Optimal Portfoilio Selection

Project on course Optimization Methods and NLA

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Introduction

1. **Asset** — an investment instrument that can be bought and sold.
2. Suppose we purchase an asset for $x(0)$ dollars on one date and then later sell it for $x(1)$ dollars and there are N risky assets, whose **rates of returns** are given by random Variables $R_1, \dots, R_N$:

$$R_n = \frac{x_n(1) - x_n(0)}{x_n(0)}, \ n = 1, \dots, N$$

1. Let $w = (w_1, \dots, w_N)^T, \ w_n$ denotes the proportion of wealth invested in asset n. The **rate of return of the portfolio** is

$$R_P = \sum_{n=1}^{N} w_n R_n, \ \text{where} \ \sum_{n=1}^{N} w_n \leq 1, \ w_n \geq 0$$

The goal: to choose the portfolio weighting factors **optimally** !

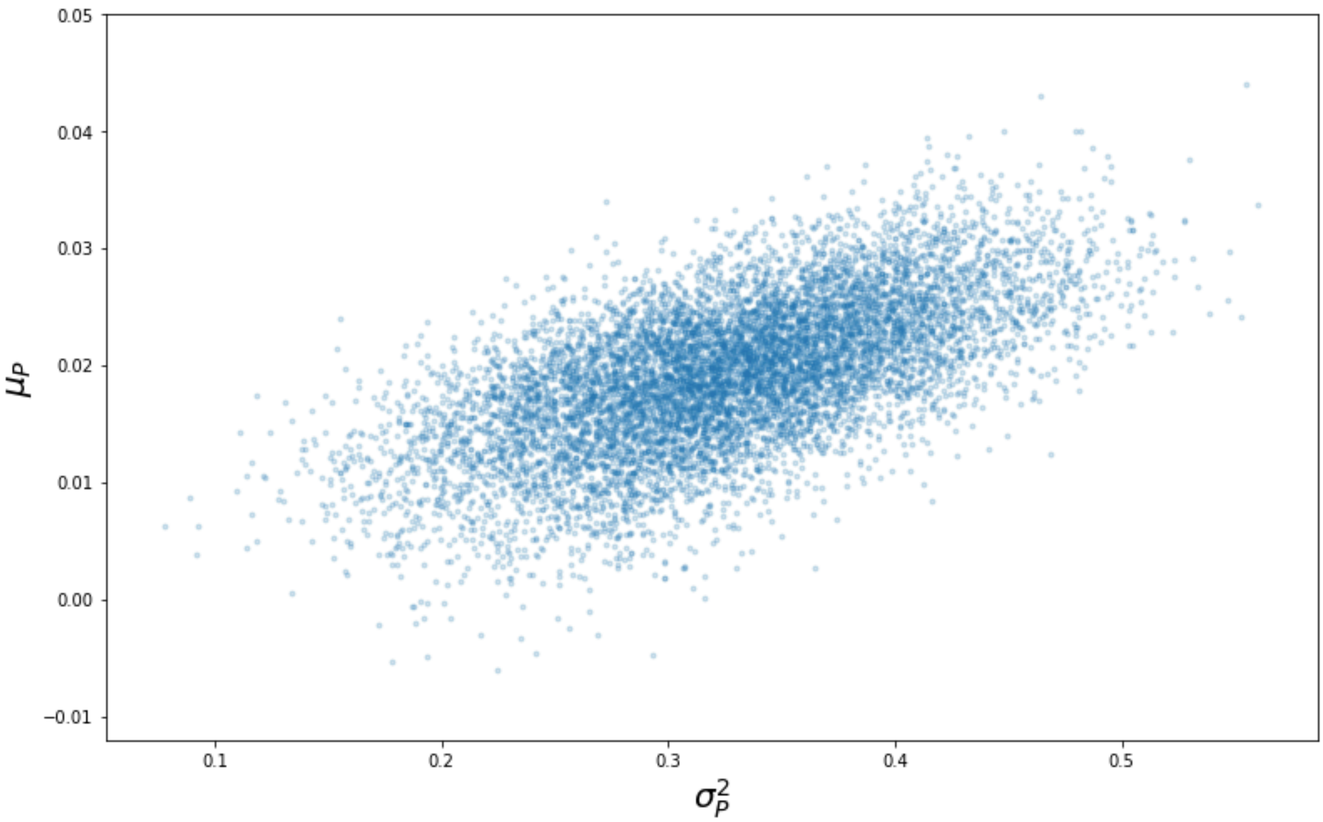
Markowitz Mean-Variance Analysis

For a given portfolio w, we can compute the mean and the variance of the portfolio return as:

$$\mathbb{E}[R] = \sum_{i=1}^{N} w_i R_i = w^T R$$

$$\mathrm{Var[R]} = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i \sigma_{ij} w_j = w^T \Sigma w$$

*For **arbitrary** weigths we will get the following results of means and variances:*



