Beckmann's approach to multi-item multi-bidder auctions

arXiv:2203.06837

Alexander V. Kolesnikov (HSE)
Fedor Sandomirskiy (Caltech, fsandomi@caltech.edu, https://www.fedors.info/)
Aleh Tsyvinski (Yale)
Alexander P. Zimin (MIT)

Most profitable way to sell m unrelated items to n agents

Bayesian setting: independent private values, seller knows distribution

Most profitable way to sell m unrelated items to n agents

Bayesian setting: independent private values, seller knows distribution

What is known?

	$m = 1$:	$m \geq 2$:
$n = 1$:		
$n \geq 2$:		

Most profitable way to sell m unrelated items to n agents

Bayesian setting: independent private values, seller knows distribution

What is known?

	$m = 1$:	$m \geq 2$:
$n = 1$:	**Single-item monopolist** Myerson (1981): posted price is optimal	
$n \geq 2$:		

Most profitable way to sell m unrelated items to n agents

Bayesian setting: independent private values, seller knows distribution

What is known?

	$m = 1$:	$m \geq 2$:
$n = 1$:	**Single-item monopolist** Myerson (1981): posted price is optimal	
$n \geq 2$:	**Classic auctions** Myerson (1981): 2nd-price auction with reserve is optimal	

Most profitable way to sell m unrelated items to n agents

Bayesian setting: independent private values, seller knows distribution

What is known?

	$m = 1$:	$m \geq 2$:
$n = 1$:	**Single-item monopolist** Myerson (1981): posted price is optimal	**Multi-item monopolist** • optimal mechanisms known in particular cases • connection to optimal transport
$n \geq 2$:	**Classic auctions** Myerson (1981): 2nd-price auction with reserve is optimal	

Most profitable way to sell m unrelated items to n agents

Bayesian setting: independent private values, seller knows distribution

What is known?

	$m = 1$:	$m \geq 2$:
$n = 1$:	**Single-item monopolist** Myerson (1981): posted price is optimal	**Multi-item monopolist** • optimal mechanisms known in particular cases • connection to optimal transport[a] [a] C.Daskalakis, A.Deckelbaum, C.Tzamos (2017) Strong duality for a multiple-good monopolist Econometrica
$n \geq 2$:	**Classic auctions** Myerson (1981): 2nd-price auction with reserve is optimal	

Most profitable way to sell m unrelated items to n agents

Bayesian setting: independent private values, seller knows distribution

What is known?

	$m = 1$:	$m \geq 2$:
$n = 1$:	**Single-item monopolist** Myerson (1981): posted price is optimal	**Multi-item monopolist** • optimal mechanisms known in particular cases • connection to optimal transport[a] [a] C.Daskalakis, A.Deckelbaum, C.Tzamos (2017) Strong duality for a multiple-good monopolist Econometrica
$n \geq 2$:	**Classic auctions** Myerson (1981): 2nd-price auction with reserve is optimal	**Multi-item auctions** • almost nothing known about optimal mechanisms[a] • **Our paper:** connection to optimal transport [a] Combined obstacles of multidimensional screening and multidimensional majorization

Most profitable way to sell m unrelated items to n agents

Bayesian setting: independent private values, seller knows distribution

What is known?

	$m = 1$:	$m \geq 2$:
$n = 1$:	**Single-item monopolist** Myerson (1981): posted price is optimal	**Multi-item monopolist** • optimal mechanisms known in particular cases • connection to optimal transport[a] [a] C.Daskalakis, A.Deckelbaum, C.Tzamos (2017) Strong duality for a multiple-good monopolist Econometrica
$n \geq 2$:	**Classic auctions** Myerson (1981): 2nd-price auction with reserve is optimal	**Multi-item auctions** • almost nothing known about optimal mechanisms[a] • **Our paper:** connection to optimal transport [a] Combined obstacles of multidimensional screening and multidimensional majorization

What will we see?

Strong duality (informal)

For $n \geq 1$ bidders with additive utilities over $m \geq 1$ items

$$\max_{\text{BIC IR mechanisms}} \text{Revenue} = \min_{\text{transport flows}} \text{Cost}$$

- formal statement later
- left-hand side is intuitive $\Rightarrow$ discuss the right-hand side

What will we see?

Strong duality (informal)

For $n \geq 1$ bidders with additive utilities over $m \geq 1$ items

$$\max_{\text{BIC IR mechanisms}} \text{Revenue} = \min_{\text{transport flows}} \text{Cost}$$

- formal statement later
- left-hand side is intuitive $\Rightarrow$ discuss the right-hand side

What will we see?

