Extreme Equilibria:

The Benefits of Correlation

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Introduction

If there is intelligent life on other planets, in a majority of them, they would have discovered correlated equilibrium before Nash equilibrium

Roger Myerson (allegedly)

Introduction

Correlated Equilibria (Aumann, 1974) generalize Nash equilibria to allow correlation

- capture pre-play communication, intermediation, and joint randomization

  Bárány (1992), Lehrer and Sorin (1997), Ben-Porath (1998), Gerardi (2004)
- result from natural learning dynamics

  Foster and Vohra (1997), Fudenberg and Levine (1999), Hart and Mas-Colell (2000)
- efficiently computable

  Papadimitriou and Roughgarden (2008)

Our Question

When is there potential value in correlation?

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In context:

- CE capture adding a recommendation system on top of any existing interaction
- $\implies$ What interactions can be improved by a recommendation system?

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In context:

- CE capture adding a recommendation system on top of any existing interaction
- $\Longrightarrow$ What interactions can be improved by a recommendation system?

- CE capture outcomes of arbitrary communication protocols
- $\Longrightarrow$ What strategic interactions are susceptible to communication?

Our Question

When is there potential value in correlation?

Main Message: Correlation potentially valuable for Nash equilibria with sufficient randomization

Outline

- Extremality and Improvability (+ Literature)
- Conditions for General Improvability
- Utilitarian and Pareto Improvements
- Application to Specific Games

Extremality and Improvability

Normal-form game

$$\Gamma = \left(N,\ (A_i)_{i \in N},\ (u_i)_{i \in N}\right)$$

- $N = \{1, \ldots, n\}$ is finite set of agents
- A_i is a finite set of actions of player i
- $A = \prod_{i \in N} A_i$ is the set of action profiles
- $u_i : A \to \mathbb{R}$ is utility of player i

Correlated Equilibria

Definition (Aumann, 1974)

A distribution $\mu \in \Delta(A)$ is a CE if

$$\sum_{a_{-i} \in A_{-i}} \mu(a_i, a_{-i})\, u_i(a_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \mu(a_i, a_{-i})\, u_i(a_i', a_{-i})$$

for all $i \in N$ and all $a_i, a_i' \in A_i$

Interpretation: μ generated by a mediator and agents best respond by adhering

Remark: Nash Equilibria (NE) are CE of the form $\mu = \mu_1 \times \ldots \times \mu_n$

Formalizing the Question

- The set of correlated equilibria is a convex polytope
- A polytope is the convex hull of its vertices, aka **extreme points**

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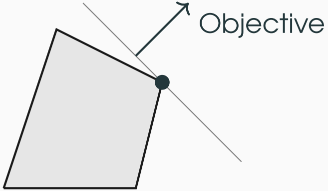
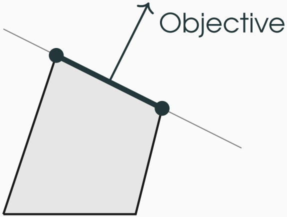
Our Question: When is a Nash equilibrium extreme?

Improvability of non-extreme equilibria

Maximization of a linear objective over a polytope P:





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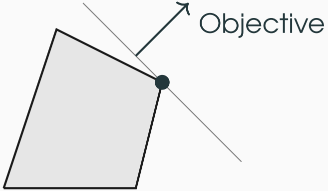

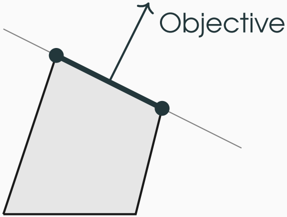

Bauer's Maximum Principle

Generically, any linear or convex objective attains its unique maximum at an extreme point

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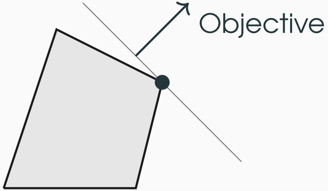

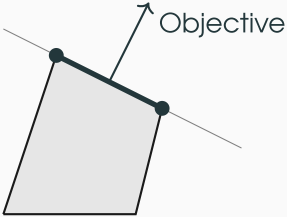

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Generically, any linear or convex objective attains its unique maximum at an extreme point

