Notes on Arc Elasticities of Demand

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Sources

- This material has been drawn from, or influenced by, a number of sources.
- Some of these include those listed below.
  - Gans, J, S King, M Byford, and NG Mankiw (2015), *Principles of microeconomics (sixth edition)*, Cengage Learning Australia, China: Chapter 5.
  - Leftwich, RH (1960), *The price system and resource allocation (revised edition)*, Holt, Rinehart and Winston, USA: Chapter 3.

Definition of an Arc Elasticity

- Consider the function $y = f(x)$.
- Suppose that there is a change in the value taken by the variable x.
- This will in turn induce a change in the value taken by the variable y.
- The arc elasticity of y with respect to x is defined to be:

$$\epsilon_x^y = \frac{\text{Percentage Change in } y \text{ that is Induced by the Change in } x}{\text{Percentage Change in } x}$$

$$= \frac{\text{Percentage Change in } f(x)}{\text{Percentage Change in } x}.$$

The Mid-Point Formula for an Arc Elasticity Part 1

- Consider the function $y = f(x)$.
- Suppose that the value taken by x changes from x_1 to x_2.
  - Denote this change by $\Delta x = x_2 - x_1$.
  - The mid-point between x_1 and x_2 is

$$x_{MP} = x_1 + \frac{1}{2}(x_2 - x_1) = \left(\frac{x_1 + x_2}{2}\right).$$

  - The mid-point measure of the percentage change in x is

$$\% \text{ change in } x = \left(\frac{\Delta x}{x_{MP}}\right) 100.$$

The Mid-Point Formula for an Arc Elasticity Part 2

- The change in the value taken by x induces a change in the value taken by y from $y_1 = f(x_1)$ to $y_2 = f(x_2)$.
  - Denote this change by $\Delta y = y_2 - y_1 = f(x_2) - f(x_1)$.
  - The mid-point between y_1 and y_2 is

$$y_{MP} = y_1 + \frac{1}{2}(y_2 - y_1) = \left(\frac{y_1 + y_2}{2}\right) = \left(\frac{f(x_1) + f(x_2)}{2}\right).$$

  - The mid-point measure of the percentage change in y is

$$\% \text{ change in } y = \left(\frac{\Delta y}{y_{MP}}\right) 100.$$

The Mid-Point Formula for an Arc Elasticity Part 3

- The arc elasticity of y with respect to x is thus given by

$$\epsilon_x^y = \frac{\%\text{ change in } y}{\%\text{ change in } x}$$

$$= \frac{\left(\frac{\Delta y}{y_{MP}}\right) 100}{\left(\frac{\Delta x}{x_{MP}}\right) 100}$$

$$= \frac{\left(\frac{\Delta y}{y_{MP}}\right)}{\left(\frac{\Delta x}{x_{MP}}\right)}$$

$$= \left(\frac{x_{MP}}{y_{MP}}\right)\left(\frac{\Delta y}{\Delta x}\right).$$

Why use the Mid-Point Formula? Part 1

- Recall that the mid-point formula for the arc elasticity of y with respect to x is

$$\epsilon_x^y = \frac{\%\text{ change in } y}{\%\text{ change in } x} = \left(\frac{x_{MP}}{y_{MP}}\right)\left(\frac{\Delta y}{\Delta x}\right).$$

- The main alternatives to the mid-point formula are the initial point formula and the final point formula.
  - The initial point formula for the arc elasticity of y with respect to x is

$$\eta_x^y = \frac{\%\text{ change in } y}{\%\text{ change in } x} = \left(\frac{x_1}{y_1}\right)\left(\frac{\Delta y}{\Delta x}\right).$$

  - The final point formula for the arc elasticity of y with respect to x is

$$\theta_x^y = \frac{\%\text{ change in } y}{\%\text{ change in } x} = \left(\frac{x_2}{y_2}\right)\left(\frac{\Delta y}{\Delta x}\right).$$