Strong duality (informal)

For $n \geq 1$ bidders with additive utilities over $m \geq 1$ items

$$\max_{\text{BIC IR mechanisms}} \text{Revenue} = \min_{\text{transport flows}} \text{Cost}$$

- formal statement later
- left-hand side is intuitive $\Rightarrow$ discuss the right-hand side

Transportation problem

For given geographical distribution of production $\pi_+ \in \Delta(X)$ and consumption $\pi_- \in \Delta(X)$, $X \subset \mathbb{R}^d$, find least costly way of transportation

Transportation problem

For given geographical distribution of production $\pi_+ \in \Delta(X)$ and consumption $\pi_- \in \Delta(X)$, $X \subset \mathbb{R}^d$, find least costly way of transportation

Monge-Kantorovich problem

Transportation problem

For given geographical distribution of production $\pi_+ \in \Delta(X)$ and consumption $\pi_- \in \Delta(X)$, $X \subset \mathbb{R}^d$, find least costly way of transportation

Monge-Kantorovich problem

Given:

π_+, π_- and transport costs $c(x, y)$

Transportation problem

For given geographical distribution of production $\pi_+ \in \Delta(X)$ and consumption $\pi_- \in \Delta(X)$, $X \subset \mathbb{R}^d$, find least costly way of transportation

Monge-Kantorovich problem

Given:

π_+, π_- and transport costs $c(x, y)$

Find:

transportation plan γ solving

$$\min_{\substack{\gamma \in \Delta(X \times X) : \\ \text{marginals } \pi_+, \pi_-}} \int c(x, y) \, \mathrm{d}\gamma(x, y)$$

Transportation problem

For given geographical distribution of production $\pi_+ \in \Delta(X)$ and consumption $\pi_- \in \Delta(X)$, $X \subset \mathbb{R}^d$, find least costly way of transportation

Monge-Kantorovich problem

Given:

π_+, π_- and transport costs $c(x, y)$

Find:

transportation plan γ solving

$$\min_{\substack{\gamma \in \Delta(X \times X): \\ \text{marginals } \pi_+, \pi_-}} \int c(x, y) \, \mathrm{d}\gamma(x, y)$$

- archetypal coupling problem, many econ applications
- appears for $n = 1$ bidder
- transportation is immediate

Transportation problem

For given geographical distribution of production $\pi_+ \in \Delta(X)$ and consumption $\pi_- \in \Delta(X)$, $X \subset \mathbb{R}^d$, find least costly way of transportation

Monge-Kantorovich problem

Given:

π_+, π_- and transport costs $c(x, y)$

Find:

transportation plan γ solving

$$\min_{\substack{\gamma \in \Delta(X \times X):\\ \text{marginals } \pi_+, \pi_-}} \int c(x, y)\, \mathrm{d}\gamma(x, y)$$

- archetypal coupling problem, many econ applications
- appears for $n = 1$ bidder
- transportation is immediate

Transportation problem

For given geographical distribution of production $\pi_+ \in \Delta(X)$ and consumption $\pi_- \in \Delta(X)$, $X \subset \mathbb{R}^d$, find least costly way of transportation

Monge-Kantorovich problem

Given:

π_+, π_- and transport costs $c(x, y)$

Find:

transportation plan γ solving

$$\min_{\substack{\gamma \in \Delta(X \times X): \\ \text{marginals } \pi_+, \pi_-}} \int c(x, y)\, \mathrm{d}\gamma(x, y)$$

- archetypal coupling problem, many econ applications
- appears for $n = 1$ bidder
- transportation is immediate

Transportation problem

For given geographical distribution of production $\pi_+ \in \Delta(X)$ and consumption $\pi_- \in \Delta(X)$, $X \subset \mathbb{R}^d$, find least costly way of transportation

Monge-Kantorovich problem

Given:

π_+, π_- and transport costs $c(x, y)$

Find:

transportation plan γ solving

$$\min_{\substack{\gamma \in \Delta(X \times X) : \\ \text{marginals } \pi_+, \pi_-}} \int c(x, y) \, \mathrm{d}\gamma(x, y)$$

Beckmann's problem[a]

- archetypal coupling problem, many econ applications
- appears for $n = 1$ bidder
- transportation is immediate

[a] M.Beckmann (1952) A continuous model of transportation Econometrica

Transportation problem

For given geographical distribution of production $\pi_+ \in \Delta(X)$ and consumption $\pi_- \in \Delta(X)$, $X \subset \mathbb{R}^d$, find least costly way of transportation

Monge-Kantorovich problem

Given:

π_+, π_- and transport costs $c(x, y)$

Find:

transportation plan γ solving

$$\min_{\substack{\gamma \in \Delta(X \times X) : \\ \text{marginals } \pi_+, \pi_-}} \int c(x, y) \, \mathrm{d}\gamma(x, y)$$

Beckmann's problem[a]

Given:

π_+, π_-, costs $\Phi(x)$

- archetypal coupling problem, many econ applications
- appears for $n = 1$ bidder
- transportation is immediate