- $\Rightarrow$ Non-extreme equilibria are generically improvable
- A conservative improvability notion, agnostic to the designer's objective

Literature

- **Value of correlation in** 2**-player games**: Cripps (1995), Evangelista and Raghavan (1996), Canovas, Hansen, and Jaumard (1999), Peeters and Potters (1999), Nau, Canovas, and Hansen (2004), Calvó-Armengol (2006), Ashlagi, Monderer, and Tennenholtz (2008)
- **Extreme-point approach in info & mech. design:** Manelli and Vincent (2007), Kleiner, Moldovanu, and Strack (2021), Nikzad (2022), Arieli, Babichenko, Smorodinsky, and Yamashita (2023), Kleiner, Moldovanu, Strack, and Whitmeyer (2024), Lahr and Niemeyer (2024), Yang and Zentefis (2024)

Conditions for Extremality

Extremality of Nash Equilibria

Theorem

In a generic n-player game, a mixed NE is extreme $\iff$ ≤ 2 agents randomize

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  $\Rightarrow$ 2-player games not representative

Genericity Assumption

- A **generic** game is a game from an open everywhere dense set with complement of 0 Lebesgue measure
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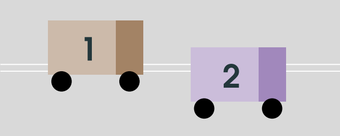
Example: 2 agents vs 3 agents

A version of the Game of Chicken by Aumann (1974):



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Risky	6, 6	10, 7
Safe	7, 10	9, 9

Example: 2-Player Games

	Risky	Safe
Risky	6, 6	10, 7
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	p	$1 - p$

- Mixed NE: $(1/2, 1/2)$ for both agents

  Solves linear equation: $6p + 10(1 - p) = 7p + 9(1 - p) \implies p = 1/2$

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- Aumann (1974): CE can increase utilitarian welfare by shifting weight from (6,6)
- However, the mixed NE is an **extreme point**
- Indeed, it is the optimum for a non-degenerate objective

$$\text{weight of (Risky, Risky) \& (Safe, Safe)} \to \max$$

3-Player Games: A Nod to Nash

Looking at Nash's Princeton dissertation (and 1951 paper):

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Looking at Nash's Princeton dissertation (and 1951 paper):

- Introduces the concept of a non-cooperative game and develops methods for the mathematical analysis of such games
- Analyzes a three-person poker game application

Nash's Dissertation

A DISSERTATION

Presented to the Faculty of Princeton
University in Candidacy for the Degree
of Doctor of Philosophy

Recommended for Acceptance by the
Department of Mathematics

May, 1950

Nash's Dissertation: Abstract

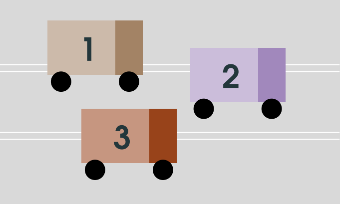
Abstract

This paper introduces the concept of a non-cooperative game and develops methods for the mathematical analysis of such games. The games considered are n-person games represented by means of pure strategies and pay-off functions defined for the combinations of pure strategies.

⋮

As an illustration of the possibilities for application a treatment of a simple three-man poker model is included.

Example: 3-Player Games



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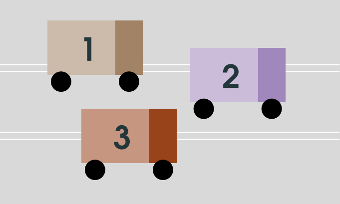

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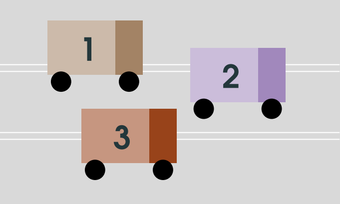

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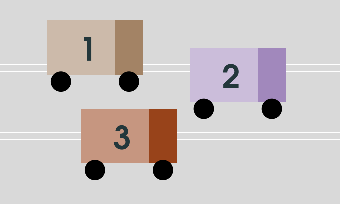

