Vector and Matrix Arithmetic

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Sources and References Part 1

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What is a matrix?

- A matrix is an array of numbers or variables. It is organised into rows and columns. These form its dimensions.
- An $(n \times m)$ matrix has n rows and m columns. Note that, while it is possible that $n = m$, it is also possible that $n \neq m$. When $n = m$, we say that the matrix is a square matrix.
- An $(n \times m)$ matrix takes the following form:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm} \end{pmatrix}.$$

Scalars and vectors

- A scalar is a real number ($a \in \mathbb{R}$).
- A column vector is an $(n \times 1)$ matrix of the form

$$A = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{n1} \end{pmatrix}.$$

- A row vector is a $(1 \times m)$ matrix of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \end{pmatrix}.$$

Economic examples of matrices and vectors Part 1

- The $(L \times 1)$ price vector for an economy there are L commodities:

$$p = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_L \end{pmatrix}.$$

- The $(L \times 1)$ Marshallian demand vector for a consumer whose preferences are defined over bundles of L commodities:

$$x(p, y) = \begin{pmatrix} x_1(p, y) \\ x_2(p, y) \\ x_3(p, y) \\ \vdots \\ x_L(p, y) \end{pmatrix}.$$

Economic examples of matrices and vectors Part 2

- The $(n \times k)$ "design matrix" (that is, matrix of independent variables) in a multivatiate linear regression model of the form

$$y_i = \beta_0 + \sum_{j=1}^{k-1} \beta_j x_{i,j} + \epsilon_i,$$

where there are observations of all of the observable variables for each of n sample points.

- This matrix takes the form

$$X = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & x_{1,(k-1)} \\ 1 & x_{2,2} & x_{2,3} & \cdots & x_{2,(k-1)} \\ 1 & x_{3,2} & x_{3,3} & \cdots & x_{3,(k-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n,2} & x_{n,3} & \cdots & x_{n,(k-1)} \end{pmatrix}.$$

An overview of matrix arithmetic

- Scalar multiplication of a matrix.
- Matrix addition.
- Matrix subtraction.
- Matrix multiplication: The inner, or dot, product.
- The transpose of a matrix and matrix symmetry.
- The additive inverse of a matrix and the null matrix.
- The multiplicative inverse of a matrix and the identity matrix.
- Idempotent matrices.
- Vector inequalities.

Scalar Multiplication Part 1

- Suppose that $c \in \mathbb{R}$ and A is an $(n \times m)$ matrix takes the following form:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm} \end{pmatrix}.$$

- We will assume that $a_{ij} \in \mathbb{R}$ for all

$$(i, j) \in \{1, 2, \cdots, n\} \times \{1, 2, \cdots, m\}.$$

Scalar Multiplication Part 2

- The scalar pre-product of this constant with this matrix is given by

$$
\begin{aligned}
cA &= c\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm} \end{pmatrix} \\
&= \begin{pmatrix} ca_{11} & ca_{12} & ca_{13} & \cdots & ca_{1m} \\ ca_{21} & ca_{22} & ca_{23} & \cdots & ca_{2m} \\ ca_{31} & ca_{32} & ca_{33} & \cdots & ca_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ca_{n1} & ca_{n2} & ca_{n3} & \cdots & ca_{nm} \end{pmatrix}.
\end{aligned}
$$

Scalar Multiplication Part 3

- The scalar post-product of the matrix with constant is given by

$$
\begin{aligned}
Ac &= \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm} \end{pmatrix} c \\
&= \begin{pmatrix} ca_{11} & ca_{12} & ca_{13} & \cdots & ca_{1m} \\ ca_{21} & ca_{22} & ca_{23} & \cdots & ca_{2m} \\ ca_{31} & ca_{32} & ca_{33} & \cdots & ca_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ca_{n1} & ca_{n2} & ca_{n3} & \cdots & ca_{nm} \end{pmatrix}.
\end{aligned}
$$

- Note that $cA = Ac$. As such, we can just talk about the scalar product of a constant with a matrix, without specifying the order in which the multiplication takes place.

