Sets, Numbers, Coordinates, and Distances

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Reading Guide 1

- Introductory level references.
  - Haeussler, EF Jr, and RS Paul (1987), *Introductory mathematical analysis for business, economics, and the life and social sciences (fifth edition)*, Prentice-Hall, USA:Chapter 0.2 (pp. 1–3).
  - Sydsaeter, K, P Hammond, A Strom, and A Carvajal (2016), *Essential mathematics for economic analysis (fifth edition)*, Pearson Education, Italy: Chapters 1.1, 2.1, and 5.5 (pp. 1–12, 19–22, and 160–163).
  - Shannon, J (1995), *Mathematics for business, economics and finance*, John Wiley and Sons, Brisbane: Chapter 1.2 and 1,3 (pp.2–11).

Reading Guide 2

- More advanced references.

  - Banks, J, G Elton and J Strantzen (2009), *Topology and analysis: Unit text for MAT3TA (2009 and 2010 edition)*, Department of Mathematics and Statistics, La Trobe University, Bundoora, February.
  - Corbae, D, MB Stinchcombe and J Zeman (2009), *An introduction to mathematical analysis for economic theory and econometrics*, Princeton University Press, USA: Chapters 1 and 2 (pp. 1-71).
  - Halmos, PR (1960), *Naive set theory*, The University Series in Undergraduate Mathematics, D Van Nostrand Company, USA.
  - Kolmogorov, AN and SV Fomin (1970), *Introductory real analysis*, Translated and Edited by RA Silverman, The 1975 Dover Edition (an unabridged, slightly corrected republication of the original 1970 Prentice-Hall edition), Dover Publications, USA: Chapter 1 (pp. 1-36).
  - Simon, C, and L Blume (1994), *Mathematics for economists*, WW Norton and Co, USA: Appendix A1 (pp. 847-858).

Sets and Elements

- A set (X) is a collection of objects.
- A particular object within a set (x) is known as an element of that set.
- The idea that x is an element of X is written in mathematical notation as $x \in X$.
- Suppose that there are elements that do not belong to X.
  - Let y be one such element.
  - The idea that y is not an element of X is written in mathematical notation as $y \notin X$.
- A set that does not contain any elements is said to be empty.
  - An empty set is denoted by either $\varnothing$ or $\{\}$.

Set Definition

- There are two fundamental ways of defining a particular set.
- The first of these is to exhaustively list all of the elements of the set.
  - Example 1: $X = \{1, 2, 3\}$.
  - Example 2: $Y = \{1, 2, 3, \cdots, 100\}$.
  - Example 3: $\mathbb{N} = \{1, 2, 3, \cdots\}$.
- The second of these is to specify one or more properties that characterise all of the elements in the set.
  - Example 4: $X = \{n \in \mathbb{N} : n < 4\}$.
  - Example 5: $Y = \{n \in \mathbb{N} : n \leqslant 100\}$.

Russell's Paradox 1

- It would be nice if we could always associate some type of set with any particular property that we might consider.
- In other words, it would be nice if for any property $\mathbb{A}$, we could form a set $\{x \in X : \mathbb{A}(x) \text{ is true}\}$ that consisted of all of the elements that satisfy this property.
- Unfortunately, this is not the case.
- This was established by Bertrand Russell. He did this by developing the following paradox.
- Let $\mathbb{A}$ be the property "is a set and does not belong to itself". Suppose that A is the set of all sets that possess property $\mathbb{A}$. Is $A \in A$?

Russell's Paradox 2

- If $A \in A$, then it must be the case that A possesses property $\mathbb{A}$. This means that $A \notin A$.
- Contradiction! Thus it must be the case that $A \notin A$.
- But if A is a set and $A \notin A$, then it clearly possesses property $\mathbb{A}$. Thus $A \in A$.
- Contradiction. Thus it must be the case that $A \in A$.
- We have a paradox. It cannot be the case that both $A \in A$ and $A \notin A$.
- One possible resolution to Russell's paradox is to not allow mathematical objects like this particular A to be considered a set.

