Equations, Inequalities, and Binary Relations

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Equations, Inequalities, and Binary Relations

- Consider two real-valued functions that have the same domain (X), $f : X \longrightarrow \mathbb{R}$ and $g : X \longrightarrow \mathbb{R}$.
- An equation takes the form $f(x) = g(x)$. The solution to this equation consists of the set of values of $x \in X$ that ensure the equation is satisfied.
- An inequality takes one of the forms $f(x) \leqslant g(x)$, or $f(x) < g(x)$, or $f(x) \geqslant g(x)$, or $f(x) > g(x)$. The solution to an inequality consists of the set of values of $x \in X$ that ensure the inequality is satisfied.
- Note that if we let $h(x) = f(x) - g(x)$, then we can rewrite these various equations and inequalities as $h(x) = 0$, $h(x) \leqslant 0$, $h(x) \geqslant 0$, $h(x) < 0$, and $h(x) > 0$.
- The mathematical concepts of $=$, $\leqslant$, $\geqslant$, $<$, and $>$ are examples of binary relations. Some other examples of binary relations that you have already seen include $\subseteq$ and $\subset$.

Polynomial Equations

- A polynomial equation is an equation of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0.$$

- Note that we can rewrite this equation as

$$x^n + \frac{a_{n-1}}{a_n} x^{n-1} + \cdots + \frac{a_1}{a_n} x + \frac{a_1}{a_n} = 0.$$

- This in turn can be written as

$$x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0 = 0,$$

where

$$b_k = \frac{a_k}{a_n} \text{ for all } k \in \{0, 1, 2, \cdots, n-1\}.$$

Solutions of Polynomial Equations Part 1

- According to the "fundamental theorem of algebra", every polynomial equation of degree $n > 0$ has exactly n solutions.
- There are two important clarifications for this result.
    - First, some of the solutions might be the same. (This is the case of "repeated roots".)
    - Second, some of the solutions might be complex numbers with a non-zero imaginary component.

Solutions of Polynomial Equations Part 2

- If the solutions to the polynomial equation

$$x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0 = 0$$

are denoted by $\{x_1, x_2, \cdots, x_n\}$, then we can write the polynomial function on the left-hand side of the equation as

$$\begin{aligned} f(x) &= x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0 \\ &= (x - x_1)(x - x_2)\cdots(x - x_n) \\ &= \prod_{k=1}^{n} (x - x_k). \end{aligned}$$

- The solutions to a polynomial equation are known as the roots of the corresponding polynomial function.

Some Examples

- $f_1(x) = x^2 + 10x + 25 = (x+5)(x+5)$. In this case $x_1 = x_2 = -5$.
- $f_2(x) = x^2 - 25 = (x+5)(x-5)$. In this case $x_1 = -5$ and $x_2 = 5$.
- $f_3(x) = x^2 + 10x + 9 = (x+9)(x+1)$. In this case $x_1 = -9$ and $x_2 = -1$.
- $f_4(x) = x^2 + 2x + 2 = (x - (-1+i))(x - (-1-i))$. In this case $x_1 = -1 + i$ and $x_2 = -1 - i$. (Note that $i = \sqrt{-1}$.)

Is there a solution formula?

- Consider the polynomial

$$f(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0.$$

- Is there a formula that will provide all of the roots of a polynomial of degree n? (We will assume that $n \in \mathbb{N}$.)
- The answer to this question is "sometimes". If $n \in \{1, 2, 3, 4\}$, then there is a formula (different for each value of n) that will provide you with all of the roots of the polynomial. If $n > 4$, then no such formula exists (if we require the solution to involve "radicals"). This result is known as "Abel's Theorem".
  - Note that this does not mean that there are no roots for such polynomials. We know from the fundamental theorem of algebra that such roots do indeed exist.

Linear Equations

- Consider the following polynomial of degree one:

$$f(x) = ax + b, \text{ where } a \neq 0.$$

  - This is a linear (or, more precisely, an affine) function.
  - Strictly speaking, linear functions are affine functions for which $b = 0$.

- The corresponding polynomial equation is

$$ax + b = 0.$$

  - This is a linear equation.

Solving Linear Equations

- We can solve the linear equation as follows:

$$\begin{aligned} ax + b &= 0 \\ \iff \quad ax &= -b \\ \iff \quad x &= \tfrac{-b}{a}. \end{aligned}$$

- The last step is allowed because $a \neq 0$.
- Thus we have a formula that provides the unique solution for any single linear equation.