Some examples of scalar multiplication of a matrix Part 1

- The following examples come from Asano (2013, pp. 222-224).
- Example 1:

$$2X = 2\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2(1) & 2(1) \\ 2(1) & 2(2) \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}.$$

- Example 2:

$$3Y = 3\begin{pmatrix} 2 & 1 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 3(2) & 3(1) \\ 3(2) & 3(4) \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ 6 & 12 \end{pmatrix}.$$

- Example 3:

$$2Z = 2\begin{pmatrix} -1 & -2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 2(-1) & 2(-2) \\ 2(3) & 2(1) \end{pmatrix} = \begin{pmatrix} -2 & -4 \\ 6 & 2 \end{pmatrix}.$$

Some examples of scalar multiplication of a matrix Part 2

- The following examples come from Sydsaeter and Hammond (2006, pp. 555-556).
- Example 4:

$$3A = 2\begin{pmatrix} 1 & 2 & 0 \\ 4 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 3(1) & 3(2) & 3(0) \\ 3(4) & 3-(3) & 3(1) \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 6 & 0 \\ 12 & -9 & 3 \end{pmatrix}.$$

- Example 5:

$$\left(\frac{-1}{2}\right) B = 2\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} \left(\frac{-1}{2}\right)(0) & \left(\frac{-1}{2}\right)(1) & \left(\frac{-1}{2}\right)(2) \\ \left(\frac{-1}{2}\right)(1) & \left(\frac{-1}{2}\right)(0) & \left(\frac{-1}{2}\right)(2) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{-1}{2} & -1 \\ \frac{-1}{2} & 0 & -1 \end{pmatrix}.$$

Matrix Addition Part 1

- The sum of two matrices is only defined if the two matrices have exactly the same dimensions.
- Suppose that A is an $(n \times m)$ matrix that takes the following form:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm} \end{pmatrix}.$$

- Suppose that B is an $(n \times m)$ matrix that takes the following form:

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1m} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2m} \\ b_{31} & b_{32} & b_{33} & \cdots & b_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nm} \end{pmatrix}.$$

Matrix Addition Part 2

- The matrix sum $(A + B)$ is an $(n \times m)$ matrix that takes the following form:

$$
\begin{aligned}
A + B \\
&= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{pmatrix} \\
&= \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1m} + b_{1m} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2m} + b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} + b_{n1} & a_{n2} + b_{n2} & \cdots & a_{nm} + b_{nm} \end{pmatrix}.
\end{aligned}
$$

- Note that $A + B = B + A$. (Exercise: Convince yourself of the validity of this claim.)

Some examples of matrix addition Part 1

- Suppose that A is an $(m \times n)$ matrix, B is an $(n \times m)$ matrix and C is an $(n \times p)$ matrix, where $m \neq n$, $m \neq p$ and $n \neq p$.
- Example 1: Neither the matrix sum $A + B$ nor the matrix sum $B + A$ are defined.
- Example 2: Neither the matrix sum $A + C$ nor the matrix sum $C + A$ are defined.
- Example 3: Neither the matrix sum $B + C$ nor the matrix sum $C + B$ are defined.

Some examples of matrix addition Part 2

- The following examples come from Asano (2013, pp. 222-224).
- Example 4:

$$X + Y = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1+1 & 1+0 \\ 1+1 & 2+2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & 4 \end{pmatrix}.$$

- Example 5:

$$X + Z = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} -1 & 3 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1+(-1) & 1+3 \\ 1+(-2) & 2+1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 4 \\ -1 & 3 \end{pmatrix}.$$

Some examples of matrix addition Part 3

- The following example comes from Sydsaeter and Hammond (2006, pp. 555-556).
- Example 6:

$$M + N = \begin{pmatrix} 1 & 2 & 0 \\ 4 & -3 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1+0 & 2+1 & 0+2 \\ 4+1 & -3+0 & -1+2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 & 2 \\ 5 & -3 & 1 \end{pmatrix}.$$

An economic example of matrix addition Part 1

- Suppose that there are I consumers in an economy, each of whom has preferences defined over bundles of the same L commodities.
- Suppose that consumer i's Marshallian demand vector is given by the column vector

$$x^i(p, y^i) = \begin{pmatrix} x_1^i(p, y^i) \\ x_2^i(p, y^{i)} \\ x_3^i(p, y^i) \\ \vdots \\ x_L^i(p, y^i) \end{pmatrix}.$$

- The aggegate vector of Marshallian demands (or, if you prefer, the vector of market demands) for this economy is given by

$$x(p, y^1, y^2, \cdots, y^I) = \sum_{i=1}^{I} x^i(p, y^i)$$

- Continued on the next page.

