An Introduction to Optimisation

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Reading Guide 1

- Introduction-level references.
  - Bradley, T (2008), *Essential mathematics for economics and business (third edition)*, John Wiley and Sons, Great Britain: Chapter 7 (Section 4) (pp. 408-420).
  - Haeussler, EF Jr and RS Paul (1987), *Introductory mathematical analysis for business, economics, and the life and social sciences (fifth edition)*, Prentice-Hall International Edition, Prentice-Hall, USA: Chapter 17 (Section 8) (pp. 706-714).
  - Shannon, J (1995), *Mathematics for business, economics and finance*, John Wiley and Sons, Brisbane: Chapter 10, Section 7 (pp. 489-501).
  - Sydsaeter, K, P Hammond, A Strom, and A Carvajal (2016), *Essential mathematics for economic analysis (fifth edition)*, Pearson Education, United Kingdom: Chapters 8, 13, and 14.

Reading Guide 2

- Intermediate-level references.
    - Chiang, AC and K Wainwright (2005), *Fundamental methods of mathematical economics (fourth edition)*, McGraw-Hill, Singapore: Chapters 9 and 11-13 (pp. 220-254 and 291-442).
    - Dixit, AK (1990), *Optimization in economic theory (second edition)*, Oxford University Press, Great Britain: Chapters 1 to 9 and the Appendix (pp. 1-144 and 181-185).
    - Leonard, D, and NV Long (1992), *Optimal control theory and static optimization in economics*, Cambridge University Press, USA: Chapter 1 (pp. 1-86).
    - Silberberg, E (1990), *The structure of economics: A mathematical approach (second edition)*, McGraw-Hill, Singapore: Chapters 4, 6-14 and 17 (pp. 107-134, 156-490 and 573-612).
    - Simon, C, and L Blume (1994), *Mathematics for economists*, WW Norton and Co, USA: Chapters 16-22, 23 (Section 8) and 30 (pp. 375-576, 626-629 and 822-844).
- Continued on the next slide.

Reading Guide 3

- Intermediate-level references (continued from the previous slide).
  - Starr, RM (1997), *General equilibrium theory: An introduction*, Cambridge University Press, USA: (pp. 207-209).
  - Takayama, A (1993), *Analytical methods in economics*, The University of Michigan Press, USA: Chapter 2-5, Appendix B and Appendix C (pp. 75-321 and 605-647).
- Advanced-level references.
  - Ausubel, LM and RJ Deneckere (1993), "A generalized theorem of the maximum", *Economic Theory 3(1)*, January, pp. 99-107.
  - Corbae, D, MB Stinchcombe and J Zeman (2009), *An introduction to mathematical analysis for economic theory and econometrics*, Princeton University Press, USA: Chapters 5 and 6 (pp. 172-354).
  - Sundaram, RK (1996), *A first course in optimization theory*, Cambridge University Press, USA: Chapters 2-10 (pp. 74-267).
  - Takayama, A (1985), *Mathematical economics (second edition)*, Cambridge University Press, USA: Chapters 1 and 2 (pp. 59-294).

Why do economists care about optimisation? Part 1

Economists try to explain social phenomena in terms of the behaviour of an individual who is confronted with scarcity and the interaction of that individual with other individuals who also face scarcity. This is perhaps best captured by Malinvaude's definition of economics:

> *"$\cdots$ economics is the science which studies how scarce resources are employed for the satisfaction of the needs of men living in society: on the one hand, it is interested in the essential operations of production, distribution and consumption of goods, and on the other hand, in the institutions and activities whose object it is to facilitate these operations." (Italics in original.)*

(From page one of Malinvaude, E. (1972), Lectures on microeconomic theory, Advanced Textbooks in Economics Volume 2, North Holland Publishing Company, Scotland, translated by Mrs. A. Silvey.)

Why do economists care about optimisation? Part 2

- Scarcity:
  - This is the defining feature of economics.
  - It is this feature that distinguishes economics from other social sciences.
- The behaviour of an individual who is faced with scarcity:
  - Often modelled using "constrained optimisation" techniques.
- The interaction of individuals that face scarcity:
  - Economic equilibrium (eg competitive equilibrium and Nash equilibrium).
  - When does a system of equations have at least one solution?
  - How do we find such a solution (if it exists)?
  - Use techniques from linear algebra and (for nonlinear cases) fixed point theorems.

Why do economists care about optimisation? Part 3

- According to Ausubel and Deneckere (1993, p. 99):

  "Almost every economic problem involves the study of an agent's optimal choice as a function of certain parameters or state variables. For example, demand theory is concerned with an agent's optimal consumption as a function of prices and income, while capital theory studies the optimal investment rule as a function of the existing capital stock."

Maximisation or Minimisation?

