A Further Comment on Comparative Statics

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A very useful approach 1

- In this set of lecture notes, we will explore a very useful approach to comparative static analysis.
- This approach starts with a system of simultaneous equations that jointly characterise some economic phenomena of interest. The solution to this system of equations will be a set of functions (or possibly correspondences) that express the outcome values taken by various economic choice variables as functions of the values taken by the economic parameters in the system.
- It is sometimes convenient to be able to obtain expressions for the impact of a change in one of the economic parameters, holding all of the other economic parameters constant, on one or more of the economic choice variables, without explicitly finding the functions (or correspondences) that are the solution to the original system of simultaneous equations.
- This can sometimes be achieved by the following process.

A very useful approach 2

- First, write each of the $n \in \mathbb{N}$ equations in the initial simultaneous equations system in the form

$$f^i\left(x_1, x_2, \cdots, x_n; \alpha_1, \alpha_2, \cdots, \alpha_m\right) = 0,$$

where

  - the i superscript denotes a particular equation in this n equation system, with $i \in \{1, 2, \cdots, n\}$;
  - the economic choice variables are x_1, x_2, $\cdots$, and x_n; and
  - the economic parameters are α_1, α_2, $\cdots$, and α_m.

A very useful approach 3

- Second, calculate the total differential for each of the $f^i(\cdot)$ functions on the left-hand side of the (possibly rewritten) system of simultaneous equations.
  - The total differential of a function is the linear, or first-order, differential approximation of the change in the value that is taken by the function that is induced by changes in the economic variables and economic parameters that make up the arguments of the function.
  - The total differential for $f^i(\cdot)$ is given by

$$df^i(\cdot) \approx \sum_{j=1}^{n} \left(\frac{\partial f^i(\cdot)}{\partial x_j} \right) dx_j + \sum_{k=1}^{m} \left(\frac{\partial f^i(\cdot)}{\partial \alpha_k} \right) d\alpha_k,$$

    where $df^i(\cdot)$ is the change in the value of the $f^i(\cdot)$ function, dx_j is the change in the value of the x_j variable, and $d\alpha_k$ is the change in the value of the α_k parameter.
  - If the changes in the economic variables and the economic parameters that induce the changes in the $f^i(\cdot)$ functions are sufficiently small, then this approximation should be reasonably good.

A very useful approach 4

- Third, note that, since the economic phenomenon in which we are interested requires that

$$f^{i}\left(x_1, x_2, \cdots, x_n; \alpha_1, \alpha_2, \cdots, \alpha_m\right) = 0$$

for all $i \in \{1, 2, \cdots, n\}$, any combination of changes in the economic parameters of the system that preserves the phenomenon of interest must not move the value of the $f^{i}(\cdot)$ functions away from zero (that is, it must not induce any changes in the values that are taken by the $f^{i}(\cdot)$ functions).
  - In other words, we require that $df^{i}(\cdot) = 0$ for all $i \in \{1, 2, \cdots, n\}$, which is approximately equivalent to a requirement that

$$\sum_{j=1}^{n}\left(\frac{\partial f^{i}(\cdot)}{\partial x_j}\right) dx_j + \sum_{k=1}^{m}\left(\frac{\partial f^{i}(\cdot)}{\partial \alpha_k}\right) d\alpha_k = 0$$

for all $i \in \{1, 2, \cdots, n\}$, when the changes in the economic variables and economic parameters are sufficiently small.

A very useful approach 5

- Fourth, rearrange each of the equations in the system of simultaneous "total differential equations" that have just been obtained so that all of the terms involving changes in the economic variables appear on the left-hand-side of the equations, and all of the terms involving changed in the economic parameters appear on the right-hand-side of the equations.
  - This yields a system of simultaneous equations of the form

$$\sum_{j=1}^{n} \left(\frac{\partial f^{i}\left(\cdot\right)}{\partial x_j} \right) dx_j = - \left(\sum_{k=1}^{m} \left(\frac{\partial f^{i}\left(\cdot\right)}{\partial \alpha_k} \right) d\alpha_k \right)$$

for all $i \in \{1, 2, \cdots, n\}$.