Some Common Number Sets

- The set of natural numbers: $\mathbb{N} = \{1, 2, 3, \cdots\}$;
- The set of non-negative intergers: $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$;
- The set of integers: $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$;
- The set of rational numbers: $\mathbb{Q} = \left\{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}\right\}$;
  - The set of non-negative rational numbers: $\mathbb{Q}_+ = \{x \in \mathbb{Q} : x \geqslant 0\}$;
  - The set of positive rational numbers: $\mathbb{Q}_{++} = \{x \in \mathbb{Q} : x > 0\}$;
- The set of real numbers: $\mathbb{R} = (-\infty, \infty)$;
  - The set of non-negative real numbers: $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geqslant 0\}$;
  - The set of positive real numbers: $\mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$; and
- The set of complex numbers:

$$\mathbb{C} = \left\{a + bi : a \in \mathbb{R}, b \in \mathbb{R}, i = \sqrt{-1}\right\}.$$

Subsets

- Consider two sets, X and Y.
- Suppose that every element of X also belongs to Y. If this is the case, then we say that X is a subset of Y.
  - This is written is mathematical notation as $X \subseteq Y$.
- Suppose that in addition to every element of X also belonging to Y, there is at least one element of Y that does not belong to X. If this is the case, then we say that X is a proper subset of Y.
  - This is written is mathematical notation as $X \subset Y$.
  - Sometimes $X \subset Y$ is used (rather loosely) to mean $X \subseteq Y$. If this meaning of the notation is employed, then $X \subsetneq Y$ would need to be used to indicate that X is a proper subset of Y.
- Suppose that both every element of X also belongs to Y, and every element of Y also belongs to X. If this is the case, then we say that X is equal to Y.
  - This is written in mathematical notation as $X = Y$.

Subset Relationships Between Number Sets

- Recall the following sets of numbers:
  - The set of natural numbers: $\mathbb{N} = \{1, 2, 3, \cdots\}$;
  - The set of non-negative intergers: $\mathbb{Z}_{+} = \{0, 1, 2, \cdots\}$;
  - The set of integers: $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$;
  - The set of rational numbers: $\mathbb{Q} = \left\{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}\right\}$;
  - The set of real numbers: $\mathbb{R} = (-\infty, \infty)$; and
  - The set of complex numbers:

$$\mathbb{C} = \left\{a + bi : a \in \mathbb{R}, b \in \mathbb{R}, i = \sqrt{-1}\right\}.$$

- The following "nesting" relationship exists between these common sets of numbers:

$$\mathbb{N} \subset \mathbb{Z}_{+} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

The Nesting of Number Sets

- Note that $\mathbb{N} \subset \mathbb{Z}_+$ because $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.
- Note that $\mathbb{Z}_+ \subset \mathbb{Z}$ because $\mathbb{Z} = \mathbb{Z}_+ \cup \{\cdots, -3, -2, -1\}$.
- Note that $\mathbb{Z} \subset \mathbb{Q}$ because any $m \in \mathbb{Z}$ can be written as $\frac{m}{1}$ and $1 \in \mathbb{N}$, but there are fractions that do not belong to $\mathbb{Z}$ (for example $\frac{1}{2} \notin \mathbb{Z}$).
- Note that $\mathbb{Q} \subset \mathbb{R}$ because $\frac{m}{n} \in (-\infty, \infty)$ for all $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, but there are numbers on the real line that cannot be expressed as fractions (for example $\sqrt{2}$, π and e).
  - Real numbers that cannot be expressed as fractions are known as "irrational numbers".
- Note that $\mathbb{R} \subset \mathbb{C}$ because

$$\mathbb{R} = \left\{a + bi : a \in \mathbb{R}, b = 0, i = \sqrt{-1}\right\}$$

  and $0 \in \mathbb{R}$, but $(a + bi) \notin \mathbb{R}$ if $b \neq 0$.
  - Complex numbers in which $a = 0$ are known as (purely) "imaginary numbers".

