Quadratic Forms

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Reading Part 1

- Anton, H (1987), *Elementary linear algebra (fifth edition)*, John Wiley and Sons, USA: Chapter 6 and Chapter 7 (Section 3).
- Chiang, AC (1984), *Fundamental methods of mathematical economics (third edition)*, McGraw-Hill, Singapore: Chapter 11 (Section 3).
- Debreu, G (1952), "Definite and semidefinite quadratic forms", *Econometrica 20(2)*, April, pp. 295–300.
- Hicks, JR (1939), *Value and capital: an inquiry into some fundamental principles of economic theory*, Oxford University Press, Great Britain, February: Mathematical Appendix (pp. 303–328).
- Hicks, JR (1946), *Value and capital: an inquiry into some fundamental principles of economic theory (second edition)*, Oxford University Press, Great Britain: Mathematical Appendix (pp. 303–328).

Reading Part 2

- Mandy, DM (2013), “On second order conditions for equality constrained extremum problems”, *Economics Letters 121(3)*, December, pp. 440–443.
- Mandy, DM (2018), “Leading principal minors and semidefiniteness”, *Economic Inquiry 56(2)*, April, pp. 1396–1398.
- Mann, HB (1943), “Quadratic forms with linear constraints”. *The American Mathematical Monthly 50(7)*, August to September, pp. 430–433.
- Samuelson, PA (1947), *Foundations of Economic Analysis*, Harvard University Press, USA: Mathematical Appendix A (pp. 357–379).
- Silbeberg, E, and W Suen (2001), *The structure of economics: A mathematical analysis (third edition)*, Irwin McGraw-Hill, Singapore: Chapter 6.

Reading Part 3

- Simon, CP, and L Blume (1994), *Mathematics for economists*, WW Norton and Company, USA: Chapter 13 (Section 3), Chapter 16, Chapter 17, Chapter 19, Chapter 21, and Chapter 23.
- Sundaram, RK (1996), *A first course in optimization theory*, Cambridge University Press, USA: Chapter 1 (Section 5).
- Takayama, A (1993), *Analytical methods in economics*, The University of Michigan Press, USA: Chapter 1 (Section 4).

What is a Quadratic Form? 1

- Suppose that

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

  is an $(n \times n)$ square matrix whose elements are all fixed parameters (constants) and

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

  is an $(n \times 1)$ column vector whose elements are all variables.
- Consider the function $f(x) = x^T A x$.

What is a Quadratic Form? 2

- Note that

$$
\begin{aligned}
x^T A x &= \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\
&= \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} \sum_{i=1}^n a_{1i} x_i \\ \sum_{i=1}^n a_{2i} x_i \\ \vdots \\ \sum_{i=1}^n a_{ni} x_i \end{pmatrix} \\
&= \sum_{j=1}^n \left(x_j \left(\sum_{i=1}^n a_{ji} x_i \right) \right)
\end{aligned}
$$

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What is a Quadratic Form? 3

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$$
\begin{aligned}
x^T A x &= \sum_{j=1}^{n} \left(x_j \left(\sum_{i=1}^{n} a_{ji} x_i \right) \right) \\
&= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_{ji} x_i x_j \right) \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ji} x_i x_j \\
&= \left(\sum_{j=1}^{n} a_{jj} x_j^2 \right) + \left(\sum_{j \neq i} \sum_{i=1}^{n} a_{ji} x_i x_j \right).
\end{aligned}
$$

- Clearly $f(x) = x^T A x$ is a quadratic function of the variables in the x vector. It is for this reason that $x^T A x$ is known as a quadratic form.

Symmetric and Non-Symmetric Matrices 1

- Suppose that a square matrix A is not symmetric, so that $a_{ij} \neq a_{ji}$ for at least one (i, j) pair for which $i \neq j$.
- We have already shown that

$$x^T A x = \left(\sum_{j=1}^{n} a_{jj} x_j^2 \right) + \left(\sum_{j \neq i} \sum_{i=1}^{n} a_{ji} x_i x_j \right).$$

- Note that the second component of the right hand side of this expression includes only terms for which either $j < i$ or $j > i$. If we collect like terms, we can rewrite this component as

$$\sum_{j \neq i} \sum_{i=1}^{n} a_{ji} x_i x_j = \sum_{j<i} \sum_{i=1}^{n} \left(a_{ji} + a_{ij} \right) x_i x_j.$$

- Suppose that we let $b_{ii} = a_{ii}$ and $b_{ij} = \frac{(a_{ji}+a_{ij})}{2} = b_{ji}$.