- Optimisation problems come in two broad varieties: maximisation problems and minimisation problems.
  - A maximisation problem involves finding a maximum for some function $f : S \longrightarrow \mathbb{R}$, where $S \subseteq \mathbb{R}^n$, on some subset of its domain $C \subseteq S$.
  - A minimisation problem typically involves finding a minimum for some function $f : S \longrightarrow \mathbb{R}$, where $S \subseteq \mathbb{R}^n$, on some subset of its domain $C \subseteq S$.
- Some terminology.
  - The function f is known as the objective function.
  - The set C is known as the constraint set (or the feasible set).
- It turns out that the problem of finding a minimum value for f on the set C is equivalent to the problem of finding a maximum value for $(-f)$ on the set C.
  - Thus we can restrict our attention to maximisation problems without loss of generality.
  - If we are faced with a minimisation problem, we can simply multiply the objective function by (-1) and treat it like a maximisation problem.

Unconstrained or Constrained?

- Consider an optimisation problem that involves finding a maximum for some function $f : S \longrightarrow \mathbb{R}$, where $S \subseteq \mathbb{R}^n$, on some subset of its domain $C \subseteq S$.
- If $C = S$, then the problem is an unconstrained optimisation problem.
- If $C \subset S$ (that is, if $C \subseteq S$ and $C \neq S$), then the problem is a constrained optimisation problem.
- In this course, we will only look at unconstrained problems and equality-constrained problems. But please note that inequality-constraints are also possible. These are covered in more advanced courses.

The existence and properties of a solution

- As we shall see, there are some optimisation problems that have no solutions.
- It would be nice to know that the problem we are trying to solve has a solution before we attempt to solve it.
- In cases where f is a function, a set of conditions that are sufficient to guarantee the existence of a solution to an optimisation problem is provided by Weierstrass' Theorem of the Maximum.
- A set of conditions that are sufficient to guarantee that that the solution set and the set of optimal choice variables in an optimisation problem have some desirable properties is provided by Berge's Theorem of the maximum.
- These two theorems are beyond the scope of this course. However, it is comforting to know that they exist!

The potential non-existence of a solution

- We will illustrate the potential non-existence of a solution by means of some examples.
- Consider the following optimisation problems.
  - Find a maximum for $f(x) = x$ when $x \in (-\infty, \infty)$.
    - No solution exists because $f(x) \to \infty$ as $x \to \infty$.
  - Find a minimum for $f(x) = x$ when $x \in (-\infty, \infty)$.
    - No solution exists because $f(x) \to -\infty$ as $x \to -\infty$.
  - Find a maximum for $f(x) = x$ when $x \in (a, b)$ where $-\infty < a < b < \infty$.
    - No solution exists because of the constraint set is open.
    - $f(x) \to f(b)$ as $x \to b$, but never quite gets there.
    - What is the largest real number that is strictly less than b?
  - Find a minimum for $f(x) = x$ when $x \in (a, b)$ where $-\infty < a < b < \infty$.
    - No solution exists because of the constraint set is open.
    - $f(x) \to f(a)$ as $x \to a$, but never quite gets there.
    - What is the largest real number that is strictly greater than a?
- Illustrate these examples on the white-board.

Solutions and Control Variables Part 1

- The solution to a constrained maximisation problem is the maximum value of the objective function $f(x_1, x_2, \cdots, x_n)$ over the constraint set C
- It is denoted by the maximum value function

$$f^*(\cdot) \equiv \max\left\{f(x) : x \in C\right\}.$$

- The optimal choices of the x_i variables are known as "arg max"s:

$$\begin{aligned} \{x^*(\cdot)\} &= \left\{ \begin{pmatrix} x_1^*(\cdot) \\ x_2^*(\cdot) \\ \vdots \\ x_n^*(\cdot) \end{pmatrix} \right\} \\ &= \arg\max\left\{f(x) : x \in C\right\}. \end{aligned}$$

Solutions and Control Variables Part 2

- We can find the maximum value function from the "arg max"s as follows:

$$f^* (\cdot) = f (x^*)$$

where

$$x^* \in \arg\max \{ f (x) : x \in C \} .$$

- If the "arg max" is unique, then this becomes

$$f^* (\cdot) = f (x^*)$$

where

$$x^* = \arg\max \{ f (x) : x \in C \} .$$

Solutions and Control Variables Part 3

- The solution to a constrained minimisation problem is the minimum value of the objective function $f\left(x_1, x_2, \cdots, x_n\right)$ over the constraint set C.
- It is denoted by the minimum value function

$$f^*\left(\cdot\right) \equiv \min\left\{f\left(x\right) : x \in C\right\}.$$

- The optimal choices of the x_i variables are known as "arg min"s:

$$\begin{aligned} \left\{x^*\left(\cdot\right)\right\} &= \left\{\begin{pmatrix} x_1^*\left(\cdot\right) \\ x_2^*\left(\cdot\right) \\ \vdots \\ x_n^*\left(\cdot\right) \end{pmatrix}\right\} \\ &= \arg\min\left\{f\left(x\right) : x \in C\right\}. \end{aligned}$$

