EMET7001

Contents

Preliminary schedule

Week	Date	Topic	Notes
1	Feb 19	Introduction	
2	Feb 26		Tutorials start
3	Mar 4		
4	Mar 11		
5	Mar 18		
6	Mar 25		
Break			2 weeks
7	Apr 15		
8	Apr 22		
9	Apr 29		TBA
10	May 6		
11	May 13		
12	May 20		

ANU course pages

Course Wattle page Schedule, announcements, teaching team contacts, recordings, assignement, grades

Course overview and Class summary General course description in ANU Programs and Courses

Assessment

TBA

Skip to main content

TBA

Welcome

Course title: **“Mathematical Techniques for Economic Analysis”**

- Compulsory first math course in the *Master of Economics* program

Plan for this lecture

1. Organization
2. Administrative topics
3. Course content
4. Self-learning materials

Instructor

Fedor Iskhakov Professor of Economics at RSE

- Office: 1021 HW Arndt Building
- Email: fedor.iskhakov@anu.edu.au
- Web: fedor.iskh.me
- Contact hours: TBA

Timetable

Face-to-face:

- Lectures: TBA

Online:

- Echo-360 recordings on Wattle

Skip to main content

Face-to-face is strictly preferred

Course web pages

- Wattle Schedule, announcements, teaching team contacts, recordings, assignment, grades
- [Online notes](domain TBA) Lecture notes, slides, assignment tasks
- Lecture slides should appear online the previous day before the lecture
- Details on assessment including the exam instructions will appear on Wattle

Tutorials

- Enrollments open on *Wattle*

Tutorial questions

- posted on the course website
- not assessed, help you learn and prepare

Tutorials start on week 2

Tutors

TBA

Prerequisites

None specifically; Course overview and Class summary

Prior knowledge of maths, however, will be helpful:

- basic algebra
- basic calculus
- some idea of what a matrix is, etc.

This course will teach you the foundational mathematical concepts that you will need for other courses in your degree. For example, [Optimisation for Economics and Financial Economics (ECON6102)] is the compulsory second maths course in the *Master of Economics* program. ECON6102 is a general course on mathematical modeling for economics and financial economics, but optimization will be an important and recurring theme.

Assessment

TBA

Questions

1. Administrative questions: RSE admin

- **Bronwyn Cammack** Senior School Administrator
- Email: enquiries.rse@anu.edu.au
- "I can not register for the tutorial group"

2. Content related questions: please, refer to the tutors

- "I don't understand why this function is convex"

3. Other questions: to Fedor

- "I'm working hard but still can not keep up"
- "Can I please have extra assignment for more practice"

Attendance

- Please, **do not** use email for *instructional* questions\Instead make use of the office hours
- Attendance of tutorials is *very highly* recommended
  You will make your life much easier this way
- Attendance of lectures is *highly* recommended
  But not mandatory

- Cover exactly what you are required to know
- Code inserts are the exception, they are not assessable

In particular, you need to know:

- The definitions from the notes
- The facts from the notes
- How to apply facts and definitions

If a concept in not in the lecture notes, it is not assessable

Definitions and facts

The lectures notes/slides are full of definitions and facts.

> ⚠ **Definition**
>
> Functions $f : \mathbb{R} \to \mathbb{R}$ is called *continuous at x* if, for any sequence $\{x_n\}$ converging to x, we have $f(x_n) \to f(x)$.

Possible exam question: "Show that if functions f and g are continuous at x, so is $f + g$."

You should start the answer with the definition of continuity:

"Let $\{x_n\}$ be any sequence converging to x. We need to show that $f(x_n) + g(x_n) \to f(x) + g(x)$. To see this, note that …"

Facts

In the lecture notes/slides you will often see

> **Fact**
>
> The only N-dimensional subset of $\mathbb{R}^N$ is $\mathbb{R}^N$.

- theorem
- proposition
- lemma
- true statement

All well known results. You need to remember them, have some intuition for, and be able to apply.

Note on Assessments

Assessable = definitions and facts + a few simple steps of logic

Exams and tests will award:

- Hard work
- Deeper understanding of the concepts

In each question there will be an *easy* path to the solution

An introduction to economics

Sources and reading guide

- Alchian, AA and WR Allen (1972), *University economics: Elements of inquiry*, Wadsworth Publishing Company, USA: Chapters 1, 2, 3, 4, and 20 (pp. 2–52 and 384–404).
- Alchian, AA, and WR Allen (1983), *Exchange and production: Competition, coordination and control (third edition)*, Wadsworth Publishing Company, USA: Chapter 1 (pp. 1–12).
- Ausubel, LM, and RJ Deneckere (1993), “A generalized theorem of the maximum”, *Economic Theory 3(1)*, January, pp. 99–107.
- Case, KE, RC Fair, and SM Oster (2017), *Principles of Economics (Twelfth Edition) (Global Edition)*, Pearson Education, Italy: Chapters 1 and 2 (pp. 35–75).
- Frank, RH (2006), *Microeconomics and behavior (sixth edition)*, McGraw-Hill, USA: Chapter 1 (pp. 3–26).

Skip to main content

- Gravelle, H, and R Rees (2004), *Microeconomics (third edition)*, Pearson Education, United Kingdom: Chapter 1 (pp. 1–10).
- Hamermesh, DS (2006), *Economics is everywhere (second edition)*, McGraw-Hill-Irwin, USA: Chapter 1 (pp. 3–14).
- Heyne, P (2000), *A student's guide to economics*, Edited by JA Eglarz, Intercollegiate Studies Institute (ISI) Books, USA.
- Heyne, PL, PJ Boettke, and DL Prychitko (2014), *The economic way of thinking (thirteenth edition)*, The Pearson New International Edition, Pearson Education, USA: Chapters 1 and 2 (pp. 1–44).
- Hirshleifer, J, A Glazer, and D Hirshleifer (2005), *Price theory and applications: Decisions, markets, and information (seventh edition)*, Cambridge University Press, USA: Chapter 1 (pp. 2–26).
- Kunimoto, T (2010), *Lecture notes on mathematics for economists*, Unpublished, McGill University, Canada, 18 May, Page 6.
- Malinvaude, E. (1972), *Lectures on microeconomic theory*, Advanced Textbooks in Economics Volume 2, North Holland Publishing Company, Scotland, Translated by Mrs. A. Silvey: Page 1.
- Mankiw, NG (2003), *Macroeconomics (fifth edition)*, Worth Publishers, USA: Chapter 1 (pp. 2-14).
- Perloff, JM (2014), *Microeconomics with calculus (third edition) (global edition)*, Pearson Education Limited, USA: Chapter 1 (pp. 23–30).
- Robbins, LC (1984), *An essay on the nature and significance of economic science (third edition)*, With a foreword by WJ Baumol, New York University Press, Hong Kong. (The first edition of this book was published in 1932.)
- Vohra, RV (2005), *Advanced mathematical economics*, Routledge, The United Kingdom: The Preface only.
- Waud, RN, P Maxwell and J Bonnici (1989), *Macroeconomics (Australian edition)*, Harper and Row Publishers, Australia: Chapters 8-10 (pp. 169-249).

What is economics?

Economists try to explain social phenomena in terms of the behaviour of an individual who is confronted with scarcity and the interaction of that individual with other individuals who also face scarcity. This is perhaps best captured by Malinvaude's definition of economics:

Skip to main content

> *needs of men living in society: on the one hand, it is interested in the essential operations of production, distribution and consumption of goods, and on the other hand, in the institutions and activities whose object it is to facilitate these operations." (Italics in original.)*

– (From page one of Malinvaude, E. (1972), Lectures on microeconomic theory, Advanced Textbooks in Economics Volume 2, North Holland Publishing Company, Scotland, translated by Mrs. A. Silvey.)

> **Note**
>
> A definition of economics along these lines (that is, one that emphasises the importance of scarcity) can be traced back at least as far as Lord Lionel Robbins' justifiably famous "essay on the nature and significance of economic science". Chapter one of this essay contains a very nice discussion of the definition of economics and its history.
>
> - The first edition of this essay was published in 1932.
> - The third edition of this essay was published in 1984.

Core components of economics

The presence of scarcity

- This is the defining feature of economics.
- It is this feature that distinguishes economics from other social sciences.
- In the absence of scarcity, economics would either not exist or look very different.

The behaviour of an individual who is faced with scarcity

- This involves the individual making a choice from a set of available (or feasible) alternatives.
- The need to make a choice implies the existence of foregone alternatives and hence a cost.
- The opportunity cost of something is the value of the best of the foregone alternatives.
- Individual choice is often modelled using "constrained optimisation" techniques.

- Economic equilibrium (eg competitive equilibrium and Nash equilibrium).
- When does a system of equations have at least one solution?
- How do we find such a solution (if it exists)?
- How does any such solution vary with changes in the parameters (exogenous variables) of the economic system being studied? (This is known as comparative statics.)
- Use techniques from linear algebra and (for nonlinear cases) fixed point theorems.

What is scarcity?

Scarcity basically means that the availability of an item is limited relative to the desired uses of that item. Some important examples include:

- Scarcity of income or wealth (a budget constraint);
- Scarcity of time (a time constraint);
- Scarcity of productive resources and technological limitations (a production possibilities constraint).

A budget constraint

Suppose that there are two goods: Good one and good two. The price per unit of good one is p_1 and the price per unit of good two is p_2. The quantity of good one purchased by a consumer is q_1 and the quantity of good two purchased by a consumer is q_2. Note that the consumer's total expenditure is $p_1q_1 + p_2q_2$.

Suppose that the consumer's income is y. Ignoring the possibility of borrowing money from somewhere, the consumer cannot spend more than his or her income. This restriction is known as the budget constraint for the consumer. It can be represented mathematically by the inequality $p_1q_1 + p_2q_2 \leqslant y$. We typically also impose non-negativity constraints of the form $q_1 \geqslant 0$ and $q_2 \geqslant 0$.

