Worked Examples of Univariate Integration

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Questions

1. (Sydsaeter et al 2016, Chapter 9, p. 324, Questions 1, 4, and 8.) Find the following indefinite integrals.

   (a) $\int x^{13} dx$.

   (b) $\int x\sqrt{x} dx$.

   (c) $\int \frac{1}{\sqrt{x}} dx$.

   (d) $\int \sqrt{x\sqrt{x\sqrt{x}}} dx$.

   (e) $\int e^{-x} dx$.

   (f) $\int e^{\frac{x}{4}} dx$.

   (g) $\int 3e^{-2x} dx$.

   (h) $\int 2^x dx$.

   (i) $\int \frac{x^2}{(x+1)} dx$.

   (j) $\int 2x \ln\left(x^2 + a^2\right) dx$.

2. (Sydsaeter et al 2016, Chapter 9, p. 324, Question 2.) The marginal cost of manufacturing q units of some product is given $C'(q)$ and the fixed cost of manufacturing this product is given by $C(0)$. Find the total cost $C(q)$ of manufacturing q units of this product in each of the following cases.

   (a) $C'(q) = 3q + 4$ and $C(0) = 40$.

   (b) $C'(q) = aq + b$ and $C(0) = C_0$.

3. (Sydsaeter et al 2016, Chapter 9, p. 346, Question 3.) Use integration by substitution to find the following indefinite integrals.

   (a) $\int (x^3+1)^8 2x dx$.

   (b) $\int (x+2)^{10} dx$.

   (c) $\int \frac{(2x-1)}{(x^2-x+8)} dx$.

4. Questions from the final exam for EMET1001 in Semester Two of 2017.

   (a) Find $\int_0^{\frac{1}{2}} \frac{3x^2}{8} dx$.

   (b) Find $\int_0^1 e^{-x} dx$.

   (c) Solve the following equation for the parameter c: $\int_0^1 cx^2 dx = 1$.

   (d) Find $\int_1^x \frac{2}{w^3} dw$.

   (e) Find $\int_0^1 2x(1-x) dx$.

   (f) Find the indefinite integral $\int x^{-1} \ln(x) dx$. (You may assume that $x > 0$ for this part of the question.)

   (g) Find the indefinite integral $\int x^n \ln(x) dx$ when $n \notin \{-1, 0\}$. (You may assume that $x > 0$ for this part of the question.)

   (h) Find the indefinite integral $\int 3^{\sqrt{2x+1}} dx$. (You may assume that $x \geqslant -\left(\frac{1}{2}\right)$ for this part of the question.)

   (i) Find the indefinite integral $\int \frac{6-x}{(x-3)(2x+5)} dx$. (You may assume that $x > 3$ for this part of the question.)

5. Questions from the final exam for EMET1001 in Semester One of 2017.

   The inverse demand curve for a product describes how the maximum price that any consumer is prepared to pay for an additional unit of the product varies with the amount of the product that is currently purchased. This can be interpreted as the marginal benefit curve for the product. The demand curve for a product describes how the amount of the product that is purchased varies with the per-unit price of the product.

   The inverse supply curve for a product describes how the minimum price that any producer is prepared to accept for supplying an additional unit of the product varies with the amount of the product that is currently supplied. This can be interpreted as the marginal cost curve for the product. The supply curve for a product describes how the amount of the product that is supplied varies with the per-unit price of the product.

Suppose that the inverse demand function for widgets is given by

$$P^D(Q) = \frac{50}{(Q+5)},$$

and the inverse supply function for widgets is given by

$$P^S(Q) = 4.5 + 0.1Q.$$

(a) Invert the inverse demand function to obtain the demand function $(Q^D(P))$.

(b) Invert the inverse supply function to obtain the supply function $(Q^S(P))$.

(c) Impose the market clearing condition that $Q^D = Q^S = Q$ on the demand and supply functions and solve the resulting system of two equations in two unknown variables to find the equilibrium price and quantity.

(d) The consumers surplus that results from purchasing x widgets at a price of p_x per widget is given by the formula

$$CS(x, p_x) = \int_0^x P^D(Q)dQ - p_x x.$$

Calculate the consumer surplus associated with the equilibrium price and quantity in this market for widgets.

