CS/SS/EC 149: Introduction to economic design[1][2]

for students with CS/math background

Fedor Sandomirskiy (Caltech)

[1] These lecture notes are (unintentionally) full of typos and inaccuracies. Please, shoot me a message if you spot some or if you have other comments & suggestions: fsandomi@caltech.edu.

[2] I am thankful to Sumit Goel for being an outstanding TA.

Contents

Chapter 1

About this course

The course is an introduction to an active research area at the intersection of Econ and CS sometimes referred to as algorithmic economics. It deals with multi-agent systems, where agents differ in their preferences and may behave strategically to improve the outcome for themselves. We will discuss applications to voting, crowd-sourcing, rating design, revenue-maximization and auctions, allocation of resources without money, and matching markets.

Algorithmic economics can be roughly divided into algorithmic game theory (rules of interaction are given and we ask how the system behaves) and mechanism design (rules are to be designed). As there is a full-fledged course on game theory per se (PS/EC 172), we will focus on the design perspective and try to avoid intersections as much as possible. The broad topic of matching markets will be touched on briefly; an extensive discussion they deserve is given in EC 117. Some topics will be developed in homework assignments that are included in the appendix.

1.0.1 Prerequisites

The course has no prerequisites, in particular, prior knowledge of game theory is not needed. However, it is assumed that you are comfortable with mathematical proofs and know calculus and algebra basics (derivatives, integration, and vectors). Familiarity with the basics of probability theory (random variables, distributions, independence, the Bayes formula) will also be useful.

1.0.2 Goals

The course will teach you the basic building blocks and insights used by economists to design multi-agent systems. You will learn the language and the model needed to formulate design goals, tools to achieve them, and some workarounds allowing you to find a reasonable solution if the goals are incompatible.

By the end of the course, you will be able to understand more than half[1] of talks at the

[1] To understand the other 30%, you need a game-theory course. The remaining 5-10% nobody understands :-)

top conferences at the intersection of economics and computation such as the annual ACM Conference on Economics and Computation (2022 edition) or start your own research on one of the cutting-edge topics.

1.0.3 Materials

You may find the following sources useful, all freely available online. I also used them to prepare the lectures:

- Game Theory, Alive by Anna Karlin and Yuval Peres, in my opinion, is the best introduction to both game theory and mechanism design for readers with math or CS background.

- Handbook of computational social choice edited by Felix Brandt, Vincent Conitzer, Ulle Endriss, Jerome Lang, and Ariel Procaccia
  Topics: voting (Section 2), judgment aggregation (Section 17), fair division (Section 13).

- Handbook of Algorithmic Game Theory edited by Noam Nisan, Tim Roughgarden, Eva Tardos, and Vijay Vazirani
  Topics: competitive equilibrium (Chapter 5), social choice (Chapters 9,10)

- Multiagent Systems: Algorithmic, Game-Theoretic, and Logical Foundations by Yoav Shoham and Kevin Leyton-Brown, Cambridge University Press, 2009
  Topics: voting (Section 9), VCG (Section 10), auctions (Section 11).

I also recommend amazing lecture notes by Tim Roughgarden's "Incentives in Computer Science" from where I borrowed quite a few ideas. Some inspiration came from courses by Ariel Procaccia; slides are available on his website.

Chapter 2

Multi-agent systems: economics or CS?

This course is about multi-agent systems. But who are the agents? Originally, economics was about people producing, selling, and buying goods for money. These people — and, more generally, their groups such as firms or countries — are economic agents. However, the key feature of these economic agents is not the type of interaction they engage in but the fact that **different agents may pursue different goals, may differ in their preferences and characteristics**. A device in a network having its own local objective, a car following a route suggested by Google Maps, or processes sharing common computational resources can all be modeled as such agents.

Game theory aims to understand how systems of agents behave when the rules of interaction are given. We will be interested in the opposite questions: How to design multi-agent systems? How to take differences in preferences and agents' incentives into account? These questions belong to the field of economic design also known as mechanism design.

Originally, mechanism design was a part of economics but now it can be considered interdisciplinary research at the interface of economics, CS, and math. We will discuss applications of mechanism design to ranking aggregation, auctions including sponsored search auctions used by search engines such as Google, platform markets, crowdsourcing, fair resource allocation in sharing economy, and others.

There is a group of key goals that a designer can have:

- Efficiency: The outcome of a mechanism must be optimal in some sense, e.g., maximize a certain numerical objective (revenue or welfare) or be non-improvable in some other sense.
- Robustness to strategic behavior (strategy-proofness): Agents' incentives to game the system should not ruin the outcome.
- Fairness: Agents must be treated fairly, e.g., enjoy equal opportunities.

While the formalization of these goals depends on the particular application, the rule of thumb is that they are typically incompatible. This incompatibility discovered early by economists (famous Arrow's theorem and the Gibbard-Sattherwaite theorem that we will discuss soon) became a major obstacle and discouragement to the economic theory of mechanism design. The current research on mechanism design is largely affected by new ideas from CS offering a workaround for impossibilities and allowing you to design good mechanisms if you must.

Chapter 3

Ranking aggregation (aka Voting or Social Choice)

The most fundamental problem of economic design: how to aggregate a collection of distinct rankings of alternatives into one collective ranking or one collective decision? It is usually referred to as the social choice problem or voting by its most immediate application. Applications:

- Voting in political elections (voters have different preferences over candidates and the goal is to choose the "best" candidate)
- Collective decision-making within an organization (the board of directors decides on a marketing campaign).
- Aggregating experts' opinions, crowd-sourcing (workers on Mechanical Turk are asked to rank several interface options by their usability), and peer grading (students are asked to rank works of their peers).
- Aggregating algorithmic recommendations (experts=algorithms), e.g., setting up an ensemble of neural networks (aka boosting)
- Ranking aggregation (sports, movies)
- Exotic examples: Blockchain governance and DAOs (decentralized autonomous organizations).

Apart from these direct applications, social choice problem is important as you can think of any economic design problem as that of social choice. The idea is as follows. Think of an auction. The set of outcomes consists of possible allocations of the goods and payments. Clearly, each bidder prefers to get the auctioned good and to pay less. In other words, agents are having different preferences over a common set of "outcomes" and the auction can be seen as a mechanism for choosing an outcome given these conflicting preferences.

In particular, social choice is a benchmark problem illustrating all the obstacles that we will see later in other economic design problems.

The problem of social choice There is a set $N = \{1, 2, \ldots, n\}$ of agents (we will sometimes refer to them as "voters") and a set A of $|A| = m$ alternatives ("candidates" or "outcomes"). We will denote generic alternatives by a, b, c and generic voters by i, j, k.

Each agent's preferences (ranking) are given by a list of alternatives, from best to worst. We will denote agent i's preference by $\succ_i$ and write $a \succ_i b$ to indicate that voter i prefers the alternative a to b. We assume transitivity: if i prefers a to b and b to c, then i prefers a to c i.e., $\succ_i$ is a transitive relation.[1] In the basic setting discussed in the first two lectures, we also assume that the voter is never indifferent (has strict preferences): for any two distinct alternatives a, b either $a \succ_i b$ or $b \succ_i a$.

A collection of preferences of each of the agents is called a **preference profile** and is denoted by $\pi = (\succ_1, \ldots, \succ_n) = (\succ_i)_{i \in N}$.

A **voting rule** is a map f assigning a winning alternative $f(\pi) \in A$ to each preference profile π.

A **ranking rule** is a map F assigning a ranking of alternatives $F(\pi) =\succ_\pi$ to each preference profile π.

3.1 Some (de)motivating examples

3.1.1 Success story: two alternatives.

First consider the case of just two candidates, $|A| = 2$ and odd number of voters $|N|$.

The will of the majority is the idea underlying modern democracy and is rooted in ancient Greece. The **majority rule** which outputs the candidate preferred by the majority[2].

In what sense the majority rule is good? It treats voters symmetrically and gives no incentive for them to "game the rule" by misreporting their preferences to make their favorite candidate win. Formally, these properties are captured by the following two notions that are also applicable to voting rules with any number of alternatives.

A voting rule F is said to be

- **strategy-proof (aka truthful aka non-manipulable)** if no agent i with preferences $\succ_i$ can make her more preferable candidate win by pretending that her preferences are given by some $\succ_i'$. Formally, for any profile $(\succ_i)_{i \in N}$ of preferences and any agent i, there is no $\succ_i'$ such that[3]

$$f\left(\succ_1, \ldots, \succ_{i-1}, \succ_i', \succ_{i+1}, \ldots, \succ_n\right) \succ_i f\left(\succ_1, \ldots, \succ_n\right).$$

[1] This requirement captures the idea that agents are rational. There are experiments showing that real people's preferences may violate this requirement, but this is still a natural approximation to the incomprehensible complexity of real life.

[2] It is uniquely defined thanks to the assumption that there is an odd number of voters.

[3] For those of you familiar with game theory: Equivalently, a rule f is strategy-proof if submitting the true preferences is a weakly dominant strategy in the game defined by f.

- **voter-symmetric (aka anonymous)**[4] if voters' names do not matter. Formally, for any profile $(\succ_i)_{i\in N}$ and any permutation[5] σ of voters,

$$f\left((\succ_i)_{i\in N}\right) = f\left((\succ_{\sigma(i)})_{i\in N}\right).$$

Such requirements on a voting rule are usually called axioms. Are there any other rules satisfying the two axioms?

Theorem 1 (a version of May's theorem)**.** *For two alternatives $A = \{0,1\}$ and an odd number of voters $|N|$, any strategy-proof anonymous voting rule f is a threshold rule: there is a number $t \in \{0,1,2,\ldots,n\}$ such that*

$$f((\succ_i)_{i\in N}) = \begin{cases} 1, & |\{i \in N\colon 1 \succ_i 0\}| \geq t \\ 0, & \textit{otherwise.} \end{cases}$$

Corollary 1. *If we additionally assume that the rule treats the two alternatives in a symmetric way, i.e., candidate's names do not matter*[6] *(the axiom called **neutrality**), then we recover the majority rule corresponding to $t = \frac{|N|}{2}$.*

To prove May's theorem note that we can represent any profile of preferences over two alternatives $A = \{0,1\}$ via the set of voters $N_1 = \{i \in N\colon 1 \succ_i 0\}$ preferring the alternative 1. In other words, any rule f can be seen as a function of N_1. With this representation, the two axioms are reformulated handily.

Claim 1. *A rule $f = f(N_1)$ with $A = \{0,1\}$ is*

- *strategy-proof if and only if f is monotone, i.e., $f(N_1) \leq f(N_1 \cup \{i\})$ for any $i \in N$.*

- *anonymous if and only if $f(N_1) = g(|N_1|)$, i.e., f depends only on the number of elements in $|N_1|$.*

Now we can prove the theorem.

Proof. We know that $f = g(|N_1|)$, where g is a monotone function on $\{0,1,\ldots,n\}$ with values in $\{0,1\}$. Any such function is a step function, i.e., $g(x) = 1$ for $x \geq t$ and 0, otherwise. This step function gives the threshold rule from the theorem. $\square$

[4] This assumption is reasonable in political elections but may not be so in crowd-sourcing, where the weight of a "voter" may depend on their performance, e.g., on the average number of mistakes they make.

[5] A permutation of a set B is a bijection $B \to B$.

[6] If the alternatives differ in their nature (e.g., implementing a project or not), there is no reason to assume that they must be treated symmetrically.

3.1.2 This irrational majority

What should we do if there are more than two alternatives? The idea goes back to Marquise de Condorcet[7] is using the majority rule to compare each pair of alternatives.

Define the preference of the majority as follows:

$$a \succ_{maj} b \quad \text{if and only if the majority of voters prefers } a \text{ to } b.$$

To focus on the essence of the problem, we assume that there is an odd number of voters n.

An alternative a is called the **Condorcet winner** if $a \succ_{maj} b$ for any $b \in A$. The problem with this definition is that the Condorcet winner may fail to exist because of the fundamental problem with majority preferences, illustrated by the following example.

Example 1 (Condorcet paradox)**.** Consider the 3-voter profile:

$\succ_1$	$\succ_2$	$\succ_3$
a	b	c
b	c	a
c	a	b

Then $a \succ_{maj} b$ by the majority (voters 1,3) and $b \succ_{maj} c$ by the majority (voters 1,2). We conclude that

$$a \succ_{maj} b \succ_{maj} c.$$

But wait, $c \succ_{maj} a$ by the majority (voters 2,3)?!? We end up with the so-called Condorcet cycle and see that the Condorcet winner may fail to exist.

The example demonstrates that the majority preferences may violate transitivity, i.e., can be irrational. The reason is that the "majority" is not a well-defined set of agents and it depends on the question we ask. In particular, if sequential pairwise comparisons are used to come up with a non-binary decision, the outcome may depend on the sequence of comparisons. Another related irrationality phenomenon is known in the field of *judgment aggregation*[8], which studies voting over logically related issues.

Example 2 (Doctrinal paradox in judgment aggregation)**.** Consider 3 judges that need to decide on 3 logically related issues: whether the defendant was obliged to complete the task, whether she failed, and whether she is liable. The judges are rational, which is captured by the following logical relation obliged&failed $\Rightarrow$ liable. Judges' individual judgments are as follows:

	obliged	failed	liable
Judge 1	yes	yes	yes
Judge 1	yes	no	no
Judge 1	no	yes	no
majority	yes	yes	no

[7] He was a remarkable 18th-century thinker and politician defending humanistic ideas two centuries before this became mainstream. Read about him on Wikipedia.

[8] If you are interested to learn more, check the chapter on judgment aggregation in the Handbook of Computational Social Choice.

As we see, the majority's judgments are irrational.

While the Condorcet winner may fail to exist, it often does both in real-life elections[9] and in theory.[10]

When the Condorcet winner exists, it is natural to require that she gets elected. Voting rules satisfying this requirement are called **Condorcet consistent** or satisfying the Condorcet winner property. While this requirement seems to be mild and unambiguous, most of the voting rules we are familiar with fail to pass this test.

A seemingly must-have property is that the alternative defeated by any other alternative in the majority vote must never be chosen. Formally, an alternative a is the **Condorcet loser** if $a \prec_{maj} b$ for any $b \in A$. A rule is said to have the **Condorcet loser property** if such a loser is never selected by the rule. As we will see, even this test is failed by the most popular voting rule that we discuss next.

3.2 Plurality rule

The **plurality rule** selects the alternative that is top-ranked by the maximal number of voters. It is the most common voting rule. This rule and its variants such as plurality with runoff[11] are omnipresent in politics. This rule can be seen as another attempt to extend the majority rule to $|A| > 2$ alternatives. But is it a good extension?

Example 3. Consider the following profile of preferences:

21% of voters	19% of voters	20% of voters	20% of voters	20% of voters
a	c	d	e	f
b	b	b	b	b
c	f	e	d	c
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	a	a	a	a

The alternative a selected by the plurality is hated by 79% of the population. The flaw of the plurality rule is that it does not take into account these "dislikes" and fails to find a compromise such as the alternative b in this profile. In general, the plurality rule favors candidates with extreme views strongly supported by a relatively small group of fanatics.

Formally, these concerns are captured by the fact that a is the Condorcet loser, i.e., the plurality rule does not have the Condorcet loser property.

[9] As argued, for example, by Eric Maskin, a Nobel laureate for his works on mechanism design and proponent of majority-based voting rules.

[10] Various results are showing that, if you fix the number of alternatives $|A|$ and take voters' preferences at random, then (depending on a particular model of randomness), the Condorcet winner either exists with probability approaching 1 or approaching some number strictly above 0. See a survey.

[11] Plurality with runoff selects the winner in two rounds. In the first round, the two candidates with the maximal number of first places get selected. In the second round, the majority selects the winner among the two.

Another phenomenon that can be illustrated using this example is the manipulability of the plurality rule. If, say, 3% of voters from the last group report the preference

$$\begin{array}{c} c \\ f \\ b \\ \vdots \end{array}$$

instead of their sincere preferences, the alternative c gets selected instead of a, i.e., by this misreport they make their 3rd best alternative c win whereas a was their worst alternative. We conclude that the plurality rule is not strategy-proof.[12]

All the flaws of the plurality are shared by plurality with runoff. Plurality is omnipresent but is arguably the worst among popular voting rules.

3.3 Scoring rules

How could we take into account both likes and dislikes and their strength so that the compromise alternative b in the above example is selected?

Consider a family of **scoring rules**, where each voter gives s_k points to the alternative she ranks k'th, and the alternative with the maximal total number of points wins. The resulting rule is parameterized by the vector of scores $s = (s_1, s_2, \ldots, s_{|A|})$.

Note that the plurality rule is a particular case corresponding to $s = (1, 0, 0, \ldots, 0)$. The **Borda rule** named after chevalier de Borda, a contemporary of Condorcet, corresponds to

$$s = (|A| - 1, |A| - 2, |A| - 3, \ldots, 0)$$

and is used to determine the winner of the Eurovision Song Contest.

The Borda rule balances likes and dislikes. In particular, it selects b in the example above and never chooses the Condorcet loser.[13] As you will see in the first homework, the Borda rule is not strategy-proof and may not select the Condorcet winner even if it exists.

[12] Formally, the definition of strategy-proofness requires that only one voter misreports her preference. How can we modify the example so even a single voter has a profitable manipulation?

[13] Try proving this. It is not immediate but is a good puzzle to think about.

Chapter 4

Impossibility results

In the last lecture, we saw that the problem of ranking aggregation with three or more alternatives is prone to paradoxes and does not admit a simple solution. For two alternatives, the combination of non-manipulability and a fairness requirement pined down the family of threshold rules. Fixing the requirements (axioms) and exploring the set of rules satisfying them is known as the normative approach. In this lecture, we explore what the normative approach gives in the case of more than two alternatives.

We will start by discussing the ranking rules and prove the famous Arrow's theorem and then discuss voting rules and formulate the Gibbard-Sattherwaite theorem. These two results are known as impossibility theorems as they show that even a combination of mild requirements leads to an almost vacuous set of rules. These results show that often there are no "ideal" mechanisms; in future lectures, we will see how one can circumvent impossibilities and design good ones.

The model. Let us recall the setting. That there is a set $N = \{1, 2, \dots, n\}$ of agents (aka voters) and a set A of alternatives ("candidates" or "outcomes"). We will denote generic alternatives by a, b, c and generic voters by i, j, k. Each agent's preferences (ranking) are given by a list of alternatives, from best to worst. We will denote agent i's preference by $\succ_i$ and write $a \succ_i b$ to indicate that voter i prefers the alternative a to b. A collection of preferences of each of the agents is called a **preference profile** and is denoted by $\pi = (\succ_1, \dots, \succ_n) = (\succ_i)_{i\in N}$. A **voting rule** is a map f assigning a winning alternative $f(\pi) \in A$ to each preference profile π. A **ranking rule** is a map F assigning a ranking of alternatives $F(\pi) = \succ_\pi$ to each preference profile π.

4.1 Irrelevant alternatives

To motivate the problem, similarly to scoring voting rules, one can define **scoring ranking rules**. Each voter gives s_k points to the alternative she ranks k'th and then the alternatives are ordered by the maximal number of points.

These rules are especially popular in sports to aggregate athletes' results over several competitions as these rules have very transparent mechanics easy-to-follow for spectators. In this application, $i \in N$ indexes different competitions, and $\succ_i$ is the ranking of athletes in this particular competition. The goal is, given $\pi = (\succ_1, \dots, \succ_n)$, to determine the overall ranking of athletes $\succ_\pi$.

The following historical example[1] illustrates what are irrelevant alternatives and why we would like the ranking rule to be independent of them.

Example 4. The Women's Pursuit category of 2014/2015 IBU Biathlon World Cup consisted of seven races. To aggregate the results over the races, IBU uses a scoring ranking rule with the following weights

$$(s_1, s_2, s_3, \dots) = (60, 54, 48, 43, 40, 38, 36, 34, 32, 31, 30, 29, 28, \dots, 1).$$

Kaisa Mäkäräinen came first with two first-place finishes, two second places, a third, a fourth, and a twelfth, for a total score of 348 points. Second was Darya Domracheva, with four first-place finishes, one fourth, a seventh, and a thirteenth, for a total score of 347. In tenth place was Ekaterina Glazyrina, well out of the running with 190 points.

athlete	1	2	3	4	5	6	7	total score
Mäkäräinen	60	60	54	48	54	29	43	348
Domracheva	43	28	60	60	60	36	60	347
...	...	...	...	...	...	...	...	
Glazyrina	32	54	10	26	38	20	10	190

Four years later, Glazyrina was disqualified for doping violations and all her results from 2013 onwards were annulled. This bumped Domracheva's thirteenth-place finish in race two into a twelfth, and her total score to 348. The number of first-place finishes is used as a tie-breaker, and in March 2019 the official results implied that Mäkäräinen will be stripped of the trophy in favor of Domracheva. Because the tenth-place competitor was disqualified for doping four years after the fact.

There is something dissatisfactory about this example: ideally, we would like the comparison of the two athletes to depend on their relative performance only but not on the performance or presence of some other athlete. This is captured by the following axiom.

A ranking rule $F : \pi \to \succ_\pi$ is **independent of irrelevant alternatives (IIA)** if for any pair of alternatives, their ranking $\succ_\pi$ depend only on voters' preferences over these two alternatives. Formally, for any $a \neq b$ and $\pi = (\succ_1, \dots, \succ_n)$ and $\pi' = (\succ'_1, \dots, \succ'_n)$

$$(\forall i\ a \succ_i b \Leftrightarrow a \succ'_i b) \Rightarrow (a \succ_\pi b \Leftrightarrow a \succ_{\pi'} b).$$

Of course, IIA can be easily satisfied, for example, by choosing $\succ_\pi$ independent of π. There is, however, one situation, where the question of ranking aggregation is unambiguous:

[1] We borrow this example from a fun-to-read paper (Kondratev et al., 2019) containing multiple examples of voting paradoxes emerging in athletes ranking and finding a subfamily of scoring rules that is the least paradoxical.

when athletes ranking in each competition is the same. A ranking rule F is said to be **unanimous** if for

$$\pi = (\succ, \ldots, \succ) \implies \succ_\pi = \succ .$$

Are there ranking rules satisfying both IIA and unanimity? Yes, but they do not really aggregate information as the ranking of athletes is determined by one given race. A rule F is said to be **dictatorial** if there exists $i \in N$ such that $\succ_\pi = \succ_i$ for any $\pi = (\succ_1, \ldots, \succ_n)$.

Theorem 2 (Kenneth Arrow 1951)**.** *If $|A| \geq 3$, then any unanimous F satisfying independence of irrelevant alternatives is a dictatorship.*

As we see, the paradox illustrated by the biathlon example was not specific to the particular scoring rule but is unavoidable.

There are many proofs of Arrow's theorem. In the class, we followed one of the shortest of them; its version can be found in Chapter 9.2 of Handbook of Algorithmic Game Theory.

There is another perspective on Arrow's theorem connecting it to Condorcet cycles. Assume that for each pair of alternatives, we want to decide on the pairwise comparison $a \succ_\pi b$ or $a \prec_\pi b$ as a function of the set of voters $\{i \in N : a \succ_i b\}$ preferring a to b. For example, we can use the majority rule but, as we already know, the resulting relation $\succ_\pi$ may exhibit Condorcet cycles for some preference profiles.

Theorem 3 (an equivalent form of Arrow's theorem)**.** *If $|A| \geq 3$ and the pairwise comparison $\succ_\pi$ of a and b is determined by $\{i \in N : a \succ_i b\}$ and is unanimous*[2]*, then there is π such that the relation $\succ_\pi$ has a cycle unless $\succ_\pi$ is dictatorial.*

A proof of Arrow's theorem by Gil Kalai relies on this reinterpretation. He computes the probability that $\succ_\pi$ exhibits a cycle on a random preference profile π. His elegant computation uses the Fourier analysis on the binary cube and shows that the probability of the cycle is non-zero unless the comparison depends on just one coordinate, i.e., is dictatorial.[3]

4.2 Manipulable rules

Arrow's theorem is related to an equally important impossibility result for voting rules, the Gibbard-Sattherwaite theorem.

Now we are back to considering **voting rules**, maps f assigning a winning alternative $f(\pi) \in A$ to each preference profile π. Recall that a voting rule f is said to be **strategy-proof (SP)** if no agent i with preferences $\succ_i$ can make her more preferable candidate win by pretending that her preferences are given by some $\succ_i'$. Formally, for any profile $(\succ_i)_{i\in N}$ of preferences and any agent i, there is no $\succ_i'$ such that

$$f\left(\succ_1, \ldots, \succ_{i-1}, \succ_i', \succ_{i+1}, \ldots, \succ_n\right) \succ_i f\left(\succ_1, \ldots, \succ_n\right).$$

[2] Whenever $\{i \in N : a \succ_i b\} = N$, we have $a \succ_\pi b$.

[3] You are not assumed to read this paper or have any prior knowledge of Fourier analysis for our course. However, I encourage you to learn it as it is a powerful technique appearing in various problems of probability, combinatorics, and algorithms.

What are the examples of SP voting rules? A rule selecting the same alternative a for any preference profile. What if we want a rule that non-trivially depends on voters' preferences? Then we can use the majority vote (or any other threshold rule) between any two predefined alternatives $\{a, b\} \subset A$. Unfortunately, this rule will not select an alternative $c \neq a, b$ even if all voters top rank it. Are there any SP rules that can select any alternative $c \in A$ depending on the preference profile? A non-satisfactory answer is a **dictatorial** f that selects voter i's top alternative for any profile π.

Theorem 4 (Gibbard (1973), Sattherwaite (1975))**.** *If f is strategy-proof and takes at least 3 different values, then f is dictatorial.*

The proof sketch that we discussed in the classroom was also from Chapter 9.2 of Handbook of Algorithmic Game Theory.

As we see, the manipulability of the plurality and the Borda rules are not their faults but an unavoidable consequence of being non-dictatorial. Can we escape this impossibility by allowing randomization, i.e., considering voting rules that output a probability distribution over A from which then the winner is sampled? As it was shown by Gibbard (1978), randomization alone does not open any new possibilities and any randomized strategy-proof rule can be decomposed as randomization over already familiar rules: those that take at most two values or dictatorships.

In the subsequent lectures, we will discuss workarounds for today's impossibilities.

Chapter 5

Escaping impossibilities: domain restriction

At the end of the previous lecture, we formulated the Gibbard-Sattherwaite theorem stating that there are no non-trivial non-manipulable rules for non-binary decisions. There are several ways to escape this impossibility result:

- *domain restriction:* Let's add some extra structure on preferences that the agents can have.
- *complexity-theoretic barriers:* Maybe a rule is manipulable but finding a profitable manipulation is nearly impossible as it requires unlimited computational resources or detailed knowledge of others' preferences.
- *typical behavior aka non-worst-case approach:* Maybe there are preference profiles where the rule behaves poorly — e.g., is manipulable — but they are rare.
- *quantitative relaxation of requirements:* Maybe a rule is manipulable but the manipulator cannot gain much by the manipulation.
- *randomization:* Let's allow for rules that sample the outcome at random with a distribution depending on the profile of preferences.

There is synergy between these ideas. For example, as we discussed, randomization alone does not help but it does if combined with a domain restriction.