A Linear Break-Even Problem Part 1

- Consider a firm that produces a single good. Suppose that the firm faces a demand curve for this good that is perfectly elastic at the price P. The firm has a fixed cost of F and a constant marginal cost of c per unit of the good that is produced. We will assume that $0 \leqslant c < P$.
- The firm's total revenue is given by

$$R(Q) = PQ.$$

- The firm's variable cost is given by

$$V(Q) = cQ.$$

- The firm's total cost is given by

$$\begin{aligned} C(Q) &= F + V(Q) \\ &= F + cQ. \end{aligned}$$

A Linear Break-Even Problem Part 2

- The firm's profit is given by

$$\begin{aligned}\pi(Q) &= R(Q) - C(Q) \\ &= PQ - (F + cQ) \\ &= PQ - F - cQ \\ &= (P - c)Q - F.\end{aligned}$$

A Linear Break-Even Problem Part 3

- The firm will break-even when

$$\begin{aligned} \pi(Q) &= 0 \\ \iff (P-c)Q - F &= 0 \\ \iff (P-c)Q &= F \\ \iff Q &= \frac{F}{(P-c)}. \end{aligned}$$

The Equation of a Straight Line

- A straight line in Euclidean two-space ($\mathbb{R}^2$) can be expressed as a set of ordered pairs of the form

$$\left\{(x, y) \in \mathbb{R}^2 : y = ax + b\right\}, \text{ where } a \neq 0.$$

- The equation of this straight line is given by

$$y = ax + b,$$

  where a is the gradient (or slope) of the line and b is the y–intercept.
- Illustrate this on the whiteboard.
- We will now look at two alternative methods for finding the equation of a straight line and the relationship between them.

Point-Slope Formula

- Suppose you are told that a straight line has a slope equal to a and that the point (x_1, y_1) lies on the line.
- You can find the equation of this straight-line by using the following formula:

$$y - y_1 = a\left(x - x_1\right).$$

- This formula is known as the "point-slope formula".
- We can rearrange the point-slope formula to obtain

$$y = ax + \left(y_1 - ax_1\right).$$

- Thus the $y-$intercept for this straight line is given by $b = (y_1 - ax_1)$.

Two-Point Formula

- Suppose you are told that two distinct points, (x_1, y_1) and (x_2, y_2), both lie on a straight line.
- You can find the equation of this straight-line by using the following formula:
$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}.$$
- This formula is known as the "two-point formula".
- We can rearrange the point-slope formula to obtain
$$y = \left(\frac{y_2 - y_1}{x_2 - x_1} \right) x + \left(y_1 - \left(\frac{y_2 - y_1}{x_2 - x_1} \right) x_1 \right).$$
- Thus the gradient (or slope) of this straight line is given by $a = \left(\frac{y_2 - y_1}{x_2 - x_1} \right)$ and the y–intercept for this straight line is given by $b = \left(y_1 - \left(\frac{y_2 - y_1}{x_2 - x_1} \right) x_1 \right)$.

The Slope of a Straight Line

- The slope of a straight line is given by

$$a = \frac{\text{rise}}{\text{run}},$$

  where "rise" is the vertical distance travelled in a northerly (not southerly) direction and "run" is the horizontal distance travelled in an easterly (not westerly) direction.
  - Travel in a southerly direction is measured as negative travel in a northerly direction.
  - Travel in a westerly direction is measured as negative travel in an easterly direction.
- The formula for the slope of a straight line that connects the two distinct points (x_1, y_1) and (x_2, y_2) is given by

$$a = \left(\frac{y_2 - y_1}{x_2 - x_1} \right).$$

- See the whiteboard.

These Formulae are Related

- Recall that the point-slope formula is

$$y - y_1 = a\left(x - x_1\right).$$

- If we substitute the slope formula into the point-slope formula, we obtain

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)\left(x - x_1\right).$$

- If we divide both sides of this equation by $\left(x - x_1\right)$, we obtain

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}.$$

- Recall that this is the two-point formula.

Playing with Straight Lines

- Consider the straight line that is given by the equation $y = ax + b$.
- The slope of this straight line is a.
- The $y-$intercept of this this straight line is b (or, more accurately, it is the point $(0, b)$). This is obtained by setting $x = 0$ in the above equation and solving for y.
- The $x-$intercept of this this straight line is $\frac{-b}{a}$ (or, more accurately, it is the point $\left(\frac{-b}{a}, 0\right)$). This is obtained by setting $y = 0$ in the above equation and solving for x.

Some Economic Applications of Straight Lines

- A budget line.
- A linear demand curve and a linear inverse demand (marginal benefit) curve.
- A linear supply curve and a linear inverse supply (marginal cost) curve.
- A linear total cost curve.
- A linear total revenue curve.
- A linear marginal revenue curve.
- A linear consumption function.
- A linear savings function.

A Budget Line

- A budget line consists of the set of commodity bundles that ensure that a consumer's expenditure exactly matches his or her income.
- The equation for a budget line in a two-commodity world is

$$P_1 Q_1 + P_2 Q_2 = Y.$$

- We can rearrange the budget-line equation to obtain

$$Q_2 = -\left(\frac{P_1}{P_2}\right) Q_1 + \left(\frac{Y}{P_2}\right).$$

- Thus we know that the slope of the budget line is equal to $-\left(\frac{P_1}{P_2}\right)$ and the Q_2–intercept is the point $\left(0, \frac{Y}{P_2}\right)$.
- We can also show that the Q_1–intercept is the point $\left(\frac{Y}{P_1}, 0\right)$.
- See the whiteboard.