Intervals as Subsets of the Real Line

- Some (but not all) of the subsets of the real line take the form of an interval.
- There are four types of interval.
- Let $a \in \mathbb{R}$, $b \in \mathbb{R}$ and $a < b$. The four types of interval are as follows.
- $[a, b] = \{x \in \mathbb{R} : a \leqslant x \leqslant b\}$. (If $a > b$, then $[a, b] = \varnothing$.) (If $a = b$, then $[a, b] = \{a\} = \{b\}$.)
- $(a, b) = \{x \in \mathbb{R} : a < x < b\}$. (If $a \geqslant b$, then $(a, b) = \varnothing$.)
- $[a, b) = \{x \in \mathbb{R} : a \leqslant x < b\}$. (If $a \geqslant b$, then $[a, b) = \varnothing$.)
- $(a, b] = \{x \in \mathbb{R} : a < x \leqslant b\}$. (If $a \geqslant b$, then $(a, b] = \varnothing$.)

The Real Number System

- The system of real numbers is an algebraic structure known as a complete ordered field.
- Indeed, in a sense, it is the only complete ordered field in existence. Any other complete ordered field turns out to be "isomoporhic" to the real number system. (The term "isomorphic" is a mathematical version of "essentially the same").
- The system of real numbers is formally denoted by $(\mathbb{R}, \mathbb{R}_{++}, +, \times)$, where $+$ and $\times$ are the familiar addition and multiplication operations for real numbers.
- The set of real numbers $\mathbb{R}$ can be viewed as the completion of the the set of rational numbers $\mathbb{Q}$ because it involves filling in the "holes" that exist in the set of rational numbers. These holes take the form of irrational numbers like $\sqrt{2}$, e, and π.

Why aren't the rationals enough?

- Why aren't the rational numbers enough? What makes us think that they contain "holes"?
- This is a very good question. Especially when you realise that any numerical calculation on a computer will only generate a rational number.
- A geometric argument for the existence of irrational numbers is perhaps the easiest way to convince yourself of their existence.
- Think about a right-angled triangle with a base (horizontal side) that is one metre long and whose (perpendicular) height (which is also its vertical side) is also one metre long. We know from Pythagorus' Theorem that the length of the hypotenuse for this triangle is "the square root of two" metres long. But it can be shown that $\sqrt{2}$ is an irrational number!

Algebraic Rules for the Real Number System

- Consider any three real numbers: $a \in \mathbb{R}$, $b \in \mathbb{R}$, and $c \in \mathbb{R}$. Let $0 \in \mathbb{R}$ be the additive identity element and $1 \in \mathbb{R}$ be the multiplicative identity element.
- The real numbers obey the following algebraic rules.
  - Commutative Law for Addition: $(a + b) = (b + a)$.
  - Associative Law for Addition: $a + (b + c) = (a + b) + c$.
  - Existence of the Additive Identity (0): $a + 0 = a$.
  - Existence of an Additive Inverse $(-a)$: $a + (-a) = 0$.
  - Commutative Law for Multiplication: $a \times b = b \times a$.
  - Associative Law for Multiplication: $a \times (b \times c) = (a \times b) \times c$.
  - Existence of the Multiplicative Identity (1): $a \times 1 = a$.
  - Existence of a Multiplicative Inverse (a^{-1}): $a \times a^{-1} = 1$ for all $a \neq 0$.
  - Left Distributive Law: $a \times (b + c) = a \times b + a \times c$.
  - Right Distributive Law: $(a + b) \times c = a \times c + b \times c$.
- The real numbers also possess the Archimedean property. This simply says that for any $x \in \mathbb{R}$, there exists some $n \in \mathbb{N}$ such that $x < n$.

