Extreme Equilibria:

The Benefits of Correlation

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Introduction

Correlated Equilibria (Aumann, 1974) generalize Nash equilibria to allow correlation

- capture pre-play communication, intermediation, and joint randomization
  Bárány (1992), Ben-Porath (1998), Gerardi (2004), Lehrer and Sorin (1997)
- result from natural learning dynamics
  Foster and Vohra (1997), Fudenberg and Levine (1999), Hart and Mas-Colell (2000)
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Broad question: When is there potential value in correlation?

Question in context

CE $\simeq$ adding a recommendation system on top of the existing interaction

- $\Longrightarrow$ What interactions can be improved by a recommendation system?

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CE $\simeq$ adding a recommendation system on top of the existing interaction

- $\implies$ What interactions can be improved by a recommendation system?

CE $\simeq$ outcomes of arbitrary pre-play communication protocols

- $\implies$ What strategic interactions are susceptible to communication / collusion?

Normal-form game

$$\Gamma = \Big(N,\ (A_i)_{i\in N},\ (u_i)_{i\in N}\Big)$$

- $N = \{1, \ldots, n\}$ is finite set of players
- A_i is a finite set of actions of player i
- $A = \prod_{i\in N} A_i$ is the set of action profiles
- $u_i : A \to \mathbb{R}$ is utility of player i

Correlated Equilibria

Definition (Aumann, 1974)

A distribution $\mu \in \Delta(A)$ is a CE if

$$\sum_{a_{-i} \in A_{-i}} \mu(a_i, a_{-i})\, u_i(a_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \mu(a_i, a_{-i})\, u_i(a_i', a_{-i})$$

for all $i \in N$ and all $a_i, a_i' \in A_i$

Interpretation: μ generated by a mediator and players best respond by adhering

Remark: Nash Equilibria (NE) are CE of the form $\mu = \mu_1 \times \ldots \times \mu_n$

Formalizing the Question

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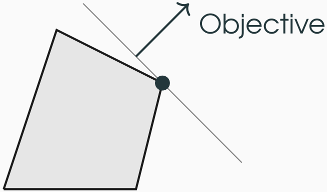
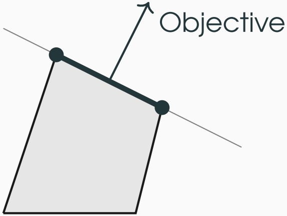
Our Question: When is a Nash equilibrium extreme?

Improvability of non-extreme equilibria

Maximization of a linear objective over a polytope P:





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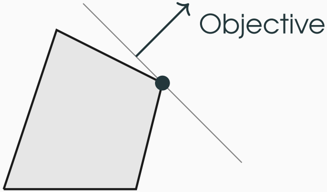

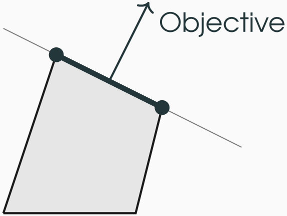

Bauer's Maximum Principle

Generically, any linear or convex objective attains its unique maximum at an extreme point

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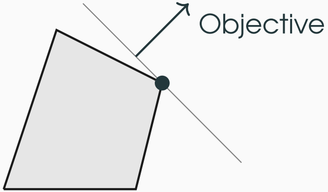

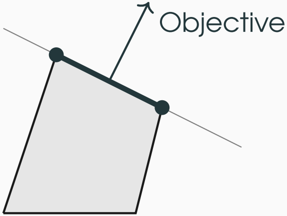

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Main Insight: Improvable, or **non-extreme**, NE are prevalent