[a] M.Beckmann (1952) A continuous model of transportation Econometrica

Transportation problem

For given geographical distribution of production $\pi_+ \in \Delta(X)$ and consumption $\pi_- \in \Delta(X)$, $X \subset \mathbb{R}^d$, find least costly way of transportation

Monge-Kantorovich problem

Given:

π_+, π_- and transport costs $c(x, y)$

Find:

transportation plan γ solving

$$\min_{\substack{\gamma \in \Delta(X \times X):\\ \text{marginals } \pi_+, \pi_-}} \int c(x, y)\,\mathrm{d}\gamma(x, y)$$

Beckmann's problem[a]

Given:

π_+, π_-, costs $\Phi(x)$

Find:

flow $f: X \to \mathbb{R}^n$ solving

$$\min_{\substack{f \text{ s.t.}\\ \mathrm{div}[f] = \pi_+ - \pi_-}} \int \Phi(f(x))\,\mathrm{d}x$$

- archetypal coupling problem, many econ applications
- appears for $n = 1$ bidder
- transportation is immediate

[a] M.Beckmann (1952) A continuous model of transportation Econometrica

Transportation problem

For given geographical distribution of production $\pi_+ \in \Delta(X)$ and consumption $\pi_- \in \Delta(X)$, $X \subset \mathbb{R}^d$, find least costly way of transportation

Monge-Kantorovich problem

Given:

π_+, π_- and transport costs $c(x, y)$

Find:

transportation plan γ solving

$$\min_{\substack{\gamma \in \Delta(X \times X): \\ \text{marginals } \pi_+, \pi_-}} \int c(x, y)\, \mathrm{d}\gamma(x, y)$$

- archetypal coupling problem, many econ applications
- appears for $n = 1$ bidder
- transportation is immediate

Beckmann's problem[a]

Given:

π_+, π_-, costs $\Phi(x)$

Find:

flow $f : X \to \mathbb{R}^n$ solving

$$\min_{\substack{f \text{ s.t.} \\ \mathrm{div}[f] = \pi_+ - \pi_-}} \int \Phi(f(x))\, \mathrm{d}x$$

- transportation is continuous
- surprisingly, we get Beckmann not Monge-Kantorovich

[a] M.Beckmann (1952) A continuous model of transportation Econometrica

Transportation problem

For given geographical distribution of production $\pi_+ \in \Delta(X)$ and consumption $\pi_- \in \Delta(X)$, $X \subset \mathbb{R}^d$, find least costly way of transportation

Monge-Kantorovich problem

Given:

π_+, π_- and transport costs $c(x, y)$

Find:

transportation plan γ solving

$$\min_{\substack{\gamma \in \Delta(X \times X) : \\ \text{marginals } \pi_+, \pi_-}} \int c(x, y)\, \mathrm{d}\gamma(x, y)$$

- archetypal coupling problem, many econ applications
- appears for $n = 1$ bidder
- transportation is immediate

Beckmann's problem[a]

Given:

π_+, π_-, costs $\Phi(x)$

Find:

flow $f : X \to \mathbb{R}^n$ solving

$$\min_{\substack{f \text{ s.t.} \\ \operatorname{div}[f] = \pi_+ - \pi_-}} \int \Phi(f(x))\, \mathrm{d}x$$

- transportation is continuous
- surprisingly, we get Beckmann not Monge-Kantorovich

[a] M.Beckmann (1952) A continuous model of transportation Econometrica

Related literature

- **Econ applications of optimal transport**
  - Monge-Kantorovich: Daskalakis et al. (2017), Kleiner, Manelli (2019), Boerma et al.(2021), Chiapporiet et al. (2010), Galichon (2021), Steinerberger, Tsyvinski (2019), Gensbittel (2015), Arieli et al.,(2022), Guo, Shmaya (2021)
  - Beckmann: Fajgelbaum, Schaal (2020), Allenand, Arkolakis (2014), Santambrogio (2015)
- **Non-transport duality in auction design** Giannakopoulos, Koutsoupias (2018), Cai et al. (2019), Bergemann et al. (2016)
- **Simple mechanisms with good revenue guaratees** Hart, Reny (2019), Haghpanah, Hartline (2021), Babaioff et al. (2020,2021), Hart, Nisan (2017), Jehiel et al. (2007), Yao (2017), and many more...
- **Majorization in economics** Hart and Reny (2015), Kleiner et al. (2021), Arieli et al. (2019), Candogan, Strack (2021), Nikzad (2022)

Outline

- Known results: monopolist's problem and its dual
- The case of $n \geq 2$ bidders
    - Similarities and differences
    - Formal statement of duality theorem
- Applications and simulations

Warm-up: $n = m = 1$

- How to sell one good to one agent with the value $v \sim \rho(v)\,\mathrm{d}v$?