A Linear Demand Curve Part 1

- Suppose that we have a linear demand curve:

$$Q = a - bP,$$

where $a > 0$ and $b > 0$.

- The corresponding inverse-demand curve can be found as follows:

$$\begin{aligned} & Q && = && a - bP \\ \iff & Q + bP && = && a \\ \iff & bP && = && a - Q \\ \iff & P && = && \frac{a}{b} - \left(\frac{1}{b}\right) Q. \end{aligned}$$

- Note the inverse demand curve is also linear when the demand curve itself is linear.

A Linear Demand Curve Part 2

- Suppose that we have a linear demand curve:

$$Q = a - bP,$$

  where $a > 0$ and $b > 0$.
- The slope of this demand curve is equal to $(-b)$ and the $Q-$intercept is the point $(0, a)$ in (P, Q)–space.
- It can be shown that the P–intercept is the point $\left(\frac{a}{b}, 0\right)$ in (P, Q)–space.
- Illustrate on the whiteboard using the standard mathematical convention for choice of axes.

A Linear Demand Curve Part 3

- The corresponding inverse demand curve is

$$P = \frac{a}{b} - \left(\frac{1}{b} \right) Q,$$

  where $a > 0$ and $b > 0$.
- The slope of this demand curve is equal to $\left\{ - \left(\frac{1}{b} \right) \right\}$ and the P–intercept is the point $\left(0, \frac{a}{b} \right)$ in (Q, P)–space.
- It can be shown that the P–intercept is the point $(a, 0)$ in (Q, P)–space.
- Illustrate on the whiteboard using the standard mathematical convention for choice of axes.

A Linear Demand Curve Part 4

- VERY IMPORTANT: Note the unusual "Marshallian convention" that is used when graphing demand curves in economics.
- Normally, we think that consumers take prices as given and choose quantities. This means that price is the independent variable and quantity is the dependent variable. As such, demand functions express quantity as a function of price.
- The standard mathematical graphing convention is to place the independent variable on the horizontal axis and the dependent variable on the vertical axis.
- However, in economics, we typically place price (the independent variable) on the vertical axis and quantity (the dependent variable) on the horizontal axis when graphing demand curves.

A Linear Demand Curve Part 5

- The Marshallian convention means that, while we talk about demand curves, we actually draw inverse demand curves.
- This means that you need to be careful when moving from the equation for a demand curve to the graph of that demand curve if you are going to follow the Marshallian convention when graphing demand curves.
- Illustrate on the whiteboard.

A Linear Demand Curve Part 6

- The "point" own-price elasticity of demand is given by the formula

$$\varepsilon_P^D(P) = \left(\frac{P}{Q^D(P)}\right)\left(slope\left(Q^D(P)\right)\right).$$

- Suppose that we have a linear demand curve of the form

$$Q = a - bP,$$

  where $a > 0$ and $b > 0$.

- The slope of this demand curve is equal to $(-b)$.
- As such, the point own-price elasticity of demand in this case is given by

$$\begin{aligned}\varepsilon_P^D(P) &= \left(\frac{P}{a - bP}\right)(-b) \\ &= \left(\frac{-bP}{a - bP}\right).\end{aligned}$$

A Linear Demand Curve Part 7

- Note that

$$\begin{aligned}
\left|\varepsilon_P^D(P)\right| &= 1 \\
\iff \left|\frac{-bP}{a-bP}\right| &= 1 \\
\iff \frac{bP}{a-bP} &= 1 \\
\iff bP &= a - bP \\
\iff 2bP &= a \\
\iff P &= \frac{a}{2b}.
\end{aligned}$$

A Linear Demand Curve Part 8

- More generally, we can show that, for the linear demand cuve above, we have

$$\left|\varepsilon_P^D(P)\right| \begin{cases} \in (0,1) & \text{if } P < \frac{a}{2b}, \\ = 1 & \text{if } P = \frac{a}{2b}, \\ \in (1,\infty) & \text{if } P > \frac{a}{2b}. \end{cases}$$

- Since the demand curve above is downward sloping, this means that

$$\varepsilon_P^D(P) \begin{cases} \in (-1,0) & \text{if } P < \frac{a}{2b}, \\ = -1 & \text{if } P = \frac{a}{2b}, \\ \in (-\infty,-1) & \text{if } P > \frac{a}{2b}. \end{cases}$$

- Illustrate this on the whiteboard.

Quadratic Equations

- Consider the following polynomial of degree two:

$$f\left(x\right) = ax^2 + bx + c, \text{ where } a \neq 0.$$

  - This is a quadratic function.
- The corresponding polynomial equation is

$$ax^2 + bx + c = 0.$$

  - This is a quadratic equation.