Skip to main content

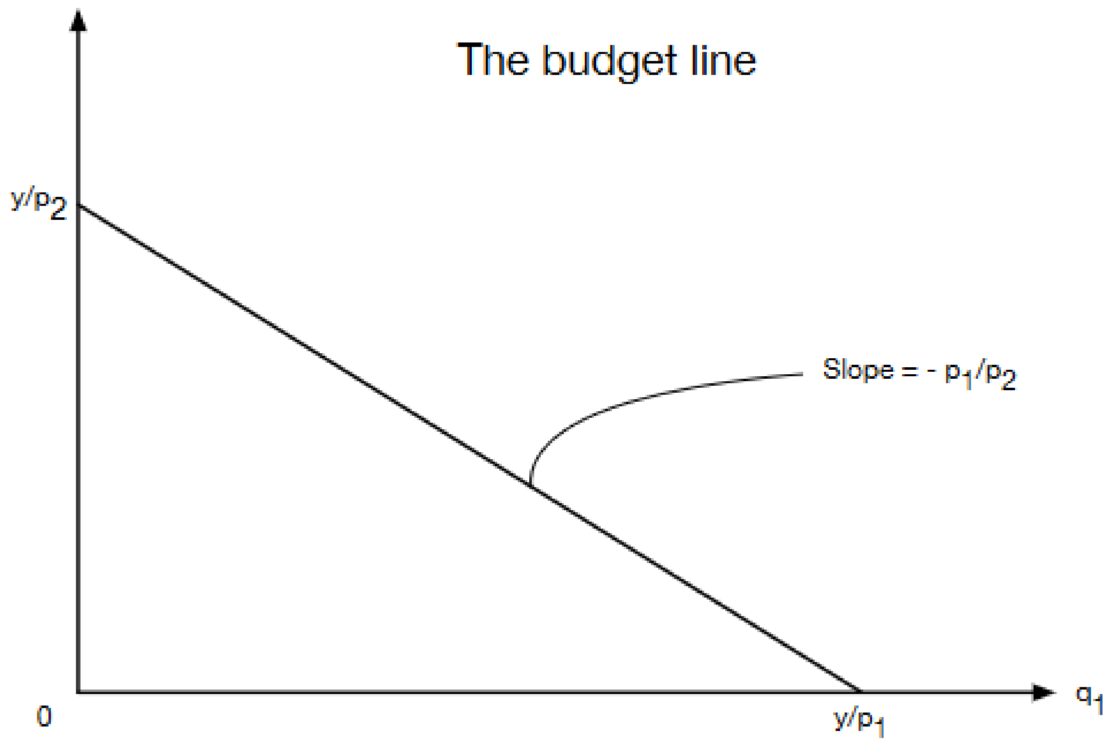
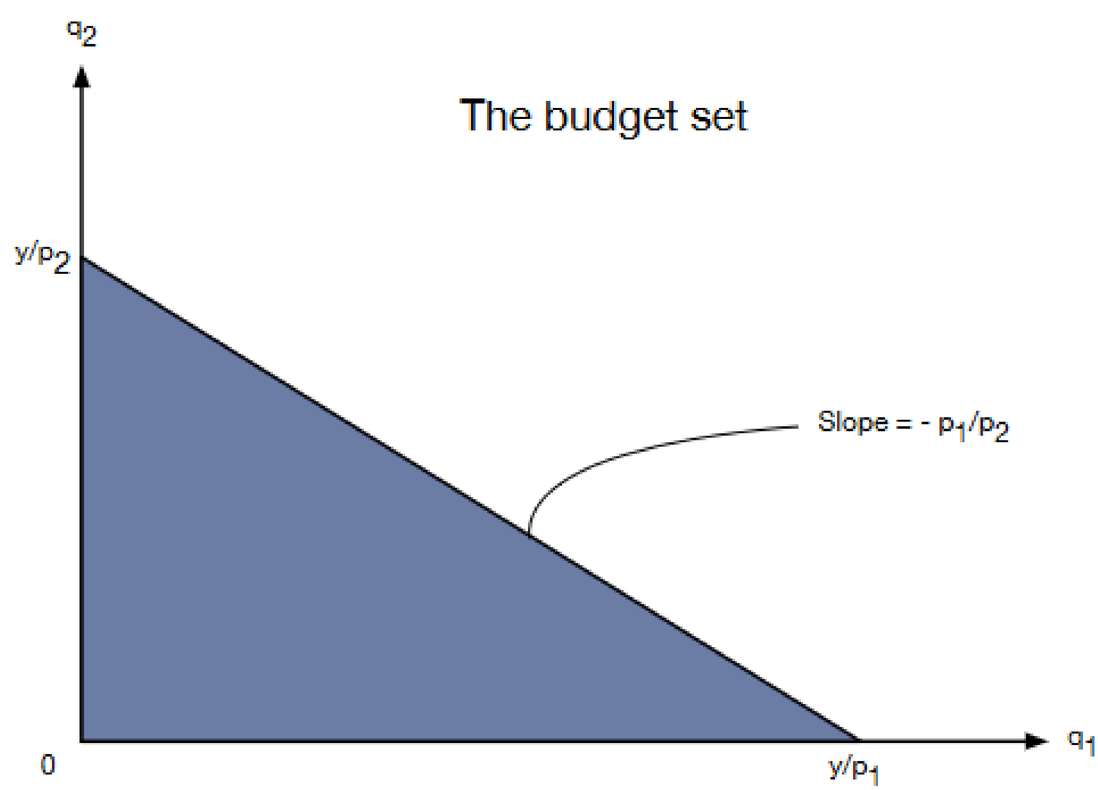


Fig. 1 The budget line



Skip to main content

A time constraint

Suppose that there are activities that can be undertaken: Labour (N) and Leisure (L). The amount of time spent working is t_N and the amount of time spent relaxing is t_L. Note that the total time that an individual spends on the various activities is $t_N + t_L$.

Suppose that the total time available to the individual is T. If we assume that every unit of time must be used for some activity, and that only one activity can be undertaken at any point in time, then the individual faces a time constraint of the form $t_N + t_L = T$. We typically also impose non-negativity constraints of the form $t_N \geqslant 0$ and $t_L \geqslant 0$.

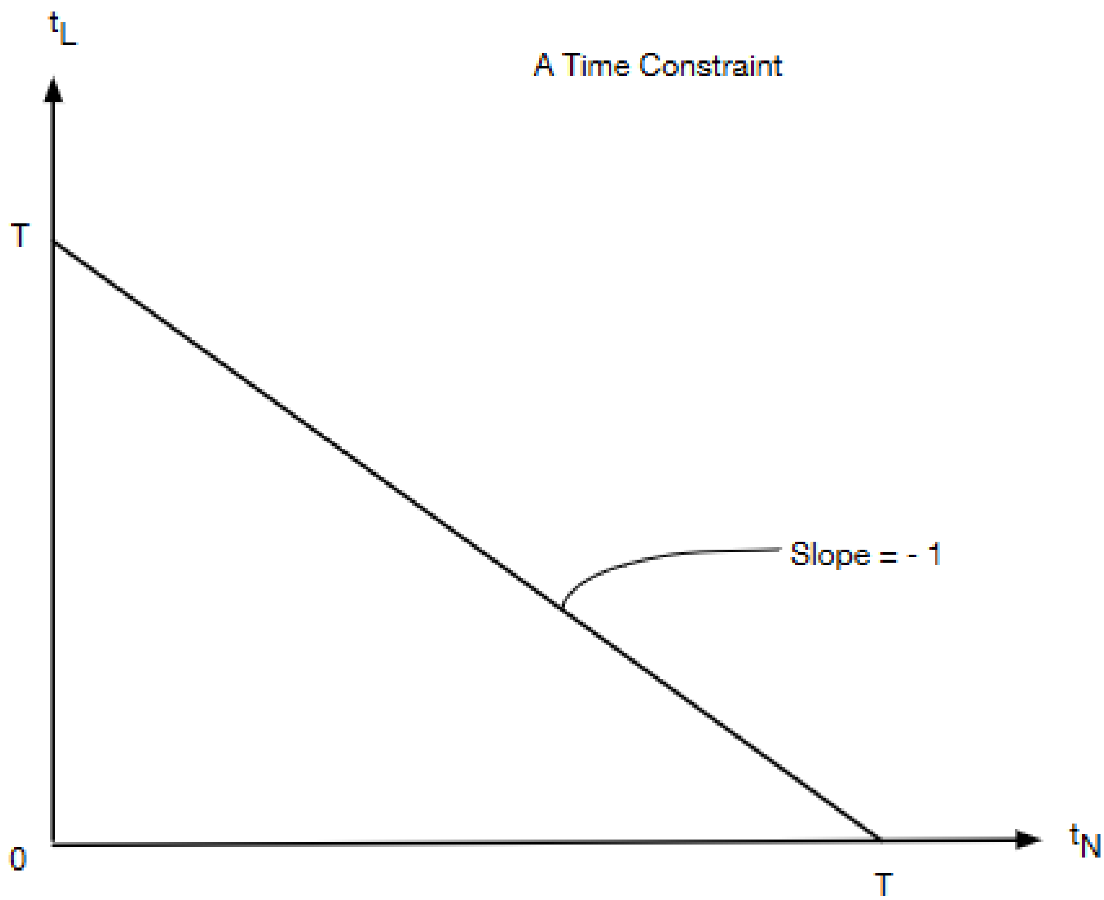


Fig. 3 A time constraint

Skip to main content

Suppose that there are only two goods that can be produced: Food and housing. Given the current state of technology and the current stock of productive resources, for any given feasible amount of one of the goods that is produced, there is a limit on the maximum amount of the other good that can also be produced.

If we graph the maximum amount of housing that can be produced for each feasible choice of an amount of food , then the resulting curve is known as the production possibilities frontier (PPF). This frontier can be represented by an equation of the form $F(Q_F, Q_H) = 0$, or alternatively by an equation of the form $Q_H = G(Q_F)$.

The PPF, along with the area under it (bounded by the axes because of non-negativity constraints on production), is known as the Production Possibilities Set. This is the set of all possible feasible combinations of food and housing that could be produced under the current technology and resource constraints.

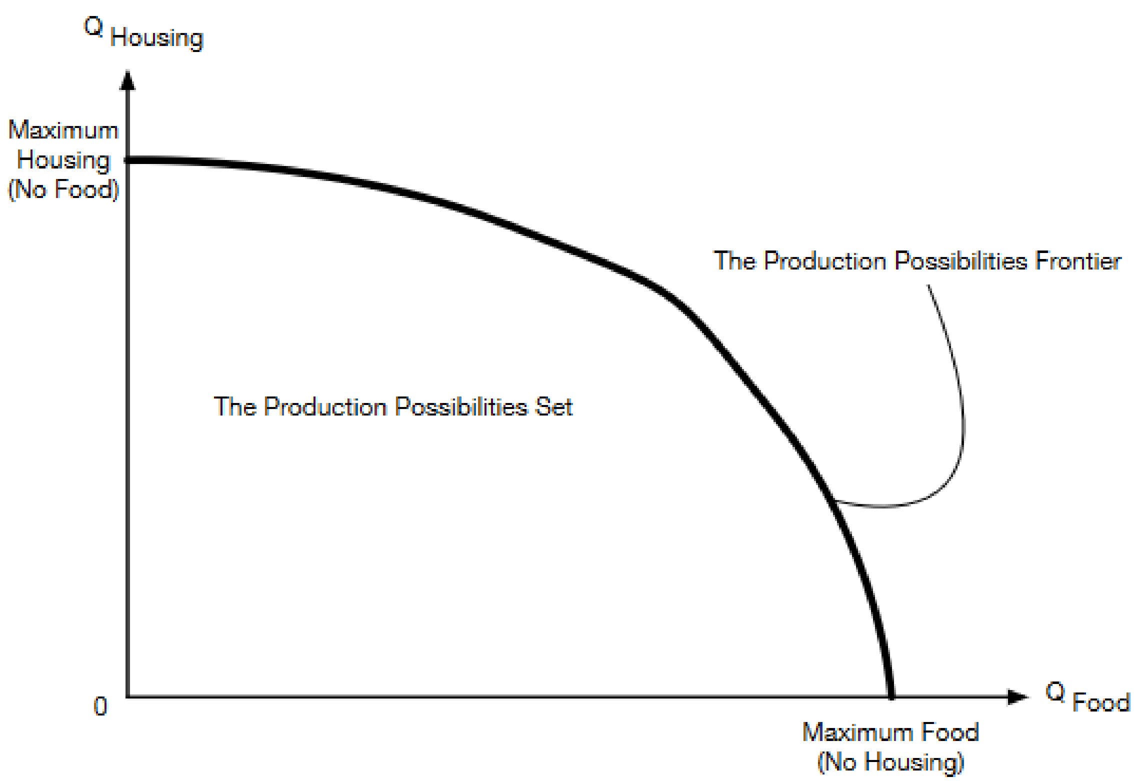


Fig. 4 A production possibilities constraint

Scarcity and choice

Skip to main content

not select in order to obtain the opportunity that you do select.