In this lecture, we will see examples of how a domain restriction works. This is the main method to cope with impossibilities.

5.1 Domain restriction and utilities

Previously, we've assumed that each agent can have an *arbitrary strict* preference $\succ_i$ over the set of alternatives A. A domain restriction refers to altering one or both "arbitrary" and "strict".

A useful tool to describe a domain restriction is using utility functions. We say that a function $u\colon A \to \mathbb{R}$ is a utility function representing $\succ$ if and only if

$$a \succ b \Leftrightarrow u(a) > u(b).$$

We say that the two utility functions u and v are equivalent if there is a strictly monotone-increasing function $g\colon \mathbb{R} \to \mathbb{R}$ such that $v(a) = g(u(a))$ for all alternatives a. Equivalent utility functions define the same preferences and one can think of preferences as a class of equivalent utility functions.

Utility functions can represent not only strict preferences but also preferences with indifferences. For example, if $u(c) = 3$, $u(a) = u(b) = 1$, and $u(d) = 0$, the corresponding preferences are as follows:

$\succ$
c
$\{a, b\}$
d

in other words, the agent is indifferent between a and b, prefers c to both of them, and prefers both a and b to d. In what follows, we will write $e \succeq f$ to indicate that e is either strictly preferred to f or that the agent is indifferent between the two or that $e = f$.

To describe a domain restriction, we will specify the functional form of utilities.

5.2 Dichotomous domain

In the dichotomous domain of preferences, each agent's preference $\succ_i$ corresponds to a utility function $u_i\colon A \to \{0, 1\}$, i.e., each agent either approves or disapproves each of the alternatives and is indifferent between all those approved and all those disapproved.

The approval voting rule f selects an alternative approved by the maximal number of voters. It has the following properties:

- it treats all the voters and all the alternatives in a symmetric way (is anonymous and neutral)
- if all the voters have the same preferences and approve just one alternative, this alternative gets selected (a version of unanimity).

In particular, any alternative $a \in A$ can be selected for some preference profile.

Recall the definition of strategy-proofness. It remains unchanged even though now we allow for indifferences. A rule f is strategy-proof if for any profile $(\succ_i)_{i \in N}$ of preferences and any agent i, there is no $\succ_i'$ such that

$$f\left(\succ_1, \ldots, \succ_{i-1}, \succ_i', \succ_{i+1}, \ldots, \succ_n\right) \succ_i f\left(\succ_1, \ldots, \succ_n\right).$$

Proposition 1. *The approval voting rule is strategy-proof.*

Proof. Consider a profile $\pi = (\succ_i)_{i \in N}$ and a voter i. If the alternative $a = f(\pi)$ is approved by i, then i has no profitable manipulation $\succ_i'$ as she is already maximally happy (she gets an alternative from her top indifference class). If a is not approved by i, then by pretending that she likes alternatives that she actually disapproves or pretending that she dislikes alternatives that she actually approves, she can only make her disliked alternatives win. In other words, if the rule selects i's disapproved alternative a, i's manipulation can only result in another disapproved alternative a' being selected. □

In 2016, the specialists on computational social choice (COMSOC) conducted voting over voting rules to rank them by attractiveness. The approval voting won. By coincidence, the procedure used was a version of the approval voting that ranks the alternatives by the number of approvals. No causality is suspected though :-)

Note that the rules designed for a particular domain restriction can be used outside of this domain but may lose their good properties. In particular, approval voting can also be used if agents have strict preferences over alternatives A. Then, however, each voter faces a strategic decision of choosing a threshold such that all the alternatives above are approved and below are disapproved.

5.3 Single-peaked preferences

Assume that the set of alternatives A is a subset of the real line $\mathbb{R}$ (or the real line itself). Each agent i has an ideal point a_i on this line called i's peak and the closer an alternative is to i's peak, the more it is preferred by i. Denote by $>$ the natural ordering on the real line. Formally, we say that i has single-peaked preferences if there is an alternative a_i such that

$$x < y < a_i \Rightarrow x \prec_i y \prec_i a_i \qquad \text{and} \qquad a_i < x < y \Rightarrow a_i \succ_i x \succ_i y.$$

Equivalently, the utility function u_i has a peak at a_i and decreases to the left of a_i and to the right of it. The notion of single-peakedness restricts the structure of preferences on both sides of the peak but does not restrict how the alternatives to the left are compared to alternatives to the right.

Single-peaked preferences are natural if we think of candidates located on the conservative-liberal scale, decide where to locate a facility on the road, or choose the temperature in the room, tax rate, or production level of any other public good.

What would be a good mechanism for finding a compromise in a profile of single-peaked preferences? The first one that comes to mind is the mean: it uses the information about the peaks only and outputs

$$\frac{a_1 + a_2 + \ldots + a_n}{n}.$$

However, it is easy to see that the mean is manipulable. Indeed, if the mean equals m and $a_i > m$, then i can pretend that her peak is $a_i' = a_i + n(m - a_i)$ so that the median after the manipulation coincides with her true peak a_i.[1]

[1] Even if the data is not generated by strategic agents, the mean is sensitive to outliers in the data-generating process.

What can we use instead of the mean? The median! For a profile $(\succ_1, \dots, \succ_n)$, the median is the alternative $a = \text{median}(\succ_1, \dots, \succ_n)$ such that

$$|\{i\colon a_i \geq a\}| \geq \frac{n}{2} \qquad \text{and} \qquad |\{i\colon a_i \leq a\}| \geq \frac{n}{2}.$$

We assume that n is odd so that the median is uniquely defined.

Proposition 2. *The median rule is strategy-proof.*

Proof. Consider a voter with $a_i \neq \text{median}(\succ_1, \dots, \succ_n)$. If she reports a_i' instead of a_i and a_i' is on the same side of the median as a_i, then nothing changes. The only way i can affect the median is by choosing a_i' on the other side of the median. In this case, however, the median can only move further away from the true peak a_i. Hence, there are no profitable manipulations. □

Any k'th order statistic also gives a strategy-proof rule for a similar reason. The median corresponds to $k = \left[\frac{n+1}{2}\right]$. Chapter 10 of Handbook of Algorithmic Game Theory contains a characterization of even a more general class of strategy-proof rules called generalized median rules.

Even in non-strategic data, to make robust predictions, the rule of thumb is to use the median instead of the mean whenever possible.

5.4 House-allocation domain

This domain captures allocation of private goods without money transfers, e.g., public housing, charity, or organ transplants.

There is a set $H = (h_1, h_2, \dots, h_m)$ of "houses." For simplicity, we assume that m is bigger or equal to the number of agents n. Each agent i has strict preferences $\succ_i$ over H and is interested in receiving exactly one house.[2]

The set of alternatives A consists of injective maps $\mu\colon \{1, \dots, n\} \to H$ that specify a house received by each agent. We will refer to μ as allocation or matching. The preferences over allocations are defined as follows:

$$\mu \succ_i \mu' \Leftrightarrow \mu(i) \succ_i \mu'(i).$$

In other words, an agent cares only about her house and is indifferent to what other agents receive.

Serial Dictatorship (SD) aka the priority mechanism[3] is omnipresent in practice: There is a fixed order of agents σ (a permutation of $\{1, \dots, n\}$), agents come sequentially according to this order and pick their best house among available ones until every agent is allocated a house.

[2]The assumption of strict preferences is not very realistic and is imposed for simplicity. Usually, there are different types of housing and an agent is indifferent between houses of the same type.

[3]For resource allocation, rules are often called mechanisms or procedures.

This procedure defines a map $(\succ_1, \ldots, \succ_n) \to \mu$. Note that there is no need to implement this procedure in this dynamic way: agents can submit their preferences, then we run the serial dictatorship algorithm on these preferences and so determine who gets what.[4]

Proposition 3. *Serial Dictatorship is strategy-proof.*

Proof. Assume an agent i misreports her preference over houses. Then either the house that she gets remains the same, or at the time she picks the house from the set of available houses $H' \subset H$, she selects some other house instead of her most preferred one $\mu(i)$. We conclude that i cannot be better off by a manipulation. □

Another reason to like Serial Dictatorship is that it produces non-improvable outcomes. This is captured by the following notion of Pareto Optimality which is the central efficiency concept in economics and is applicable beyond the house allocation domain.

Definition 1. *For a profile of preferences (not necessarily strict) $(\succ_1, \ldots, \succ_n)$ over A, an alternative a is Pareto Optimal if there is no other alternative b such that $b \succeq_i a$ for all i and $b \succ_j a$ for some j.*

A rule is Pareto Optimal if it always selects a Pareto Optimal alternative.

In the context of house allocation, Pareto Optimality of μ means that we cannot reallocate houses so that no agent is harmed (either strictly prefers the new allocation or receives the same house) and some agent strictly prefers her new house.

Proposition 4. *Serial Dictatorship is Pareto Optimal.*[5]

Proof. The first agent to pick $i = \sigma(1)$ receives her best house $h = \mu(i)$ and so her allocation cannot be improved. Eliminate $(i, \mu(i))$ and repeat the argument. We conclude that without harming the allocation of agents $\sigma(1), \ldots, \sigma(t-1)$, the allocation of $\sigma(t)$ cannot be improved. Thus Serial Dictatorship is Pareto Optimal. □

5.5 Open problem: characterize random serial dictatorship (RSD)

Sometimes there is a natural priority ordering σ, say, by the application date for social housing or urgency of transplantation needed. But what should we do if there is no natural ordering and all the agents have equal rights for resources? Take σ uniformly at random! The outcome is known as Random Serial Dictatorship (RSD). Denote by $p_\mu(\succ_1, \ldots, \succ_n)$ the probability that an allocation μ is chosen by RSD.

Why is this a good idea? RSD is:

[4] This equivalence is a particular case of the general phenomenon called the revelation principle: any mechanism — no matter how complicated, e.g., dynamic — is equivalent to a "direct revelation mechanism", where agents submit their preferences and then the original mechanism is emulated inside a black box.

[5] The converse is also true. Namely, for any Pareto Optimal μ there is an ordering σ of agents producing μ. You can try proving it yourself (Hint: show that for any Pareto Optimal μ there is an agent receiving her most preferred house; she must be the first dictator) or check the paper by Abdulkadiroğlu and Sönmez (1998).

- Anonymous (agent-symmetric): treats all agents the same way.
- Efficient: only Pareto optimal allocations μ appear with non-zero probability, i.e., $p_\mu > 0 \Rightarrow \mu$ is Pareto Optimal.[6]
- Strategy-Proof:[7] For any profile of preferences $(\succ_1, \ldots, \succ_n)$, any agent i and utility function u_i representing her true preference $\succ_i$, there is no misreport $\succ_i'$ such that

$$\sum_\mu p_\mu(\succ_1, \ldots, \succ_i', \ldots, \succ_n) u_i(\mu(i)) \geq \sum_\mu p_\mu(\succ_1, \ldots, \succ_i, \ldots, \succ_n) u_i(\mu(i)).$$

  In other words, the expected utility from misreporting never exceeds that from staying truthful.

A long-standing high-stake easy-to-formulate hard-to-solve conjecture in economic design is to show that these properties characterize RSD. We say that two randomized mechanisms are equivalent if for each profile, each agent i, and each house h, the probability that $\mu(i) = h$ is the same under these two mechanisms.

Conjecture 1. *If $n = m$, then any randomized mechanism satisfying Anonymity, Efficiency, and Strategy-proofness is equivalent to RSD.*

For $n = 3$, this conjecture is proved in the influential paper by Bogomolnaia and Moulin (2001). If you find the conjecture interesting, check this thread on Twitter as well as several related threads by Nick Arnosti.

[6] This type of efficiency of randomized mechanisms is known as ex-post efficiency. It requires that the outcome is efficiency *after* the lottery has been realized.

[7] We need to adjust the familiar definition to allow for randomized outcomes. There are different ways to do that and the one we use is known as Stochastic-Dominance-Strategy-Proofness.

Chapter 6

Escaping impossibilities: complexity barriers

The Gibbard-Sattherwaite theorem demonstrates that any non-trivial voting rule defined on the full domain of strict preferences is manipulable. In the first part of this lecture we will discuss complexity-theoretic barriers to manipulation. The idea is that some rules are hard to manipulate just because finding a manipulation requires solving a computationally a hard problem.

In the second part of the lecture we will discuss an alternative non-axiomatic approach to design of voting rules based on maximal likelihood estimation.

6.1 Complexity of manipulations

The study of manipulation complexity was pioneered by Bartholdi et al. (1989). Their paper attracted the attention to computer scientists to the problems of social choice giving birth to the field of computational social choice or COMSOC. Our discussion is largely based on two sources: Chapter 6 of Handbook of Algorithmic Game Theory offers a technical survey of results and Faliszewski and Procaccia (2010)[1] give an informal introduction to manipulation complexity.

Today we consider the case of a finite number of alternatives $A = \{a, b, c, \ldots\}$ and n voters having a profile of strict preferences $\pi = (\succ_1, \ldots, \succ_n)$ over A. A voting rule is a map assigning a winning alternative to each π.

6.1.1 Hard-to-compute rules.

A rule can be hard to manipulate just because its outcome is hard to compute even if all agents are truthful.

We will need to measure how different the two preferences $\succ$ and $\succ'$ over A are. For this purpose, we use the Kendall tau distance between two permutations. In our context, it is

[1]The paper is available via the link.

defined as follows:

$$d_K(\succ, \succ') = \left|\{(a,b) \in A \times A \colon a \neq b, a \succ b, a \prec' b\}\right|,$$

i.e., the distance is the total number of pair of alternatives, where the two preferences disagree. An equivalent definition of d_K is the minimal number of flips of adjacent alternatives in the ranking $\succ$ needed to obtain $\succ'$ out of it. This number is equal to the number of steps made by the bubble-sort algorithm and, for this reason, d_K is sometimes called bubble-sort distance.

To define the Dodgson[2] voting rule on a profile $\pi = (\succ_1, \ldots, \succ_n)$, we compute the Dodgson score of an alternative a as the minimal sum of distances $\sum_{i=1}^n d_K(\succ_i, \succ_i')$ over profiles $\pi' = (\succ_1', \ldots, \succ_n')$ such that a is the unique Condorcet winner for π'. The Dodgson rule selects an alternative with the minimal score.

By the construction, the Dodgson rule is Condorcet consistent, i.e., the rule selects a Condorcet winner whenever it exists.[3]

Theorem 5 (Bartholdi et al. (1989)). *Computing the outcome of the Dodgson rule is NP-hard.*

In practice, NP-hardness means that there are instances of the problem (preference profiles) such that the solution cannot be found substantially faster than exhaustively checking the whole space of parameters.[4] The hardness, however, does not exclude easy-to-solve instances[5] and the presence of efficient heuristics finding/approximating the answer for all/most of all instances. One such approach is based on using integer linear programming solvers. We will illustrate it for the next rule we discuss.

Consider a ranking rule $\pi = (\succ_1, \ldots, \succ_n) \to \succ_\pi$ that outputs the ranking $\succ_\pi = \succ$ minimizing the sum of distances

$$\sum_{i=1}^n d_K(\succ_i, \succ) \to \min.$$

This rule is known as the Kemeny-Young rule.

The Kemeny-Young rule is often used in practice for ranking aggregation. The corresponding voting rule which outputs the top-ranked alternative is Condorcet consistent and never selects a Condorcet loser.

Theorem 6 (Bartholdi et al. (1989)). *Computing the outcome of the Kemeny-Young rule is NP-hard.*

This hardness makes the rule hard to manipulate but also hard to compute in problems with many alternatives. Note that the hardness result holds even if the number of agents

[2] A mathematician Charles Dodgson is also known as Lewis Carroll, the author of Alice's adventures in wonderland.

[3] The Dodgson rule may, however, select a Condorcet loser.

[4] For the Dodgson rule, this means that we need to compute the score for each of the alternatives and, to do that, we need to check all the possible profiles π' (a set of hyper-exponential size!).

[5] For example, profiles admitting a Condorcet winner.

$n = 4$ is fixed. Usually, the Kemeny-Young rule is computed via the following optimization problem.

Let x_{ab} be the indicator of $a \succ_\pi b$. A collection $(x_{ab})_{a,b\in A, a\neq b}$ corresponds to a valid ranking if and only if

$$x_{ab} \in \{0,1\}, \qquad x_{ab} + x_{ba} = 1, \qquad x_{ab} + x_{bc} + x_{ca} \leq 2.$$

The second condition means that the ranking is strict and the third one, that it is transitive. Finding the outcome of the Kemeny-Young rule boils down to minimizing

$$\sum_{i=1}^{n} d_K(\succ_i, \succ) = \sum_{a\neq b} x_{ab} \, |\{i \colon b \succ_i a\}|$$

over such collections $(x_{ab})_{a,b\in A, a\neq b}$. This problem can be fed to any commercial integer LP solver.

6.1.2 Hard-to-manipulate easy-to-compute rules.

Plurality, Borda, and, more generally, all the scoring rules are easy to compute, namely, there is a polynomial algorithm computing the outcome for any profile and this algorithm is given by the definition of the rule. Are these rules easy to manipulate? Let us first formalize this question.

Manipulation problem:
Input: a set of alternatives A, a voting rule f, preferences of all voters except for the first one $(\succ_2, \succ_3, \ldots, \succ_n)$, an alternative $a \in A$
Question: Does there exist $\succ_1$ such that $a = f(\succ_1, \ldots, \succ_n)$?

If f is computable in polynomial time, then this problem belongs to the class NP as the positive answer comes with a simple proof: the preferences $\succ_1$.

For Plurality, the manipulation problem has a trivial solution: select arbitrary $\succ_1$ with a at the top and check if a wins. For Borda and all the scoring rules, the following greedy algorithm solves the manipulation problem in polynomial time (Bartholdi et al., 1989):

- Fill the ranking $\succ_1$ sequentially, from top to bottom:
- Place a at the top.
- While there is an unranked alternative b
  - If there is b that can be placed next without preventing a from winning, place it next. Otherwise, return "no."

  Repeat.
- If all alternatives are ranked, return "yes".

There are only three voting rules that are known to be computable in polynomial time but the manipulation problem is hard for them (STV, Copeland, and Ranked Pairs). We will discuss STV and Copeland. The third one, Ranked Pairs relies on pairwise majority comparison similarly to Copeland; you can read about RP on Wikipedia.

The single transferable vote (STV) also known as Plurality with instant runoff works as follows:

- Find an alternative b with the minimal number of agents who top-rank it. Eliminate b.
- Repeat until only one alternative is left. This alternative is the winner.

STV is quite popular in practice and is used in political elections in Australia and New Zealand. It inherits flaws of Plurality: may not select a Condorcet winner and may select a loser, however, it has one clear advantage.

Theorem 7 (Bartholdi and Orlin (1991)). *The manipulation problem for STV is NP-hard.*

The Copeland score of an alternative $C(a)$ is equal to the total number of alternatives defeated by a in the majority comparison: $C(a) = |\{b \in A : a \succ_{maj} b\}|$. The Copeland rule selects the alternative with the highest score.

Theorem 8 (Bartholdi et al. (1989)). *The manipulation problem for Copeland is NP-hard.*

The Copeland rule is prone to ties and the hardness result is specific to a particular tie-breaking: if there are several alternatives with the maximal score, the one with the highest "second-order Copeland score" $C_2(a) = \sum_{b \prec_{maj} a} C(b)$ is selected.

This nuance highlights a weakness of the plain complexity-theoretic approach. Hardness is defined as the worst-case notion and the manipulation may be deemed hard because it is hard for a particular knife-edge class of instances: for example, in the case of the Copeland rule hardness comes from instances with lots of ties. There are results indicating that for manipulable rules even a random manipulation is likely to be successful; see (Faliszewski and Procaccia, 2010) for the modern debate.

6.2 Voting rules as instruments to discover ground truth

We have not discussed the origin of agents preferences $\succ_i$. Are they subjective opinions or noisy signals about some ground truth reflecting the underlying true ranking $\succ$? While the answer is not apparent in the context of political elections, it is apparent in crowdsourcing.

Think of a group of n workers on Amazon Mechanical Turk who are asked to perform the data labeling, say, are asked whether there is a cat in the photo $\theta = 1$ or not $\theta = 0$. Assume that each worker i gives a correct answer $a_i = \theta$ with probability $1 - \epsilon_i$ and mistakes are independent across workers.

Given the observed answers $(a_1, \ldots, a_n)$, the maximal likelihood estimate[6] for θ is obtained as follows. We compute the probability that $(a_1, \ldots, a_n)$ appear if $\theta = 1$

$$\mathbb{P}((a_1, \ldots, a_n) \mid \theta = 1) = \prod_{i: a_i = 1} (1 - \epsilon_i) \prod_{i: a_i = 0} \epsilon_i$$

and the same probability if $\theta = 0$

$$\mathbb{P}((a_1, \ldots, a_n) \mid \theta = 0) = \prod_{i: a_i = 1} \epsilon_i \prod_{i: a_i = 0} (1 - \epsilon_i).$$

Then the maximal likelihood estimate $\hat{\theta}$ corresponds to the highest of these two probabilities. Defining the weight of worker i by $w_i = \ln\left(\frac{1-\epsilon}{\epsilon_i}\right)$, we obtain that

$$\hat{\theta} = \begin{cases} 1, & \text{if } \sum_{i:\, a_i=1} w_i \geq \sum_{i:\, a_i=0} w_i \\ 0, & \text{if } \sum_{i:\, a_i=1} w_i \leq \sum_{i:\, a_i=0} w_i \end{cases}$$

In other words, the maximal likelihood estimator can be seen as the outcome of the weighted majority rule,[7] where the weight of a voter is equal to

$$\ln\left(\frac{1}{\text{probability of mistake}} - 1\right).$$

In practice, the probability of a mistake can be computed using historical data. One approach is to include the so-called "golden" questions where the dataset designer knows the answer. Alternatively, one can try to compute ϵ_i on a large dataset as a fixed point of the following process: initialize ϵ_i all equal; compute the maximal likelihood estimate; for each worker update ϵ_i to be the fraction of questions where she disagrees with the estimate; repeat until the process stabilizes.

Now consider a less elementary example of ranking aggregation. Again $\succ$ is the underlying true ranking and each voter's preference $\succ_i$ is a random perturbation of $\succ$. The commonly used model for random perturbation of a given ranking is the Mallows model:

$$\mathbb{P}(\succ_i | \succ) = C \cdot \lambda^{d_K(\succ_i, \succ)},$$

where $\lambda \in (0,1)$ is a fixed parameter, d_K is the Kendall tau distance, and C is a constant chosen so that the sum of probabilities is equal to 1.

Assuming that the perturbations are independent across agents, the probability to observe a profile $(\succ_1, \ldots, \succ_n)$ is equal to

$$\mathbb{P}\left((\succ_1, \ldots, \succ_n) \mid \succ\right) = C^n \cdot \lambda^{\sum_{i=1}^n d_K(\succ_i, \succ)}.$$

The maximal likelihood estimate for $\succ$ is the ordering $\hat{\succ}$ that minimizes $\sum_{i=1}^n d_K(\succ_i, \hat{\succ})$. We obtain another justification for using the Kemeny-Young rule.

[6] Recall that for a data-generating process depending on an unknown parameter $\theta \in \Theta$, the maximal likelihood estimate for θ is the value $\hat{\theta} \in \Theta$ that maximizes the probability $\mathbb{P}(\text{observed data} \mid \theta = \hat{\theta})$.

[7] We assume that the set of alternatives is $A = \{0, 1\}$ and the answer a_i of worker i is represented via the preference $a_i \succ_i 1 - a_i$.

Chapter 7

Quasilinear domain, introduction to auctions

In this lecture, we start discussing auctions, one of the first practical successes of the theory economic design. In our course, the treatment of this topic is close to that in lectures 13—16 of Incentives in Computer Science by Tim Roughgarden. The order and terminology is different though.

From a more general perspective, we will be talking about making collective decisions (e.g., deciding who gets the item) in the presence of monetary transfers. This is captured by arguably the most important domain restriction: *the quasilinear domain of preferences*. In this lecture, we will focus on a problem of allocating one good to n agents and will now describe the quasilinear domain in this case.

7.1 How to model selling 1 item to n agents?

We have an item — a good g — to allocate to one of n agents. Agents may have different valuations for the good. We denote by $v_i(g)$ the valuation of agent i, i.e., her maximal willingness to pay for this good. If i receives g and pays p_i, her utility is $v_i(g) - p_i$. Such utilities are called quasilinear as they are linear in money.

Let us put this into the familiar context of alternatives and preferences. An allocation is a function $\mu\colon \{1, \ldots, n\} \to \{g, \emptyset\}$ such that $|\{i : \mu_i = g\}| \leq 1$. The interpretation: $\mu_i = g$ means that agent i receives the good, $\mu_i = \emptyset$ means that i receives nothing, and there is at most one agent who gets the good. The set of alternatives is the set of all possible allocations accompanied by the vector of payments:

$$A = \{(\mu, p_1, \ldots, p_n)\}.$$

The preferences are defined using utility functions. The utility of agent i is equal to

$$u_i(\mu, p_1, \ldots, p_n) = v_i(\mu_i) - p_i.$$

We assume the normalization $v_i(\emptyset) = 0$. Agent i prefers one alternative to another if it gives her higher utility, i.e.,

$$(\mu, p_1, \ldots, p_n) \succ_i (\mu', p_1', \ldots, p_n')$$

if and only if

$$u_i(\mu, p_1, \ldots, p_n) > u_i(\mu', p_1', \ldots, p_n').$$

As we see, an agent cares only about her allocation (whether she receives a good or not) and her payment. Compare this construction to the house-allocation domain: it is similar but the new element is the presence of payments.

Note that the preference of i is captured by v_i. So we will identify a profile of preferences $\pi = (\succ_1, \ldots, \succ_n)$ and $(v_1, \ldots, v_n)$. A rule or a mechanism (the latter term is used more often in the context of resource allocation) is a map $\pi \to (\mu(\pi), p_1(\pi), \ldots, p_n(\pi))$. The standard definition of strategy-proofness rewrites as follows: there is no profile $(v_1, \ldots, v_n)$, agent i, and "misreport" v_i' such that

$$v_i(\mu_i(v_1, \ldots, v_{i-1}, v_i', v_{i+1}, \ldots, v_n)) - p_i(v_1, \ldots, v_{i-1}, v_i', v_{i+1}, \ldots, v_n) >$$

$$> v_i(\mu_i(v_1, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_n)) - p_i(v_1, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_n).$$

The case of $n = 1$ agent. How should we sell a good to one agent? Consider the following mechanism:

$$(\mu_1, p_1) = (g, v_1).$$

The interpretation is that we ask an agent how much she values the good, give her the good, and charge her the amount she told us. This mechanism is, clearly, not strategy-proof: for example, if true $v_1(g) = 1$, why not pretend that $v_1(g) = 0$ and get the good for free?

Consider the so-called posted price mechanism with price p:

$$(\mu_1, p_1) = \begin{cases} (g, p), & v_1(g) > p \\ (\emptyset, 0), & \text{otherwise} \end{cases}$$

This mechanism is omnipresent in practice: if agent's value exceeds the posted price p, the agent gets the item and pays the posted price. Otherwise, she gets nothing and pays nothing.