More Algebraic Rules for the Real Number System

- Consider any three real numbers: $a \in \mathbb{R}$, $b \in \mathbb{R}$, and $c \in \mathbb{R}$. Let $0 \in \mathbb{R}$ be the additive identity element and $1 \in \mathbb{R}$ be the multiplicative identity element.
- The real numbers also obey the following algebraic rules.
  - Multiplication by Zero: $a \times 0 = 0$.
  - Non-Existence of a Multiplicative Inverse for Zero: $0^{-1} = \frac{1}{0}$ is undefined.
  - Multiplication of a Positive by a Negative: $a \times (-b) = -(a \times b)$.
  - Multiplication of a Negative by a Negative: $(-a) \times (-b) = (a \times b)$.
  - Multiplication of Inequalities by (-1).
    - $a \leqslant b \Longleftrightarrow (-a) \geqslant (-b)$.
    - $a < b \Longleftrightarrow (-a) > (-b)$.
    - $a > b \Longleftrightarrow (-a) < (-b)$.
    - $a \geqslant b \Longleftrightarrow (-a) \leqslant (-b)$.
  - Order Reversal for Multiplicative Inverses (Fractions).
    - Assume that both $a \neq 0$ and $b \neq 0$ in the following two statements.
    - $a \leqslant b \Longleftrightarrow a^{-1} \geqslant b^{-1}$ (that is, $\frac{1}{a} \geqslant \frac{1}{b}$).
    - $a < b \Longleftrightarrow a^{-1} > b^{-1}$ (that is, $\frac{1}{a} > \frac{1}{b}$).

Power Sets

- The power set (2^X) of a set (X) is the set of all subsets of that set.
  - Note that the elements of a power set are sets themselves.
- If there are $n < \infty$ elements (that is, a finite number of elements) in the set X, then the number of subsets of X will be equal to 2^n.
  - As such, there will be 2^n elements in the set 2^X.
  - This might be the reason that the power set for some set X is often denoted by 2^X.
- Example: Suppose that $X = \{1, 2, 3\}$.
  - The power set for the set X in this example is

$$2^X = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

  - Note that there are three elements in the set A and eight elements in the power set for A. Note also that $2^3 = 8$.

Cartesian Products: Definition

- The Cartesian product of two sets is defined to be the set of all ordered pairs (or doublets) that contain one component from each of the two sets in the order that the sets were specified. This can be formally expressed as

$$X \times Y = \{(x, y) : x \in X, y \in Y\}.$$

- The Cartesian product of n sets is defined to be the set of all ordered n-tuples that contain one component from each of the n sets in the order that the sets were specified. This can be formally expressed as

$$\times_{i \in \{1,2,\cdots,n\}} X_i = \{(x_1, x_2, \cdots, x_n) : x_i \in X_i \; \forall i \in \{1, 2, \cdots, n\}\}.$$

- Note that the order of the sets matters here. Cartesian products generate sets of "ordered" n-tuples.

Cartesian Products: Examples Part 1

- The standard two-dimensional Euclidean coordinate plane from high school:
$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}\}.$$
- The n dimensional Euclidean coordinate plane:
$$\mathbb{R}^n = \times_{i \in \{1,2,\cdots,n\}} \mathbb{R} = \{(x_1, x_2, \cdots, x_n) : x_i \in \mathbb{R} \; \forall i \in \{1, 2, \cdots, n\}\}.$$
- A discrete-continuous example:
$$\mathbb{N} \times \mathbb{R} = \{(n, y) : n \in \mathbb{N}, y \in \mathbb{R}\}.$$
- A continuous-discrete example:
$$\mathbb{R} \times \mathbb{N} = \{(x, n) : x \in \mathbb{R}, n \in \mathbb{N}\}.$$

Cartesian Products: Examples Part 2

- If $X = \{1, 2, 3\}$, then the Cartesian product of X with itself is given by
$$X^2 = X \times X = \{(x, y) : x \in X, y \in X\}.$$
- This set can also be written out as an exhaustive list of possible cases as follows:

$$\begin{aligned} X^2 &= \{(1,1), (1,2), (1,3), (2,1), \\ &\quad (2,2), (2,3), (3,1), (3,2), (3,3)\}. \end{aligned}$$

