Some notes on marginal revenue and monopoly

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1 Linear demand curves

Let $Q^D(P)$ be a demand curve and $P^D(Q)$ be an inverse demand curve. If the demand curve is linear, it has the following form:

$$Q^D = \alpha - \beta P^D. \tag{1}$$

This can be rearranged to yield:

$$P^D = \frac{\alpha}{\beta} - \frac{1}{\beta} Q^D = a - bQ^D, \tag{2}$$

where $a = \frac{\alpha}{\beta}$ and $b = \frac{1}{\beta}$. Thus if the demand curve is linear, then the inverse demand curve is also linear.

2 An algebraic derivation of marginal revenue when demand is linear

Total revenue for a monopolist is given by the formula:

$$TR(Q) = P^D(Q)Q. \tag{3}$$

If the demand curve is linear, this becomes:

$$TR(Q) = (a - bQ)\, Q = aQ - bQ^2. \tag{4}$$

The marginal revenue that a monopolist recieves if it increases output from Q_1 to Q_2 is given by the formula:

$$MR(Q) = \frac{\Delta TR(Q)}{\Delta Q} = \frac{TR(Q_2) - TR(Q_1)}{Q_2 - Q_1}. \tag{5}$$

For linear demand curves we have:

$$TR(Q_1) = aQ_1 - bQ_1^2. \tag{6}$$

Note that $\Delta Q = (Q_2 - Q_1)$, so that $Q_2 = (Q_1 + \Delta Q)$. Hence we have:

$$TR(Q_2) = a(Q_1 + \Delta Q) - b(Q_1 + \Delta Q)^2, \tag{7}$$

which can be expanded out to yield:

$$TR(Q_2) = aQ_1 + a\Delta Q - bQ_1^2 - 2bQ_1\Delta Q - b(\Delta Q)^2. \tag{8}$$

This can be rearranged to obtain:

$$TR(Q_2) = a\Delta Q - b(\Delta Q)^2 - 2bQ_1\Delta Q + (aQ_1 - bQ_1^2), \tag{9}$$

which simplifies to:

$$TR(Q_2) = a\Delta Q - b(\Delta Q)^2 - 2bQ_1\Delta Q + TR(Q_1). \tag{10}$$

The change in total revenue that results from an increase in output from Q_1 to Q_2 is:

$$\Delta TR(Q_1) = TR(Q_2) - TR(Q_1), \tag{11}$$

which is:

$$\Delta TR(Q_1) = a\Delta Q - b(\Delta Q)^2 - 2bQ_1\Delta Q + TR(Q_1) - TR(Q_1). \tag{12}$$

This can be simplified to obtain:

$$\Delta TR(Q_1) = a\Delta Q - b(\Delta Q)^2 - 2bQ_1\Delta Q. \tag{13}$$

Upon substituting this into the formula for marginal revenue, we obtain:

$$MR(Q_1) = \frac{a\Delta Q - b(\Delta Q)^2 - 2bQ_1\Delta Q}{\Delta Q} = a - b\Delta Q + 2bQ_1. \tag{14}$$

Dropping the subscripts for convenience, we have:

$$MR(Q) = a - b\Delta Q + 2bQ. \tag{15}$$

Since the marginal revenue concept is really about the change in total revenue in response to a very small increase in output, we will take the limit of our marginal revenue expresion as the change in output approaches zero This yields:

$$MR(Q) = \lim_{\Delta Q \to 0}(a - b\Delta Q + 2bQ) = a - 2bQ. \tag{16}$$

3 A much shorter derivation of marginal revenue when demand is linear that uses calculus

Recall that total revenue is given by:

$$R(Q) = aQ - bQ^2. \tag{17}$$

The marginal revenue function for this monopolist can be found by differentiating this total revenue function with respect to quantity. Doing this yields:

$$MR(Q) = \frac{dR(Q)}{dQ} = a - 2bQ. \tag{18}$$

The moral of this story is that calculus can sometimes be a very useful tool!!!