Literature

- **Value of correlation in** 2**-player games**: Cripps (1995), Evangelista and Raghavan (1996), Canovas, Hansen, and Jaumard (1999), Nau, Canovas, and Hansen (2004), Peeters and Potters (1999), Calvó-Armengol (2006), Ashlagi, Monderer, and Tennenholtz (2008)
- **Extreme-point approach in info & mech. design:** Manelli and Vincent (2007), Kleiner, Moldovanu, and Strack (2021), Arieli, Babichenko, Smorodinsky, and Yamashita (2023), Yang and Zentefis (2024), Nikzad (2022), Kleiner, Moldovanu, Strack, and Whitmeyer (2024), Lahr and Niemeyer (2024)

Rough Outline

- **Conditions for extremality:**
  in the space of action distributions and payoff space
- **Particular classes of games:**
  symmetric, having unique CE

Conditions for Extremality

Extremality of Nash Equilibria

Theorem

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  $\Rightarrow$ 2-player games not representative

Genericity Assumption

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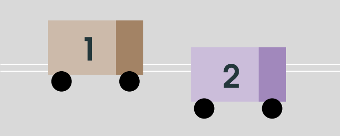
- In a generic game, any NE is regular (Harsanyi, 1973)
- Hence, Theorem $1' \Rightarrow$ Theorem 1

Example: 2 Players vs 3 Players

Example: 2-Player Games

A version of the Game of Chicken by Aumann (1974):



	Risky	Safe
Risky	6, 6	10, 7
Safe	7, 10	9, 9

Example: 2-Player Games

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  Solves linear equation: $6p + 10(1-p) = 7p + 9(1-p) \implies p = 1/2$

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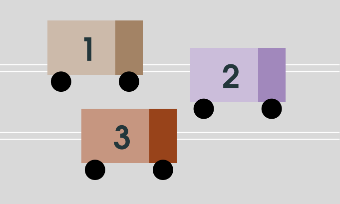
- Indeed, it is the optimum for a non-degenerate objective

$$\text{weight of (Risky, Risky) \& (Safe, Safe)} \to \max$$

Example: 3-Player Games



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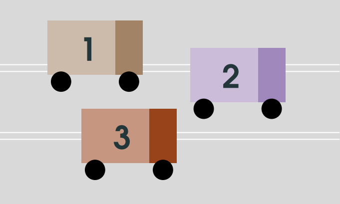

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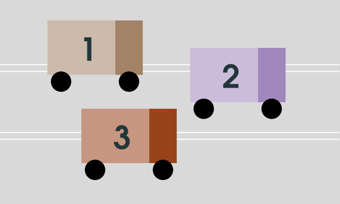

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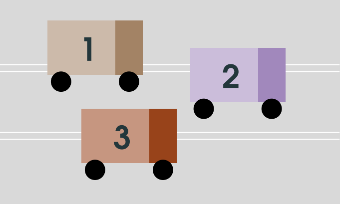

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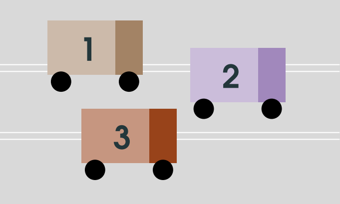

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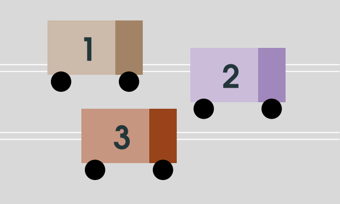

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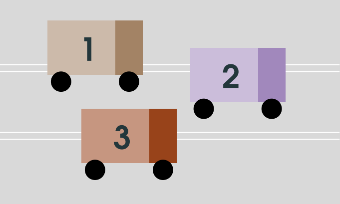

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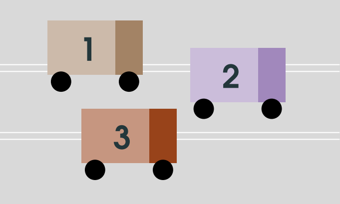

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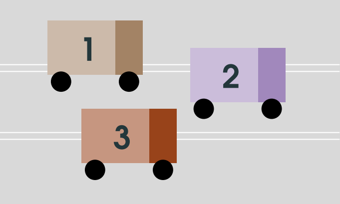

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More than 2 players mixing makes a difference...