Solutions and Control Variables Part 4

- We can find the minimum value function from the "arg min"s as follows:

$$f^* (\cdot) = f (x^*)$$

where

$$x^* \in \arg\min \{ f (x) : x \in C \} .$$

- If the "arg min" is unique, then this becomes

$$f^* (\cdot) = f (x^*)$$

where

$$x^* = \arg\min \{ f (x) : x \in C \} .$$

Global Optima and Local Optima Part 1

- Let $f : S \longrightarrow \mathbb{R}$ and $C \subseteq S$, where $S \subseteq \mathbb{R}^n$.
- Global and local maxima.
  - $f(x^*)$ is a global maximum of f on C if $x^* \in C$ and $f(x^*) \geqslant f(x)$ for all $x \in C$.
  - $f(x^*)$ is a local maximum of f on C if $x^* \in C$ and $f(x^*) \geqslant f(x)$ for all $x \in C$ that are "sufficiently close to" x^*.
- Global and local minima.
  - $f(x^*)$ is a global minimum of f on C if $x^* \in C$ and $f(x^*) \leqslant f(x)$ for all $x \in C$.
  - $f(x^*)$ is a local minimum of f on C if $x^* \in C$ and $f(x^*) \leqslant f(x)$ for all $x \in C$ that are "sufficiently close to" x^*.

Global Optima and Local Optima Part 2

- Note that every global maximum is a local maximum, but not all local maxima will be a global maximum.
    - Illustrate this on the white-board.
- Note that every global minimum is a local minimum, but not all local minima will be a global minimum.
    - Illustrate this on the white-board.

Unconstrained Optimisation

- Let $f : S \longrightarrow \mathbb{R}$ be a function that is at least twice continuously differentiable and $C = S$, where $S \subseteq \mathbb{R}^n$ is an open set.
- Since $C = S$, this an unconstrained optimisation proiblem.
- The point $x^* \in S$ will yield a local maximum of f if both of the following sets of conditions are satisfied.
  - (First-Order Conditions) (FOC): The gradient vector $D_x f$ equals the null vector at the point x^*. (In other words, $D_x f(x^*) = 0$.
    - This requires that $\frac{\partial f(x^*)}{\partial x_i} = 0$ for all $i \in \{1, 2, \cdots, n\}$.
  - (Second-Order Conditions) (SOC): The Hessian matrix $D^2_{xx^T} f(x^*)$ is negative definite at the point x^*
- The point $x^* \in S$ will yield a local minimum of f if both of the following sets of conditions are satisfied.
  - (First-Order Conditions) (FOC): The gradient vector $D_x f$ equals the null vector at the point x^*. (In other words, $D_x f(x^*) = 0$.
    - This requires that $\frac{\partial f(x^*)}{\partial x_i} = 0$ for all $i \in \{1, 2, \cdots, n\}$.
  - (Second-Order Conditions) (SOC): The Hessian matrix $D^2_{xx^T} f(x^*)$ is positive definite at the point x^*

Univariate Unconstrained Optimisation Examples

- Work through the following examples on the white-board.
  - Minimise $f(x) = (x-1)^2$ on $\mathbb{R}$. (A minimum of $f = 0$ at $x = 1$.)
  - Maximise $f(x) = -(x-1)^2$ on $\mathbb{R}$. (A maximum of $f = 0$ at $x = 1$.)
  - Maximise $f(x) = \left(\frac{1}{3}\right) x^3 - x + 1$ on $\mathbb{R}$. (A "minimum" turning point at $x = 1$ and a "maximum" turning point at $x = -1$. A local maximum of $f = \frac{5}{3}$ at $x = -1$.)
  - Minimise $f(x) = \left(\frac{1}{3}\right) x^3 - x + 1$ on $\mathbb{R}$. (A "minimum" turning point at $x = 1$ and a "maximum" turning point at $x = -1$. A local minimum of $f = \frac{1}{3}$ at $x = 1$.)