Proposition 5. *The posted price mechanism is strategy-proof.*

Proof. Assume the true value $v_1(g) > p$. Then the truthful agent gets the utility $u_1 = v_1(g) - p > 0$. Imagine she pretends that her value is $v_1(g)'$. The only way this can affect the outcome of the mechanism is if $v_1'(g) \le p$. Then she does not get the good and pays nothing so that her utility $u' = 0 - 0 = 0$. Thus a manipulation either changes nothing or decreases agent's utility. The case $v_1(g) \le p$ can be analyzed similarly. $\square$

Several agents. The posted price mechanism extends to $n \geq 1$ in a straightforward way. For example, we can look at all the agents whose values are above the posted price and allocate the good to the one with the lowest number:

$$(\mu_i, p_i) = \begin{cases} (g, p), & i = \min\{j : v_j(g) > p\} \\ (\emptyset, 0), & \text{otherwise} \end{cases}$$

Alternatively, we could select a winner among $\{j : v_j(g) > p\}$ at random or break the tie in any other way independent of the values reported by $\{j : v_j > p\}$. Mimicking the proof above we see that this mechanism is also strategy-proof.

This mechanism has another good feature. No truthful agent gets negative utility[1]; this property is called individual rationality and means that no agent will regret participating the mechanism. Posted price mechanism is also easy to implement. For all these reasons, it is a default choice to sell goods in practice.

Why do people need any other mechanisms such as auctions? There are several reasons:

- We may not know what price p to set. What if we sell something unique such as an art-piece or a bandwidth of 5G spectrum? These objects do not have a well-defined market price p as there is no free market where they are traded, e.g., they are of interest to just a few potential buyers. For this reason, we need a mechanism eliciting buyers' values.

- In some cases the market is so volatile and the number of goods to price is so big that we just cannot price them all (or if we do, this results in under/overpricing). This is the case for search engines such as Google selling advertisement slots for each possible search query.

- When a government sells resources to private sector (e.g., 5G spectrum or fishing quotas) the revenue may be of secondary importance or at least not the only objective. The government may aim to allocate resources efficiently, namely, to those who value them the most: to an agent with the highest $v_i(g)$. Posted price mechanism may fail to do so: it may misallocate the resource to an arbitrary agent who value it above the posted price, or fail to allocate the resource at all if the posted price is too high.

7.2 Auctions for 1 item and n bidders

Auctions are price-discovery mechanisms that help with all the above concerns.

When discussing auctions, we will refer to the profile of values $(v_1, \ldots, v_n)$ submitted by agents as "bids". The first-price auction allocates the item to the highest bidder and charges her bid:

$$(\mu_i, p_i) = \begin{cases} (g, v_i(g)), & v_i(g) > v_j(g)\ \forall j \neq i \\ (\emptyset, 0), & \text{otherwise} \end{cases}$$

[1] Check!

The interpretation is that each agent privately submits their bid to the auctioneer and the auctioneer decides who gets the good and how much agents pay. Such auctions are referred to as "sealed-bid" auctions.

The first-price auction generalizes the one-agent mechanism that we've started the lecture with. So it is not surprising that the first-price auction is not strategy-proof. Indeed, the winner i whose true value is v_i is better off pretending that her value is only slightly above the second bid $\max_{j\neq i} v_j(g)$, i.e., submitting $v_i'(g) = \max_{j\neq i} v_j(g) + \epsilon$, where ϵ is a small positive number.

Bidding below your true value is known as bid-shading. The optimal degree of shading depends on the information that the bidder knows about values and rationality of others. This makes bidding in the first-price auction a complex strategic decision and makes the outcome sensitive to bidders' beliefs about their competitors and thus unpredictable. We will come back to bidding in the first-price auction in subsequent lectures.

Consider the all-pay auction, where each agent pays her bid:

$$(\mu_i, p_i) = \begin{cases} (g, v_i(g)), & v_i(g) > v_j(g) \ \forall j \neq i \\ (\emptyset, v_i(g)), & \text{otherwise} \end{cases}$$

It is very unsafe to participate as an agent may end up paying money while getting nothing in return: a truthful agent may get negative utility and so the all-pay auction is not individually rational. As a result, all-pay auctions are even more prone to shading than the first-price ones. This auction format is not used in practice but is extremely important in theory, where it is used to model contests (e.g., competition between athletes or between teams in crowd-sourcing competitions such as the Netflix prize): the bid $v_i(g)$ is interpreted as the amount of effort that i decides to exert, the agent with the highest effort gets rewarded while others just waste their effort.

The second-price auction. The most important auction format is the second price auction:

$$(\mu_i, p_i) = \begin{cases} (g, \max_{j\neq i} v_j(g)), & v_i(g) > v_j(g) \ \forall j \neq i \\ (\emptyset, 0), & \text{otherwise} \end{cases}$$

As in the first-price auction, the good goes to the highest bidder, but she pays the second highest bid. It was first studied by William Vikrey (Nobel prize in economics 1996) and is often called the Vikrey auction. The second-price auction was used in practice before, e.g., the famous German writer Johann Wolfgang Goethe set up a second-price auction to determine how much publishers valued his manuscripts.[2]

[2] The second price auction is equivalent to another format that looks quite different. This is the so-called English auction, the format that first comes to mind when we hear the word "auction": bidders are all in one room shouting to increase their bids to overbid the current highest bidder. In other words, this is a dynamic process, where the bidders all observe the current maximal bid and have an option to increase it. When nobody wants to increase, the highest bidder is awarded the good and pays her bid. To win this auction, the highest-value agent i does not need to raise her bid to her value $v_i(g)$ but just to overbid the second highest-value bidder by a bit. As a result the good goes to the highest bidder for the price close of the second-highest value. A similar auction was the main selling format on early eBay.

Theorem 9 (Vikrey)**.** *The second-price auction is strategy-proof.*

The intuition is as follows: bidding slightly above the second-highest bid is the optimal bid-shading in the first-price auction. The price in the second-price auction already incorporates this optimal shading.

Strategy-proofness of the Vikrey auction makes is easy to participate. The bidders do not need to reason about the values of others: no matter what others do, posting the true value is the best for an agent.

Proof. Consider an agent i with the true value $v_i(g)$ who considers to report $v_i'(g)$ instead.

If $v_i(g) > v_j(g)$ for all $j \neq i$, then being truthful gives her strictly positive utility $u_i = v_i(g) - \max_{j\neq i} v_j(g) > 0$. The only way the misreport $v_i'(g)$ can affect the outcome is that i does not get the good anymore and so her utility is $u_i' = 0 - 0 = 0$. Thus this manipulation is not profitable.

If $v_i(g) \leq \max_{j\neq i} v_j(g)$, then truthful i does not get the good and $u_i = 0 - 0 = 0$. The manipulation can change the outcome only if $v_i'(g) > \max_{j\neq i} v_j(g)$. Then i gets the good and pays $\max_{j\neq i} v_j(g)$ and so her new utility is $u_i' = v_i(g) - \max_{j\neq i} v_j(g) \leq 0$.

To summarize: a manipulation either does not affect agent's utility or decreases it, i.e., the mechanism is strategy-proof. □

In addition to strategy-proofness, the Vikrey auction is individually rational. Indeed, the winners utility $u_i = v_i(g) - \max_{j\neq i} v_j(g) > 0$ and the loser's utility $u_j = 0 - 0 = 0$, so no truthful agent gets negative utility.

Let us stress how surprising is what we've seen:

- We've described a mechanism that extracts the information about their true values from strategic agents.
- It is efficient: always allocates the good to an agent who values it the most.
- Knowing nothing about agents' values, the revenue received by the auctioneer is close to the highest bidder's value.[3]

[3] We will see later that a revenue-seeking auctioneer having some statistical information about the bidders can slightly improve upon the second price auction but not by much.

Chapter 8

Auctions: sponsored search and VCG

In the previous lecture, we considered selling 1 good item to n agents with quasi-linear preferences and discussed the second-price auction that achieves a surprising combination of properties: incentivizes truth-telling, is efficient (allocates the good to an agent with the highest value), and is easy to implement.

In this lecture, we start discussing multi-item auctions. Such auctions play the central role in the Internet economy, where they are used to sell advertisement slots on web-pages.

8.1 Sponsored search and GSP auction

The canonical example is sponsored search: When you type the search query "health insurance" in Google, the page you see combines the organic search results with paid advertisements. The major part of Google's revenue comes from selling these slots.

How to price these slots? The difficulty is that we are selling as many goods as there are word combinations, some of them are highly demanded and some are not, moreover this demand may frequently change as new firms enter the market. Originally, the posted price mechanism was used, which led to over/under-pricing and overall congested market. In 2002, Google replaced the posted price mechanism that failed to aggregate rapidly-changing information by the second-price auction that is run every time you google something. Soon, instead of one advertisement slot for each query, Google started selling several slots and needed to generalize the second price option. They came up with the generalized second price auction or GSP that we are going to discuss.

Consider the following toy model of sponsored search (we follow the analysis of Varian (2007); Hal Varian is the chief economist at Google). For a given query, there are k advertisement slots $G = \{g_1, \ldots, g_k\}$ ordered according to the way they appear on the web page from top to bottom. The probability that a user clicks on a slot l is equal to $\alpha_l \in [0, 1]$. Users are more likely to click on top slots and so $\alpha_1 > \alpha_2 > \ldots > \alpha_k$.

There are n advertisers. Advertiser i attributes value β_i for being clicked. Hence, her value for a slot g_l is equal to

$$v_i(g_l) = \alpha_l \cdot \beta_i.$$

Hence, the profile of preferences can be identified with the vector $\pi = (\beta_1, \dots, \beta_n)$

An allocation μ is a function $\{1, \dots, n\} \to \{g_1, \dots, g_k, \emptyset\}$ such that $|\{i\colon \mu_i = g_l\}| \leq 1$ for all l. A mechanism maps a profile of preferences π to $(\mu(\pi), p_1(\pi), \dots, p_n(\pi))$. The utility of agent i is

$$u_i = v_i(\mu_i) - p_i$$

and, as usual, we assume that $v_i(\emptyset) = 0$.

Generalized second-price auction (GSP). In 2002, Google started using the following natural[1] generalization of the 1-item second-price auction in their advertisement product Google Adwords (later, renamed as Google Ads):

- Ask agents to submit their "bids" $\beta_1, \dots, \beta_n$.
- Agents are ordered by their bids. Without loss of generality $\beta_1 > \dots > \beta_n$.
- Each agent i is allocated the slot g_i and charged β_{i+1} if clicked (i.e., $p_i = \alpha_i \beta_{i+1}$ on average).[2]

Formally,

$$(\mu_i, p_i) = (g_i, \alpha_i \beta_{i+1}).$$

If there is just one slot with $\alpha_1 = 1$, this auction coincides with the Vikrey auction. Unfortunately, GSP turned out to be a wrong generalization of the Vikrey auction to several slots: it is not strategy-proof and prone to bid-shading. Indeed, consider two slots

$$\alpha_1 = 0.5, \qquad \alpha_2 = 0.25$$

and three advertisers with values

$$\beta_1 = 9, \qquad \beta_2 = 7, \qquad \beta_3 = 1.$$

If all are sincere, agent 1 gets the best slot and pays 7 with probability 0.5. So her utility is

$$u_1 = \alpha_1 \beta_1 - \alpha_1 \beta_2 = 1.$$

Now imagine that she bids $\beta_1' = 6$ instead of her true value. Then she gets the second slot and pays $\beta_3 = 1$ for a click. So her utility is

$$u_1' = \alpha_2 \beta_1 - \alpha_2 \beta_3 = 2.$$

Hence bid-shading allows our agent to get a lower click rate for so low price that this manipulation becomes profitable.

[1] Natural but flawed.

[2] Here and below we assume that there are enough slots and bidders so that all formulas make sense. We add dummy bidders with $\beta_j = 0$ or slots with $\alpha_m = 0$ if necessary.

Does there exist a strategy-proof mechanism, the right generalization of the Vikrey auction? The answer to this question is positive and is given by the VCG mechanism that we discuss below. The use of VCG for advertisement was pioneered by Facebook.

Has Google replaced GSP by VCG? No, and the reason is the trap of manipulable mechanisms. When agents are used to shading their bids, some of them will continue doing that by inertia, even if a mechanism is replaced by a strategy-proof one. As a result, the revenue in a short run will drop. The way Google resolved this issue is by launching a new product, Google Adsence, with a new interface, extended functionality (advertising on partners' websites in addition to sponsored search), and based on VCG. Sometimes it is easier to build a new product then to repair the old one.

8.2 VCG

VCG stands for the Vikrey-Clarke-Groves mechanism. This mechanism extends Vikrey's construction of the second-price auction to the general quasi-linear domain. We will discuss the general construction and adapt it to sponsored search in the next section.

Let Ω be the set of outcomes (e.g., different ways to allocate goods to agents). Agent i has value $v_i(\omega)$ for $\omega \in \Omega$. The set of alternatives is

$$A = \{(\omega, p_1, \ldots, p_n)\}.$$

A mechanism maps a profile $\pi = (v_1, \ldots, v_n)$ to $(\omega(\pi), p_1(\pi), \ldots, p_n(\pi))$. Agent i's utility is

$$u_i = v_i(\omega) - p_i.$$

For example, in the case of an auction, an outcome ω is an allocation μ and $v_i(\omega) = v_i(\mu_i)$.

Let us generalize the single-item Vikrey auction to this domain. The Vikrey auction is strategy-proof. Is there a strategy-proof mechanism in the general quasi-linear? Of course, for example, the one selecting the same outcome no matter what the preferences are. Perhaps, this is not what we are looking for. What are other attractive properties of the Vikrey auction that we want to keep? It is efficient, namely it allocates the good to a bidder with the highest value. The way to extend this efficiency requirement to the general domain is to require that the outcome $\omega = \omega^*(v_1, \ldots, v_n)$ chosen by a mechanism to maximize the sum of values

$$\omega^*: \quad \sum_{i=1}^n v_i(\omega) \to \max.$$

The sum of agents' values is called the social welfare.[3]

Are there strategy-proof welfare-maximizing mechanisms? This question is non-trivial. Let's try to construct one. The choice of $\omega = \omega^*$ determines[4] the outcome as a function of

[3]The social welfare objective does not include payments as the idea is to generate as much value as possible by giving the resources to those who value them the most. Then this value can be redistributed via payments. In HW2, you will explore the connection of welfare-maximization and Pareto optimality.

[4]As usual, in the interest of time, we do not discuss ties.

agents' preferences $(v_1, \ldots, v_n)$. So, to define the mechanism, it remains to come up with payments $(p_1, \ldots, p_n)$ so that the resulting mechanism is strategy-proof.

We want to come up with a payment rule generalizing the second-price payment of the Vikrey auction. But it is unclear what the second price in the general quasilinear domain is and who should pay it. To find the right generalization, we need a concept of an externality.

An externality of an agent i is the cost that the presence of this agent imposes on other agents:

$$\text{(optimal welfare of agents } j \neq i \text{ if } i \text{ was absent)} - \text{(welfare of } j \neq i \text{ if } i \text{ is present)}.$$

Formally, i's externality is equal to

$$\max_{\omega} \sum_{j \neq i} v_j(\omega) - \sum_{j \neq i} v_j(\omega^*).$$

For example, consider the Vikrey auction and compute the externality of an agent i. If i is not the winner, then the presence of i does not affect the allocation of the good and so the two sums cancel out and the externality is zero. If i is the winner, i.e., the highest bidder, then, without i, the good is allocated to the second-highest bidder and so

$$\max_{\omega} \sum_{j \neq i} v_j(g) = \max_{j \neq i} v_j(g)$$

while

$$\sum_{j \neq i} v_j(\omega^*) = 0$$

as none of the agents $j \neq i$ get the good when i is present. So the externality of the winner i is equal to the second bid $\max_{j \neq i} v_j(g)$. We see that in the Vikrey auction every agent pays her externality. This perspective allows us to extend the auction to the general quasilinear domain.

VCG mechanism operates as follows:

- Agents submit their "bids"[5] $(v_1, \ldots, v_n)$
- The outcome $\omega = \omega^*$ with maximal welfare

  $$\sum_{i=1}^{n} v_i(\omega)$$

  is chosen.
- Each agent i pays her externality

  $$p_i = \max_{\omega} \sum_{j \neq i} v_j(\omega) - \sum_{j \neq i} v_j(\omega^*).$$

[5] These bids are not numbers but functions specifying the value for each possible outcome $\omega \in \Omega$.

Theorem 10. *VCG is strategy-proof.*

Proof. If agent i is truthful, her utility is

$$u_i = v_i(\omega^*) - p_i = v_i(\omega^*) + \sum_{j \neq i} v_j(\omega^*) - \max_{\omega} \sum_{j \neq i} v_j(\omega) =$$

$$= \sum_{j=1}^{n} v_j(\omega^*) - \max_{\omega} \sum_{j \neq i} v_j(\omega).$$

If agent i reports v_i' instead of v_i, this may change the outcome of a mechanism to some ω'. As a result, i's utility becomes

$$u_i' = \sum_{j=1}^{n} v_j(\omega') - \max_{\omega} \sum_{j \neq i} v_j(\omega).$$

The second sum remains unchanged and the first sum can only decrease as $\omega = \omega^*$ maximizes welfare $\sum_{j=1}^{n} v_j(\omega)$ over all ω. We conclude that no manipulation can increase i's utility. □

Charging an agent her externality aligns individual incentives with social ones, i.e., with maximizing welfare. This was noted by an English economist Arthur Cecil Pigou in the early XX century, much before the birth of VCG.[6]

In addition to strategy-proofness, VCG is individually rational no outcome has negative value (e.g., if we auction good items and not bad ones such as tasks or other liabilities).

Proposition 6. *Provided that $v_i(\omega) \geq 0$ for all ω, no truthful agent gets negative utility in VCG.*

Proof. We already know that

$$u_i = \sum_{j=1}^{n} v_j(\omega^*) - \max_{\omega} \sum_{j \neq i} v_j(\omega).$$

Let ω' be the outcome maximizing the second sum. Then by the choice of ω^*, the first sum can only decrease if we replace ω^* by ω'. Therefore,

$$u_i \geq \sum_{j=1}^{n} v_j(\omega') - \sum_{j \neq i} v_j(\omega') = v_i(\omega') \geq 0.$$

Thus i gets a non-negative utility. □

[6]His idea now known as "Pigovian taxes" was that charging each citizen consuming a public good her externality incentivizes the socially-optimal level of consumption and gives a universal recipe to cope with socially-suboptimal over-consumption of public resources (the phenomenon called the tragedy of commons).

For example, each driver taking a congested road imposes an extra delay on other drivers (this is her externality) which is monotone with the overall congestion. Toll roads with a toll depending on current congestion reflect Pigou's idea.

8.3 Back to sponsored search

Let us see how VCG adapts to sponsored search.

For this purpose we need to determine what the welfare-maximizing way to allocate slots to agents is: We order slots by the click rates $\alpha_1 > \alpha_2 > \ldots$ and agents according to their bids $\beta_1 > \beta_2 > \ldots$ and allocate slot g_i to agent i, i.e., $\mu_i^* = g_i$.[7] As before, we add dummy zero slots or dummy zero agents if needed.

VCG for sponsored search is as follows:

- Ask agents to submit their “bids” $\beta_1, \ldots, \beta_n$.
- Agents are ordered by their bids. Without loss of generality $\beta_1 > \ldots > \beta_n$.
- Each agent i is allocated the slot $\mu_i^* = g_i$ and is charged

  $$\frac{1}{\alpha_i} \cdot (\text{externality of } i)$$

  if clicked.[8]

Let us compute the externality of i explicitly:

$$\begin{aligned}
\max_{\mu} \sum_{j \neq i} v_j(\mu_j) - \sum_{j \neq i} v_j(\mu_j^*) = \\
= (v_1(g_1) + \ldots + v_{i-1}(g_{i-1}) + v_{i+1}(g_i) + \ldots + v_n(g_{n-1})) - \\
- (v_1(g_1) + \ldots + v_{i-1}(g_{i-1}) + v_{i+1}(g_{i+1}) + \ldots + v_n(g_n)) = \\
= (\alpha_1\beta_1 + \ldots + \alpha_{i-1}\beta_{i-1} + \alpha_{i+1}\beta_i + \ldots + \alpha_{n-1}\beta_n) - \\
- (\alpha_1\beta_1 + \ldots + \alpha_{i-1}\beta_{i-1} + \alpha_{i+1}\beta_{i+1} + \ldots + \alpha_n\beta_n) = \\
= \sum_{j=i+1}^{n} (\alpha_{j-1} - \alpha_j)\beta_j.
\end{aligned}$$

Thus agent i pays the price

$$\frac{1}{\alpha_i} \sum_{j=i+1}^{n} (\alpha_{j-1} - \alpha_j)\beta_j$$

per click.

From the general properties of VCG, this mechanism is strategy-proof and individually rational. Another advantage of VCG over GSP is the flexibility and universality of this approach. It can be easily adapted to the case with different types of ads, different layout, and advertisers bidding for different types of events (clicks, scrolling, watching an embedded video and so on).

However, as we will discuss next time, using VCG in practice is not as straightforward as it may now seem.

[7] Exercise: show that this allocation indeed maximizes the welfare $\sum_{i=1}^{n} v_i(\mu_i)$.

[8] An agent is charged with probability α_i so that the expected payment is equal to the externality.

Chapter 9

Combinatorial auctions

In the previous lecture, we considered sponsored search auction which gives an example of an auction, where several good items (advertisement slots) are auctioned simultaneously. Although there were multiple goods, agent's preferences over them were simple: captured by just one number = the advertiser's value for receiving a click. In this lecture, we will discuss auctions where agents have complex preferences over multiple goods. Such auctions are called *combinatorial*.

The model is a natural extension of the one discussed before. There are k goods $G = \{g_1, \ldots, g_k\}$ for sale and n bidders. A subset of goods is called a bundle. Agent i attributes value $v_i(B)$ to each bundle $B \subset G$, where

$$v_i \colon 2^G \to \mathbb{R}_+.$$

An allocation μ is a function $\{1, \ldots, n\} \to 2^G$ such that $\mu_i \cap \mu_j = \emptyset$ for all $i \neq j$. A mechanism maps a profile of preferences $\pi = (v_1, \ldots, v_n)$ to $(\mu(\pi), p_1(\pi), \ldots, p_n(\pi))$. The utility of agent i is

$$u_i = v_i(\mu_i) - p_i.$$

The simplest case is when agent's value for a good does not depend on the presence of other goods in the bundle, i.e.,

$$v_i(C \cup \{g\}) = v_i(C) + v_i(\{g\})$$

for all C and g. Consequently,

$$v_i(B) = \sum_{g \in B} v_i(\{g\}).$$

Such preferences are called additive. In real-life problems, additivity is often violated. We say that two goods g and g' are complements for i if

$$v_i(\{g, g'\}) > v_i(\{g\}) + v_i(\{g'\})$$

and substitutes if

$$v_i(\{g, g'\}) \leq v_i(\{g\}) + v_i(\{g'\}).$$

Combinatorial auctions are those auctions where preferences exhibit complicated patterns of strong complementarity or substitutability. These definitions extend to two disjoint bundles B and B' in a straightforward way.

The first historical example of a combinatorial auction is the auction for takeoff and landing slots at various airports.[1] Agents are airlines. Takeoff and landing slots are strong complements as getting one without another is useless. On the other hand, the two close takeoff slots at the same airport are substitutes as a scheduled flight can use one or another but does not require both. Similar phenomena arise in auctions for railway routes.

The second and the most important example is spectrum auctions. In nineties, many countries auctioned GSM spectrum among telecom companies; similar auctions have been run for $1G, 2G, 3G, 4G, 5G, \ldots$ spectra. The goods are the rights to use a particular bandwidth over a particular region. Roughly, these rights over different regions are complements and over the same region are substitutes.

9.1 Why not VCG?

VCG mechanism discussed in the previous lecture is immediately applicable to combinatorial auctions and is strategy-proof, efficient, and individually-rational. It may seem to be a universal off-the-shelf recipe. However, until recently it has not been used in practice because of the following complexity impediments:

- **Preferences elicitation complexity:** Imagine we are selling $k = 20$ items.[2] Assuming that agents' value function v_i is generic, to run VCG, each agent has to report an impossible number $2^{|G|} = 2^{20}$ of values.

  Hence, to use VCG, we needs to restrict preferences that agents can report to those having a particular structure, both flexible enough to capture the variety of real agents' preferences and simple enough to make reporting possible. This boils down to designing a bidding interface or a "bidding language", a tricky practical design problem having no universal answers and requiring deep understanding of a particular market and agents' goals.[3] For this reason, VCG is by no means an off-the-shelf solution.

- **Winner determination complexity:** To use VCG, we must be able to compute an allocation $\mu = \mu^*$ maximizing welfare

$$\mu^*: \quad \sum_{i=1}^{n} v_i(\mu_i) \to \max .$$

  This problem is referred to as winner determination.

[1] Caltech researchers Grether et al. (1979) contributed to developing these first auctions. Their proposal was along the lines of the VCG mechanism.

[2] In a typical spectrum auction, there are hundreds of items.

[3] Bidding interface for advertisers on Facebook and Google Adsence are examples of how this problem can be resolved.

> Surprisingly, winner determination is computationally hard for seemingly innocent classes of value functions v_i (see an example below). As a result, in large or frequently run auctions such as those in advertising, one is forced to rely on heuristic algorithms that do not guarantee to find the exact solution but an approximation. This not only creates an extra algorithmic complication but also has a side effect: the approximate VCG may fail not only the requirement of efficiency, but also strategy-proofness and, most importantly, individual rationality.

To illustrate computational complexity, consider the so-called single-minded agents. An agent i is single-minded if she has a desired bundle $B_i \subset G$ such that $v_i(B) = \gamma_i > 0$ whenever B contains B_i and is zero, otherwise. So each agent's preferences are captured by the pair (B_i, γ_i).

Proposition 7. *Winner determination for single-minded agents is NP-hard.*

Proof. To prove hardness it is enough to show that, if we had a fast algorithm for winner determination, we would also get a fast algorithm for another problem that is known to be hard. As this benchmark problem, we use the NP-hard problem "weighted independent set": given a graph (V, E) and vertex weights $(\alpha_v)_{v\in V}$, find an independent set[4] with the maximal total weight.

Given (V, E) and $(\alpha_v)_{v\in V}$, we consider the following winner-determination problem with single-minded agents. We enumerate vertices V so that $\{1, \ldots, n\} \simeq V$, goods are edges $G \simeq E$, agent i's desired bundle B_i consists of all the edges incident to i, the intensity $\gamma_i = \alpha_i$. Winner-determination problem for this instance is equivalent to finding a weighted independent set of the maximal weight. Hence, winner determination is at least as hard as "weighted independent set." □

9.2 What if not VCG? Simultaneous ascending auctions.

If you need to sell k goods, you may try selling them separately, say, by running k sequential second-price auctions. This is, however, a notoriously bad idea if agents' preferences exhibit strong complementarity/substitutability.[5] Indeed, this sequential format is prone to the so-called exposure problem: an agent may end up with a useless bundle (e.g., a single-minded agent fails to get her desired set of goods) but pay for it. For this reason, selling goods separately makes participation extremely risky and an agent may regret participating (individual rationality is violated) and may result in inefficient allocation and unpredictable

[4] A set of vertices is independent if there are no edges between any two of them.

