Some Notes on Solving Linear Equation Systems

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Contents

1 A note on sources

The material in this handout is drawn from Chapter eight (pp. 238-309) of Haeussler and Paul (1987). The title of that chapter is "Matrix Algebra". Please note that the material in this handout does not originate with me.

2 Introduction

This handout covers Guass-Jordan elimination, matrix inversion, determinants and Cramer's rule.

3 Gauss-Jordan elimination

In this section, we consider some examples that illustrate how to solve a system of simultaneous linear equations by applying the process of Gauss-Jordan elimination to an appropriate augmented matrix.

3.1 Augmented row matrices and row-reduction

Consider an arbitrary system of n linear equations in n unknown variables. This system is given by

$$\left\{ \begin{array}{rcll} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 & \text{(Equation 1)} \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 & \text{(Equation 2)} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n & = & b_n & \text{(Equation n)} \end{array} \right\}.$$

This equation system can be represented as an augmented row matrix of the form

$$(A\,|b) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array} \right).$$

Note that each row of this augmented row matrix corresponds to one of the equations in the system under consideration. The first row corrsponds to equation (1), the second row corresponds to equation (2), and so on and so forth, with the nth row corresponding to equation (n). Note also that each column on the left-hand side of the augmented row-matrix corresponds to a particular variable in the equations. The first column corresponds to the x_1

variable, the second column corresponds to the x_2 variable, and so on and so forth, with the nth column corresponding to the x_n variable. Finally, note that the last colum, which is on right-hand side of the augmented row matrix, corresponds to the constant terms on the right-hand side of the equations in the system. Thus this augmented row-matrix is one way of representing the equation system under consideration.

We will use the technique of Gauss-Jordan elimination in an attempt to solve this system of equations. This involves using a sequence of elementary row operations to attempt to transform the augmented row-matrix into the form $(I\,|c)$. If this can be done, then we will have found the unique solution to the system of equations. If this cannot be done, then there is either no solution to the system of equations or infinitely many solutions to the system of equations.

Elementary row operations come in three basic varieties. The first involves interchanging rows in the augmented row-matrix. This is equivalent to changing the ordering of the equations in the system. The second involves multiplying one of the rows in the augmented-row matrix by a constant. This is equivalent to multiplying both sides of one of the equations by a constant. The third involves adding one row to another row. This equivalent to adding one equation to another. Sometimes the last two steps are combined, in the sense that we add a multiple of one row to another row. This is equivalent to adding a multiple of one equation to another equation. If these steps are carried out correctly, they will not change the underlying system of equations in any fundamental way. As such, they will not alter the solution set for the underlying system og equations.

3.2 Example one

This example is based on a coefficient matrix that is considered in Haeussler and Paul (1987, p. 280). Consider the following system of two linear equations in two unknown variables:

$$\left\{ \begin{array}{rcll} 1x_1 + 0x_2 & = & 1 & \text{(Equation 1)} \\ 2x_1 + 2x_2 & = & 1 & \text{(Equation 2)} \end{array} \right\}.$$

The augmented row-matrix for this equation system is

$$(A\,|b) = \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 2 & 2 & 1 \end{array} \right).$$

The first step is obtain a leading one in the "first-row, first column" position of the augmented row matrix. In this case, we already have this first leading one. As such we can skip to the next step. The second step is to transform all of the entries below the first leading one to zeroes. This requires that we add (-2) times the first row to the second row. Upon doing this, we obtain

$$\begin{aligned}(A\,|b) &\rightarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 2 & 2 & 1 \end{array}\right) \ R2 \rightarrow R2 + (-2)\,R1 \\ &\rightarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 2 & -1 \end{array}\right).\end{aligned}$$

The third step is to obtain a leading one in the "first-row, first column" position of the augmented row matrix. This requires us to multiply the second row by $\left(\frac{1}{2}\right)$. Upon doing this, we obtain

$$\begin{aligned}(A\,|b) &\rightarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 2 & 2 & 1 \end{array}\right) \ R2 \rightarrow R2 + (-2)\,R1 \\ &\rightarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 2 & -1 \end{array}\right) \ R2 \rightarrow \left(\tfrac{1}{2}\right) R2 \\ &\rightarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -\left(\frac{1}{2}\right) \end{array}\right).\end{aligned}$$

