal fraction $1/(n+2)$ for each covered simplex).

$(n+2)^{-1} \cdot \mathbb{Z}$. One thing this shows us is that homotoping relative X^{n-1} will not get us anywhere - suppose for each relative degree value we found some Lipschitz map on the simplex with that degree. We could then produce a h^n that would be Lipschitz on that skeleton, but we could not bound the constant - since degrees can go arbitrarily high the simplex could have to stretch over arbitrarily many simplices, making the Lipschitz constant arbitrarily high.

If we relaxed the notion of homotoping relative boundary to homotoping between maps that agree on the boundary agree on the boundary, we could do much better! Consider the case $n = 1$. Given a simplex map of relative degree, say $100\frac{1}{3}$ if we relax the condition of homotoping it relative boundary we could unwrap the endpoint against the orientation 100 full circles to achieve the relative degree of $1/3$. But this would of course affect relative degree of the neighboring edges. To develop this idea further we first have to look into how exactly relative degrees of neighboring edges are related.

Since $m \geq n$, there are $n+1$ simplices in B^{m+1}. The image of the boundary of an $n+1$-simplex Δ^{n+1} is contractible (since the simplex is), therefore it must have degree zero. Hence, $deg(\partial\Delta^{n+1}) = 0 = \omega_\lambda(\partial\Delta^{n+1}) = \sum_{\Delta_i}(-1)^i\omega_\lambda(\Delta_i)$, the latter being the oriented sum of relative degrees of the boundary simplices. This is a good time to notice that ω_λ is a real-valued cochain. The last expression is by definition $\delta\omega$ - the co-boundary of ω. Hence, what we have just shown is that ω_λ is a cocycle. Let us restate the same fact in the language of differential forms using this classical result:

Theorem 2.2 (Stokes)**.** Let ω be a differential n-form on an orientable manifold with boundary Ω^{n+1}, then

$$\int_{\partial\Omega^n} \omega = \int_{\Omega^n} \mathrm{d}\omega$$

Claim. *Let $\lambda : \Delta^n \to S^n$, $\omega_\lambda(\Delta)$ the relative degree, then ω_λ is a co-cycle.*

Proof. We give a second proof, this time using Stokes theorem:

$$\delta\omega_\lambda(\Delta^{n+1}) = \int_{\partial\Delta^{n+1}} \lambda^* dvolS^n \overset{\text{Stokes}}{=\!=\!=} \int_{\Delta^{n+1}} \mathrm{d}\lambda^* dvol_{S^n} = \int_{\Delta^{n+1}} \lambda^* \mathrm{d}dvol_{S^n} = 0,$$

since $dvolS^n$ is a closed form because it is an n-form on $S^n \Rightarrow \mathrm{d}dvolS^n = 0$. ☐

This claims tells us that degrees on the boundary of every Dˆ{n+1} must add up to zero. Let us continue our Sˆ1 unwrapping example with a specific triangulation and specific relative degrees:

Sadly, unwrapping does not always allow us to homotope maps that are simplicial on the n-skeleton. Consider, for instance the case $n = 2$. If any map on the interior has relative degree $1/2$ modulo $\mathbb{Z}$ then it cannot be unwrapped to a simplicial map. But it turns out

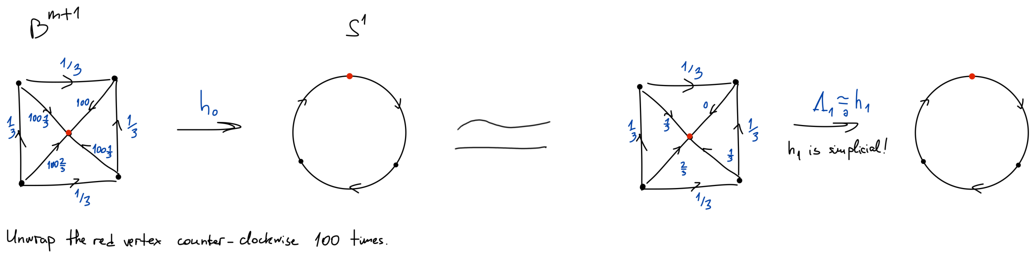


Figure 2.1: Unwrapping a map

that unwrapping does allows us to bound the relative degree sufficiently. The rest of the section develops this idea into a proof.