In the context of a budget constraint, if an individual purchases one more unit of good X , then he must give up (P_X/P_Y) units of good Y. In the context of a time constraint, if an individual takes an additional hour of leisure, then he must reduce the time that he works by one hour. This means that he will earn less income. As such, he will have to reduce his consumption by some amount. In the context of a production possibilities constraint, if a society that is currently on its PPF decides to produce one more unit of good Y , then it will need to produce fewer units of good X.

Choice and cost

When you select your most preferred option from the available alternatives, you are effectively giving up all of the other options that were available to you. The fact that you must give up alternative outcomes when you choose a particular option means that there is a cost associated with your choice.

Opportunity cost

As we have seen, scarcity requires choices and choices impose costs. In effect, the existence of costs is intimately related to the presence of scarcity. Costs arise because you must give up some option (or options) in order to obtain something that you want.

Definition

The value to you of the best (most preferred by you) of the alternative options that you give up is known as the **opportunity cost** of the option that you select.

Note

In economics, the word “cost” should always be taken to mean “opportunity cost”. Note that this might not be the same as the accounting cost or the monetary cost of something.

The importance of constrained optimisation

"bread and butter" skill for economists. It would difficult to make a living as an economist without some knowledge of constrained optimisation techniques.

According to Ausubel and Deneckere (1993, p. 99):

> *"Almost every economic problem involves the study of an agent's optimal choice as a function of certain parameters or state variables. For example, demand theory is concerned with an agent's optimal consumption as a function of prices and income, while capital theory studies the optimal investment rule as a function of the existing capital stock."*

Economic interactions between individuals

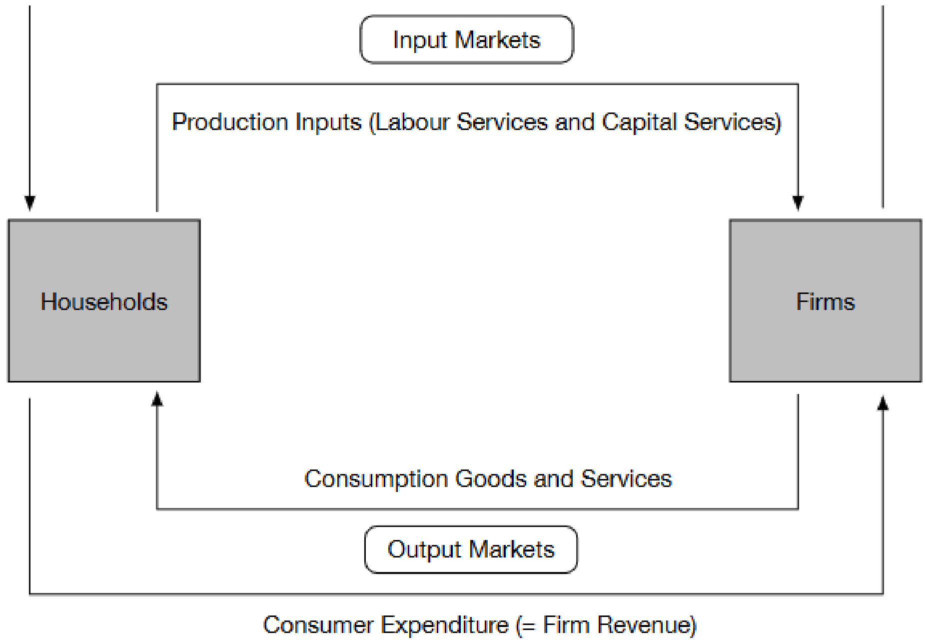
Most people are not hermits. In general, individuals interact with other individuals on at least some occasions. These interactions are an important component of the subject matter of economics. Economics is, after all, one of the "social" sciences.

The interactions that occur between individuals that are important for economics take many forms and occur in many places. They include interactions in output markets, input markets and various institutions. Some of these interactions can be illustrated in a stylised diagrammatic representation of an economy known as a "circular flow of economic activity" diagram. These "circular flow" diagrams can incorporate varying levels of detail.

The next four slides contain some examples for a closed economy. While not being a formal economic model itself, a sufficiently detailed circular flow diagram can provide a very useful guide to the construction of a formal general equilibrium economic model of an economy.

Circular flow diagram examples

Skip to main content



Source 1: Hirsheifer, J (1988), Price theory and applications (fourth edition), With the assistance of M Sproul, Prentice-Hall, USA: Figure 1.1 on page 17.
Source 2: Mankiw, NG (2003), Macroeconomics (fifth edition), Worth Publishers, USA: Figure 2.1 on page 17.
Source 3: Waud, RN, P Maxwell and J Bonnici (1986), Macroeconomics (Australian Edition), Harper and Row Publishers, Australia: Figure 5.1 on page 105.

Fig. 5 Circular flow example 1: households and firms

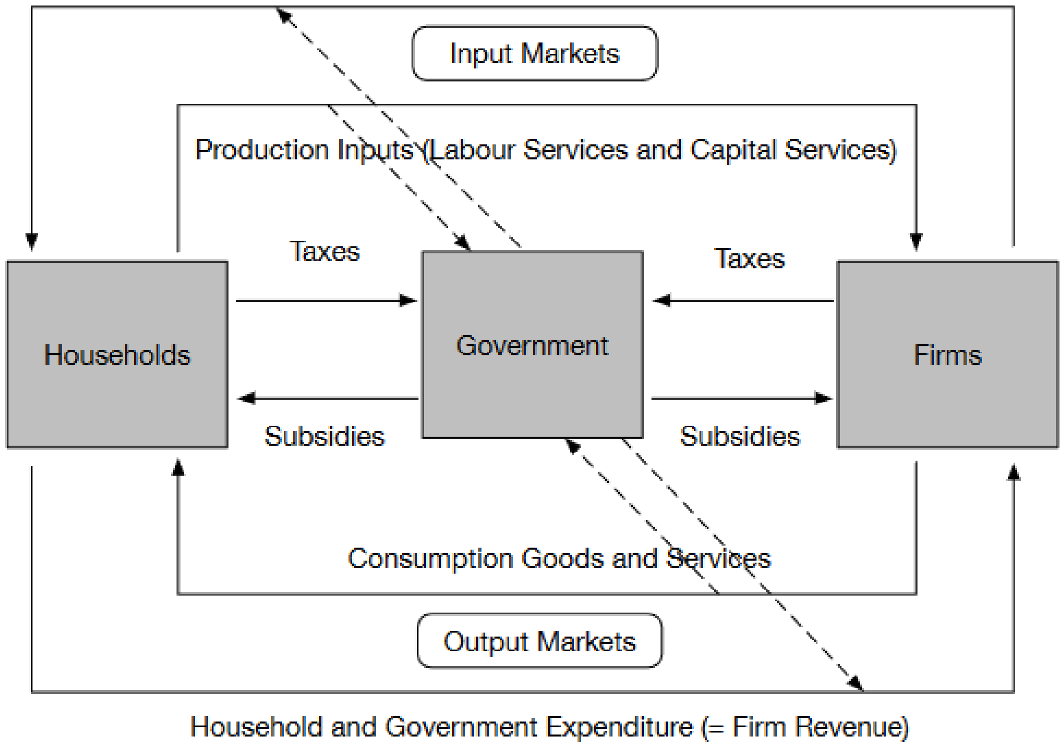
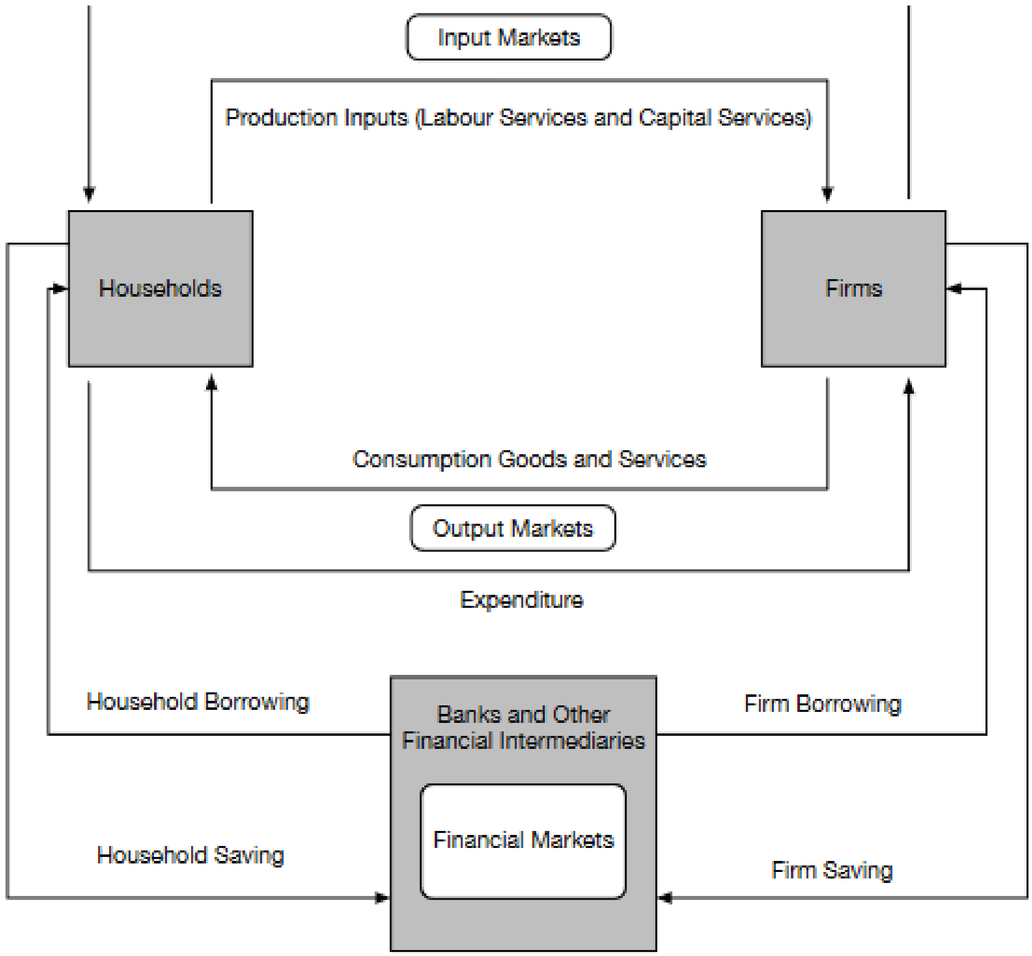