Symmetric and Non-Symmetric Matrices 2

- This yields

$$\begin{aligned}
x^T A x &= \left(\sum_{j=1}^{n} a_{jj} x_j^2 \right) + \left(\sum_{j \neq i} \sum_{i=1}^{n} a_{ji} x_i x_j \right) \\
&= \left(\sum_{j=1}^{n} a_{jj} x_j^2 \right) + \left(\sum_{j < i} \sum_{i=1}^{n} \left(a_{ji} + a_{ij} \right) x_i x_j \right) \\
&= \left(\sum_{j=1}^{n} b_{jj} x_j^2 \right) + \left(\sum_{j < i} \sum_{i=1}^{n} \left(\frac{\left(a_{ji} + a_{ij} \right)}{2} + \frac{\left(a_{ji} + a_{ij} \right)}{2} \right) x_i x_j \right) \\
&= \left(\sum_{j=1}^{n} b_{jj} x_j^2 \right) + \left(\sum_{j < i} \sum_{i=1}^{n} \left(b_{ji} + b_{ji} \right) x_i x_j \right)
\end{aligned}$$

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Symmetric and Non-Symmetric Matrices 3

- Continued from the previous slide.

$$
\begin{aligned}
x^T A x &= \left(\sum_{j=1}^{n} b_{jj} x_j^2 \right) + \left(\sum_{j \neq i} \sum_{i=1}^{n} b_{ji} x_i x_j \right) \\
&= x^T B x,
\end{aligned}
$$

  where

$$
B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{12} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1n} & b_{2n} & \cdots & b_{nn} \end{pmatrix}
$$

  is a symmetric matrix.

- Thus any quadratic form $x^T A x$ in which the matrix A is not symmetric can also be expressed as a quadratic form $x^T B x$ in which the matrix B is symmetric.

The Definiteness of a Matrix 1

- The "definiteness" of a square matrix A is related to the sign of the quadratic form $x^T Ax$ when the x vector is not a null vector.
  - Trivially, when x is a null vector (that is, a vector of zeros), the quadratic form $x^T Ax$ must be equal to zero.
- The matrix A is said to be "positive definite" if $x^T Ax > 0$ for all $x \neq 0$.
- The matrix A is said to be "positive semi-definite" if $x^T Ax \geqslant 0$ for all $x \neq 0$.
- The matrix A is said to be "negative semi-definite" if $x^T Ax \leqslant 0$ for all $x \neq 0$.
- The matrix A is said to be "negative definite" if $x^T Ax < 0$ for all $x \neq 0$.
- The matrix A is said to be "indefinite" if $x^T Ax > 0$ for at least one $x \neq 0$ **and** $x^T Ax < 0$ for at least one $x \neq 0$.

The Definiteness of a Matrix 2

- Sometimes the definition of the various types of matrix definiteness can be used to establish the definiteness of a particular square matrix.
    - But often this is not a particularly convenient method for doing this.
- When a square matrix is symmetric, there are two indirect approaches to determining its definiteness that are often much more convenient than attempting to directly employ the definition itself.
    - One of these indirect methods involves an examination of the "eigenvalues" of the matrix.
    - The other of these indirect methods involves an examination of the "leading principal minors" of the matrix.
- But what if a square matrix is not symmetric?
    - We can always use the technique discussed above to construct a symmetric square matrix that will have an identical definiteness to the original non-symmetric square matrix.

Eigenvalues and Definiteness 1

- Suppose that, in addition to being a square matrix, the matrix A is a symmetric matrix, so that $A^T = A$.
  - The matrix A will be positive definite if and only if all of its eigenvalues are strictly positive (> 0).
  - The matrix A will be positive semi-definite if and only if all of its eigenvalues are non-negative ($\geqslant 0$).
  - The matrix A will be negative semi-definite if and only if all of its eigenvalues are non-positive ($\leqslant 0$).
  - The matrix A will be negative definite if and only if all of its eigenvalues are strictly negative (< 0).
  - The matrix A will be indefinite if and only if it has both at least one strictly positive eigenvalue and at least one strictly negative eigenvalue.
- But what are the eigenvalues of a matrix and how do we find them?

Eigenvalues and Definiteness 2

- The characteristic matrix associated with a square matrix A is defined to be the square matrix $(\lambda I - A)$, where λ is a scalar variable and I is the identity matrix that has the same dimensions as A.
  - Sometimes the characteristic matrix is defined to be $(A - \lambda I)$.
- The characteristic polynomial associated with the matrix A is defined to be $\det(\lambda I - A)$.
  - Sometimes the characteristic polynomial associated with the matrix A is defined to be $\det(A - \lambda I)$.
  - If A is an $(n \times n)$ matrix for some $n \in \mathbb{N}$, then the characteristic polynomial will be an nth degree polynomial function of the scalar variable λ.
- The characteristic equation associated with the matrix A is defined to be $\det(\lambda I - A) = 0$.
  - Sometimes the characteristic equation associated with the matrix A is defined to be $\det(A - \lambda I) = 0$.

Eigenvalues and Definiteness 3

- The eigenvalues (λ) of a square matrix (A) are the solutions to the characteristic equation.
  - Both versions of the characteristic equation will yield the same set of solution values for λ.
- If A is a symmetric matrix, then all of its eigenvalues will be real numbers.
  - This is Part (a) of Theorem 23.16 in Simon and Blume (1994, p. 621).