(e) The producers surplus that results from selling x widgets at a price of p_x per widget is given by the formula

$$PS(x, p_x) = p_x x - \int_0^x P^S(Q)dQ.$$

Calculate the producers surplus associated with the equilibrium price and quantity in this market for widgets.

Answer to Question 1

Answer to Question 1 Part (a)

$$\int x^{13} dx = \int \left(\frac{14}{14}\right) x^{13} dx = \left(\frac{1}{14}\right) \int 14x^{13} dx = \left(\frac{1}{14}\right) \int \left(\frac{dx^{14}}{dx}\right) dx$$
$$= \left(\frac{1}{14}\right) x^{14} + C.$$

Answer to Question 1 Part (b)

$$\int x\sqrt{x} dx = \int x^1 x^{\frac{1}{2}} dx = \int x^{\frac{3}{2}} dx = \int \left(\frac{\frac{5}{2}}{\frac{5}{2}}\right) x^{\frac{3}{2}} dx = \left(\frac{1}{\frac{5}{2}}\right) \int \left(\frac{5}{2}\right) x^{\frac{3}{2}} dx$$
$$= \left(\frac{2}{5}\right) \int \left(\frac{dx^{\frac{5}{2}}}{dx}\right) dx = \left(\frac{2}{5}\right) x^{\frac{5}{2}} + C.$$

Answer to Question 1 Part (c)

$$\int \frac{1}{\sqrt{x}} dx = \int \frac{1}{x^{\frac{1}{2}}} dx = \int x^{\frac{-1}{2}} dx = \int \left(\frac{\frac{1}{2}}{\frac{1}{2}}\right) x^{\frac{-1}{2}} dx = \left(\frac{1}{\frac{1}{2}}\right) \int \left(\frac{1}{2}\right) x^{\frac{-1}{2}} dx$$
$$= 2 \int \left(\frac{dx^{\frac{1}{2}}}{dx}\right) dx = 2x^{\frac{1}{2}} + C = 2\sqrt{x} + C.$$

Answer to Question 1 Part (d)

Note that

$$\sqrt{x\sqrt{x\sqrt{x}}} = \left(x\sqrt{x\sqrt{x}}\right)^{\frac{1}{2}} = x^{\frac{1}{2}} \left(\sqrt{x\sqrt{x}}\right)^{\frac{1}{2}} = x^{\frac{1}{2}} \left(\left(x\sqrt{x}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} = x^{\frac{1}{2}} \left(x\sqrt{x}\right)^{\frac{1}{4}}$$
$$= x^{\frac{1}{2}} x^{\frac{1}{4}} \left(\sqrt{x}\right)^{\frac{1}{4}} = x^{\frac{3}{4}} \left(x^{\frac{1}{2}}\right)^{\frac{1}{4}} = x^{\frac{3}{4}} x^{\frac{1}{8}} = x^{\frac{7}{8}}.$$

Thus we have

$$\int \sqrt{x\sqrt{x\sqrt{x}}} dx = \int x^{\frac{7}{8}} dx = \int \left(\frac{\frac{9}{8}}{\frac{9}{8}}\right) x^{\frac{7}{8}} dx = \left(\frac{1}{\frac{9}{8}}\right) \int \left(\frac{9}{8}\right) x^{\frac{7}{8}} dx$$
$$= \frac{8}{9} \int \left(\frac{dx^{\frac{9}{8}}}{dx}\right) dx = \frac{8}{9} x^{\frac{9}{8}} + C.$$

Answer to Question 1 Part (e)

$$\int e^{-x} dx = \int (-1)(-1)e^{-x} dx = -\int (-1)e^{-x} dx = -\int \left(\frac{de^{-x}}{dx}\right) dx = -e^{-x} + C.$$

Answer to Question 1 Part (f)

$$\int e^{\frac{x}{4}} dx = \int \left(\frac{4}{4}\right) e^{\frac{x}{4}} dx = 4\int \left(\frac{1}{4}\right) e^{\frac{x}{4}} dx = 4\int \left(\frac{de^{\frac{x}{4}}}{dx}\right) dx = 4e^{\frac{x}{4}} + C.$$