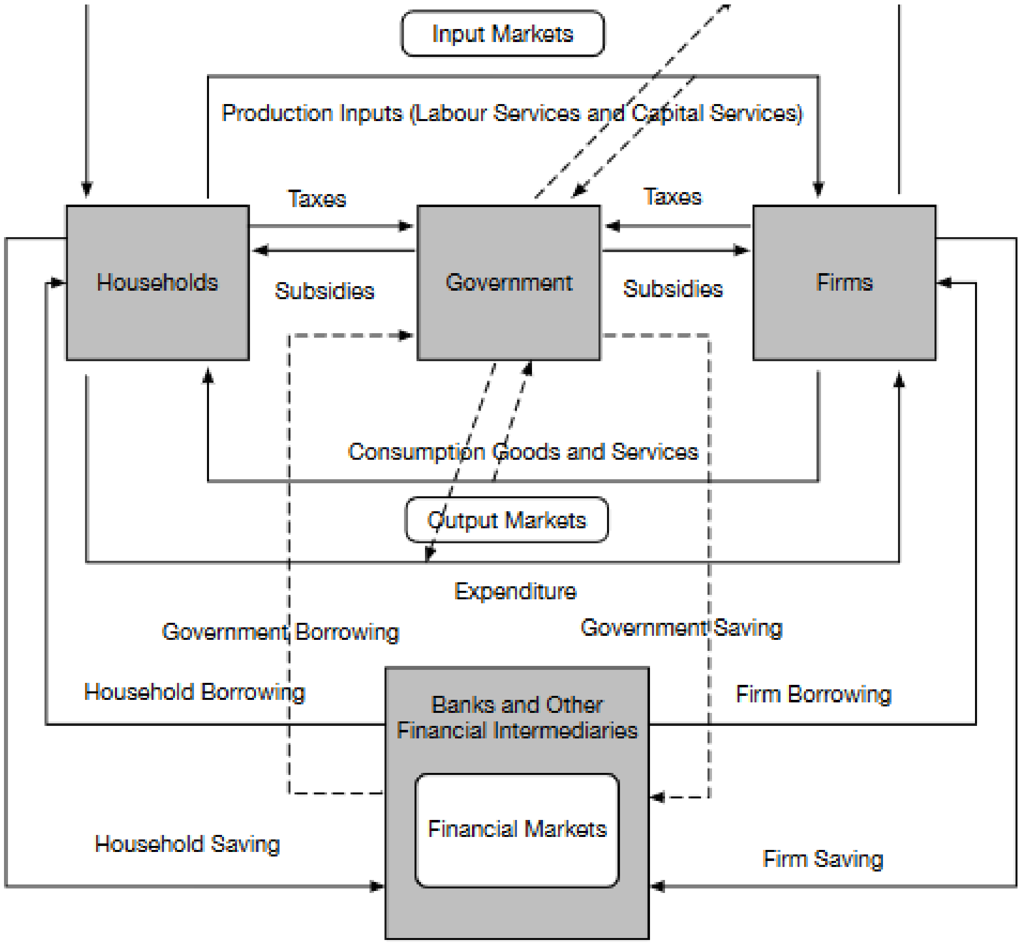
Fig. 6 Circular flow example 2: adding government

Skip to main content



Source: Waud, RN, P Maxwell and J Bonnici (1986), Macroeconomics (Australian Edition), Harper and Row Publishers, Australia: Figure 5.2 on page 107.

Fig. 7 Circular flow example 3: households, firms and financial markets



Source 1: Mankiw, NG (2003), Macroeconomics (Fifth Edition), Worth Publishers, USA: Figure 3.1 on page 43.

Source 2: Waud, RN, P Maxwell and J Bonnici (1986), Macroeconomics (Australian Edition), Harper and Row Publishers, Australia: Figure 5.3 on page 108.

Fig. 8 Circular flow example 4: households, firms, financial markets and government

The three main components of mathematical (micro-)economics

Takashi Kunimoto (Unpublished lecture notes on mathematical economics, 18 May 2010, page 6) notes that, according to Rakesh Vohra (2005, Preface), the three core technical components that underlie much of economic theory are as follows.

- Feasibility questions
- Optimality questions
- Equilibrium (fixed-point) questions

Skip to main content

(and various other related questions).

Optimisation in economics: motivational quotes

> "For since the fabric of the universe is most perfect and the work of a most wise Creator, nothing at all takes place in the universe in which some rule of maximum or minimum does not appear."

– Leonhard Euler in the introduction to *De Curvis Elasticis*, Additamentum 1 to *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive Solutio problematis isoperimetrici latissimo sensu accepti* (1744).

> "Constrained-maximization problems are mother's milk to the well-trained economist."

– From page 88 of Caves, Richard E (1980), "Industrial organisation, corporate strategy and structure", *The Journal of Economic Literature 18(1)*, March, pp. 64–92.

> "Almost every economic problem involves the study of an agent's optimal choice as a function of certain parameters or state variables. For example, demand theory is concerned with an agent's optimal consumption as a function of prices and income, while capital theory studies the optimal investment rule as a function of the existing capital stock."

– From page 99 of Ausubel, LM, and RJ Deneckere (1993), "A generalized theorem of the maximum", *Economic Theory 3(1)*, January, pp. 99–107.

> "The very name of my subject, economics, suggests economizing or maximising. · · · So at the very foundations of our subject maximization is involved."

– From page 249 of Samuelson, (1972), "Maximum principles in analytical economics", *The American Economic Review 62(3)*, June, pp. 249–262. This journal article is the text of Paul Samuelson's Nobel

Sets, numbers, coordinates, and distances

Reading guide

Introductory level references:

- Haeussler, EF Jr, and RS Paul (1987), *Introductory mathematical analysis for business, economics, and the life and social sciences (fifth edition)*, Prentice-Hall, USA:Chapter 0.2 (pp. 1–3).
- Sydsaeter, K, P Hammond, A Strom, and A Carvajal (2016), *Essential mathematics for economic analysis (fifth edition)*, Pearson Education, Italy: Chapters 1.1, 2.1, and 5.5 (pp. 1–12, 19–22, and 160–163).
- Shannon, J (1995), *Mathematics for business, economics and finance*, John Wiley and Sons, Brisbane: Chapter 1.2 and 1,3 (pp.2–11).

More advanced references:

- Banks, J, G Elton and J Strantzen (2009), *Topology and analysis: Unit text for MAT3TA (2009 and 2010 edition)*, Department of Mathematics and Statistics, La Trobe University, Bundoora, February.
- Corbae, D, MB Stinchcombe and J Zeman (2009), *An introduction to mathematical analysis for economic theory and econometrics*, Princeton University Press, USA: Chapters 1 and 2 (pp. 1-71).
- Halmos, PR (1960), *Naive set theory*, The University Series in Undergraduate Mathematics, D Van Nostrand Company, USA.
- Kolmogorov, AN and SV Fomin (1970), *Introductory real analysis*, Translated and Edited by RA Silverman, The 1975 Dover Edition (an unabridged, slightly corrected republication of the original 1970 Prentice-Hall edition), Dover Publications, USA: Chapter 1 (pp. 1-36).
- Simon, C, and L Blume (1994), *Mathematics for economists*, WW Norton and Co, USA: Appendix A1 (pp. 847-858).

Sets and elements

A **set** (X) is a collection of objects. A particular object within a set (x) is known as an **element** of that set. The idea that x is an element of X is written in mathematical notation as $x \in X$.

Skip to main content

A set that does not contain any elements is said to be empty. An **empty set** is denoted by either $\varnothing$ or $\{\}$.

There are two fundamental ways of defining a particular set:

- The first of these is to exhaustively list all of the elements of the set.
  - Example 1: $X = \{1, 2, 3\}$
  - Example 2: $Y = \{1, 2, 3, \cdots, 100\}$
  - Example 3: $\mathbb{N} = \{1, 2, 3, \cdots\}$
- The second of these is to specify one or more properties that characterise all of the elements in the set.
  - Example 4: $X = \{n \in \mathbb{N} : n < 4\}$
  - Example 5: $Y = \{n \in \mathbb{N} : n \leqslant 100\}$

Russell's Paradox

It would be nice if we could always associate some type of set with any particular property that we might consider. In other words, it would be nice if for any property $\mathbb{A}$, we could form a set $\{x \in X : \mathbb{A}(x) \text{ is true}\}$ that consisted of all of the elements that satisfy this property. Unfortunately, this is not the case.

This was established by Bertrand Russell. He did this by developing the following paradox.

Let $\mathbb{A}$ be the property "is a set and does not belong to itself". Suppose that A is the set of all sets that possess property $\mathbb{A}$. Is $A \in A$?

If $A \in A$, then it must be the case that A possesses property $\mathbb{A}$. This means that $A \notin A$. Contradiction! Thus it must be the case that $A \notin A$. But if A is a set and $A \notin A$, then it clearly possesses property $\mathbb{A}$. Thus $A \in A$. Contradiction. Thus it must be the case that $A \in A$.

We have a paradox. It cannot be the case that both $A \in A$ and $A \notin A$.

One possible resolution to Russell's paradox is to not allow mathematical objects like this particular A to be considered a set.

- The set of natural numbers: $\mathbb{N} = \{1, 2, 3, \cdots\}$
- The set of non-negative intergers: $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$
- The set of integers: $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$
- The set of rational numbers: $\mathbb{Q} = \{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}\}$;
    - The set of non-negative rational numbers: $Q_+ = \{x \in \mathbb{Q} : x \geqslant 0\}$
    - The set of positive rational numbers: $Q_{++} = \{x \in \mathbb{Q} : x > 0\}$;
- The set of real numbers: $\mathbb{R} = (-\infty, \infty)$
    - The set of non-negative real numbers: $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geqslant 0\}$
    - The set of positive real numbers: $\mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$
- The set of complex numbers:

$$\mathbb{C} = \{a + bi : a \in \mathbb{R},\ b \in \mathbb{R},\ i = \sqrt{-1}\}$$

Subsets

Consider two sets, X and Y.

Suppose that every element of X also belongs to Y. If this is the case, then we say that X is a **subset** of Y. This is written in mathematical notation as $X \subseteq Y$.

Suppose that in addition to every element of X also belonging to Y, there is at least one element of Y that does not belong to X. If this is the case, then we say that X is a **proper subset** of Y. This is written is mathematical notation as $X \subset Y$.

Sometimes $X \subset Y$ is used (rather loosely) to mean $X \subseteq Y$. If this meaning of the notation is employed, then $X \subsetneq Y$ would need to be used to indicate that X is a proper subset of Y.

Suppose that both every element of X also belongs to Y, and every element of Y also belongs to X. If this is the case, then we say that X is equal to Y. This is written in mathematical notation as $X = Y$.

Recall the common number sets from above. The following “nesting” relationship exists between these common sets of numbers:

$$\mathbb{N} \subset \mathbb{Z}_+ \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

The nesting of number sets

Note that $\mathbb{N} \subset \mathbb{Z}_+$ because $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

Note that $\mathbb{Z}_+ \subset \mathbb{Z}$ because $\mathbb{Z} = \mathbb{Z}_+ \cup \{\cdot\cdot\cdot, -3, -2, -1\}$.

Note that $\mathbb{Z} \subset \mathbb{Q}$ because any $m \in \mathbb{Z}$ can be written as $\frac{m}{1}$ and $1 \in \mathbb{N}$, but there are fractions that do not belong to Z (for example $\frac{1}{2} \notin \mathbb{Z}$).

Note that $\mathbb{Q} \subset \mathbb{R}$ because $\frac{m}{n} \in (-\infty, \infty)$ for all $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, but there are numbers on the real line that cannot be expressed as fractions (for example $\sqrt{2}$, π and e). Real numbers that cannot be expressed as fractions are known as “irrational numbers”.

Note that $\mathbb{R} \subset \mathbb{C}$ because

$$\mathbb{R} = \{a + bi : a \in \mathbb{R},\ b = 0,\ i = \sqrt{-1}\}$$

and $0 \in \mathbb{R}$, but $(a + bi) \notin \mathbb{R}$ if $b \neq 0$.