General Proof Intuition

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High-level idea: When many players randomize, there are too many ways to correlate their actions $\implies$ one must be beneficial

Focus on a particular example to illustrate

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- Game with n players, each with 2 actions

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- $2n$ constraints
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- $\Rightarrow$ support of an extreme CE μ is bounded by $2n+1$

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- The main difficulty is handling very asymmetric equilibria ▸ details

Utilitarian and Pareto Improvements

Particular Objectives

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Remark: For 2 agents mixing, the NE may or may not be extreme

- Example: the game of chicken

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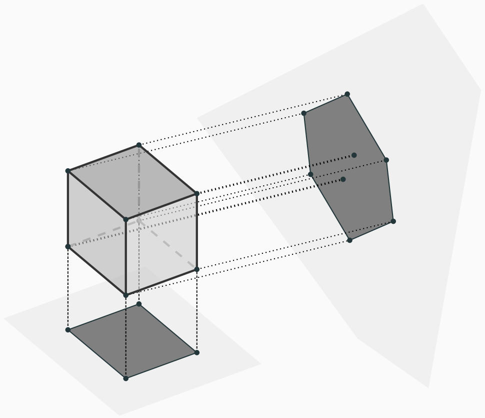

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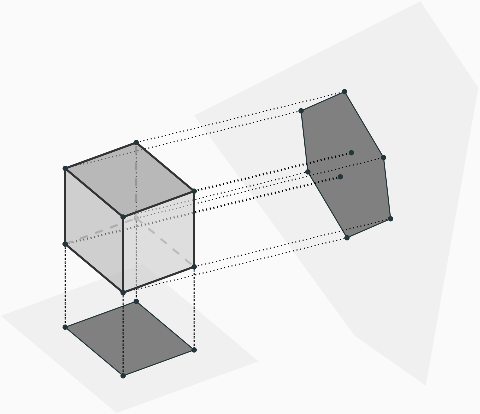

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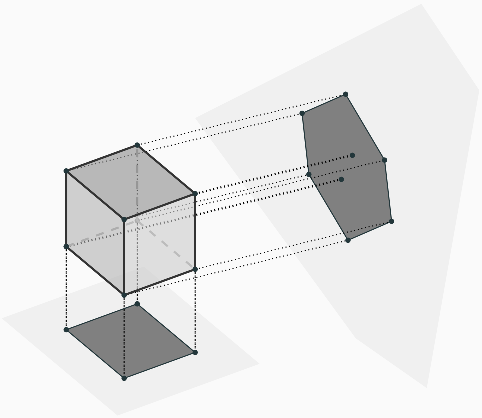

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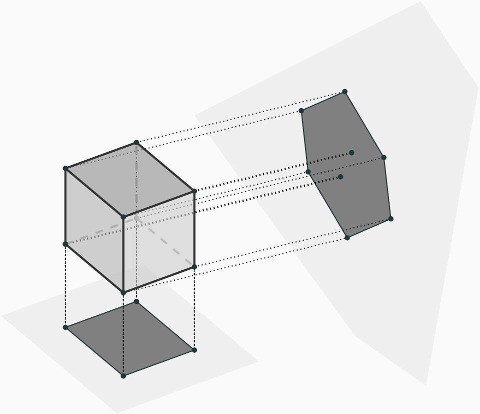
- Projections to the payoff space $\simeq$ generic projection

Geometric Intuition

- CE payoffs $=$ projection of CE to a lower-dimensional space



- For a generic projection, extreme points only originate from extreme points
- Projections to the payoff space $\simeq$ generic projection
- $\Rightarrow$ NE with ≥ 3 mixers cannot lead to extreme payoffs

Pareto Improvability

Proposition

In a generic game, any NE with at least $9 + \log_2(n+1)$ mixers is Pareto dominated