Answer to Question 1 Part (g)

$$\int 3e^{-2x} dx = 3\int e^{-2x} dx = 3\int \left(\frac{-2}{-2}\right) e^{-2x} dx = -\left(\frac{3}{2}\right)\int (-2)e^{-2x} dx$$
$$= -\left(\frac{3}{2}\right)\int \left(\frac{de^{-2x}}{dx}\right) dx = -\left(\frac{3}{2}\right) e^{-2x} + C.$$

Answer to Question 1 Part (h)

$$\int 2^x dx = \int \left(\frac{\ln(2)}{\ln(2)}\right) 2^x dx = \left(\frac{1}{\ln(2)}\right)\int \ln(2) 2^x dx = \left(\frac{1}{\ln(2)}\right)\int \left(\frac{d2^x}{dx}\right) dx$$
$$= \left(\frac{1}{\ln(2)}\right) 2^x + C = \frac{2^x}{\ln(2)} + C.$$

Answer to Question 1 Part (i)

Note that

$$\begin{array}{r}
x - 1. \\
x+1 \big)\overline{\;\; x^2 \qquad\qquad} \\
\underline{-\, x^2 - x} \\
-\, x \\
\underline{x+1} \\
1
\end{array}$$

This means that

$$\frac{x^2}{(x+1)} = (x-1) + \frac{1}{(x+1)}.$$

Thus we have (assuming that $x+1 > 0$)

$$\int \frac{x^2}{(x+1)} dx = \int \left((x-1) + \frac{1}{(x+1)}\right) dx = \int \left(x - 1 + \frac{1}{(x+1)}\right) dx$$

$$= \int x dx - \int 1 dx + \int \left(\frac{1}{(x+1)} \right) dx = \left(\frac{1}{2} \right) x^2 - x + \ln(x+1) + C.$$

Note that if we instead used the weaker restriction that $x + 1 \neq 0$, then we would have obtained

$$\int \frac{x^2}{(x+1)} dx = \left(\frac{1}{2} \right) x^2 - x + \ln(|x+1|) + C.$$

(Can you explain why?)

Answer to Question 1 Part (j)

Consider the indefinite integral

$$\int 2x \ln \left(x^2 + a^2 \right) dx.$$

We will find this indefinite integral by using the technique of integration by parts. Let

$$u'(x) = 2x$$

and

$$v(x) = \ln \left(x^2 + a^2 \right).$$

This means that (ignoring arbitrary constants of integration for the moment)

$$u(x) = x^2$$

and

$$v'(x) = \frac{2x}{(x^2 + a^2)}.$$

As such, we know that

$$\int 2x \ln \left(x^2 + a^2 \right) dx = \int u'(x) v(x) dx$$

$$= u(x)v(x) - \int v'(x) u(x) dx$$

$$= x^2 \ln \left(x^2 + a^2 \right) - \int \left(\frac{2x}{(x^2 + a^2)} \right) x^2 dx$$

$$= x^2 \ln \left(x^2 + a^2 \right) - \int \frac{2x^3}{(x^2 + a^2)} dx.$$

Note that

$$
\begin{array}{r|l}
 & \quad\quad\quad\quad\quad\quad\quad 2x. \\
\cline{2-2}
x^2 + (0 + 1a^2) & \; 2x^3 \\
 & -2x^3 - 2\left(0 + 1a^2\right)x \\
\cline{2-2}
 & \quad\quad -2\left(0 + 1a^2\right)x
\end{array}
$$