Warm-up: $n = m = 1$

- How to sell one good to one agent with the value $v \sim \rho(v)\,\mathrm{d}v$?
- **posted-price mechanism:** chose the best option from the menu
  - pay 0 and get nothing
  - pay t and get the good

Warm-up: $n = m = 1$

- How to sell one good to one agent with the value $v \sim \rho(v)\,\mathrm{d}v$?
- **posted-price mechanism:** chose the best option from the menu
  - pay 0 and get nothing
  - pay t and get the good

  best price $t^* = \arg\max_t t \cdot \int_t^\infty \rho(v)\,\mathrm{d}v$

Warm-up: $n = m = 1$

- How to sell one good to one agent with the value $v \sim \rho(v)\,\mathrm{d}v$?
- **posted-price mechanism:** chose the best option from the menu
  - pay 0 and get nothing
  - pay t and get the good

  best price $t^* = \arg\max_t t \cdot \int_t^\infty \rho(v)\,\mathrm{d}v$

Theorem (Myerson (1981))

Posted price mechanism with t^* maximizes revenue

$m \geq 2$ goods, $n = 1$ agent: menu mechanisms

- agent's values $v = (v_1, \ldots, v_m) \sim \rho(v)\,\mathrm{d}v$
- if agent gets a bundle $x = (x_1, \ldots, x_m) \in [0,1]^m$ and pays t, her utility is $\langle x, v \rangle - t$
- Is selling each good separately always optimal?
- Is bundling all goods together always optimal?
- Is $x \in \{0,1\}^m$ enough?
- **menu mechanism:** chose the best option from the menu
  - pay 0 get 0
  - pay t get x
  - pay t' pay x'
  - ....

$m \geq 2$ goods, $n = 1$ agent: menu mechanisms

- agent's values $v = (v_1, \ldots, v_m) \sim \rho(v)\,\mathrm{d}v$
- if agent gets a bundle $x = (x_1, \ldots, x_m) \in [0,1]^m$ and pays t, her utility is $\langle x, v \rangle - t$
- Is selling each good separately always optimal?
- Is bundling all goods together always optimal?
- Is $x \in \{0,1\}^m$ enough?
- **menu mechanism:** chose the best option from the menu
    - pay 0 get 0
    - pay t get x
    - pay t' pay x'
    - ....

$m \geq 2$ goods, $n = 1$ agent: menu mechanisms

- agent's values $v = (v_1, \ldots, v_m) \sim \rho(v)\,\mathrm{d}v$
- if agent gets a bundle $x = (x_1, \ldots, x_m) \in [0,1]^m$ and pays t, her utility is $\langle x, v \rangle - t$
- Is selling each good separately always optimal?
- Is bundling all goods together always optimal?
- Is $x \in \{0,1\}^m$ enough?
- **menu mechanism:** chose the best option from the menu
  - pay 0 get 0
  - pay t get x
  - pay t' pay x'
  - ....

$m \geq 2$ goods, $n = 1$ agent: menu mechanisms

- agent's values $v = (v_1, \ldots, v_m) \sim \rho(v)\,\mathrm{d}v$
- if agent gets a bundle $x = (x_1, \ldots, x_m) \in [0,1]^m$ and pays t, her utility is $\langle x, v \rangle - t$
- Is selling each good separately always optimal? **No**[1]
- Is bundling all goods together always optimal?
- Is $x \in \{0,1\}^m$ enough?
- **menu mechanism:** chose the best option from the menu
  - pay 0 get 0
  - pay t get x
  - pay t' pay x'
  - ....

[1] Hart, Reny (2015) Maximal revenue with multiple goods: Nonmonotonicity TE

$m \geq 2$ goods, $n = 1$ agent: menu mechanisms

- agent's values $v = (v_1, \ldots, v_m) \sim \rho(v)\,\mathrm{d}v$
- if agent gets a bundle $x = (x_1, \ldots, x_m) \in [0,1]^m$ and pays t, her utility is $\langle x, v \rangle - t$
- Is selling each good separately always optimal? **No**[1]
- Is bundling all goods together always optimal?
- Is $x \in \{0,1\}^m$ enough?
- **menu mechanism:** chose the best option from the menu
  - pay 0 get 0
  - pay t get x
  - pay t' pay x'
  - ....

[1] Hart, Reny (2015) Maximal revenue with multiple goods: Nonmonotonicity TE

$m \geq 2$ goods, $n = 1$ agent: menu mechanisms

- agent's values $v = (v_1, \ldots, v_m) \sim \rho(v)\,\mathrm{d}v$
- if agent gets a bundle $x = (x_1, \ldots, x_m) \in [0,1]^m$ and pays t, her utility is $\langle x, v \rangle - t$
- Is selling each good separately always optimal? **No**[1]
- Is bundling all goods together always optimal? **No**[1]
- Is $x \in \{0,1\}^m$ enough?
- **menu mechanism:** chose the best option from the menu
  - pay 0 get 0
  - pay t get x
  - pay t' pay x'
  - ....