Non-Negative and Positive Real Orthants

- The set of non-negative real numbers is given by

$$\mathbb{R}_{+} = \{x \in \mathbb{R} : x \geqslant 0\} = [0, \infty).$$

- The Euclidean L-dimensional non-negative orthant is

$$\mathbb{R}_{+}^{L} = \times_{l=1}^{L} \mathbb{R}_{+}.$$

- The set of positive real numbers is given by

$$\mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\} = (0, \infty).$$

- The Euclidean L-dimensional positive orthant is

$$\mathbb{R}_{++}^{L} = \times_{l=1}^{L} \mathbb{R}_{++}.$$

Non-Positive and Negative Real Orthants

- The set of non-positive real numbers is given by

$$\mathbb{R}_{-} = \{x \in \mathbb{R} : x \leqslant 0\} = (-\infty, 0].$$

- The Euclidean L-dimensional non-positive orthant is

$$\mathbb{R}^L_{-} = \times_{l=1}^{L} \mathbb{R}_{-}.$$

- The set of negative real numbers is given by

$$\mathbb{R}_{--} = \{x \in \mathbb{R} : x < 0\} = (-\infty, 0).$$

- The Euclidean L-dimensional negative orthant is

$$\mathbb{R}^L_{--} = \times_{l=1}^{L} \mathbb{R}_{--}.$$

Distances in Euclidean Spaces

- Suppose that $x, y \in \mathbb{R}^n$, where $n \in \mathbb{N}$. The coordinates of the two points will take the form $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$
- The Euclidean distance between these two points is given by the Euclidean distance metric:

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (y_i - x_i)^2}.$$

- Note that when $n = 1$, so that $\mathbb{R}^n = \mathbb{R}$, we have

$$d(x, y) = \sqrt{(y - x)^2} = |(y - x)|.$$

Properties of Distance Metrics

- The Euclidean distance metric is just one example of many possible distance metrics on sets of the form $\mathbb{R}^n$ for some $n \in \mathbb{N}$. It is the metric that is most commonly used for these sets in economics (and, I suspect, a number of other disciplines). But it is not the only possible distance metric.
- All distance metrics must satisfy a number of properties to ensure that they are valid measures of distance.
- To be precise, any distance metric $d(x, y)$ on a set S must satisfy each of the following properties.
  - (DM1: Non-Negativity): $d(x, y) \geqslant 0$ for all $x, y \in S$.
  - (DM2: Separability): $d(x, y) = 0$ if and only if $y = x$.
  - (DM3: Symmetry): $d(x, y) = d(y, x)$ for all $x, y \in S$.
  - (DM4: The Triangle Inequality): $d(x, y) \leqslant d(x, z) + d(z, y)$ for all $x, y, z \in S$.

Set Operations 1: Union and Intersection

- Suppose that U is some universal set, $X \subseteq U$ and $Y \subseteq U$.
- The union of X and Y, which is denoted by $X \cup Y$, is the set

$$X \cup Y = \{a \in U : a \in X \text{ or } a \in Y\}.$$

  - Note that the "or" in this definition is not exclusive. If the element a belongs to either X only, or Y only, or both X and Y, then $a \in X \cup Y$.
- The intersection of X and Y, which is denoted by $X \cap Y$, is the set

$$X \cap Y = \{a \in U : a \in X \text{ and } a \in Y\}.$$

  - If $X \cap Y = \varnothing$, then the sets X and Y are said to be disjoint.
- Illustrate set union and set intersection on the whiteboard using Venn diagrams for both the case of disjoint sets and the case of overlapping sets.

Set Operations 2: Union Example

U = {1,2,3}.

2^U = {∅,{1},{2},{3},{1,2},{1,3},{2,3},{1,2,3}}.

A ∈ 2^U are row set headings in the following table.

B ∈ 2^U are column set headings in the following table.

A∪B	**∅**	**{1}**	**{2}**	**{3}**	**{1,2}**	**{1,3}**	**{2,3}**	**{1,2,3}**
∅	∅	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
{1}	{1}	{1}	{1,2}	{1,3}	{1,2}	{1,3}	{1,2,3}	{1,2,3}
{2}	{2}	{1,2}	{2}	{2,3}	{1,2}	{1,2,3}	{2,3}	{1,2,3}
{3}	{3}	{1,3}	{2,3}	{3}	{1,2,3}	{1,3}	{2,3}	{1,2,3}
{1,2}	{1,2}	{1,2}	{1,2}	{1,2,3}	{1,2}	{1,2,3}	{1,2,3}	{1,2,3}
{1,3}	{1,3}	{1,3}	{1,2,3}	{1,3}	{1,2,3}	{1,3}	{1,2,3}	{1,2,3}
{2,3}	{2,3}	{1,2,3}	{2,3}	{2,3}	{1,2,3}	{1,2,3}	{2,3}	{1,2,3}
{1.2,3}	{1.2,3}	{1,2,3}	{1,2,3}	{1,2,3}	{1.2,3}	{1,2,3}	{1,2,3}	{1,2,3}

Set Operations 3: Intersection Example

$U = \{1,2,3\}$.