This means that

$$\frac{2x^3}{(x^2 + a^2)} = 2x - \frac{2a^2x}{(x^2 + a^2)}.$$

Hence we have (ignoring arbitrary constants for the moment)

$$\int \frac{2x^3}{(x^2 + a^2)}dx = \int \left(2x - \frac{2a^2x}{(x^2 + a^2)}\right) dx$$

$$= \int 2xdx - \int \left(\frac{2a^2x}{(x^2 + a^2)}\right) dx$$

$$= x^2 - a^2 \int \left(\frac{2x}{(x^2 + a^2)}\right) dx$$

$$= x^2 - a^2 \ln\left(x^2 + a^2\right).$$

Thus we have

$$\int 2x \ln\left(x^2 + a^2\right) dx = x^2 \ln\left(x^2 + a^2\right) - \int \frac{2x^3}{(x^2 + a^2)}dx$$

$$= x^2 \ln\left(x^2 + a^2\right) - \left(x^2 - a^2 \ln\left(x^2 + a^2\right)\right) + C$$

$$= x^2 \ln\left(x^2 + a^2\right) - x^2 + a^2 \ln\left(x^2 + a^2\right) + C$$

$$= \left(x^2 + a^2\right) \ln\left(x^2 + a^2\right) - x^2 + C.$$

Answer to Question 2

Answer to Question 2 Part (a)

Note that

$$C(q) = \int C'(q) = \int (3q+4)dq = \left(\frac{3}{2}\right) q^2 + 4q + C.$$

Hence we have

$$C(0) = 40 \Longleftrightarrow \left(\frac{3}{2}\right) 0^2 + 4(0) + C = 40 \Longleftrightarrow 0 + 0 + C = 40 \Longleftrightarrow C = 40.$$

Thus the firm's total cost function is

$$C(q) = \left(\frac{3}{2}\right) q^2 + 4q + 40.$$

Answer to Question 2 Part (b)

Note that

$$C(q) = \int C'(q) = \int (aq+b)dq = \left(\frac{a}{2}\right) q^2 + bq + C.$$

Hence we have

$$C(0) = C_0 \Longleftrightarrow \left(\frac{a}{2}\right) 0^2 + b(0) + C = C_0 \Longleftrightarrow 0 + 0 + C = 4C_0 \Longleftrightarrow C = C_0.$$

Thus the firm's total cost function is

$$C(q) = \left(\frac{a}{2}\right) q^2 + bq + C_0.$$

Answer to Question 3

Answer to Question 3 Part (a)

Consider the indefinite integral $\int (x^2+1)^8 2x dx$. Let $g(t) = t^8$ and $t(x) = x^2+1$. This means that $t'(x) = 2x$. Thus we have

$$\int (x^2+1)^8 2x dx = \int g(x(t))t'(x)dx = \int g(t)dt = \int t^8 dt = \left(\frac{1}{9}\right) t^9 + C$$

$$= \left(\frac{1}{9}\right)(t(x))^9 + C = \left(\frac{1}{9}\right)(x^2+1)^9 + C.$$

Answer to Question 3 Part (b)

Consider the indefinite integral $\int (x+2)^{10} dx$. Let $g(t) = t^{10}$ and $t(x) = x+2$. This means that $t'(x) = 1$. Thus we have

$$\int (x+2)^{10} dx = \int (x+2)^{10}(1) dx = \int g(x(t))t'(x)dx = \int g(t)dt = \int t^{10} dt$$

$$= \left(\frac{1}{11}\right) t^{11} + C = \left(\frac{1}{11}\right)(t(x))^{11} + C = \left(\frac{1}{11}\right)(x+2)^{11} + C.$$

Answer to Question 3 Part (c)

Consider the indefinite integral $\int \frac{(2x-1)}{(x^2-x+8)} dx$. Let $g(t) = \frac{1}{t}$ and $t(x) = x^2 - x + 8$. This means that $t'(x) = 2x - 1$. Thus we have, assuming that we restrict attention to cases where $t > 0$ (so that $x^2 - x + 8 > 0$),

$$\int \frac{(2x-1)}{(x^2-x+8)} dx = \int g(t)t'(x)dx = \int g(t)dt = \int \left(\frac{1}{t}\right) dt$$

$$= \ln(t) + C = \ln(x^2 - x + 8) + C.$$

Note that if we had instead used the weaker restriction that $t \neq 0$ (so that $x^2 - x + 8 \neq 0$), we would obtain

$$\int \frac{(2x-1)}{(x^2-x+8)} dx = \ln(|x^2 - x + 8|) + C.$$

(Can you explain why?)