Solving Quadratic Equations

- Consider the quadratic equation given by

$$ax^2 + bx + c = 0,$$

where $a \neq 0$.

- The two solutions to this quadratic equation are given by the quadratic formula:

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

- In other words, we have

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The Discriminant

- The discriminant is the part of the quadratic formula that appears under the square root sign. We will denote the discriminant by D.
- In other words, we have

$$D = b^2 - 4ac.$$

- The discriminant is important because it tells us whether or not we have (i) two repeated real roots, or (ii) two distinct real roots, or (iii) no real roots.
    - In case (iii), there will be two distinct complex roots, both of which will have non-zero imaginary components.
- These three possibilities are the only ones that can occur for a quadratic equation.

The Nature of the Solutions

- There will be two repeated real roots when
$$b^2 - 4ac = 0.$$
- There will be two distinct real roots when
$$b^2 - 4ac > 0.$$
- There will be no real roots when
$$b^2 - 4ac < 0.$$
- In this unit, we will not cover complex numbers. As such, we will only be interested in real solutions to quadratic equations.

Deriving the Quadratic Formula Part 1

$$\begin{aligned}
ax^2 + bx + c &= 0 \\
\iff \quad x^2 + \tfrac{b}{a}x + \tfrac{c}{a} &= 0 \\
\iff \quad x^2 + \tfrac{b}{a}x &= \tfrac{-c}{a} \\
\iff \quad x^2 + \tfrac{2b}{2a}x &= \tfrac{-c}{a} \\
\iff \quad x^2 + 2\tfrac{b}{2a}x &= \tfrac{-c}{a} \\
\iff \quad x^2 + 2\tfrac{b}{2a}x + \left(\tfrac{b}{2a}\right)^2 &= \tfrac{-c}{a} + \left(\tfrac{b}{2a}\right)^2 \\
\iff \quad \left(x + \tfrac{b}{2a}\right)\left(x + \tfrac{b}{2a}\right) &= \tfrac{-4ac}{4a^2} + \tfrac{b^2}{4a^2}
\end{aligned}$$

Deriving the Quadratic Formula Part 2

$$
\begin{aligned}
&\iff & \left(x+\tfrac{b}{2a}\right)\left(x+\tfrac{b}{2a}\right) &= \frac{-4ac}{4a^2}+\frac{b^2}{4a^2} \\
&\iff & \left(x+\tfrac{b}{2a}\right)^2 &= \frac{b^2-4ac}{4a^2} \\
&\iff & x+\frac{b}{2a} &= \pm\sqrt{\frac{b^2-4ac}{4a^2}} \\
&\iff & x &= \frac{-b}{2a}\pm\sqrt{\frac{b^2-4ac}{4a^2}} \\
&\iff & x &= \frac{-b}{2a}\pm\frac{\sqrt{b^2-4a}}{\sqrt{4a^2}} \\
&\iff & x &= \frac{-b}{2a}\pm\frac{\sqrt{b^2-4a}}{2a} \\
&\iff & x &= \frac{-b\pm\sqrt{b^2-4a}}{2a}.
\end{aligned}
$$

The Equation of a Parabola

- A parabola in Euclidean two-space ($\mathbb{R}^2$) can be expressed as a set of ordered pairs of the form

$$\left\{(x, y) \in \mathbb{R}^2 : y = ax^2 + bx + c\right\},$$

  where $a \neq 0$.

- The equation of this parabola is given by

$$y = ax^2 + bx + c.$$

Playing with Parabolas Part 1

- The sign of the coefficient on x^2 (a) tells us whether the parabola is bucket shaped (opens up) or umbrella shaped (opens down).
- The magnitude (that is, absolute value) of a dominates the slope of the parabola for sufficiently large values of x. (You should reconsider this after we have covered univariate differential calculus.)
- If we fix the values of b and c, then a parabola will get steeper as the value taken by a gets larger.
- See the whiteboard for details.

Playing with Parabolas Part 2

- Note that c is the y-coordinate of the y-intercept of the parabola.
- If $a > 0$, then increases in the constant term (c) move the y-intercept of the parabola up the y-axis and the lowest point on the parabola up a vertical straight line (known as the axis of symmetry of the parabola), while decreases in c move the y-intercept of the parabola down the y-axis and the lowest point down the axis of symmetry.
- If $a < 0$, then increases in the constant term (c) move the y-intercept of the parabola up the y-axis and the highest point on the parabola up the axis of symmetry of the parabola, while increases in c move the y-intercept of the parabola down the y-axis and the highest point point down the axis of symmetry.
- Note that the axis of symmetry of a parabola will either be the y-axis itself, or a line that is parallel to the y-axis.
- See the whiteboard for details.

Some Economic Applications

- A quadratic marginal cost curve.
- A quadratic total cost curve.
- A quadratic average variable cost curve.
- A quadratic total revenue curve.
- A quadratic average revenue curve.
- A quadratic break-even problem.