$2^U = \{\varnothing,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$.

$A \in 2^U$ are row set headings in the following table.

$B \in 2^U$ are column set headings in the following table.

A∩B	**∅**	**{1}**	**{2}**	**{3}**	**{1,2}**	**{1,3}**	**{2,3}**	**{1,2,3}**
∅	∅	∅	∅	∅	∅	∅	∅	∅
{1}	∅	{1}	∅	∅	{1}	{1}	∅	{1}
{2}	∅	∅	{2}	∅	{2}	∅	{2}	{2}
{3}	∅	∅	∅	{3}	∅	{3}	{3}	{3}
{1,2}	∅	{1}	{2}	∅	{1,2}	{1}	{2}	{1,2}
{1,3}	∅	{1}	∅	{3}	{1}	{1,3}	{3}	{1,3}
{2,3}	∅	∅	{2}	{3}	{2}	{3}	{2,3}	{2,3}
{1.2,3}	∅	{1}	{2}	{3}	{1.2}	{1,3}	{2,3}	{1,2,3}

Set Operations 4: Exclusion and Complementation

- Suppose that U is some universal set, $X \subseteq U$ and $Y \subseteq U$.
- The set "X excluding Y", which is denoted by $X \diagdown Y$, is the set

$$X \diagdown Y = \{a \in X : a \notin Y\}.$$

- Set complementation is a special case of set exclusion. The complement of the set X, which is denoted by X^C, is defined as $X^C = U \diagdown X$.
  - Note that $X \cup X^C = U$ and $X \cap X^C = \varnothing$.
- Illustrate set exclusion and set complementation on the whiteboard using Venn diagrams for both the case of disjoint sets and the case of overlapping sets.

Set Operations 5: Exclusion Example

$U = \{1,2,3\}$.

$2^U = \{\varnothing,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$.

$A \in 2^U$ are row set headings in the following table.

$B \in 2^U$ are column set headings in the following table.

A\B	**∅**	**{1}**	**{2}**	**{3}**	**{1,2}**	**{1,3}**	**{2,3}**	**{1,2,3}**
∅	∅	∅	∅	∅	∅	∅	∅	∅
{1}	{1}	∅	{1}	{1}	∅	∅	{1}	∅
{2}	{2}	{2}	∅	{2}	∅	{2}	∅	∅
{3}	{3}	{3}	{3}	∅	{3}	∅	∅	∅
{1,2}	{1,2}	{2}	{1}	{1,2}	∅	{2}	{1}	∅
{1,3}	{1,3}	{3}	{1,3}	{1}	{3}	∅	{1}	∅
{2,3}	{2,3}	{2,3}	{3}	{2}	{3}	{2}	∅	∅
{1,2,3}	{1,2,3}	{2,3}	{1,3}	{1,2}	{3}	{2}	{1}	∅

Set Operations 6: A Complementation Example

- Suppose that the universal set is $U = \{1, 2, 3\}$.
- The power set for the set U in this example is

$$2^U = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

- The elements of this power set are the subsets of the universal set.
- The complements for each of these subsets are $\varnothing^C = U$, $\{1\}^C = \{2, 3\}$, $\{2\}^C = \{1, 3\}$, $\{3\}^C = \{1, 2\}$, $\{1, 2\}^C = \{3\}$, $\{1, 3\}^C = \{2\}$, $\{2, 3\}^C = \{1\}$ and $U^C = \varnothing$.