Since there are no rows below row two, the next step is to transform all of the entries above the leading one in row two to zeroes. Since this is already the case, we can skip this step. Since there are no rows above row one, we are finished. We have obtained the "reduced row-echelon" form of the augmented row matrix $(A\,|b)$. Note that

$$\begin{aligned}(A\,|b) &\rightarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 2 & 2 & 1 \end{array}\right) \ R2 \rightarrow R2 + (-2)\,R1 \\ &\rightarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 2 & -1 \end{array}\right) \ R2 \rightarrow \left(\tfrac{1}{2}\right) R2 \\ &\rightarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -\left(\frac{1}{2}\right) \end{array}\right) \\ &\rightarrow (I\,|c)\,.\end{aligned}$$

Thus we know that the "reduced row-echelon" form of the augmented row matrix $(A\,|b)$ is given by $(I\,|c)$. This means that there is a unique solution to this system of equations. This solution is $x_1 = 1$ and $x_2 = -\left(\frac{1}{2}\right)$.

3.3 Example two

This example is based on a coefficient matrix that is considered in Haeussler and Paul (1987, pp. 281-282). Consider the system of three linear equations in three unknown variables that is given by

$$\left\{ \begin{array}{rcll} 1x_1 + 0x_2 - 2x_3 & = & 1 & \text{(Equation 1)} \\ 4x_1 - 2x_2 + 1x_3 & = & 1 & \text{(Equation 2)} \\ 1x_1 + 2x_2 - 10x_3 & = & 1 & \text{(Equation 3)} \end{array} \right\}.$$

The augmented row-matrix for this equation system is

$$(A\,|b) = \left(\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 4 & -2 & 1 & 1 \\ 1 & 2 & -10 & 1 \end{array} \right).$$

Note that

$$\begin{aligned}
(A\,|b) &\to \left(\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 4 & -2 & 1 & 1 \\ 1 & 2 & -10 & 1 \end{array} \right) \begin{array}{c} \\ R2 \to R2 + (-4)\,R1 \\ R3 \to R3 + (-1)\,R1 \end{array} \\
&\to \left(\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & -2 & 9 & -3 \\ 0 & 2 & -8 & 0 \end{array} \right) \begin{array}{c} \\ R2 \to \left(-\left(\frac{1}{2}\right)\right) R2 \\ R3 \to R3 + R2 \end{array} \\
&\to \left(\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & -\left(\frac{9}{2}\right) & \left(\frac{3}{2}\right) \\ 0 & 0 & 1 & -3 \end{array} \right) \begin{array}{c} R1 \to R1 + 2R3 \\ R2 \to R2 + \left(\frac{9}{2}\right) R3 \end{array} \\
&\to \left(\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & \left(\frac{-24}{2}\right) \\ 0 & 0 & 1 & -3 \end{array} \right) \\
&\to \left(\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -12 \\ 0 & 0 & 1 & -3 \end{array} \right) \\
&\to (I\,|c)\,.
\end{aligned}$$

Thus we know that the "reduced row-echelon" form of the augmented row matrix $(A\,|b)$ is given by $(I\,|c)$. This means that there is a unique solution to this system of equations. This solution is $x_1 = -5$, $x_2 = -12$ and $x_3 = -3$.

4 Matrix inversion by row-reduction

In this section, we consider some examples that illustrate how to obtain the inverse matrix (A^{-1}) of some original matrix (A) by applying the process of Gauss-Jordan elimination to an augmented matrix of the form $(A\,|I\,)$. If we can use elementary row operations to transform this augmented matrix into reduced row-echelon form (that is, $(I\,|B\,)$), then the matrix A is non-singular (that is, the matrix A is invertible) and the inverse matrix is given by $A^{-1} = B$.

4.1 Example one

This example comes from Haeussler and Paul (1987, p. 280).

Consider the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}.$$

We want to find A^{-1}, if it exists, by using row-reduction. First, we will set up the augmented matrix $(A\,|I\,)$. We will then use elementary row operations in an attempt to transform this matrix into another augmented matrix of the form $(I\,|B\,)$. If this can be done, and if we have not made any mistakes in the process of row-reduction, then it should be the case that $B = A^{-1}$. If we are able to transform $(A\,|I\,)$ to $(I\,|B\,)$, we will check that $B = A^{-1}$ is indeed the case by confirming that

$$AB = BA = I.$$

We have

$$\begin{array}{rcll} (A\,|I\,) & \longrightarrow & \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{array}\right) & R2 \longrightarrow R2 - 2R1 \\ & \longrightarrow & \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 2 & -2 & 1 \end{array}\right) & R2 \longrightarrow \left(\frac{1}{2}\right) R2 \\ & \longrightarrow & \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & \frac{1}{2} \end{array}\right) & \\ & \longrightarrow & (I\,|B\,). & \end{array}$$