An economic example of matrix addition Part 2

- Continued from the previous page.

$$= \sum_{i=1}^{I} \begin{pmatrix} x_1^i(p, y^i) \\ x_2^i(p, y^i) \\ x_3^i(p, y^i) \\ \vdots \\ x_L^i(p, y^i) \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=1}^{I} x_1^i(p, y^i) \\ \sum_{i=1}^{I} x_2^i(p, y^i) \\ \sum_{i=1}^{I} x_3^i(p, y^i) \\ \vdots \\ \sum_{i=1}^{I} x_L^i(p, y^i) \end{pmatrix}.$$

Matrix Subtraction Part 1

- Matrix subtraction involves a combination of (i) scalar multiplication of a matrix, and (ii) matrix addition.
- As with matrix addition, the difference of two matrices is only defined if the two matrices have exactly the same dimensions.
- Suppose that A and B are both $(n \times m)$ matrices. The difference between A and B is defined to be

$$A - B = A + (-1)\,B.$$

Matrix Subtraction Part 2

- Since A and B are both $(n \times m)$ matrices, the matrix difference $(A - B)$ is an $(n \times m)$ matrix that takes the following form:

$$
\begin{aligned}
& A - B \\
= \ & A + (-1)\, B \\
= \ & \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} + (-1) \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{pmatrix}
\end{aligned}
$$

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Matrix Subtraction Part 3

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$$= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} + \begin{pmatrix} -b_{11} & -b_{12} & \cdots & -b_{1m} \\ -b_{21} & -b_{22} & \cdots & -b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ -b_{n1} & -b_{n2} & \cdots & -b_{nm} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} - b_{11} & a_{12} - b_{12} & \cdots & a_{1m} - b_{1m} \\ a_{21} - b_{21} & a_{22} - b_{22} & \cdots & a_{2m} - b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} - b_{n1} & a_{n2} - b_{n2} & \cdots & a_{nm} - b_{nm} \end{pmatrix}.$$

- In general, $A - B \neq B - A$. Thus matrix subtraction does not share all of the properties of matrix addition.
  - Exercise: Under what circumstances will $A - B = B - A$?

A matrix subtraction example

- The following example comes from Asano (2013, pp. 222-224).

$$
\begin{aligned}
X - Y &= \begin{pmatrix} 6 & 3 \\ 6 & 12 \end{pmatrix} - \begin{pmatrix} -2 & -4 \\ 6 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 6 & 3 \\ 6 & 12 \end{pmatrix} + (-1) \begin{pmatrix} -2 & -4 \\ 6 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 6 & 3 \\ 6 & 12 \end{pmatrix} + \begin{pmatrix} (-1)(-2) & (-1)(-4) \\ (-1)(6) & (-1)(2) \end{pmatrix} \\
&= \begin{pmatrix} 6 & 3 \\ 6 & 12 \end{pmatrix} + \begin{pmatrix} 2 & 4 \\ -6 & -2 \end{pmatrix} \\
&= \begin{pmatrix} 6 + 2 & 3 + 4 \\ 6 + (-6) & 12 + (-2) \end{pmatrix} \\
&= \begin{pmatrix} 8 & 7 \\ 0 & 10 \end{pmatrix}.
\end{aligned}
$$

Matrix Multiplication Part 1

- The standard matrix product is the dot, or inner, product of two matrices.
- The dot product of two matrices is only defined for cases in which the number of columns of the first listed matrix is identical to the number of rows of the second listed matrix.
- If the dot product is defined, the solution matrix will have the same number of rows as the first listed matrix and the same number of columns as the second listed matrix.

Matrix Multiplication Part 2

- Suppose that X is an $(m \times n)$ matrix, Y is an $(n \times m)$ matrix and Z is an $(n \times p)$ matrix, where $m \neq n$, $m \neq p$ and $n \neq p$.
- The matrix product XY is defined and will be an $(m \times m)$ matrix.
- The matrix product YX is defined and will be an $(n \times n)$ matrix.
- The matrix product XZ is defined and will be an $(m \times p)$ matrix.
- The matrix products ZX, YZ and ZY are not defined.