A very useful approach 6

- Fifth, note that in comparative statics exercises, we typically focus on the impact of a change in only one of the economic parameters on the economic phenomenon of interest, while holding all of the other economic parameters constant.
    - Suppose that we want to analyse the impact of a change in the α_s parameter alone. In this case, we should assume that $d\alpha_s \neq 0$, and set $d\alpha_k = 0$ for all $k \neq s$, in each of the equations in the system of simultaneous equations that we obtained in step four.
    - Upon doing this, we obtain a system of simultaneous equations of the form
    $$\sum_{j=1}^{n} \left(\frac{\partial f^i (\cdot)}{\partial x_j} \right) dx_j = - \left(\frac{\partial f^i (\cdot)}{\partial \alpha_s} \right) d\alpha_s$$
    for all $i \in \{1, 2, \cdots, n\}$.

A very useful approach 7

- Sixth, note that we can divide both sides of each of the equations in in the system of simultaneous equations that we obtained in step five by $d\alpha_s$, because $d\alpha_s \neq 0$.
    - Upon doing this, we obtain a system of simultaneous equations of the form
    $$\sum_{j=1}^{n} \left(\frac{\partial f^i(\cdot)}{\partial x_j} \right) \left(\frac{dx_j}{d\alpha_s} \right) = - \left(\frac{\partial f^i(\cdot)}{\partial \alpha_s} \right)$$
    for all $i \in \{1, 2, \cdots, n\}$.
- Seventh, note that
$$\lim_{d\alpha_s \longrightarrow 0} \left(\left. \frac{dx_j}{d\alpha_s} \right|_{d\alpha_k = 0 \ \forall \ k \neq 0} \right) = \frac{\partial x_j}{\partial \alpha_s}$$
for all $i \in \{1, 2, \cdots, n\}$, where $\frac{\partial x_j}{\partial \alpha_s}$ is the partial derivative of the implicitly defined function $x_j(\alpha_1, \alpha_2, \cdots, \alpha_m)$ with respect to α_s.

A very useful approach 8

- Eighth, note that if we take the limit as $d\alpha_s \longrightarrow 0$ of both sides of each of the equations in in the system of simultaneous equations that we obtained in step six, we obtain

$$\lim_{d\alpha_s \longrightarrow 0} \left\{ \sum_{j=1}^{n} \left(\frac{\partial f^i(\cdot)}{\partial x_j} \right) \left(\frac{dx_j}{d\alpha_s} \right) \right\} = \lim_{d\alpha_s \longrightarrow 0} \left\{ - \left(\frac{\partial f^i(\cdot)}{\partial \alpha_s} \right) \right\}$$

$$\iff \sum_{j=1}^{n} \left(\frac{\partial f^i(\cdot)}{\partial x_j} \right) \left\{ \lim_{d\alpha_s \longrightarrow 0} \left(\left. \frac{dx_j}{d\alpha_s} \right|_{d\alpha_k = 0 \ \forall \ k \neq 0} \right) \right\} = - \left(\frac{\partial f^i(\cdot)}{\partial \alpha_s} \right)$$

$$\iff \sum_{j=1}^{n} \left(\frac{\partial f^i(\cdot)}{\partial x_j} \right) \left(\frac{\partial x_j}{\partial \alpha_s} \right) = - \left(\frac{\partial f^i(\cdot)}{\partial \alpha_s} \right)$$

for all $i \in \{1, 2, \cdots, n\}$.