Note that

$$\begin{aligned}
AB &= \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & \frac{1}{2} \end{pmatrix} \\
&= \begin{pmatrix} (1)(1)+(0)(-1) & (1)(0)+(0)\left(\frac{1}{2}\right) \\ (2)(1)+(2)(-1) & (2)(0)+(2)\left(\frac{1}{2}\right) \end{pmatrix} \\
&= \begin{pmatrix} 1+0 & 0+0 \\ 2-2 & 0+1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= I.
\end{aligned}$$

Similarly,

$$\begin{aligned}
BA &= \begin{pmatrix} 1 & 0 \\ -1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix} \\
&= \begin{pmatrix} (1)(1)+(0)(2) & (1)(0)+(0)(2) \\ (-1)(1)+\left(\frac{1}{2}\right)(2) & (-1)(0)+\left(\frac{1}{2}\right)(2) \end{pmatrix} \\
&= \begin{pmatrix} 1+0 & 0+0 \\ -1+1 & 0+1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= I.
\end{aligned}$$

Thus we can conclude that

$$A^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & \frac{1}{2} \end{pmatrix}.$$

4.2 Example two

This example comes from Haeussler and Paul (1987, pp. 281-282).

Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 4 & -2 & 1 \\ 1 & 2 & -10 \end{pmatrix}.$$

Once again, we want to attempt to find A^{-1} by using row-reduction. We have

$$
\begin{aligned}
(A\,|I\,) &\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 4 & -2 & 1 & 0 & 1 & 0 \\ 1 & 2 & -10 & 0 & 0 & 1 \end{array}\right) & \begin{array}{c} R2 \longrightarrow R2 - 4R1 \\ R3 \longrightarrow R3 - R1 \end{array} \\
&\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & -2 & 9 & -4 & 1 & 0 \\ 0 & 2 & -8 & -1 & 0 & 1 \end{array}\right) & \begin{array}{c} R2 \longrightarrow \left(\frac{-1}{2}\right) R2 \\ R3 \longrightarrow R3 + R2 \end{array} \\
&\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & \frac{-9}{2} & 2 & \frac{-1}{2} & 0 \\ 0 & 0 & 1 & -5 & 1 & 1 \end{array}\right) & \begin{array}{c} R1 \longrightarrow R1 + 2R3 \\ R2 \longrightarrow R2 + \left(\frac{9}{2}\right) R3 \end{array} \\
&\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -9 & 2 & 2 \\ 0 & 1 & 0 & \frac{-41}{2} & 4 & \frac{9}{2} \\ 0 & 0 & 1 & -5 & 1 & 1 \end{array}\right) \\
&\longrightarrow \qquad (I\,|B\,)\,.
\end{aligned}
$$

Thus we can conclude that

$$
A^{-1} = \begin{pmatrix} -9 & 2 & 2 \\ \frac{-41}{2} & 4 & \frac{9}{2} \\ -5 & 1 & 1 \end{pmatrix}.
$$

(As an exercise, you should confirm that this is indeed the case by verifying that $AB = BA = I$.)

5 Determinants

The determinant of a matrix is a very important function. A determinant is only defined for square matrices (that is, for $(n \times n)$ matrices). Square matrices are matrices that have the same number of rows and columns. In general, we will find the determinant of a matrix by using a technique that is known as a "cofactor expansion". The only exception to this is when we find the determinant of a (1×1) matrix. Note that a (1×1) matrix is simply a scalar. We will define the determinant of a scalar to be the scalar number itself. Thus if we have a (1×1) matrix of the form $A = (a_{11})$, then the determinant of A is given by $\det(A) = a_{11}$.