Matrix Multiplication Part 3

- Suppose that A is an $(n \times m)$ matrix that takes the following form:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm} \end{pmatrix}.$$

- Suppose that B is an $(m \times p)$ matrix that takes the following form:

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1p} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2p} \\ b_{31} & b_{32} & b_{33} & \cdots & b_{3p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & b_{m3} & \cdots & b_{mp} \end{pmatrix}.$$

Matrix Multiplication Part 4

- The matrix product AB is defined and will be an $(n \times p)$ matrix. The solution matrix is given by

$$
\begin{aligned}
& AB \\
= & \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mp} \end{pmatrix} \\
= & \begin{pmatrix} \sum_{k=1}^{m} a_{1k} b_{k1} & \sum_{k=1}^{m} a_{1k} b_{k2} & \cdots & \sum_{k=1}^{m} a_{1k} b_{kp} \\ \sum_{k=1}^{m} a_{2k} b_{k1} & \sum_{k=1}^{m} a_{2k} b_{k2} & \cdots & \sum_{k=1}^{m} a_{2k} b_{kp} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^{m} a_{nk} b_{k1} & \sum_{k=1}^{m} a_{nk} b_{k2} & \cdots & \sum_{k=1}^{m} a_{nk} b_{kp} \end{pmatrix}.
\end{aligned}
$$

Matrix Multiplication Part 5

- Note that, while it is possible for $AB = BA$ in some cases, in general we will have $AB \neq BA$.
- There are three reasons for this.
  - First, BA will not necessarily be defined even if AB is defined.
  - Second, even when both AB and BA are defined, they might have different dimensions.
  - Third, even when both AB and BA are defined and have the same dimensions, they might have one or more different entries.

Some examples of matrix multiplication Part 1

- These examples come from Bradley (2008, pp. 490-492).
- Example 1:

$$AB = \begin{pmatrix} 1 & 2 \\ -2 & 4 \end{pmatrix} \cdot \begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} (1)(0)+(2)(1) & (1)(2)+(2)(0) & (1)(2)+(2)(5) \\ (-2)(0)+(4)(1) & (-2)(2)+(4)(0) & (-2)(2)+(4)(5) \end{pmatrix}$$

$$= \begin{pmatrix} 0+2 & 2+0 & 2+10 \\ 0+4 & -4+0 & -4+20 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 2 & 12 \\ 4 & -4 & 16 \end{pmatrix}.$$

- Example 2: The matrix product BA is undefined because the number of columns in B (which is three) does not equal the number of rows in A (which is two).
- Note that $AB \neq BA$.

Some examples of matrix multiplication Part 2

- Example 3:

$$AC = \begin{pmatrix} 1 & 2 \\ -2 & 4 \end{pmatrix} \cdot \begin{pmatrix} 3 & -2 \\ 5 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} (1)(3) + (2)(5) & (1)(-2) + (2)(0) \\ (-2)(3) + (4)(5) & (-2)(-2) + (4)(0) \end{pmatrix}$$

$$= \begin{pmatrix} 3 + 10 & -2 + 0 \\ -6 + 20 & 4 + 0 \end{pmatrix}$$

$$= \begin{pmatrix} 13 & -2 \\ 14 & 4 \end{pmatrix}.$$

Some examples of matrix multiplication Part 3

- Example 4:

$$
\begin{aligned}
CA &= \begin{pmatrix} 3 & -2 \\ 5 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ -2 & 4 \end{pmatrix} \\
&= \begin{pmatrix} (3)(1) + (-2)(-2) & (3)(2) + (-2)(4) \\ (5)(1) + (0)(-2) & (5)(2) + (0)(4) \end{pmatrix} \\
&= \begin{pmatrix} 3 + 4 & 6 + (-8) \\ 5 + 0 & 10 + 0 \end{pmatrix} \\
&= \begin{pmatrix} 7 & -2 \\ 5 & 10 \end{pmatrix}.
\end{aligned}
$$

- Note that $AC \neq CA$.