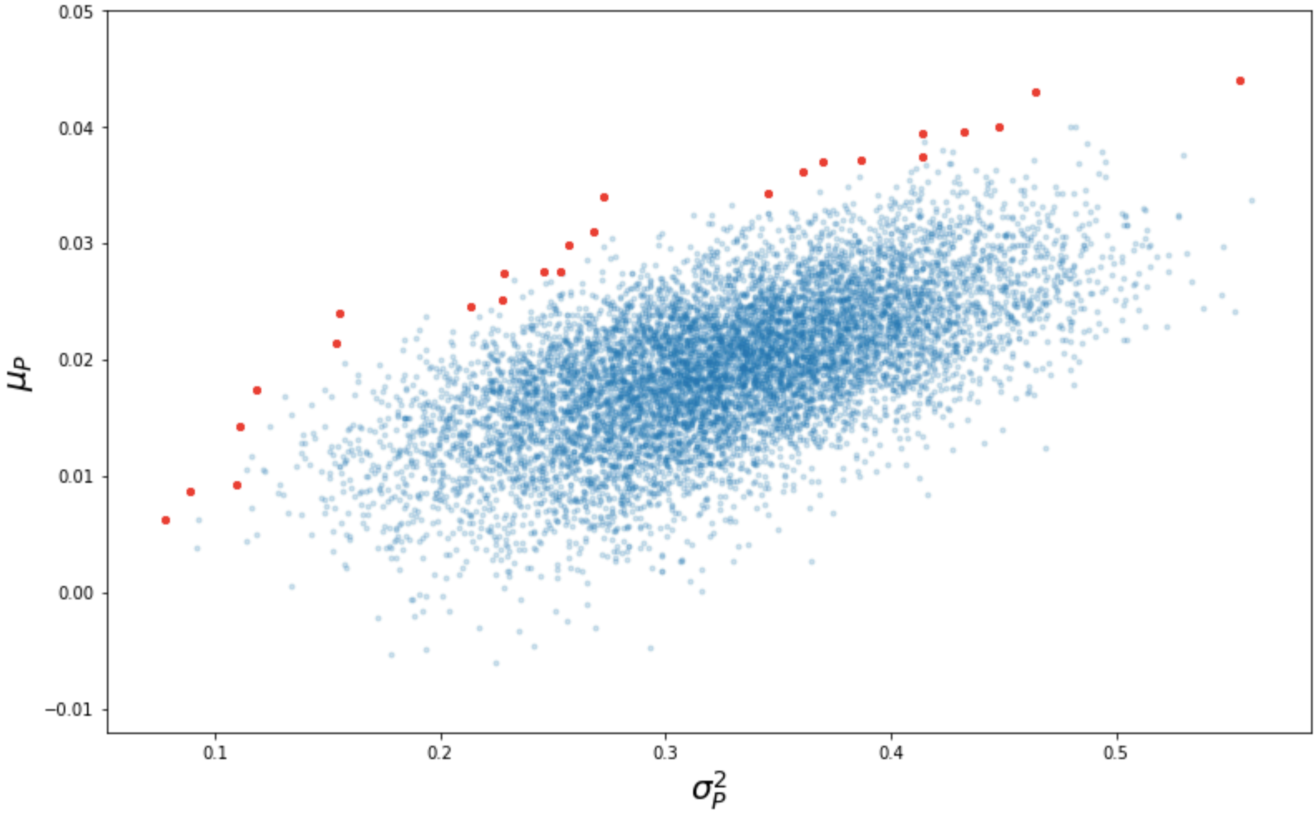
What is the efficient portfolio?

Mean-Variance efficient portfolio

A portfolio w^* is said to be **mean-variance efficient** if there exists no portfolio w with:

$$\mu_P \geq \mu^* \text{ and } \sigma_P^2 \leq {\sigma^*}^2$$

That is, you cannot find a portfolio that has a higher return and lower risk than those for an efficient portfolio.

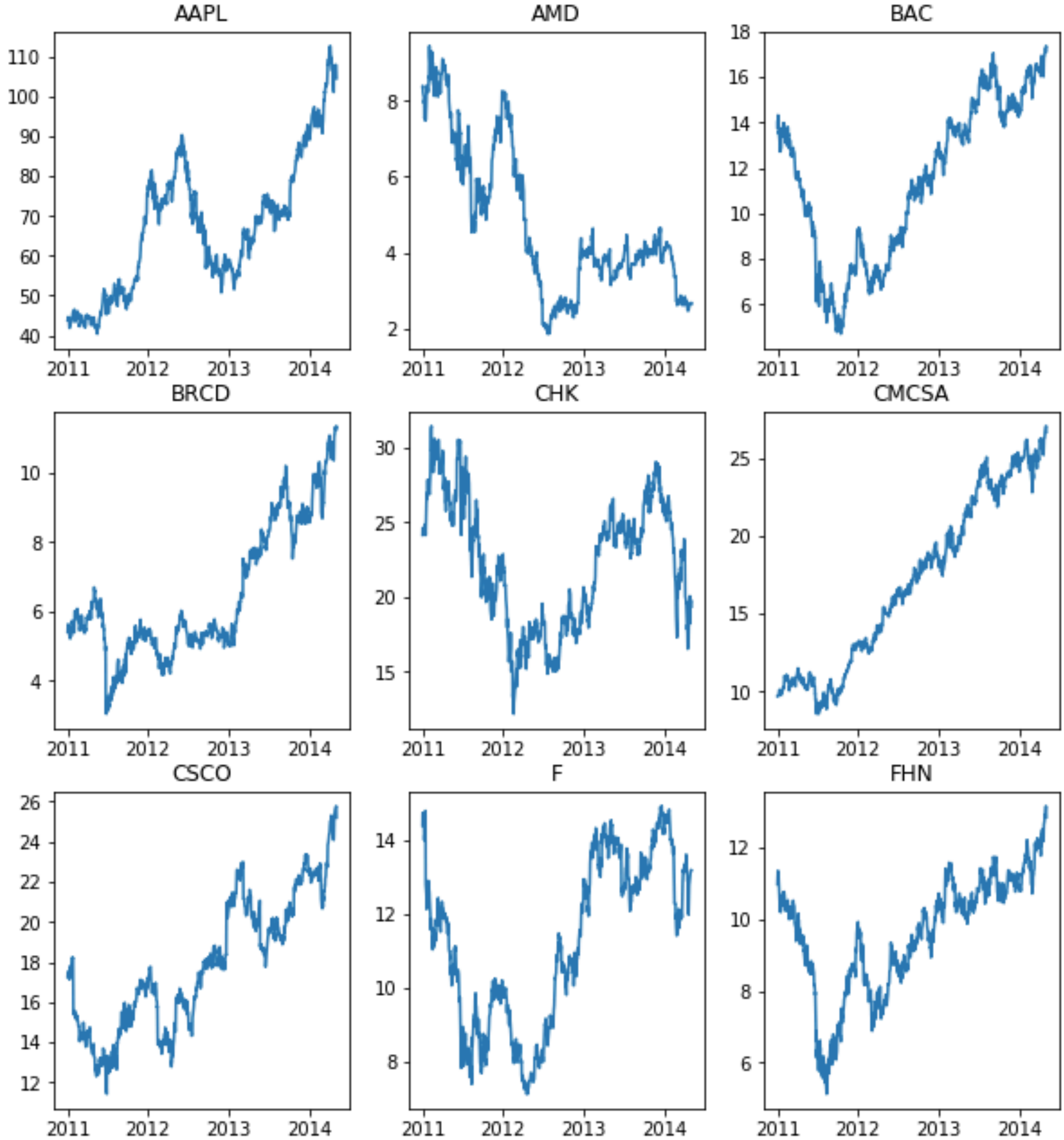


In [1]:
```
import numpy as np
import pandas as pd
import os
import matplotlib.pyplot as plt
from datetime import datetime
import cvxpy as cvx
from cvxpy import *
%matplotlib inline
```