Answer to Question 4

Answer to Question 4 Part (1)

$$\int_0^{\frac{1}{2}} \frac{3x^2}{8} dx = \left(\frac{1}{8}\right) \int_0^{\frac{1}{2}} 3x^2 dx = \left(\frac{1}{8}\right) \left[x^3\right]_0^{\frac{1}{2}} = \left(\frac{1}{8}\right) \left[\left(\frac{1}{2}\right)^3 - 0^3\right]$$
$$= \left(\frac{1}{8}\right) \left[\frac{1}{8} - 0\right] = \left(\frac{1}{8}\right)\left(\frac{1}{8}\right) = \frac{1}{64}.$$

Answer to Question 4 Part (2)

$$\int_0^1 e^{-x} dx = -\int_0^1 -e^{-x} dx = -\left[e^{-x}\right]_0^1 = -\left[e^{-1} - e^0\right] = -\left[e^{-1} - 1\right] = 1 - e^{-1}.$$

Answer to Question 4 Part (3)

$$\int_0^1 cx^2 dx = 1 \iff \left(\frac{c}{3}\right) \int_0^1 3x^2 dx = 1 \iff \left(\frac{c}{3}\right) \left[x^3\right]_0^1 = 1$$
$$\iff \left(\frac{c}{3}\right)\left[1^3 - 0^3\right] = 1 \iff \left(\frac{c}{3}\right)[1 - 0] = 1 \iff \left(\frac{c}{3}\right)(1) = 1 \iff \left(\frac{c}{3}\right) = 1$$
$$\iff c = 3.$$

Answer to Question 4 Part (4)

$$\int_1^x \frac{2}{w^3} dw = \int_1^x 2w^{-3} dw = -\int_1^x -2w^{-3} dw = -\left[w^{-2}\right]_1^x = - = -\left[x^{-2} - 1^{-2}\right]$$
$$= -\left[x^{-2} - 1\right] = 1 - x^{-2}.$$

Answer to Question 4 Part (5)

$$\int_0^1 2x(1-x) dx = \int_0^1 2x - 2x^2) dx = \int_0^1 2x dx - \int_0^1 2x^2 dx$$
$$= \int_0^1 2x dx - \left(\frac{2}{3}\right) \int_0^1 3x^2 dx = \left[x^2\right]_0^1 - \left(\frac{2}{3}\right)\left[x^3\right]_0^1 = \left[1^2 - 0^2\right] - \left(\frac{2}{3}\right)\left[1^3 - 0^3\right]$$
$$= [1 - 0] - \left(\frac{2}{3}\right)\left[1^3 - 0^3\right] = 1 - \left(\frac{2}{3}\right)(1) = 1 - \left(\frac{2}{3}\right) = \left(\frac{1}{3}\right).$$

Answer to Question 4 Part (6)

Note that

$$\int x^{-1}\ln(x)dx = \int \frac{\ln(x)}{x}dx.$$

We can obtain this indefinite integral in at least two ways. One of these is through the use of "integration by substitution". The other is through the use of "integration by parts".

Integration by Substitution: Let $g(t) = t$ and $t(x) = \ln(x)$. This means that $g(t(x)) = t(x) = \ln(x)$ and $t'(x) = \frac{1}{x}$. Note that $f(x) = \frac{\ln(x)}{x} = g(t(x))t'(x)$ and $\int g(t)dt = \int tdt = \frac{t^2}{2} + C$. Thus we have

$$F(x) + C = \frac{(t(x))^2}{2} + C = \frac{(\ln(x))^2}{2} + C = \frac{\ln^2(x)}{2} + C.$$

Integration by Parts: Let $u(x) = \ln(x)$ and $v(x) = \ln(x)$. This means that $u'(x) = \frac{1}{x}$ and $v'(x) = \frac{1}{x}$. This allows us to express the required integral as $\int \frac{\ln(x)}{x}dx = \int u'(x)v(x)dx$. Integrating by parts, and ignoring any arbitrary constants, we obtain

$$\begin{aligned}\int \frac{\ln(x)}{x}dx &= \int u'(x)v(x)dx = u(x)v(x) - \int v'(x)u(x)dx \\ &= (\ln(x))(\ln(x)) - \int \frac{\ln(x)}{x}dx = (\ln(x))^2 - \int \frac{\ln(x)}{x}dx \\ &= \ln^2(x) - \int \frac{\ln(x)}{x}dx.\end{aligned}$$