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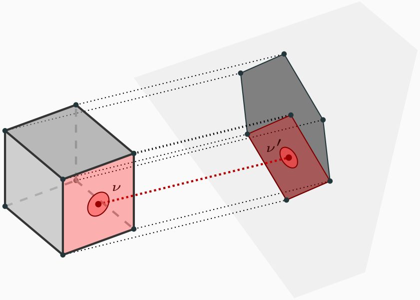

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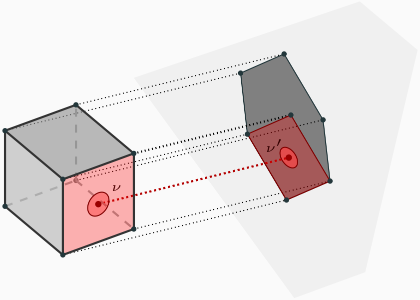

$\geq 9 + \log_2(n+1)$ agents randomizing $\Rightarrow$ in a generic game, NE $\in$ a face of dimension at least n of the CE polytope.

What Extreme CE Look Like

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For a non-extreme NE, any non-degenerate objective can be strictly improved by switching to an extreme CE

Question: What is the structure of extreme CE?

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For a non-extreme NE, any non-degenerate objective can be strictly improved by switching to an extreme CE

Question: What is the structure of extreme CE?

- For general games, we only know that extreme CE have small support
- For symmetric games, we can say more

Symmetric Games

- In many applications, strategic interactions are symmetric
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Take-away: symmetric mixed equilibria in symmetric games are inherently suboptimal

Symmetric CE and Exchangability

Observation:

- For a symmetric CE, the random variables $a_1, \ldots, a_n$ are exchangeable

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Theorem (de Finetti)

Any infinite exchangeable sequence $a_1, a_2, a_3 \ldots$ is a mixture of i.i.d. distributions

Extreme Symmetric CE with Many Players

- Consider a symmetric game with m actions per player
- Assume the number of players n is large

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- A version of Proposition 2 holds: sampling without replacement instead of i.i.d.

details

Example: Symmetric Binary Action Games

- n agents, actions from $\{0, 1\}$
  - Take route A or route B, vote or not, protest or not

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$$u_i(a_i, a_{-i}) = \begin{cases} f\left(\frac{|\{j \in N:\ a_j = 1\}|}{n}\right), & a_i = 1 \\ 0, & a_i = 0 \end{cases}$$

where f is continuous, takes both positive and negative values, and $f(1) < 0$

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- Focus on large-population behavior

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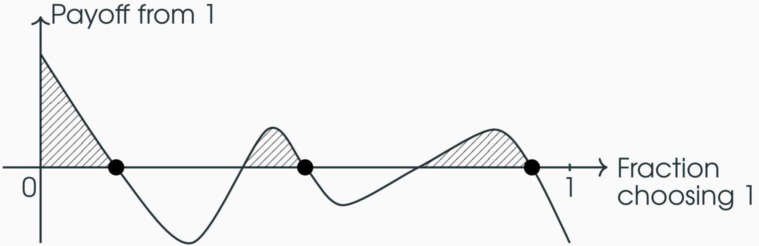

- In shaded areas, incentive to deviate from 0 to 1
- In blank areas, incentive to deviate from 1 to 0

Finding Optimal Correlated Equilibrium

De Finetti's: A symmetric CE $\simeq$ a mixture of i.i.d. distributions

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$\max W = \mathbb{E}[Xf(X)]$ over distributions on [0, 1] subject to $IC = \mathbb{E}[(1-X)f(X)] \leq 0$

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$$\Phi = \Big\{\varphi(x),\ \ x \in [0,1]\Big\} \subset \mathbb{R}^2$$