[5] Exercise: Check that if agents' preferences are additive, than VCG mechanism is, in fact, equivalent to k independent second-price auctions for each of the goods.

Note also that additive preferences are relatively easy for agents to report as they just need to come up with k (not 2^k) numbers. In particular, none of the two complexity barriers of VCG are present for additive preferences.

outcomes. Excessive risk makes agents cautious and results in excessive bid-shading and overall low revenue.

In practice, the standard recipe of selling several goods to agents with complex preferences is by running *simultaneous ascending auctions*: all the goods are sold simultaneously over a period of time, agents observe the current bid for each of the items, can increase it, and the highest bidder gets the good.

Throughout this dynamic process, bidders learn some information about their competitors' preferences by observing their previous bidding behavior, may reassess their chances to get the desired bundle and adapt the bidding strategy accordingly. Arguably, this procedure leads to more efficient allocation of resources and partially eliminates the exposure problem.

Again a simultaneous ascending auction cannot be seen as an off-the-shelf solution as the way it performs depends dramatically on nuances of its implementation. For example, each bidder has an incentive to learn about other bidders' preferences before bidding herself. To facilitate bidding, one needs to impose activity rules that do not allow a bidder to wait until the very last moment.[6] Another pitfall is that observability of bids facilitates tacit collusion: agent's first bids may signal which goods she is interested in and which she is ready to leave to competitors. Particularly, when the number of competitors is small and the market can be partitioned into bundles-substitutes, one per competitor, there is a high chance that the competitors will get their bundles for a minimal price.

All these nuances make practical auction design both science and art. Spectrum auctions run in different countries indicate how important these nuances are for success or failure of the auction. For stark historical examples, see lecture 8 of Algorithmic Game Theory by Tim Roughgarden. The insights on how to deal with pitfalls of simultaneous ascending auctions are discussed in this concise survey by Peter Cramton. The first successful spectrum auctions were designed in 1990ies by Preston McAfee[7] and Nobel laureates, Paul Milgrom and Robert Wilson. I recommend the insightful and easy-to-read book Discovering Prices by Paul Milgrom.

[6] When everybody has an incentive to wait until the very last moment, the outcome becomes unpredictable; many markets with a strict deadline suffer from such failure. A well-known example is sniping on eBay (Roth and Ockenfels, 2002).

[7] Preston McAfee was a professor at Caltech! Now he works at Google.

Chapter 10

Revenue maximization with one good to sell

In our previous discussion of auctions, the central role was played by efficiency (the resources must be allocated to those who value them the most), non-manipulability (no need to strategize and predict the behavior of others $\Rightarrow$ easy to participate), and individual rationality (nobody regrets participation $\Rightarrow$ safe to participate). These concerns are central if we think of a government allocating resources, but not for a seller whose main concern is revenue. In this lecture, we will discuss revenue maximization when selling one good item g to n agents. We will still impose individual rationality to ensure participation but will care less about strategy-proofness.

Recall the model. There is one good g and n agents. If i receives g and pays p_i, her utility is $v_i(g) - p_i$, where $v_i(g) \geq 0$. If i doesn't get the good, her utility is $v_i(\emptyset) - p_i$ with $v_i(\emptyset) = 0$. An allocation is a function $\mu\colon \{1, \ldots, n\} \to \{g, \emptyset\}$ such that $|\{i : \mu_i = g\}| \leq 1$. A mechanism is a map $\pi = (v_1, \ldots, v_n) \to (\mu(\pi), p_1(\pi), \ldots, p_n(\pi))$.

10.1 Optimal posted-price mechanism

Consider the case of $n = 1$ agent and the posted-price mechanism with price p:

$$(\mu_1, p_1) = \begin{cases} (g, p), & v_1(g) > p \\ (\emptyset, 0), & \text{otherwise} \end{cases}$$

Seller's revenue is p if $v_1(g) > p$ and 0, otherwise. What p should the seller choose? The answer depends on the information she has. If she happens to know the exact buyer's value (which is very unrealistic), then she can post $p = v_1(g) - \epsilon$ with some small $\epsilon > 0$ and extract the whole surplus. Another extreme is when the seller has no idea about how valuable the item is. Then it is unclear what p to select and it is better to run an auction to extract some information about agents' values.

The middle ground is a situation, where the seller knows the cumulative distribution function $F = F(x) = \mathbb{P}[v_1(g) \leq x]$ of the value $v_1(g)$ over the population of agents. We will assume that F has a density f and use the uniform distribution $U([0,1])$ as a running example.

The goal is to find p maximizing the expected revenue r

$$r = \mathbb{E}[p_1] = p \cdot \mathbb{P}[v_1(g) > x] = p\,(1 - F(p))\,.$$

The trade-off is between selling at a higher price and selling with higher probability. The first order conditions for the optimal price $p = p^*$ give[1]

$$p^* = \frac{1 - F(p^*)}{f(p^*)}.$$

A distribution F is called *regular* if $x - \frac{1-F(x)}{f(x)}$ is strictly increasing in x. For regular distributions the optimal price is unique.

For example, consider the uniform distribution $U([0,1])$. Then $F(x) = x$, $f(x) = 1$, and we obtain $p^* = 1 - p^*$. So the optimal price $p^* = \frac{1}{2}$ and the optimal revenue is $\frac{1}{4}$.

It turns out that one cannot improve upon the outcome of the posted price mechanism in the case of 1 bidder.

Theorem 11 (Myerson (1981)). *Posted-price mechanism with* $p^*\colon p(1 - F(p)) \to \max$ *achieves the highest revenue among strategy-proof individually-rational mechanisms.*[2]

Let us stress an interesting phenomenon. If a seller was interested in efficient allocation of resources, she would always give the good to the buyer by setting $p = 0$ and deriving zero revenue. To improve revenue, the seller needs to sacrifice efficiency. We will see this phenomenon in auctions as well.

10.2 Revenue for the standard auction formats

Now there are $n \geq 2$ bidders and we assume that the bidder's values $v_i(g)$ are independent identically distributed random variables with some distribution F having f. The interpretation is that each bidder knows the realization of her own value and believes that values of all others are distributed according to the distribution F (also known to the auctioneer).[3] The auctioneer's goal is to maximize the expected revenue

$$r = \mathbb{E}[p_1 + \ldots + p_n] \to \max$$

by choosing a format for the auction.

[1] The quantity $H(x) = \frac{f(x)}{1-F(x)}$ is called the hazard rate of a distribution F and is omnipresent in risk analysis. If F represents the distribution of time until the first failure of a machine, then $H(x)$ is a failure intensity at time x provided that the machine has not yet failed. We see that the optimal price equals the inverse hazard rate.

[2] Myerson proves a more general result allowing for randomization and lack of strategy-proofness (in this case, it is assumed that the agent chooses her optimal manipulation). In particular, Myerson's result captures a sequential process of offers and counteroffers.

[3] The assumption that the values are independent represents the case where values are determined by bidders' individual tastes and not by the objective item's characteristics in which case the values would be correlated.

10.2.1 The second price auction

We know that the second-price auction

$$(\mu_i, p_i) = \begin{cases} (g, \max_{j\neq i} v_j(g)), & v_i(g) > v_j(g)\ \forall j \neq i \\ (\emptyset, 0), & \text{otherwise} \end{cases}$$

is strategy-proof and so each agent bids her true value.

Let us compute the revenue assuming that there are $n = 2$ bidders with $v_i(g) \sim U([0,1])$. The payment to the auctioneer is equal to

$$r = \mathbb{E}[\min\{v_1(g), v_2(g)\}].$$

To compute it, recall how to compute the expected minimum and maximum of i.i.d. random variables $\xi_1, \dots, \xi_n$ with distribution F. For this purpose, let us compute the distribution functions $F_{\max}$ of $\max\{\xi_1, \dots, \xi_n\}$ and $F_{\min}$ of $\min\{\xi_1, \dots, \xi_n\}$:

$$F_{\max}(x) = \mathbb{P}[\max\{\xi_1, \dots, \xi_n\} \leq x] = \mathbb{P}[\xi_i \leq x\ \forall i] = \prod_i \mathbb{P}[\xi_i \leq x] = (F(x))^n .$$

$$\begin{aligned} F_{\min}(x) = \mathbb{P}[\min\{\xi_1, \dots, \xi_n\} \leq x] = 1 - \mathbb{P}[\min\{\xi_1, \dots, \xi_n\} > x] = \\ = 1 - \prod_i \mathbb{P}[\xi_i > x] = 1 - (1 - F(x))^n . \end{aligned}$$

By taking the derivatives, we get densities $f_{\max}$ and $f_{\min}$. Now we can compute the expectations:

$$\mathbb{E}[\max\{\xi_1, \dots, \xi_n\}] = \int_{\mathbb{R}} x \cdot f_{\max}(x)dx = n \cdot \int_{\mathbb{R}} x \cdot f(x)\,(F(x))^{n-1}\,dx$$

$$\mathbb{E}[\min\{\xi_1, \dots, \xi_n\}] = \int_{\mathbb{R}} x \cdot f_{\min}(x)dx = n \cdot \int_{\mathbb{R}} x \cdot f(x)\,(1 - F(x))^{n-1}\,dx$$

Coming back to the revenue of the second-price auction with two uniform bidders, we obtain

$$r = \mathbb{E}[\min\{v_1(g), v_2(g)\}] = 2\int_0^1 x(1-x)dx = \frac{1}{3}.$$

We see that running the second price auction with two bidders is more profitable than using the one-bidder posted price mechanism with the optimal price.

10.2.2 The first-price auction

Consider the first-price auction:

$$(\mu_i, p_i) = \begin{cases} (g, v_i(g)), & v_i(g) > v_j(g)\ \forall j \neq i \\ (\emptyset, 0), & \text{otherwise} \end{cases}$$

How to predict its revenue assuming that there are $n \geq 2$ bidders with i.i.d. values $v_i(g)$ distributed according to F?

Since the first-price auction is not strategy-proof, to answer this question we need to learn how to predict bidders' behavior. We expect that bidders are going to shade their values and agent i with value $v_i(g)$ will report some $v_i' = \beta(v_i(g))$; we will refer to v_i' as i's bid and to β, as a bid-function.

How can we predict what β will be used? One seemingly mild requirement is that this prediction must not be self-contradictory: if agent i learns our prediction that her competitors $j \neq i$ are going to use a bid function β (or just comes up with this prediction herself), it is still in her best interest to bid $v_i' = \beta(v_i(g))$. It turns out that this lack of self-contradiction often pins down unique β.

Definition 2. *We call β an equilibrium bid function*[4] *if for any agent i and any realization of her value $v_i(g) = x$, her expected utility is maximized if she bids $v_i' = \beta(x)$ assuming that all other agents j bid $v_j' = \beta(v_j(g))$.*

Formally, the expected utility of agent i with $v_i(g) = x$ posting a bid b and assuming that others post $v_j' = \beta(v_j(g))$ is equal to

$$\mathbb{E}\left[u_i \mid v_i(g) = x\right] =$$

$$= x \cdot \mathbb{P}\left[\mu_i(v_1', \ldots, v_{i-1}', b, v_{i+1}', \ldots, v_n') = g\right] - \mathbb{E}\left[p_i(v_1', \ldots, v_{i-1}', b, v_{i+1}', \ldots, v_n')\right]$$

and must be maximized by $b = \beta(x)$ for any $x \geq 0$.

Let's find equilibrium β for the first-price auction. Expected utility of an agent i with value x and bid b is equal to

$$x \cdot \mathbb{P}\left[b > \beta(v_j)\ \forall j \neq i\right] - b \cdot \mathbb{P}\left[b > \beta(v_j)\ \forall j \neq i\right]. \tag{10.1}$$

This expression must be maximized by $b = \beta(x)$. In particular, choosing $b = \beta(y)$ for some $y \neq x$ can only lower the expected utility. Hence,

$$x \cdot \mathbb{P}\left[\beta(y) > \beta(v_j)\ \forall j \neq i\right] - \beta(y) \cdot \mathbb{P}\left[\beta(y) > \beta(v_j)\ \forall j \neq i\right]$$

must be maximized over y at $y = x$.

Let us guess β imposing some natural extra assumptions: β is strictly increasing (the higher the value, the higher the bid), smooth, and $\beta(0) = 0$. By monotonicity,

$$\mathbb{P}\left[\beta(y) > \beta(v_j)\ \forall j \neq i\right] = \mathbb{P}\left[y > v_j\ \forall j \neq i\right] = \left(F(y)\right)^{n-1}.$$

Thus the maximum of

$$x \cdot \left(F(y)\right)^{n-1} - \beta(y)\left(F(y)\right)^{n-1}$$

over y must be attained at $y = x$. Hence, the derivative with respect to y

$$x \cdot f'(y)(n-1)\left(F(y)\right)^{n-2} - \left(\beta(y)\left(F(y)\right)^{n-1}\right)_y'$$

[4] Readers familiar with game theory will recognize that this is just a definition of a symmetric Bayes-Nash equilibrium.

must be zero at $y = x$. Therefore,

$$\left(\beta(x)\left(F(x)\right)^{n-1}\right)'_x = x \cdot f'(x)(n-1)\left(F(x)\right)^{n-2}.$$

Integrating from 0 to x we get

$$\beta(x)\left(F(x)\right)^{n-1} = \int_0^x t \cdot f'(t)(n-1)\left(F(t)\right)^{n-2}\, dt$$

and thus[5]

$$\beta(x) = \frac{n-1}{\left(F(x)\right)^{n-1}} \int_0^x t \cdot f'(t)\left(F(t)\right)^{n-2}\, dt.$$

Consider the case of the uniform distribution $U([0,1])$. We get

$$\beta(x) = \frac{n-1}{x^{n-1}} \int_0^x t^{n-1} dt = \frac{n-1}{n} \cdot x.$$

We see that in the first-price auction with two bidders each should post half of her true value! We also see that agents shade their bids less when the competition gets tougher and shading disappears in the limit $n \to \infty$. Let us compute the revenue

$$r = \mathbb{E}[p_1 + \ldots + p_n] = \mathbb{E}\left[\max\{v'_1, \ldots, v'_n\}\right] = \frac{n-1}{n}\mathbb{E}\left[\max\{v_1, \ldots, v_n\}\right] =$$

$$= \frac{n-1}{n}\left(n\int_0^1 t \cdot t^{n-1} dt\right) = \frac{n-1}{n+1}.$$

Surprisingly, for two bidders we obtain the same revenue of $\frac{1}{3}$ as in the case of the second-price auction.

10.3 The all-pay auction

Let us derive an equilibrium bid function for the all-pay auction with n bidders:

$$(\mu_i, p_i) = \begin{cases} (g, v_i(g)), & v_i(g) > v_j(g)\ \forall j \neq i \\ (\emptyset, v_i(g)), & \text{otherwise} \end{cases}$$

The expected utility of an agent i with value x and bid b is equal to

$$x \cdot \mathbb{P}\left[b > \beta(v_j)\ \forall j \neq i\right] - b.$$

Substituting $b = \beta(y)$ and using monotonicity of β, we obtain that

$$x \cdot \left(F(y)\right)^{n-1} - \beta(y)$$

[5] Now, when the explicit expression for β is derived, we should have come back to (10.1) and check that placing $b = \beta(x)$ is indeed the best choice for i (this must be checked as the first order conditions that we used are only necessary for the optimum but may not be sufficient) but we will not do that...

attains its maximum over y at $y = x$. Hence,

$$x \cdot f(y)(n-1)\left(F(y)\right)^{n-2} - \beta'(y)$$

must be zero at $y = x$. We get

$$\beta'(x) = x \cdot f(x)(n-1)\left(F(x)\right)^{n-2}$$

and

$$\beta(x) = (n-1)\int_0^x t \cdot f(t)\left(F(t)\right)^{n-2}\, dt.$$

Consider the uniform distribution $U([0,1])$. Then

$$\beta(x) = (n-1)\int_0^x t \cdot t^{n-2}\, dt = \frac{n-1}{n}x^n.$$

We see that, in contrast to the first-price auction, the tougher the competition is the more substantial is the shading. What is the revenue? It is easy to compute as every agent pays her bid:

$$r = \mathbb{E}[p_1 + \ldots + p_n] = \mathbb{E}\left[\frac{n-1}{n}(v_1(g))^n + \ldots + \frac{n-1}{n}(v_n(g))^n\right] = n \cdot \int_0^1 \frac{n-1}{n}x^n dx = \frac{n-1}{n+1}.$$

We get the same answer as for the revenue of the first-price auction (and the second-price auction for $n = 2$). Is it a coincidence?

10.4 Revenue-equivalence theorem

Previously we defined efficiency only for strategy-proof mechanisms. For example, the second price auction is efficient in the sense that the item gets allocated to the agent with the highest value. We can extend this notion to non-strategy-proof auctions as follows. A mechanism is efficient if there exists an equilibrium bid function β such that the item gets allocated to the highest-value agent provided that all agents bid $v_i' = \beta(v_i)$.

For example, both the first-price and the all-pay auctions are efficient as β is strictly increasing and the good is awarded to the highest bidder.

Theorem 12 (Myerson (1981))**.** *Consider n bidders with independent values $v_i(g) \sim F_i$. Then all efficient mechanisms such that the expected payment of an agent with $v_i(g) = 0$ is zero*[6] *have the same revenue.*

As a corollary of the revenue-equivalence theorem, we conclude that the second-price auction has revenue $\frac{n-1}{n+1}$ for any number of bidders n and the uniform distribution $U([0,1])$. Is this good or bad revenue? A natural benchmark is provided by a seller who knows each

[6]This condition rules out mechanisms charging each bidder \$1000000 no matter what her value is.

bidder's value and so can allocate the good to the highest-value bidder and charge her exactly her value. Such full surplus extraction gives

$$r = \mathbb{E}[\max\{v_1, \dots, v_n\}] = n \int_0^1 t \cdot t^{n-1} dt = \frac{n}{n+1}.$$

Hence, the seller not knowing the agents' values loses only

$$\frac{n}{n+1} - \frac{n-1}{n+1} = \frac{1}{n+1},$$

i.e., $O(\frac{1}{n})$ fraction of the revenue of the omniscient seller.

We see that auctions with large number of competitors do a good job of surplus extraction. Can a seller do better?

10.5 Revenue-maximizing auctions

As suggested by the revenue-equivalence theorem, the only way to improve revenue is to sacrifice efficiency. This should not be surprising as we already saw this phenomenon in the case of the posted-price mechanism.

Consider the second price auction with a reserve price p, which can be seen as a hybrid of the posted price mechanism and the second price auction:

$$(\mu_i, p_i) = \begin{cases} (g, \max_{j \neq i} v_j(g)), & v_i(g) > \max_{j \neq i} v_j(g) > p \\ (g, p), & v_i(g) > p > \max_{j \neq i} v_j(g) \\ (\emptyset, 0), & \text{otherwise} \end{cases}$$

So the auctioneer retain the good if all the bids are below p. This mechanism is strategy-proof and individually-rational as it can be thought as the standard second-price auction with an auxiliary bidder $i = 0$ who always bids p (and never pays).

Let us compute its revenue for $n = 2$ agents and $v_i(g) \sim U([0, 1])$:

$$r = \mathbb{E}\left[\min\{v_1, v_2\} 1_{\min\{v_1, v_2\} > p}\right] + \mathbb{E}\left[p \cdot 1_{v_1 > p > v_2} + p \cdot 1_{v_2 > p > v_1}\right].$$

Recall that $\min\{v_1, v_2\}$ is distributed with the density $f(x) = 2(1 - x)$. Thus

$$r = \int_p^1 x \cdot 2(1 - x) dx + 2p(p(1 - p)) = \frac{1}{3} + p^2 - \frac{4}{3} p^3.$$

By taking maximum over p we obtain $p^* = \frac{1}{2}$ (the familiar optimal posted price!) and revenue

$$r = \frac{5}{12},$$

which exceeds $\frac{1}{3} = \frac{4}{12}$ obtained without a reserve price.

Theorem 13 (Myerson (1981)). *Consider n bidders with independent identically distributed values $v_i(g) \sim F$ with a regular*[7] *distribution F. The optimal revenue over all individually-rational strategy-proof mechanisms*[8] *is achieved by the second-price auction with the reserve price $p = p^*$ equal to the optimal price in the posted-price mechanism.*

10.6 Prior-free guarantees

Note that p^* in the revenue-maximizing auction does not depend on the number of agents but does depend on the distribution F. How much revenue does the seller lose if she is not aware of F and runs the standard second-price auction (with zero reserve)? These type of questions is a subject of prior-free auction design literature. Here is a canonical result.

Theorem 14 (Bulow and Klemperer (1994)). *Consider bidders with independent identically distributed values $v_i(g) \sim F$ for a regular distribution F. The optimal revenue for n bidders does not exceed the revenue of the second-price auction (with zero reserve) for $n+1$ bidders.*

In other words, attracting one more bidder is more important than tailoring the optimal reserve price. The intuition behind the theorem is that the second-price auction with reserve p^* for n bidders can be thought as an $(n+1)$-bidder auction with an auxiliary bidder who bids p^* and never pays if wins. It turns out that replacing this auxiliary bidder by a real bidder can only benefit the auctioneer.

[7] See the discussion of regularity for posted-price mechanisms.

[8] The conclusion remains the same if we drop the condition of strategy-proofness and individual rationality and assume instead that each agent uses an equilibrium bid function β and no agent i with $v_i(g) = x$ gets a negative expected utility no matter what x is.

Chapter 11

Revenue maximization with multiple goods

In contrast to the case of one item considered in the previous lecture, revenue maximization with multiple items is a much less tractable problem. We will focus on the benchmark case of one buyer often referred to as the "monopolist's problem." Even for $k = 2$ items, no explicit answers are known except for particular examples. This lecture is inspired by the survey Daskalakis (2015).

The model is as follows. There is a set $G = \{g_1, \ldots, g_k\}$ of k goods and $n = 1$ buyer. An allocation is captured by a bundle of goods $\mu \subset G$ allocated to the buyer. The set of alternatives consists of bundle-payment pairs $A = \{(\mu, p)\}$, where $p \in \mathbb{R}$.

The buyer's value for a bundle μ is additive in goods entering the bundle[1]

$$v(\mu) = \sum_{g \in \mu} v(g)$$

and the utility is quasilinear in the payment

$$u((\mu, p)) = v(\mu) - p.$$

A mechanism maps the preferences captured by the vector $v = (v(g_1), \ldots, v(g_k)) \in \mathbb{R}_+^G$ to an "alternative" $(\mu(v), p(v))$.

11.1 Menu mechanisms

A menu is a list of bundle-payment pairs offered to the buyer so that she selects the most preferred option from the menu. Formally, a menu $\mathcal{M}$ is a subset of $2^G \times \mathbb{R}$.

Definition 3. *A mechanism $(\mu(v), p(v))$ is a menu mechanism if there exists a menu $\mathcal{M}$ such that for any v the mechanism's outcome $(\mu(v), p(v))$ maximizes the utility $v(\mu) - p$ over $(\mu, p) \in \mathcal{M}$.*

[1] We focus on this benchmark case, where agent's preferences exhibit neither complementarity nor substitutability.

Proposition 8. *A mechanism $(\mu(v), p(v))$ is strategy-proof* $\Longleftrightarrow$ $(\mu(v), p(v))$ *is a menu mechanism.*

Proof. Assume that $(\mu(v), p(v))$ is a menu mechanism with menu $\mathcal{M}$ and show that it is strategy-proof. In other words, we need to show that

$$v(\mu(v)) - p(v) \geq v(\mu(v')) - p(v')$$

for any true values v and misreport v'. This inequality holds since the pair $(\mu(v'), p(v'))$ belongs to the menu $\mathcal{M}$ and the pair $(\mu(v), p(v))$ maximizes the utility $v(\mu) - p$ over all $(\mu, p) \in \mathcal{M}$.

For the opposite direction, we start from a strategy-proof mechanism $(\mu(v), p(v))$ and show that it is induced by some menu. Define $\mathcal{M}$ as follows

$$\mathcal{M} = \{(\mu(v), p(v)) \colon v \in \mathbb{R}_+^G\}.$$

By strategy-proofness,

$$v(\mu(v)) - p(v) \geq v(\mu(v')) - p(v')$$

for any v'. Hence,

$$v(\mu(v)) - p(v) \geq v(\mu) - p$$

for any $(\mu, p) \in \mathcal{M}$, i.e., $(\mu(v), p(v))$ maximizes the utility over the menu. $\square$

Proposition 9. *A menu mechanism $(\mu(v), p(v))$ is individually rational if and only if it can be represented*[2] *via a menu $\mathcal{M}$ that contains $(\emptyset, 0)$.*

Proof. Assume that $(\emptyset, 0) \in \mathcal{M}$ and show individual rationality. As $(\mu(v), p(v))$ is a menu mechanism, the pair $(\mu(v), p(v))$ must give the utility at least as high as any other element of $\mathcal{M}$. Hence,

$$v(\mu(v)) - p(v) \geq v(\emptyset) - 0 = 0,$$

i.e., the mechanism is individually rational.

Let us now show that if $(\mu(v), p(v))$ is individually rational with menu $\mathcal{M}$, then it can be also represented by $\mathcal{M}' = \mathcal{M} \cup \{(\emptyset, 0)\}$. By individual rationality,

$$v(\mu(v)) - p(v) \geq 0 = v(\emptyset) - 0.$$

Since $(\mu(v), p(v))$ maximized the utility over the menu $\mathcal{M}$, we conclude that it maximizes it over $\mathcal{M}'$ as well, i.e., both $\mathcal{M}$ and $\mathcal{M}'$ represent $(\mu(v), p(v))$. $\square$

[2]Different menus may represent the same mechanism as some menu items may be redundant and never selected.

11.2 Revenue maximization

Assume that the vectors of values $v = (v(g_1), \ldots, v(g_k))$ are distributed over the population of potential buyers according to some distribution F on $\mathbb{R}_+^G$ known to the seller. What is a mechanism maximizing the seller's expected revenue $r = \mathbb{E}[p(v)]$ over strategy-proof individually-rational mechanisms?

We know that any such mechanism can be represented via a menu $\mathcal{M}$ containing $(\emptyset, 0)$. Hence, revenue-maximization boils down to designing an optimal menu.

Let us simplify the problem further and assume that the components of the vector v are independent, i.e., the distribution function $F(x_1, \ldots, x_k) = F_1(x_1) \cdot \ldots \cdot F_k(x_k)$. In other words, the buyers value for one good tells nothing about her value for another. Intuitively, in this case, it must be optimal to sell the goods separately using the optimal posted price mechanism for each of them. Surprisingly, sometimes the seller can do better.