A Quadratic Break-Even Problem Part 1

- This is Worked Example 4.9 in Bradley, T (2008), *Essential mathematics for economics and business (third edition)*, John Wiley and Sons, Great Britain (on pp. 162-163).
- Consider a monopolist that produces a single good. Suppose that the demand function for this good is $Q = 65 - 5P$. The monopolist has a fixed cost of \$30 and a constant marginal cost of \$2 per unit of the good that is produced.

A Quadratic Break-Even Problem Part 2

- The monopolist's variable cost function is

$$V(Q) = cQ.$$

- The monopolist's total cost function is

$$\begin{aligned} C(Q) &= F + V(Q) \\ &= F + cQ \\ &= 30 + 2Q. \end{aligned}$$

A Quadratic Break-Even Problem Part 3

- The inverse demand function is

$$P = 13 - \frac{1}{5}Q.$$

- The monopolist's total revenue function is given by

$$\begin{aligned} R(Q) &= P^D(Q)\,Q \\ &= \left(13 - \frac{1}{5}Q\right)Q \\ &= 13Q - \frac{1}{5}Q^2. \end{aligned}$$

A Quadratic Break-Even Problem Part 4

- The monopolist's profit function is

$$\begin{aligned}\pi(Q) &= R(Q) - C(Q) \\ &= 13Q - \frac{1}{5}Q^2 - (30 + 2Q) \\ &= 13Q - \frac{1}{5}Q^2 - 30 - 2Q \\ &= -\left(\frac{1}{5}\right)Q^2 + 11Q - 30.\end{aligned}$$

- Note that this monopolist's profit function is an "upside down" parabola.

A Quadratic Break-Even Problem Part 5

- The firm will break-even when it makes exactly zero profit. This requires that $\pi(Q) = 0$, which in turn requires that

$$-\left(\frac{1}{5}\right) Q^2 + 11Q - 30 = 0.$$

- Upon multiplying both sides of this quadratic equation by (-5), we obtain another quadratic equation of the form

$$Q^2 - 55Q + 150 = 0.$$

- This second quadratic equation is equivalent to the first quadratic equation in the sense that it will have identical solution values for the variable Q.

A Quadratic Break-Even Problem Part 6

- We know from the quadratic formula that

$$
\begin{aligned}
Q_1, Q_2 &= \frac{-(-55) \pm \sqrt{(-55)^2 - 4(1)(150)}}{2(1)} \\
&= \frac{55 \pm \sqrt{3,025 - 600}}{2} \\
&= \frac{55 \pm \sqrt{2,425}}{2} \\
&\approx \frac{55 \pm 49.24}{2} \\
&\approx \frac{104.24}{2}, \frac{5.76}{2} \\
&\approx 52.12,\ 2.88.
\end{aligned}
$$

A Quadratic Break-Even Problem Part 7

- Thus the monopolist will make:
  - negative profits when $Q \in (0, 2.88) \cup (52.12, \infty)$ (approximately),
  - zero profits when $Q \in \{2.88, 52.12\}$ (approximately), and
  - positive profits when $Q \in (2.88, 52.12)$ (approximately).
- The break-even points are $Q \approx 2.88$ units of output and $Q \approx 52.12$ units of output.
- See the whiteboard for an illustration.

The Binomial Theorem

- Suppose that $x, y \in \mathbb{R}$ and $n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.
- We have

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{(n-k)} y^k,$$

  where

$$\binom{n}{k} = C_{n,k} = \frac{n!}{k!\,(n-k)!}$$

  is known as a binomial coefficient.
- When $m \in \mathbb{N}$, the factorial function is defined as

$$\begin{aligned} m! &= (m)\,(m-1)\,(m-2)\cdots(2)\,(1) \\ &= \prod_{n=1}^{m} n. \end{aligned}$$

- When $m = 0$, we define $m! = 0! = 1$.

Pascal's Triangle Part 1

- If n is not too large, the binomial coefficients can be conveniently found by using Pascal's triangle.
- The first eleven rows of Pascal's triangle are provided in the following table. Can you see the relationship between the entries in Pascal's triangle?

n	**The binomial coefficients**										
0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	254	210	120	45	10	1

Pascal's Triangle Part 2

- Note that the boundary entries of Pascal's triangle are all ones. The reason for this is that they are either of the form $\binom{n}{0}$ or $\binom{n}{n}$ and we have
$$\binom{n}{0} = \binom{n}{n} = 1$$
for all $n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.
- The relationship that generates the interior entries of Pascal's triangle is provided by the following equation:
$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k},$$
where $n \in \mathbb{N}$ and, for each choice of n, we have $k \in \{1, 2, \cdots n\}$.