Consider an arbitrary $(n \times n)$ matrix of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Each element of this matrix has a minor. Note that the element a_{ij} is the (i,j)th element of the matrix, or the element that lies in both the ith row and the jth column of the matrix. We will denote the minor of the element a_{ij} by M_{ij}. The minor of the element a_{ij} is simply the determinant of the sub-matrix of A that is formed by deleting both the ith row of the matrix A and the jth column of the matrix A. We will denote this sub-matrix by A_{ij}. Note that A_{ij} is the $((n-1) \times (n-1))$ matrix given by

$$A_{ij} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1(j-1)} & a_{1(j+1)} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2(j-1)} & a_{2(j+1)} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{(i-1)1} & a_{(i-1)2} & \cdots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ a_{(i+1)1} & a_{(i+1)2} & \cdots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \cdots & a_{(i+1)n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n(j-1)} & a_{n(j+1)} & \cdots & a_{nn} \end{pmatrix}.$$

Thus the minor of the element a_{ij} is given by $\det(A_{ij})$. The cofactor of the element a_{ij} is given by $C_{ij} = (-1)^{(i+j)} \det(A_{ij})$.

The determinant of the $(n \times n)$ matrix A can be found by taking a cofactor expansion of the matrix along any given row or along any given column. Suppose that we choose to take a cofactor expansion along the ith row of the matrix A. In this case, we would have

$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij}.$$

If instead we wanted to calculate the determinant of A by taking a cofactor expansion along the jth column of the matrix A, we would have

$$\det(A) = \sum_{i=1}^{n} a_{ij} C_{ij}.$$

You should get the same value for the determinant, regardless of which row or column along which you choose to take the cofactor expansion.

5.1 Determinants of (2×2) matrices

Consider an arbitrary (2×2) matrix of the form

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Note that

$$C_{11} = (-1)^{(1+1)} \det(A_{11}) = (-1)^2 \det((a_{22})) = (1)(a_{22}) = a_{22}$$

and

$$C_{12} = (-1)^{(1+2)} \det(A_{12}) = (-1)^3 \det((a_{21})) = (-1)(a_{21}) = -a_{21}.$$

Upon taking a cofactor expansion along the first row of the matrix A, we obtain

$$\begin{aligned} \det(A) &= a_{11}C_{11} + a_{12}C_{12} \\ &= a_{11}a_{22} + a_{12}(-a_{21}) \\ &= a_{11}a_{22} - a_{12}a_{21}. \end{aligned}$$

Thus the determinant of a (2×2) matrix can be found by subtracting the product of its off-diagonal terms from the product of its main-diagonal terms.

5.2 Example one continued

Consider, once again, the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}.$$

The determinant of this matrix is given by

$$\det(A) = (1)(2) - (0)(2) = 2 - 0 = 2.$$

The cofactors for this matrix are given by

$$C_{11} = (-1)^{(1+1)} \det(A_{11}) = (-1)^2 \det((2)) = (1)(2) = 2,$$

$$C_{12} = (-1)^{(1+2)} \det(A_{12}) = (-1)^3 \det((2)) = (-1)(2) = -2,$$

$$C_{21} = (-1)^{(2+1)} \det(A_{21}) = (-1)^3 \det((0)) = (-1)(0) = 0$$

and

$$C_{22} = (-1)^{(2+2)} \det(A_{22}) = (-1)^4 \det((a_{11})) = (1)(1) = 1.$$

5.3 Example two continued

Consider, once again, the matrix

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 4 & -2 & 1 \\ 1 & 2 & -10 \end{pmatrix}.$$

The cofactors for this matrix are

$$\begin{aligned} C_{11} &= (-1)^{(1+1)} \det(A_{11}) \\ &= (-1)^2 \det\left(\begin{pmatrix} -2 & 1 \\ 2 & -10 \end{pmatrix}\right) \\ &= (1)\{(-2)(-10) - (1)(2)\} \\ &= \{20 - 2\} \\ &= 18, \end{aligned}$$

$$\begin{aligned} C_{12} &= (-1)^{(1+2)} \det(A_{12}) \\ &= (-1)^3 \det\left(\begin{pmatrix} 4 & 1 \\ 1 & -10 \end{pmatrix}\right) \\ &= (-1)\{(4)(-10) - (1)(1)\} \\ &= (-1)\{-40 - 1\} \\ &= (-1)(-41) \\ &= 41, \end{aligned}$$

$$
\begin{aligned}
C_{13} &= (-1)^{(1+3)} \det (A_{13}) \\
&= (-1)^4 \det \left(\begin{pmatrix} 4 & -2 \\ 1 & 2 \end{pmatrix} \right) \\
&= (1) \{(4)(2) - (-2)(1)\} \\
&= \{8 + 2\} \\
&= 10,
\end{aligned}
$$