Example 5. Assume there are $k = 2$ goods, the buyer's values $v(g_1), v(g_2)$ are independent and are equal 1 or 2 equally likely. If the goods are sold separately, the optimal price for each is $p^* = 1$ ($p^* = 2$ is also optimal) and the total revenue is 2. The corresponding menu is as follows:

$$\mathcal{M} = \left\{ \begin{array}{c} (\{g_1\}, p^*), \\ (\{g_2\}, p^*), \\ (\{g_1, g_2\}, 2p^*), \\ (\emptyset, 0) \end{array} \right\}.$$

What if instead we bundle the goods together and sell this grand bundle $g = \{g_1, g_2\}$ using the optimal posted price mechanism for this auxiliary good g? The value $v(g)$ equals $2, 3, 4$ with probabilities $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$. Hence, the optimal price is $q^* = 3$ and the revenue is $3 \cdot \frac{3}{4} = \frac{9}{4} > 2$. The menu is

$$\mathcal{M} = \left\{ \begin{array}{c} (\{g_1, g_2\}, q^*), \\ (\emptyset, 0) \end{array} \right\}.$$

We conclude that bundling increases the revenue in this example.

Is bundling always optimal?

Example 6. Consider a minor modification of the previous example so that $v(g_1), v(g_2)$ are equal 0 or 1 equally likely (instead of 1 and 2). Then the optimal price for selling separately is $p^* = 1$ and the total revenue is 1.

The value for the grand bundle $g = \{g_1, g_2\}$ is equal $0, 1, 2$ with probabilities $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$. Hence, the optimal price for the bundle is $q^* = 1$ which gives revenue of $\frac{3}{4}$. Selling separately is more profitable than bundling.

Is either selling separately or bundling always optimal? The next example shows that sometimes the revenue can be improved by offering the two goods separately together with the grand bundle for a discounted price.

Example 7. We are again selling two goods, $v(g_1)$ and $v(g_2)$ are independent but now they can take three different values: 0, 1, and 3 equally likely. One can check that selling separately and selling the grand bundle result in the same revenue of $\frac{4}{3}$. The menu mechanism with the menu

$$\mathcal{M} = \left\{ \begin{array}{c} (\{g_1\}, 2), \\ (\{g_2\}, 2), \\ (\{g_1, g_2\}, 3), \\ (\emptyset, 0) \end{array} \right\}$$

improves the revenue to $\frac{13}{9}$.

What about continuous distributions?

Example 8. Consider two goods and independent $v(g_1), v(g_2)$ uniformly distributed on $[0, 1]$. The optimal menu was found by Manelli and Vincent (2006):

$$\mathcal{M} = \left\{ \begin{array}{c} (\{g_1\}, \frac{2}{3}), \\ (\{g_2\}, \frac{2}{3}), \\ (\{g_1, g_2\}, \frac{4-\sqrt{2}}{3}), \\ (\emptyset, 0) \end{array} \right\}.$$

These examples convince us that to sell two goods with independent values, we need to price each good separately and, perhaps, offer the bundle for a discounted price. So revenue-maximization seems to reduce to a 3-dimensional optimization problem which looks quite tractable. This is true if we focus on *deterministic* mechanisms but, surprisingly, one can increase revenue by pricing lotteries.[3] We discuss this phenomenon in the next section.

11.3 Randomized mechanisms

Let us extend the model by allowing fractional bundles $\mu \in [0, 1]^G$ and interpreting $\mu(g)$ as the probability to receive the good[4] g. The buyer's utility is

$$u_i((\mu, p)) = \sum_{g \in G} \mu(g) v(g) - p.$$

A randomized mechanism maps the vector of values v to $(\mu(v), p(v)) \in [0, 1]^G \times \mathbb{R}$.

The deterministic model considered above corresponds to $\mu \in \{0, 1\}^G$. Defining a (randomized) menu as a subset $\mathcal{M} \subset [0, 1]^G \times \mathbb{R}$, one can extend Propositions 8 and 9 to randomized mechanisms in a straightforward way.

The next example shows that randomization may improve revenue.

Example 9. There are two goods and the values $v(g_1), v(g_2)$ are independent but not identically distributed: $v(g_1)$ equals 1 or 2 equally likely and $v(g_2)$ equals 1 or 3 equally likely.

[3]The result of Manelli and Vincent (2006) claims optimality of the mechanism from Example 7 even if lotteries are allowed. This is a highly non-trivial result.

[4]Alternatively, if g is divisible, one can think of $\mu(g)$ as the amount of g allocated.

The optimal deterministic mechanism is selling the two goods separately. The optimal price for the first good is 1 and 3 for the second so that the revenue equals $\frac{5}{2}$.

Consider the following randomized menu:

$$\mathcal{M} = \left\{ \begin{array}{c} ((1,1),\, 4-2\epsilon), \\ ((1,0.5),\, 2.5-\epsilon), \\ ((0,0),\, 0) \end{array} \right\},$$

where ϵ is a negligibly small positive number. What option will the buyer select depending on her values v? If $v = (1,1)$, she chooses the last option; if $v = (2,1)$, she picks the lottery, where the second good is allocated with probability 0.5 (the second option); in the two remaining cases, she buys both goods (the first option). The resulting revenue of $\frac{21}{8}$ exceeds that of the optimal deterministic mechanism.

How dramatic is the revenue loss if we stick to deterministic mechanisms? Counter-intuitively, this gap can be large.

Theorem 15 (S.Hart and N.Nisan (2013))**.** *For any number N, there exists a distribution $F(x_1, x_2) = H(x_1)H(x_2)$ of values for two goods such that the optimal revenue of a randomized mechanism is at least N times higher than that of the optimal deterministic.*

Another phenomenon showing that intuition is of little help in multi-item auctions is a failure of a natural monotonicity: a population with higher values for goods may bring lower revenue!

A distribution H on $\mathbb{R}$ first-order stochastically dominates G (denoted by $H \succ_1 G$) if there exists two random variables $\xi \sim H$ and $\eta \sim G$ defined on the same probability space such that $\xi \geq \eta$ with probability one. A more handy equivalent condition is that

$$H(x) \leq G(x) \quad \forall x \in \mathbb{R},$$

i.e., the distribution H is obtained by shifting the mass in G to the right.

Theorem 16 (S.Hart and N.Nisan (2012))**.** *There exists a pair of distributions $H \succ_1 G$ on $\mathbb{R}$ such that the optimal revenue for a pair of goods whose values are independent and distributed according to H is lower than the optimal revenue if both values have distribution G.*

The reason why monotonicity fails is that the strategy-proofness constraint for H may be harder to satisfy than for G, which pushes the revenue down.

Chapter 12

Complexity of multi-good revenue maximization

We continue discussing mechanisms for selling k good items to one buyer. At the end of the last lecture, we observed that selling lotteries over goods instead of goods themselves may substantially improve revenue.

Recall the model: There is a set $G = \{g_1, \dots, g_k\}$ of k goods and $n = 1$ buyer. An allocation is captured by a fractional bundle $\mu \in [0,1]^G$ where $\mu(g)$ is the probability to receive the good g. The buyer's utility is

$$u_i((\mu, p)) = \sum_{g \in G} \mu(g) v(g) - p.$$

A randomized mechanism maps the vector of values $v = (v(g_1), \dots, v(g_k)) \in \mathbb{R}_+^G$ to $(\mu(v), p(v)) \in [0,1]^G \times \mathbb{R}$.

12.1 Menu-size complexity

We know that each strategy-proof individually-rational mechanism can be represented by a menu $\mathcal{M} \subset [0,1]^G \times \mathbb{R}$ of bundle-payment pairs such that an agent with value vector v selects the most preferred option from this menu.

When we considered deterministic mechanisms, the size of the menu was bounded by 2^k, the number of possible bundles of at most k goods (still a large number if k is big). When lotteries are allowed, we can potentially have a continuum of options. On the other hand, for practical purposes we would like to have small finite menus. Are finite menus enough for revenue-maximization? For some continuous distributions, the answer is negative.

Example 10. There are two goods, the values $v(g_1)$ and $v(g_2)$ are independent and have the beta distribution:[1] $v_1(g_1) \sim Beta(3,3)$ and $v_2(g_2) \sim Beta(3,4)$. Daskalakis et al. (2013) show that revenue-maximizing mechanism for this example requires a continual menu.

[1] The beta distribution $Beta(a,b)$ is the distribution on $[0,1]$ with density $C_{a,b}x^{a-1}(1-x)^{b-1}$, where $C_{a,b}$ is a constant.

This example motivates the literature on menu-size complexity which studies how well the optimal revenue can be approximated within the class of "simple" mechanisms, e.g., those that have a bounded number of entries in the menu.

It turns out, that mechanisms with large finite menus can approximate the optimal revenue arbitrarily well, moreover, the bound on the menu size can be chosen to be independent of the distribution.

Theorem 17 (Babaioff et al. (2021)). *For any number of goods k and $\epsilon > 0$, there exists a constant $C_{k,\epsilon}$ such that for any problem with k goods and independent values, there exists a mechanism with menu of size at most $C_{k,\epsilon}$ and revenue at least $(1-\epsilon)$ fraction of the optimal.*

This constant admits the following bound:

$$C_{k,\epsilon} = \left(\frac{\log k}{\epsilon}\right)^{O(k)}.$$

Still, to get the revenue close to the optimal, one needs large menus. What can be guaranteed if we insist on elementary mechanisms?

Theorem 18 (Babaioff et al. (2020)). *For k goods with independent values, one of the two mechanisms "selling separately" or "selling as a grand bundle" guarantees at least $\frac{1}{6}$ of the optimal revenue.*

12.2 Automated mechanism design

In practice, we may not be satisfied with getting just $\frac{1}{6}$ of the optimal revenue. How should we design a reasonable finite menu?

Assume that there is just a finite number of possible vectors of values $v = v^m \in \mathbb{R}_+^G$, $m = 1, \dots, M$ appearing with probabilities q^m such that $\sum_{m=1}^M q^m = 1$. If the actual distribution of values is continuous, there is a way to show that the optimal mechanism for fine enough discretization will approximate the optimal revenue for the original distribution.[2]

Then finding the optimal mechanism boils down to finding the bundle μ^m and payment p^m for an agent with the vector of values v^m, i.e., we need to determine a finite number of parameters. These parameters must be such that the resulting mechanism satisfies the requirements of strategy-proofness and individual rationality. Revenue maximization boils down to solving the following linear optimization problem:

$$\sum_{m=1}^{M} q^m p^m \to \max$$

$$\text{over menus } (\mu^m, p^m)_{m=1}^M \subset [0,1]^G \times \mathbb{R} \text{ such that}$$

$$\sum_{g\in G} \mu^m(g) v^m(g) - p^m \geq \sum_{g\in G} \mu^l(g) v^m(g) - p^l, \quad \forall m, l = 1, \dots n$$

[2]The construction of such a discretization follows from (Babaioff et al., 2020).

$$\sum_{g\in G} \mu^m(g)v^m(g) - p^m \geq 0, \quad \forall m = 1, \dots n.$$

This linear program can be fed to your favorite LP solver to determine a revenue-maximizing menu. This approach is known as automated mechanism design and is now gaining popularity. It is applicable to mechanism-design problems very broadly: e.g., not only to menu design for one buyer but also to multi-bidder multi-item auctions.

The main obstacle is that the dimension of this linear program explodes fast. Indeed, if the value for each of k goods can take at least D possible values, the number of variables in the linear program is of the order of D^k and the number of strategy-proofness constraints is of the order of $D^k \cdot D^k$. Wile modern LP-solvers can handle linear programs with millions of variables, we need just 100 points and just four goods to make these solvers useless. If there are several agents n, the dimension is of the order of D^{kn} that makes even the problem with two bidders and two goods with $D = 100$ out of reach.

A recent idea is to use deep neural networks to solve the optimization problem approximately (Dütting et al., 2019). As indicated by Kolesnikov et al. (2022), this approach does not completely avoid the curse of dimensionality and the outcomes for several bidders are not free of smoothing artefacts. An alternative approach that may help with multi-agent automated mechanism design is the multi-to-single-agent reduction developed by Hartline et al. (2012). Combining the two approaches seem to be a fruitful unexplored direction.

12.3 Geometry of revenue maximization and complexity of extreme points

In this part of the lecture, we will outline the convex-geometry approach to optimal menu design. A version of this approach lies at the heart of the most of modern analytic results.

Recall the geometric interpretation of a linear program (e.g., the one considered above). The constraints define a set X of feasible solutions that is a polytope in a finite-dimensional Euclidean space. Maximizing a linear objective over such a polytope we obtain a point, where the hyperplane defined by the objective (its level set) touches the polytope. You cannot touch a polytope by a hyperplane without touching its vertex. Thus there is always a vertex solution to the linear program. The standard simplex algorithm always finds such a vertex solution.

These geometric observations extend to general convex sets (instead of polytopes), infinite-dimensional spaces, and convex objectives (instead of just linear). Recall that a subset X of a vector space is convex if with each two points it contains a segment connecting them. A point $x \in X$ is an extreme point if it cannot be represented as a convex combination of two distinct points from X. Extreme points are a generalization of vertices to general convex sets. An objective function $f = f(x)$ is convex if for any pair of points, the value at their convex combination is at most the convex combination of values. In particular, linear objectives are convex.

Theorem 19 (Bauer's principle[3]). *A continuous[4] convex function over a compact convex subset of a vector space[5] attains its maximum at an extreme point of this set.*

The takeaway is that to understand the structure of solutions to a convex optimization problem, one needs to understand extreme points of the feasible set.[6]

Now let us apply these insights to the menu-design problem. Consider a menu $\mathcal{M}$ containing $(0,0)$.[7] With each such menu, we can associate a function $U_{\mathcal{M}} = U_{\mathcal{M}}(x)$ that is equal to the expected utility of the buyer with vector of values $v = x$:

$$U_{\mathcal{M}}(x) = \sup_{(\mu,p)\in\mathcal{M}} \left(\langle \mu, x\rangle - p\right),$$

where the scalar product is defined in the standard way $\langle \mu, x\rangle = \sum_{g\in G} \mu(g) x_g$.

The function $U = U_{\mathcal{M}}$ has the following properties:

1. *U is convex:* Indeed, it is a pointwise maximum of functions linear in x.

2. *U is non-decreasing:* This follows from the fact that each of the linear functions are non-decreasing (coefficients $\mu(g)$ are non-negative) and the fact that the property of being non-decreasing is preserved under pointwise maximum.

3. *U is non-negative:* This is a corollary of the presence of $(0,0)$ in the menu.

4. *U is 1-Lipshitz, namely, $|U(x) - U(y)| \le \sum_{g\in G} |x_g - y_g|$ for any x, y:* Since $|\mu(g)| \le 1$, each linear function has this 1-Lipshitz property and this property is again preserved under pointwise maximum.

Let $\mathcal{U}$ be the set of all functions U satisfying properties 1-4. We assigned an element of $\mathcal{U}$ to each menu. It turns out that each element of $\mathcal{U}$ can be obtained this way.

Theorem 20 (Rochet and Choné (1998)). *For any $U \in \mathcal{U}$ there exists a menu $\mathcal{M} \ni \{(0,0)\}$ such that $U = U_{\mathcal{M}}$.*

This theorem is constructive and a strategy-proof individually-rational mechanism $(\mu(v), p(v))$, corresponding to $U \in \mathcal{U}$ can be recovered as follows. Imagine for a moment that U is just linear: $U(x) = \left(\sum_{g\in G} \mu(g) x_g - p\right)$. How can we recover μ and p if we just know U? The answer is easy to obtain:

$$\mu(x) = \left(\frac{\partial}{\partial x_1} U(x), \ldots, \frac{\partial}{\partial x_g} U(x)\right) = \nabla U(x) \tag{12.1}$$

$$p(x) = \langle x, \nabla U(x)\rangle - U(x). \tag{12.2}$$

[3]See Theorem 7.69 in (Charalambos and Aliprantis, 2013).

[4]Upper semicontinuity is enough.

[5]One needs to make a technical assumptions that the space is a Hausdorff locally convex topological vector space.

[6]To learn more about these functional-analytic/geometric tools, the keywords are "Krein-Milman theorem" covered by Charalambos and Aliprantis (2013) and a (more advanced) "Choquet theory."

[7]As we showed in the previous lecture, such menus represent all strategy-proof individually-rational mechanisms.

It turns out that the same formulas[8] remain true for any $U \in \mathcal{U}$.[9]

We conclude that the problem of revenue-maximization for a given distribution F of values v with density f boils down to the following optimization problem:

$$\int_{\mathbb{R}_+^G} \left(\langle x, \nabla U(x)\rangle - U(x)\right) f(x) dx \to \max$$

$$\text{over} \quad u \in \mathcal{U}.$$

Without loss of generality, one can focus on those functions U that are zero at zero as replacing $U(x)$ by $U'(x) = U(x) - U(0)$ again gives a function from $\mathcal{U}$ and improves the objective. Denote the set of functions $U \in \mathcal{U}$ such that $U(0) = 0$ by $\mathcal{U}_0$.

Revenue is a linear functional and $\mathcal{U}_0$ is a convex set. Hence, by Bauer's principle, the maximum is attained at an extreme point.

Thus, to understand the optimal mechanisms, one needs to understand extreme points of the set $\mathcal{U}_0$ of non-decreasing convex 1-Lipshitz functions on $\mathbb{R}_+^G$ that are equal 0 at 0. For $k = 1$ good, the extreme points are the following functions[10] indexed by a threshold $p \in \mathbb{R}_+$:

$$U(x) = \begin{cases} x - p & x \geq p \\ 0 & x \leq p \end{cases}.$$

These threshold functions lead to the following mechanisms:

$$(\mu(v), p(v)) = \begin{cases} (1, p) & v > p \\ (0, 0) & v \leq p \end{cases}.$$

We obtain that posted-price mechanisms correspond to extreme points when we sell one good to one buyer. This observation implies the Myerson theorem about optimality of posted-price mechanism for this setting. The modern proof of his theorem about multi-bidder auctions also relies on a similar extreme-point argument: there the extreme points correspond to second-price auctions with reserve and "ironing" (Kleiner et al., 2021).

The situation is totally different in the multi-item case. Even for $k = 2$ goods, finding a tractable characterization of the extreme points of $\mathcal{U}$ is an open problem in math. Most likely, there is no such characterization. This mathematical difficulty provides another — now geometric — explanation for hardness of multi-item revenue maximization.

[8] We did not assume differentiability of U so the gradient may not be well-defined for some x. However, it is known that any convex function is (twice!) differentiable for all x except for zero-measure set (the Alexandrov theorem). Alternatively, one can refer to the Rademacher theorem: any Lipshitz function is differentiable for all x except for zero-measure set.

[9] This result requires a useful tool known as the "envelope theorem." It gives a general formula for differentiating an optimum with respect to a parameter.

[10] To show this, consider the set $\mathcal{U}_0'$ containing derivatives of functions from $\mathcal{U}_0$. The set $\mathcal{U}_0'$ is the set of non-decreasing functions with values in $[0, 1]$. Demonstrate that the extreme points of $\mathcal{U}_0'$ are step functions. Then integrate.

Chapter 13

Introduction to fair division

Auctions considered in the last few lectures offer a simple and efficient way to distribute resources. However, there are situations where auctions are not feasible as monetary payments are ruled out. For example, they may be ruled out for ethical reasons; think of government programs (e.g., education or social housing), charity, allocation of seats at over-demanded courses to students, or allocation of organ transplants. Selling these items would create bias towards the rich and is broadly considered unacceptable. Payments may be ruled out for institutional reasons, e.g., if it is unclear who could play the role of a seller. Think of division of a common property (e.g., partners dissolving their partnership or siblings dividing inheritance), allocation of tasks or resources within a firm, or of computational resources in a cloud among users.

Fair division studies mechanisms for allocation of resources when fairness and efficiency are the main concerns, as in the above examples. This field originated in works of Polish mathematicians on "cake-cutting", then developed as a part of welfare economics intersecting with political philosophy, and currently has become a hot topic at the interface of economics, CS, and AI; see the survey by Moulin (2019).

Fairness questions were central for philosophy since ancient times:

> *Equals should be treated equally, and unequals unequally, in proportion to relevant similarities and differences.* (Aristotle, Ethics.)

What are these relevant similarities and differences? Agents may differ in their

- Rights for the resources: For example, one partner has contributed to a project more than others and hence deserves a bigger share of surplus.

- Tastes (preferences): For example, siblings may have different emotional attachment to particular items in the inheritance.

Fair division literature has mainly focused on the extreme case of equal rights but different tastes, while another extreme of unequal rights but identical tastes is the subject of cooperative game theory.

We will discuss the case of equal rights but different tastes where even the basic questions of what fairness is and how it can be be taken into account are non-trivial. The fairness concepts that we will see are universal and applicable more generally than just to the model that we use to illustrate them.

13.1 Fair division of divisible goods under additive utilities

We deal with the following benchmark model. There is a set of goods $G = \{g_1, \dots, g_k\}$ that are to be allocated to n agents. The bundle of goods received by agent i is $\mu_i \in [0,1]^G$, where $\mu_i(g)$ is the amount of g in this bundle.[1] An allocation $\mu = (\mu_1, \dots, \mu_n)$ is a collection of bundles such that all the goods are distributed,[2] i.e., $\sum_i \mu_i = (1, 1, \dots, 1)$. The set of alternatives coincides with the set of all allocations $\{\mu\}$ (in contrast to the quasilinear domain, there are no payments!). We assume that agent's utilities are additive

$$u_i(\mu) = u_i(\mu_i) = \sum_{g \in G} \mu_i(g) v_i(g).$$

Agent i's preferences are captured by the vector of values $v_i = (v_i(g_1), \dots, v_i(g_k)) \in \mathbb{R}_+^G$. A mechanism (or a fair division rule) maps the profile of values $v = (v_1, \dots, v_n)$ to an allocation $\mu = (\mu_1, \dots, \mu_n)$.

Note that $v_i = (v_i(g_1), \dots, v_i(g_k))$ and $v_i' = \alpha v_i$ with $\alpha > 0$ lead to exactly the same preferences over bundles: $u_i(\mu_i) > u_i(\nu_i)$ if and only if $u_i'(\mu_i) > u_i'(\nu_i)$. This allows us to fix a particular normalization: $u_i((1, \dots, 1)) = \text{const}$, i.e., the sum of agent's values is equal to a constant independent of i (e.g., $\sum_g v_i(g) = 1$ or 100 for all i). Additive utilities are easy to report[3] but they rules out complementarity between the goods.

To understand the roles of efficiency and fairness, consider the following toy example

	CS books	Econ books
v_{Alice} :	70	30
v_{Bob} :	10	90

What can we say about the equal division?

μ_{Alice} :	0.5	0.5
μ_{Bob} :	0.5	0.5

[1] We assume that the goods are divisible. If they are not, they can be made divisible by allowing lotteries or time-sharing: then $\mu_i(g)$ is interpreted as the probability that i gets the good g or as the right to use the good $\mu_i(g)$ fraction of time.

[2] This is a convenient technical assumption made for the sake of simplicity. It can easily be relaxed to $\sum_i \mu_i \leq (1, 1, \dots, 1)$.

[3] We can just give an agent 100 of points to distribute among the goods as it is done on http://www.spliddit.org/.

It is ultimately fair but inefficient. The difference in preferences could be exploited to make both agents better off, i.e., this allocation is not Pareto optimal. Recall the definition of Pareto optimality in application to the fair division setting.

Definition 4. *An allocation μ is Pareto optimal if there are no allocation μ' such that $u_i(\mu_i) \leq u_i(\mu'_i)$ and for some agent the inequality is strict.*

Inefficiency gives an opportunity to trade: both agents are happy to exchange less wanted items (a deep idea to be discussed at the end of the lecture, where free trade will be formally related to Pareto optimality).

Now consider an allocation, where everything is given to Alice

$$\begin{array}{rcc} \mu_{Alice}: & 1 & 1 \\ \mu_{Bob}: & 0 & 0 \end{array}$$

It is clearly efficient (Pareto optimal) but, intuitively, unfair. How can we make this intuition formal? There are two notions of fairness dominant in the literature.

Definition 5 (Fair Share Guaranteed (FSG) also known as Equal Split Lower Bound or Proportionality)**.** *Every agent prefers her bundle to the equal division:*

$$u_i(\mu_i) \geq u_i\left(\left(\frac{1}{n}, \ldots, \frac{1}{n}\right)\right).$$

Definition 6 (Envy-Freeness (E-F))**.** *Every agent prefers her bundle to the bundle of any other agent:*

$$u_i(\mu_i) \geq u_i(\mu_j)$$

for all $i, j \in N$.

Note that both definitions are applicable to agents with arbitrary preferences, not necessary represented by additive utilities.

Proposition 10. *For additive utilities:*

- *E-F $\Rightarrow$ FSG.*
- *If $n = 2$, E-F $\Leftrightarrow$ FSG.*

Proof.

- By E-F, $u_i(\mu_i) \geq u_i(\mu_j)$. Let us sum up these inequalities over j. We get

  $$n \cdot u_i(\mu_i) \geq \sum_j u_i(\mu_j) = u_1((1, \ldots, 1)),$$

  where we used additivity. Dividing by n, we get

  $$u_i(\mu_i) \geq \frac{1}{n} u_1((1, \ldots, 1)) = u_i\left(\left(\frac{1}{n}, \ldots, \frac{1}{n}\right)\right)$$

  and conclude that E-F implies FSG.

- By FSG,
$$u_i(\mu_i) \geq u_i\left(\left(\frac{1}{2}, \ldots, \frac{1}{2}\right)\right) = \frac{1}{2}u_i((1, \ldots, 1)) = \frac{1}{2}\left(u_i(\mu_i) + u_i(\mu_{3-i})\right).$$
Thus,
$$u_i(\mu_i) \geq u_i(\mu_{3-i}),$$
i.e., the allocation is envy-free.

□

The above example convinces us that both fairness and efficiency are important. Can we always combine envy-freeness with Pareto optimality? Let us look at some examples of fair division rules.

13.2 Social Welfare maximizers: Utilitarian, Egalitarian, and the Nash rules

To achieve Pareto optimality, the standard idea is to maximize social welfare. Let $G = G(x_1, \ldots, x_n)$ be a continuous function strictly increasing in each of the coordinates. Define the social welfare of an allocation μ by
$$SW_G(\mu) = G\left(u_1(\mu_1), \ldots, u_n(\mu_n)\right).$$
We are interested in those rules that pick an allocation
$$\mu^*: \quad SW_G(\mu) \to \max.$$

Proposition 11. *μ^* is Pareto optimal.*

Proof. Indeed, if μ^* was not Pareto optimal, we could find μ' such that $u_i(\mu_i^*) \leq u_i(\mu_i')$ and for some agent the inequality is strict. Hence, $SW_G(\mu') > SW_G(\mu^*)$. This would contradict the definition of μ^*. □

In your second homework, you proved that in quasi-linear domain Pareto optimality is equivalent to maximization of the so-called utilitarian social welfare corresponding to $G(x) = x_1 + \ldots + x_n$. As we see, when payments are not allowed, the Pareto frontier has a richer structure.

13.2.1 The Utilitarian rule

The utilitarian rule corresponds to $G(x) = x_1 + \ldots + x_n$, i.e., we aim to maximize the sum of agents' utilities. Note that this objective is sensitive to the normalization and recall that we assume that the sum of all values is the same across agents.