Principal Minors and Definiteness 1

- Once again, suppose that A is a symmetric square matrix.
  - Symmetry requires that $a_{ij} = a_{ji}$ for all $i \neq j$.
  - In other words, if A is an an $(n \times n)$ square matrix, then it takes the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}.$$

- The leading principal sub-matrices for A are given by

$$A_1 = (a_{11}),$$

$$A_2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix},$$

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Principal Minors and Definiteness 2

- Continued from the previous slide.

$$A_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix},$$

  and so on and so forth up until

$$A_n = A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}.$$

- The leading principal minors for A are given by $\det(A_1)$, $\det(A_2)$, $\det(A_3)$, and so on and so forth up until $\det(A_n) = \det(A)$.

Principal Minors and Definiteness 3

- The matrix A will be positive definite if and only if $\det(A_i) > 0$ for all $i \in \{1, 2, \cdots, n\}$.
- The matrix A will be negative definite if and only if both $\det(A_i) < 0$ for all odd i and $\det(A_i) > 0$ for all even i.
- Using the leading principle minors of a matrix to determine whether or not it is either positive semi-definite or negative semi-definite is slightly more complicated.
  - Unfortunately, we cannot just directly modify the strict inequalities in the positive definite and negative definite tests to incorporate weak inequalities.
  - Instead, we need the pattern implied by doing just that to hold for all possible permutations of the entries in the matrix.
  - This makes this approach to determining positive semi-definiteness or negative semi-definiteness somewhat cumbersome.
  - Further details can be found in Sundaram (1996, pp. 50–55) and Mandy (2018, pp. 1396–1398).
- In any other circumstance, the matrix A is indefinite.

Example 1 Part 1

- Consider the matrix

$$H = \begin{pmatrix} \frac{\alpha}{x} & 0 & 0 \\ 0 & \frac{\beta}{y} & 0 \\ 0 & 0 & \frac{\gamma}{z} \end{pmatrix}.$$

  where $x > 0$, $y > 0$, $z > 0$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$ and $(\alpha + \beta + \gamma) = 1$.

- Note that H is a symmetric matrix.

Example 1 Part 2

- First, let us see if we can determine the definiteness of the matrix H by using the leading principal minors approach.
- Note that the first leading principle minor of H is

$$\det\left(H_1\right) = \det\left(\left(\frac{\alpha}{x}\right)\right) = \frac{\alpha}{x} > 0$$

because $x > 0$ and $\alpha > 0$.

Example 1 Part 3

- Note that the second leading principle minor of H is

$$\begin{aligned}
\det\left(H_2\right) &= \det\left(\left(\begin{array}{cc} \frac{\alpha}{x} & 0 \\ 0 & \frac{\beta}{y} \end{array}\right)\right) \\
&= \left(\frac{\alpha}{x}\right)\left(\frac{\beta}{y}\right) - (0)(0) \\
&= \frac{\alpha\beta}{xy} - 0 \\
&= \frac{\alpha\beta}{xy} \\
&> 0 \text{ (because } x > 0, y > 0, \alpha > 0 \text{ and } \beta > 0\text{).}
\end{aligned}$$

Example 1 Part 4

- Note that the third (and final) leading principle minor of H is simply the determinant of the matrix H itself.
- Upon employing a cofactor expansion along the first row of matrix H, we obtain

$$
\begin{aligned}
\det(H_3) &= \det(H) \\
&= \left(\frac{\alpha}{x}\right)(-1)^{1+1}\det\left(\begin{pmatrix} \frac{\beta}{y} & 0 \\ 0 & \frac{\gamma}{z} \end{pmatrix}\right) + 0 + 0 \\
&= \left(\frac{\alpha}{x}\right)(-1)^{2}\left[\left(\frac{\beta}{y}\right)\left(\frac{\gamma}{z}\right) - (0)(0)\right] \\
&= \left(\frac{\alpha}{x}\right)(1)\left[\frac{\beta\gamma}{yz} - 0\right].
\end{aligned}
$$

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Example 1 Part 5

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$$\begin{aligned}
\det(H_3) &= \det(H) \\
&= \left(\frac{\alpha}{x}\right)(1)\left[\frac{\beta\gamma}{yz} - 0\right] \\
&= \left(\frac{\alpha}{x}\right)\left(\frac{\beta\gamma}{yz}\right) \\
&= \frac{\alpha\beta\gamma}{xyz} \\
&> 0
\end{aligned}$$

  because $x > 0$, $y > 0$, $z > 0$, $\alpha > 0$, $\beta > 0$, and $\gamma > 0$.

- Since $\det(H_1) > 0$, $\det(H_2) > 0$ and $\det(H_3) > 0$, we can conclude that the matrix H is positive definite.