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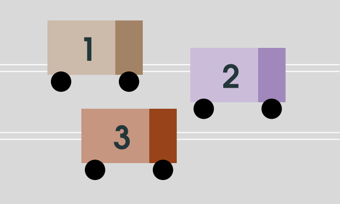

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- Symmetric Mixed NE: $(\sqrt{3/2}-1,\ 2-\sqrt{3/2})$ for each player

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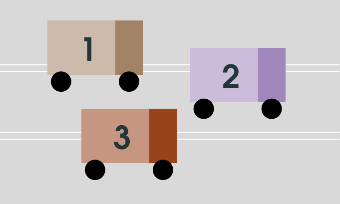

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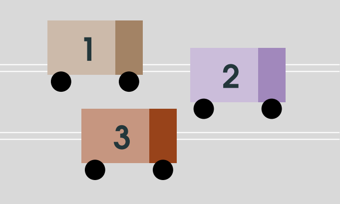

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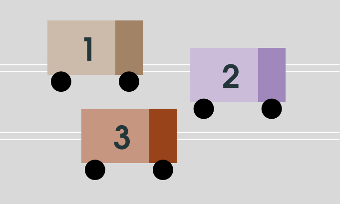

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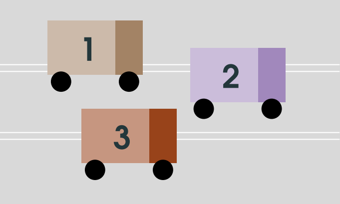

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More than 2 agents mixing makes a difference...

General Proof Intuition

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High-level idea: When many agents randomize, there are too many ways to correlate their actions $\implies$ one must be beneficial

Focus on a particular example to illustrate

General Proof Intuition

- Game with n agents, each with 2 actions

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- At most $2n$ constraints
- Winkler (1988): if k linear constraints are imposed on the set of all distributions $\Delta(A)$, extreme distributions have support $\leq k + 1$
- $\Rightarrow$ Support of an extreme CE μ is bounded by $2n + 1$

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- The main difficulty is handling very asymmetric equilibria ▸ details

Utilitarian and Pareto Improvements

Particular Objectives

- So far, take agnostic perspective: improvability for generic objectives in distribution space
- What about specific objectives like utilitarian welfare or Pareto?

Payoff Objectives

- Use extremality approach as before
- Relevant information is captured in **payoff space**
  $\Rightarrow$ represent equilibria via payoff vectors in $\mathbb{R}^n$, not distributions in $\Delta(A)$

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Definition

A Nash equilibrium is **payoff-extreme** if its payoff vector is an extreme point of the set of CE payoffs

Payoff Extremality and Utilitarian Efficiency

Proposition

In a generic game, any NE ν with three or more agents randomizing:

- not extreme in payoff space
- its utilitarian welfare $\sum_i u_i(\nu)$ can be strictly improved

Geometric Intuition

- CE payoffs $=$ projection of CE to a lower-dimensional space

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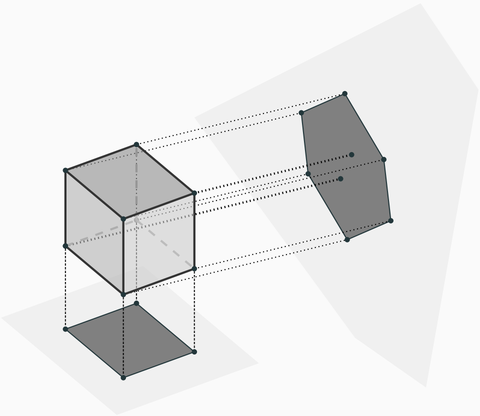

Geometric Intuition (continued)

- Show that in a generic game, each extreme point of CE payoffs corresponds to *unique* extreme CE

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- Show that in a generic game, each extreme point of CE payoffs corresponds to *unique* extreme CE
- Consider NE with more than two agents mixing $\Rightarrow$ not extreme (Theorem)
- In a generic game, if payoff-extreme, must map to an extreme CE
- $\Rightarrow$ Contradiction!