$$
\begin{aligned}
C_{21} &= (-1)^{(2+1)} \det (A_{21}) \\
&= (-1)^3 \det \left(\begin{pmatrix} 0 & -2 \\ 2 & -10 \end{pmatrix} \right) \\
&= (-1) \{(0)(-10) - (-2)(2)\} \\
&= (-1) \{0 + 4\} \\
&= (-1)(4) \\
&= -4,
\end{aligned}
$$

$$
\begin{aligned}
C_{22} &= (-1)^{(2+2)} \det (A_{22}) \\
&= (-1)^4 \det \left(\begin{pmatrix} 1 & -2 \\ 1 & -10 \end{pmatrix} \right) \\
&= (1) \{(1)(-10) - (-2)(1)\} \\
&= \{-10 + 2\} \\
&= -8,
\end{aligned}
$$

$$
\begin{aligned}
C_{23} &= (-1)^{(2+3)} \det(A_{23}) \\
&= (-1)^5 \det\left(\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}\right) \\
&= (-1)\{(1)(2) - (0)(1)\} \\
&= (-1)\{2 - 0\} \\
&= (-1)(2) \\
&= -2,
\end{aligned}
$$

$$
\begin{aligned}
C_{31} &= (-1)^{(3+1)} \det(A_{31}) \\
&= (-1)^4 \det\left(\begin{pmatrix} 0 & -2 \\ -2 & 1 \end{pmatrix}\right) \\
&= (1)\{(0)(1) - (-2)(-2)\} \\
&= \{0 - 4\} \\
&= -4,
\end{aligned}
$$

$$
\begin{aligned}
C_{32} &= (-1)^{(3+2)} \det(A_{32}) \\
&= (-1)^5 \det\left(\begin{pmatrix} 1 & 4 \\ -2 & 1 \end{pmatrix}\right) \\
&= (-1)\{(1)(1) - (4)(-2)\} \\
&= (-1)\{1 + 8\} \\
&= (-1)(9) \\
&= -9
\end{aligned}
$$

and

$$\begin{aligned}
C_{33} &= (-1)^{(3+3)} \det(A_{33}) \\
&= (-1)^6 \det\left(\begin{pmatrix} 1 & 0 \\ 4 & -2 \end{pmatrix}\right) \\
&= (1)\{(1)(-2) - (0)(4)\} \\
&= \{-2 - 0\} \\
&= -2.
\end{aligned}$$

Using a cofactor expansion along the first row of the matrix A, we find that the determinant of this matrix is given by

$$\begin{aligned}
\det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\
&= (1)(18) + (0)(41) + (-2)(10) \\
&= 18 + 0 - 20 \\
&= -2.
\end{aligned}$$

6 Matrix inversion using cofactors

The cofactors of a matrix, along with the elements of the matrix, can be used to find the inverse of the matrix. Let A be an arbitrary $(n \times n)$ matrix. The inverse matrix for A is given by

$$A^{-1} = \left(\frac{1}{\det(A)}\right) Adj\,(A),$$

where $Adj\,(A)$ is the adjoint matrix for A. The adjoint matrix for A is given by

$$Adj\,(A) = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}^T = \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix}.$$

Thus the adjoint matrix for A can be found from the cofactors of the elements of A. Recall that the determinant can be found by using a cofactor expansion. Thus the determinant of A can be found from a combination of the elements of A and the cofactors of the elements of A.

6.1 When is a matrix invertible?

Some square matrices are not invertible. A square matrix that is not invertible is called a singular matrix. Correspondingly, a square matrix that is invertible is called a non-singular matrix. We can use the cofactor approach to finding an inverse matrix to suggest a simple criterion that will identify whether or not a matrix is singular (and hence not invertible). Recall that

$$A^{-1} = \left(\frac{1}{\det(A)}\right) Adj\,(A)\,.$$

Note that this expression is only defined when $\det(A) \neq 0$. If $\det(A) = 0$, then this expression for A^{-1} is undefined. It turns out that a matrix A is invertible if and only if $\det(A) \neq 0$. (Note that this argument does not constitute a formal proof of this result. It is merely suggests the possibility that the result might be true. However, you may take it as given that the result is indeed true.)