Why use the Mid-Point Formula? Part 2

- For really small changes in the value of x, the there is not much difference between the measure of the arc elasticity of y with respect to x that is provided by the three alternative formulae.
  - When the change in the variable x is very small, the difference between x_{MP}, x_2, and x_1 will be very small.
  - Furthermore, if the function $y = f(x)$ is continuous, the difference between y_{MP}, y_1, and y_2 will also be very small when the change in the variable x is very small.
  - Indeed, if we take the limit as x_2 approaches x_1, then the three formulae will provide identical measures of the arc elasticity of y with respect to x. In this case, all three measures will be equal to the point elasticity of y with respect to x at the point (x_1, y_1), which is given by the formula

$$\varepsilon_x^y = \left(\frac{x_1}{y_1}\right)\left(\frac{dy}{dx}\right) = \frac{d\ln(y)}{d\ln(x)}.$$

- However, for discrete (non-infinitesimal) changes in the value of x, the measure of the arc elasticity of y with respect to x that is provided by the three alternative formulae will be different.

Why use the Mid-Point Formula? Part 3

- The main reason for using the mid-point approach to calculating an arc elasticity is the directional mis-match that occurs with the initial point and final point approaches, but does not occur for the midpoint approach.
  - If we calculate the arc elasticity of y with respect to a change in x, then the direction in which we move will matter for both the initial point measure and the final point measure, but it will not matter for the midpoint measure. This can be seen from the following example.
- Example: Consider the function $y = x^2$, along with the points $(1, 1)$ and $(5, 25)$ that lie on the graph of the function.
  - First, let $(x_1, y_1) = (1, 1)$ and $(x_2, y_2) = (5, 25)$. Calculate the arc elasticity of demand using each of the three formulae.
  - Second, let $(x_1, y_1) = (5, 25)$ and $(x_2, y_2) = (1, 1)$. Calculate the arc elasticity of demand using each of the three formulae.
  - Third, compare your answers.

Why use the Mid-Point Formula? Part 4

- Case 1: $(x_1, y_1) = (1, 1)$ and $(x_2, y_2) = (5, 25)$.
  - Here we have $\Delta x = (5 - 1) = 4$, $\Delta y = (25 - 1) = 24$, and $\frac{\Delta y}{\Delta x} = \frac{24}{4} = 6$.
  - We also have $x_{MP} = \frac{(1+5)}{2} = \frac{6}{2} = 3$, and $y_{MP} = \frac{(1+25)}{2} = \frac{26}{2} = 13$.
- Thus the arc elasticity measures in this case are as follows.
  - Mid-Point Approach: $\epsilon_x^y = \left(\frac{3}{13}\right)(6) = \frac{18}{13} \approx 1.38$.
  - Initial Point Approach: $\eta_x^y = \left(\frac{1}{1}\right)(6) = \frac{6}{1} = 6$.
  - Final Point Approach: $\theta_x^y = \left(\frac{5}{25}\right)(6) = \frac{30}{25} = \frac{6}{5} = 1.2$.

Why use the Mid-Point Formula? Part 5

- Case 2: $(x_1, y_1) = (5, 25)$ and $(x_2, y_2) = (1, 1)$.
  - Here we have $\Delta x = (1 - 5) = -4$, $\Delta y = (1 - 25) = -24$, and $\frac{\Delta y}{\Delta x} = \frac{-24}{-4} = 6$.
  - We also have $x_{MP} = \frac{(5+1)}{2} = \frac{6}{2} = 3$, and $y_{MP} = \frac{(25+1)}{2} = \frac{26}{2} = 13$.
- Thus the arc elasticity measures in this case are as follows.
  - Mid-Point Approach: $\epsilon_x^y = \left(\frac{3}{13}\right)(6) = \frac{18}{13} \approx 1.38$.
  - Initial Point Approach: $\eta_x^y = \left(\frac{5}{25}\right)(6) = \frac{30}{25} = \frac{6}{5} = 1.2$.
  - Final Point Approach: $\theta_x^y = \left(\frac{1}{1}\right)(6) = \frac{6}{1} = 6$.