A very useful approach 9

- Ninth, note that the system of simultaneous equations that we obtained in step eight can be expressed in the form of a matrix equation as

$$\begin{pmatrix} \frac{\partial f^1(\cdot)}{\partial x_1} & \frac{\partial f^1(\cdot)}{\partial x_2} & \cdots & \frac{\partial f^1(\cdot)}{\partial x_n} \\ \frac{\partial f^2(\cdot)}{\partial x_1} & \frac{\partial f^2(\cdot)}{\partial x_2} & \cdots & \frac{\partial f^2(\cdot)}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f^n(\cdot)}{\partial x_1} & \frac{\partial f^n(\cdot)}{\partial x_2} & \cdots & \frac{\partial f^n(\cdot)}{\partial x_n} \end{pmatrix} \begin{pmatrix} \frac{\partial x_1}{\partial \alpha_s} \\ \frac{\partial x_2}{\partial \alpha_s} \\ \vdots \\ \frac{\partial x_n}{\partial \alpha_s} \end{pmatrix} = \begin{pmatrix} -\left(\frac{\partial f^1(\cdot)}{\partial \alpha_s}\right) \\ -\left(\frac{\partial f^2(\cdot)}{\partial \alpha_s}\right) \\ \vdots \\ -\left(\frac{\partial f^n(\cdot)}{\partial \alpha_s}\right) \end{pmatrix}.$$

A very useful approach 10

- Tenth, note that if

$$
\det \left\{ \begin{pmatrix} \frac{\partial f^1(\cdot)}{\partial x_1} & \frac{\partial f^1(\cdot)}{\partial x_2} & \cdots & \frac{\partial f^1(\cdot)}{\partial x_n} \\ \frac{\partial f^2(\cdot)}{\partial x_1} & \frac{\partial f^2(\cdot)}{\partial x_2} & \cdots & \frac{\partial f^2(\cdot)}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f^n(\cdot)}{\partial x_1} & \frac{\partial f^n(\cdot)}{\partial x_2} & \cdots & \frac{\partial f^n(\cdot)}{\partial x_n} \end{pmatrix} \right\} \neq 0,
$$

  so that the the coefficient matrix on the left-hand-side of the matrix equation that was obtained in step nine is non-singular (that is, it is invertible), then we can solve that matrix equation to obtain expressions for the n comparative static effects in the vector

$$
\begin{pmatrix} \frac{\partial x_1}{\partial \alpha_s} & \frac{\partial x_2}{\partial \alpha_s} & \cdots & \frac{\partial x_n}{\partial \alpha_s} \end{pmatrix}^T .
$$

The budget-constrained utility maximisation problem 1

- We will illustrate this process for a simple two commodity version of an individual's budget-constrained utility maximisation problem.
- The individual's problem is to choose $x_1 \geqslant 0$ and $x_2 \geqslant 0$ to maximise $U(x_1, x_2)$, subject to $p_1 x_1 + p_2 x_2 \leqslant y$.
- We will assume that the individual's preferences are sufficiently well behaved to ensure that he or she will optimally choose to both consume strictly positive amounts of both commodities and exhaust his or her budget.
- The Lagrangean function for this constrained optimisation problem is

$$\mathcal{L}(x_1, x_2, \lambda; p_1, p_2, y) = U(x_1, x_2) + \lambda\left[y - p_1 x_1 - p_2 x_2\right].$$

The budget-constrained utility maximisation problem 2

- The first-order conditions that characterise any critical points of the Lagrangean function for this constrained maximisation problem are

$$\left\{ \begin{array}{ccccc} \frac{\partial \mathcal{L}}{\partial x_1} & = & \frac{\partial U}{\partial x_1} - \lambda p_1 & = & 0, \\ \frac{\partial \mathcal{L}}{\partial x_2} & = & \frac{\partial U}{\partial x_2} - \lambda p_2 & = & 0, \\ \frac{\partial \mathcal{L}}{\partial \lambda} & = & y - p_1 x_1 - p_2 x_2 & = & 0. \end{array} \right\}$$

- In order to simplify the notation somewhat, let $\mathcal{L}_1 = \frac{\partial \mathcal{L}}{\partial x_1}$, $\mathcal{L}_2 = \frac{\partial \mathcal{L}}{\partial x_2}$, $\mathcal{L}_\lambda = \frac{\partial \mathcal{L}}{\partial \lambda}$, $U_1 = \frac{\partial U}{\partial x_1}$, and $U_2 = \frac{\partial U}{\partial x_2}$.
- Note that both U_1 and U_2 are functions of x_1 and x_2.