Definition of a Binary Relation

- A binary relation R on a set A compares two elements from the set A.
- Formally, a binary relation on the set A is a non-empty subset of the Cartesian Product $A \times A$.
  - In other words, $R \subseteq A \times A$ such that $R \neq \varnothing$.
- If the binary relation holds for the particular elements $a \in A$ and $b \in A$, this is denoted by either $(a, b) \in R$ or aRb.
  - If $(a, b) \in R$, then (a, b) is called a "labelled pair" for the binary relation R.

"Less Than Or Equal To" Relation Example

Table: Graphical Representation of the Relation "Less Than or Equal To" ($\leq$) on the set $\{1,2,3,4,5\}\times\{1,2,3,4,5\}$.

$x \leq y$		y				
		1	**2**	**3**	**4**	**5**
x	**1**	✓	✓	✓	✓	✓
	2		✓	✓	✓	✓
	3			✓	✓	✓
	4				✓	✓
	5					✓

“Strictly Less Than” Relation Example

Table: Graphical Representation of the Relation “Strictly Less Than” (<) on the set $\{1, 2, 3, 4, 5\}\times\{1, 2, 3, 4, 5\}$.

$x < y$		y				
		1	**2**	**3**	**4**	**5**
x	**1**		✓	✓	✓	✓
	2			✓	✓	✓
	3				✓	✓
	4					✓
	5					

"Equal To" Relation Example

Table: Graphical Representation of the Relation "Equal To" (=) on the set $\{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4, 5\}$.

$x = y$		y				
		1	**2**	**3**	**4**	**5**
x	**1**	✓				
	2		✓			
	3			✓		
	4				✓	
	5					✓

Proper Subset Relation Example

$U = \{1,2,3\}$.

$2^U = \{\varnothing,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$.

$A \in 2^U$ are row set headings in the following table.

$B \in 2^U$ are column set headings in the following table.

$A \subseteq B$	**$\varnothing$**	**{1}**	**{2}**	**{3}**	**{1,2}**	**{1,3}**	**{2,3}**	**{1,2,3}**
$\varnothing$	Y	Y	Y	Y	Y	Y	Y	Y
{1}		Y			Y	Y		Y
{2}			Y		Y		Y	Y
{3}				Y		Y	Y	Y
{1,2}					Y			Y
{1,3}						Y		Y
{2,3}							Y	Y
{1.2,3}								Y

Weak Subset Relation Example

$U = \{1,2,3\}$.

$2^U = \{\varnothing,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$.

$A \in 2^U$ are row set headings in the following table.

$B \in 2^U$ are column set headings in the following table.

A⊂B	**∅**	**{1}**	**{2}**	**{3}**	**{1,2}**	**{1,3}**	**{2,3}**	**{1,2,3}**
∅		Y	Y	Y	Y	Y	Y	Y
{1}					Y	Y		Y
{2}					Y		Y	Y
{3}						Y	Y	Y
{1,2}								Y
{1,3}								Y
{2,3}								Y
{1.2,3}								

Set Equality Relation Example

$U = \{1,2,3\}$.

$2^U = \{\varnothing,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$.

$A \in 2^U$ are row set headings in the following table.

$B \in 2^U$ are column set headings in the following table.

A=B	**∅**	**{1}**	**{2}**	**{3}**	**{1,2}**	**{1,3}**	**{2,3}**	**{1,2,3}**
∅	Y							
{1}		Y						
{2}			Y					
{3}				Y				
{1,2}					Y			
{1,3}						Y		
{2,3}							Y	
{1.2,3}								Y

Potential Properties of Binary Relations

- There are many potential properties that might be satisfied by some binary relations but not by others.
  - Some of these properties are inconsistent with others, so that no one binary relation can satisfy them all.
- Some of these properties include the following.
  - Weak Completeness: If $(a, b) \in A \times A$ such that $a \neq b$, then either $(a, b) \in R$ or $(b, a) \in R$ or both.
  - Strong Completeness: If $(a, b) \in A \times A$, then either $(a, b) \in R$ or $(b, a) \in R$ or both.
  - Reflexivity: If $(a, a) \in A \times A$, then $(a, a) \in R$.
  - Irreflexivity: If $(a, a) \in A \times A$, then $(a, a) \notin R$.
  - Symmetry: If $(a, b) \in R$ and $a \neq b$, then $(b, a) \in R$.
  - Anti-Symmetry: If $(a, b) \in R$ and $a \neq b$, then $(b, a) \notin R$.
  - Transitivity: If $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.