Why use the Mid-Point Formula? Part 6

- Note the following.
  - The mid-point measure of the arc elasticity is identical in both cases.
  - The initial point measure of the arc elasticity is different in the two cases.
  - The final point measure of the arc elasticity is different in the two cases.
- Key Point: You should always use the mid-point approach to measuring arc elasticities.

Own-Price Elasticity of Demand Example

- Suppose that the demand curve for widgets is described by the equation $Q^D = \frac{100}{P}$.
- Consider the cases where price of widgets rises: (i) from \$1 to\$5, (ii) from \$5 to \$10, (iii) from \$10 to \$25.
- Calculate the arc own-price elasticity of demand for each of these cases.

Table: Arc Own Price Elasticity of Demand Calculations For $Q^D = {}^{100}\!/_{P}$.

Case	P_1	P_2	$Q_1 = Q^D(P_1)$	$Q_2 = Q^D(P_2)$	P_{MP}	Q_{MP}	ΔP	ΔQ	$\frac{\Delta Q}{\Delta P}$	$\frac{P_{MP}}{Q_{MP}}$	$\in_P^D$
(i)	1	5	100	20	3	60	4	– 80	– 20	(1/20)	– 1
(ii)	5	10	20	10	7.50	15	5	– 10	– 2	(1/2)	– 1
(iii)	10	25	10	4	17.50	7	15	– 6	– (2/5)	(5/2)	– 1

Cross-Price Elasticity of Demand Example

- Suppose that the demand curve for widgets is described by the equation $Q_W^D = \left(\frac{100}{P_W}\right) - 2P_D$, where P_D is the price per unit of another commodity known as a doodangle.
- Suppose that the price of widgets is fixed at $P_W = \$1$. This means that the demand function becomes $Q_W^D = 100 - 2P_D$.
- Consider the cases where price of doodangles rises: (i) from \$1 to\$5, (ii) from \$5 to \$10, (iii) from \$10 to \$25.
- Calculate the arc cross-price elasticity of demand for each of these cases.

Table: Arc Cross Price Elasticity of Demand Calculations For $Q_W^D = 100 - 2P_D$ **.**

Case	P_1	P_2	$Q_1 = Q^D(P_1)$	$Q_2 = Q^D(P_2)$	P_{MP}	Q_{MP}	ΔP	ΔQ	$\frac{\Delta Q}{\Delta P}$	$\frac{P_{MP}}{Q_{MP}}$	ϵ_P^D
(i)	1	5	98	90	3	94	4	-8	-2	(3/94)	$-(3/47)$
(ii)	5	10	90	80	7.50	85	5	-10	-2	(3/34)	$-(3/17)$
(iii)	10	25	80	50	17.50	65	15	-30	-2	(3/26)	$-(3/13)$

Income Elasticity of Demand Example

- Suppose that the demand curve for widgets is described by the equation $Q_W^D = \left(\frac{0.5Y}{P_W}\right)$, where Y is the consumer's income.
- Suppose that the price of widgets is fixed at $P_W = \$1$. This means that the demand function becomes $Q_W^D = 0.5Y$.
- Consider the cases where income rises: (i) from \$1 to\$5, (ii) from \$5 to \$10, (iii) from \$10 to \$25.
- Calculate the arc income elasticity of demand for each of these cases.

Table: Arc Income Elasticity of Demand Calculations For $Q_W^D = 0.5Y$.

Case	Y_1	Y_2	$Q_1 = Q^D(Y_1)$	$Q_2 = Q^D(Y_2)$	Y_{MP}	Q_{MP}	ΔY	ΔQ	$\frac{\Delta Q}{\Delta Y}$	$\frac{Y_{MP}}{Q_{MP}}$	ϵ_Y^D
(i)	1	5	0.5	2.5	3	1.5	4	2	(1/2)	2	1
(ii)	5	10	2.5	5	7.50	3.75	5	2.5	(1/2)	2	1
(iii)	10	25	5	12.5	17.50	8.75	15	7.5	(1/2)	2	1