In [2]:

```
# Name of train/test dataset. 'data', 'nyse_each_50','nyse_each_10' or 'n
yse'
DATASET_NAME = 'data'

# Period, on which we will invest.
PERIOD = 60

# Set of lambda parameters, which we will check.
LAMBDA_RANGE = [0.01, 0.1, 0.5, 1, 2]
```

Dataset: historical 2014 prices for p = 22 securities



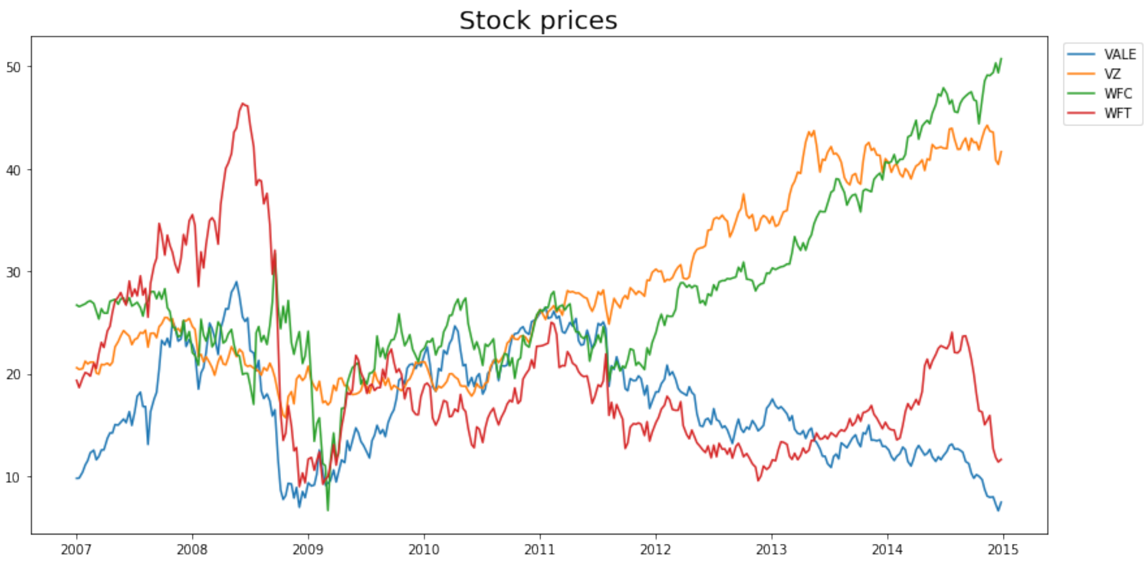
In [3]:
```
data_folder = os.path.join(os.path.dirname(os.getcwd()), 'data')
train_file = os.path.join(data_folder, DATASET_NAME + '_train.csv')
test_file = os.path.join(data_folder, DATASET_NAME + '_test.csv')
```

In [4]:
```
# Reading data.
data = pd.read_csv(train_file)
```

In [5]:
```
# Extract prices and dates.

sequrity_ids = data.columns[1:]
dates = [datetime.strptime(d, '%Y-%m-%d') for d in data['Date']]
prices = np.array(data[sequrity_ids]).T

# Covariation matrix.
cov = np.corrcoef(prices)

# Number of sequrities.
N = len(sequrity_ids)

# Number of time moments in train data.
T = len(dates)
```

Data: prices from 2007 to 2015 years

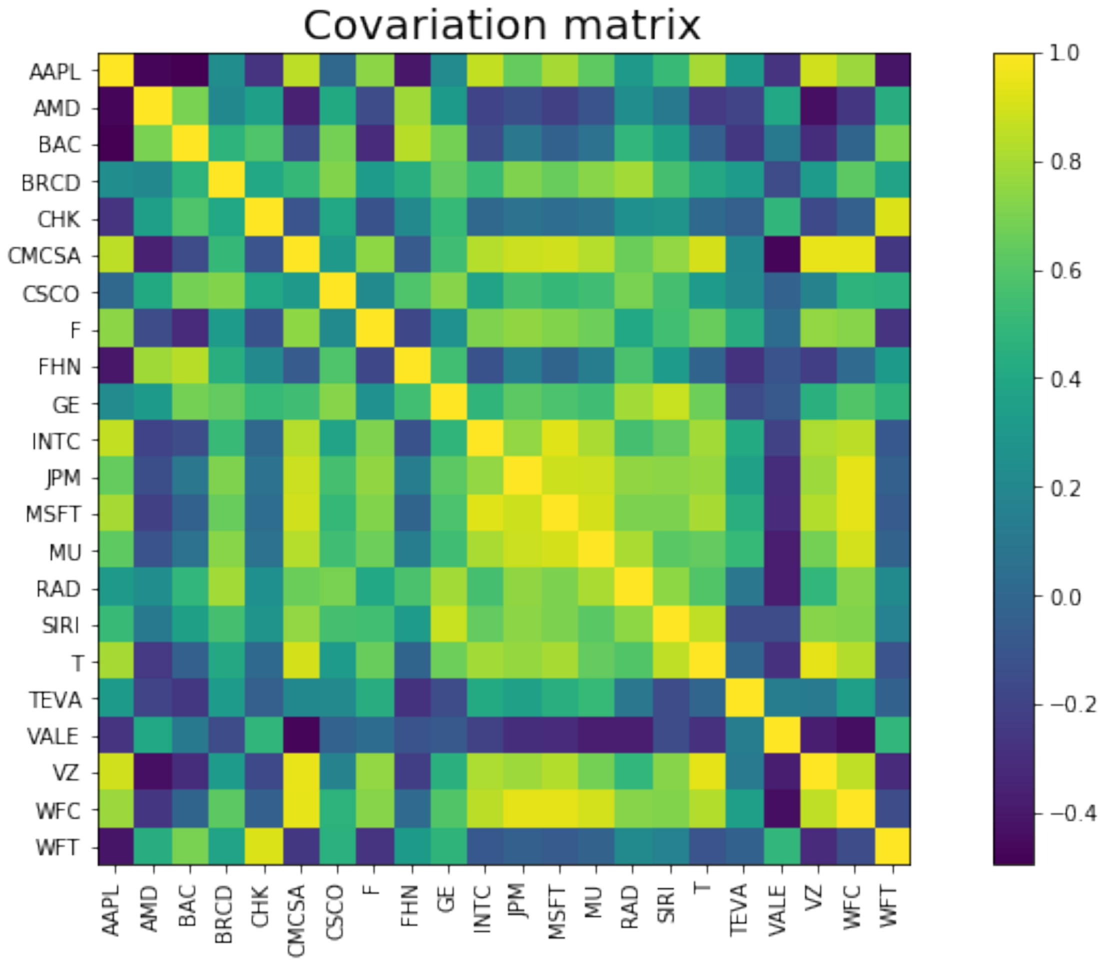
In [6]:
```
# Visualize some prices.

plt.figure(figsize=(14,7))
seq_to_show = sequrity_ids[-4:]

for seq in seq_to_show:
    plt.plot(dates[::6], data[seq][::6], label = seq)
plt.legend( bbox_to_anchor=(1.1, 1))
plt.title("Stock prices", fontsize=20)
plt.show()
```