The utilitarian rule is in line with the political philosophy ideas of Jeremy Bentham (1748 — 1832) who argued that

> *"The goal of a society must be the greatest happiness of the greatest number of its members."*

In particular, the utilitarian rule may sacrifice the happiness of a minority for the happiness of the majority. It can be very unfair as we can see in the following example.

	book	*flower*	*bicycle*	*laptop*	*armchair*
u_{Alice} :	1	11	21	31	36
u_{Bob} :	5	10	20	30	35

The outcome of the utilitarian rule is easy to compute: we allocate each good to an agent with the highest value. As a result, Bob gets the first good only which is, from his perspective, worth 5% of the total value of 100. FSG requires that each agent get at least 50% and so the utilitarian rule drastically violates both FSG and E-F. In the third homework, you will also see that the utilitarian rule favors single-minded agent at the expense of flexible ones. In particular, for $n \geq 3$, a flexible agent (e.g., the one assigning the same value to all the goods) may get nothing.

To summarize, the utilitarian is not to be used if fairness is a concern unless monetary payments are allowed and can be used to compensate unlucky agents.

Note that the utilitarian rule is also easy to manipulate. In the example above, Bob could increase his value for last good by 2 and decrease that for the next to the last one by 2 and get a more preferable allocation.

13.2.2 The Egalitarian rule

The egalitarian rule introduced by Pazner and Schmeidler (1978) corresponds to $G(x) = \min\{x_1, \ldots, x_n\}$, in other words, it outputs an allocation maximizing the minimal of agents' utilities. It reflects the philosophical views of John Rawls (1921 – 2002):

> *"The goal of a society must be the greatest happiness of the least happy members".*

Let us analyze the properties of the Egalitarian rule under the assumption that all values $v_i(g) > 0$:

- μ^* is equitable, namely $u_i(\mu_i^*) = u_j(\mu_j^*)$

  Proof. If it was not the case, we could transfer a small amount of some good from the agent with the highest utility to others thus increasing the minimal utility (here we use that there are no zero values and so the utility of each of the agents who gets an extra good strictly improves). □

- μ^* is Pareto optimal. This is to be proved as G is not strictly increasing and we cannot apply Proposition 11. The proof mimics that for the previous item.

Proof. If μ^* was not Pareto optimal, we could find an allocation μ', where all agents are at least as happy as before and one agent j is strictly happier. By transferring a small amount of some good from j to others, we could increase the welfare. Contradiction. $\square$

- μ^* is FSG

  Proof. For μ^* the minimal utility is at least as high as the minimal utility for any other allocation, in particular, for the equal division. Hence,

$$u_i(\mu_i^*) \geq \min_j u_j(\mu_j^*) \geq \min_j u_j \left(\left(\frac{1}{n}, \dots, \frac{1}{n} \right) \right) = u_i \left(\left(\frac{1}{n}, \dots, \frac{1}{n} \right) \right),$$

  where in the last equality we used the normalization. $\square$

Although the Egalitarian rule may seem to be the most fair rule, μ^* may not be envy-free for $n \geq 3$ agents. You will see such an example in the third homework.

Egalitarian rule is manipulable: an agent may pretend that her values for the goods that *she does not get* is higher than it actually is, and the rule will compensate her regret by improving her allocation.

13.2.3 The Nash rule

The Nash rule is a compromise between the utilitarian and the egalitarian objectives. It was introduced by a Nobel and Abel laureate and one of founding fathers of game theory, Nash Jr (1950).[4] The Nash rule corresponds to the "Nash product" $G(x) = (x_1 \cdot x_2 \cdot \ldots \cdot x_m)^{\frac{1}{n}}$, i.e., we aim to maximize the product of utilities (or, equivalently, their geometric mean). The outcome μ^* of the Nash rule has the following properties:

- μ^* is Pareto optimal by Proposition 11.

- μ^* satisfies FSG, which follows from the next item.

- μ^* is envy-free.

  Proof. Consider a pair of agents i and j and transfer an ϵ-fraction of j's bundle to i for some $\epsilon > 0$. This cannot increase the product and so

$$u_i(\mu_i^*) u_j(\mu_j^*) \geq \left(u_i(\mu_i^*) + \epsilon u_i(\mu_j^*) \right) \left((1-\epsilon) u_j(\mu_j^*) \right)$$

  for any $\epsilon > 0$. Cancelling $u_j(\mu_j^*)$ on both sides and opening the brackets, we get:

$$u_i(\mu_i^*) \geq u_i(\mu_i^*) + \epsilon u_i(\mu_j^*) - \epsilon u_i(\mu_i^*) - \epsilon^2 u_i(\mu_j^*).$$

[4] He considered a different but related bargaining problem.

Cancelling $u_i(\mu_i^*)$ and dividing both sides by ϵ leads to

$$u_i(\mu_i^*) \geq u_i(\mu_j^*) - \epsilon u_i(\mu_j^*).$$

Since $\epsilon > 0$ was arbitrary, we conclude that

$$u_i(\mu_i^*) \geq u_i(\mu_j^*)$$

and so the allocation is envy-free. □

Corollary 2. *For agents with additive utilities, there always exists an envy-free Pareto optimal allocation.*

The maximization of the Nash product is a convex optimization problem and so the outcome can be approximately computed using the standard algorithms such as the constrained gradient descent. Surprisingly,[5] there are exact polynomial algorithms that rely on the connection between the Nash rule and exchange economies which we discuss below; see Chapters 5 and 6 of Handbook of Algorithmic Game Theory.

The Nash rule is manipulable but this is unavoidable as, for agents with additive utilities, any fair division rule satisfying FSG and Pareto optimality is known to be manipulable (Cho and Thomson, 2013). On the other hand, there are various justifications that the Nash rule is "not so manipulable." In particular, in contrast to the egalitarian rule, it cannot be manipulated by lying about goods that an agent doesn't get (Bogomolnaia et al., 2016). Intuitively, envy-freeness itself limits manipulability since the fact that an agent prefers her bundle to bundles of other means that, in particular, she has no incentive to pretend that her preferences are similar to preferences of others.[6]

There are many confirmations that the Nash rule is the best rule to divide goods under additive utilities. Why is the Nash product so specific? A possible answer is its connection to the fundamental concept, the competitive equilibrium of exchange economies, which we discuss next.

13.3 Envy-freeness as equal choice opportunities. CEEI.

The following example offers an alternative perspective on envy-freeness. Alice and Bob each of whom has \$100. Each goes to a supermarket and spends money on the most preferred goods. Will they envy each other? No, because both select the best bundle of goods from the same choice set (their budget set).

[5] Usually, non-linear optimization problems have irrational solution even if the input consists of rational numbers. Consequently, there is no hope for a polynomial algorithm (unless a certain short representation for the resulting irrationals is fixed). The surprising property of the Nash rule is that the maximization always leads to rational numbers on a rational input.

[6] A formal equivalence between envy-freeness and strategy-proofness can be obtained in the "large market limit" where there are agents with any possible vector of values (Azevedo and Budish, 2019).

This example shows that offering agents the same choice opportunities (a menu of bundles to select from) leads to envy-freeness. Varian (1973) combined this observation with theory of general equilibrium of exchange economies and defined the Competitive Equilibrium with Equal Incomes (CEEI) also known as the Competitive or the Pseudo-Market mechanism; in CS literature it is known as the Fisher market equilibrium. The informal description is as follows:

- give every agent a unit amount of “virtual” money;
- select prices such that the market clears, when everybody buys the best bundle she can afford.

From market equilibrium perspective, the resulting allocation represents what happens if we first divide the goods equally among agents (this equal division is captured by equal budgets) and then allow free trade.

The formal definition of CEEI is as follows.

Definition 7. *An allocation μ is a CEEI if there exists a vector of prices $p \in \mathbb{R}_+^G$ such that for any agent i*

$$x = \mu_i \text{ maximizes } u_i(x) \text{ over the budget set } B(p) = \left\{ x \in \mathbb{R}_+^G : \sum_{g \in G} p_g x_g \leq 1 \right\}.$$

Note that each agent selects the best bundle from the budget set $B(p)$ as if there were no other agents; magically, these choices do not conflict with each other which requires tailoring the price p.

From the definition it is unclear whether a CEEI exists. It turns out that it does very generally even beyond additive utilities.

Theorem 21 (Arrow-Debreu)**.** *If utility functions u_i are concave, strictly-increasing, and continuous, then a CEEI exists.*

This proof is non-constructive;[7] see Chapter 15 in (Intriligator and Arrow, 1987).

Proposition 12. *Any CEEI μ is envy-free.*[8]

Proof. The utility $u_i(\mu_i)$ is at least as high as $u_i(x)$ for any x from the budget set $B(p)$. Picking, $x = \mu_j$, we obtain that i does not envy j. □

The next proposition is known as the first fundamental theorem of welfare economics or the “invisible hand” of Adam Smith.

[7]The idea is to define an improvement map that takes an allocation-price pair and returns a “more-equilibrium” pair by identifying over-demanded goods and increasing prices for them and decreasing for under-demanded ones. CEEI are fixed points of this map and their existence can be proved using topological arguments such as the Brouwer fixed-point theorem.

[8]No assumptions on utilities needed provided that CEEI exists.

Proposition 13. *Assuming that each u_i is strictly increasing, any CEEI μ is Pareto optimal.*

To prove this proposition, we need the following auxiliary result.

Lemma 1. *Assume u_i is strictly increasing. If μ is a CEEI with a price vector p, then for any $y \in \mathbb{R}_+^G$ such that $u_i(y) \geq u_i(\mu_i)$, the price of y is at least 1, i.e., $\sum_g y_g p_g \geq 1$. Similarly, if $u_i(y) > u_i(\mu_i)$, then $\sum_g y_g p_g > 1$.*

Proof. If the price of y was below 1, then $\mu_i' = y + \epsilon(1, \ldots, 1)$ would belong to $B(p)$ for positive ϵ small enough and $u_i(\mu_i') > u_i(\mu_i')$ by strict monotonicity. Contradiction with the fact that μ_i maximizes u_i over $B(p)$.

Similarly, if $u_i(y) > u_i(\mu_i)$, then $\sum_g y_g p_g > 1$ as otherwise y would belong to $B(p)$ contradicting the choice of μ_i. □

Proof of the the first fundamental theorem of welfare economics. Towards a contradiction, suppose there is μ' such that $u_i(\mu_i') \geq u_i(\mu_i)$ and this inequality is strict for some j. Let p be the vector of prices corresponding to p. By the lemma, each μ_i' must have a price of at least 1 and μ_j must have a price strictly higher that 1. Hence,

$$\sum_i \sum_g p_g \mu_i'(g) > n.$$

On the other hand,

$$\sum_i \sum_g p_g \mu_i'(g) = \sum_g p_g \sum_i \mu_i'(g) = \sum_g p_g = \sum_g p_g \sum_i \mu_i(g) = \sum_i \sum_g p_g \mu_i(g) \leq n$$

since each bundle μ_i has a price of at most 1. This contradiction completes the proof. □

Combining the Arrow-Debreu theorem with the two propositions, we obtain the following general corollary.

Corollary 3. *Assume all u_i are concave, strictly-increasing, and continuous. Then there exists an envy-free Pareto optimal allocation.*[9]

It turns out that there is a connection between the Nash rule and the CEEI.

Theorem 22 (Eisenberg-Gale)**.** *Assume all u_i are concave, continuous, and 1-homogeneous ($u_i(\alpha x) = \alpha u_i(x)$ for $\alpha > 0$). Then the set of CEEI coincides with the outcome of the Nash rule.*

The proof can be found in (Eisenberg, 1961).

[9] An interesting open question is to find a simple direct proof for this result without a detour through the theory of general equilibrium.

Chapter 14

Fair division of computational resources

In the previous lecture, we considered the problem of fair division of private good items under additive utilities. This captures the case of no complementarity between the goods. In this lecture, we consider the opposite extreme of the strongest complementarity, where one good becomes useless if not combined with a proper combination of others. Such extreme complementarity arises if the goods serve as resources needed to solve a certain task (the so-called intermediate goods), e.g., materials needed to build a house or computational resources (CPU, RAM, hard drive, bandwidth, etc) needed for a program to run.

We will describe a solution for division of computational resources known as **dominant resource fairness** (DRF) and introduced by Ghodsi et al. (2011). We will consider a simple static model which may not seem very realistic. However, DRF can be made practical and a version of it[1] is used by Microsoft Azure (a cloud-computing service).

The model is as follows. There is a set of goods interpreted as computational resources in a cloud $G = \{g_1, \ldots, g_k\} = \{\text{CPU, RAM}, \ldots\}$. We normalize the total availability of each of the resources to be 1. There are n agents. Each agent comes with a series of identical computational tasks demanding a combination of computational resources. We denote the per-task demand of agent i by $d_i = (d_i(g_1), \ldots, d_i(g_k)) \in \mathbb{R}_+^G$ and assume that each coordinate is strictly positive. If an agent i is allocated a bundle $\mu_i \in [0,1]^G$ of resources, her utility is assumed to be equal to the total number of tasks that she can perform using this amount of resources:

$$u_i(\mu_i) = \text{maximal } \alpha \in \mathbb{R}_+ \text{ such that } \alpha \cdot d_i \leq \mu_i.$$

As we see, the tasks are assumed to be divisible, e.g., an agent prefers being able to run 1.5 tasks to running just one. We can equivalently write

$$u_i(\mu_i) = \min_{g \in G} \frac{\mu_i(g)}{d_i(g)} = \min_{g \in G} v_i(g)\mu_i(g),$$

[1] Developed by Adam Wierman (Caltech) and his student (name?).

where the values $v_i(g)$ are defined as $\frac{1}{d_i(g)}$. This representation demonstrates that that "$\sum$" that we had for additive utilities is replaces by "min"; such utilities are called Leontief utilities. An allocation μ is a collection of bundles $(\mu_1, \ldots, \mu_n)$ such that

$$\sum_{i=1}^{n} \mu_i \leq (1, 1, \ldots, 1).$$

In contrast to the setting from the last lecture, we do not insists on distributing all the resources.[2]

Let us call g a dominant resource for i if $d_i(g) = \max_{g' \in G} d_i(g')$. The **DRF rule** operates as follows:

- Agents submit their demand vectors d_i normalized so that the demand for the dominant resource is 1, i.e., $\max_{g \in G} d_i(g) = 1$ for all i. Equivalently, agents submit the vectors values v_i normalized so that $u_i((1, \ldots, 1)) = 1$.[3]

- Find maximal $\alpha \geq 0$ such that

$$\alpha \cdot \sum_{i=1}^{n} d_i \leq (1, 1, \ldots, 1).$$

- allocate the bundle

$$\mu_i = \alpha \cdot d_i$$

  to agent i.[4]

The DRF is a version of the Egalitarian rule rediscovered in the context of Leontief preferences.

Proposition 14. *The DRF allocation μ maximizes the Egalitarian welfare*

$$\min_i u_i(\mu_i') \to \max$$

over all allocations μ'.

Proof. Towards a contradiction, assume there is some μ' with strictly higher welfare. For each i, define $\mu_i'' = \alpha_i \cdot d_i$ with the maximal α_i such that $\alpha_i \cdot d_i \leq \mu_i'$. Then the welfare of μ' and μ'' is the same and is higher than that of μ. Thus $\alpha_i > \alpha$ for each i, where α is from the definition of DRF. Taking $\alpha' = \min_i \alpha_i$, we conclude that $\alpha' > \alpha$ and $\alpha' \cdot \sum_{i=1}^{n} d_i \leq (1, 1, \ldots, 1)$. This contradicts the fact that α is the maximal number with this property. $\square$

[2] This is a reasonable assumption from practical perspective as the leftover may be assigned to newly arriving agents (of course, this concern is, formally, beyond our model).

[3] This normalization is familiar from the last lecture.

[4] For any bundle μ_i there is a bundle proportional to the demand $\alpha_i d_i \leq \mu_i$ that gives agent i the same utility $u_i(\mu_i) = u_i(\alpha \cdot d_i)$. For this reason, restricting attention to bundles proportional to i's demand is natural.

Repeating the proof for the Egalitarian rule from the previous lecture, we can show that DRF is efficient, i.e., its outcome is Pareto optimal. A surprising feature of DRF is that it is also non-manipulable.

Proposition 15. *DRF is strategy-proof.*

Proof. Suppose that agent i reports d'_i instead of d_i. As a result the factor α becomes α'. To show that this manipulation is not profitable, we need to demonstrate that

$$u_i(\alpha \cdot d_i) \geq u_i(\alpha' \cdot d'_i).$$

Since we assume that the demand vectors have no zeros, this inequality is equivalent to the existence of a good g such that

$$\alpha \cdot d_i(g) \geq \alpha' \cdot d'_i(g).$$

Consider the two cases:

- If $\alpha' \leq \alpha'$, let g be a dominant resource with respect to d_i. We get

$$\alpha' \cdot d'_i(g) \leq \alpha' \cdot 1 \leq \alpha \cdot 1 = \alpha \cdot d_i(g).$$

- If $\alpha' > \alpha'$, consider a good g that is fully allocated under the true preferences:[5] $\alpha \sum_i d_i(g) = 1$. We obtain

$$\alpha \cdot d_i(g) = 1 - \alpha \sum_{j \neq i} d_j(g) > 1 - \alpha' \sum_{j \neq i} d_j(g) \geq \alpha' \cdot d'_i(g),$$

  where the last inequality follows from the fact that $\sum_j \mu'_j(g) \leq 1$.

We conclude that a manipulation is never profitable. □

Proposition 16. *DRF is envy-free.*

Proof. We need to show that $u_i(\alpha \cdot d_i) \geq u_i(\alpha \cdot d_j)$. It is enough to find a good g such that $\alpha \cdot d_i(g) \geq \alpha \cdot d_j(g)$. Taking g to be i's dominant resource completes the proof. □

The only feature of DRF that makes it different from a straightforward extension of the Egalitarian rule to Leontief preferences is that DRF does not insist on allocating all the goods.[6] This feature is crucial: as demonstrated by Nicoló (2004), no rule allocating all the goods to agents with Leontief preferences can be simultaneously efficient, envy-free, and strategy-proof. Allowing for "wasteful" rules which may not allocate some of the goods helps to combine fairness, strategy-proofness, and some efficiency guarantees for additive utilities too (Cole et al., 2013; Abebe et al., 2020).

[5] Such a good exists as, otherwise, we could increase α by some $\epsilon > 0$ and still $(\alpha + \epsilon) \sum_i d_i \leq (1, \ldots, 1)$ contradicting the definition of α.

[6] If these goods cannot improve agents' utilities anyway.

Chapter 15

Fair division of indivisible goods

We are back to the setting of fair division of private goods under additive utilities considered in Lecture 12. The main example is the case of former partners dividing common assets consisting of unrelated non-complementary goods. The new twist is that now these goods are assumed to be indivisible and so each of them must be entirely allocated to one of the agents.

There are n agents and a set of indivisible goods $G = \{g_1, \ldots, g_k\}$. The bundle of indivisible goods received by agent i is a vector μ_i from the k-dimensional binary cube $\{0,1\}^G$, where $\mu_i(g)$ is the amount of g in this bundle that is equal to either 0 or 1. Abusing the notation, we will sometimes identify a bundle μ_i with the set of goods $\{g \in G\colon \mu_i(g) = 1\}$. An allocation of indivisible goods $\mu = (\mu_1, \ldots, \mu_n)$ is a collection of bundles such that all the goods are distributed, i.e., $\sum_i \mu_i = (1, 1, \ldots, 1)$ or, equivalently, $\mu_i \cap \mu_j = \emptyset$ and $\cup_i \mu_i = G$. Agent's utilities are additive

$$u_i(\mu) = u_i(\mu_i) = \sum_{g \in G} \mu_i(g) v_i(g) = \sum_{g \in \mu_i} v_i(g).$$

Agent i's preferences are captured by the vector of values $v_i = (v_i(g_1), \ldots, v_i(g_k)) \in \mathbb{R}_+^G$ and a fair division rule maps the profile of values $v = (v_1, \ldots, v_n)$ to an allocation of indivisible goods $\mu = (\mu_1, \ldots, \mu_n)$.

Note that this model can be seen as a particular case of the one with divisible goods where we additionally enforce that each bundle must be integral. So the notions of fairness (FSG and E-F) extends in a straightforward way to the model with indivisible goods. The immediate bad news is that a fair allocation may fail to exist, e.g., think of a problem with two agents dividing two indivisible goods and each agent preferring the same good the other:

	Econ book	CS book
v_{Alice} :	80	20
v_{Bob} :	60	40

Clearly, an agent who does not get the desired good will be envious to the one who does. FSG cannot be satisfied too as it is equivalent to E-F for two additive agents.

There is a pair of ideas of how to circumvent such non-existence, both ideas come from the CS community:

1. If there are no mechanisms with desired worst-case behavior, let's switch to the average case. In our setting, the question becomes: do fair allocations of indivisible goods typically exist?

2. If there is no mechanism that satisfies the desired properties exactly, let's try to satisfy them approximately. In our case, the question becomes: can we find a reasonable relaxation of fairness axioms that is compatible with the existence?

We will explore both routes.

15.1 Typical existence of fair allocations

Theorem 23 (Dickerson et al. (2014))**.** *If a preference profile consists of independent identically distributed*[1] *random variables $v_i(g)$ having a continuous distribution F with bounded variance*

$$\mathbb{P}\left(\textit{E-F allocation exists}\right) \to 1$$

as the number of goods k becomes large.

In other words, for problems with large number of goods with similar statistical properties, the existence is not an issue.

Proof. Instead of proving the abstract "existence", show that the utilitarian allocation (the one that gives each good to an agent with the highest utility) is envy-free with high probability.[2][3] Since the goods are statistically identical, each agent gets on average the same number $\frac{k}{n}$ of her most preferred goods. For the expected utilities, we get

$$\mathbb{E}\left[u_i(\mu_i)\right] = \frac{k}{n}\mathbb{E}\left[v_i(g_1)\mathbb{1}_{\{v_i(g_1)=\max_l v_l(g_1)\}}\right],$$

$$\mathbb{E}\left[u_i(\mu_j)\right] = \frac{k}{n}\mathbb{E}\left[v_i(g_1)\mathbb{1}_{\{v_i(g_1)\neq\max_l v_l(g_1)\}}\right] \quad \text{for } j \neq i.$$

For continuous distributions, the expected values of higher order statistics are strictly above lower ones and, hence, there is no envy in expectation:

$$\mathbb{E}\left[u_i(\mu_i)\right] > \mathbb{E}\left[u_i(\mu_j)\right].$$

[1] The original result is more general and allows for correlation across agents.

[2] We know that the Utilitarian rule can be very unfair and the fact that it behaves well for large number of goods with similar statistical properties should not be considered as a justification for its practical use.

[3] In fact, we get a stronger statement: $\mathbb{P}$ (E-F Pareto optimal allocation exists) $\to 1$.

To show that there is no envy with high probability, it remains to apply the law of large numbers which gives that

$$\mathbb{P}\left[\left|\frac{1}{k}u_i(\mu_l) - \frac{1}{k}\mathbb{E}[u_i(\mu_l)]\right| < \epsilon\right] \to 1$$

for any $\epsilon > 0$ as $k \to \infty$. Choosing ϵ to be less than half of the difference between the two expectations implies that

$$\mathbb{P}\left[u_i(\mu_i) > u_i(\mu_j)\right] \to 1$$

and completes the proof. □

It is important to know that hard instances are rare but still the results on typical behavior are of limited practical value. Indeed, the fact that a rule behaves well on average does not eliminate dissatisfaction if it has behaved poorly in a particular instance of interest. Robust non-probabilistic guarantees applicable to each instance are more practical.

15.2 Approximating FSG

The following approximate variant of FSG has been proposed by Budish (2011). For a given profile v, define the maximin share of agent i by

$$MMS_i = \max_{\mu} \min_{j} u_i(\mu_j).$$

One can think of it as the outcome of the following procedure: an agent divides all the goods into n piles and then picks the worst pile from her perspective. MMS_i is the utility that i gets for the division that is best for her.

Definition 8. *An allocation μ is an MMS-allocation if*

$$u_i(\mu_i) \geq MMS_i \quad \forall i.$$

Note that if, in the definition of MMS_i, we allowed for divisible allocations μ, it would be optimal for i to divide the available goods into n identical bundles $\mu_j = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$. Hence, for divisible goods, a natural analog of MMS-allocations, are those satisfying

$$u_i(\mu_i) \geq u_i\left(\left(\frac{1}{n}, \ldots, \frac{1}{n}\right)\right),$$

i.e., FSG allocations. For indivisible goods, we conclude that

$$MMS_i \leq u_i\left(\left(\frac{1}{n}, \ldots, \frac{1}{n}\right)\right).$$

Do MMS allocations exist? In the third homework, you will check that they always do for $n = 2$ agents; the argument relies on the cut-and-choose procedure, where one agent divides the available resources into two piles and the other agent selects her best pile.

For a couple of years, it had been conjectured that MMS allocations exist for any number of agents. This conjecture was supported by numerous simulations and real-life data.

Theorem 24 (Procaccia and Wang (2014)).

- *For $n \geq 3$ agents an MMS-allocation may fail to exist.*
- *One can relax the MMS notion in a multiplicative way to guarantee existence. Namely, for any number of agents and any profile of preferences, there exists μ such that*

$$u_i(\mu_i) \geq \frac{2}{3} MMS_i.$$

All known counterexamples to the existence of MMS allocations are knife-edge (the one by Procaccia and Wang (2014) requires 12 goods and cleverly tailored values with a multi-scale structure), in particular, there is almost no chance to discover such a counterexample by taking a random instance. Moreover, from practical perspective, one can safely assume that MMS always exists. This teaches a lesson about economic design: if a mechanism has a certain property on all random or real-life inputs it does not necessarily mean that it has it in the worst case.

15.3 Approximating envy-freeness

Another idea of how to adapt a fairness notion to indivisible goods belongs to Lipton et al. (2004) who suggested the following relaxation of envy-freeness.

Definition 9. *An allocation μ is envy-free up to one good or EF-1 if for any pair of agents i and j there is $g \in \mu_j$ such that*

$$u_i(\mu_i) \geq u_i(\mu_j \setminus \{g\}).$$

In other words, i can envy j but this envy can be eliminated after deleting some good from j's bundle. It is easy to show that the round-robin procedure always outputs an EF-1 allocation, in particular, an EF-1 allocation always exists.[4]

EF-1 can also be combined with Pareto optimality using the familiar Nash social welfare.

Definition 10. *An allocation μ of indivisible goods is Pareto optimal if there no other allocation of indivisible goods[5] μ' such that $u_i(\mu'_i) \geq u_i(\mu_i)$ for all agents i and for some i the inequality is strict.*

Theorem 25 (Caragiannis et al. (2019)). *An allocation μ^* of indivisible goods that maximizes the Nash social welfare*

$$\mu = \mu^*: \quad \prod_{i=1}^{n} u_i(\mu_i) \to \max$$

is EF-1 and Pareto optimal.

[4] Lipton et al. (2004) demonstrate the existence much more generally.