An economic example of matrix multiplication Part 1

- Suppose that a consumer whose preferences are defined over bundles of L commodities faces a price vector given by the row vector $p = (p_1, p_2, \cdots, p_L)$ and chooses to purchase the quantities of each commodity that are given by the column vector

$$q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \\ q_L \end{pmatrix}.$$

An economic examples of matrix multiplication Part 2

- The consumer's total expenditure will be equal to

$$pq = (p_1, p_2, \cdots, p_L) \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \\ q_L \end{pmatrix}$$

$$= p_1 q_1 + p_2 q_2 + \cdots + p_L q_L$$

$$= \sum_{l=1}^{L} p_l q_l.$$

Matrix Transposition Part 1

- Suppose that A is an $(n \times m)$ matrix. The transpose of the matrix A, which is denoted by A^T, is the $(m \times n)$ matrix that is formed by taking the rows of A and turning them into columns, without changing their order.
- In other words, the ith column of A^T is the ith row of A.
- This also means that the jth row of A^T is the jth column of A.

Matrix Transposition Part 2

- Suppose that A is the $(n \times m)$ matrix that takes the following form:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm} \end{pmatrix}.$$

- The transpose of the matrix A is the $(m \times n)$ matrix that takes the following form:

$$A^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{n1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{n2} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{n3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1m} & a_{2m} & a_{3m} & \cdots & a_{nm} \end{pmatrix}.$$

Some examples of matrix transposition Part 1

- These examples come from Shannon (1995, p. 139, Example 8).
- Example 1: If

$$X = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix},$$

then $X^T = (1, 3, 5)$.

- Example 2: If

$$Y = \begin{pmatrix} 2 & 3 \\ 5 & 9 \\ 7 & 6 \end{pmatrix},$$

then

$$Y^T = \begin{pmatrix} 2 & 5 & 7 \\ 3 & 9 & 6 \end{pmatrix}.$$

Some examples of matrix transposition Part 2

- Example 3: If

$$Z = \begin{pmatrix} 1 & 3 & 7 \\ 4 & 5 & 11 \\ 6 & 8 & 10 \end{pmatrix},$$

then

$$Z^T = \begin{pmatrix} 1 & 4 & 6 \\ 3 & 5 & 8 \\ 7 & 11 & 10 \end{pmatrix}.$$

Symmetric Matrices

- In general, $A^T \neq A$.
- There are two reasons for this.
  - First, unless A is a square matrix (that is, unless it has the same number of rows and columns), the dimensions of the matrix A^T will be different to the dimensions of the matrix A.
  - Second, even if A is a square matrix, in general the ith row of A will not be identical to the ith column of A. As such, in general we will have $A^T \neq A$ even for square matrices.
- If it is the case that $A^T = A$, then we say that A is a symmetric matrix.

Some examples of symmetric matrices

- Example 1:

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = A^T.$$

- Example 2:

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = B^T.$$

- Example 3:

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = C^T.$$

- Example 4:

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = D^T.$$

Null Matrices

- A null matrix (or vector) is a matrix that consists solely of zeroes.
- For example, the (2×2) null matrix is

$$0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

- The $(n \times m)$ null matrix is the ADDITIVE identity matrix for the space of all $(n \times m)$ null matrices.
- This means that if A is an $(n \times m)$ matrix, then $A + 0 = 0 + A = A$.

Additive Inverses

- Suppose that A is an $(n \times m)$ matrix and 0 is the $(n \times m)$ null matrix.
- The $(n \times m)$ matrix B is the additive inverse of A if and only if $A + B = B + A = 0$.
- Suppose that
$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}.$$
- The additive inverse of A is
$$B = -A = \begin{pmatrix} -a_{11} & -a_{12} & \cdots & -a_{1m} \\ -a_{21} & -a_{22} & \cdots & -a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & -a_{nm} \end{pmatrix}.$$

An example of an additive inverse matrix

- Note that
$$A + B = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} + \begin{pmatrix} -1 & -2 \\ -4 & -3 \end{pmatrix}$$
$$= \begin{pmatrix} 1+(-1) & 2+(-2) \\ 4+(-4) & 3+(-3) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0_{(2\times 2)}.$$
- Note also that
$$B + A = \begin{pmatrix} -1 & -2 \\ -4 & -3 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} -1+1 & -2+2 \\ -4+4 & -3+3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0_{(2\times 2)}.$$
- Since $A + B = B + A = 0_{(2\times 2)}$, we can conclude that B is the additive inverse for A.

Identity Matrices

- An identity matrix is a square matrix that has ones on the main (north-west to south-east) diagonal and zeros everywhere else.
- For example, the (2×2) identity matrix is

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

- The $(n \times n)$ identity matrix is the MULTIPLICATIVE identity matrix for any relevant space of matrices:
  - If A is an $(n \times n)$ matrix, then $AI = IA = A$.
  - If A is an $(m \times n)$ matrix, then $AI = A$.
  - If A is an $(n \times m)$ matrix, then $IA = A$.