The budget-constrained utility maximisation problem 3

- The simplified notation allows us to rewrite the first-order conditions as

$$\left\{\begin{array}{rcccl} \mathcal{L}_1 & = & U_1 - \lambda p_1 & = & 0, \\ \\ \mathcal{L}_2 & = & U_2 - \lambda p_2 & = & 0, \\ \\ \mathcal{L}_\lambda & = & y - p_1 x_1 - p_2 x_2 & = & 0. \end{array}\right\}$$

- Suppose that there is a unique critical point, $(x_1^*, x_2^*, \lambda^*)$, of the Lagrangean function for this budget-constrained utility maximisation problem. If an appropriate rank condition and an appropriate second-order condition are satisfied for this constrained optimisation problem, then $x_1^* = x_1^D(p_1, p_2, y)$ will be the individual's ordinary demand function for commodity one, $x_2^* = x_2^D(p_1, p_2, y)$ will be the individual's ordinary demand function for commodity two, and $\lambda^* = \lambda(p_1, p_2, y)$ will be the individual's marginal utility of money income function.

- At this point, it is convenient to introduce some more notation.
- The second-order partial derivatives of the utility function will be denoted by $U_{11} = \frac{\partial U_1}{\partial x_1} = \frac{\partial^2 U}{\partial x_1^2}$, $U_{12} = \frac{\partial U_1}{\partial x_2} = \frac{\partial^2 U}{\partial x_2 \partial x_1}$, $U_{21} = \frac{\partial U_2}{\partial x_1} = \frac{\partial^2 U}{\partial x_1 \partial x_2}$, and $U_{22} = \frac{\partial U_2}{\partial x_2} = \frac{\partial^2 U}{\partial x_2^2}$.
- The second-order partial derivatives of the Lagrangean function for this constrained maximisation problem will be denoted by $\mathcal{L}_{11} = \frac{\partial \mathcal{L}_1}{\partial x_1} = \frac{\partial^2 \mathcal{L}}{\partial x_1^2}$, $\mathcal{L}_{12} = \frac{\partial \mathcal{L}_1}{\partial x_2} = \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1}$, $\mathcal{L}_{1\lambda} = \frac{\partial \mathcal{L}_1}{\partial \lambda} = \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1}$, $\mathcal{L}_{21} = \frac{\partial \mathcal{L}_2}{\partial x_1} = \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2}$, $\mathcal{L}_{22} = \frac{\partial \mathcal{L}_2}{\partial x_2} = \frac{\partial^2 \mathcal{L}}{\partial x_2^2}$, $\mathcal{L}_{2\lambda} = \frac{\partial \mathcal{L}_2}{\partial \lambda} = \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2}$, $\mathcal{L}_{\lambda 1} = \frac{\partial \mathcal{L}_\lambda}{\partial x_1} = \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda}$, $\mathcal{L}_{\lambda 2} = \frac{\partial \mathcal{L}_\lambda}{\partial x_2} = \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda}$, and $\mathcal{L}_{\lambda\lambda} = \frac{\partial \mathcal{L}_\lambda}{\partial \lambda} = \frac{\partial^2 \mathcal{L}}{\partial \lambda^2}$.

The budget-constrained utility maximisation problem 5

- The Hessian matrix (that is, the matrix of second-order partial derivatives) for the Lagrangean function for this budget-constrained utility maximisation problem is a bordered Hessian matrix for the individual's utility function, where the borders take the form of the partial derivatives of the budget constraint function $(b(x_1, x_2) = y - p_1x_1 - p_2x_2)$ with respect to x_1, x_2, and λ.
- This bordered Hessian matrix is

$$H = \begin{pmatrix} \mathcal{L}_{\lambda\lambda} & \mathcal{L}_{\lambda 1} & \mathcal{L}_{\lambda 2} \\ \mathcal{L}_{1\lambda} & \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{2\lambda} & \mathcal{L}_{21} & \mathcal{L}_{22} \end{pmatrix} = \begin{pmatrix} 0 & -p_1 & -p_2 \\ -p_1 & U_{11} & U_{12} \\ -p_2 & U_{21} & U_{22} \end{pmatrix}.$$

- Note that if the utility function is at least twice continuously differentiable in all of its arguments, then we know from Young's Theorem that $U_{12} = U_{21}$.