[5] For indivisible goods, the set of alternatives is the set of indivisible allocations. The Pareto optimality is defined with respect to this set and so the counterfactual allocation μ' in the definition of Pareto optimality is also indivisible. If we allowed for fractional μ' we would get a non-equivalent definition of "fractional Pareto otpimality" which is, however, also compatible with EF-1 (Barman et al., 2018).

Proof. The proof resembles the one for the analogous result that we saw in the divisible case.

Since $\mu = \mu^*$ maximizes the product, transferring any good $g \in \mu_j$ from agent j to agent i, cannot increase the product. Thus

$$u_i(\mu_i)u_j(\mu_j) \geq (u_i(\mu_i) + v_i(g))\,(u_j(\mu_j) - v_j(g))\,.$$

Dividing both sides by $u_i(\mu_i)u_j(\mu_j)$, opening the brackets, and simplifying, we get:

$$\frac{v_j(g)}{u_j(\mu_j)} + \frac{v_j(g)}{u_j(\mu_j)}\frac{v_i(g)}{u_i(\mu_i)} \geq \frac{v_i(g)}{u_i(\mu_i)}. \tag{15.1}$$

Now, let's select a particular good $g = g^* \in \mu_j$. The idea is that we need the good that is relatively more preferred by i than by j:

$$g = g^* \quad \text{that minimizes} \quad \frac{v_j(g)}{v_i(g)} \quad \text{over } g \in \mu_j.$$

We obtain

$$v_j(\mu_j) = \sum_{g \in \mu_j} v_j(g) = \sum_{g \in \mu_j} \frac{v_j(g)}{v_i(g)} \cdot v_i(g) \geq \frac{v_j(g^*)}{v_i(g^*)} \cdot \sum_{g \in \mu_j} v_i(g) = \frac{v_j(g^*)}{v_i(g^*)} \cdot u_i(\mu_j).$$

Dividing both sides by $\frac{u_i(\mu_j)}{v_i(g^*)}$ gives

$$\frac{v_i(g^*)}{u_i(\mu_j)} \geq \frac{v_j(g^*)}{u_j(\mu_j)}.$$

Combining this inequality with (15.1) results in

$$\frac{v_i(g^*)}{u_i(\mu_j)} + \frac{v_i(g^*)}{u_i(\mu_j)}\frac{v_i(g^*)}{u_i(\mu_i)} \geq \frac{v_i(g^*)}{u_i(\mu_i)}.$$

Multiplying both sides by $\frac{u_i(\mu_i)u_i(\mu_j)}{v_i(g^*)}$ gives

$$u_i(\mu_i) \geq u_i(\mu_j) - v_i(g^*)$$

and thus i does not envy j after eliminating g^* from j's bundle. As $i \neq j$ are arbitrary, the allocation is EF-1. □

15.4 Open problem: EFX

A natural strengthening of EF-1 is the concept, where possible envy between i and j must disappear after eliminating *any* good from j's bundle.

Definition 11. *An allocation μ is envy-free up to any good or EFX if for any pair of agents i and j and any good $g \in \mu_j$, we have*

$$u_i(\mu_i) \geq u_i(\mu_j \setminus \{g\}).$$

For $n = 2$ agents, it is easy to see that an allocation μ maximizing the minimal utility (the outcome of the Egalitarian rule) satisfies EFX (Plaut and Roughgarden, 2020). Thus EFX allocations always exist for $n = 2$. The existence of EFX allocations for $n \geq 3$ and its compatibility with Pareto optimality remains the main open problems in fair division of indivisible goods. A breakthrough paper by Chaudhury et al. (2020) shows the existence for $n = 3$.

15.5 Open problem: EF-1 for bads

One can consider a version of the same model where agents divide bads instead of goods, e.g., some tiring tasks or liabilities. Formally, we just need to assume that $v_i(g) \leq 0$ for all items. The notions of envy-freeness, FSG, and Pareto optimality for divisible bads repeat those for goods. The extension of indivisible notions is also straightforward.

Definition 12. *An allocation μ of indivisible bads is EF-1 if for any pair of agents i and j there is $g \in \mu_i$ such that*

$$u_i(\mu_i \setminus \{g\}) \geq u_i(\mu_j).$$

As we see, now a possibly envious agent i needs to eliminate an item from *her own* bundle to stop envying agent j. Indeed, eliminating a bad from j's bundle would only increase envy.

One can show that EF-1 allocation exist via round robin, as in the case of goods. Interestingly, the existence of EF-1 Pareto optimal allocations of bads is an open problem. We cannot mimic the proof of Theorem 25, e.g., minimizing the product of absolute values of the utilities leads to an unfair allocation where one agent is allocated no bads, while maximizing the product pushes agents' utilities in the wrong direction and so violates Pareto optimality.

Presumably, a combination of the fallowing approaches may help to resolve the conjecture: Barman et al. (2018) describes how to perturb equal budgets in a CEEI for divisible goods to get an EF-1 allocation of indivisible ones, and Bogomolnaia et al. (2017) extend the notion of CEEI to bads.

Chapter 16

Envy-free pricing and matching markets with money

In this lecture, we consider a hybrid setting, where indivisible goods are to be allocated in the presence of one divisible good: money. While the setting is similar to that of multi-item auctions, the central role will be played by fairness instead of revenue maximization. In the end of the lecture, we will relate the model to two-sided matching markets. A more detailed discussion can be found in (Karlin and Peres, 2017, Chapter 17).

16.1 Competitive equilibrium and envy-free pricing

The model is as follows. There are n agents and n indivisible goods $G = \{g_1, \ldots, g_n\}$; each agent needs only one good.[1] Hence, an allocation μ is a bijection $\{1, \ldots, n\} \to G$. The set of alternatives consists of allocation-price pairs (μ, p), where $p\colon G \to \mathbb{R}_+$ defines a price for each good g. The utility of agent i is quasilinear in money:

$$u_i = v_i(\mu_i) - p(\mu_i).$$

Given prices p, define the demand of agent i by

$$D_i(p) = \{g \in G \;\; : \;\; v_i(g) - p(g) \geq v_i(g') - p(g') \;\; \forall g' \in G \quad \text{and} \quad v_i(g) - p(g) \geq 0\}\,.$$

So the demand consists of those goods that bring the highest utility provided that this utility is non-negative. The non-negativity captures individual rationality: an agent would prefer to receive no good if all the goods are too pricey.

Definition 13. *A pair (μ, p) is a competitive equilibrium (CE) if*

$$\mu_i \in D_i(p)$$

for any agent i.

[1] We assume that the numbers of goods and agents are equal for the sake of simplicity. One can easily reduce the problem with unequal numbers to this benchmark setting by either adding zero goods or dummy agents that have zero values for all the goods.

The prices p are referred to as envy-free prices. Indeed, if (μ, p) is a CE we have

$$v_i(\mu_i) - p(\mu_i) \geq v_i(\mu_j) - p(\mu_j)$$

for any pair of agents i and j, i.e., i does not envy j. One can think of a CE as a menu of item-price pairs such that all agents, when offered this menu, select different options and so the market clears. An analogy with the definition of CEEI for markets without money must be clear.[2]

Example 11 (Rent division)**.** Consider the following problem: n students are renting a house with n non-identical rooms $G = \{g_1, \ldots, g_n\}$. How should they divide the total rent of R? On the Internet, you can find various rent-division calculators. Some just propose equal division $p(g) = \frac{R}{n}$. Others go further and determine the prices as a function of various amenities such as room area, windows, private bathroom and so on. Such an approach is flawed as no matter how many parameters are introduced to the calculator, it fails to capture that different tenants may have different attitude to these amenities.

Instead of measuring the objective parameters of the rooms, an economic design approach would be to take into account agents' subjective preferences. This is how the rent-division calculator on http://www.spliddit.org/ works. Each tenant i reports her values $v_i(g)$ for the rooms with the condition that $\sum_{g\in G} v_i(g) = R$, i.e., she proposes a way to split the rent among the rooms in a way that reflects attractiveness of each of the rooms from her perspective. Then the calculator determines who gets what and how much they pay.

Formally, a pair (μ, q) is an envy-free rent division if

$$\sum_{g\in G} q(g) = R$$

and

$$v_i(\mu_i) - q(\mu_i) \geq v_i(\mu_j) - q(\mu_j).$$

Below we will show that a CE exists. Let us check that this implies the existence of an envy-free rent division. If (μ, p) is a CE. Define

$$q(g) = p(g) + \frac{R - \sum_{g'} p(g')}{n}.$$

Thus (μ, q) is an envy-free rent division as adding a constant to all the prices does not ruin the property of envy-freeness. Note that envy-free rent division may not be unique and selecting "the best one" is a separate non-trivial problem (Gal et al., 2017).

[2]The crucial difference between the two models is that, in CEEI, agents have no value for money but there is a budget constraint instead: equivalently, the contribution of money to utility is 0 if an agent is within her budget and $-\infty$, otherwise. The value for money exhibits threshold behavior, not quasilinear.

16.2 Properties of competitive equilibria

The following result is an analog of the first fundamental theorem of welfare economics familiar by our discussion of CEEI.

Theorem 26. *If (μ, p) is a CE, then μ maximizes the utilitarian social welfare*

$$\sum_{i=1}^{n} v_i(\mu_i).$$

In the second homework, you proved that for agents with quasi-linear preferences, an alternative is Pareto optimal if and only if it maximizes the sum of utilities. Thus the theorem can be equivalently stated as follows: a CE is Pareto optimal.

Proof. Consider some other allocation μ' and show that its welfare is bounded by that of μ. We have

$$\sum_{i=1}^{n} v_i(\mu_i') = \sum_{i=1}^{n} (v_i(\mu_i') - p(\mu_i')) + p(\mu_i') \leq \sum_{i=1}^{n} (v_i(\mu_i) - p(\mu_i)) + p(\mu_i'),$$

by envy-freeness of CE. Since

$$\sum_{i=1}^{n} (v_i(\mu_i) - p(\mu_i)) + p(\mu_i') = \sum_{i=1}^{n} v_i(\mu_i) - \sum_{g \in G} p(g) + \sum_{g \in G} p(g) = \sum_{i=1}^{n} v_i(\mu_i),$$

we obtain that

$$\sum_{i=1}^{n} v_i(\mu_i') \leq \sum_{i=1}^{n} v_i(\mu_i),$$

which concludes the proof. □

This theorem suggests a recipe for finding a CE. We first compute μ maximizing $\sum_{i=1}^{n} v_i(\mu_i)$. This problem is equivalent to finding the maximal-weight matching in the complete bipartite graph with parts $\{1, \dots, n\}$ and G and an edge (i, g) having weight of $v_i(g)$; this can be done, e.g., using Edmonds' algorithm. Once this matching is computed, the prices can be determined as a solution to the following linear system:

$$\begin{aligned} v_i(\mu_i) - p(\mu_i) &\geq v_i(\mu_j) - p(\mu_j) \quad \forall i \neq j \\ v_i(\mu_i) - p(\mu_i) &\geq 0 \quad \forall i \\ p &\in \mathbb{R}_+^G. \end{aligned}$$

However, we still do not know if this system always has a solution. We will see that it always does by considering an alternative way to find a CE relating it to simultaneous ascending auctions. As a byproduct, we will obtain an auction-based algorithm for computing the maximal-weight matching in a bipartite graph.

16.2.1 Computing a CE via simultaneous ascending auctions

The following algorithm is by Demange et al. (1986) and is close to earlier ideas of Bertsekas that, however, were published later (Bertsekas, 1988):

- The input is an $n \times n$ matrix $v_i(g)$ of non-negative *integer numbers*[3] and an approximation parameter $0 < \delta < \frac{1}{n}$ playing a role of a bid increment
- Initialize $\mu_i = \emptyset$ for all i and $p(g) = 0$ for all g
- While there exists an agent i with $\mu_i = \emptyset$
  - Pick $g \in D_i(p)$
  - If there is j such that $\mu_j = g$, update $\mu_j = \emptyset$.
  - Allocate g to i, i.e., define $\mu_i = g$.
  - Update $p(g) = p(g) + \delta$
- End.

Theorem 27. *The algorithm terminates. The resulting pair (μ^*, p^*) is such that μ^* is the maximal-weight matching and p^* satisfies*

$$\begin{aligned} v_i(\mu_i) - p^*(\mu_i) &\geq v_i(\mu_j) - p^*(\mu_j) - \delta \quad \forall i \neq j \\ v_i(\mu_i) - p^*(\mu_i) &\geq -\delta \quad \forall i \\ p^* &\in \mathbb{R}_+^G. \end{aligned}$$

Corollary 4. *By letting $\delta \to +0$, we obtain that a CE exists. Indeed, we know that, for any δ, the vector of prices $p^* = p^*_\delta$ belongs to the cube $\left[0,\ \max_{i,g} v_i(g) + \frac{1}{n}\right]^G$. Thus the sequence p^*_δ has a limit point. Any such limit point corresponds to a CE by the continuity of envy-freeness constraints.*

Proof. First, let us demonstrate correctness, namely that the demand set $D_i(p)$ in the while-cycle is never empty and so we can always pick a good from $D_i(p)$. Note that once a good g is allocated to an agent, it will always be allocated to some agent at all future iterations of the cycle. As the numbers of agents and goods are the same, whenever there is an agent with $\mu_i = \emptyset$, there is a good g that has never been allocated. Such a good has zero price and thus $D_i(p)$ cannot be empty.

If a good $g \in D_i(p)$ is selected, it must be that the current price $p(g) \leq v_i(g)$. As only the prices of selected goods increase, we see that a price of any good must satisfy

$$p(g) \leq \max_{i,g} v_i(g) + \delta$$

throughout the execution of the algorithm. Since each iteration of the while-cycle increases the price of one good by δ, we conclude that the algorithm terminates after a finite number of iterations.

[3] If $v_i(g)$ are rationals, multiply all the values by the common denominator.

At the iteration when i is finally allocated $g = \mu_i^*$, this good g was contained in $D_i(p)$ before its price was increased by δ. Thus

$$v_i(g) - p(g) \geq v_i(g') - p(g') - \delta$$

for all g' and

$$v_i(g) - p(g) \geq -\delta.$$

Since that iteration, the price of g hasn't changed while the prices of other goods have only increased. Thus the same inequalities hold if we replace p by p^*.

It remains to demonstrate that μ^* is the maximal-weight matching. The argument is similar to the proof of Theorem 26 up to the nuance that now envy-freeness holds with a slack δ. For any μ',

$$\begin{aligned}\sum_{i=1}^n v_i(\mu_i') = \sum_{i=1}^n \left(v_i(\mu_i') - p^*(\mu_i')\right) + p^*(\mu_i') \leq \\ \leq \sum_{i=1}^n \left(v_i(\mu_i^*) - p^*(\mu_i^*) + \delta\right) + p^*(\mu_i') = n\delta + \sum_{i=1}^n v_i(\mu_i^*).\end{aligned}$$

Since values are integral and $n\delta \in (0,1)$, this inequality holds if and only if

$$\sum_{i=1}^n v_i(\mu_i') \leq \sum_{i=1}^n v_i(\mu_i^*)$$

and thus μ^* is the maximal-weight matching. □

In (Karlin and Peres, 2017, Section 17.2.1), you can read about another exciting and unexpected connection between CE and auctions. It turns out, that the allocation and prices of the VCG mechanism are envy-free. In particular, a CE can be found in a strategy-proof way.

16.3 Matching markets with money

The model that we've just considered was introduced by Shapley and Shubik (1971) and is sometimes referred to as the Shapley-Shubik assignment game. It has an alternative important interpretation of a *two-sided market* with money. By two-sided we mean that "resources" also have preferences over whom they are allocated. Examples of two-sided markets with money include labor markets matching workers with firms, gig-economy platforms matching users with service-providers (Taskrabbit, Uber, etc) or platforms matching buyers and sellers (Amazon). Such markets are also called *matching markets*.

Imagine that the set $\{1, \ldots, n\}$ are firms each with one vacancy and $\{g_1, \ldots, g_n\}$ are workers. If worker g works for firm i, she generates $v_i(g)$ units of value which are to be split between i and g into i's revenue $u_i \geq 0$ and g's wage $p(g) \geq 0$ so that $v_i(g) = u_i + p(g)$.

Definition 14. *A matching with money is a triplet* (μ, u, p)*, where* $\mu\colon \{1, \dots, n\} \to G$ *is a bijection,* $u \in \mathbb{R}^n_+$ *is a vector of revenues,* $p \in \mathbb{R}^G_+$ *is a vector of wages such that*

$$v_i(\mu_i) = u_i + p(\mu_i)$$

for any firm i*.*

Definition 15. *Given* (μ, u, p)*, a pair* (i, g) *such that* $\mu_i \neq g$ *is a blocking pair if*

$$v_i(g) > u_i + p(g).$$

A blocking pair (i, g) makes the matching unstable as both prefer to be matched with each other compared to what they get in (μ, u, p): indeed, i and g can agree to divide $v_i(g) = u_i' + p'(g)$ in a way that $u_i' > u_i$ and $p'(g) > p(g)$.[4]

Definition 16. (μ, u, p) *is stable if there are no blocking pairs.*

An unstable matching will evolve over time, agents rematch, and/or wages adjust until it reaches the steady state of a stable matching. Does this steady state always exist? It turns out that this question is almost equivalent to the existence of a CE.

Theorem 28. *A stable matching matching with money* (μ, u, p) *exists.*

Proof. By the definition of a blocking pair, (μ, u, p) is stable if and only if

$$u_i + p(g) \geq v_i(g) \tag{16.1}$$

for any i and g.

Let's imagine for a moment that firms are agents and workers are goods with values $v_i(g)$ and consider a CE (μ, p). A pricing in a CE is envy-free and so

$$v_i(\mu_i) - p(\mu_i) \geq v_i(\mu_j) - p(\mu_j)$$

for all i and j or, equivalently,

$$v_i(\mu_i) - p(\mu_i) \geq v_i(g) - p(g)$$

for all i and g. Defining $u_i = v_i(\mu_i) - p(\mu_i)$ we obtain

$$u_i \geq v_i(g) - p(g)$$

for all i and g. Comparing this inequality to (16.1), we see that (μ, u, p) is stable. □

[4]This lack of stability is similar to the way in which violation of Pareto optimality creates opportunities to trade.

Chapter 17

Matching markets without money

In the previous lecture, we considered stability in two-sided markets with money, e.g., labor markets. The match and prices/wages may be determined in a centralized way as in the case of ride-sharing services such as Uber. Alternatively, the match and the prices can be discovered in a decentralized way as it happens on Taskrabbit: each worker can announce her skills and expected wage and the other side of the market may contact the preferred worker given this information. Overdemanded workers will increase their wage expectations, underdemanded decrease pushing the market towards the competitive equilibrium. These market forces work especially well because the match is governed by prices which aggregate the information about market participants' preferences in a simple finite-dimensional way.

Market forces are less efficient in matching markets without money and they usually fail to operate well without centralized clearinghouses. Markets without money are omnipresent: assigning students to public schools,[1] college admission,[2] women-men marriage market,[3] assigning engineers to teams within a firm,[4] or interns to internships.[5] Roth (2015) is an insightful and engaging popular book about challenges and successes matching-market design in practice.

The model is as follows. There is a set $S = \{s_1, \ldots, s_n\}$ of students and $C = \{c_1, \ldots, c_n\}$ of colleges, each having one seat.[6] Each student s has strict preference $\succ_s$ over C and each

[1]Schools have "preferences" over students captured by priorities, e.g., students having a siblings in the school or living close by get higher priority. These preferences usually have huge indifference classes and choosing the right tie-breaking rule is a separate hotly debated topic.

[2]In countries having a nation-wide exam, colleges' "preferences" over students may be quite fine. Note that different programs may be interested in different aspects of students' performance and so the college preferences are rather aligned but not identical.

[3]This is a standard metaphor for a two-sided market. Arguably, this market does not require a centralized clearinghouse (dating app developers may disagree though) but the stability issues which we will discuss make perfect sense.

[4]Engineers have preferences over teams and so do team leads over engineers.

[5]National Resident Matching Program (NRMP) assigning medical graduates to internships was the first matching market redesigned by economists.

[6]One can chose any other metaphor for the two sides of the market. Extension to non-unit capacities and unequal numbers of seats and students is also straightforward (e.g, add auxiliary colleges $\emptyset_i$ where all unmatched students go). We stick to the most basic model for the sake of simplicity.

college c has strict preference $\succ_c$ over S.[7] A matching μ determines which college each student is assigned to. The formal definition may look confusing at first glance as we want to be able to write $\mu(s)$ for the college where s is assigned and $\mu(c)$ for the student assigned to c. Formally, a matching μ is a bijection $S \cup C \to S \cup C$ such that $\mu(S) = C$ and $\mu(s) = c$ implies $\mu(c) = s$. A matching mechanism maps a profile of preferences $\big((\succ_s)_{s\in S}, (\succ_c)_{c\in C}\big)$ to a matching μ.

Example 12 (Serial dictatorship)**.** Consider a special case of this model, where preferences on one side of the market, say, colleges are all the same (e.g., students are ranked by the number of points earned in a nation-wide exam). Assume that each college ranks the students as $s_1, s_2, s_3, \ldots$. In this case, a reasonable mechanism to use is the familiar Serial Dictatorship: let s_1 choose her best college, then let s_2 choose one among those with vacant seats, and so on. For example, this mechanism is strategy-proof and Pareto optimal for students.

What should we do if preferences are distinct on both sides of the market?

17.1 Immediate Acceptance aka Boston mechanism

The following mechanism had been used in Boston to assign students to public schools until 2005. This mechanism referred to as the Immediate Acceptance (IA) is very intuitive and, as a result, is ubiquitous. It operates as follows:

- Initialization: empty matching μ
- While (there are unmatched students and colleges)
  - Each unmatched s applies to s's best c among those that have not yet been filled
  - Each c accepts the best applicant, rejects others
- End.

This procedure can be implemented as a black box (students submit their preferences and the algorithm determines the matching) or as a dynamic procedure with several application "waves" where yet unmatched literally apply to unfilled colleges. No matter what implementation is used, IA is deeply flawed.

Example 13. Consider 3 students and 3 colleges with the following preferences:

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
c_1	c_1	c_2
c_2	c_2	c_3
c_3	c_3	c_1

$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$
s_1	s_2	s_3
s_2	s_1	s_1
s_3	s_3	s_2

[7] Non-strict preferences on one side of the market, say, colleges are also easy to accommodate.

At the first wave of IA, both s_1 and s_2 apply to their most preferred college c_1 and s_3 applies to c_2. The student s_1 gets admitted to c_1, the student s_3 is admitted to c_2, the student s_2 gets rejected. At the second wave s_2 is forced to apply to c_3 (the only remaining college with an empty seat) and gets admitted. We obtain the following match:

$$s_1 - c_1, \quad s_2 - c_3, \quad s_3 - c_2.$$

In this example, it is easy to see that **IA is manipulable**: By top-ranking c_2 instead of c_1, the second student would get into her second-best college instead of her third-best.

Manipulability is a consequence of the fact that IA penalizes those students who fail to get admitted at the first wave: the only option remaining for them are unpopular colleges that still have empty seats. This opens a possibility for manipulation: a student needs to top-rank a school that she both likes and where she has high chances to be accepted.[8] The outcome of the Boston school-choice mechanism was unpredictable, depended on the strategic abilities of participants, and suffered from poor match quality.

There is another crucial flaw of IA: **lack of stability**. The definition of a blocking pair that we saw for markets with money naturally adapts to the present setting.

Definition 17. *Unmatched (s, c) form a blocking pair in μ if*

$$c \succ_s \mu(s) \quad \textit{and} \quad s \succ_c \mu(c),$$

i.e., both would prefer to be matched together to their matches in μ.

Definition 18. *A matching μ is stable if there are no blocking pairs.*

In our example, (s_2, c_2) is a blocking pair so, indeed, IA does not guarantee stability. Lack of stability leads to unravelling of the market as participants can find better matches "outside of the mechanism."

Because of these flaws, the Boston mechanism was replaced in 2005 (Abdulkadiroglu et al., 2006). Now we will discuss the replacement.

17.2 Deferred Acceptance aka Gale-Shapley algorithm

The Deferred Acceptance algorithm was introduced by Gale and Shapley (1962). Alvin Roth was the first to realize the practical value of this theoretical result which has lead to improving many real markets and, in fact, led to creation of the very field of market design. Roth and Shapley received the 2012 Nobel prize.[9]

DA is identical to IA up to a seemingly minor detail: all the acceptances throughout the execution of the algorithm become tentative so that a college may reject a student that has been accepted at first stages in favor of a more preferred student that applies at later stages. DA runs as follows:

[8] Boston school district even issued the recommendations of how to strategize.

[9] David Gale passed away in 2008 and the Nobel prize is not awarded posthumously.

- Initialization: empty matching μ
- While (there are unmatched students and colleges)
  - Each unmatched s applies to s's best c among those that **have not yet rejected** s
  - Each c **tentatively** accepts the best s **who has ever applied**, rejects other applicants
- Tentative acceptances are finalized.

Example 14. Consider the preference profile from Example 13. At the first wave, both s_1 and s_2 apply to c_1 and s_3 applies to c_2. The students s_1 and s_3 get tentatively accepted and s_2 is rejected. At the second wave, s_2 applies to her second-best college c_2. Now c_2 must select between tentatively accepted s_3 and newly applied s_2. The college rejects s_3 and tentatively accepts s_3. At the third wave, s_3 applies to g_3 and gets tentatively accepted. As a result, the algorithm terminates with the following matching:

$$s_1 - c_1, \quad s_2 - c_2, \quad s_3 - c_3.$$

One can check that this matching is stable and this is not a coincidence.

Theorem 29 (Gale and Shapley (1962))**.** *DA terminates with a stable matching μ.*

Proof. First let's see that the algorithm terminates with a matching. Indeed, imagine that a student s remains unmatched. As students go down the list of their preferences, it means that s has been rejected by all the colleges. Since, colleges reject a student only in favor of another student, s can get rejected by all the colleges only if $|S| > |C|$ but the two sets are assumed to be of equal size. Similarly, no college can remain unmatched as this unavoidably creates an unmatched student and we just showed that there are no such students. We conclude that DA terminates with a matching μ.

Let us check that μ is stable. Consider a pair (s, c) such that $\mu(s) \neq c$. Throughout the algorithm students apply to less and less preferred colleges (if have been rejected by more preferred ones) and colleges tentatively accept more and more preferred students. Hence, either s has not applied to c in which case $\mu(s) \succ_s c$, or s has got rejected by c in which case $\mu(c) \succ_c s$. Thus (s, c) is not a blocking pair and so μ is stable. $\square$

Corollary 5. *A stable matching exists and can be found in polynomial time.*

Note that DA treats students and colleges asymmetrically and the version of the algorithm that we considered is usually referred to as *student-proposing* DA: students go down the list of their preferences applying to less and less preferred colleges while colleges receive better and better applications. One can reverse the roles of students and colleges and get the *college-proposing* DA, where colleges go down the list of their preferences and students tentatively accept more and more attractive offers. The notion of stability is symmetric with respect to the two sides of the market and so college-proposing DA also returns a stable matching. Is it the same matching as in the case of student-proposing DA?