A Multivariate Unconstrained Optimisation Example

- Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $f(x, y) = -x^2 - y^2$.
- The first-order conditions for either a maximum or a minimum are:
  - $f_x(x, y) = -2x = 0 \Longrightarrow x^* = 0$; and
  - $f_y(x, y) = -2y = 0 \Longrightarrow y^* = 0$.
  - Thus $(x^*, y^*) = (0, 0)$ is the unique critical point for $f(x, y)$.
- The Hessian matrix for $f(x, y)$ is

$$H = \begin{pmatrix} f_{xx}(x, y) & f_{xy}(x, y) \\ f_{yx}(x, y) & f_{yy}(x, y) \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}.$$

- Note that:
  - H is a symmetric matrix,
  - $\det(H_1) = -2 < 0$, and
  - $\det(H_2) = \det(H) = 4 > 0$.
- Thus H is negative definite for all $(x, y) \in \mathbb{R}^2$.
- This means that $(x^*, y^*) = (0, 0)$ yields a global maximum of $f(x, y)$.
- The maximum value of $f(x, y)$ is $f^* = f(x^*, y^*) = f(0, 0) = 0$.

Univariate Profit Maximisation by a Price-Taking Firm Part 1

- Suppose that a single-product price-taking firm faces an output price of P per unit and has a (total) cost function that is given by $C(Q)$.
  - We will assume that the cost function is at least twice continuously differentiable.
- The firm's profits are given by the function $\Pi(Q) = PQ - C(Q)$.
- Suppose that this firm wants to choose Q to maximise its profits.
  - We will ignore the shut-down condition as a matter of convenience.
- The first-order condition for profit maximisation is

$$\frac{d\Pi}{dQ} = P - \frac{dC}{dQ} = 0.$$

- This can be rearranged to obtain $P = MC(Q)$, where $MC(Q) = \frac{dC}{dQ}$.
- This first-order condition implicitly defines one or more critical values of Q for the function $\Pi(Q)$.

Univariate Profit Maximisation by a Price-Taking Firm Part 2

- Suppose that Q^* is a critical value of Q for the function $\Pi(Q)$.
- The second-order condition for a local maximum requires that

$$\frac{d^2\Pi}{dQ^2} = -\frac{d^2 C}{dQ^2} = -\frac{dMC}{dQ} < 0,$$

when it is evaluated at the point Q^*.

  - Note that this requires that $\frac{dMC}{dQ} > 0$ when it is evaluated at the point Q^*.
  - In other words, we need marginal cost to be an increasing function of the output quantity in a non-empty neighbourhood around the point Q^*.

Bivariate Profit Maximisation by a Price-Taking Firm Part 1

- Suppose that a two-product price-taking firm faces output prices of P_1 per unit for one of its products and and P_1 per unit for its other product.
- Suppose also that this firm has a (total) cost function that is given by $C(Q_1, Q_2)$, where Q_1 is its output of good one and Q_2 is its output of good two.
  - We will assume that the cost function is at least twice continuously differentiable in all of its arguments.
- The firm's profits are given by the function $\Pi(Q_1, Q_2) = P_1 Q_1 + P_2 Q_2 - C(Q_1, Q_2)$.
- Suppose that this firm wants to choose Q_1 and Q_2 to maximise its profits.
  - We will ignore the shut-down condition as a matter of convenience.

Bivariate Profit Maximisation by a Price-Taking Firm Part 2

- The first-order conditions for profit maximisation by the firm are

$$\frac{\partial \Pi}{\partial Q_1} = P_1 - \frac{\partial C}{\partial Q_1} = 0$$

and

$$\frac{\partial \Pi}{\partial Q_2} = P_2 - \frac{\partial C}{\partial Q_2} = 0.$$

- These two conditions can be rearranged to obtain

$$P_1 = MC_1(Q_1, Q_2)$$

and

$$P_2 = MC_2(Q_1, Q_2),$$

where $MC_i(Q_1, Q_2) = \frac{\partial C}{\partial Q_i}$ for $i \in \{1, 2\}$.

Bivariate Profit Maximisation by a Price-Taking Firm Part 3

- The two first-order conditions for this bivariate profit maximisation problem jointly define one or more critical points for the function $\Pi(Q_1, Q_2)$.
- Let $Q^* = (Q_1^*, Q_2^*)$ be one such point.
- The second-order condition for Q^* to yield a local maximum of the function $\Pi(Q_1, Q_2)$ requires that the Hessian matrix for $\Pi(Q_1, Q_2)$ be negative definite when it is evaluated at the point $Q^* = (Q_1^*, Q_2^*)$.