We now want to try and capture the idea of unwrapping formally. First we construct a toy homotopy - it will allow us to develop all the concepts we require to finish the proof. We have an orientation on S^n, and we pick some $n-1$-simplex in the interior of B^{m+1} and unwrap it against the orientation of S^n 100 times relative its boundary. Let us denote this dummy homotopy by $\Lambda : B^{m+1} \times [n, n+1] \to S^n$. It is a homotopy rel $X^{n-2} \cup S^m$. For Λ restricted to an $n-1$-simplex we would like to know its degree - it would tell us how much this simplex has wrapped around the target sphere. We use prism decomposition (as defined in proof of 2.10 in [2]) to view the prism of our $n-1$-simplex as a sum of oriented n-simplices. This means that relative degree is defined on $\Delta^{n-1} \times [n-1, n]$. We denote this degree by $\alpha_\Lambda(\Delta^{n-1})$. One possible value of $\alpha_L ambda(\Delta^{n-1})$ is always zero, for instance if Λ does not move the $n-1$-simplex, which implies that $\alpha_L ambda$ is an integral cochain, i.e. it can only take integer values (since a difference of relative degrees must be an integer). Additionally we know that $\alpha_L ambda$ vanishes on the boundary of B^{m+1}.

Claim: Let $\Lambda : B^{m+1} \times [n-1, n] \to S^n$ be a homotopy relative $X^{n-2} \cup S^m$. Then

$$\omega_{\Lambda_n} = \omega_{\Lambda_{n-1}} - \delta\alpha_\Lambda.$$

This formula is very intuitive. Coboundary of α just counts the total relative degree of wrapping/unwrapping done by each $n-1$-simplex of the boundary $\partial\Delta^n$ (the sum of relative degrees takes orientation into account). The formula basically states that the new relative degree is equal to the old relative degree minus unwrapping.

Proof. We prove the formula using Stokes theorem. We isolate the change in relative degree of a simplex Δ through unwrapping by homotopy Λ by looking at the boundary of the homotopy cylinder of Δ^n. The following integral is again zero by Stokes theorem:

$$0 = \int_{\partial(\Delta^n \times [n-1,n])} \Lambda^* dvol_{S^n}.$$

The integral breaks into three parts - the top $\omega_{\Lambda_n}(\Delta^n)$, the bottom $\omega_{\Lambda_{n-1}}(\Delta^n)$ and the sides add up to $\delta\omega$, which comes out with the minus sign for the repeating simplices on top and bottom to cancel out. So if the top Δ^n has one orientation on the boundary the top of the cylinder will have the opposite to avoid double counting. □

Unfortunately, we cannot guarantee that unwrapping will allow to reduce the absolute value of all relative degrees below 1. But the following lemma shows that unwrapping can bound relative degrees, that is, absolute values of all relative degrees are $\lesssim 1$.

Lemma 2.3. *Suppose ω is a real cocycle in $C^n(Tri_L, \mathbb{R})$, and $\|\omega\|_{L^\infty(\partial B^{m+1})} \leq 1$, then there is an integral $(n-1)$-cochain α which vanishes on ∂B^{m+1} so that*

$$\|\omega - \delta\alpha\|_{L^\infty} \lesssim 1.$$

Sadly, proving this lemma using estimates for extensions and primitives ended up being out of scope of this theses. We refer to Guth [1] for the proof. Let us use the lemma to finish the proof.