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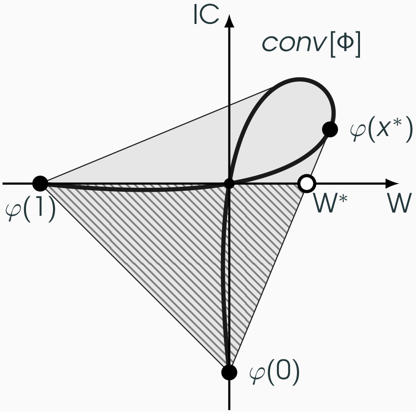

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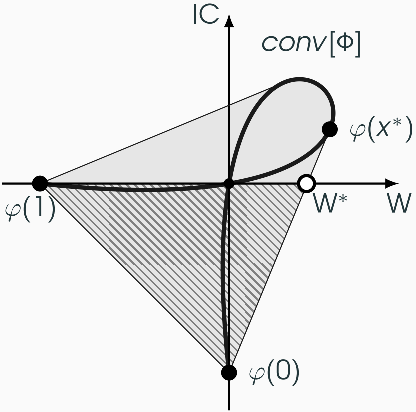

Optimum:
randomize between $x = 0$ and some $x = x^* > 1/2$ with weights making IC bind

Games with Unique Correlated Equilibrium

▸ skip

Games with Unique Correlated Equilibrium

- Unique CE $\implies$ CE=NE
- Such NE is robust to communication/collusion, incomplete information about payoffs, and can be computed without fixed points (Einy et al., 2022)
- **Examples:** games with dominant strategies, some congestion games, Cournot competition

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Corollary

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- a pure Nash, or
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- No genericity assumption since games with a unique CE form an open set (Viossat, 2010)

Conclusions

Several papers effectively show extremality of NE in specific contexts:

- Tullock contests, Cournot and Bertrand, patent races, location games (Einy, Haimanko, and Lagziel, 2022)
- First-price auctions (Feldman, Lucier, and Nisan, 2016)
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Our paper:

- a general tension between equilibrium randomness and extremality
- detail-free criterion for extremality in various settings

Main takeaway: inherent suboptimality of equilibria with a lot of mixing

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Thank you!

Coarse Correlated Equilibria

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Consider situations where agents commit **ex-ante** to a correlating device, before receiving recommendations. For example,

- firms entering binding collusive agreements (McAfee and McMillan, 1992)
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Definition (Hannan, 1957; Moulin and Vial, 1978)

A distribution $\mu \in \Delta(A)$ is a coarse correlated equilibrium (CCE) if, for all $i \in N$,

$$\sum_{a \in A} \mu(a) u_i(a) \geq \max_{a_i' \in A_i} \sum_{a \in A} u_i(a_i', a_{-i}) \mu(a),$$

i.e., the expected payoff from following the correlated strategy is at least as high as the best deterministic deviation

- CCE $\supseteq$ CE $\supseteq$ NE

Coarse Correlated Equilibria: Extremality

Proposition

A NE an extreme point of the set of CCE $\Leftrightarrow$

- it is pure
- or 2 players randomize over 2 actions each and this subgame is strategically equivalent to "matching pennies"

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- No genericity assumption
- The tension between randomness and optimality is even stronger for CCE than for CE
- Consistent with the prevalence of examples where CCE improves over NE, even in two-player games (Moulin and Vial, 1978; Gérard-Varet and Moulin, 1978; Moulin, Ray, and Gupta, 2014; Dokka, Moulin, Ray, and SenGupta, 2023)

Key Lemmas

Key Lemmas

Support Size of Extreme Correlated Equilibria (follows from Winkler (1988))

If μ is an extreme correlated equilibrium, then

$$\text{supp}(\mu) \leq 1 + \sum_{i \in N} |A_i| \cdot (|A_i| - 1)$$

Support Size of Regular Nash Equilibria (McKelvey and McLennan, 1997)

For a regular Nash equilibrium, $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$:

$$\text{supp}(\nu_i) - 1 \leq \sum_{j \neq i} (\text{supp}(\nu_j) - 1), \quad \text{for any player } i$$