This equation can be rearranged to obtain

$$2\int \frac{\ln(x)}{x}dx = \ln^2(x),$$

so that, ignoring any arbitrary constants, we have

$$\int \frac{\ln(x)}{x}dx = \left(\frac{1}{2}\right)\ln^2(x).$$

Thus we know that

$$\int \frac{\ln(x)}{x}dx = \left(\frac{1}{2}\right)\ln^2(x) + C,$$

where $C \in \mathbb{R}$ is an arbitrary constant.

Answer to Question 4 Part (7)

Consider the indefinite integral $\int x^n \ln(x)dx$ where $n \notin \{-1, 0\}$. We will use the technique of integration by parts to find this indefinite integral. Let $u'(x) = x^n$ and $v(x) = \ln(x)$. This means that $u(x) = \frac{x^{n+1}}{(n+1)}$ and $v'(x) = \frac{1}{x}$. This allows us to express the required integral as $\int x^n \ln(x)dx = \int u'(x)v(x)dx$. Integrating by parts, and ignoring any arbitrary constants, we obtain

$$\begin{aligned}
\int x^n \ln(x)dx &= \int u'(x)v(x)dx \\
&= u(x)v(x) - \int v'(x)u(x)dx \\
&= \left(\frac{x^{n+1}}{(n+1)}\right)\ln(x) - \int \left(\frac{1}{x}\right)\left(\frac{x^{n+1}}{(n+1)}\right)dx \\
&= \left(\frac{x^{n+1}}{(n+1)}\right)\ln(x) - \int \frac{x^n}{(n+1)}dx \\
&= \left(\frac{x^{n+1}}{(n+1)}\right)\ln(x) - \left(\frac{1}{(n+1)^2}\right)\int (n+1)x^n dx \\
&= \left(\frac{x^{n+1}}{(n+1)}\right)\ln(x) - \frac{x^{n+1}}{(n+1)^2} \\
&= \frac{x^{n+1}}{(n+1)}\left(\ln(x) - \frac{1}{(n+1)}\right).
\end{aligned}$$

Thus we know that

$$\int x^n \ln(x)dx = \frac{x^{n+1}}{(n+1)}\left(\ln(x) - \frac{1}{(n+1)}\right) + C,$$

where $C \in \mathbb{R}$ is an arbitrary constant.

Answer to Question 4 Part (8)

Consider the indefinite integral $\int 3^{\sqrt{2x+1}}dx$. We will use a combination of both "integration by substitution" and "integration by parts" to find this indefinite integral. Let $t(x) = \sqrt{2x+1}$. Note that $x(t) = \left(\frac{1}{2}\right)t^2 - \left(\frac{1}{2}\right)$, since

$$t = \sqrt{2x+1} \iff t^2 = 2x+1 \iff 2x = t^2 - 1 \iff x = \left(\frac{1}{2}\right)t^2 - \left(\frac{1}{2}\right).$$

This means that

$$dx = (2)\left(\frac{1}{2}\right)tdt = tdt.$$

Thus we have

$$\int 3^{\sqrt{2x+1}} dx = \int 3^t t dt.$$

Let $u'(t) = 3^t$ and $v(t) = t$. This means that $u(t) = \left(\frac{1}{\ln(3)}\right) 3^t$ and $v'(t) = 1$. Thus, ignoring any arbitrary constants, we have

$$\begin{aligned}
\int 3^t t dt &= \int u'(t)v(t)dt \\
&= u(t)v(t) - \int v'(t)u(t)dt \\
&= \left(\frac{1}{\ln(3)}\right) 3^t t - \int (1) \left(\frac{1}{\ln(3)}\right) 3^t dt \\
&= \left(\frac{1}{\ln(3)}\right) 3^t t - \frac{1}{(\ln(3))^2} \int \ln(3) 3^t dt \\
&= \left(\frac{1}{\ln(3)}\right) 3^t t - \frac{1}{(\ln(3))^2} \left(3^t\right) \\
&= \left(\frac{3^t}{\ln(3)}\right) \left(t - \frac{1}{\ln(3)}\right).
\end{aligned}$$