[1] Hart, Reny (2015) Maximal revenue with multiple goods: Nonmonotonicity TE

$m \geq 2$ goods, $n = 1$ agent: menu mechanisms

- agent's values $v = (v_1, \ldots, v_m) \sim \rho(v)\,\mathrm{d}v$
- if agent gets a bundle $x = (x_1, \ldots, x_m) \in [0, 1]^m$ and pays t, her utility is $\langle x, v\rangle - t$
- Is selling each good separately always optimal? **No**[1]
- Is bundling all goods together always optimal? **No**[1]
- Is $x \in \{0, 1\}^m$ enough?
- **menu mechanism:** chose the best option from the menu
  - pay 0 get 0
  - pay t get x
  - pay t' pay x'
  - ....

[1] Hart, Reny (2015) Maximal revenue with multiple goods: Nonmonotonicity TE

$m \geq 2$ goods, $n = 1$ agent: menu mechanisms

- agent's values $v = (v_1, \ldots, v_m) \sim \rho(v)\,\mathrm{d}v$
- if agent gets a bundle $x = (x_1, \ldots, x_m) \in [0,1]^m$ and pays t, her utility is $\langle x, v\rangle - t$
- Is selling each good separately always optimal? **No**[1]
- Is bundling all goods together always optimal? **No**[1]
- Is $x \in \{0,1\}^m$ enough? **No**[1]
- **menu mechanism:** chose the best option from the menu
  - pay 0 get 0
  - pay t get x
  - pay t' pay x'
  - ....

[1] Hart, Reny (2015) Maximal revenue with multiple goods: Nonmonotonicity TE

$m \geq 2$ goods, $n = 1$ agent: menu mechanisms

- agent's values $v = (v_1, \dots, v_m) \sim \rho(v)\,\mathrm{d}v$
- if agent gets a bundle $x = (x_1, \dots, x_m) \in [0,1]^m$ and pays t, her utility is $\langle x, v \rangle - t$
- Is selling each good separately always optimal? **No**[1]
- Is bundling all goods together always optimal? **No**[1]
- Is $x \in \{0,1\}^m$ enough? **No**[1]
- **menu mechanism:** chose the best option from the menu
    - pay 0 get 0
    - pay t get x
    - pay t' pay x'
    - ....

[1] Hart, Reny (2015) Maximal revenue with multiple goods: Nonmonotonicity TE

$m \geq 2$ goods, $n = 1$ agent: menu mechanisms

- agent's values $v = (v_1, \dots, v_m) \sim \rho(v)\,\mathrm{d}v$
- if agent gets a bundle $x = (x_1, \dots, x_m) \in [0,1]^m$ and pays t, her utility is $\langle x, v \rangle - t$
- Is selling each good separately always optimal? **No**[1]
- Is bundling all goods together always optimal? **No**[1]
- Is $x \in \{0,1\}^m$ enough? **No**[1]
- **menu mechanism:** chose the best option from the menu
  - pay 0 get 0
  - pay t get x
  - pay t' pay x'
  - ....

Revelation principle

Any mechanism is equivalent to a menu mechanism

[1] Hart, Reny (2015) Maximal revenue with multiple goods: Nonmonotonicity TE

$m \geq 2$ items, $n = 1$ agent: optimal menus

- a menu $M \subset \mathbb{R}_+ \times [0,1]^m$
- utility obtained by an agent with values $v = (v_1, \ldots, v_m)$:

$$u_M(v) = \max_{(t,x)\in M} \langle x, v\rangle - t,$$

- u_M is convex and

$$x(v) = \partial u_M(v), \quad t(v) = \big\langle x(v), v\big\rangle - u_M(v)$$

$m \geq 2$ items, $n = 1$ agent: optimal menus

- a menu $M \subset \mathbb{R}_+ \times [0,1]^m$
- utility obtained by an agent with values $v = (v_1, \ldots, v_m)$:

$$u_M(v) = \max_{(t,x) \in M} \langle x, v \rangle - t,$$

- u_M is convex and

$$x(v) = \partial u_M(v), \quad t(v) = \big\langle x(v), v \big\rangle - u_M(v)$$

$m \geq 2$ items, $n = 1$ agent: optimal menus

- a menu $M \subset \mathbb{R}_+ \times [0,1]^m$
- utility obtained by an agent with values $v = (v_1, \ldots, v_m)$:

$$u_M(v) = \max_{(t,x) \in M} \langle x, v \rangle - t,$$

- u_M is convex and

$$x(v) = \partial u_M(v), \quad t(v) = \big\langle x(v), v \big\rangle - u_M(v)$$