Bivariate Profit Maximisation by a Price-Taking Firm Part 4

- The Hessian matrix for the function $\Pi(Q_1, Q_2)$ is

$$\begin{aligned} H &= D^2\Pi(Q_1, Q_2) \\ &= \begin{pmatrix} \frac{\partial^2 \Pi}{\partial Q_1^2} & \frac{\partial^2 \Pi}{\partial Q_2 \partial Q_1} \\ \frac{\partial^2 \Pi}{\partial Q_1 \partial Q_2} & \frac{\partial^2 \Pi}{\partial Q_2^2} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{\partial^2 C}{\partial Q_1^2} & -\frac{\partial^2 C}{\partial Q_2 \partial Q_1} \\ -\frac{\partial^2 C}{\partial Q_1 \partial Q_2} & -\frac{\partial^2 C}{\partial Q_2^2} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{\partial MC_1}{\partial Q_1} & -\frac{\partial MC_1}{\partial Q_2} \\ -\frac{\partial MC_2}{\partial Q_1} & -\frac{\partial MC_2}{\partial Q_2} \end{pmatrix}. \end{aligned}$$

Bivariate Profit Maximisation by a Price-Taking Firm Part 5

- A sufficient condition for the critical point $Q^* = (Q_1^*, Q_2^*)$ to yield a local maximum of the function $\Pi(Q_1, Q_2)$ is that the Hessian matrix (H) be negative definite at the point $Q^* = (Q_1^*, Q_2^*)$
- Assuming that the function $\Pi(Q_1, Q_2)$ is at least twice continuously differentiable in a non-empty neighbourhood around the point $Q^* = (Q_1^*, Q_2^*)$ (so that Young's theorem applies and the Hessian matrix is symmetric at the critical point being considered), this requires that:
  1. $\det(H_1) = -\frac{\partial MC_1}{\partial Q_1} < 0$ at $Q^* = (Q_1^*, Q_2^*)$, and
  2. $\det(H_2) = \det(H) = \left(\frac{\partial MC_1}{\partial Q_1}\right)\left(\frac{\partial MC_2}{\partial Q_2}\right) - \left(\frac{\partial MC_1}{\partial Q_2}\right)\left(\frac{\partial MC_2}{\partial Q_1}\right) = \left(\frac{\partial MC_1}{\partial Q_1}\right)\left(\frac{\partial MC_2}{\partial Q_2}\right) - \left(\frac{\partial MC_1}{\partial Q_2}\right)^2 > 0$ at $Q^* = (Q_1^*, Q_2^*)$.

Bivariate Profit Maximisation by a Price-Taking Firm Part 6

- This in turn requires that:
  1. $\frac{\partial MC_1}{\partial Q_1} > 0$ at $Q^* = (Q_1^*, Q_2^*)$,
  2. $\frac{\partial MC_2}{\partial Q_2} > 0$ at $Q^* = (Q_1^*, Q_2^*)$, and
  3. $\left(\frac{\partial MC_1}{\partial Q_1}\right)\left(\frac{\partial MC_2}{\partial Q_2}\right) > \left(\frac{\partial MC_1}{\partial Q_2}\right)^2$ at $Q^* = (Q_1^*, Q_2^*)$.

Equality Constrained Optimisation

- Let $f : S \longrightarrow \mathbb{R}$ be a function that is at least twice continuously differentiable and $C \subseteq S$, where $S \subseteq \mathbb{R}^n$ is an open set.
- Suppose that the constraint set C consists only of equality restrictions. In other words, suppose that

$$C = \left\{ x \in \mathbb{R}^n : g^i(x) = b^i \text{ for all } i \in \{1, 2, \cdots, m\} \right\},$$

  where $b^i \in \mathbb{R}$ for $i \in \{1, 2, \cdots, m\}$ are all constant terms.
- Consider the problem of maximising f on C.
    - We can resrict our attention to a constrained maximisation problem without loss of generality.
    - The reason for this is that minimising f on C is equivalent to maximising $(-f)$ on C.
- Note that a necessary condition for there to be a solution to a constrained optimisation problem is that the constraint set be non-empty.

The Lagrangean Function

- Let $S \subseteq \mathbb{R}^n$ be an open set, $f : S \longrightarrow \mathbb{R}$ be a function that is at least twice continuously differentiable, and

$$C = \left\{ x \in \mathbb{R}^n : g^i(x) = b^i \text{ for all } i \in \{1, 2, \cdots, m\} \right\},$$

  be a constraint set.
- Consider the problem of maximising f on C.
- The Lagrangean function for this constrained maximisation problem is

$$\mathcal{L}(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i \left[b^i - g^i(x) \right].$$

- The λ_i variables are known as Lagrange multipliers.

Equality Constrained Optimisation Solutions

- Theorem: $x^* \in C$ yields a local maximum of f if the following three sets of conditions are satisfied.
  - (FOC Part 1): $D_x \mathcal{L}(x^*, \lambda^*) = 0$.
  - (FOC Part 2): $D_\lambda \mathcal{L}(x^*, \lambda^*) = 0$.
  - An appropriate rank condition and appropriate second-order conditions are satisfied. (These are beyond the scope of this course.)