- Complex numbers in which a = 0 are known as (purely) “imaginary numbers”.

Intervals as subsets of the real line

Some (but not all) of the subsets of the real line take the form of an interval. There are four types of interval. Let $a \in \mathbb{R}, b \in \mathbb{R}$ and $a < b$. The four types of interval are as follows:

- $[a, b] = x \in \mathbb{R} : a \leqslant x \leqslant b$. (If $a > b$, then $[a, b] = \varnothing$.) (If $a = b$, then $[a, b] = \{a\} = \{b\}$.)
- $(a, b) = x \in \mathbb{R} : a < x < b$. (If $a > b$, then $[a, b] = \varnothing$.)
- $[a, b) = x \in \mathbb{R} : a \leqslant x < b$. (If $a > b$, then $[a, b] = \varnothing$.)

Skip to main content

The real number system

The system of real numbers is an algebraic structure known as a **complete ordered field**. Indeed, in a sense, it is the only complete ordered field in existence. Any other complete ordered field turns out to be "isomoporhic" to the real number system. (The term "isomorphic" is a mathematical version of "essentially the same").

The system of real numbers is formally denoted by $(\mathbb{R}, \mathbb{R}_{++}, +, \times)$, where $+$ and $\times$ are the familiar addition and multiplication operations for real numbers.

The set of real numbers $\mathbb{R}$ can be viewed as the completion of the the set of rational numbers $\mathbb{Q}$ because it involves filling in the "holes" that exist in the set of rational numbers. These holes take the form of irrational numbers like $\sqrt{2}$, π and e.

Why aren't the rationals enough?

Why aren't the rational numbers enough? What makes us think that they contain "holes"? This is a very good question. Especially when you realise that any numerical calculation on a computer will only generate a rational number.

A geometric argument for the existence of irrational numbers is perhaps the easiest way to convince yourself of their existence. Think about a right-angled triangle with a base (horizontal side) that is one metre long and whose (perpendicular) height (which is also its vertical side) is also one metre long. We know from Pythagoras' Theorem that the length of the hypotenuse for this triangle is "the square root of two" metres long. But it can be shown that $\sqrt{2}$ is an irrational number!

Algebraic rules for the real number system

Consider any three real numbers: $a \in \mathbb{R}$, $b \in \mathbb{R}$, and $c \in \mathbb{R}$. Let $0 \in \mathbb{R}$ be the additive identity element and $1 \in \mathbb{R}$ be the multiplicative identity element. The real numbers obey the following algebraic rules:

- Commutative Law for Addition: $(a + b) = (b + a)$.
- Associative Law for Addition: $a + (b + c) = (a + b) + c$.
- Existence of the Additive Identity (0): $a + 0 = a$.

Skip to main content

- Commutative Law for Multiplication: $a \times b = b \times a$.
- Associative Law for Multiplication: $a \times (b \times c) = (a \times b) \times c$.
- Existence of the Multiplicative Identity (1): $a \times 1 = a$.
- Existence of a Multiplicative Inverse (a^{-1}): $a \times a^{-1} = 1$ for all $a \neq 0$.
- Left Distributive Law: $a \times (b + c) = a \times b + a \times c$.
- Right Distributive Law: $(a + b) \times c = a \times c + b \times c$.

The real numbers also possess the Archimedean property. This simply says that for any $x \in \mathbb{R}$, there exists some $n \in \mathbb{N}$ such that $x < n$.

The real numbers also obey the following algebraic rules.

- Multiplication by Zero: $a \times 0 = 0$.
- Non-Existence of a Multiplicative Inverse for Zero: $0^{-1} = \frac{1}{0}$ is undefined.
- Multiplication of a Positive by a Negative: $a \times (-b) = -(a \times b)$.
- Multiplication of a Negative by a Negative: $(-a) \times (-b) = (a \times b)$.
- Multiplication of Inequalities by (-1).
    - $a \leqslant b \iff (-a) \geqslant (-b)$.
    - $a < b \iff (-a) > (-b)$.
    - $a > b \iff (-a) < (-b)$.
    - $a \geqslant b \iff (-a) \leqslant (-b)$.
- Order Reversal for Multiplicative Inverses (Fractions).
    - Assume that both $a \neq 0$ and $b \neq 0$ in the following two statements.
    - $a \leqslant b \iff a^{-1} > b^{-1}$ (that is, $\frac{1}{a} \geqslant \frac{1}{b}$).
    - $a < b \iff a^{-1} > b^{-1}$ (that is, $\frac{1}{a} > \frac{1}{b}$).

More on sets

Power sets

The **power set** (2^X) of a set (X) is the set of all subsets of that set. Note that the elements of a power set are sets themselves.

that the power set for some set X is often denoted by 2^X.

Example: Suppose that $X = \{1, 2, 3\}$. The power set for the set X in this example is

$$2^X = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

Note that there are three elements in the set X and eight elements in the power set for X. Note also that $2^3 = 8$.

Cartesian products

The **Cartesian product** of *two* sets is defined to be the set of all ordered *pairs* (or doublets) that contain one component from each of the two sets in the order that the sets were specified. This can be formally expressed as

$$X \times Y = \{(x, y) : x \in X,\ y \in Y\}$$

The Cartesian product of n sets is defined to be the set of all ordered n-tuples that contain one component from each of the n sets in the order that the sets were specified. This can be formally expressed as

$$\times_{i \in \{1,2,\cdots,n\}} X_i = \{(x_1, x_2, \cdots, x_n) : x_i \in X_i\ \forall i \in \{1, 2, \cdots, n\}\}$$

Note that the order of the sets matters here. Cartesian products generate sets of “ordered” n-tuples.

Examples

- The standard two-dimensional Euclidean coordinate plane from high school:

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) : x \in \mathbb{R},\ y \in \mathbb{R}\}$$

- The n-dimensional Euclidean coordinate plane:

$$\mathbb{R}^n = \times_{i \in \{1,2,\cdots,n\}} \mathbb{R} = \{(x_1, x_2, \cdots, x_n) : x_i \in \mathbb{R}\ \forall i \in \{1, 2, \cdots, n\}\}$$

$$\mathbb{N} \times \mathbb{R} = \{(n, y) : n \in \mathbb{N},\ y \in \mathbb{R}\}$$

- A continuous-discrete example:

$$\mathbb{R} \times \mathbb{N} = \{(x, n) : x \in \mathbb{R},\ n \in \mathbb{N}\}$$

- If X = {1, 2, 3}, then the Cartesian product of X with itself is given by

$$X^2 = X \times X = \{(x, y) : x \in X,\ y \in X\}$$

- This set can also be written out as an exhaustive list of possible cases as follows:

$$X^2 = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$$

Non-negative and positive real orthants

- The set of non-negative real numbers is given by

$$R_+ = \{x \in \mathbb{R} : x \geqslant 0\} = [0, \infty)$$

- The Euclidean L-dimensional non-negative orthant is

$$R_+^L = \times_{I=1}^{L} \mathbb{R}_+$$

- The set of positive real numbers is given by

$$R_{++} = \{x \in \mathbb{R} : x > 0\} = (0, \infty)$$

- The Euclidean L-dimensional positive orthant is

$$R_{++}^L = \times_{I=1}^{L} \mathbb{R}_{++}$$

- The set of non-positive real numbers is given by

$$R_{-} = \{x \in \mathbb{R} : x \leqslant 0\} = (-\infty, 0]$$

- The Euclidean L-dimensional non-positive orthant is

$$R_{-}^{L} = \times_{I=1}^{L} \mathbb{R}_{-}$$

- The set of negative real numbers is given by

$$R_{--} = \{x \in \mathbb{R} : x < 0\} = (-\infty, 0)$$

- The Euclidean L-dimensional negative orthant is

$$R_{--}^{L} = \times_{I=1}^{L} \mathbb{R}_{--}$$

Distances in Euclidean spaces

Suppose that $x, y \in \mathbb{R}^n$, where $n \in \mathbb{N}$. The coordinates of the two points will take the form $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$.

The Euclidean distance between these two points is given by the Euclidean distance metric:

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (y_i - x_i)^2}$$

Note that when $n = 1$, so that $\mathbb{R}^n = \mathbb{R}$, we have

$$d(x, y) = \sqrt{(y - x)^2} = |(y - x)|$$

Properties of distance metrics

suspect, a number of other disciplines). But it is not the only possible distance metric.

All distance metrics must satisfy a number of properties to ensure that they are valid measures of distance. To be precise, any distance metric $d(x, y)$ on a set S must satisfy each of the following properties.

- (DM1: Non-Negativity): $d(x, y) \geqslant 0$ for all $x, y \in S$.
- (DM2: Separability): $d(x, y) = 0$ if and only if $y = x$.
- (DM3: Symmetry): $d(x, y) = d(y, x)$ for all $x, y \in S$.
- (DM4: The Triangle Inequality): $d(x, y) \leqslant d(x, z) + d(z, y)$ for all $x, y, z \in S$.

Operations on sets

Union and intersection

Suppose that U is some universal set, $X \subseteq U$ and $Y \subseteq U$.

The **union** of X and Y, which is denoted by $X \cup Y$, is the set

$$X \cup Y = \{a \in U : a \in X \text{ or } a \in Y\}$$

Note that the "or" in this definition is not exclusive. If the element a belongs to either X only, or Y only, or both X and Y , then $a \in X \cup Y$.

The **intersection** of X and Y, which is denoted by $X \cap Y$, is the set

$$X \cap Y = \{a \in U : a \in X \text{ and } a \in Y\}$$

If $X \cap Y = \varnothing$, then the sets X and Y are said to be **disjoint**.

Illustrate set union and set intersection on the whiteboard using Venn diagrams for both the case of disjoint sets and the case of overlapping sets.

Examples

$2^{U} = \{\varnothing,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$.

$A \in 2^{U}$ are row set headings in the following table.

$B \in 2^{U}$ are column set headings in the following table.

A∪B	**∅**	**{1}**	**{2}**	**{3}**	**{1,2}**	**{1,3}**	**{2,3}**	**{1,2,3}**
∅	∅	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
{1}	{1}	{1}	{1,2}	{1,3}	{1,2}	{1,3}	{1,2,3}	{1,2,3}
{2}	{2}	{1,2}	{2}	{2,3}	{1,2}	{1,2,3}	{2,3}	{1,2,3}
{3}	{3}	{1,3}	{2,3}	{3}	{1,2,3}	{1,3}	{2,3}	{1,2,3}
{1,2}	{1,2}	{1,2}	{1,2}	{1,2,3}	{1,2}	{1,2,3}	{1,2,3}	{1,2,3}
{1,3}	{1,3}	{1,3}	{1,2,3}	{1,3}	{1,2,3}	{1,3}	{1,2,3}	{1,2,3}
{2,3}	{2,3}	{1,2,3}	{2,3}	{2,3}	{1,2,3}	{1,2,3}	{2,3}	{1,2,3}
{1,2,3}	{1.2,3}	{1,2,3}	{1,2,3}	{1,2,3}	{1.2,3}	{1,2,3}	{1,2,3}	{1,2,3}

Fig. 9 Examples: union

$U = \{1,2,3\}$.

$2^{U} = \{\varnothing,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$.

$A \in 2^{U}$ are row set headings in the following table.

$B \in 2^{U}$ are column set headings in the following table.