6.2 Example one continued

Consider, once again, the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}.$$

We have already found the determinant of this matrix and the cofactors for this matrix. Using those cofactors, we find that the adjoint matrix for A is given by

$$\begin{aligned} Adj\,(A) &= \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}^T \\ &= \begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix}^T \\ &= \begin{pmatrix} 2 & 0 \\ -2 & 1 \end{pmatrix}. \end{aligned}$$

Thus the inverse matrix for A is given by

$$\begin{aligned} A^{-1} &= \left(\frac{1}{\det(A)}\right) Adj\,(A) \\ &= \left(\frac{1}{2}\right)\begin{pmatrix} 2 & 0 \\ -2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ -1 & \frac{1}{2} \end{pmatrix}. \end{aligned}$$

6.3 Example two continued

Consider, once again, the matrix

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 4 & -2 & 1 \\ 1 & 2 & -10 \end{pmatrix}.$$

We have already found the determinant of this matrix and the cofactors for this matrix. Using those cofactors, we find that the adjoint matrix for A is given by

$$\begin{aligned} Adj\,(A) &= \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T \\ &= \begin{pmatrix} 18 & 41 & 10 \\ -4 & -8 & -2 \\ -4 & -9 & -2 \end{pmatrix}^T \\ &= \begin{pmatrix} 18 & -4 & -4 \\ 41 & -8 & -9 \\ 10 & -2 & -2 \end{pmatrix}. \end{aligned}$$

Thus the inverse matrix for A is given by

$$\begin{aligned} A^{-1} &= \left(\frac{1}{\det(A)}\right) Adj\,(A) \\ &= \left(\frac{1}{-2}\right)\begin{pmatrix} 18 & -4 & -4 \\ 41 & -8 & -9 \\ 10 & -2 & -2 \end{pmatrix} \\ &= \begin{pmatrix} -9 & 2 & 2 \\ \frac{-41}{2} & 4 & \frac{9}{2} \\ -5 & 1 & 1 \end{pmatrix}. \end{aligned}$$

7 An application of inverse matrices

Consider an arbitrary system of n linear equations in n unknown variables. This system is given by

$$\left\{ \begin{array}{ccccc} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 & \text{(Equation 1)} \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 & \text{(Equation 2)} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n & = & b_n & \text{(Equation n)} \end{array} \right\}.$$

This equation system can be represented as a matrix equation of the form

$$Ax = b,$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

and

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

If the matrix A is non-singular, so that A^{-1} exists, then we have

$$\begin{array}{rccc} & Ax & = & b \\ \Longleftrightarrow & A^{-1}Ax & = & A^{-1}b \\ \Longleftrightarrow & Ix & = & A^{-1}b \\ \Longleftrightarrow & x & = & A^{-1}b. \end{array}$$

Thus, if A is a non-singular matrix, then we can use its inverse matrix A^{-1}, along with the vector b, to calculate the solution value for each unknown variable in the vector x.

7.1 Example one continued

Consider the system of two linear equations in two unknown variables that is given by

$$\left\{ \begin{array}{rcll} 1x_1 + 0x_2 & = & 1 & \text{(Equation 1)} \\ 2x_1 + 2x_2 & = & 1 & \text{(Equation 2)} \end{array} \right\}.$$

This system can be represented in matrix form as

$$Ax = b,$$

where

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix},$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and

$$b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We have already shown that

$$A^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & \frac{1}{2} \end{pmatrix}.$$

Hence we have

$$\begin{aligned} x &= A^{-1}b \\ &= \begin{pmatrix} 1 & 0 \\ -1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} (1)(1) + (0)(1) \\ (-1)(1) + \left(\frac{1}{2}\right)(1) \end{pmatrix} \\ &= \begin{pmatrix} 1 + 0 \\ -1 + \left(\frac{1}{2}\right) \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ \frac{-1}{2} \end{pmatrix}. \end{aligned}$$

Thus we can conclude that

$$x_1 = 1$$

and

$$x_2 = \frac{-1}{2}.$$

7.2 Example two continued

Consider the system of three linear equations in three unknown variables that is given by

$$\left\{ \begin{array}{rcll} 1x_1 + 0x_2 - 2x_3 & = & 1 & \text{(Equation 1)} \\ 4x_1 - 2x_2 + 1x_3 & = & 1 & \text{(Equation 2)} \\ 1x_1 + 2x_2 - 10x_3 & = & 1 & \text{(Equation 3)} \end{array} \right\}.$$

This equation can be represented as a matrix equation of the form

$$Ax = b,$$

where

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 4 & -2 & 1 \\ 1 & 2 & -10 \end{pmatrix},$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

and

$$b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

We have already shown that

$$A^{-1} = \begin{pmatrix} -9 & 2 & 2 \\ \frac{-41}{2} & 4 & \frac{9}{2} \\ -5 & 1 & 1 \end{pmatrix}.$$