Lemma 2.4. *Given a relative degree ω there is a Lipschitz map $\lambda : \Delta^n \to S^n$ that realizes ω.*

Proof. One way to see that such maps exist in any degree is to start with a Lipschitz map $\lambda : \Delta^n \to S^n$ of a given relative degree $\omega_0 \in (-1, 1)$ - there are precisely $2(n+2)+1$ possible values (as there are $n+2$ n-faces in S^n). We can then pick an $n-1$-face and a Lipschitz homotopy W that wraps it once around the sphere (along the positive orientation). Now, for an arbitrary relative degree $\omega = \omega_0 + a$, $a \in \mathbb{Z}$ the map $\lambda \cup aW$ is a Lipschitz map of the desired relative degree. □

Lemma 2.5 (bounded relative degree)**.** *Fix m,n and let Λ be the homotopy that realizes the cochains from the previous lemma. Let $b(m,n)$ be the upper bound for the relative degree on all simplices of B^{m+1}, i.e. $\omega_{\Lambda_1}(\Delta^n) \leq b$. Then there is a map $h^n : (B^{m+1}, Tri_L) \to \partial\Delta^{n+1}$ that is Lipschitz with Lipschitz constant $\lesssim L$.*

Proof. Our goal is to take the map Λ_n and improve it to satisfy the Lipschitz constant. We essentially use the same approach as we did for all the lower dimensional skeleta: we homotope maps on simplices of B^{m+1} to "controlled" maps relative $S^m \cup X^{n-1}$, then extend those homotopies to the whole ball, obtaining a homotopy $\bar{g}^n$ between maps Λ_n and h^n. The main difference is that this time the "controlled" maps are not (in general) simplicial, but merely some fixed maps of a given relative degree.
For each possible relative degree we fix a representative Lipschitz map. We can show preference for maps with a lower Lipschitz constant, but we can also choose completely arbitrary Lipschitz maps. Now let A be the maximum over all Lipschitz constants of the representative maps. Then the map h^n we aim to contruct will have Lipschitz constant

$\lesssim AL$ (with the approximation coming from $\psi^n(L)$ - the maximum Lipschitz constant of homeomorhisms $\Delta^n \to \sigma$, where σ is an n simplex in Tri_L). Thus we have a list of representative maps of all possible relative degrees. For each n-simplex Δ in B^{m+1} we homotope $\Lambda_1|\Delta$ to the map from the list of matching relative degree (the homotopy is relative $S^m \cup X^{n-1}$). We denote the union of such homotopies with domain in $S^m \cup X^n$ by g^n. The usual application of the H.E.P. yields the desired h^n. □

2.3.1 Higher dimensional skeleta

Now, suppose n is odd. By Serre's finiteness theorem, higher homotopy groups of the target sphere $\pi_k(S^n)$ for $k > n$ are all finite. We use this fact to make a similar finiteness argument as in the Lemma above.

The n+1-st skeleton

We know that h^n restricted to an n-simplex comes from a finite list of maps that are simplicial on the boundary of $\partial\Delta^n$. It suffices to only consider maps that agree with one selected simplex of $S^n = \partial\Delta^{n+1}$, considering a different simplex would be the same up to rotation. So really the size of our list here is the possible values of relative degree of such maps.
Now, we want to consider a map from an n+1-simplex of B^{m+1}. The restriction to its boundary $h^n_{|\partial\Delta^{n+1}}$ comes from a finite list of maps. For each of the maps g_a that agree with it on the boundary, the gluing at the equator $h^n_{|\Delta^{n+1}} \cup -g_a : S^{n+1} \to S^n$ represents an element of $\pi_{n+1}(S^n)$. The homotopy group is finite, meaning we only need $|\pi_{n+1}(S^n)|$ many maps that agree with $h^n_{|\Delta^{n+1}}$ on the boundary to guarantee that there is a g_a such that $h^n_{|\Delta^{n+1}} \cup -g_a$ maps to zero in the homotopy group. Then by Lemma 2.1 the two maps are homotopic rel boundary, and we can homotope $h^n_{|\Delta^{n+1}}$ to a map with a controlled Lipschitz constant from a finite pre-determined list of maps. The usual H.E.P. procedure yields the desired map h^{n+1} that is $\lesssim L$-Lipschitz on the {n+1}-skeleton.