The budget-constrained utility maximisation problem 6

- The second-order conditions for a constrained maximisation problem in which there are n choice variables (in the context of a budget-constrained utility maximisation problem, this would mean n commodities) and $m < n$ constraints place restrictions of the signs of the last $(n - m)$ leading principle minors of the bordered Hessian matrix for the problem (when the borders are placed on the top rows and left columns of the bordered Hessian matrix).
- Specifically, the last $(n - m)$ leading principal minors of the bordered Hessian matrix must alternate in sign, with the last leading principle minor having the same sign as $(-1)^n$.
- In the two commodity version of a budget-constrained utility maximisation problem that is being considered here, there are $n = 2$ choice variables (namely, x_1 and x_2) and there is $m = 1$ constraint (namely, $b(x_1, x_2) = 0$). Since $n - m = 2 - 1 = 1$ in this case, the second-order condition for this problem requires that the determinant of the bordered Hessian matrix for the problem be strictly positive.

The budget-constrained utility maximisation problem 7

- Note that

$$
\begin{aligned}
\det(H) &= \begin{vmatrix} 0 & -p_1 & -p_2 \\ -p_1 & U_{11} & U_{12} \\ -p_2 & U_{21} & U_{22} \end{vmatrix} \\
&= 0 + (-p_1)(-1)^{1+2} \begin{vmatrix} -p_1 & U_{12} \\ -p_2 & U_{22} \end{vmatrix} \\
&\quad + (-p_2)(-1)^{1+3} \begin{vmatrix} -p_1 & U_{11} \\ -p_2 & U_{21} \end{vmatrix}
\end{aligned}
$$

- (Continued on the next slide.)

The budget-constrained utility maximisation problem 8

- (Continued from the previous slide.)

$$
\begin{aligned}
\det(H) &= (-p_1)(-1)^3\{-p_1 U_{22} - (-p_2 U_{12})\} \\
&\quad + (-p_2)(-1)^4\{-p_1 U_{21} - (-p_2 U_{11})\} \\
&= (-p_1)(-1)\{p_2 U_{12} - p_1 U_{22}\} \\
&\quad + (-p_2)(1)\{p_2 U_{11} - p_1 U_{21}\} \\
&= p_1\{p_2 U_{12} - p_1 U_{22}\} - p_2\{p_2 U_{11} - p_1 U_{21}\} \\
&= p_1 p_2 U_{12} - p_1^2 U_{22} - p_2^2 U_{11} + p_1 p_2 U_{21} \\
&= p_1 p_2 (U_{12} + U_{21}) - \left(p_1^2 U_{22} + p_2^2 U_{11}\right).
\end{aligned}
$$

The budget-constrained utility maximisation problem 9

- Recalling that $U_{21} = U_{12}$, this becomes

$$\begin{aligned} \det(H) &= p_1 p_2 (U_{12} + U_{21}) - \left(p_1^2 U_{22} + p_2^2 U_{11}\right) \\ &= 2 p_1 p_2 U_{12} - \left(p_1^2 U_{22} + p_2^2 U_{11}\right). \end{aligned}$$

- Thus the second-order condition for this two commodity budget-constrained utility maximisation problem is

$$\begin{aligned} \det(H) &> 0 \\ \iff \quad 2 p_1 p_2 U_{12} - \left(p_1^2 U_{22} + p_2^2 U_{11}\right) &> 0 \\ \iff \quad 2 p_1 p_2 U_{12} &> p_1^2 U_{22} + p_2^2 U_{11}. \end{aligned}$$