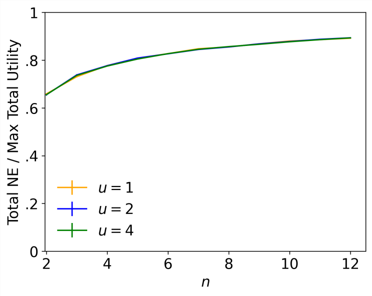
Regularity of Generic games (Harsanyi, 1973)

In a generic game, any Nash equilibrium is regular

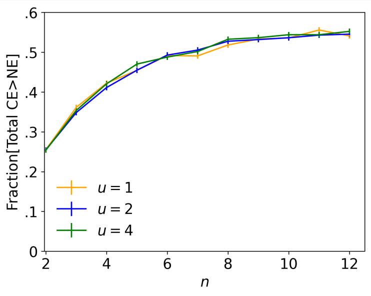
back

Simulations

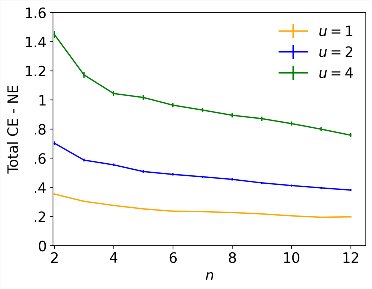
Simulations

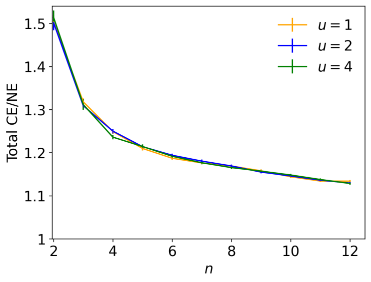


(a) NE/2u



(b) Fraction of CE > NE*





Extreme Symmetric CE with Any Number of Players

Consider n players with m actions each

Proposition

Any extreme symmetric CE can be obtained as follows:

- there are M urns, each with n balls labeled by actions

$$1 \leq M \leq m(m-1)+1$$

- an urn is selected at random according to $p \in \Delta_M$, secretly from players
- players draw balls sequentially without replacement
- i's action = her ball's label, no incentive to deviate

Remark: If n is large, sampling without replacement can be approximated by i.i.d.

back

Bayesian Correlated Equilibria

skip

Bayesian Correlated Equilibria

General games with incomplete information (Bergemann and Morris, 2019):

- Common payoff uncertainty: a finite set of states Θ
- Private information: finite sets of types T_i
- Prior $\pi \in \Delta(\Theta \times T)$

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Definition

A distribution $\psi \in \Delta(A \times \Theta \times T)$ is a BCE if

- its marginal on $\Theta \times T$ coincides with π
- no agent can gain by deviating from a recommended action a_i to another action a_i', given her private type t_i

$$\sum_{a_{-i} \in A_{-i},\ \theta \in \Theta,\ t_{-i} \in T_{-i}} \psi(a, \theta, t)\, u_i(a, \theta, t) \geq \sum_{a_{-i} \in A_{-i},\ \theta \in \Theta,\ t_{-i} \in T_{-i}} \psi(a, \theta, t)\, u_i(a_i', a_{-i}, \theta, t)$$

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A Bayesian Nash equilibrium (BNE) is a BCE where a_i is independent of (θ, a_{-i}, t_{-i}) conditional on t_i for each agent i

Bayesian Correlated Equilibria: Extremality

Theorem

For a generic game with either:

- non-trivial common payoff uncertainty ($|\Theta| \geq 2$), or
- non-trivial private information ($|T_i| \geq 2$ for at least 3 agents),

a BNE is an extreme point of BCE $\Leftrightarrow$ it is pure

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Intuition: Randomness in the state and private types provides more degrees of freedom for correlation to be beneficial

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