The First-Order Conditions for Equality-Constrained Maximisation Problems

- Recall that the Lagrangean function for the constrained maximisation problem under consideration is

$$\mathcal{L}(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i \left[b^i - g^i(x) \right].$$

- The first-order conditions for this problem are:
  - $\frac{\partial \mathcal{L}}{\partial x_k} = \frac{\partial f}{\partial x_k} + \sum_{i=1}^{m} \left(-\lambda_i \frac{\partial g}{\partial x_k} \right) = 0$ for each $k \in \{1, 2, \cdots, n\}$, and
  - $\frac{\partial \mathcal{L}}{\partial \lambda_j} = b^j - g^j(x) = 0$ for each $j \in \{1, 2, \cdots, m\}$.

The Second-Order Conditions for Equality-Constrained Maximisation Problems

- It is possible to express the second-order conditions for this problem in terms of something known as a bordered Hessian matrix. However, this is beyond the scope of this course.
  - We will instead look at converting equality-constrained optimisation problems into unconstrained optimisation problems in fewer variables by using the constraints to express some variables in terms of others.
  - This will allow us to use the second-order conditions for uncontrained optimisation problems.

Interpreting Lagrange Multipliers

- The optimal value of the Lagrange multiplier for the ith constraint (that is, the value of that Lagrange multiplier when all of the first-order conditions are satisfied) tells us the degree of sensitivity of the optimal value of the objective function over the constraint set, $f^*(x^*, b)$, to a change in the ith constraint value b^i, where $b = \{b^1, b^2, \cdots, b^m\}$ is the vector of constraint values.
- It is, in effect, a "shadow price".
- The optimal value of the Lagrange multiplier in a standard utility maximisation problem can be interpreted as the marginal utility of income.
- The optimal value of the Lagrange multiplier in a cost minimisation problem can be interpreted as marginal cost.

Some Economic Applications

- Budget-contrained utility maximisations problems.
- Utlity-constrained expenditure minimisation problems.
- Output-constrained cost minimisation problems.
- Technology-constrained profit maximisation problems.
- Pareto efficiency problems.

Equality Constrained Optimisation Examples

- Work through the following examples on the board or from other notes.
  - Maximise $f(x, y) = 3x - y + 6$ subject to the constraint that $x^2 + y^2 = 4$, where $(x, y) \in \mathbb{R}^2$.
  - Solve some UMP examples.
  - Solve a cost-minimisation problem for a particular two-input version of a Cobb-Douglas production function and show that the optimal value of the Lagrange multiplier on the "minimum output" constraint is equal to marginal cost.

Converting Equality Constrained Optimisation Problems into Unconstrained Optimisation Problems Part 1

- Consider an equality constrained optimisation problem of the form

$$\text{Max } f(x_1, x_2, \cdots, x_n) \text{ subject to}$$

$$\begin{aligned}
g^1(x_1, x_2, \cdots, x_n) &= b_1, \\
g^2(x_1, x_2, \cdots, x_n) &= b_2, \\
&\vdots \\
g^m(x_1, x_2, \cdots, x_n) &= b_m.
\end{aligned}$$

- The constraint set for this problem is $C =$

$$\left\{(x_1, x_2, \cdots, x_n) : g^j(x_1, x_2, \cdots, x_n) = b_j \text{ for all } j \in \{1, 2, \cdots, m\}\right\}.$$

- Assume that there are fewer constraints than there are choice variables ($m < n$) and that the constraint set is nonempty ($C \neq \varnothing$).

Converting Equality Constrained Optimisation Problems into Unconstrained Optimisation Problems Part 2

- Since $m < n$ and $C \neq \varnothing$, we know that if the constraint equations are all linear and the domain of each choice variable is an infinite subset of $\mathbb{R}$, then there will be an infinite number of solutions for the system of equations given by

$$\begin{gathered} g^1(x_1, x_2, \cdots, x_n) = b_1, \\ g^2(x_1, x_2, \cdots, x_n) = b_2, \\ \vdots \\ g^m(x_1, x_2, \cdots, x_n) = b_m. \end{gathered}$$

- These solutions would be parametric. They would pin down the value of m of the choice variables in terms of the other $(n-m)$ choice variables and the constraint values. However, each of the remaining $(n-m)$ choice variables would be free to vary over their entire domain.