Example 1 Part 6

- Now let us now establish that H is positive definite by using the eigenvalue approach.
- The characteristic matrix associated with H is

$$\lambda I - H = \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{\alpha}{x} & 0 & 0 \\ 0 & \frac{\beta}{y} & 0 \\ 0 & 0 & \frac{\gamma}{z} \end{pmatrix}$$

$$= \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} - \begin{pmatrix} \frac{\alpha}{x} & 0 & 0 \\ 0 & \frac{\beta}{y} & 0 \\ 0 & 0 & \frac{\gamma}{z} \end{pmatrix}.$$

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Example 1 Part 7

- Continued from the previous slide.

$$\lambda I - H = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} - \begin{pmatrix} \frac{\alpha}{x} & 0 & 0 \\ 0 & \frac{\beta}{y} & 0 \\ 0 & 0 & \frac{\gamma}{z} \end{pmatrix}$$

$$= \begin{pmatrix} \lambda - \frac{\alpha}{x} & 0 & 0 \\ 0 & \lambda - \frac{\beta}{y} & 0 \\ 0 & 0 & \lambda - \frac{\gamma}{z} \end{pmatrix}.$$

Example 1 Part 8

- It is straight-forward to establish that the characteristic polynomial for H is

$$\det(\lambda I - H) = \left(\lambda - \frac{\alpha}{x}\right)\left(\lambda - \frac{\beta}{y}\right)\left(\lambda - \frac{\gamma}{z}\right).$$

  - You should establish the validity of this claim as a form of revision of the calculation of determinants.

- Thus the characteristic equation associated with H is

$$\det(\lambda I - H) = \left(\lambda - \frac{\alpha}{x}\right)\left(\lambda - \frac{\beta}{y}\right)\left(\lambda - \frac{\gamma}{z}\right) = 0.$$

Example 1 Part 9

- Clearly the eigenvalues for H are $\lambda_1 = \frac{\alpha}{x}$, $\lambda_2 = \frac{\beta}{y}$, and $\lambda_3 = \frac{\gamma}{z}$.
- Since $x > 0$, $y > 0$, $z > 0$, $\alpha > 0$, $\beta > 0$, and $\gamma > 0$, we know that $\lambda_1 > 0$, $\lambda_2 > 0$, and $\lambda_3 > 0$.
- Thus we can conclude that H is a positive definite matrix.

Example 2

- Consider the matrix
$$A = \begin{pmatrix} 2 & 3 \\ 3 & 7 \end{pmatrix}.$$
- Note that A is a symmetric matrix.
- The leading principal sub-matrices for this matrix are $A_1 = (2)$ and $A_2 = A$.
- As such, the leading principle minors for this matrix are
$$\det(A_1) = 2 > 0$$
and
$$\det(A_2) = \det(A) = (2)(7) - (3)(3) = 14 - 9 = 5 > 0.$$
- Thus we can conclude that A is a positive definite matrix.

Example 3 Part 1

- Consider the matrix
$$A = \begin{pmatrix} 2 & 4 \\ 4 & 7 \end{pmatrix}.$$
- Note that A is a symmetric matrix.
- The leading principal sub-matrices for this matrix are $A_1 = (2)$ and $A_2 = A$.
- As such, the leading principle minors for this matrix are
$$\det(A_1) = 2 > 0$$
and
$$\det(A_2) = \det(A) = (2)(7) - (4)(4) = 14 - 16 = -2 < 0.$$

Example 3 Part 2

- Since $\det(A_1) > 0$ and $\det(A_2) = \det(A) < 0$, we know that A is neither negative definite nor positive definite.
  - But is it negative semi-definite, positive semi-definite, or indefinite?
- Let us find the eigenvalues of A.
- The characteristic matrix for A is

$$\begin{aligned} \lambda I - A &= \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 4 \\ 4 & 7 \end{pmatrix} \\ &= \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 2 & 4 \\ 4 & 7 \end{pmatrix} \\ &= \begin{pmatrix} \lambda - 2 & -4 \\ -4 & \lambda - 7 \end{pmatrix}. \end{aligned}$$

Example 3 Part 3

- This means that the characteristic polynomial for A is

$$\begin{aligned} \det(\lambda I - A) &= (\lambda - 2)(\lambda - 7) - (-4)(-4) \\ &= \lambda^2 - 7\lambda - 2\lambda + 14 - 16 \\ &= \lambda^2 - 9\lambda - 2. \end{aligned}$$

- Thus the characteristic equation for A is

$$\det(\lambda I - A) = \lambda^2 - 9\lambda - 2 = 0.$$

- Note that the characteristic equation for A is a quadratic equation in the variable λ.

Example 3 Part 4

- Upon applying the quadratic formula to the characteristic equation for A, we find that the eigenvalues for the matrix A are $\lambda_1 = \frac{9+\sqrt{89}}{2}$ and $\lambda_2 = \frac{9-\sqrt{89}}{2}$.
- Since $9^2 = 81 < 89 < 100 = 10^2$, we know that $9 < \sqrt{89} < 10$.
- This means that $\lambda_1 = \frac{9+\sqrt{89}}{2} > 0$ and $\lambda_2 = \frac{9-\sqrt{89}}{2} < 0$.
- Thus we can conclude that the A is an indefinite matrix.

Example 4

- Consider the matrix
$$A = \begin{pmatrix} 0 & 0 \\ 0 & 7 \end{pmatrix}.$$
- Note that $x^T A x = 7x_2^2 \geqslant 0$ for all $(x_1, x_2)^T \neq (0,0)^T$.
- Thus this matrix is positive semi-definite.
  - This follows directly from the definition of positive semi-definiteness of a matrix.
  - It is not positive definite because $x^T A x = 7x_2^2 = 0$ when $(x_1, x_2)^T = (1,0)$ and $(1,0) \neq (0,0)$.