Pareto Improvability

Pareto improvability more demanding than utilitarian improvability

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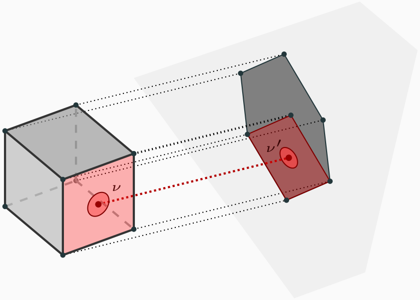
In fact, can show that such NE can be strictly improved for *all* agents

▸ symmetric games ▸ finito

(Handwaving) Intuition

For a generic projection of a polytope to an *n*-dimensional space, the interior of *n*-dimensional faces maps to the interior of the image



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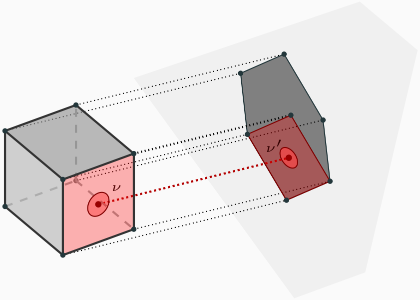

$\geq 9 + \log_2(n+1)$ agents randomizing $\Rightarrow$ in a generic game, NE $\in$ a face of dimension at least n of the CE polytope.

What Extreme CE Look Like

skip

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For a non-extreme NE, any non-degenerate objective can be strictly improved by switching to an extreme CE

Question: What is the structure of extreme CE?

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For a non-extreme NE, any non-degenerate objective can be strictly improved by switching to an extreme CE

Question: What is the structure of extreme CE?

- For general games, we only know that extreme CE have small support
- For symmetric games, we can say more

Symmetric Games

- In many applications, strategic interactions are symmetric
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Theorem (de Finetti)

Any infinite exchangeable sequence $a_1, a_2, a_3 \ldots$ is a mixture of i.i.d. distributions

Extreme Symmetric CE with Many agents

- Consider a symmetric game with m actions per player
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- For an exact result, sampling without replacement instead of i.i.d.

details

Example: Symmetric Binary Action Games

- n agents, actions from $\{0, 1\}$
  - Take route A or route B, vote or not, protest or not

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where f is continuous, takes both positive and negative values, and $f(1) < 0$

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- Focus on large-population behavior and utilitarian welfare

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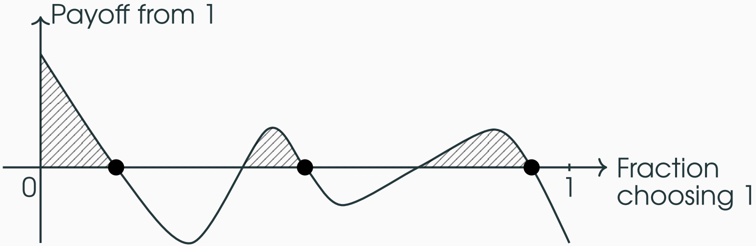

- In shaded areas, incentive to deviate from 0 to 1
- In blank areas, incentive to deviate from 1 to 0

Finding Optimal Correlated Equilibrium

De Finetti: A symmetric CE $\simeq$ a mixture of i.i.d. distributions

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Optimization Problem for Utilitarian Optimal CE

$\max W = \mathbb{E}[Xf(X)]$ over distributions on $[0, 1]$ subject to $IC = \mathbb{E}[(1-X)f(X)] \leq 0$

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$$\Phi = \Big\{\varphi(x),\ \ x \in [0, 1]\Big\} \subset \mathbb{R}^2$$

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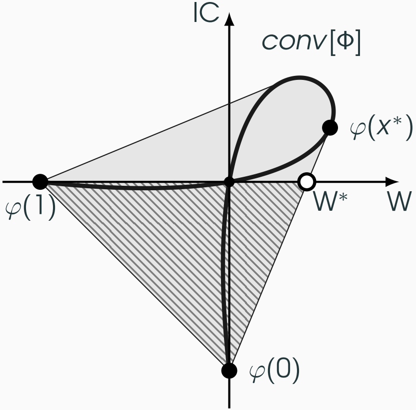
Optimization Problem for Utilitarian Optimal CE 2

$$\max W \quad \text{over} \quad (W, IC) \in conv[\Phi], \;\; IC \leq 0$$

Geometric Solution

Assume:

- f symmetric around $1/2$
- $f(1/2) > 0$
- $f(0) = f(1) < 0$



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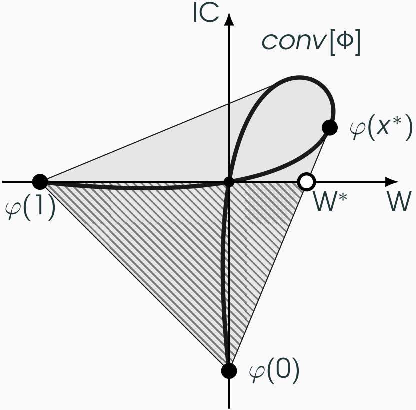

Optimum: Randomize between $x = 0$ and some $x = x^* > 1/2$ with weights making IC bind

Games with Unique Correlated Equilibrium

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Games with Unique Correlated Equilibrium

- Unique CE $\Longrightarrow$ CE=NE
- Such NE is robust to communication/collusion, incomplete information about payoffs, and can be computed without fixed points (Einy et al., 2022)
- **Examples:** games with dominant strategies, some congestion games, Cournot competition

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Corollary

If a game has a unique CE, then it is either:

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Corollary

If a game has a unique CE, then it is either:

- a pure Nash, or
- a Nash where exactly two agents randomize

- No genericity assumption since games with a unique CE form an open set (Viossat, 2010)

Conclusions

Several papers effectively show extremality of NE in specific contexts:

- Tullock contests, Cournot and Bertrand, patent races, location games (Einy, Haimanko, and Lagziel, 2022)
- First-price auctions (Feldman, Lucier, and Nisan, 2016)
- Convex potential games (Neyman, 1997; Ui, 2008)
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Our paper:

- A tension between equilibrium randomness and extremality
- Detail-free criterion for extremality in various settings

The End

Coarse Correlated Equilibria

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Coarse Correlated Equilibria

Consider situations where agents commit **ex-ante** to a correlating device, before receiving recommendations. For example,

- firms entering binding collusive agreements (McAfee and McMillan, 1992)
- users opting in to algorithmic recommendation systems

Coarse Correlated Equilibria

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- users opting in to algorithmic recommendation systems

Definition (Hannan, 1957; Moulin and Vial, 1978)

A distribution $\mu \in \Delta(A)$ is a coarse correlated equilibrium (CCE) if, for all $i \in N$,

$$\sum_{a \in A} \mu(a) u_i(a) \geq \max_{a_i' \in A_i} \sum_{a \in A} u_i(a_i', a_{-i}) \mu(a),$$

i.e., the expected payoff from following the correlated strategy is at least as high as the best deterministic deviation

- CCE $\supseteq$ CE $\supseteq$ NE

Coarse Correlated Equilibria: Extremality

Proposition

A NE an extreme point of the set of CCE $\Leftrightarrow$

- it is pure
- or 2 agents randomize over 2 actions each

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- No genericity assumption
- The tension between randomness and optimality is even stronger for CCE than for CE
- Consistent with the prevalence of examples where CCE improves over NE, even in two-player games (Moulin and Vial, 1978; Gérard-Varet and Moulin, 1978; Moulin, Ray, and Gupta, 2014; Dokka, Moulin, Ray, and SenGupta, 2023)

Key Lemmas

Key Lemmas

Support Size of Extreme Correlated Equilibria (follows from Winkler (1988))

If μ is an extreme correlated equilibrium, then

$$\operatorname{supp}(\mu) \leq 1 + \sum_{i \in N} |A_i| \cdot (|A_i| - 1)$$

Support Size of Regular Nash Equilibria (McKelvey and McLennan, 1997)