Multiplicative Inverses Part 1

- Only square matrices have any chance of having a multiplicative inverse. Some, but not all, square matrices will have a multiplicative inverse.
- Suppose that A is an $(n \times n)$ matrix and I is the $(n \times n)$ identity matrix.
- The $(n \times n)$ matrix B is the multiplicative inverse (usually just referred to as the inverse) of A if and only if $AB = BA = I$.

Multiplicative Inverses Part 2

- A square matrix that has an inverse is said to be non-singular.
- A square matrix that does not have an inverse is said to be singular.
- We will talk about methods for determining whether or not a matrix is non-sigular later in this unit.
- We will talk about methods for finding an inverse matrix, if it exists, later in this unit.
- Useful fact: "The transpose of the inverse is equal to the inverse of the transpose".
  - If A is a non-singular square matrix whose multiplicative inverse is A^{-1}, then we have $\left(A^{-1}\right)^T = \left(A^T\right)^{-1}$.

An example of a multiplicative inverse matrix Part 1

- This example comes from Hauessler and Paul (1987, p. 278, Example 1).
- Note that

$$
\begin{aligned}
AB &= \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix} \cdot \begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix} \\
&= \begin{pmatrix} (1)(7)+(2)(-3) & (1)(-2)+(2)(1) \\ (3)(7)+(7)(-3) & (3)(-2)+(7)(1) \end{pmatrix} \\
&= \begin{pmatrix} 7+(-6) & -2+2 \\ 21+(-21) & -6+7 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= I.
\end{aligned}
$$

- Continued on the next page.

An example of a multiplicative inverse matrix Part 2

- Continued from the previous page.
- Note that

$$BA = \begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} (7)(1) + (-2)(3) & (7)(2) + (-2)(7) \\ (-3)(1) + (1)(3) & (-3)(2) + (1)(7) \end{pmatrix}$$

$$= \begin{pmatrix} 7 + (-6) & 14 + (-14) \\ -3 + 3 & -6 + 7 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= I.$$

- Since $AB = BA = I$, we can conclude that $A^{-1} = B$.

Idempotent Matrices

- A matrix A is said to be idempotent if and only if $AA = A$.
- Clearly a NECESSARY condition for matrix to be idempotent is that A be a square matrix. (Exercise: Explain why this is the case.)
- However, this is NOT a SUFFICIENT condition for a matrix to be idempotent. In general, $AA \neq A$, even for square matrices.
- Two examples of idempotent matrices that you have already encountered are square null matrices and identity matrices.
- We will shortly encounter two more examples. These are the Hat matrix (P) and the residual-making matrix ($M = I - P$) from statistics and econometrics.

Some examples of idempotent matrices Part 1

- The (2×2) identity matrix:

$$I_2 \cdot I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (1)(1) + (0)(0) & (1)(0) + (0)(1) \\ (0)(1) + (1)(0) & (0)(0) + (1)(1) \end{pmatrix}$$

$$= \begin{pmatrix} 1+0 & 0+0 \\ 0+0 & 0+1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= I_2.$$

Some examples of idempotent matrices Part 2

- The (2×2) null matrix:

$$0_{(2\times 2)} \cdot 0_{(2\times 2)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} (0)(0)+(0)(0) & (0)(0)+(0)(0) \\ (0)(0)+(0)(0) & (0)(0)+(0)(0) \end{pmatrix}$$

$$= \begin{pmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= 0_{(2\times 2)}.$$

Econometric Example: The Classical Linear Regression Model

- One of simplest models that you will encounter in statistics and econometrics is the classical linear regression model (CLRM).
- This model takes the form

$$Y = X\beta + \epsilon,$$

where Y is an $(n \times 1)$ vector of n observations on a single dependent variable, X is an $(n \times k)$ matrix of n observations on k independent variables, β is a $(k \times 1)$ vector of unknown parameters and ϵ is an $(n \times 1)$ vector of random disturbances.
- In the CLRM, the joint distribution of the random disturbances, conditional on X, is given by

$$\epsilon \,|X \sim N\left(0, \sigma^2 I\right),$$

where 0 is an $(n \times 1)$ null vector, I is an $(n \times n)$ identity matrix and σ^2 is an unknown parameter.