Converting Equality Constrained Optimisation Problems into Unconstrained Optimisation Problems Part 3

- Suppose that this is the case even when the constraint equations are nonlinear. (This may impose some restrictions on the constraint equations. Think about the implicit function theorem and the inverse function theorem, for example.)
- In this case, we can use the system of constraint equations to express m of the choice variables as implicit functions of the remaining $(n-m)$ choice variables and the m constraint values.
- These implicit functions might take the form

$$\begin{gathered}
x_{n-m+1} = h^{n-m+1}(x_1, x_2, \cdots, x_{n-m}, b_1, b_2, \cdots, b_m), \\
x_{n-m+2} = h^{n-m+2}(x_1, x_2, \cdots, x_{n-m}, b_1, b_2, \cdots, b_m), \\
\vdots \\
x_n = h^n(x_1, x_2, \cdots, x_{n-m}, b_1, b_2, \cdots, b_m).
\end{gathered}$$

Converting Equality Constrained Optimisation Problems into Unconstrained Optimisation Problems Part 4

- Having used the system of constraint equations to express m of the choice variables as implicit functions of the remaining $(n-m)$ choice variables and the m constraint values, we can now substitute these implicit functions back into the objective function to obtain the following "reduced-form" unconstrained optimisation problem:

$$\text{Max } f(x_1, x_2, \cdots, x_{n-m}, h^{n-m+1}(\cdot), h^{n-m+2}(\cdot), \cdots, h^n(\cdot)).$$

- Note that there are only $(n-m)$ choice variables in this "reduced-form" unconstrained optimisation problem.
- But each of these variables potentially enters the objective function multiple times; once directly and up to m times indirectly. The indirect entries come through the implicit functions that solve the set of constraint equations (in other words, the h functions).

Converting Equality Constrained Optimisation Problems into Unconstrained Optimisation Problems Part 5

- The technique for solving unconstrained optimisation problems that we have already covered can now be applied to this 'reduced-form" unconstrained optimisation problem in order to solve the oringinal equality-constrained optimisation problem.
- Once optimal values for the $(n - m)$ choice variables in this "reduced-form" unconstrained optimisation problem have been obtained, the optimal values of remaining m choice variables from the original equality-constrained optimisation problem can be obtained by substituting them into the h functions. This will give us the optimal choice values for all n of the choice variables.
- The solution to the equality-constrained optimisation problem can then be found by substituting the optimal values for all n of the choice variables into the objective function.

Conversion Example Part 1

- Consider the following UMP:

$$\text{Max } \; U(x_1, x_2, x_3) = \alpha \ln(x_1) + \beta \ln(x_2) + (1 - \alpha - \beta)\ln(x_3)$$

$$\text{subject to } \; p_1 x_1 + p_2 x_2 + p_3 x_3 = y,$$

where $0 < \alpha < 1$, $0 < \beta < 1$, and $\alpha + \beta < 1$.

- We will assume throughout that $x_1 > 0$, $x_2 > 0$, and $x_3 > 0$.
- We can use the budget constraint to express x_3 as a function of x_1 and x_2 as follows:

$$x_3 = \frac{y}{p_3} - \frac{p_1}{p_3} x_1 - \frac{p_2}{p_3} x_2.$$

- Upon substituting this into the utility function, we obtain the following unconstrained optimisation problem:

$$\text{Max } \; \hat{U}(x_1, x_2) =$$

$$\alpha \ln(x_1) + \beta \ln(x_2) + (1 - \alpha - \beta)\ln\left(\frac{y}{p_3} - \frac{p_1}{p_3} x_1 - \frac{p_2}{p_3} x_2\right).$$

Conversion Example Part 2

- The first-order conditions for this problem are

$$\frac{\partial \hat{U}}{\partial x_1} = \frac{\alpha}{x_1} - \frac{(1-\alpha-\beta)\frac{p_1}{p_3}}{\left(\frac{y}{p_3} - \frac{p_1}{p_3}x_1 - \frac{p_2}{p_3}x_2\right)} = 0,$$

  and

$$\frac{\partial \hat{U}}{\partial x_2} = \frac{\beta}{x_2} - \frac{(1-\alpha-\beta)\frac{p_2}{p_3}}{\left(\frac{y}{p_3} - \frac{p_1}{p_3}x_1 - \frac{p_2}{p_3}x_2\right)} = 0.$$

- These can be rearranged and simplified to obtain

$$(1-\alpha-\beta)p_1x_1 = \alpha(y - p_1x_1 - p_2x_2),$$

  and

$$(1-\alpha-\beta)p_2x_2 = \beta(y - p_1x_1 - p_2x_2).$$

Conversion Example Part 3

- These can be further rearranged and simplified to obtain the following system of two equations in two unknowns:

$$(1-\beta)p_1x_1 + \alpha p_2x_2 = \alpha y,$$

and

$$\beta p_1x_1 + (1-\alpha)p_2x_2 = \beta y.$$

- It is probably easiest to solve this system of equations by using expenditure on each commodity as the variables. Upon doing this, we obtain $p_1x_1 = \alpha y$ and $p_2x_2 = \beta y$, so that

$$x_1 = \frac{\alpha y}{p_1} \text{ and } x_2 = \frac{\beta y}{p_2}.$$

- This means that

$$x_3 = \frac{y}{p_3} - \frac{p_1}{p_3}\frac{\alpha y}{p_1} - \frac{p_2}{p_3}\frac{\beta y}{p_2} = \frac{(1-\alpha-\beta)y}{p_3}.$$