$m \geq 2$ items, $n = 1$ agent: optimal menus

- a menu $M \subset \mathbb{R}_+ \times [0,1]^m$
- utility obtained by an agent with values $v = (v_1, \ldots, v_m)$:

$$u_M(v) = \max_{(t,x) \in M} \langle x, v \rangle - t,$$

- u_M is convex and

$$x(v) = \partial u_M(v), \quad t(v) = \big\langle x(v), v \big\rangle - u_M(v)$$

Theorem (Rochet and Chone (1998))

$M \leftrightarrow u_M$ is a bijection between menus and convex u_M with $u_M(0) = 0$ and $\partial u_M \in [0,1]^m$

$m \geq 2$ items, $n = 1$ agent: optimal menus

- a menu $M \subset \mathbb{R}_+ \times [0,1]^m$
- utility obtained by an agent with values $v = (v_1, \ldots, v_m)$:

$$u_M(v) = \max_{(t,x) \in M} \langle x, v \rangle - t,$$

- u_M is convex and

$$x(v) = \partial u_M(v), \quad t(v) = \big\langle x(v), v \big\rangle - u_M(v)$$

Theorem (Rochet and Chone (1998))

$M \leftrightarrow u_M$ is a bijection between menus and convex u_M with $u_M(0) = 0$ and $\partial u_M \in [0,1]^m$

Revenue maximization becomes:

$$R_m(\rho) = \max_{\substack{\text{convex } u \\ u(0)=0,\ \partial u \in [0,1]^m}} \int_{\mathbb{R}_+^m} \Big(\big\langle \partial u(v), v \big\rangle - u(v) \Big) \rho(v) \, \mathrm{d}v.$$

$m \geq 2$ items, $n = 1$ agent: optimal menus and transportation

$$R_m(\rho) = \max_{\substack{\text{convex } u \\ u(0)=0,\ \partial u \in [0,1]^m}} \int_{\mathbb{R}_+^m} \Big(\langle \partial u(v), v \rangle - u(v)\Big)\rho(v)\,\mathrm{d}v$$

$m \geq 2$ items, $n = 1$ agent: optimal menus and transportation

$$R_m(\rho) = \max_{\substack{\text{convex } u \\ u(0)=0,\ \partial u \in [0,1]^m}} \int_{\mathbb{R}^m_+} \Big(\langle \partial u(v), v \rangle - u(v) \Big) \rho(v) \, \mathrm{d}v =$$

$\Big[$integrating by parts$\Big]$

$m \geq 2$ items, $n = 1$ agent: optimal menus and transportation

$$R_m(\rho) = \max_{\substack{\text{convex } u \\ u(0)=0,\ \partial u \in [0,1]^m}} \int_{\mathbb{R}_+^m} \Big(\langle \partial u(v), v \rangle - u(v) \Big) \rho(v) \, \mathrm{d}v =$$

[integrating by parts]

$$= \max_{\substack{\text{convex } u \\ u(0)=0,\ \partial u \in [0,1]^m}} \int_{\mathbb{R}_+^m} u(v) \, \mathrm{d}\psi,$$

where $\mathrm{d}\psi = \big((m+1)\rho(v) + \sum_{j=1}^m v_i \partial_{v_i} \rho \big) \, \mathrm{d}v$ (not necessary positive!)

$m \geq 2$ items, $n = 1$ agent: optimal menus and transportation

$$R_m(\rho) = \max_{\substack{\text{convex } u \\ u(0) = 0,\ \partial u \in [0,1]^m}} \int_{\mathbb{R}_+^m} u(v) \, \mathrm{d}\psi$$

$m \geq 2$ items, $n = 1$ agent: optimal menus and transportation

$$R_m(\rho) = \max_{\substack{\text{convex } u \\ u(0)=0,\ \partial u \in [0,1]^m}} \int_{\mathbb{R}_+^m} u(v)\,\mathrm{d}\psi$$

What is the dual?

$m \geq 2$ items, $n = 1$ agent: optimal menus and transportation

$$R_m(\rho) = \max_{\substack{\text{convex } u \\ u(0)=0,\ \partial u \in [0,1]^m}} \int_{\mathbb{R}_+^m} u(v) \, \mathrm{d}\psi$$

What is the dual?

Definition: 2nd-order stochastic dominance aka majorization

$$\mu \succeq \nu \iff \int g \, \mathrm{d}\mu \geq \int g \, \mathrm{d}\nu \ \text{ for any convex monotone g}$$

$m \geq 2$ items, $n = 1$ agent: optimal menus and transportation

$$R_m(\rho) = \max_{\substack{\text{convex } u \\ u(0)=0,\ \partial u \in [0,1]^m}} \int_{\mathbb{R}_+^m} u(v)\,\mathrm{d}\psi$$

What is the dual?