Equivalence Relations

- A binary relation $R \subseteq A \times A$ is called an "equivalence relation" on A if it is reflexive, symmetric and transitive.
- If $x \in A$ and R is an equivalence relation on A, then the "equivalence class for x" is defined as $E_x = \{a \in A : (a, x) \in R\}$.
- If E_x and E_y are both equivalence classes for an equivalence relation R defined on a set A, then either $E_x = E_y$ or $E_X \cap E_y = \varnothing$.
- We will denote the collection of all unique (that is "non-equal") equivalence classes in the set A that are generated by the equivalence relation R as A/R.
  - Note that this expression uses a forward-slash, not a back-slash. The latter denotes set exclusion, which is a very different concept.
- A partition of a set A is a collection of non-empty disjoint subsets of A whose union is all of A.
  - If R is an equivalence relation on A, then the collection A/R is a partition of A

Examples of Equivalence Relations

- The equality relation $(=)$ is an equivalence relation on $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$.
- The set equality relation $(=)$ is an equivalence relation on the power set 2^A, where $A = \{1, 2, 3\}$.
- The indifference relation $(\sim)$ for an individual with rational preferences is an equivalence relation on the individual's consumption set (X).
  - This is an important equivalence relation in economics.

Ordered Sets

- A binary relation $R \subseteq A \times A$ is said to be a partial ordering on the set A if it is reflexive, transitive and anti-symmetric.
  - If this is the case, then the set A is said to partially ordered by the binary relation R.
- A binary relation $R \subseteq A \times A$ is said to be an ordering on the set A if it is reflexive, transitive, anti-symmetric and weakly complete.
  - Equivalently, a binary relation $R \subseteq A \times A$ is said to be an ordering on the set A if it is strongly complete, transitive and anti-symmetric.
  - Note than an ordering is a partial ordering that is weakly complete.
  - If this is the case, then the set A is said to ordered by the binary relation R.
- An ordered set A is said to be "well ordered" if every non-empty subset of A has a unique smallest element.
  - That is, if $X \subseteq A$ such that $X \neq \varnothing$, then there exists $\hat{x} \in X$ such that $\hat{x} < x$ for all $x \in X$.

Examples of Ordered Sets

- Partially ordered sets.
  - The sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are partially ordered by the equality relation $(=)$.
  - The power set 2^A for any finite set A is partially ordered by the weak subset relation $(\subseteq)$.
- Ordered sets.
  - The sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are ordered by the weak inequality relation $\geqslant$.
  - he sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are not ordered by the equality relation $(=)$.
  - The power set 2^A for any finite set A is not ordered by the weak subset relation $(\subseteq)$.
- Well ordered sets.
  - The set $\mathbb{N}$ is well-ordered by the weak inequality relation $\geqslant$.
  - The set $\{x \in \mathbb{Q} : 0 < x < 1\}$ is not well-ordered because there is no smallest rational number in this set.
  - The set $\{x \in \mathbb{Q} : 0 \leqslant x \leqslant 1\}$ is not well-ordered because $\{x \in \mathbb{Q} : 0 < x < 1\} \subset \{x \in \mathbb{Q} : 0 \leqslant x \leqslant 1\}$ and there is no smallest rational number in the set $\{x \in \mathbb{Q} : 0 < x < 1\}$.

Some Useful Results about Ordered Sets

- Every non-empty subset of a well-ordered set is itself well-ordered.
- The Cartesian product of two well-ordered sets is itself a well ordered set.
- The Cartesian product of a finite number of well-ordered sets is itself a well-ordered set.
- (The Well-Ordering Theorem): Every set can be well-ordered. (This was proved by Zermelo in 1904.)
- (The Axiom of Choice): Given any set A, there is a "choice function" f such that $f(X) \in X$ for every non-empty $X \subseteq A$

Rational Relations

- A binary relation $R \subseteq A \times A$ is "rational" if it is weakly complete, reflexive and transitive.
  - Equivalently, a binary relation $R \subseteq A \times A$ is "rational" if it is strongly complete and transitive.
    - The reason for this is that a binary relation is strongly complete if it is both weakly complete and reflexive.
- Some examples.
  - The weak inequality "less than or equal to" ($\leqslant$) is rational on $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$.
  - The weak inequality "less than or equal to" ($\geqslant$) is rational on $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$.
  - The strict inequality "less than" ($\leqslant$) is not rational on $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ because it is not reflexive (and hence not strongly complete).
  - The strict inequality "greater than" ($\geqslant$) is not rational on $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ because it is not reflexive (and hence not strongly complete).
  - Equality ($=$) is not rational on $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ because it is not weakly complete (and hence not strongly complete).

Economic Application: Weak Preference Relations Part 1

- Suppose that $\succsim$ is a weak preference relation on a consumption set X.
  - Most of the time, we will want to assume that $\succsim$ is a rational binary relation on X.
  - The reason for this is that rational weak preference relations provide a theory of consistent choice behaviour.
- Strong completeness of $\succsim$ guarantees that the individual can always directly compare any two consumption bundles.
  - Strong completeness combines two other subsidiary properties that are known as weak completeness and reflexivity.
- Transitivity of $\succsim$ guarantees that the individual can indirectly directly compare any two consumption bundles.
- Strong completeness and transitivity together allow the individual to construct a ranking of all possible consumption bundles.