Matrices Associated with the CLRM

- The ordinary least squares estimator (and, in the case of the CLRM, maximum likelihood estimator) of the parameter vector β in the CLRM is given by

$$b = \left(X^T X\right)^{-1} X^T Y.$$

- The hat matrix for the CLRM is given by

$$P = X\left(X^T X\right)^{-1} X^T.$$

- The residual-making matrix for the CLRM is given by

$$\begin{aligned} M &= I - P \\ &= I - X\left(X^T X\right)^{-1} X^T. \end{aligned}$$

The Hat Matrix is Symmetric

$$
\begin{aligned}
P^T &= \left(X\left(X^TX\right)^{-1}X^T\right)^T \\
&= \left(X^T\right)^T\left(\left(X^TX\right)^{-1}\right)^T(X)^T \\
&= X\left(\left(X^TX\right)^T\right)^{-1}X^T \\
&= X\left((X)^T\left(X^T\right)^T\right)^{-1}X^T \\
&= X\left(X^TX\right)^{-1}X^T \\
&= P.
\end{aligned}
$$

The Hat Matrix is Idempotent

$$
\begin{aligned}
PP &= \left(X\left(X^TX\right)^{-1}X^T\right)\left(X\left(X^TX\right)^{-1}X^T\right) \\
&= X\left(X^TX\right)^{-1}X^TX\left(X^TX\right)^{-1}X^T \\
&= X\left(X^TX\right)^{-1}IX^T \\
&= X\left(X^TX\right)^{-1}X^T \\
&= P.
\end{aligned}
$$

The Residual-Making Matrix is Symmetric

$$
\begin{aligned}
M^T &= (I - P)^T \\
&= I^T - P^T \\
&= I - P \\
&= M.
\end{aligned}
$$

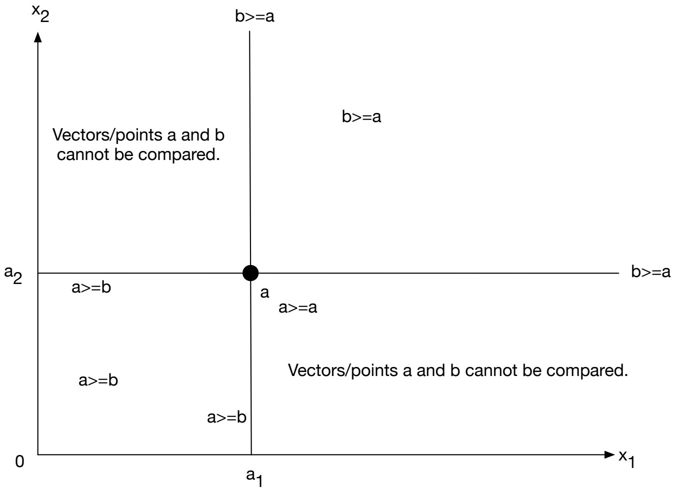
The Residual-Making Matrix is Idempotent

$$
\begin{aligned}
MM &= (I - P)(I - P) \\
&= II - IP - PI + PP \\
&= I - P - P + P \\
&= I - P \\
&= M.
\end{aligned}
$$