Example 5

- Consider the matrix
$$A = \begin{pmatrix} 0 & 0 \\ 0 & -7 \end{pmatrix}.$$
- Note that $x^T Ax = -7x_2^2 \leqslant 0$ for all $(x_1, x_2)^T \neq (0,0)^T$.
- Thus this matrix is negative semi-definite.
    - This follows directly from the definition of negative semi-definiteness of a matrix.
    - It is not negative definite because $x^T Ax = 7x_2^2 = 0$ when $(x_1, x_2)^T = (1,0)$ and $(1,0) \neq (0,0)$.)

Example 6 Part 1

- Consider the matrix
$$A = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}.$$
- Note that A is a symmetric matrix.
- The leading principal sub-matrices for this matrix are $A_1 = (2)$ and $A_2 = A$.
- As such, the leading principle minors for this matrix are
$$\det(A_1) = 2 > 0$$
and
$$\det(A_2) = \det(A) = (2)(-1) - (2)(2) = -2 - 4 = -6 < 0.$$

Example 6 Part 2

- Since $\det(A_1) > 0$ and $\det(A_2) = \det(A) < 0$, we know that A is neither negative definite nor positive definite.
    - But is it negative semi-definite, positive semi-definite, or indefinite?
- Let us find the eigenvalues of A.
- The characteristic matrix for A is

$$
\begin{aligned}
\lambda I - A &= \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} \\
&= \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} \\
&= \begin{pmatrix} \lambda - 2 & -2 \\ -2 & \lambda + 1 \end{pmatrix}.
\end{aligned}
$$

Example 6 Part 3

- This means that the characteristic polynomial for A is

$$\begin{aligned} \det(\lambda I - A) &= (\lambda - 2)(\lambda + 1) - (-2)(-2) \\ &= \lambda^2 + \lambda - 2\lambda - 2 - 4 \\ &= \lambda^2 - \lambda - 6 \\ &= (\lambda + 2)(\lambda - 3). \end{aligned}$$

- Thus the characteristic equation for A is

$$\det(\lambda I - A) = (\lambda + 2)(\lambda - 3) = 0.$$

- This means that the eigenvalues for the matrix A are $\lambda_1 = -2$ and $\lambda_2 = 3$.
- Since $\lambda_1 < 0$ and $\lambda_2 > 0$, we can conclude that A is an indefinite matrix.

Example 7 Part 1

- Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{pmatrix}.$$

- Note that A is a symmetric matrix.
- The leading principal sub-matrices for this matrix are $A_1 = (1)$,

$$A_2 = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

and $A_3 = A$.

Example 7 Part 2

- As such, the leading principle minors for this matrix are $\det(A_1) = 1$,

$$\det(A_2) = (1)(4) - (2)(2) = 4 - 4 = 0$$

and

$$\begin{aligned}
\det(A_3) &= \det(A) \\
&= (1)(-1)^{1+1} \det\left(\begin{pmatrix} 4 & 5 \\ 5 & 6 \end{pmatrix}\right) \\
&\quad + (2)(-1)^{1+2} \det\left(\begin{pmatrix} 2 & 5 \\ 0 & 6 \end{pmatrix}\right) \\
&\quad + (0)(-1)^{1+3} \det\left(\begin{pmatrix} 2 & 4 \\ 0 & 5 \end{pmatrix}\right) \\
&= (1)(-1)^2 \{(4)(6) - (5)(5)\} \\
&\quad + (2)(-1)^3 \{(2)(6) - (5)(0)\} + 0
\end{aligned}$$

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Example 7 Part 3

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$$\begin{aligned}
&= (1)(-1)^2\{(4)(6)-(5)(5)\} \\
&\quad + (2)(-1)^3\{(2)(6)-(5)(0)\}+0 \\
&= (1)(1)\{24-25\}+(2)(-1)\{12-0\} \\
&= (1)(1)(-1)+(2)(-1)(12) \\
&= -1+(-24) \\
&= -1-24 \\
&= -25.
\end{aligned}$$

- Since $\det(A_1)>0$, $\det(A_2)=0$, and $\det(A_3)=\det(A)<0$, we know that A is neither negative definite nor positive definite.
  - But is it negative semi-definite, positive semi-definite, or indefinite?
- Let us find the eigenvalues of A.