Set Operations 7: The Symmetric Difference

- The symmetric difference of X and Y, which is denoted by $X \triangle Y$, is the set
$$X \triangle Y = (X \setminus Y) \cup (Y \setminus X) \,.$$
- It is interesting to compare the operations of union and symmetric difference.
    - They relate to alternative interpreations of the phrase "belongs to either X or Y".
    - The set $X \cup Y$ consists of all elements that are either in set X only, or in set Y only, or in both of these sets.
    - The set $X \triangle Y$ consists of all elements that are either in set X only, or in set Y only, but are not in both of these sets.
- Illustrate the symmetric difference on the whiteboard using Venn diagrams for both the case of disjoint sets and the case of overlapping sets.

Set Operations 8: Symmetric Difference Example

U = {1,2,3}.

2^U = {∅,{1},{2},{3},{1,2},{1,3},{2,3},{1,2,3}}.

A ∈ 2^U are row set headings in the following table.

B ∈ 2^U are column set headings in the following table.

AΔB	**∅**	**{1}**	**{2}**	**{3}**	**{1,2}**	**{1,3}**	**{2,3}**	**{1,2,3}**
∅	∅	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
{1}	{1}	∅	{1,2}	{1,3}	{2}	{3}	{1,2,3}	{2,3}
{2}	{2}	{1,2}	∅	{2,3}	{1}	{1,2,3}	{3}	{1,3}
{3}	{3}	{1,3}	{2,3}	∅	{1,2,3}	{1}	{2}	{1,2}
{1,2}	{1,2}	{2}	{1}	{1,2,3}	∅	{2,3}	{1,3}	{3}
{1,3}	{1,3}	{3}	{1,2,3}	{1}	{2,3}	∅	{1,2}	{2}
{2,3}	{2,3}	{1,2,3}	{3}	{2}	{1,3}	{1,2}	∅	{1}
{1,2,3}	{1,2,3}	{2,3}	{1,3}	{1,2}	{3}	{2}	{1}	∅

De Morgan's Law's: Statement

- Simple version.
  - Suppose that X is some set, $A \subseteq X$ and $B \subseteq X$.
  - According to De Morgan's laws, we have

$$X \diagdown (A \cup B) = (X \diagdown A) \cap (X \diagdown B)$$

    and

$$X \diagdown (A \cap B) = (X \diagdown A) \cup (X \diagdown B) .$$

- General version.
  - Let I be some index set. Note that while I is allowed to be finite, it does not have to be finite.
  - Let X be some set and suppose that $A_i \subseteq U$ for all $i \in I$.
  - According to De Morgan's laws, we have

$$X \diagdown (\cup_{i \in I} A_i) = \cap_{i \in I} (X \diagdown A_i)$$

    and

$$X \diagdown (\cap_{i \in I} A_i) = \cup_{i \in I} (X \diagdown A_i) .$$

De Morgan's Law's: Example

- Let $X = \{1, 2, 3\}$, $A = \{1, 2\} \subset X$ and $B = \{2, 3\} \subset X$.
- Note that $A \cup B = \{1, 2, 3\} = X$ and $A \cap B = \{2\}$.
- Note also that $X \setminus A = \{3\}$, $X \setminus B = \{1\}$, $X \setminus (A \cup B) = X \setminus X = \varnothing$ and $X \setminus (A \cap B) = X \setminus \{2\} = \{1, 3\}$.
- We have

$$(X \setminus A) \cap (X \setminus B) \quad = \quad \{3\} \cap \{1\} = \varnothing = X \setminus (A \cup B)$$

and

$$(X \setminus A) \cup (X \setminus B) \quad = \quad \{3\} \cup \{1\} = \{1, 3\} = X \setminus (A \cap B)\,.$$

Examples of Sets in Economics

- Some sets that you might encounter during your study of economics include:
  - Budget sets;
  - Weak preference sets;
  - Indifference sets;
  - Input requirement sets;
  - Isoquants;
  - Isocosts; and
  - Price simplices (simplexes).
- We will briefly consider some of these examples below.