Example 15. Consider the following two-student two-college profile:

$\succ_{s_1}$	$\succ_{s_2}$
c_1	c_2
c_2	c_1

$\succ_{c_1}$	$\succ_{c_2}$
s_2	s_1
s_1	s_2

Student-proposing DA results in

$$s_1 - c_1, \quad s_2 - c_2$$

while college-proposing gives

$$s_1 - c_2, \quad s_2 - c_1.$$

Based on this example, we conclude that a stable matching may not be unique. We also see that students are better off in the student-proposing DA while colleges are better off in college-proposing DA. We will see that this is a general phenomenon.

Denote by $\mu^*(s)$ the most preferred college of s among those colleges c that there exists a stable matching μ' where $\mu'(s) = c$.

Theorem 30 (student-optimality of student-proposing DA)**.** *Student-proposing DA matches each s with $\mu^*(s)$.*

Proof. Towards a contradiction, assume that some some s is rejected by $\mu^*(s)$ throughout the execution of DA. Consider the first such rejection: a student s is rejected by $c = \mu^*(s)$ in favor of some s'. Hence,

$$s' \succ_c s.$$

Since s' has not yet been rejected by $\mu^*(s')$ and students go down their preference list, we conclude that

$$c \succ_{s'} \mu^*(s').$$

Now consider a stable matching μ' where s is matched to c. Hence, $\mu'(s') = c' \neq c$. By the definition of $\mu^*(s')$,

$$\mu^*(s') \succeq_{s'} c'$$

and so

$$c \succ_{s'} c'.$$

Therefore, $c \succ_{s'} c' = \mu'(s')$ and $s' \succ_c s = \mu'(c)$. Thus (s', c) is a blocking pair in μ' which contradicts its stability. This contradiction completes the proof. □

Remark 1. An analogous argument shows that student-proposing DA matches each college with the worst student it can get in a stable matching. Similarly, the college-proposing DA gives the best stable outcome for colleges and the worst for students. This optimality concern, however, is almost irrelevant in practice as, in large markets, both in theory and practice, a stable matching is likely to be unique (Ashlagi et al., 2017).

Student-optimality of DA can be seen as restricted Pareto optimality.

Definition 19. *Consider a profile of preferences of a set N of agents over a set A of alternatives. Given $N' \subset N$ and $A' \subset A$, an alternative $a \in A$ is (N', A')-restricted Pareto optimal if there is no $a' \in A'$ such that $a' \succeq_i a$ for all $i \in N'$ and $a' \succ_j a$ for some $j \in N'$.*

Hence, Theorem 30 can be reformulated as follows: student-proposing DA is (student,stable)-restricted Pareto optimal. The following example demonstrates that one cannot drop the restriction to stable counterfactual matchings as there may be unstable matchings that are preferred by the students.

Example 16 (Pathak (2011))**.** Consider the following preference profile:

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
c_2	c_1	c_1
c_1	c_2	c_2
c_3	c_3	c_3

$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$
s_1	s_2	s_3
s_3	s_1	s_1
s_2	s_3	s_2

In the student-proposing DA, all the students get rejected by their top-choice colleges:

$$s_1 - c_1, \quad s_2 - c_2 \quad s_3 - c_3.$$

The following matching

$$s_1 - c_2, \quad s_2 - c_1 \quad s_3 - c_3$$

is strictly preferred to the DA outcome by s_1 and s_2 while s_3 is indifferent. Note, however, that this improvement is unstable with a blocking pair (s_3, c_1).

In addition to stability, students do not need to strategize.

Theorem 31. *The student-proposing DA is strategy-proof for students.*[10]

Given Theorem 30, strategy-proofness seems rather intuitive. Indeed, if each student already gets her best match, there is no way she can improve it by misreporting. There is, however, a nuance: the manipulation may change the set of stable matchings. Because of this subtlety, the formal proof is rather complicated. An elegant version of it can be found in (Karlin and Peres, 2017, Chapter 10).

[10]Colleges can manipulate though. Try to find an example with $n = 3$.

Chapter 18

Information design

Information design is a sub-field of economic design that deals with the questions of how information available to agents affects their behavior and, most importantly, how to induce the desired behavior by optimally choosing the availability.

To approach these questions, we need to discuss how to model agent's (lack of) information. We already saw the basic idea in the context of revenue-maximizing auctions. There we assumed that agent i's value $v_i(g)$ for the good is a random variable with some distribution F. Agent i is informed of her own value: this is captured by the assumption that she observes *the realization* of $v_i(g)$. Other agents including the seller do not know the value and so they do not observe *the realization* of $v_i(g)$. However, they are not totally agnostic and have some guesses about what this value could be: this is captured by the assumption that *all agents know* F.

In the auction context, we dealt with the extreme cases where some agents have full information (i knows her value) or have no information. How to model agents that are partially informed?

18.1 Signals, beliefs, and the splitting lemma

Let $\theta \in \Theta$ be a state of interest. For simplicity, we will focus on the case of a binary state $\Theta = \{0, 1\}$, for example, it may indicate a quality of a product (low or high) or whether the exchange rate USD/JPY will go down or up tomorrow. The state is uncertain: $\theta = 1$ with probability $p \in [0, 1]$. Imagine that we know p and, additionally, observe a noisy signal m about θ.

A signal is a random variable m with values in some set of messages M, possibly, correlated with θ. We will give all the definitions assuming that M is a finite set even though this assumption can be easily relaxed. Denote by π_0 the distribution of m conditional on $\theta = 0$ and, by π_1, the distribution of m conditional on $\theta = 1$:

$$\pi_0(x) = \mathbb{P}(m = x \mid \theta = 0), \qquad \pi_1(x) = \mathbb{P}(m = x \mid \theta = 1)$$

for all $x \in M$.

Definition 20. *A triplet* (M, π_0, π_1) *is called an information structure.*

Before observing a signal m, our best guess about the probability of $\theta = 1$ is p, usually referred to as prior belief or just prior. If m is not independent of θ (equivalently, $\pi_0 \neq \pi_1$), then observing the realization of m carries some information about the realization of θ. Imagine that we get a signal m equal to some $x \in M$. Our new best guess about θ is called a posterior belief (or posterior for short) and is given by the Bayes formula:[1]

$$p'(x) = \mathbb{P}(\theta = 1 \mid m = x) = \frac{\pi_1(x) \cdot p}{\pi_1(x) \cdot p + \pi_0(x) \cdot (1 - p)}.$$

Since the signal m is a random variable, the posterior belief $p' = p'(m)$ is also a random variable with values in $[0, 1]$.

Example 17. Let the prior be $p = \frac{1}{2}$. Consider a binary signal $m \in M = \Theta = \{0, 1\}$ that matches the state with probability $\frac{1}{2} + \epsilon$, i.e.,

$$\mathbb{P}(m = 1 \mid \theta = 1) = \mathbb{P}(m = 0 \mid \theta = 0) = \frac{1}{2} + \epsilon$$

or, equivalently,

$$\pi_1(1) = \pi_0(0) = \frac{1}{2} + \epsilon.$$

We obtain

$$p'(1) = \frac{\left(\frac{1}{2} + \epsilon\right)\frac{1}{2}}{\left(\frac{1}{2} + \epsilon\right)\frac{1}{2} + \left(\frac{1}{2} - \epsilon\right)\frac{1}{2}} = \frac{1}{2} + \epsilon$$

and

$$p'(0) = \frac{\left(\frac{1}{2} - \epsilon\right)\frac{1}{2}}{\left(\frac{1}{2} - \epsilon\right)\frac{1}{2} + \left(\frac{1}{2} + \epsilon\right)\frac{1}{2}} = \frac{1}{2} - \epsilon.$$

Since $\mathbb{P}(m = 1) = \mathbb{P}(m = 0) = \frac{1}{2}$, the posterior $p'(m)$ takes the values $\frac{1}{2} \pm \epsilon$ equally likely.

For $\epsilon = 0$, m is independent of θ and so totally uninformative. For uninformative signals, the posterior equals the prior. If $m = \theta$ or $m = -\theta$, the signal is fully informative: knowing the realization of m one can recover the realization of θ. Fully informative signals correspond to $\epsilon = \pm\frac{1}{2}$ and the induced posteriors are either 0 or 1, i.e., we either think that $\theta = 1$ with probability 0 or with probability 1 depending on the observed signal. As $|\epsilon|$ increases, m becomes more and more (anti-)correlated with θ and, hence, more informative, and the distribution of posteriors becomes more and more spread.

The following example shows how one can heuristically handle continuous signals despite that the conditional probabilities may not be well-defined when the conditioning is on a zero-probability event.

[1] $\mathbb{P}(A \mid B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B|A)\mathbb{P}(A) + \mathbb{P}(B|\overline{A})\mathbb{P}(\overline{A})}$, where $\overline{A}$ is the complementary event to A.

Example 18. The prior is $\frac{1}{2}$ again but the set of messages M is continuous: $M = [0,1]$. If $\theta = 1$, the signal's distribution π_1 has density $2t$ on $[0,1]$ and, if $\theta = 0$, the signal's distribution π_0 has density $2(1-t)$.

The conditional probability $\mathbb{P}(\theta = 1 \mid m = x)$ is not well-defined as $m = x$ has probability zero. One can still compute this conditional probability by replacing $\{m = x\}$ by the event $\{m \in [x-\delta, x+\delta]\}$ and then letting δ go to $+0$.[2] We get

$$p'(m = x) = \lim_{\delta \to +0} \mathbb{P}(\theta = 1 \mid m \in [x-\delta, x+\delta]) =$$

$$= \lim_{\delta \to +0} \frac{\int_{x-\delta}^{x+\delta} 2t\, dt \cdot \frac{1}{2}}{\int_{x-\delta}^{x+\delta} 2t\, dt \cdot \frac{1}{2} + \int_{x-\delta}^{x+\delta} 2(1-t)\, dt \cdot \frac{1}{2}} = \frac{2x}{2x + 2(1-x)} = x,$$

where we got rid of the integrals by applying L'Hospital's rule.

We conclude that $p'(m) = m$, in other words, the induced posterior coincides with the signal. What is the distribution of the signal? By the formula of total probability,

$$\mathbb{P}(m \in [a,b]) = \frac{1}{2}\mathbb{P}(m \in [a,b] \mid \theta = 1) + \frac{1}{2}\mathbb{P}(m \in [a,b] \mid \theta = 0) =$$

$$= \frac{1}{2}\int_a^b 2t\, dt + \frac{1}{2}\int_a^b 2(1-t)\, dt = \int_a^b 1\, dt$$

and so m is uniformly distributed on $[0,1]$. Thus the posterior $p'(m)$ is also uniformly distributed on $[0,1]$.

In both examples, the average posterior is equal to the prior. This phenomenon is general and is known as the martingale property.[3]

Theorem 32. *For any prior p and any information structure (M, π_0, π_1),*

$$\mathbb{E}[p'(m)] = p.$$

The martingale property states that, on average, beliefs do not change when we get new information. This can be seen as rationality property capturing the time-consistency of beliefs. Indeed, $\mathbb{E}[p'(m)]$ is the best today's prediction for tomorrow's belief. If today, an agent expected that her belief will drift in a particular direction tomorrow, that would be rational to shift the today's belief in this direction as well.

[2] This trick works if we deal with random variables having continuous densities. An alternative more general approach is based on functional analysis and allows to define "regular conditional probability" that is pinned down uniquely for almost all x. For continuous densities the two approaches are equivalent.

[3] In probability theory, martingales are those random processes where the best prediction of the value in the next moment of time is the current value. An example of such a process is the simple random walk. Theory of martingales extends that of random walks and allows for processes whose increments are not necessarily independent. Multistage martingales arise in the context of dynamic learning, where agents observe one signal after another and update their beliefs sequentially.

Proof. In the proof, we assume that the set of messages M is finite. The expected belief can be computed as follows:

$$\mathbb{E}[p'(m)] = \sum_{x \in M} p'(x)\mathbb{P}(m = x) = \sum_{x \in M} \mathbb{P}(\theta = 1 \mid m = x)\mathbb{P}(m = x) = \mathbb{P}(\theta = 1),$$

where the last identity follows from the formula of total probability.[4] Thus $\mathbb{E}[p'(m)] = p$. □

It turns out that the martingale property is not only necessary but also sufficient for a distribution F on $[0, 1]$ to be a distribution of posterior beliefs for some information structure.

Theorem 33 ("splitting lemma" (Aumann et al., 1995))**.** *For any distribution F on $[0, 1]$ such that the mean $\int t\, dF(t)$ is equal to p, there exists an information structure (M, π_0, π_1) such that the posterior belief $p'(m)$ is distributed according to F.*

As we will see, a byproduct of the proof is that any such distribution F can be generated via (M, π_0, π_1) such that $p'(m) = m$, i.e., the belief and the signal coincide. Example 18 is a particular case of the below construction for the uniform distribution F.

Proof. We will prove the result for a discrete F, which allows us to focus on the essence of the argument and avoid technicalities.

Let F place a weight $w_k > 0$ on a point $x_k \in [0, 1]$, $k = 1, \ldots, K$. Hence, $\sum_k w_k = 1$ as F is a probability distribution and $\sum_k x_k \cdot w_k = p$ as the mean equals p by the assumption.

Consider an information structure (M, π_0, π_1) where $M = \{x_1, \ldots, x_K\}$ and

$$\pi_1(x_k) = \frac{x_k}{p} w_k$$
$$\pi_1(x_k) = \frac{1 - x_k}{1 - p} w_k.$$

Both π_1 and π_0 are probability measures as $\sum_k x_k \cdot w_k = p$.

Let us compute the unconditional distribution of the message m. By the formula of total probability,

$$\mathbb{P}(m = x_k) = \pi_1(x_k) \cdot p + \pi_0(x_k) \cdot (1 - p) = w_k.$$

Hence, m has distribution F. Now we find the posterior belief induced by m:

$$p'(x_k) = \frac{\pi_1(x) \cdot p}{\pi_1(x) \cdot p + \pi_0(x) \cdot (1 - p)} = x_k.$$

Thus the posterior belief $p'(m)$ coincides with the signal m and so has distribution F. □

Now we are prepared to discuss information design.

[4] $\mathbb{P}(A) = \sum_{k=1}^{K} \mathbb{P}(A \mid B_k)\mathbb{P}(B_k)$ for any disjoint partition $\Omega = \cup_k B_k$ of the probability space.

18.2 Bayesian persuasion

The model of Bayesian persuasion represents an archetypal situation where two agents have a conflict of interest, one is informed about a payoff-relevant state and can affect the action of the uninformed one by revealing some information to her.

The model was introduced by Kamenica and Gentzkow (2011) and builds on earlier insights of Aumann et al. (1995). Bayesian persuasion is the most popular model of information design and some authors use information design and persuasion as synonyms. The reason for the popularity is the combination of tractability and the ability to capture a wide range of real-life phenomena; see (Kamenica, 2019) for a survey of both the theory and applications.

We will describe the simplest version of the model with a binary state. There are two agents, Sender (S) and Receiver (R) and a random state $\theta \in \Theta = \{0, 1\}$. Both agents know the prior $p = \mathbb{P}(\theta = 1)$.

- Sender selects an information structure (M, π_0, π_1), observes the realization of θ and sends a signal m with distribution π_θ to Receiver.
- Receiver knows the information structure chosen by Sender, observes m but not θ, and selects an action a from her set of actions A.
- The utilities of the agents are $u_S(a, \theta)$ and $u_R(a, \theta)$. Both agents aim to maximize their expected utilities.

We will illustrate the model via the following toy example of a court problem. Other (more practical but less fun) interpretations are discussed afterwards.

Example 19. 75% of defendants are innocent ($\theta = 0$) and 25% are guilty ($\theta = 1$), so $p = \frac{1}{4}$. The prosecutor P knows the realization of θ and can send a signal about it to the judge J who knows p only. The judge decides whether to acquit ($a = 0$) or to convict ($a = 1$). The utilities are

$$u_P(a, \theta) = a, \qquad u_J(a, \theta) = \mathbb{1}_{\{\theta = a\}},$$

i.e., the prosecutor is interested in maximizing the fraction of convicted defendants (no matter guilty or not) and the judge aims to maximize the fraction of fair decisions (acquitted innocent and convicted guilty defendants).

If the judge believes that $\theta = 1$ with probability p', then her best action is

$$a^*(p') = \begin{cases} \text{convict}, & p' \geq \frac{1}{2} \\ \text{acquit}, & \text{otherwise.} \end{cases}$$

Thus the prosecutor's goal is to choose an information structure maximizing $\mathbb{E}[a^*(p')]$ (the fraction of convicted) or, equivalently, $\mathbb{E}\left[\mathbb{1}_{\{p' \geq \frac{1}{2}\}}\right]$.

What should the prosecutor do? Let's look at some ideas:

- *Reveal no information:* Hence, $p' = p = \frac{1}{4}$ and so no defendants are convicted. Prosecutor's utility is zero.

- *Reveal full information* ($m = \theta$)*:* Therefore, the judge learns θ and convicts whenever $\theta = 1$, i.e., $\frac{1}{4}$ of defendants. Prosecutor's utility is $\frac{1}{4}$.

- *Send a noisy signal:* It turns out that the prosecutor can achieve the utility of $\frac{1}{2}$ by sending an optimally-tailored noisy signal.

The optimal information structure will be described once we develop the general method for solving persuasion problems. Here we describe the intuition. The prosecutor either tells "innocent" (which means that the defendant is indeed innocent and so will be acquitted by the judge) or "maybe guilty". This latter signal pools together all the guilty defendants and a certain fraction of innocent ones. This fraction is selected so that it is still optimal for the judge to convict whenever she gets this signal. As a result, all the guilty defendants and a fraction of innocent ones get convicted.

It may seem surprising that the judge will convict $\frac{1}{2}$ of the defendants knowing that only $\frac{1}{4}$ of defendants are guilty. To understand why this happens, note that the judge could ignore the signal sent by the prosecutor if this reduced the fraction of mistakes. But making a decision on her own, the judge would acquit $\frac{1}{4}$ of guilty defendants and so would make 25% of mistakes anyway. Thus the judge listens to the prosecutor whenever this guarantees no more than 25% of mistakes. This is exactly what happens in the optimum: the prosecutor tailors the information so that all the mistakes become in her favor (25% convicted innocent defendants).

Remark 2. The same problem as in the prosecutor-judge example can be used to argue about seemingly unrelated phenomena:

- **universities and employers:** θ is performance of a student (low/high). University wants a good placement for any student, while employers want only high-performance candidates.

  It is optimal for the universities to pull the very best students with good ones:

  > When recruiters call me up and ask me for the three best people, I tell them, "No! I will give you the names of the six best.
  >
  > Robert J. Gordon,
  > Director of Graduate Placement,
  > Northwestern University, Department of Economics

  Bayesian persuasion explains coarse grading ("A,B,C.." instead of a complete ranking) by schools, universities, and industry (Ostrovsky and Schwarz, 2010).[5]

- **sellers and buyers:** θ is the quality of a product (low/high). A seller wants to sell any product and a buyer wants a high-quality product only.

[5] Ostrovsky and Schwarz (2010) considered a particular Bayeian persuasion problem before the general model was introduced by Kamenica and Gentzkow (2011).

Bayesian persuasion explains why ordering by price (or by rating) of goods on Amazon is noisy or why you cannot order apartments by their rating or total price on Airbnb.[6]

- **law enforcement, e.g., road police and drivers:** θ determines whether a given neighbourhood is patrolled (no/yes). Police wants drivers to obey the parking rules while drivers want to obey only if the neighborhood is patrolled. Bayesian persuasion suggests that it may be optimal for police to announce a noisy patrolling schedule: when patrolled, send a signal "maybe patrolled" and also send the same signal in fraction of cases when it is not. This fraction is to be determined so that the signal "maybe patrolled" incentivizes most of the drivers to obey the rules.

18.3 How to solve Bayesian persuasion problems?

We will focus on the case where Sender's utility $u_S = u_S(a)$, i.e., it depends on Receiver's action only as in the examples discussed above. We will avoid discussing the questions of whether maxima are attained.[7] Below we describe the geometric approach to persuasion developed by Kamenica and Gentzkow (2011).[8]

Receiver's problem. Assume that Sender has already chosen the information structure (M, π_0, π_1). What action should the receiver take upon receiving a signal $m = x$? She selects an action $a = a(m)$ maximizing her expected utility:

$$\begin{aligned}\mathbb{E}[u_R(a,\theta) \mid m = x] &= \mathbb{E}\left[u_R(a,1)\mathbb{1}_{\{\theta=1\}} + u_R(a,0)\mathbb{1}_{\{\theta=0\}} \mid m = x\right] = \\ &= u_R(a,1)\mathbb{E}\left[\mathbb{1}_{\{\theta=1\}}\right] + u_R(a,0)\mathbb{E}\left[\mathbb{1}_{\{\theta=0\}}\right] = \\ &= u_R(a,1)p'(x) + u_R(a,0)(1-p'(x)).\end{aligned}$$

We conclude that the optimal action depends on the signal through the induced posterior only, i.e., $a = a^*(p')$, where

$$a^*(p') = \arg\max_{a \in A} \Big(u_R(a,1)p' + u_R(a,0)(1-p')\Big).$$

Sender's problem. Sender can compute the function a^* that transforms induced Receiver's beliefs into actions. Thus the Sender's goal is to maximize

$$\mathbb{E}\left[u_S(a^*(p'))\right] \to \max$$

[6] For platforms, there are many reason to withhold the information: they may want to incentivize users to explore underexplored options (yet unrated new goods or service providers), avoid herding on current high-rated options, support small firms, or to reduce incentives for rating manipulation (e.g., coarse ordering of restaurants by their rating on Google maps). These concerns can also be captured in persuasion models.

[7] Maxima are attained under standard (upper-semi)continuity assumptions. If these assumptions are not satisfied, maxima are to be replaced by suprema.

[8] For an alternative "action-recommendation" approach to information design (more general but often less tractable than the geometric one), see a survey by Bergemann and Morris (2019).

over all information structures (M, π_0, π_1). Denote $U(t) = u_S(a^*(t))$ and let F be the distribution of beliefs p' induced by (M, π_0, π_1). Then

$$\max_{(M,\pi_0,\pi_1)} \mathbb{E}\left[u_S(a^*(p'))\right] = \max_{(M,\pi_0,\pi_1)} \int_0^1 U(t)\, dF(t).$$

By the splitting lemma (Theorem 33), the set of possible distributions F that we can get is exactly the set of distributions with mean p. Thus

$$\max_{(M,\pi_0,\pi_1)} \mathbb{E}\left[u_S(a^*(p'))\right] = \max_{F\colon \int t\, dF(t)=p} \int_0^1 U(t)\, dF(t). \tag{18.1}$$

The latter optimization problem has an elegant geometric solution.

Definition 21. *The concavification of a function f on $[0,1]$ is the pointwise minimal concave function above f:*

$$\mathrm{cav}[f](x) = \min\{\varphi(x)\colon \varphi \text{ is concave on } [0,1] \text{ and } f(y) \leq \varphi(y)\ \forall y \in [0,1]\}.$$

Lemma 2 (concavification as martingale optimization)**.**

$$\mathrm{cav}[U](p) = \max_{F\colon \int t\, dF(t)=p} \int_0^1 U(t)\, dF(t).$$

Proof. Denote the right-hand side by $\psi(p)$. Let us check that ψ is concave. For any F_1 with mean p_1 and F_2 with mean p_2, their convex combination $F = \alpha F_1 + (1-\alpha)F_2$ has mean $p = \alpha p_1 + (1-\alpha)p_2$. Hence,

$$\alpha \int_0^1 U(t)\, dF_1(t) + (1-\alpha)\int_0^1 U(t)\, dF_2(t) = \int_0^1 U(t)\, dF(t) \leq \psi(p).$$

Taking maximum over F_1 and F_2 with fixed mean, we obtain

$$\alpha\psi(p_1) + (1-\alpha)\psi(p_2) \leq \psi(p)$$

and conclude that ψ is concave. Since the point mass at p is also a feasible F, we get that

$$U(p) \leq \psi(p).$$

Hence, ψ is a concave function above U and thus

$$\mathrm{cav}[U](p) \leq \psi(p).$$

On the other hand, for any F with mean p

$$\int_0^1 U(t)\, dF(t) \leq \int_0^1 \mathrm{cav}[U](t)\, dF(t) \leq \mathrm{cav}[U](p),$$

where the first inequality holds as we increase the integrand pointwise and the second follows from the Jensen inequality as the integrand is concave. Taking maximum over F with mean p gives

$$\psi(p) \leq \mathrm{cav}[U](p).$$

We proved that $\psi(p) \leq \mathrm{cav}[U](p)$ and $\psi(p) \geq \mathrm{cav}[U](p)$ and thus $\psi(p) = \mathrm{cav}[U](p)$. $\square$

Combining (18.1) and Lemma 2, we obtain the following theorem characterizing the optimal utility of Sender.

Theorem 34 (Kamenica and Gentzkow (2011))**.** *The maximal expected utility of Sender is equal to*

$$\mathrm{cav}[U](p),$$

i.e., it is given by the concavification of her indirect utility function $U(t) = u_S(a^*(t))$. The optimal information structure induces a distribution of posterior beliefs F such that

$$\mathrm{cav}[U](p) = \int U(t)\, dF(t).$$

Note that, given the distribution F, the information structure can be constructed explicitly as we did in the proof of the splitting lemma (Theorem 33). There is one more observation that further simplifies the solution. We say that $\mathrm{cav}[U](p)$ is supported by a distribution F if F has mean p and

$$\mathrm{cav}[U](p) = \int U(t)\, dF(t).$$

There are just two cases:

- $\mathrm{cav}[U](p) = U(p)$: In this case, $\mathrm{cav}[U](p)$ is supported by the point mass at p which corresponds to revealing no information to Receiver.

- $\mathrm{cav}[U](p) > U(p)$: In this case, one can find a two-point distribution with mean p supporting $\mathrm{cav}[U](p)$.[9]

Corollary 6. *Optimal persuasion requires at most two signals.*[10] *Indeed, in the proof of the splitting lemma, we saw that the number of points in the support of F is the number of signals needed to induce this distribution.*

Example 20. Let us get back to the persuasion problem from Example 19. Sender's indirect utility is

$$U(x) = \mathbb{1}_{\{x \geq \frac{1}{2}\}}$$

and its concavification is given by

$$\mathrm{cav}[U](x) = \begin{cases} 2x & x \leq \frac{1}{2} \\ 1 & x \geq \frac{1}{2} \end{cases}$$

The Sender's optimal utility for prior $p = \frac{1}{4}$ equals $\frac{1}{2}$; it is supported by the two-point distribution F that places equal weight on 0 and 1. Find the information structure inducing this distribution via the construction from the splitting lemma.

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[9] Convince yourself that this statement is true by drawing a picture.
A more abstract argument makes use of convex geometry tools which we introduced for multi-item auctions. A distribution F supporting $\mathrm{cav}[U](p)$ is the outcome of maximization of a linear functional $\int U(t)\, dF(t)$ over a convex set consisting of those F that have mean p. By the Bauer principle, the maximum is attained at an extreme point. One can show that the extreme points of this convex set are exactly the distributions with at most two points in the support and mean p.

[10] More generally, if the state is non-binary, the optimal persuasion requires $|\Theta|$ signals.

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