The general case

Suppose we have h^{k-1}, where $k - 1 \geq n + 1$. h^{k-1} restricted to a $k-1$-simplex comes from a finite list of maps. Thus, $h^{k-1}_{|\partial\Delta^k}$ comes from a finite list of maps. The k-th homotopy group $\pi_k(S^n)$ is also finite, meaning for each boundary mapping there are only finitely many fixed maps we need to present to choose from (multiplying the cardinalities determines the size of the new list of maps). Each of the maps on the list we select as before, aiming to minimize the Lipschitz constant. The usual procedure yields $h^k : B^{m+1} \to S^n$ that is Lipschitz on the k-skeleton with Lipschitz constant $\lesssim L$. When k equals $m + 1$ this finishes the main proof.

3 Appendix

3.1 Covering a sphere by spherical caps.

Here we provide a direct and more thourough argument to cover the sphere S^m by spherical caps. In this argument we do not drop constants. We start with a quick introduction to the topic of covering and packing:

Definition (Covering, packing)**.** Let (X, d) be a metric space, $K \subseteq X$.

A collection C of points in X is called an **ρ-covering** of K if K is contained in the union of ρ-balls around points in C, i.e. $K \subseteq \cup_{p \in C} B_\rho(p)$. In other words, for $\forall x \in K$ there is a p in C such that $d(p, x) \leq \rho$. Note that we do not require the centers of ρ-balls to lie in K. Such a covering is also called an **external ρ-covering**. The minimum ρ-covering cardinality is called the **(external) covering number** of K denoted $N(K, d, \rho)$ or simply $N(\rho)$.

A collection P of points in K is called an **ρ-packing** if for $\forall p, q \in P$ $d(p, q) > \rho$. The maximum packing cardinality is called the **packing number** of K and is denoted by $M(K, d, \rho)$ or simply $M(\rho)$.

Observation. Let P be a ρ-packing. Then the balls $B_{1/2\rho}(p)$ are pairwise disjoint (triangle inequality).
If P is maximal, then P is also an ρ-covering (by contraposition). In particular, this implies $N(\rho) \leq M(\rho)$

Claim. $M(2\rho) \leq N(\rho) \leq M(\rho)$

Proof. The second inequality follows from the observation above. To prove the first inequality, assume $M(2\rho) > N(\rho)$. Then by the pigeon-hole principle there are two points x, y of the packaging contained in the same ρ-ball of the cover. By triangle inequality this yields a contradiction. $\square$

We are now going to provide an upper bound for the covering number of a sphere. Geomtrically, we will be covering a sphere by spherical caps of equal size. We are interested in exploring the relationship between the size of the caps and the covering number.

Definition (spherical cap)**.** A closed **spherical cap** is the smaller portion of a unit sphere S^m cut off by a plane (including the boundary). Formally, the spherical cap with angle $\rho \in (0, \pi/2]$ and center $x \in S^m$ is given by

$$cap(x, \rho) = \{y \in S^m : \langle x, y\rangle \geq cos\rho\}.$$

We will call a spherical cap with a polar angle ρ a $\boldsymbol{\rho}$**-cap**. Since we are dealing with a unit sphere, the polar angle in radians is precisely the length of any geodesic from the center (the tip) of the cap to its edge.

Figure 1.1

Lemma 3.1. *The covering number of a sphere* $N(S^m, d, \rho) \lesssim \rho^{-m}$*, where* d *is the length-metric. That is, for any* $\rho > 0$*, the sphere* S^m *can be covered by at most (up to a constant)* $1/\rho^m$ ρ*-caps.*

Remark: It is sufficient for us to show the upper bound up to a constant $c(m)$. The reason for that is that in later arguments we will be able to choose the radius of the cover small enough that any constant $c(m, n)$ can be “neutralized” for our purposes, so long as the quantities we omit do not vary with ρ.