Budget Sets

- Linear prices with an income endowment:

$$B(p_1, p_2, \cdots, p_L, y) = \left\{ (x_1, x_2, \cdots, x_L) \in \mathbb{R}^L_+ : \sum_{l=1}^{L} p_l x_l \leqslant y \right\}.$$

- Linear prices with a commodity bundle endowment:

$$B(p_1, p_2, \cdots, p_L, e_1, e_2, \cdots, e_L)$$

$$= \left\{ (x_1, x_2, \cdots, x_L) \in \mathbb{R}^L_+ : \sum_{l=1}^{L} p_l (x_l - e_l) \leqslant 0 \right\}.$$

- These are examples of "lower contour sets" for expenditure by an individual.
- Illustrate these sets on the whiteboard for the two commodity case.

Weak Preference Sets

- Reference commodity bundle version:

$$U^{+}(x_1, x_2, \cdots, x_L)$$

$$= \left\{ (y_1, y_2, \cdots, y_L) \in \mathbb{R}^L_+ : U(y_1, y_2, \cdots, y_L) \geqslant U\left(x_1, x_2, \cdots, x_L\right) \right\}.$$

- Reference utility level version:

$$U^{+}(k) = \left\{ (y_1, y_2, \cdots, y_L) \in \mathbb{R}^L_+ : U(y_1, y_2, \cdots, y_L) \geqslant k \right\}.$$

- These are examples of "(weak) upper contour sets" for the utility level attained by an individual.
- Illustrate these sets on the whiteboard for the two commodity case.

Indifference Sets

- Reference commodity bundle version:

$$U^0(x_1, x_2, \cdots, x_L)$$

$$= \left\{ (y_1, y_2, \cdots, y_L) \in \mathbb{R}^L_+ : U(y_1, y_2, \cdots, y_L) = U\left(x_1, x_2, \cdots, x_L\right) \right\}.$$

- Reference utility level version:

$$U^0(k) = \left\{ (y_1, y_2, \cdots, y_L) \in \mathbb{R}^L_+ : U(y_1, y_2, \cdots, y_L) = k \right\}.$$

- These are examples of “level sets” for the utility level attained by an individual.
- Illustrate these sets on the whiteboard for the two commodity case.

Input Requirement Sets

- Consider a single output, multiple input production technology that can be represented by a production function of the form

$$y = f(x_1, x_2, \cdots, x_n).$$

- An input requirement set for this production technology takes the form

$$y^{+}(k) = \left\{ (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n_+ : f(x_1, x_2, \cdots, x_n) \geqslant k \right\}.$$

- This is an example of a "(weak) upper contour set" for the output level attained by a producer.
- Illustrate this set on the whiteboard for the two input case.

Isoquants

- Consider a single output, multiple input production technology that can be represented by a production function of the form

$$y = f(x_1, x_2, \cdots, x_n).$$

- An isoquant for this production technology takes the form

$$y^0(k) = \left\{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n_+ : f(x_1, x_2, \cdots, x_n) = k\right\}.$$

- This is an example of a "level set" or the output level attained by a producer.
- Illustrate this set on the whiteboard for the two input case.

Isocosts

- An isocost depicts the locus of all input combinations that cost the producer the same amount of money to employ.
- Suppose that there are $n \in \mathbb{N}$ distinct production inputs (or factors of production, if you prefer). An isocost for this situation takes the form

$$C^0(k) = \left\{ (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n_+ : \sum_{i=1}^{n} w_i x_i = k \right\},$$

  where w_i is the per-unit price of input i.
- This is an example of a "level set" for the expenditure on inputs by a producer.
- Illustrate this set on the whiteboard for the two input case.

Price Simplex

- In some situations in economics, it is relative prices that matter, rather than the absolute level of each individual price. In such cases, some form of price normalisation can be employed.
- Common normalisations involve choosing either a particular commodity, or a particular basket of commodities, to be the numeraire. The expenditure on the the numeraire commodity, or numeraire basket of commodities, is then set equal to one.
- If the numeraire basket consists of one unit of each of the n final commodities in an economy, then the set of possible prices is given by

$$\triangle(p_1, p_2, \cdots, p_n) = \left\{ (p_1, p_2, \cdots, p_n) \in \mathbb{R}^n_+ : \sum_{i=1}^{n} p_i = 1 \right\}.$$

- This set is known as a price "simplex".
- Illustrate this set on the whiteboard for both the two commodity case and the three commodity case.