Hence we have

$$\int 3^{\sqrt{2x+1}} dx = \int 3^t t dt = \left(\frac{3^t}{\ln(3)}\right) \left(t - \frac{1}{\ln(3)}\right) + C,$$

where $C \in \mathbb{R}$ is an arbitrary constant. Recalling that $t(x) = \sqrt{2x+1}$, we can rewrite this expression as

$$\int 3^{\sqrt{2x+1}} dx = \left(\frac{3^{\sqrt{2x+1}}}{\ln(3)}\right) \left(\sqrt{2x+1} - \frac{1}{\ln(3)}\right) + C,$$

where $C \in \mathbb{R}$ is an arbitrary constant.

Answer to Question 4 Part (9)

Note that the only divisors of $(x-3)$ are $(x-3)$ and 1, while the only divisors of $(2x+5)$ are $(2x+5)$ and 1. Thus we know that $(x-3)$ $(2x+5)$ are relatively prime (or co-prime) polynomial functions. This means that the proper rational function $R(x) = \frac{6-x}{(x-3)(2x+5)}$ has a partial fractions decomposition of the form

$$R(x) = \frac{6-x}{(x-3)(2x+5)} = \frac{A(x)}{(x-3)} + \frac{B(x)}{(2x+5)},$$

where the degree of the polynomial $A(x)$ is strictly less than the degree of the polynomial $(x-3)$ and the degree of the polynomial $B(x)$ is strictly less than the degree of the polynomial $(2x+5)$. Since both $(x-3)$ and $(2x+5)$ are degree-one (or linear) polynomial functions, we know that both $A(x)$ and $B(x)$ must be degree zero (or constant) polynomial functions. This means that $A(x) = a$ for some $a \in \mathbb{R}$ and $B(x) = b$ for some $b \in \mathbb{R}$. Thus we have

$$\frac{6-x}{(x-3)(2x+5)} = \frac{a}{(x-3)} + \frac{b}{(2x+5)}$$

$$\iff \frac{6-x}{(x-3)(2x+5)} = \frac{a(2x+5)+b(x-3)}{(x-3)(2x+5)}$$

$$\iff \frac{6-x}{(x-3)(2x+5)} = \frac{2ax+5a+bx-3b}{(x-3)(2x+5)}$$

$$\iff \frac{6-x}{(x-3)(2x+5)} = \frac{(2a+b)x+(5a-3b)}{(x-3)(2x+5)}$$

$$\iff \frac{6-x}{(x-3)(2x+5)} = \frac{(5a-3b)-(-2a-b)x}{(x-3)(2x+5)}.$$

Hence we require that $5a-3b=6$ and $-2a-b=1$. Note that

$$-2a-b=1 \iff 2a+b=-1 \iff b=-1-2a.$$

This means that

$$5a-3b=6 \iff 5a-3(-1-2a)=6 \iff 5a+3+6a=6 \iff 11a=3$$

$$\iff a = \frac{3}{11},$$

so that

$$b = -1-2a = -1-2\left(\frac{3}{11}\right) = -\left(\frac{11}{11}\right) - \left(\frac{6}{11}\right) = -\left(\frac{17}{11}\right).$$

Hence we have

$$R(x) = \frac{6-x}{(x-3)(2x+5)} = \frac{\frac{3}{11}}{(x-3)} + \frac{-\left(\frac{17}{11}\right)}{(2x+5)} = \frac{3}{11(x-3)} - \frac{17}{11(2x+5)}.$$