Definition: 2nd-order stochastic dominance aka majorization

$$\mu \succeq \nu \iff \int g\,\mathrm{d}\mu \geq \int g\,\mathrm{d}\nu \quad \text{for any convex monotone g}$$

Theorem (Daskalakis et al (2017))

$$R_m(\rho) = \min_{\substack{\text{positive measures } \gamma \\ \text{on } \mathbb{R}_+^m \times \mathbb{R}_+^m \\ \gamma_1 - \gamma_2 \succeq \psi}} \int_{\mathbb{R}_+^m \times \mathbb{R}_+^m} \|v - v'\|_1\,\mathrm{d}\gamma(v, v')$$

This is Monge-Kantorovich problem with majorization

$m \geq 2$ items, $n \geq 2$ agents: reduction to a single agent

Goal: maximize revenue over BIC, IR, symmetric n-agent mechanisms

Can we use the same approach?

- **Reduced-forms mechanism:** expected allocation and payment of an agent as a function of her vector of values
- As before, one-agent mechanism $\leftrightarrow$ convex u

- $m = 1$ proved by Hart and Reny[2]; equivalent to earlier result by K.Border. General case follows, e.g., from Kleiner et al. (2021)

[2] S.Hart, P.Reny (2015) Implementation of reduced form mechanisms ET Bulletin

$m \geq 2$ items, $n \geq 2$ agents: reduction to a single agent

Goal: maximize revenue over BIC, IR, symmetric n-agent mechanisms

Can we use the same approach? To some extent.

- **Reduced-forms mechanism:** expected allocation and payment of an agent as a function of her vector of values
- As before, one-agent mechanism $\leftrightarrow$ convex u

- $m = 1$ proved by Hart and Reny[2]; equivalent to earlier result by K.Border. General case follows, e.g., from Kleiner et al. (2021)

[2] S.Hart, P.Reny (2015) Implementation of reduced form mechanisms ET Bulletin

$m \geq 2$ items, $n \geq 2$ agents: reduction to a single agent

Goal: maximize revenue over BIC, IR, symmetric n-agent mechanisms

Can we use the same approach? To some extent.

- **Reduced-forms mechanism:** expected allocation and payment of an agent as a function of her vector of values
- As before, one-agent mechanism $\leftrightarrow$ convex u

- $m = 1$ proved by Hart and Reny[2]; equivalent to earlier result by K.Border. General case follows, e.g., from Kleiner et al. (2021)

[2] S.Hart, P.Reny (2015) Implementation of reduced form mechanisms ET Bulletin

$m \geq 2$ items, $n \geq 2$ agents: reduction to a single agent

Goal: maximize revenue over BIC, IR, symmetric n-agent mechanisms

Can we use the same approach? To some extent.

- **Reduced-forms mechanism:** expected allocation and payment of an agent as a function of her vector of values
- As before, one-agent mechanism $\leftrightarrow$ convex u

- $m = 1$ proved by Hart and Reny[2]; equivalent to earlier result by K.Border. General case follows, e.g., from Kleiner et al. (2021)

[2] S.Hart, P.Reny (2015) Implementation of reduced form mechanisms ET Bulletin

$m \geq 2$ items, $n \geq 2$ agents: reduction to a single agent

Goal: maximize revenue over BIC, IR, symmetric n-agent mechanisms

Can we use the same approach? To some extent.

- **Reduced-forms mechanism:** expected allocation and payment of an agent as a function of her vector of values
- As before, one-agent mechanism $\leftrightarrow$ convex u

New feasibility constraint

u corresponds to a symmetric n-agent mechanism if and only if

$$\partial_{v_i} u(v) \preceq z^{n-1} \quad \forall i = 1, \dots m,$$

where $v \sim \rho$ and $z \sim \mathrm{Uniform}([0,1])$.

- $m = 1$ proved by Hart and Reny[2]; equivalent to earlier result by K.Border. General case follows, e.g., from Kleiner et al. (2021)

[2] S.Hart, P.Reny (2015) Implementation of reduced form mechanisms ET Bulletin

$m \geq 2$ items, $n \geq 2$ agents: reduction to a single agent

Goal: maximize revenue over BIC, IR, symmetric n-agent mechanisms

Can we use the same approach? To some extent.

- **Reduced-forms mechanism:** expected allocation and payment of an agent as a function of her vector of values
- As before, one-agent mechanism $\leftrightarrow$ convex u

New feasibility constraint

u corresponds to a symmetric n-agent mechanism if and only if

$$\partial_{v_i} u(v) \preceq z^{n-1} \quad \forall i = 1, \ldots m,$$

where $v \sim \rho$ and $z \sim \mathrm{Uniform}([0,1])$.