4 A monopolist will not choose an output level that lies on an inelastic part of the demand curve

Assume that marginal cost is no smaller than average cost for all positive output levels. Also assume that marginal cost is positive for all positive output levels. In these circumstances, the profit maximising level of output for a monopolist will satisfy the following condition:

$$MR(Q) = MC(Q) > 0. \tag{19}$$

Thus one of the implications of profit maximisation in the circumstances outlined above is that marginal revenue must be positive. Suppose that an increase in quantity from Q_1 to Q_2 results in a decrease in the price that clears the market from $P_1 = P^D(Q_1)$ to $P_2 = P^D(Q_2)$. The change is quantity is given by $\Delta Q = (Q_2 - Q_1) > 0$, while the change in price is given by $\Delta P = (P_2 - P_1) < 0$. Note that the change in price is negative because demand curves slope down in (Q, P)-space. The change in total revenue that is induced by this change in quantity is given by:

$$\Delta TR(Q) = TR(Q_2) - TR(Q_1) = P_2 \Delta Q + Q_1 \Delta P. \tag{20}$$

Note that $P_2 \Delta Q$ is the gain in revenue from selling the additional units at the new price, while $Q_1 \Delta P$ is the loss in revenue caused by the reduction in the price the monopolist receives for each of the infra-marginal units (that is, the units that he originally sold at the higher price of P_1). (Exercise: Illustrate this in a demand curve diagram.) Thus the monopolist's marginal revenue is given by the formula:

$$MR(Q) = \frac{\Delta TR(Q)}{\Delta Q} = \frac{P_2 \Delta Q + Q_1 \Delta P}{\Delta Q} = P_2 + Q_1 \frac{\Delta P}{\Delta Q}. \tag{21}$$

Substituting (21) into (19) yields:

$$P_2 + Q_1 \frac{\Delta P}{\Delta Q} > 0, \tag{22}$$

which can be rearranged to obtain:

$$Q_1 \frac{\Delta P}{\Delta Q} > -P_2. \tag{23}$$

This can be further rearranged to obtain:

$$\frac{\Delta P}{\Delta Q} > -\left(\frac{P_2}{Q_1}\right), \tag{24}$$

which can be simplified to yield:

$$\Delta P > -\left(\frac{P_2}{Q_1}\right) \Delta Q. \tag{25}$$

Noting that $\Delta P < 0$, we can rearrange (25) to obtain:

$$1 < -\left(\frac{P_2}{Q_1}\right)\left(\frac{\Delta Q}{\Delta P}\right). \tag{26}$$

Now, if we consider really small increases in output, so that ΔQ approaches zero, then we will have Q_2 approaching Q_1 and, if the inverse demand function is continuous, $P_2 = P^D(Q_2)$ approaching $P_1 = P^D(Q_1)$. As such, we know that for small increases in quantity, the own-price elasticity of demand for the product that is produced by the monopolist can be approximated by:

$$\eta_P^D = \left(\frac{P_1}{Q_1}\right)\left(\frac{\Delta Q}{\Delta P}\right) \approx \left(\frac{P_2}{Q_1}\right)\left(\frac{\Delta Q}{\Delta P}\right). \tag{27}$$

Substituting (27) into (26) yields::

$$1 < -\eta_P^D, \tag{28}$$

which can be rewritten as

$$-\eta_P^D > 1. \tag{29}$$

Finally, note that $\eta_P^D < 0$ because demand curves slope down in (Q, P)-space. As such, we know that:

$$\left|\eta_P^D\right| = -\eta_P^D. \tag{30}$$

Substituting (30) into (29) yields:

$$\left|\eta_P^D\right| > 1. \tag{31}$$

Thus, a profit maximising monopolist will choose an output level at which the absolute value of the own-price elasticity of the demand for the monopolists product exceeds one.