Conversion Example Part 4

- Thus our candidate solution is

$$
\begin{aligned}
V(p_1, p_2, p_3, y) &= U^* \\
&= U\left(\frac{\alpha y}{p_1}, \frac{\beta y}{p_2}, \frac{(1-\alpha-\beta)y}{p_3}\right) \\
&= \alpha \ln\left(\frac{\alpha y}{p_1}\right) + \beta \ln\left(\frac{\beta y}{p_2}\right) + (1-\alpha-\beta)\ln\left(\frac{(1-\alpha-\beta)y}{p_3}\right) \\
&= \ln\left(\left(\frac{\alpha y}{p_1}\right)^{\alpha}\left(\frac{\beta y}{p_2}\right)^{\beta}\left(\frac{(1-\alpha-\beta)y}{p_3}\right)^{(1-\alpha-\beta)}\right).
\end{aligned}
$$

- What about second-order conditions?
- We need to examine the definiteness (or otherwise) of the Hessian matrix for the utility function from the "reduced-form" unconstrained optimisation problem.

Conversion Example Part 5

- Recall that

$$\hat{U}_1 = \frac{\alpha}{x_1} - \frac{(1-\alpha-\beta)\frac{p_1}{p_3}}{\left(\frac{y}{p_3} - \frac{p_1}{p_3}x_1 - \frac{p_2}{p_3}x_2\right)} = 0,$$

  and

$$\hat{U}_2 = \frac{\beta}{x_2} - \frac{(1-\alpha-\beta)\frac{p_2}{p_3}}{\left(\frac{y}{p_3} - \frac{p_1}{p_3}x_1 - \frac{p_2}{p_3}x_2\right)} = 0.$$

- Thus we have

$$\hat{U}_{11} = -\frac{\alpha}{x_1^2} - \frac{(1-\alpha-\beta)\left(\frac{p_1}{p_3}\right)^2}{\left(\frac{y}{p_3} - \frac{p_1}{p_3}x_1 - \frac{p_2}{p_3}x_2\right)^2} < 0 \text{ for all } x_1 \text{ and } x_2,$$

$$\hat{U}_{12} = \hat{U}_{21} = -\frac{(1-\alpha-\beta)\left(\frac{p_1 p_2}{p_3^2}\right)}{\left(\frac{y}{p_3} - \frac{p_1}{p_3}x_1 - \frac{p_2}{p_3}x_2\right)^2}, \text{ and}$$

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Conversion Example Part 6

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$$\hat{U}_{22} = -\frac{\beta}{x_2^2} - \frac{(1-\alpha-\beta)\left(\frac{p_2}{p_3}\right)^2}{(\frac{y}{p_3} - \frac{p_1}{p_3}x_1 - \frac{p_2}{p_3}x_2)^2}.$$

- Recall that

$$H = \begin{pmatrix} \hat{U}_{11} & \hat{U}_{12} \\ \hat{U}_{21} & \hat{U}_{22} \end{pmatrix}.$$

- Note that

$$\det(H_1) = \det(\hat{U}_{11}) = \hat{U}_{11}$$

$$= -\frac{\alpha}{x_1^2} - \frac{(1-\alpha-\beta)\left(\frac{p_1}{p_3}\right)^2}{(\frac{y}{p_3} - \frac{p_1}{p_3}x_1 - \frac{p_2}{p_3}x_2)^2} < 0 \text{ for all } x_1 \text{ and } x_2\text{, and}$$

- Continued on next page.

Conversion Example Part 7

- Continued from previous page

$$\det(H_2) = \det(H) = \hat{U}_{11}\hat{U}_{22} - \hat{U}_{12}\hat{U}_{21}.$$

- After some algebra and simplification, this becomes:

$$\det(H_2) = \frac{\alpha\beta}{x_1^2 x_2^2} + \frac{(1-\alpha-\beta)\left(\frac{p_2}{p_3}\right)^2 \alpha x_1^2}{\left(\frac{y}{p_3} - \frac{p_1}{p_3}x_1 - \frac{p_2}{p_3}x_2\right)^2} + \frac{(1-\alpha-\beta)\left(\frac{p_1}{p_3}\right)^2 \beta x_2^2}{\left(\frac{y}{p_3} - \frac{p_1}{p_3}x_1 - \frac{p_2}{p_3}x_2\right)^2}$$

$$> 0 \;\text{ for all }\; x_1 \;\text{ and }\; x_2.$$

- This means that the hessian matrix is negative definite for all combinations of ($x_1 > 0$ and $x_2 > 0$). Thus we can conclude that our candidate solution is a global maximum for the optimisation problem under consideration.