Hence we have

$$
\begin{aligned}
x &= A^{-1}b \\
&= \begin{pmatrix} -9 & 2 & 2 \\ \frac{-41}{2} & 4 & \frac{9}{2} \\ -5 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} (-9)(1) + (2)(1) + (2)(1) \\ \left(\frac{-41}{2}\right)(1) + (4)(1) + \left(\frac{9}{2}\right)(1) \\ (-5)(1) + (1)(1) + (1)(1) \end{pmatrix} \\
&= \begin{pmatrix} -9 + 2 + 2 \\ \left(\frac{-41}{2}\right) + 4 + \left(\frac{9}{2}\right) \\ -5 + 1 + 1 \end{pmatrix} \\
&= \begin{pmatrix} -5 \\ -12 \\ -3 \end{pmatrix}.
\end{aligned}
$$

Thus we can conclude that

$$x_1 = -5,$$

$$x_2 = -12$$

and

$$x_3 = -3.$$

8 Cramer's rule

An alternative way of finding the solution values of the unknown variables in a system of linear equations is provided by Cramer's rule. This approach makes use of determinants.

Once again, consider an arbitrary system of n linear equations in n unknown variables. This system is given by

$$
\left\{ \begin{array}{rcll}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &=& b_1 & \text{(Equation 1)} \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &=& b_2 & \text{(Equation 2)} \\
\vdots & \vdots & \vdots & \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &=& b_n & \text{(Equation n)}
\end{array} \right\}.
$$

As we have noted previously, this equation system can be represented as a matrix equation of the form

$$Ax = b,$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

and

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Suppose that A is a non-singular matrix, so that $\det(A) \neq 0$. According to Cramer's rule, under these circumstances, the solution value for the variable x_k is given by

$$x_k = \frac{\det\left(\overline{A}_k\right)}{\det(A)},$$

where

$$\overline{A}_k = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1(k-1)} & b_1 & a_{1(k+1)} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2(k-1)} & b_2 & a_{2(k+1)} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n(k-1)} & b_n & a_{n(k+1)} & \cdots & a_{nn} \end{pmatrix}.$$

Note that $\overline{A}_k$ is the matrix that is formed when we replace the kth column of the matrix A with the vector b.

8.1 Example one continued

Consider the system of two linear equations in two unknown variables that is given by

$$\left\{ \begin{array}{rcll} 1x_1 + 0x_2 & = & 1 & \text{(Equation 1)} \\ 2x_1 + 2x_2 & = & 1 & \text{(Equation 2)} \end{array} \right\}.$$

This system can be represented in matrix form as

$$Ax = b,$$

where

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix},$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and

$$b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We have already shown that

$$\det(A) = 2.$$

Note that

$$\overline{A}_1 = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$$

and

$$\overline{A}_2 = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}.$$

Hence we have

$$\det\left(\overline{A}_1\right) = (1)(2) - (0)(1) = 2 - 0 = 2$$

and

$$\det\left(\overline{A}_2\right) = (1)(1) - (1)(2) = 1 - 2 = -1.$$

Thus, upon applying Cramer's rule, we obtain

$$x_1 = \frac{\det\left(\overline{A}_1\right)}{\det(A)} = \frac{2}{2} = 1$$

and

$$x_2 = \frac{\det\left(\overline{A}_2\right)}{\det(A)} = \frac{-1}{2}.$$

8.2 Example two continued

Consider the system of three linear equations in three unknown variables that is given by

$$\left\{ \begin{array}{rcll} 1x_1 + 0x_2 - 2x_3 & = & 1 & \text{(Equation 1)} \\ 4x_1 - 2x_2 + 1x_3 & = & 1 & \text{(Equation 2)} \\ 1x_1 + 2x_2 - 10x_3 & = & 1 & \text{(Equation 3)} \end{array} \right\}.$$

This equation can be represented as a matrix equation of the form

$$Ax = b,$$

where

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 4 & -2 & 1 \\ 1 & 2 & -10 \end{pmatrix},$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

and

$$b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

We have already shown that

$$\det(A) = -2.$$

Note that

$$\overline{A}_1 = \begin{pmatrix} 1 & 0 & -2 \\ 1 & -2 & 1 \\ 1 & 2 & -10 \end{pmatrix},$$

$$\overline{A}_2 = \begin{pmatrix} 1 & 1 & -2 \\ 4 & 1 & 1 \\ 1 & 1 & -10 \end{pmatrix}$$

and

$$\overline{A}_3 = \begin{pmatrix} 1 & 0 & 1 \\ 4 & -2 & 1 \\ 1 & 2 & 1 \end{pmatrix}.$$