Covariation matrix

In [7]:

```
# Visualize covariation matrix.
plt.figure(figsize=(14,7))
plt.imshow(cov, interpolation='none')
plt.colorbar()
plt.xticks(range(len(sequrity_ids)), sequrity_ids, fontsize=10,rotation='
vertical')
plt.yticks(range(len(sequrity_ids)), sequrity_ids, fontsize=10,rotation='
horizontal')
plt.title("Covariation matrix", fontsize=20)
plt.show()
```



Prediction of future prices

Here we use **ARIMA** (autoregressive integrated moving average) model to predict future points in the series.

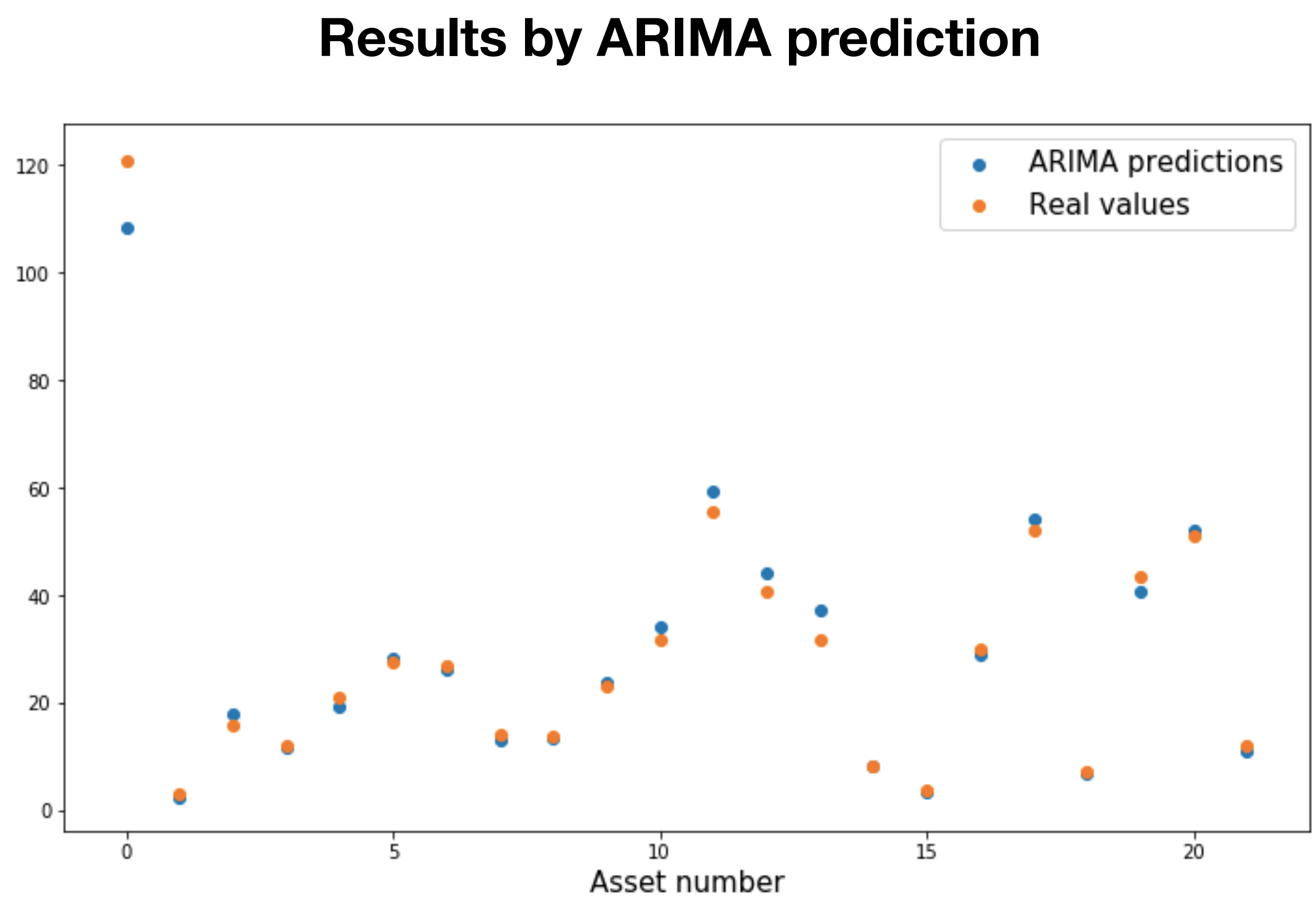
In [8]:
```
from statsmodels.tsa.arima_model import ARIMA
from statsmodels.tsa.seasonal import seasonal_decompose
from sklearn.metrics import mean_squared_error
import matplotlib.pylab as plt

def test_stationarity(timeseries):

    dftest = adfuller(timeseries, autolag='AIC')
    return (dftest[0], dftest[4]['5%'])

def stationarize(timeseries):
    timeseries_log = np.log(timeseries)
    timeseries_log_diff = timeseries_log - timeseries_log.shift()
    timeseries_log_diff.dropna(inplace=True)
    return timeseries_log_diff

def check_stationarity(data):
    for sym in list(data):
        print(test_stationarity(stationarize(data[sym])))



def predict_on_period(timeseries, period):
    model = ARIMA(timeseries, order=(2, 1, 1))
    results_ARIMA = model.fit(disp=-1)
    forecasts = results_ARIMA.forecast(period)
    return forecasts[0]

def pred_arima(data, period):

    prices_pred = []
    for sym in list(data):
        timeseries_stat = stationarize(data[sym])
        prices_pred.append(np.sum(predict_on_period(timeseries_stat, peri
od))+np.log(data[sym][-1]))

    return np.exp(prices_pred)



def to_ts(data):
    data['Date'] = pd.to_datetime(data['Date'], format = '%Y-%m-%d')
    data_indx = data.set_index('Date')
    return data_indx
```

```
/Library/Frameworks/Python.framework/Versions/3.6/lib/python3.6/site-
packages/statsmodels/compat/pandas.py:56: FutureWarning: The pandas.c
ore.datetools module is deprecated and will be removed in a future ve
rsion. Please use the pandas.tseries module instead.
  from pandas.core import datetools
```