Vector Inequalities in Euclidean n-Space

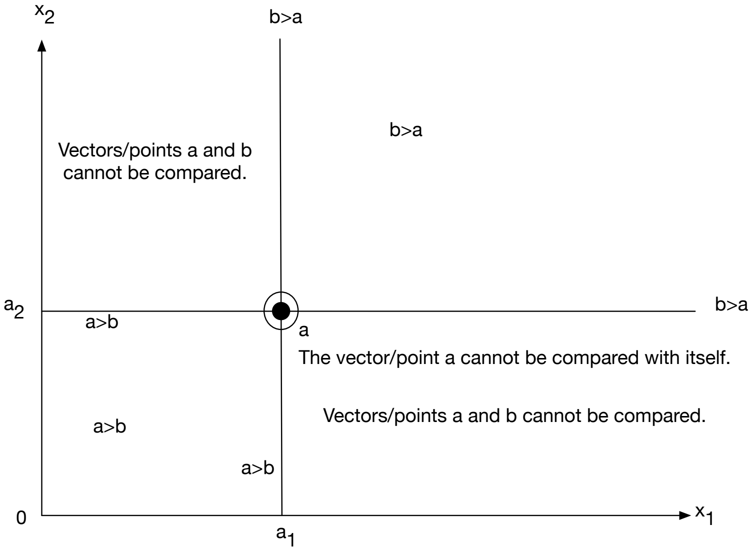
- Suppose that $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$.
    - This means that $x = (x_1, x_2, \cdots, x_n)$ where $x_i \in \mathbb{R}$ for all $i \in \{1, 2, \cdots, n\}$.
    - It also means that means that $y = (y_1, y_2, \cdots, y_n)$ where $y_i \in \mathbb{R}$ for all $i \in \{1, 2, \cdots, n\}$.
- We say that $x \geqslant y$ if $x_i \geqslant y_i$ for all $i \in \{1, 2, \cdots, n\}$.
- We say that $x > y$ if $x \geqslant y$ and $x_k > y_k$ for at least one $k \in \{1, 2, \cdots, n\}$.
- We say that $x \gg y$ if $x_i > y_i$ for all $i \in \{1, 2, \cdots, n\}$.
- Note that there are some pairs of vectors that cannot be ranked using these vector inequalities.
- These vector inequalities are illustrated for the case of $\mathbb{R}^2$ on the next three pages.

Vector Inequality Diagram 1 ($\geqslant$)

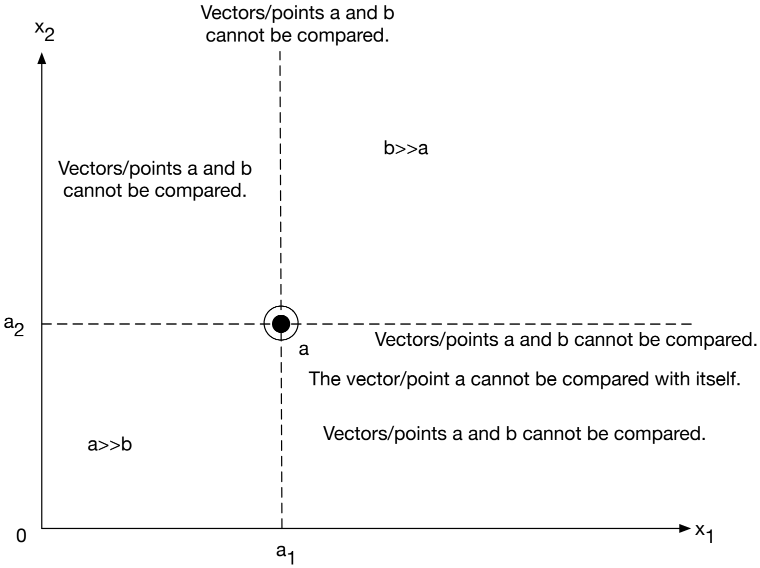


Note that ">=" means "greater than or equal to" in this diagram.

Vector Inequality Diagram 2 ($>$)



Vector Inequality Diagram 3 ($\gg$)



Some Examples of the Vector Inequalities

- Example 1: $(5,5) \gg (4,4)$, $(5,5) > (4,4)$, and $(5,5) \geqslant (4,4)$.
- Example 2: $(5,4) \gg (4,3)$, $(5,4) > (4,3)$, and $(5,4) \geqslant (4,3)$.
- Example 3: $(5,5 > (5,4)$, and $(5,5) \geqslant (5,4)$.
- Example 4: $(5,5) \geqslant (5,5)$.
- Example 5: $(5,4)$ and $(4,5)$ cannot be compared using any of the three vector inequalities.

Economic Application: Consumers who "prefer more to less"

- We sometimes want to consider the behaviour of consumers who prefer "more to less".
- Suppose that a consumer's preferences are defined over bundles of L commodities.
- There are two common versions of "more is preferred to less" for such a consumer.
  - Monotone preferences: The consumer will strictly prefer bundle x to bundle y if $x \gg y$ (that is, if $x_l > y_l$ for all $l \in \{1, 2, \cdots, L\}$).
  - Stongly monotone preferences: The consumer will strictly prefer bundle x to bundle y if $x > y$ (that is, if $x_l \geqslant y_l$ for all $l \in \{1, 2, \cdots, L\}$, and $x_k > y_k$ for at least one $k \in \{1, 2, \cdots, L\}$).