Thus, ignoring any arbitrary constants, we have

$$\int \frac{6-x}{(x-3)(2x+5)}dx = \int \left(\frac{3}{11(x-3)} - \frac{17}{11(2x+5)}\right) dx$$

$$
\begin{aligned}
&= \int \frac{3}{11(x-3)}dx - \int \frac{17}{11(2x+5)}dx \\
&= \int \frac{3}{11(x-3)}dx - \int \frac{(17)(2)}{22(2x+5)}dx \\
&= \left(\frac{3}{11}\right) \int \frac{1}{(x-3)}dx - \left(\frac{17}{22}\right) \int \frac{2}{(2x+5)}dx \\
&= \left(\frac{3}{11}\right) \ln(x-3) - \left(\frac{17}{22}\right) \ln(2x+5).
\end{aligned}
$$

This means that

$$
\int \frac{6-x}{(x-3)(2x+5)}dx = \left(\frac{3}{11}\right) \ln(x-3) - \left(\frac{17}{22}\right) \ln(2x+5) + C,
$$

where $C \in \mathbb{R}$ is an arbitrary constant.

Answer to Question 5

Answer to Question 5 Part (1)

$$P^D(Q) = \frac{50}{(Q+5)} \iff P = \frac{50}{(Q+5)} \iff (Q+5)P = 50$$
$$\iff Q + 5 = \frac{50}{P} \iff Q = \frac{50}{P} - 5 \iff Q^D(P) = \frac{50}{P} - 5.$$

Answer to Question 5 Part (2)

$$P^S(Q) = 4.5 + 0.1Q \iff P = 4.5 + 0.1Q \iff -4.5 + P = 0.1Q$$
$$\iff 0.1Q = -4.5 + P \iff Q = -45 + 10P \iff Q^S(P) = -45 + 10P.$$

Answer to Question 5 Part (3)

Upon imposing the market-clearing condition that $Q^D = Q^S = Q$, we obtain the pair of simultaneous equations consisting of $Q = \frac{50}{P} - 5$ and $Q = -45 + 10P$. Note that

$$Q = Q \iff \frac{50}{P} - 5 = -45 + 10P \iff 50 - 5P = -45P + 10P^2$$
$$\iff 10P^2 - 40P - 50 = 0 \iff P^2 - 4P - 5 = 0 \iff (P-5)(P+1) = 0$$
$$\iff P_1 = 5, P_2 = -1.$$

Since a negative price does not make economic sense, we can conclude that the equilibrium price for widgets is $P_1 = \$5$ per widget. Upon substituting this price into either the demand equation or the supply equation, we find that the equilibrium quantity of widgets that is traded is equal to five widgets. Thus we have $(P^*, Q^*) = (5, 5)$.

Answer to Question 5 Part (4)

$$\begin{aligned}
CS(Q^*, P^*) &= \int_0^{Q^*} P^D(Q)dQ - P^*Q^* \\
&= \int_0^5 \frac{50}{(Q+5)}dQ - (5)(5) \\
&= 50\int_0^5 \frac{1}{(Q+5)}dQ - 25 \\
&= 50\left[\ln(Q+5)\right]_{Q=0}^{Q=5} - 25
\end{aligned}$$

$$= 50\left(\ln(10) - \ln(5)\right) - 25$$
$$= 50\ln\left(\frac{10}{5}\right) - 25$$
$$= 50\ln(2) - 25$$
$$\approx 50(0.693147181) - 25$$
$$\approx 34.65735903 - 25$$
$$\approx 9.657359028$$
$$\approx \$9.66.$$

Answer to Question 5 Part (5)

$$PS(Q^*, P^*) = P^*Q^* - \int_0^{Q^*} P^S(Q)dQ$$
$$= (5)(5) - \int_0^5 (4.5 + 0.1Q)dQ$$
$$= 25 - \left[4.5Q + \left(\frac{0.1}{2}\right)Q^2\right]_{Q=0}^{Q=5}$$
$$= 25 - \left[4.5Q + 0.05Q^2\right]_{Q=0}^{Q=5}$$
$$= 25 - \{[4.5(5) + 0.05(25)] - [4.5(0) + 0.05(0)]\}$$
$$= 25 - \{(22.5 + 1.25) - (0 + 0)\}$$
$$= 25 - (23.75 - 0)$$
$$= 25 - 23.75$$
$$= 1.25$$
$$= \$1.25.$$