In [9]:
```
# This suppresses warnings.
import warnings
warnings.simplefilter('ignore')
```

In [10]:
```
predictions = pred_arima(to_ts(data), PERIOD)
print(predictions)
```

```
[ 112.54381181    2.43892397   18.6003847    12.27763302   18.9916865
   29.76371127   27.09299485   13.26919889   13.96294103   24.5305158
7
   35.44761556   61.21564136   45.08104417   40.01296965    9.1672790
4
    3.52334355   29.0580858    55.70364482    6.19941901   40.8417413
8
   53.92092256   10.84631203]
```



Optimization formulation

In order to forecast the best protfolio going forward:

$$\min_{w} \ w\Sigma w$$
$$s.t. \ \ R^T w \geq p$$
$$\sum_i w_i = 1, \ w_i \geq 0$$

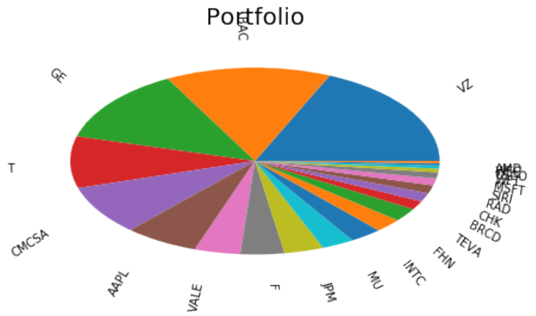
Idea: minimize risks

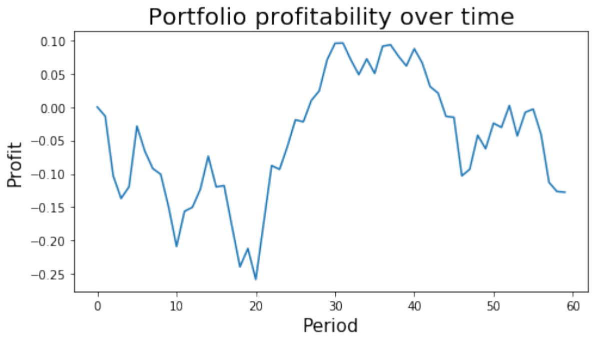
In [11]:

```
# Calculates profit if we buy sequrities at time_buy, sell them
# at time_sell, spend x[i] part of capital on i-rh security.
# Assumes that initial capital is equal to 1.
def count_profit(prices, date_buy, date_sell, x):
    prices_at_buy = prices[:, date_buy]
    prices_at_sell = prices[:, date_sell]
    capital_spent = np.sum(x)
    amount_bought = x / prices_at_buy
    revenue = np.dot(amount_bought, prices_at_sell)
    return revenue - np.sum(x)

# Calculates real profit on test data.
test_prices = None
def count_real_profit(portfolio, period):
    global test_prices
    if test_prices is None:         # This is to ensure we don't read CSV
each time we count profit
        data_folder = os.path.join(os.path.dirname(os.getcwd()), 'data')
        test_data_file = os.path.join(data_folder, DATASET_NAME + '_test.
csv')
        test_data = pd.read_csv(test_data_file)
        test_prices = np.array(test_data[sequrity_ids]).T
    return count_profit(test_prices, 0, period, portfolio)

# prices - numpy.array (N x T) of known prices (N - number of sequritites
, T - number of time moments).
# date_buy - index of row in prices, which corresponds to buying date.
# l - 'lambda' parameter in objective.
# predicted_prices - predicted/known prices on day of selling, from exter
nal predictor.
#                    Must be arranged in the same order, as columns in pr
ices.
def get_optimal_portfoilio_for_known_predictions(prices, date_buy, l, pre
dicted_prices):
    # Calculating profitabilities.
    r = np.array([
```

```
        (predicted_prices[i] - prices[i, date_buy]) / prices[i, date_buy]
        for i in range(N)
    ])
    cov = np.corrcoef(prices[:,:date_buy+1])
    return get_optimal_portfolio(r, cov, l)

def portfolio_info(portfolio, period):
    portfolio_real = portfolio.copy()
    portfolio = np.abs(portfolio)
    plt.figure(figsize=(17,4))
    # Show portfolio as pie chart.
    inv_seq = []
    for i in range(N):
        if portfolio[i] > 1e-2:
            inv_seq.append((portfolio[i], sequrity_ids[i]))
    inv_seq.sort(reverse=True)
    non_zero_inv = len(inv_seq)
    fracs = [inv_seq[i][0] for i in range(non_zero_inv)]
    labels = [inv_seq[i][1] for i in range(non_zero_inv)]
    plt.subplot(1, 2, 1)
    plt.pie(fracs, labels=labels, radius=0.9, labeldistance=1.3, rotatela
bels=True)
    plt.title('Portfolio', fontsize = 20)

    # Model that we invest according to portfolio now and sell sequrities
later,
    # varying period between buying and selling.
    period_range = range(period)
    profits = [count_real_profit(portfolio, per) for per in period_range]
    plt.subplot(1, 2, 2)
    plt.plot(period_range, profits)
    plt.xlabel("Period", fontsize = 15)
    plt.ylabel("Profit", fontsize = 15)
    plt.title('Portfolio profitability over time', fontsize = 20)
    plt.show()

    print("Profit on test data in %d steps: %f%%" % (period, count_real_p
rofit(portfolio_real, period)*100))

def get_mean_variance(prices, date_buy, predicted_prices, portfolio, peri
od):
    # Calculating profitabilities.
    r = np.array([
        (predicted_prices[i] - prices[i, date_buy]) / prices[i, date_buy]
        for i in range(N)
    ])
    cov = np.corrcoef(prices[:,:date_buy+1])
    return r.T @ portfolio, portfolio.T @ cov @ portfolio, count_real_pro
fit(portfolio, period)*100
```