Example 7 Part 4

- The characteristic matrix for A is

$$\begin{aligned} \lambda I - A &= \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{pmatrix} \\ &= \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} - \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{pmatrix} \\ &= \begin{pmatrix} \lambda - 1 & -2 & 0 \\ -2 & \lambda - 4 & -5 \\ 0 & -5 & \lambda - 6 \end{pmatrix}. \end{aligned}$$

Example 7 Part 5

- The characteristic polynomial for A is

$$\begin{aligned}
\det(\lambda I - A) &= (\lambda-1)\,(-1)^{1+1}\det\left(\begin{pmatrix} \lambda-4 & -5 \\ -5 & \lambda-6 \end{pmatrix}\right) \\
&\quad + (-2)\,(-1)^{1+2}\det\left(\begin{pmatrix} -2 & -5 \\ 0 & \lambda-6 \end{pmatrix}\right) + 0 \\
&= (\lambda-1)(-1)^2\left[(\lambda-4)(\lambda-4)-25\right] \\
&\quad +(-2)(-1)^3\left[-2(\lambda-6)-0\right] \\
&= (\lambda-1)(1)\left[\lambda^2-10\lambda+24-25\right] \\
&\quad -2(-1)\left[-2\lambda+12\right]
\end{aligned}$$

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Example 7 Part 6

- Continued from the previous slide.

$$
\begin{aligned}
\det(\lambda I - A) &= (\lambda - 1)(1)\left[\lambda^2 - 10\lambda + 24 - 25\right] \\
&\quad -2(-1)\left[-2\lambda + 12\right] \\
&= (\lambda - 1)\left[\lambda^2 - 10\lambda - 1\right] + 2\left[-2\lambda + 12\right] \\
&= \lambda^3 - 10\lambda^2 - \lambda - \left[\lambda^2 - 10\lambda - 1\right] - 4\lambda + 24 \\
&= \lambda^3 - 10\lambda^2 - \lambda - \lambda^2 + 10\lambda + 1 - 4\lambda + 24 \\
&= \lambda^3 - 11\lambda^2 + 5\lambda + 25.
\end{aligned}
$$

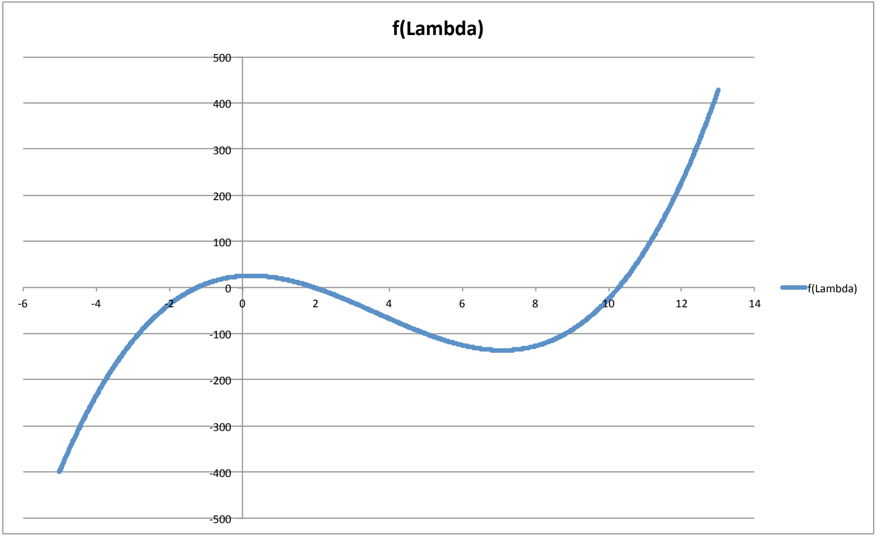
Example 7 Part 7

- The characteristic equation for the matrix A is

$$\det(\lambda I - A) = \lambda^3 - 11\lambda^2 + 5\lambda + 25 = 0.$$

- Note that this is a cubic equation in the variable λ.
- While there do not appear to be any "obvious" factorisations of this characteristic polynomial, it is possible to obtain numerical approximations to the eigenvalues of A (that is, the solutions to the characteristic equation) by using Microsoft Excel.
- Upon doing this we find that $\lambda_1 \approx -1.2395$, $\lambda_2 \approx 1.9627$, and $\lambda_3 \approx 10.277$.
- A graph of the characteristic polynomial can be found on the next slide.
- Since the matrix A has both positive and negative eigenvalues, we can conclude that it is indefinite.

Example 7 Part 8



- Consider the matrix

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

- Note that A is a symmetric matrix.
- The characteristic matrix for A is

$$(\lambda I - A) = \begin{pmatrix} \lambda+1 & -1 & 0 \\ -1 & \lambda+1 & 0 \\ 0 & 0 & \lambda+2 \end{pmatrix}.$$

Example 8 Part 2

- The characteristic polynomial for A is

$$\begin{aligned}
\det(\lambda I - A) &= 0 + 0 + (\lambda+2)\,(-1)^{3+3}\det\left(\begin{pmatrix} \lambda+1 & -1 \\ -1 & \lambda+1 \end{pmatrix}\right) \\
&= 0 + 0 + (\lambda+2)(-1)^6\left[(\lambda+1)^2 - 1\right] \\
&= (\lambda+2)(1)((\lambda+1)+1)((\lambda+1)-1) \\
&= (\lambda+2)(1)(\lambda+2)(\lambda) \\
&= (\lambda+2)^2\lambda.
\end{aligned}$$

Example 8 Part 3

- The characteristic equation for the matrix A is

$$\det(\lambda I - A) = (\lambda + 2)^2 \lambda = 0.$$

- Clearly the eigenvalues for the matrix A are $\lambda_1 = -2$, $\lambda_2 = -2$, and $\lambda_3 = 0$.
- Since the matrix A has both negative eigenvalues and a zero eigenvalue , we can conclude that it is negative semi-definite.