Proof. Let us first consider a maximal packing of our sphere with spherical caps. For any such packing the total volume of spherical caps cannot exceed the volume[^volume] of the sphere. As the caps in a packing are disjoint,

$$M(\rho) \leq \frac{\omega^m(S^m)}{\omega^m(\rho\text{-cap})}.$$

Now, S^m can be covered by exactly two $\frac{\pi}{2}$-caps, so $\omega^m(S^m) = 2\omega^m(\frac{\pi}{2}\text{-cap})$. Rewriting the inequality above we get:

$$M(\rho) \leq \frac{2\omega^m(\frac{\pi}{2}\text{-cap})}{\omega^m(\rho\text{-cap})}. \tag{3.1}$$

We would like to replace the ρ-caps in the inequality by ρ-disks, as they scale easier with ρ, and that would allow us to reduce the fraction. Projecting the cap down onto the disk at its base will reduce the volume[^cap-size], i.e. $\omega^m(\rho\text{-cap}) \geq \omega^m(\sin\rho D^m)$. Dividing both sides by the m-volume of a ρ-disk and simplifying we obtain the following inequality:

$$\frac{1}{(\frac{\pi}{2})^m} \leq \frac{\sin^m\rho}{\rho^m} = \frac{\omega^m(\sin\rho D^m)}{\omega^m(\rho D^m)} \leq \frac{\omega^m(\rho\text{-cap})}{\omega^m(\rho D^m)},$$

where $\rho \in (0, \frac{\pi}{2}]$. Multiplying by $\left(\frac{\pi}{2}\right)^m$ we get:

$$1 \leq \left(\frac{\pi}{2}\right)^m \cdot \frac{\omega^m(\rho\text{-cap})}{\omega^m(\rho D^m)} \tag{3.2}$$

Multiplying inequality (3.1) by a term (3.2) greater than 1 on the right yields:

$$N(\rho) \leq M(\rho) \leq \frac{2\omega^m(\frac{\pi}{2}\text{-cap})}{\omega^m(\rho\text{-cap})} \leq \left(\frac{\pi}{2}\right)^m \cdot \frac{2\omega^m(\frac{\pi}{2}D^m)}{\omega^m(\rho D^m)} = \left(\frac{\pi}{2}\right)^{2m} \cdot \frac{2}{\rho^m} \sim \frac{1}{\rho^m}.$$

□

For proof of next Lemma 1.2 (explicit constants):

$$|C|\omega^n(2r\text{-cap}) \leq \left(\frac{\pi}{2}\right)^{2m-n} \cdot \frac{2}{\rho^m} \cdot (2r)^n \cdot \omega^n(\tfrac{\pi}{2}\text{-cap}) \leq \omega^n(S^n) = 2\omega^n(\tfrac{\pi}{2}\text{-cap}),$$

simplified, this becomes

$$\left(\frac{\pi}{2}\right)^{2m-n} \cdot \frac{(2r)^n}{\rho^m} \leq 1.$$

Using $r = L\rho$ and $m < n$ we choose $\rho > 0$ small enough to obey

$$\rho \leq \left(\frac{L}{\pi}\right)^{-\frac{n}{n-m}} \cdot \left(\frac{\pi}{2}\right)^{-\frac{2m}{n-m}}.$$

it then follows for r

$$r = L\rho \leq \left(\frac{\pi^2 L}{4}\right)^{-\frac{m}{n-m}} \cdot \pi^{\frac{n}{n-m}}.$$