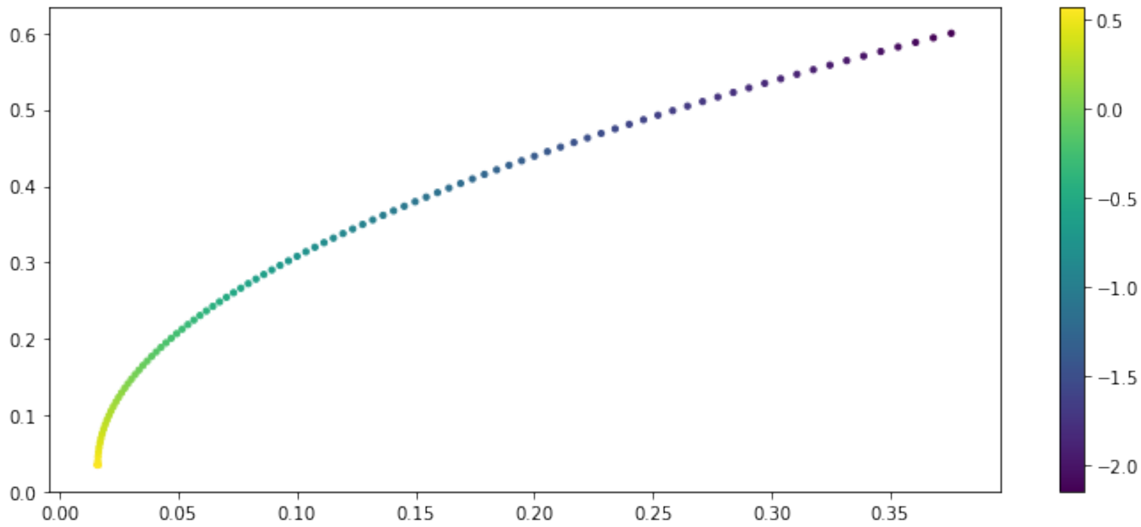
In [12]:
```python
def get_optimal_portfolio(r, cov, p):
    x = cvx.Variable(N)

    objective = cvx.Minimize(cvx.quad_form(x, cov))
    constraints = [sum_entries(cvx.diag(r) @ x) >= p, cvx.sum_entries(x)
== 1]
    prob = cvx.Problem(objective, constraints)
    obj = prob.solve()
    portfolio = np.copy(np.array(x.value).reshape(-1))
    return portfolio

#predictions = pred_arima(to_ts(data), PERIOD)

p = 0.001
opt_portfolio = get_optimal_portfoilio_for_known_predictions(prices, T-1,
p, predictions)
portfolio_info(opt_portfolio, PERIOD)
```






```
Profit on test data in 60 steps: 0.570290%
```

In [13]:
```
# Collect results

results_1 = []
for p in np.linspace(0.01, 0.6, 100):
    opt_portfolio = get_optimal_portfoilio_for_known_predictions(prices,
T-1, p, predictions)
    mean, var, profit = get_mean_variance(prices, T-1, predictions, opt_p
ortfolio, PERIOD)
    results_1.append([mean, var, profit])

results_1 = np.array(results_1)
plt.figure(figsize=(12, 5))
plt.scatter(results_1[:, 1], results_1[:, 0], c = results_1[:,2], s=10)
plt.colorbar();
```



Optimization formulation: modification 1

In order to forecast the best protfolio going forward:

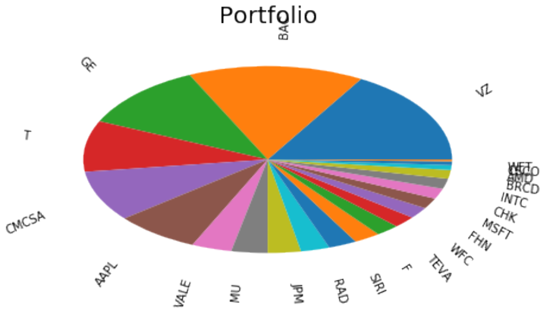
$$\max_w \; R^T w$$

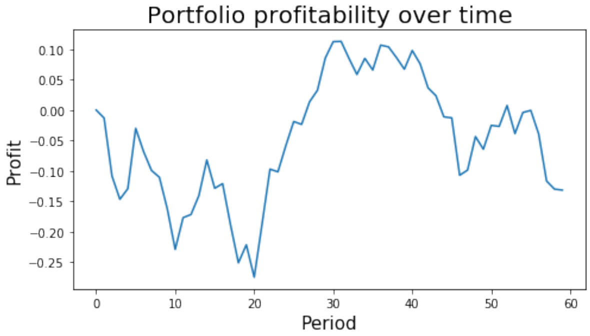
$$s.t. \quad w \Sigma w \le p$$

$$\sum_i w_i = 1, \; w_i \ge 0$$

Idea: maximize returns

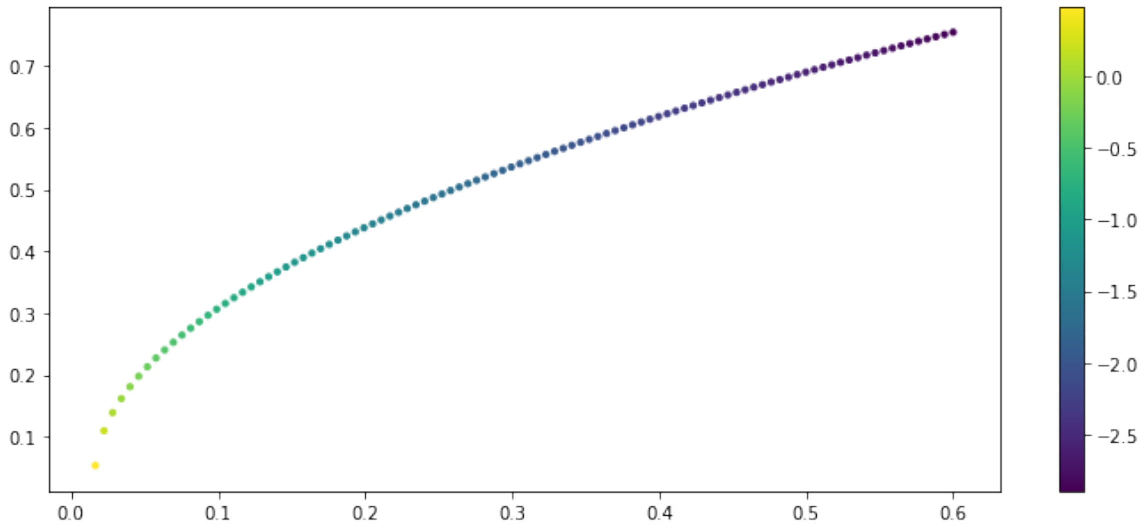
In [14]:
```python
def get_optimal_portfolio(r, cov, p):
    x = cvx.Variable(N)

    objective = cvx.Maximize(sum_entries(np.diag(r) @ x))
    constraints = [cvx.quad_form(x, cov) <= p, cvx.sum_entries(x) == 1]
    prob = cvx.Problem(objective, constraints)
    obj = prob.solve()
    portfolio = np.copy(np.array(x.value).reshape(-1))
    return portfolio

#predictions = pred_arima(to_ts(data), PERIOD)

p = 0.017
opt_portfolio = get_optimal_portfoilio_for_known_predictions(prices, T-1,
p, predictions)
portfolio_info(opt_portfolio, PERIOD)
```