Example 9 Part 1

- Consider the matrix
$$A = \begin{pmatrix} -1 & -2 \\ 4 & 3 \end{pmatrix}.$$
- Note that the matrix A is not symmetric.
  - Before proceeding, let us construct a symmetric matrix B such that $x^T A x = x^T B x$ for all $x \in \mathbb{R}^2$.
  - We will do this by setting $b_{11} = a_{11} = -1$, $b_{22} = a_{22} = 3$, and $b_{12} = b_{21} = \left(\frac{1}{2}\right)(a_{12} + a_{21}) = \frac{(-2+4)}{2} = \frac{2}{2} = 1$.
  - The resulting matrix B is
$$B = \begin{pmatrix} -1 & 1 \\ 1 & 3 \end{pmatrix}.$$
  - Note that the matrix B is symmetric.

Example 9 Part 2

- The leading principal sub-matrices for this matrix are $B_1 = (-1)$ and $B_2 = B$.
- As such, the leading principle minors for this matrix are

$$\det(B_1) = -1 < 0$$

and

$$\det(B_2) = \det(B) = (-1)(3) - (1)(1) = -3 - 1 = -4 < 0.$$

- Since $\det(B_1) < 0$ and $\det(B_2) = \det(B) < 0$, we know that B (and hence A) is neither negative definite nor positive definite.
  - But is it negative semi-definite, positive semi-definite, or indefinite?
- Let us find the eigenvalues of B.

Example 9 Part 3

- The characteristic matrix for B is

$$\begin{aligned}
\lambda I - B &= \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -1 & 1 \\ 1 & 3 \end{pmatrix} \\
&= \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} -1 & 1 \\ 1 & 3 \end{pmatrix} \\
&= \begin{pmatrix} \lambda + 1 & -1 \\ -1 & \lambda - 3 \end{pmatrix}.
\end{aligned}$$

- This means that the characteristic polynomial for B is

$$\begin{aligned}
\det(\lambda I - B) &= (\lambda + 1)(\lambda - 3) - (-1)(-1) \\
&= \lambda^2 - 3\lambda + \lambda - 3 - 1 \\
&= \lambda^2 - 2\lambda - 4.
\end{aligned}$$

Example 9 Part 4

- Thus the characteristic equation for B is

$$\det(\lambda I - B) = \lambda^2 - 2\lambda - 4 = 0.$$

- Note that the characteristic equation for B is a quadratic equation in the variable λ.
- Upon applying the quadratic formula to the characteristic equation for B, we find that the eigenvalues for the matrix B are $\lambda_1 = \frac{2+\sqrt{20}}{2}$ and $\lambda_2 = \frac{2-\sqrt{20}}{2}$.
- Since $4^2 = 16 < 20 < 25 = 5^2$, we know that $4 < \sqrt{20} < 5$.
- This means that $\lambda_1 = \frac{2+\sqrt{20}}{2} > 0$ and $\lambda_2 = \frac{2-\sqrt{20}}{2} < 0$.
- Thus we can conclude that the B (and hence A) is an indefinite matrix.

Example 10 Part 1

- Consider the matrix

$$A = \begin{pmatrix} \frac{-\alpha}{x^2} & 0 \\ 0 & \frac{-(1-\alpha)}{y^2} \end{pmatrix},$$

  where $x > 0$, $y > 0$, and $0 < \alpha < 1$.

- The leading principal sub-matrices for this matrix are $A_1 = \left(\frac{-\alpha}{x^2}\right)$ and $A_2 = A$.

- As such, the leading principle minors for this matrix are

$$\det(A_1) = \frac{-\alpha}{x^2} < 0$$

  and

$$\det(A_2) = \det(A)$$

- Continued on the next slide.

Example 10 Part 2

- Continued from the previous slide.

$$
\begin{aligned}
\det(A_2) &= \det(A) \\
&= \left(\frac{-\alpha}{x^2}\right)\left(\frac{-(1-\alpha)}{y^2}\right) - (0)(0) \\
&= \frac{\alpha(1-\alpha)}{x^2y^2} - 0 \\
&= \frac{\alpha(1-\alpha)}{x^2y^2} \\
&> 0.
\end{aligned}
$$

- Since $\det(A_1) < 0$ and $\det(A_2) = \det(A) > 0$, we can conclude that A is a negative definite matrix.

Quadratic Forms On Sets of Linear Constraints 1

- Up until now, we have been allowing the x vector to vary over all of $\mathbb{R}^n$ (for some $n \in \mathbb{N}$) and examining the sign of the quadratic form $x^T Ax$ for all such $x \neq 0$.
  - In other words, we have been examining the sign of an "unconstrained" quadratic form in x.
- In economics, however, we are often faced with situations in which the x vector will be subject to one or more constraints.
- Is it possible to determine the sign of a quadratic form for all x vectors that satisfy one or more constraints in addition to the standard constraint that $x \neq 0$?
- If so, how can this be done?
- In the remainder of these notes, we (at least partially) address these questions for the case in which the x vector is subject to one or more linear constraints (in addition to the standard constraint that $x \neq 0$).