A∩B	**∅**	**{1}**	**{2}**	**{3}**	**{1,2}**	**{1,3}**	**{2,3}**	**{1,2,3}**
∅	∅	∅	∅	∅	∅	∅	∅	∅
{1}	∅	{1}	∅	∅	{1}	{1}	∅	{1}
{2}	∅	∅	{2}	∅	{2}	∅	{2}	{2}
{3}	∅	∅	∅	{3}	∅	{3}	{3}	{3}
{1,2}	∅	{1}	{2}	∅	{1,2}	{1}	{2}	{1,2}
{1,3}	∅	{1}	∅	{3}	{1}	{1,3}	{3}	{1,3}
{2,3}	∅	∅	{2}	{3}	{2}	{3}	{2,3}	{2,3}
{1,2,3}	∅	{1}	{2}	{3}	{1.2}	{1,3}	{2,3}	{1,2,3}

Fig. 10 Examples: intersection

Exclusion and complementation

The **set difference** “X excluding Y”, which is denoted by $X \setminus Y$, is the set

$$X \setminus Y = \{a \in X : a \notin Y\}$$

Set complementation is a special case of set exclusion. The **complement** of the set X, which is denoted by X^C , is defined as $X^C = U \setminus X$.

- Note that $X \cup X^C = U$ and $X \cap X^C = \varnothing$.

Illustrate set exclusion and set complementation on the whiteboard using Venn diagrams for both the case of disjoint sets and the case of overlapping sets.

Examples

U = {1,2,3}.

2^U = {∅,{1},{2},{3},{1,2},{1,3},{2,3},{1,2,3}}.

A ∈ 2^U are row set headings in the following table.

B ∈ 2^U are column set headings in the following table.

A\B	**∅**	**{1}**	**{2}**	**{3}**	**{1,2}**	**{1,3}**	**{2,3}**	**{1,2,3}**
∅	∅	∅	∅	∅	∅	∅	∅	∅
{1}	{1}	∅	{1}	{1}	∅	∅	{1}	∅
{2}	{2}	{2}	∅	{2}	∅	{2}	∅	∅
{3}	{3}	{3}	{3}	∅	{3}	∅	∅	∅
{1,2}	{1,2}	{2}	{1}	{1,2}	∅	{2}	{1}	∅
{1,3}	{1,3}	{3}	{1,3}	{1}	{3}	∅	{1}	∅
{2,3}	{2,3}	{2,3}	{3}	{2}	{3}	{2}	∅	∅
{1,2,3}	{1,2,3}	{2,3}	{1,3}	{1,2}	{3}	{2}	{1}	∅

Fig. 11 Examples: exclusion/set difference

Suppose that the universal set is $U = 1, 2, 3$. The power set for the set U in this example is

$$2^U = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

The elements of this power set are the subsets of the universal set. The complements for each of these subsets are:

- $\varnothing^C = U$
- $\{1\}^C = \{2, 3\}$
- $\{2\}^C = \{1, 3\}$
- $\{3\}^C = \{1, 2\}$
- $\{1, 2\}^C = \{3\}$
- $\{1, 3\}^C = \{2\}$
- $\{2, 3\}^C = \{1\}$
- $U^C = \varnothing$

Symmetric difference

The **symmetric difference** of X and Y, which is denoted by $X \triangle Y$, is the set

$$X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$$

It is interesting to compare the operations of union and symmetric difference. They relate to alternative interpretions of the phrase “belongs to either X or Y”.

- The set $X \cup Y$ consists of all elements that are either in set X only, or in set Y only, or in both of these sets.
- The set $X \triangle Y$ consists of all elements that are either in set X only, or in set Y only, but are not in both of these sets.

Illustrate the symmetric difference on the whiteboard using Venn diagrams for both the case of disjoint sets and the case of overlapping sets.

Example

$2^U = \{\varnothing,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$.

$A \in 2^U$ are row set headings in the following table.

$B \in 2^U$ are column set headings in the following table.

AΔB	**∅**	**{1}**	**{2}**	**{3}**	**{1,2}**	**{1,3}**	**{2,3}**	**{1,2,3}**
∅	∅	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
{1}	{1}	∅	{1,2}	{1,3}	{2}	{3}	{1,2,3}	{2,3}
{2}	{2}	{1,2}	∅	{2,3}	{1}	{1,2,3}	{3}	{1,3}
{3}	{3}	{1,3}	{2,3}	∅	{1,2,3}	{1}	{2}	{1,2}
{1,2}	{1,2}	{2}	{1}	{1,2,3}	∅	{2,3}	{1,3}	{3}
{1,3}	{1,3}	{3}	{1,2,3}	{1}	{2,3}	∅	{1,2}	{2}
{2,3}	{2,3}	{1,2,3}	{3}	{2}	{1,3}	{1,2}	∅	{1}
{1,2,3}	{1,2,3}	{2,3}	{1,3}	{1,2}	{3}	{2}	{1}	∅

Fig. 12 Examples: symmetric difference

De Morgan’s laws

Simple version:

Suppose that X is some set, $A \subseteq X$ and $B \subseteq X$. According to De Morgan’s laws, we have

$$X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$$

and

$$X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$$

General version:

Let I be some index set. Note that while I is allowed to be finite, it does not have to be finite. Let X be some set and suppose that $A_i \subseteq U$ for all $i \in I$. According to De Morgan’s laws, we have

$$X \setminus (\cup_{i \in I} A_i) = \cap_{i \in I} (X \setminus A_i)$$

and

Skip to main content

Example

Let $X = \{1, 2, 3\}$, $A = \{1, 2\} \subset X$ and $B = \{2, 3\} \subset X$. Note that $A \cup B = \{1, 2, 3\} = X$ and $A \cap B = \{2\}$. Note also that

- $X \setminus A = \{3\}, X \setminus B = \{1\}$,
- $X \setminus (A \cup B) = X \setminus X = \varnothing$ and
- $X \setminus (A \cap B) = X \setminus \{2\} = \{1, 3\}$.

We have

$$(X \setminus A) \cap (X \setminus B) = \{3\} \cap \{1\} = \varnothing = X \setminus (A \cup B)$$

and

$$(X \setminus A) \cup (X \setminus B) = \{3\} \cup \{1\} = \{1, 3\} = X \setminus (A \cap B)$$

Examples of sets in economics

Some sets that you might encounter during your study of economics include:

- Budget sets
- Weak preference sets
- Indifference sets
- Input requirement sets
- Isoquants
- Isocosts
- Price simplices (simplexes)

We will briefly consider some of these examples below.

- Linear prices with an income endowment:

$$B(p_1, p_2, \cdots, p_L, y) = \left\{ (x_1, x_2, \cdots, x_L) \in \mathbb{R}_+^L : \sum_{I=1}^{L} p_I x_I \leqslant y \right\}$$

- Linear prices with a commodity bundle endowment:

$$B(p_1, p_2, \cdots, p_L, e_1, e_2, \cdots, e_L) = \left\{ (x_1, x_2, \cdots, x_L) \in \mathbb{R}_+^L : \sum_{I=1}^{L} p_I (x_I - e_I) \leqslant 0 \right\}$$

These are examples of "lower contour sets" for expenditure by an individual.

Illustrate these sets on the whiteboard for the two commodity case.

Weak preference sets

- Reference commodity bundle version:

$$U^+(x_1, x_2, \cdots, x_L) = \{(y_1, y_2, \cdots, y_L) \in \mathbb{R}_+^L : U(y_1, y_2, \cdots, y_L) \geqslant U(x_1, x_2, \cdots, x_L)\}$$

- Reference utility level version:

$$U^+(k) = \{(y_1, y_2, \cdots, y_L) \in \mathbb{R}_+^L : U(y_1, y_2, \cdots, y_L) \geqslant k\}$$

These are examples of "(weak) upper contour sets" for the utility level attained by an individual.

Illustrate these sets on the whiteboard for the two commodity case.

Indifference sets

- Reference commodity bundle version:

$$U^0(x_1, x_2, \cdots, x_L) = \{(y_1, y_2, \cdots, y_L) \in \mathbb{R}_+^L : U(y_1, y_2, \cdots, y_L) = U(x_1, x_2, \cdots, x_L)\}$$

Skip to main content

$$U^0(k) = \{(y_1, y_2, \cdots, y_L) \in \mathbb{R}^L_+ : U(y_1, y_2, \cdots, y_L) = k\}$$

These are examples of “level sets” for the utility level attained by an individual.

Illustrate these sets on the whiteboard for the two commodity case.

Input requirement sets

Consider a single output, multiple input production technology that can be represented by a production function of the form

$$y = f(x_1, x_2, \cdots, x_n).$$

An **input requirement set** for this production technology takes the form

$$y^+(k) = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n_+ : f(x_1, x_2, \cdots, x_n) \geqslant k\}$$

This is an example of a “(weak) upper contour set” for the output level attained by a producer.

Illustrate this set on the whiteboard for the two input case.

Isoquants

Consider a single output, multiple input production technology that can be represented by a production function of the form

$$y = f(x_1, x_2, \cdots, x_n).$$

An **isoquant** for this production technology takes the form

$$y^0(k) = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n_+ : f(x_1, x_2, \cdots, x_n) = k\}$$

This is an example of a “level set” or the output level attained by a producer. *Illustrate this set on the whiteboard for the two input case.*

Skip to main content

An **isocost** depicts the locus of all input combinations that cost the producer the same amount of money to employ.

Suppose that there are $n \in \mathbb{N}$ distinct production inputs (or factors of production, if you prefer). An isocost for this situation takes the form

$$C^0(k) = \left\{ (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n_+ : \sum_{i=1}^{n} w_i x_i = k \right\}$$

where w_i is the per-unit price of input i.

This is an example of a "level set" for the expenditure on inputs by a producer. *Illustrate this set on the whiteboard for the two input case.*

Price simplex

In some situations in economics, it is relative prices that matter, rather than the absolute level of each individual price. In such cases, some form of price normalisation can be employed.

Common normalisations involve choosing either a particular commodity, or a particular basket of commodities, to be the numeraire. The expenditure on the the numeraire commodity, or numeraire basket of commodities, is then set equal to one.

If the numeraire basket consists of one unit of each of the n final commodities in an economy, then the set of possible prices is given by:

$$\triangle(p_1, p_2, \cdots, p_n) = \left\{ (p_1, p_2, \cdots, p_n) \in \mathbb{R}^n_+ : \sum_{i=1}^{n} p_i = 1 \right\}$$

This set is known as a price "simplex".

Illustrate this set on the whiteboard for both the two commodity case and the three commodity case.

Mappings: functions and correspondences

Skip to main content

Introductory mathematical economics references:

- Haeussler, EF Jr, and RS Paul (1987), *Introductory mathematical analysis for business, economics, and the life and social sciences (fifth edition)*, Prentice-Hall, USA:Chapters 0, 3, 4, 5, and 17.1.
- Sydsaeter, K, P Hammond, A Strom, and A Carvajal (2016), *Essential mathematics for economic analysis (fifth edition)*, Pearson Education, Italy: Chapters 2, 4, 5, 9.6 (pp. 350-351 only), 11.1, and 11.5.
- Shannon, J (1995), *Mathematics for business, economics and finance*, John Wiley and Sons, Brisbane: Chapters 1, 2, and 6.

Advanced high school references:

- Coroneos, J. (Undated a), *A Higher School Certificate Course in Mathematics: Year Eleven, Three Unit Course*, Coroneos Publications, Australia: Chapters 1, 2, 4, and 7.
- Coroneos, J. (Undated b), *A Higher School Certificate Course in Mathematics: Years Eleven and Twelve, Revised Four Unit Course (for Mathematics Extension Two)*, Coroneos Publications, Australia: Chapters 1 and 2.