3.1.1 Detour: manifolds with boundaries

We want to show that we can contract the target sphere S^n in a Lipschitz way. For that we need to construct a differentiable map between the cylinder of S^n and S^n. Reminder: the (topological) cylinder is the cartesian product with the interval. So we want a map between manifolds, both equipped with a metric. For the sake of consistency, we would prefer to equip both with the length metric. Naturally, we could take the product Riemannian metric. But the interval is not a manifold, nor is the (topological) cylinder! For it is strictly speaking not Euclidean at the points on the boundary - in the interval dimension we can only move in one direction from the boundary $M \times \{0\}$. At those boundary points we do, however, have homeomorphism to the Euclidean half-space $\mathbb{R}^{m+1}$. We would like to relax the usual definition of a manifold to include manifolds with boundary:

Definition (manifold with boundary)**.** The only difference is charts are allowed to be half-spaces. A definition can be found in any reference for calculus on manifolds. See e.g. [4]

Thus, the old manifolds are just manifolds with an empty boundary. Notably, the relaxed definition encompasses basic topological objects, such as the (closed) unit disk, the Moebius strip and topological cylinders as manifolds, the latter allowing us to consider differentiable homotopies.

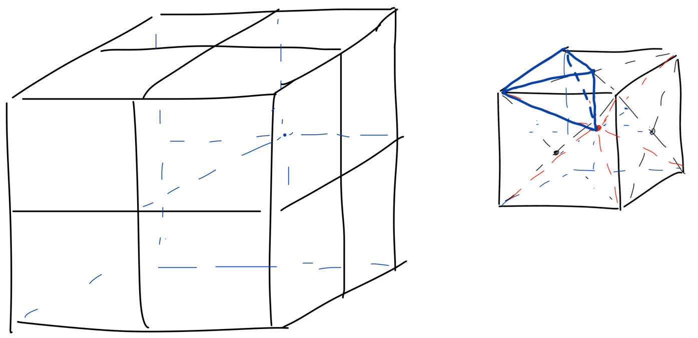
All the usual definitions of dimension, tangent spaces etc apply to manifolds with boundaries. A manifold with a boundary also always admits a Riemannian metric:

Definition (Double)**.** A double a manifold with a boundary is a union of two copies of the manifold glued along their boundaries. A double is a manifold without a boundary.

Observation. A double of a manifold M admits a Reimannian metric. Selecting a metric and restricting to M yields a Riemmanian metric on M.

3.2 On "bi-lipschitz to equilateral" triangulation of a sphere/ball

The solution is to subdivide a hypercube into smaller hypercubes (essentially breaking a cube up with a uniform grid). Then we subdivide each face by connecting barycenters to vertices until we have a triangulation. Cube is obviously bilipschitz to a ball.



One issue that arises is the grid step. It is especially noticeable for larger grids, which implies smaller Lipschitz constants. This dependency between grid jumps and Lipschitz constants allows us to write the jump off as a constant multiple: For a natural number $k \in \mathbb{N}$ we end up wanting a value of side-length $\frac{c(n)}{L}$, that sits between two available side-lengths:

$$\frac{1}{k} \leq \frac{c(n)}{L} \leq \frac{1}{k-1}$$

Instead we solve the problem $\frac{L}{k-1} \cdot C \geq 1$ to find the constant that adjusts for the jump in values.

We could probably use greedy subdivision with moving the points heuristically after to achieve uniform lengths, but it would become a stochastic/numerical problem. It would be notably harder to show and it wouldn't eliminate the problem of the step entirely, albeit it would make the jumps considerably smoother. It is also only really relevant for improving the constant multiple for the simplicial approximation of f, not of the homotopy - the degrees there will create massive constants where such small improvements are of no relevance. Nonetheless, simplicial approximation may be interesting on its own and I think the problem itself is an interesting in its own right.

Almost equilateral triangulations of spaces are called **a mesh** of that space. [5] deals with meshes for hyperspaces. There is also a software package that handles triangulation of a hypersphere in an optimal way. There are also other meshes, I haven't looked into whether it can triangulate hyperballs too. https://rdrr.io/cran/mvmesh/

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