The budget-constrained utility maximisation problem 10

- The total differentials of $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_\lambda$ are

$$\left\{ \begin{array}{rcl} d\mathcal{L}_1 & = & U_{11}dx_1 + U_{12}dx_2 - p_1 d\lambda - \lambda dp_1 + 0dp_2 + 0dy, \\ d\mathcal{L}_2 & = & U_{21}dx_1 + U_{22}dx_2 - p_2 d\lambda + 0dp_1 - \lambda dp_2 + 0dy, \\ d\mathcal{L}_\lambda & = & -p_1 dx_1 - p_2 dx_2 + 0d\lambda - x_1 dp_1 - x_2 dp_2 + 1dy, \end{array} \right\}$$

where $U_{ij} = \frac{\partial U_i}{\partial x_j} = \frac{\partial^2 U}{\partial x_j \partial x_i}$ is a function of both x_1 and x_2.

The budget-constrained utility maximisation problem 11

- The first-order-conditions that characterise the critical point for the Lagrangean must be satisfied throughout this comparative static thought experiment. This requires that $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}_\lambda = 0$, which in turn requires that

$$\left\{ \begin{array}{rcl} U_{11}dx_1 + U_{12}dx_2 - p_1 d\lambda - \lambda dp_1 + 0dp_2 + 0dy & = & 0, \\ U_{21}dx_1 + U_{22}dx_2 - p_2 d\lambda + 0dp_1 - \lambda dp_2 + 0dy & = & 0, \\ -p_1 dx_1 - p_2 dx_2 + 0d\lambda - x_1 dp_1 - x_2 dp_2 + 1dy & = & 0. \end{array} \right\}$$

The budget-constrained utility maximisation problem 12

- Rearranging this system of simultaneous equations so that the dx_1, dx_2, and $d\lambda$ terms all appear on the left-hand-side of the equations; and the dp_1, dp_2, and dy terms all appear on the right-hand-side of the equations; we obtain

$$\left\{ \begin{array}{rcl} U_{11}dx_1 + U_{12}dx_2 - p_1 d\lambda & = & \lambda dp_1 - 0dp_2 - 0dy, \\ U_{21}dx_1 + U_{22}dx_2 - p_2 d\lambda & = & -0dp_1 + \lambda dp_2 - 0dy, \\ -p_1 dx_1 - p_2 dx_2 + 0d\lambda & = & x_1 dp_1 + x_2 dp_2 - 1dy, \end{array} \right\}$$

which can be written in a slightly less cluttered fashion as

$$\left\{ \begin{array}{rcl} U_{11}dx_1 + U_{12}dx_2 - p_1 d\lambda & = & \lambda dp_1, \\ U_{21}dx_1 + U_{22}dx_2 - p_2 d\lambda & = & \lambda dp_2, \\ -p_1 dx_1 - p_2 dx_2 & = & x_1 dp_1 + x_2 dp_2 - dy. \end{array} \right\}$$

A small change in p_1 alone 1

- Suppose that we want to examine the impact of a small change in the price of commodity one alone.
- In other words, we want to consider an economic shock for which $dp_1 \neq 0$ and $dp_2 = dy = 0$.
- The system of simultaneous equations that characterises the impacts of such a shock is given by

$$\left\{ \begin{array}{rcl} U_{11}dx_1 + U_{12}dx_2 - p_1 d\lambda & = & \lambda dp_1, \\ \\ U_{21}dx_1 + U_{22}dx_2 - p_2 d\lambda & = & 0, \\ \\ -p_1 dx_1 - p_2 dx_2 & = & x_1 dp_1. \end{array} \right\}$$

A small change in p_1 alone 2

- Since $dp_1 \neq 0$, we can divide both sides of each of the equations in this system of simultaneous equations by dp_1 to obtain

$$\left\{ \begin{array}{rcl} U_{11}\left(\frac{dx_1}{dp_1}\right) + U_{12}\left(\frac{dx_2}{dp_1}\right) - p_1\left(\frac{d\lambda}{dp_1}\right) & = & \lambda, \\ U_{21}\left(\frac{dx_1}{dp_1}\right) + U_{22}\left(\frac{dx_2}{dp_1}\right) - p_2\left(\frac{d\lambda}{dp_1}\right) & = & 0, \\ -p_1\left(\frac{dx_1}{dp_1}\right) - p_2\left(\frac{dx_2}{dp_1}\right) & = & x_1. \end{array} \right\}$$