For a regular Nash equilibrium, $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$:

$$\operatorname{supp}(\nu_i) - 1 \leq \sum_{j \neq i} (\operatorname{supp}(\nu_j) - 1), \quad \text{for any player } i$$

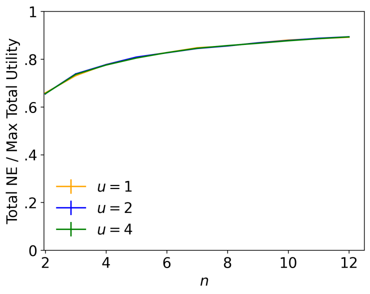
Regularity of Generic games (Harsanyi, 1973)

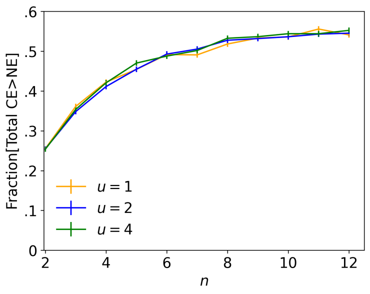
In a generic game, any Nash equilibrium is regular

back

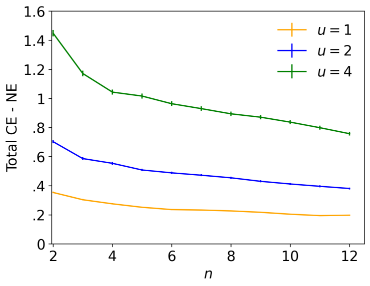
Simulations

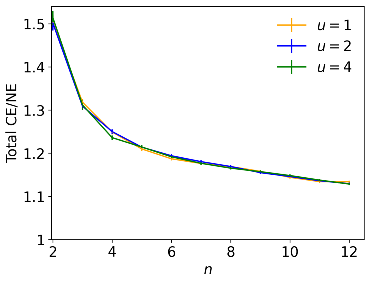
Simulations





(a) NE/2u **(b)** Fraction of CE > NE*





Extreme Symmetric CE with Any Number of agents

Consider n agents with m actions each

Proposition

Any extreme symmetric CE can be obtained as follows:

- there are M urns, each with n balls labeled by actions

$$1 \leq M \leq m(m-1)+1$$

- an urn is selected at random according to $p \in \Delta_M$, secretly from agents
- agents draw balls sequentially without replacement
- i's action = her ball's label, no incentive to deviate

Remark: If n is large, sampling without replacement can be approximated by i.i.d.

back

Bayesian Correlated Equilibria

skip

Bayesian Correlated Equilibria

General games with incomplete information (Bergemann and Morris, 2019):

- Common payoff uncertainty: a finite set of states Θ
- Private information: finite sets of types T_i
- Prior $\pi \in \Delta(\Theta \times T)$

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Definition

A distribution $\psi \in \Delta(A \times \Theta \times T)$ is a BCE if

- its marginal on $\Theta \times T$ coincides with π
- no agent can gain by deviating from a recommended action a_i to another action a_i', given her private type t_i

$$\sum_{a_{-i} \in A_{-i},\ \theta \in \Theta,\ t_{-i} \in T_{-i}} \psi(a,\theta,t)\, u_i(a,\theta,t) \geq \sum_{a_{-i} \in A_{-i},\ \theta \in \Theta,\ t_{-i} \in T_{-i}} \psi(a,\theta,t)\, u_i(a_i', a_{-i}, \theta, t)$$

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A Bayesian Nash equilibrium (BNE) is a BCE where a_i is independent of (θ, a_{-i}, t_{-i}) conditional on t_i for each agent i

Bayesian Correlated Equilibria: Extremality

Theorem

For a generic game with either:

- non-trivial common payoff uncertainty ($|\Theta| \geq 2$), or
- non-trivial private information ($|T_i| \geq 2$ for at least 3 agents),

a BNE is an extreme point of BCE $\Leftrightarrow$ it is pure

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  - Contrast with complete information games, where two agents can mix without losing extremality

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Intuition: Randomness in the state and private types provides more degrees of freedom for correlation to be beneficial

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