Quadratic Forms On Sets of Linear Constraints 2

- Let $x \in \mathbb{R}^n$ be a vector of n real-valued variables, A be an $(n \times n)$ matrix of constant real-valued coefficients, C be an $(m \times n)$ matrix of constant real-valued coefficients, $m \in \mathbb{N}$, $n \in \mathbb{N}$, and $m < n$.
- Note that $x^T Ax$ is a quadratic form in the x vector and Cx is a linear form in the x vector.
- We have already seen that if A is not a symmetric matrix, then there exists some other matrix B such that $x^T Ax = x^T Bx$ for all $x \in \mathbb{R}^n$.
    - As such, we can assume, without loss of generality, that A is a symmetric $(n \times n)$ matrix.
    - We will make this assumption for the remainder of these notes.

Quadratic Forms On Sets of Linear Constraints 3

- The linear form Cx will represent the linear constraints that we are imposing on the x vector in addition to the standard restriction that $x \neq 0$.
    - To be precise, the set of linear constraints that are being imposed are given by the matrix equation $Cx = 0$, where 0 is an $(m \times 1)$ null vector (that is, a vector consisting entirely of entries that are zeros).
    - It is important that $m < n$ because we do not want the x vector to be completely determined by the additional constraints that are being imposed.

Quadratic Forms On Sets of Linear Constraints 4

- We can use the matrices A and C to form a "bordered matrix" D that takes the partitioned form

$$D = \begin{pmatrix} 0 & C \\ C^T & A \end{pmatrix},$$

  where 0 is an $(m \times m)$ null matrix (that is, a matrix consisting entirely of entries that are zeros).
- Note that D is a symmetric square matrix.
  - It is an $((m+n) \times (m+n))$ matrix.
  - It is symmetric because we have assumed that A is a symmetric matrix.
- The theorem on the following two slides is contained within (that is, is part of) Theorem 16.4 in Simon and Blume (1994, p. 389).

- **Theorem:** Consider the quadratic form $Q(x) = x^T A x$ subject to the set of linear constraints $Cx = 0_{(m \times 1)}$ and $x \neq 0_{(n \times 1)}$, where $x \in \mathbb{R}^n$ is an $(m \times 1)$ vector of real-valued variables, A is an $(n \times n)$ symmetric matrix of constant real-valued coefficients, C is an $(m \times n)$ matrix of constant real-valued coefficients, $m \in \mathbb{N}$, $n \in \mathbb{N}$, and $m < n$. Form the $((m+n) \times (m+n))$. symmetric "bordered matrix" (in partitioned form)

$$D = \begin{pmatrix} 0_{(m \times m)} & C \\ C^T & A \end{pmatrix}.$$

- Continued on the next slide.

Quadratic Forms On Sets of Linear Constraints 6

- Continued from the previous slide.
  1. The quadratic form $Q(x)$ is negative definite on the constraint set if $\det(D)$ has the same sign as $(-1)^n$ **and** if the **last** $(n-m)$ leading principal minors alternate in sign.
  2. The quadratic form $Q(x)$ is positive definite on the constraint set if the **last** $(n-m)$ leading principal minors all have the same sign as $(-1)^m$.
  3. The quadratic form $Q(x)$ is indefinite on the constraint set if both condition (1) and condition (2) are violated by **non-zero** leading principal minors.

Example 11 Part 1

- This is Example 16.7 from Simon and Blume (1994, pp. 389–390).
- Consider the case where A is the symmetric (4×4) matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix},$$

and C is the (2×4) matrix

$$C = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & -9 & 0 & 1 \end{pmatrix}.$$

- Note that in this case there are four variables (that is, $n = 4$) and two linear constraints other than the constraint that $x \neq 0$ (that is, $m = 2$).

Example 11 Part 2

- The symmetric bordered matrix for this example is the (6×6) matrix

$$D = \begin{pmatrix} 0_{(2\times 2)} & C \\ C^T & A \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -9 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 1 & -9 & 0 & -1 & 2 & 0 \\ 1 & 0 & 0 & 2 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{pmatrix}.$$

- Since $n = 4$ and $m = 2$ in this example, we need to check the signs of the last $(n - m) = (4 - 2) = 2$ leading principal minors.
  - The last leading principle minor is $\det(D_6) = \det(D)$.
  - The second-last leading principle minor is $\det(D_5)$.

Example 11 Part 3

- According to Simon and Blume (1994, p. 390), $\det(D_6) = \det(D) = 24 > 0$ and $\det(D_5) = 77 > 0$.
  - You should confirm these values for the last two leading principle minors of D for yourself.
- Note that $(-1)^m = (-1)^2 = 1 > 0$.
  - Thus we have

$$\text{sign}(\det(D_6)) = \text{sign}(\det(D_5)) = \text{sign}\left((-1)^m\right) = \text{sign}\left((-1)^2\right).$$

  - This means that the matrix A is positive definite on the set of constraints given by both $Cx = 0$ and $x \neq 0$.