Introductory mathematics references:

- Adams, RA, and C Essex (2018), *Calculus: A complete course (ninth edition)* Pearson, Canada: Chapters P, 3, and 11.
- Kline, M (1967), *Calculus: An intuitive and physical approach (second edition)*, The 1998 Dover republication of the original John Wiley and Sons second edition, Dover Publications, USA: pp. 419–432.
- Silverman, RA (1969), *Modern calculus and analytic geometry*, The 2002 Dover corrected republication of the original 1969 Macmillan Company edition, Dover Publications, USA: Chapters 7 and 14.
- Spivak, M (2006), *Calculus (third edition)*, Cambridge University Press, The United Kingdom: Chapters 1, 2, 3, 4, 16, 18, and 19.
- Thomas, GB Jr, and RL Finney (1996), *Calculus and analytic geometry (ninth edition)*, The 1998 corrected reprint version, Addison-Wesley Publishing Company, USA: Chapters P, 6, and 12.

More advanced references:

- Banks, J, G Elton and J Strantzen (2009), *Topology and analysis: Unit text for MAT3TA (2009 and 2010 edition)*, Department of Mathematics and Statistics, La Trobe University, Bundoora, February.

Skip to main content

- Kolmogorov, AN and SV Fomin (1970), *Introductory real analysis*, Translated and Edited by RA Silverman, The 1975 Dover Edition (an unabridged, slightly corrected republication of the original 1970 Prentice-Hall edition), Dover Publications, USA: Chapter 1 (pp. 1-36).
- Simon, C, and L Blume (1994), *Mathematics for economists*, WW Norton and Co, USA: Chapters 2 and 13 (pp. 10-38 and 273-299).

Websites:

The following websites contain discussions of the concept of relatively prime, or co-prime, numbers and polynomials. This is relevant to the topic of rational functions and partial fractions.

- https://www.mathsisfun.com/definitions/relatively-prime.html
- http://mathworld.wolfram.com/RelativelyPrime.html
- https://www.varsitytutors.com/hotmath/hotmath_help/topics/relatively-prime

Mappings

Let X and Y be two sets. A rule f that assigns one or more elements of Y to each element of X is called a **mapping** from X into Y. It is denoted by $f : X \rightarrow Y$.

The set X is known as the **domain** of the mapping f. The mapping must be defined for every element of X. This means that $x \in X \implies f(x) \in Y$.

The set Y is known as the **co-domain** of the mapping f. Mappings are not required to generate Y from X. This means that there might exist one or more elements $y \in Y$ such that $y \neq f(x)$ for any $x \in X$.

The set of values $y \in Y$ that can be generated from X by the function f is known as the **image** of X under f. Sometimes the image of X under f is referred to as the **range** of f. We will denote the range of f by $f(X)$.

Skip to main content

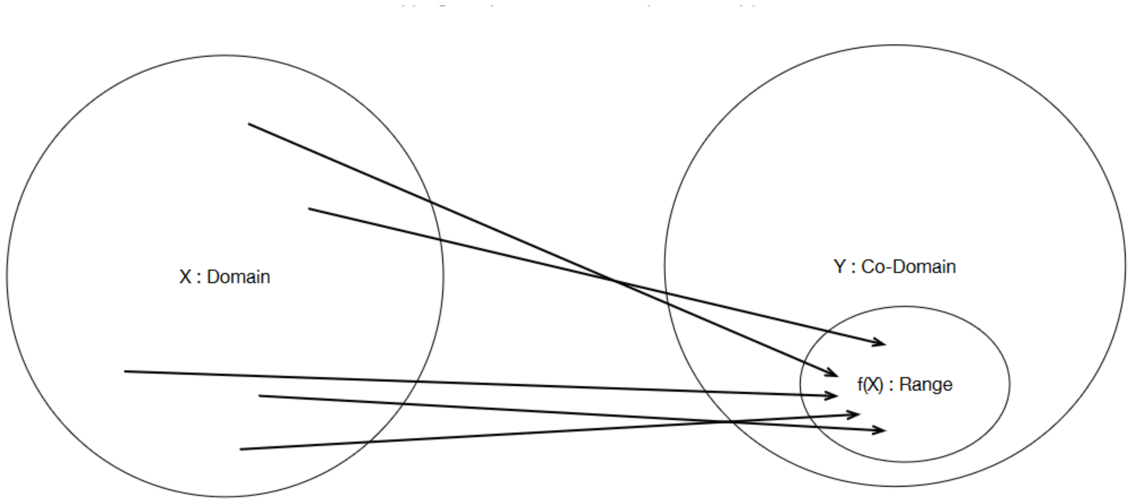


Fig. 13 Diagrammatic representation of a mapping

Images

Consider a mapping $f : X \to Y$. The *image of the point* $x \in X$ under the mapping f is the point, or collection of points, given by $f(x) \in Y$.

The *image of the set* $A \subseteq X$ under the mapping f is the set $f(A) = \{f(x) \in Y : x \in A\}$.

- Clearly $f(A) \subseteq Y$.

The *image of the domain* (X) under f is the set $f(X) = \{f(x) \in Y : x \in X\}$.

- Clearly $f(X) \subseteq Y$.

Note that if $f : X \to Y$ and $A \subseteq X$, then $f(A) \subseteq f(X) \subseteq Y$.

Pre-images

Consider a mapping $f : X \to Y$.

The **pre-image** *of the point* $y \in Y$ under the mapping f is the point, or collection of points, in X for which $y = f(x)$.

- no pre-image
- a unique pre-image, or
- multiple pre-images. If a point $y \in Y$ has a unique pre-image under the mapping f, then it is denoted by $f^{-1}(y)$.

The *pre-image of the set* $B \subseteq Y$ under f is the set $f^{-1}(B) = \{x \in X : f(x) \in B\}$.

Recall that it is possible for there to exist $y \in Y$ such that $y \neq f(x)$ for any $x \in X$. If the set B consists entirely of such points, then $f^{-1}(B) = \varnothing$.

Since $f(x) \in Y$ for all $x \in X$, it must be the case that $f^{-1}(Y) = X$.

Types of mappings

There are four basic types of mappings. These are as follows.

- A one-to-one mapping:
  - Each point in X has a unique image in Y; and
  - Each point in Y has either a unique pre-image in X or no pre-image in X.
- A many-to-one mapping:
  - Each point in X has a unique image in Y; but
  - At least one point in Y has multiple pre-images in X.
- A one-to-many mapping:
  - At least one point in X has multiple images in Y; but
  - Each point in Y has either a unique pre-image in X or no pre-image in X.
- A many-to-many mapping:
  - At least one point in X has multiple images in Y; and
  - At least one point in Y has multiple pre-images in X.

Mappings whose domain points all have unique images are known as **functions**. In other words, one-to-one mappings and many-to-one mappings are known as functions.

A one-to-one function $f : X \to Y$ is called an **injection**. The pre-image of a one-to-one function is known as its **inverse**.

Skip to main content

Note that we could express a correspondence of the form $f : X \to Y$ as a function of the form $g : X \to 2^Y$.

Into and onto

Consider a mapping $f : X \to Y$. Clearly we must have $f(X) \subseteq Y$.

- If $f(X) \subset Y$, so that $f(X) \neq Y$, we say that f maps X “into” Y.
- If $f(X) = Y$, we say that f maps X “onto” Y.
  - If $f(X) = Y$ and f is a function, then we call f a **surjection**.

A mapping that is both an injection and a surjection is called a **bijection**. In other words, a function that is *both one-to-one and onto* is called a bijection.

Examples

Consider the mapping $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$. This is sometimes called the identity map. Note that f is both onto and one-to-one. This means that f is a bijection. It also means that the pre-image of f is an inverse function.

Consider the mapping $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. Note that f is into, but not onto, because $f^{-1}(R_{--}) = \varnothing$, where $\mathbb{R}_{--} = \{x \in \mathbb{R} : x < 0\}$.

Note also that f is many-to-one, and hence not one-to-one, because $2 \in \mathbb{R}, (-2) \in \mathbb{R}, 2 \neq (-2)$, and both $2^2 = 4$ and $(-2)^2 = 4$. This means that f is neither an injection nor a surjection. Thus it cannot be a bijection. It also means that the pre-image of f is not an inverse function.

Composite functions

Let $f : X \to Y$ be a function of the form $y = f(x)$. Let $g : Y \to Z$ be a function of the form $z = g(y)$. The composite function $h = g \circ f$ is defined by $h(x) = g(f(x))$. The composite function $h = g \circ f$ is a mapping $h : X \to Z$ of the form $z = h(x)$.

Example: Let $f : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ be defined by $f(x_1, x_2) = x_1^{\alpha}\, x_2^{(1-\alpha)}$ for some fixed $\alpha \in (0, 1)$.

Skip to main content

Then the composite function $h = g \circ f$ is a mapping $h : \mathbb{R}^2_{++} \to \mathbb{R}$ that is defined by $h(x_1, x_2) = g(f(x_1, x_2)) = ln(x_1^\alpha \, x_2^{(1-\alpha)}) = \alpha \, ln(x_1) + (1-\alpha) \, ln(x_2)$

One-to-one functions

A function $f : X \to Y$ is **one-to-one** if

$$x \neq y \iff f(x) \neq f(y)$$

The contra-positive version of this condition is that

$$f(x) = f(y) \iff x = y$$

Examples:

- $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$ is a one-to-one function.
- $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is not a one-to-one function.
- $f : \mathbb{R}_+ \to \mathbb{R}$ defined by $f(x) = x^2$ is a one-to-one function.

Non-decreasing and strictly increasing functions

A function $f : X \to \mathbb{R}$, where $X \subseteq \mathbb{R}$, is said to be a **non-decreasing** function if

$$x \leqslant y \iff f(x) \leqslant f(y)$$

A function $f : X \to \mathbb{R}$, where $X \subseteq \mathbb{R}$, is said to be a **strictly increasing** function if both

- (a) $x = y \iff f(x) = f(y)$; and
- (b) $x < y \iff f(x) < f(y)$.

Note the following:

- A strictly increasing function is also a one-to-one function.
- There are some one-to-one functions that are not strictly increasing.
- A strictly increasing function is also a non-decreasing function.

Skip to main content

Economic application: utility functions are not unique

Suppose that $U : X \to \mathbb{R}$ is a utility function that represents the weak preference relation $\succsim$. This means that $x \succsim y \iff U(x) \geqslant U(y)$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly increasing function. Consider the composite function $V = f \circ U$. It can be shown that V is also a utility function that represents the weak preference relation $\succsim$. In other words, it can be shown that $x \succsim y \iff V(x) \geqslant V(y)$.

Example: A Cobb-Douglas utility function

Consider a consumer whose preferences over bundles of strictly positive amounts of each of two commodities can be represented by a utility function $U : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ of the form

$$U(x_1, x_2) = Ax_1^{\alpha} x_2^{\beta}$$

where $A > 0, \alpha > 0$, and $\beta > 0$.

Such preferences are known as Cobb-Douglas preferences.