Profit on test data in 60 steps: 0.436067%

In [15]:
```
# Collect results

results_2 = []
for p in np.linspace(0.0165, 0.6, 100):
    opt_portfolio = get_optimal_portfoilio_for_known_predictions(prices,
T-1, p, predictions)
    mean, var, profit = get_mean_variance(prices, T-1, predictions, opt_p
ortfolio, PERIOD)
    results_2.append([mean, var, profit])

results_2 = np.array(results_2)
plt.figure(figsize=(12, 5))
plt.scatter(results_2[:, 1], results_2[:, 0], c = results_2[:,2], s=10)
plt.colorbar();
```



Optimization formulation: modification 2

In order to forecast the best protfolio going forward:

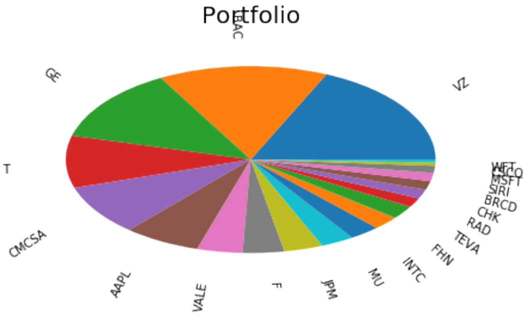
$$\max_w R^T w - \lambda w \Sigma w$$

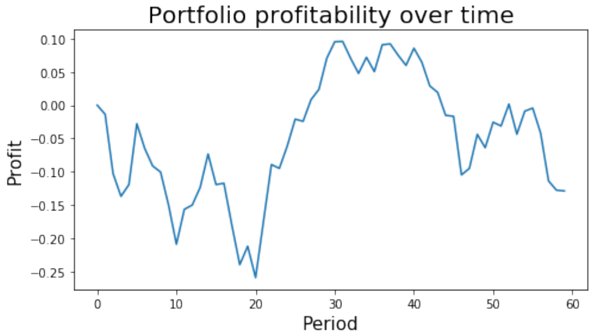
$$s.t. \sum_i w_i = 1, \ w_i \geq 0$$

Idea: try to balance previous two objectives in a single objective function!

In [16]:

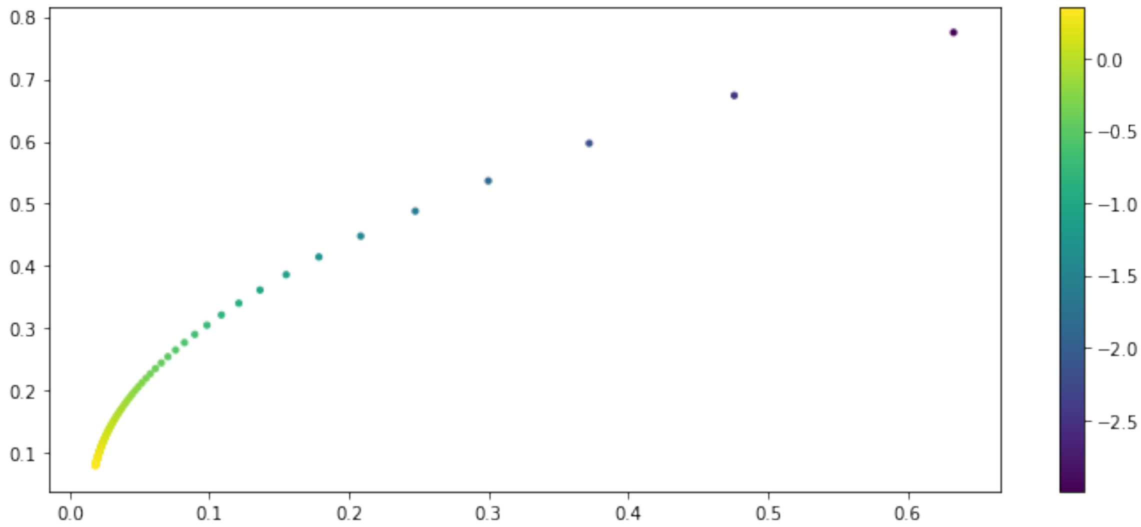
```
def get_optimal_portfolio(r, cov, l):
    x = cvx.Variable(N)

    objective = cvx.Maximize(r*x - l*cvx.quad_form(x, cov))
    constraints = [cvx.sum_entries(x) == 1]
    prob = cvx.Problem(objective, constraints)
    obj = prob.solve()
    portfolio = np.copy(np.array(x.value).reshape(-1))
    return portfolio

#predictions = pred_arima(to_ts(data), PERIOD)

l = 100
opt_portfolio = get_optimal_portfoilio_for_known_predictions(prices, T-1,
l, predictions)
portfolio_info(opt_portfolio, PERIOD)
```






```
Profit on test data in 60 steps: 0.548908%
```

In [17]:
```
# Collect results

results_3 = []
for p in np.linspace(0.6, 10, 100):
    opt_portfolio = get_optimal_portfoilio_for_known_predictions(prices, T-1, p, predictions)
    mean, var, profit = get_mean_variance(prices, T-1, predictions, opt_portfolio, PERIOD)
    results_3.append([mean, var, profit])

results_3 = np.array(results_3)
plt.figure(figsize=(12, 5))
plt.scatter(results_3[:, 1], results_3[:, 0], c = results_3[:,2], s=10)
plt.colorbar();
```



Sparse Portfolios

Practical investing requires balancing portfolio **optimality** and **simplicity**!