A small change in p_1 alone 3

- Upon taking the limit as $dp_1 \longrightarrow 0$ of both sides of each of the equations in this system of simultaneous equations, we obtain

$$\left\{ \begin{array}{rcl} U_{11}\left(\frac{\partial x_1}{\partial p_1}\right) + U_{12}\left(\frac{\partial x_2}{\partial p_1}\right) - p_1\left(\frac{\partial \lambda}{\partial p_1}\right) & = & \lambda, \\ U_{21}\left(\frac{\partial x_1}{\partial p_1}\right) + U_{22}\left(\frac{\partial x_2}{\partial p_1}\right) - p_2\left(\frac{\partial \lambda}{\partial p_1}\right) & = & 0, \\ -p_1\left(\frac{\partial x_1}{\partial p_1}\right) - p_2\left(\frac{\partial x_2}{\partial p_1}\right) & = & x_1. \end{array} \right\}$$

A small change in p_1 alone 4

- The order of the equations in this system of simultaneous equations, and the order in which the relevant variables appear within each equation, can be rearranged to obtain

$$\left\{ \begin{array}{rcl} 0\left(\frac{\partial \lambda}{\partial p_1}\right) - p_1\left(\frac{\partial x_1}{\partial p_1}\right) - p_2\left(\frac{\partial x_2}{\partial p_1}\right) & = & x_1, \\ -p_1\left(\frac{\partial \lambda}{\partial p_1}\right) + U_{11}\left(\frac{\partial x_1}{\partial p_1}\right) + U_{12}\left(\frac{\partial x_2}{\partial p_1}\right) & = & \lambda, \\ -p_2\left(\frac{\partial \lambda}{\partial p_1}\right) + U_{21}\left(\frac{\partial x_1}{\partial p_1}\right) + U_{22}\left(\frac{\partial x_2}{\partial p_1}\right) & = & 0. \end{array} \right\}$$

A small change in p_1 alone 5

- The matrix equation representation of this system of simultaneous equations is

$$\begin{pmatrix} 0 & -p_1 & -p_2 \\ -p_1 & U_{11} & U_{12} \\ -p_2 & U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} \frac{\partial \lambda}{\partial p_1} \\ \frac{\partial x_1}{\partial p_1} \\ \frac{\partial x_2}{\partial p_1} \end{pmatrix} = \begin{pmatrix} x_1 \\ \lambda \\ 0 \end{pmatrix}.$$

- Note that the coefficient matrix component of the left-hand-side of this matrix equation is simply the bordered Hessian matrix from the two-commodity budget-constrained utility maximisation problem. If the appropriate second-order condition for that problem is satisfied, then we know that its determinant is strictly positive. Since its determinant is non-zero, we know that it is a non-singular matrix. This means that it is an invertible matrix.

A small change in p_1 alone 6

- Since the coefficient matrix on the left-hand-side of this matrix equation is invertible, we can conclude that the vector of comparative statics derivatives is given by

$$\begin{pmatrix} \frac{\partial \lambda}{\partial p_1} \\ \frac{\partial x_1}{\partial p_1} \\ \frac{\partial x_2}{\partial p_1} \end{pmatrix} = \begin{pmatrix} 0 & -p_1 & -p_2 \\ -p_1 & U_{11} & U_{12} \\ -p_2 & U_{21} & U_{22} \end{pmatrix}^{-1} \begin{pmatrix} x_1 \\ \lambda \\ 0 \end{pmatrix}.$$

Some exercises

- You might like to try the following two exercises for yourself.
  1. Examine the impact of a small change in the price of commodity two alone. (In other words, conduct the above exercise for the case of an economic shock for which $dp_2 \neq 0$ and $dp_1 = dy = 0$.)
  2. Examine the impact of a small change in the individual's income alone. (In other words, conduct the above exercise for the case of an economic shock for which $dy \neq 0$ and $dp_1 = dp_2 = 0$.)