- The function $f : \mathbb{R}_{++} \to \mathbb{R}_{++}$ defined by $f(x) = \left(\frac{1}{A}\right)x$ is strictly increasing. Thus we know that another utility function that represents this consumer's preferences is

$$V(x_1, x_2) = f(U((x_1, x_2)) = \left(\frac{1}{A}\right)(Ax_1^{\alpha} x_2^{\beta}) = x_1^{\alpha} x_2^{\beta}$$

- The function $g : \mathbb{R}_{++} \to \mathbb{R}_{++}$ defined by $g(x) = x^{\frac{1}{(\alpha+\beta)}}$ is strictly increasing. (If any relevant surd expression can be either positive or negative, then we will assume that the positive option is chosen throughout this Cobb-Douglas preferences example.) Thus we know that another utility function that represents this consumer's preferences is

$$W(x_1, x_2) = g(V((x_1, x_2)) = (x_1^{\alpha} x_2^{\beta})^{\frac{1}{(\alpha+\beta)}} = x_1^{\gamma} x_2^{(1-\gamma)}$$

where $\gamma = \frac{\alpha}{\alpha+\beta} \in (0, 1)$.

Skip to main content

$$Z(x_1, x_2) = k(W((x_1, x_2)) = ln(x_1^{\gamma} x_2^{(1-\gamma)})$$

$$= \gamma\, ln(x_1) + (1-\gamma)\, ln(x_2)$$

Some types of functions

Some types of univariate functions include the following:

- Polynomial functions
  - These include constant functions, linear (and affine) functions, quadratic functions, and some power functions as special cases.
- Exponentional functions
- Power functions
- Logarithmic functions
- Trigonometric functions
  - We are unlikely to have enough time to cover trigonometric functions in this course. There are also multivariate versions of these types of functions.

Polynomial functions

A **polynomial function** (of one variable) is a function of the form

$$\begin{aligned} f(x) &= \sum_{i=0}^{n} a_i x^i \\ &= a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0 x^0 \\ &= a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \end{aligned}$$

In order to distinguish between different types of polynomials, we will typically assume that the coefficient on the term with the highest power of the variable x is non-zero. The only exception is the case in which this term involves $x^0 = 1$, in which case we will allow both $a_0 \neq 0$ and $a_0 = 0$.

Skip to main content

- A constant (degree zero) polynomial ($a_0 \neq 0$ or $a_0 = 0$):

$$f(x) = a_0$$

- A linear (degree one) polynomial ($a_1 \neq 0$):

$$f(x) = a_1 x + a_0$$

It is sometimes useful to distinguish between linear functions and affine functions. See below for details.

- A quadratic (degree two) polynomial ($a_2 \neq 0$):

$$f(x) = a_2 x^2 + a_1 x + a_0$$

- A cubic (degree three) polynomial ($a_3 \neq 0$):

$$f(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$$

Affine functions and linear functions

We often loosely speak about a linear function of one variable being a function of the form $f(x) = a_1 x + a_0$, where $a_0 \in \mathbb{R}$ and $a_1 \in \mathbb{R} \setminus \{0\}$ are fixed parameters, and $x \in \mathbb{R}$ is the single variable.

Sometimes, we want to be more precise than this in order to identify the special case in which $a_0 = 0$. In such cases, we call a function of the general form $f(x) = a_1 x + a_0$, in which $a_0 \in \mathbb{R}$, an **affine function**; and a function of the specific form $f(x) = a_1 x$, in which $a_0 = 0$, a **linear function**.

Using this more precise terminology, the family of linear functions is a proper subset of the family of affine functions. Note that we assume that $a_1 \in \mathbb{R} \setminus \{0\}$ in both cases.

Exponential functions

An **exponential function** is a non-linear function in which the independent variable appears as an exponent.

Skip to main content

$$f(x) = Ca^x$$

where C is a fixed parameter (called the **coefficient**), $a \in \mathbb{R}$ is a fixed parameter (called the **base**), and $x \in \mathbb{R}$ is an independent variable (called the **exponent**).

- Note that if $a = 0$, then $f(x)$ is only defined for $x > 0$.
- Note also that if $a < 0$, then sometimes $f(x) \notin \mathbb{R}$. An example of this is the case when $C = 1, a = (-1)$ and $x = \frac{1}{2}$. In this case, we have:

$$f(\frac{1}{2}) = (1)(-1)^{\frac{1}{2}} = \sqrt{-1} = i \notin \mathbb{R}$$

Popular choices of base

Two popular choices of base for exponential functions are $a = 10$ and $a = e$, where e denotes Euler's constant. Euler's constant is defined to be the number

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

Note that Euler's constant is an irrational number. This means that it is a real number that cannot be represented as the ratio of an integer to a natural number.

The function $f(x) = e^x$ is sometimes called "the" exponential function.

Exponential arithmetic

Assuming that the expressions are well-defined, we have the following laws of exponential arithmetic.

- The power of zero: $a_0 = 1$ if $a \neq 0$, while a^0 is undefined if $a = 0$.
- Multiplication of two exponential functions with the same base:

$$(Ca^x)(Da^y) = CDa^{(x+y)}$$

$$\frac{(Ca^x)}{(Da^y)} = \frac{C}{D}a^{(x-y)}$$

- An exponential function whose base is an exponential function:

$$(Ca^x)^y = C^y a^{xy}$$

Power functions

A **power function** takes the form

$$f(x) = Cx^a$$

where $C \in \mathbb{R}$ is a fixed parameter, $a \in \mathbb{R}$ is a fixed parameter, and $x \in \mathbb{R}$ is an independent variable.

- Note that when $a \in \mathbb{N}$, a power function can also be viewed as a polynomial function with a single term.
- Note that a power function can also be viewed as a type of exponential expression in which the base is x and the exponent is a. This means that the laws of exponential arithmetic carry over to "power function arithmetic".

Power function arithmetic

Assuming that the expressions are well-defined, we have the following laws of power function arithmetic.

- The power of zero: $x_0 = 1$ if $x \neq 0$, while x^0 is undefined if $x = 0$.
- Multiplication of two power functions with the same base:

$$(Cx^a)(Dx^b) = CDx^{(a+b)}$$

- Division of two power functions with the same base:

$$\frac{(Cx^a)}{(Dx^b)} = \frac{C}{D}x^{(a-b)}$$

$$(Cx^a)^b = C^b x^{ab}$$

A rectangular hyperbola

Consider the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $f(x) = \frac{a}{x}$, where $a \neq 0$. This is a special type of power function, as can be seen by noting that it can be rewritten as $f(x) = ax^{-1}$. The equation for the graph of this function is

$$y = \frac{a}{x}$$

Note that this equation can be rewritten as $xy = a$. This is the equation of a rectangular hyperbola.

Graph it on the whiteboard for both the case where $a > 0$ and the case where $a < 0$.

- Economic application: A constant "own-price elasticity of demand" demand curve.

Logarithms

A **logarithm** undoes an exponent. Thus we have

$$log_a(a^x) = x$$

The expression log_a stands for "log base a" or "logarithm base a". Popular choices of base are $a = 10$ and $a = e$.

The function

$$g(x) = log_a(x)$$

is the "logarithm base a" function. The "logarithm base a" function is the inverse for the "exponential base a" function. The reason for this is that

$$g(f(x)) = g(a^x) = log_a(a^x) = x$$

Skip to main content

Natural (or Naperian) logarithms

A “logarithm base *e*” is known as a natural, or Naperian, logarithm. It is named after John Napier. See Shannon (1995, pp. 270-271) for a brief introduction to John Napier. The standard notation for a **natural logarithm** is ln, although you could also use log_e.

The function

$$g(x) = ln(x)$$

is the “logarithm base e” function. The natural logarithm function is the inverse function for “the” exponential function, since

$$g(f(x)) = g(e^x) = ln(e^x) = log_e(e^x) = x$$

Illustrate the inverse (or “reflection through the 45 degree (y = x) line”) relationship between ln (x) and ex on the whiteboard.

Logarithmic arithmetic

Assuming that the expressions are well-defined, we have the following laws of logarithmic arithmetic.

- Multiplication of two logarithmic functions with the same base:

$$log_a(xy) = log_a(x) + log_a(y)$$

- Division of two logarithmic functions with the same base:

$$log_a\left(\frac{x}{y}\right) = log_a(x) - log_a(y)$$

- A logarithmic function whose argument is an exponential function:

$$log_a(x^y) = y\ log_a(x)$$

Note that

Skip to main content

Rational functions

A **rational function** $R(x)$ is simply the ratio of two polynomial functions, $P(x)$ and $Q(x)$. It takes the form

$$R(x) = \frac{P(x)}{Q(x)} = \frac{a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0}$$

where

$$P(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$$

is an m-th order polynomial (so that $a_m \neq 0$), and

$$Q(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$$

is an n-th order polynomial (so that $b_n \neq 0$).

- Note that there is no requirement that the polynomial functions $P(x)$ and $Q(x)$ be of the same order. (In other words, we do not require that $m = n$.)
- The most interesting case is when $m < n$. In such cases, the rational function $R(x)$ is called a **"proper" rational function**.
- When $m > n$, then we can always use the process of long division to write the original rational function $R(x)$ as the sum of a polynomial function $Y(x)$ and another proper rational function $R^*(x)$.

This is nicely illustrated by the following example from Chapter 14 of Silverman (1969).

Consider the rational function $R(x) = \frac{x^2+x-1}{x-1}$. Note the following.

Thus we have

$$R(x) = \frac{x^2 + x - 1}{x - 1} = (x+2) + \left(\frac{1}{x-1}\right)$$

$R(x) = \frac{x^2+x-1}{x-1}$, $Y(x) = (x+2)$, and $R^*(x) = \left(\frac{1}{x-1}\right)$, so that

$$\frac{x^2+x-1}{x-1} = (x+2) + \left(\frac{1}{x-1}\right)$$

The proper rational function component is sometimes known as a “remainder” term.

In cases where $R(x)$ is a proper rational function to begin with, then we have $Y(x) = 0$, leaving us only with the remainder term.

The remainder term $R^*(x)$ can sometimes be converted into a more convenient form by using the technique of “Partial Fractions”.

Partial fractions

Consider two rational functions, $R_1(x) = \frac{P_1(x)}{Q_1(x)}$ and $R_2(x) = \frac{P_2(x)}{Q_2(x)}$. Note that the sum of these two rational functions forms a third rational function, since

$$\begin{aligned} R_3(x) = R_1(x) + R_2(x) &= \frac{P_1(x)}{Q_1(x)} + \frac{P_2(x)}{Q_2(x)} \\ &= \frac{(P_1(x)Q_2(x) + P_2(x)Q_1(x))}{Q_1(x)Q_2(x)} \end{aligned}$$

If we reverse this process, we can write the rational function $\frac{(P_1(x)Q_2(x)+P_2(x)Q_1(x))}{Q_1(x)Q_2(x)}$ as the sum of two other rational functions, $\frac{P_1(x)}{Q_1(x)}$ and $\frac{P_2(x)}{Q_2(x)}$. This decomposition is known as a **“partial fractions” decomposition**.

Suppose that $P(x)$ and $Q(x)$ are **relatively prime** polynomials.

- Two polynomials (or, indeed, two integers) are said to be “relatively prime” (or “co-prime”) if their “highest common factor” (which is also known as their “greatest common divisor”) is the number “one”. Suppose also that the degree of the polynomial M(x) is strictly less than the degree of the polynomial P(x)Q(x). This means that the rational function R∗(x) = M(x) P(x)Q(x) is a proper rational function. Under these circumstances there exist two unique polynomial functions, A(x) and B(x), such