The determinant of $\overline{A}_1$ is given by

$$\begin{aligned}
\det\left(\overline{A}_1\right) &= (1)\,(-1)^{(1+1)}\det\left(\begin{pmatrix} -2 & 1 \\ 2 & -10 \end{pmatrix}\right) \\
&\quad + (0)\,(-1)^{(1+2)}\det\left(\begin{pmatrix} 1 & 1 \\ 1 & -10 \end{pmatrix}\right) \\
&\quad + (-2)\,(-1)^{(1+3)}\det\left(\begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix}\right) \\
&= (1)\,(-1)^2\left\{(-2)\,(-10) - (1)\,(2)\right\} \\
&\quad +0 \\
&\quad + (-2)\,(-1)^4\left\{(1)\,(2) - (-2)\,(1)\right\} \\
&= (1)\,(1)\left\{20 - 2\right\} + (-2)\,(1)\left\{2 + 2\right\} \\
&= (1)\,(18) + (-2)\,(4) \\
&= 18 - 8 \\
&= 10.
\end{aligned}$$

The determinant of $\overline{A}_2$ is given by

$$
\begin{aligned}
\det\left(\overline{A}_2\right) &= (1)\,(-1)^{(1+1)}\det\left(\begin{pmatrix} 1 & 1 \\ 1 & -10 \end{pmatrix}\right) \\
&\quad + (1)\,(-1)^{(1+2)}\det\left(\begin{pmatrix} 4 & 1 \\ 1 & -10 \end{pmatrix}\right) \\
&\quad + (-2)\,(-1)^{(1+3)}\det\left(\begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}\right) \\
&= (1)\,(-1)^2\,\{(1)\,(-10) - (1)\,(1)\} \\
&\quad + (1)\,(-1)^3\,\{(4)\,(-10) - (1)\,(1)\} \\
&\quad + (-2)\,(-1)^4\,\{(4)\,(1) - (1)\,(1)\} \\
&= (1)\,(1)\,\{-10 - 1\} \\
&\quad + (1)\,(-1)\,\{-40 - 1\} \\
&\quad + (-2)\,(1)\,\{4 - 1\} \\
&= (1)\,(-11) + (-1)\,(-41) + (-2)\,(3) \\
&= -11 + 41 - 6 \\
&= 24.
\end{aligned}
$$

The determinant of $\overline{A}_3$ is given by

$$\begin{aligned}
\det\left(\overline{A}_3\right) &= (1)\,(-1)^{(1+1)}\det\left(\begin{pmatrix} -2 & 1 \\ 2 & 1 \end{pmatrix}\right) \\
&\quad + (0)\,(-1)^{(1+2)}\det\left(\begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}\right) \\
&\quad + (1)\,(-1)^{(1+3)}\det\left(\begin{pmatrix} 4 & -2 \\ 1 & 2 \end{pmatrix}\right) \\
&= (1)\,(-1)^2\left\{(-2)\,(1)-(1)\,(2)\right\} \\
&\quad + (0)\,(-1)^3\left\{(4)\,(1)-(1)\,(1)\right\} \\
&\quad + (1)\,(-1)^4\left\{(4)\,(2)-(-2)\,(1)\right\} \\
&= (1)\,(1)\left\{-2-2\right\} \\
&\quad +0 \\
&\quad + (1)\,(1)\left\{8+2\right\} \\
&= (1)\,(-4)+(1)\,(10) \\
&= -4+10 \\
&= 6.
\end{aligned}$$

Thus, upon applying Cramer's rule, we obtain

$$x_1 = \frac{\det\left(\overline{A}_1\right)}{\det(A)} = \frac{10}{-2} = -5,$$

$$x_2 = \frac{\det\left(\overline{A}_2\right)}{\det(A)} = \frac{24}{-2} = -12$$

and

$$x_3 = \frac{\det\left(\overline{A}_3\right)}{\det(A)} = \frac{6}{-2} = -3.$$

References

[1] Haeussler, EF Jr and RS Paul (1987), *Introductory mathematical analysis for business, economics, and the life and social sciences (fifth edition)*, Prentice-Hall International Edition, Prentice-Hall, USA.