- Managing large asset positions and transacting frequently is expensive and time-consuming
- Their **choice set** for investment opportunities **is massive** and includes exchange-traded funds (ETFs), mutual funds, and thousands of individual stocks.

How does one invest optimally while keeping the simplicity (sparsity) of a portfolio in mind?

Answer: for example l_1 penalizing

$$L = \frac{1}{2} w^T \Sigma w - w^T \mu + \lambda \|w\|_1$$

This problem can reformulated to the form of **standard** sparse **regression** loss functions:

$$L = \frac{1}{2} \left\| L^T w - L^{-1} \mu \right\|_2^2 + \lambda \|w\|_1$$

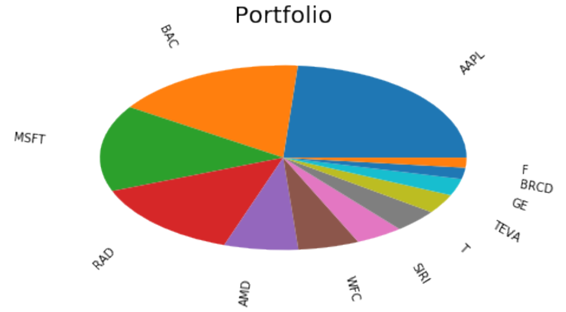
where $\Sigma = LL^T -$ Cholesky decomposition

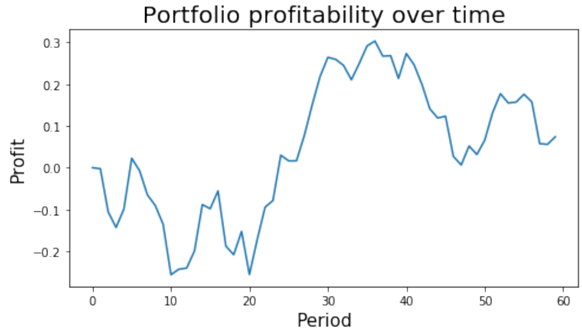
Proof:

$$\begin{aligned}
L &= \frac{1}{2} w^T \Sigma w - w^T \mu + \lambda \|w\|_1 \sim \\
&\sim \frac{1}{2} \left(w - \Sigma^{-1} \mu\right)^T \Sigma \left(w - \Sigma^{-1} \mu\right) + \lambda \|w\|_1 = \\
&= \frac{1}{2} \left(w - L^{-T} L^{-1} \mu\right)^T LL^T \left(w - L^{-T} L^{-1} \mu\right) + \lambda \|w\|_1 = \\
&= \frac{1}{2} \left[L^T \left(w - L^{-T} L^{-1} \mu\right)\right]^T \left[L^T \left(w - L^{-T} L^{-1} \mu\right)\right] + \lambda \|w\|_1 = \\
&= \frac{1}{2} \left\| L^T w - L^{-1} \mu \right\|_2^2 + \lambda \|w\|_1
\end{aligned}$$

In [18]:

```
def get_optimal_portfolio(r, cov, l):
    x = cvx.Variable(N)
    L = np.linalg.cholesky(cov)

    objective = cvx.Minimize(0.5 * cvx.norm2(L.T @ x - np.linalg.inv(L) @
r) ** 2 + l*cvx.norm1(x))
    constraints = [cvx.sum_entries(x) == 1]
    prob = cvx.Problem(objective, constraints)
    obj = prob.solve()
    portfolio = np.copy(np.array(x.value).reshape(-1))
    return portfolio

l = 0.035
opt_portfolio = get_optimal_portfoilio_for_known_predictions(prices, T-1,
l, predictions)
portfolio_info(opt_portfolio, PERIOD)
```

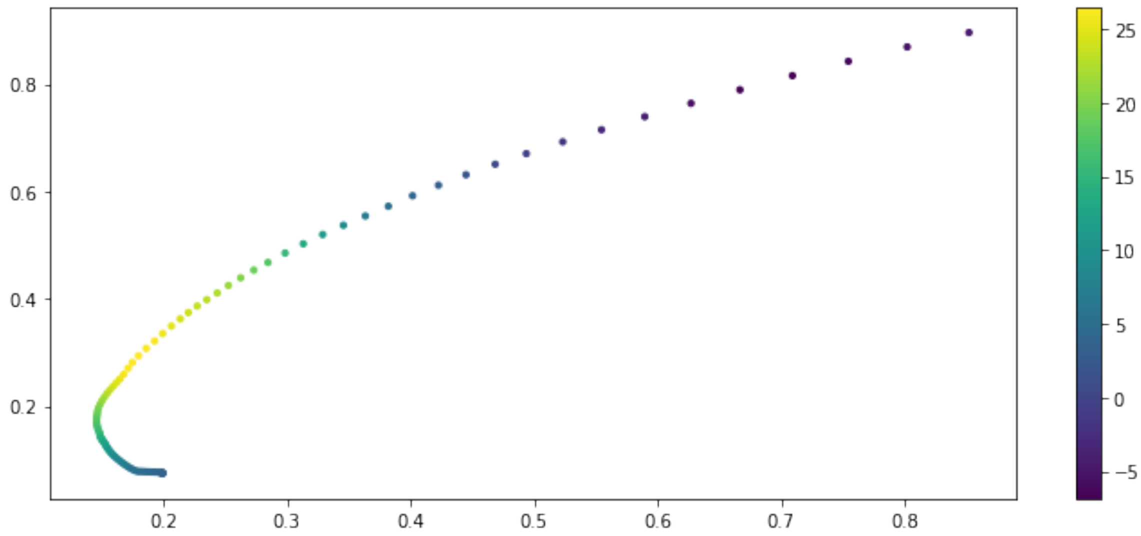





```
Profit on test data in 60 steps: 26.475352%
```

In [19]:

```
# Collect results

results_4 = []
for p in np.linspace(0.001, 0.1, 100):
    opt_portfolio = get_optimal_portfoilio_for_known_predictions(prices, 
T-1, p, predictions)
    mean, var, profit = get_mean_variance(prices, T-1, predictions, opt_p
ortfolio, PERIOD)
    results_4.append([mean, var, profit])

results_4 = np.array(results_4)
plt.figure(figsize=(12,5))
plt.scatter(results_4[:, 1], results_4[:, 0], c = results_4[:,2], s=10)
plt.colorbar();
```



What we have learned?

- Mean-Variance Optimization
- l_1 regularization to find sparse solution
- We faced with the problem of non-convex optimization