Economic Application: Weak Preference Relations Part 2

- Consider a weak preference relation $\succsim$ defined on a consumption set X (or, more precisely, defined on the Cartesian product of the consumption set with itself, $X \times X$).
- The weak preference relation is weakly complete if, for all $(x, y) \in X \times X$ such that $x \neq y$, it is the case that either (i) $x \succsim y$ but $y \not\succsim x$ or (ii) $y \succsim x$ but $x \not\succsim y$, or (iii) both $x \succsim y$ and $y \succsim x$.
- The weak preference relation is said to be reflexive if $x \succsim x$ for all $x \in X$. (Alternatively, we could write this as "the weak preference relation is said to be reflexive if $x \succsim x$ for all $(x, y) \in X \times X$.
- The weak preference relation is strongly complete if, for all $(x, y) \in X \times X$, it is the case that either (i) $x \succsim y$ but $y \not\succsim x$ or (ii) $y \succsim x$ but $x \not\succsim y$, or (iii) both $x \succsim y$ and $y \succsim x$.
- Note that strong completeness simply combines the properties of weak completeness and reflexivity.

Economic Application: Weak Preference Relations Part 3

- Consider a weak preference relation $\succsim$ defined on a consumption set X (or, more precisely, defined on the Cartesian product of the consumption set with itself, $X \times X$).
- The weak preference relation is transitive if, for all $(x, y, z) \in X \times X \times X$ such that both (i) $x \succsim y$ and (ii) $y \succsim z$, it is the case that $x \succsim z$.
- A weak preference relation that is both strongly complete and transitive is said to be rational.

Economic Application: Weak Preference Relations Part 4

- Sometimes, we will want to assume more than just rationality of the weak preference relation.
- Other properties that we will sometimes assume include the following.
  - Desirability Assumptions: Local non-satiation, monotone, strongly monotone.
    - These capture, to varying extents, the idea that "more is preferred to less".
  - Curvature Assumptions: Convexity, strict convexity.
    - These capture, to varying extents, the idea that "averages are preferred to extremes".
  - Regularity Assumptions: Continuity, Differentiability, Smoothness (n times continuously differentiable for some large $n \in \mathbb{N}$), Inada conditions.
    - These might be describes as the "picky technical details" assumptions.
- Further information about these properties can be found in Gravelle and Rees (1981), Kreps (1990), Mas-Colell et al (1995), Rubinstein (2006) and Varian (1992).

Economic Application: Weak Preference Relations Part 5

- Suppose that $\succsim$ is a rational weak preference relation on a consumption set X.
- Rational weak preference relations can sometimes be represented by utility function.
  - A function $U : X \longrightarrow \mathbb{R}$ is a utility function representation of $\succsim$ if $x \succsim y \iff f(x) \geqslant f(y)$.
  - A necessary, but not sufficient, condition for a utility function representation of $\succsim$ to exist is that $\succsim$ be rational.
    - One set of sufficient conditions for utility function existence is that X be countable (that is, either finite or countably infinite) and $\succsim$ be rational.
    - Another set of sufficient conditions for utility function existence, which will work even when X is uncountable, is that $\succsim$ be both rational and continuous. (This result, which was established by Gerard Debreu, actually guarantees the existence of a continuous utility function.)

Economic Application: Weak Preference Relations Part 6

- The weak preference relation ($\succsim$) can be used to construct the strict preference relation ($\succ$) and the indifference relation ($\sim$).
  - $x \succ y$ if and only if both (a) $x \succsim y$ and (b) $y \not\succsim x$.
  - $x \sim y$ if and only if both (a) $x \succsim y$ and (b) $y \succsim x$.
- If the weak preference relation is rational, then in general, neither the strict preference relation nor the indifference relation will be be rational.
  - The strict preference relation cannot be rational because it is not reflexive (and hence it is not strongly complete).
  - The indifference relation will, in general, not be rational because it will not be weakly complete (and hence not be strongly complete). The exception is the case where an individual is indifferent between every possible consumption bundle.

Economic Example: Lexicographic Preferences

- The weak preference relation for Lexicographic preferences on $\mathbb{R}^2_+$ is defined as follows.
  - Suppose that $x = (x_1, x_2) \in \mathbb{R}^2_+$ and $y = (y_1, y_2) \in \mathbb{R}^2_+$.
  - $x \succsim y$ if either:
    - (a) $x_1 > y_1$; or
    - (b) Both (i) $x_1 = y_1$ and (ii) $x_2 \geqslant y_2$.
- This definition can be extended to define lexicographic preferences on $\mathbb{R}^L_+$ for $L > 2$.
- Lexicographic basically means "dictionary, or alphabetic, ordering".
- The weak preference relation for lexicographic preferences is rational, strongly monotone and strictly convex. So in many respects, lexicographic preferences are very well behaved.
- However, the weak preference relation for lexicographic preferences on $\mathbb{R}^2_+$ is not continuous. Furthermore, it cannot be represented by a utility function.