- $m = 1$ proved by Hart and Reny[2]; equivalent to earlier result by K.Border. General case follows, e.g., from Kleiner et al. (2021)

[2] S.Hart, P.Reny (2015) Implementation of reduced form mechanisms ET Bulletin

$m \geq 2$ items, $n \geq 2$ agents: primal problem

multi-bidder version of Rochet-Chone theorem

$$R_{n,m}(\rho) = \max_{\substack{\text{convex monotone } u \\ u(0)=0,\ \partial_{v_i} u(v) \preceq z^{n-1}\, \forall i}} n \cdot \int_{\mathbb{R}_+^m} u(v)\, \mathrm{d}\psi(v),$$

where $\mathrm{d}\psi = \left((m+1)\rho(v) + \sum_{j=1}^m v_i \partial_{v_i} \rho\right) \mathrm{d}v$.

- Looks similar to one-agent case
- **Major obstacle:** Local feasibility constraint $\partial u \in [0,1]^m$ is replaced by a non-local non-linear majorization constraint on gradient's distribution. Cannot get rid of u's derivatives.

$m \geq 2$ items, $n \geq 2$ agents: primal problem

multi-bidder version of Rochet-Chone theorem

$$R_{n,m}(\rho) = \max_{\substack{\text{convex monotone } u \\ u(0)=0,\ \partial_{v_i} u(v) \preceq z^{n-1}\,\forall i}} n \cdot \int_{\mathbb{R}_+^m} u(v)\,\mathrm{d}\psi(v),$$

where $\mathrm{d}\psi = \left((m+1)\rho(v) + \sum_{j=1}^m v_i \partial_{v_i}\rho\right)\mathrm{d}v$.

- Looks similar to one-agent case
- **Major obstacle:** Local feasibility constraint $\partial u \in [0,1]^m$ is replaced by a non-local non-linear majorization constraint on gradient's distribution. Cannot get rid of u's derivatives.

$m \geq 2$ items, $n \geq 2$ agents: primal problem

multi-bidder version of Rochet-Chone theorem

$$R_{n,m}(\rho) = \max_{\substack{\text{convex monotone } u \\ u(0)=0,\ \partial_{v_i} u(v) \preceq z^{n-1}\ \forall i}} n \cdot \int_{\mathbb{R}_+^m} u(v)\,\mathrm{d}\psi(v),$$

where $\mathrm{d}\psi = \left((m+1)\rho(v) + \sum_{j=1}^m v_i \partial_{v_i} \rho\right) \mathrm{d}v$.

- Looks similar to one-agent case
- **Major obstacle:** Local feasibility constraint $\partial u \in [0,1]^m$ is replaced by a non-local non-linear majorization constraint on gradient's distribution. Cannot get rid of u's derivatives.

$m \geq 2$ items, $n \geq 2$ agents: primal problem

multi-bidder version of Rochet-Chone theorem

$$R_{n,m}(\rho) = \max_{\substack{\text{convex monotone } u \\ u(0)=0,\ \partial_{v_i} u(v) \preceq z^{n-1}\, \forall i}} n \cdot \int_{\mathbb{R}_+^m} u(v)\, \mathrm{d}\psi(v),$$

where $\mathrm{d}\psi = \left((m+1)\rho(v) + \sum_{j=1}^m v_i \partial_{v_i} \rho\right) \mathrm{d}v$.

- Looks similar to one-agent case
- **Major obstacle:** Local feasibility constraint $\partial u \in [0,1]^m$ is replaced by a non-local non-linear majorization constraint on gradient's distribution. Cannot get rid of u's derivatives.

What is the dual?

$m \geq 2$ items, $n \geq 2$ agents: dual problem

- **Beckmann's problem:**

$$\mathrm{Beck}_\rho(\pi, \Phi) = \min_{f:\ \mathrm{div}[\rho \cdot f] + \pi = 0} \int_{\mathbb{R}_+^m} \Phi(f(v)) \cdot \rho(v) \, \mathrm{d}v.$$

- **The choice of costs:**
  for convex monotone φ_i on $\mathbb{R}_+$ with $\varphi_i(0) = 0$ define

$$\Phi(f) = \sum_{i=1}^{m} \varphi_i^*(|f_i|) \quad \text{where } \varphi_i^*(y) = \sup_x \langle x, y \rangle - \varphi_i(x)$$

$m \geq 2$ items, $n \geq 2$ agents: dual problem

- **Beckmann's problem:**

$$\mathrm{Beck}_\rho(\pi, \Phi) = \min_{f\colon\, \mathrm{div}[\rho \cdot f] + \pi = 0} \int_{\mathbb{R}_+^m} \Phi(f(v)) \cdot \rho(v)\, \mathrm{d}v.$$

- **The choice of costs:**
  for convex monotone φ_i on $\mathbb{R}_+$ with $\varphi_i(0) = 0$ define

$$\Phi(f) = \sum_{i=1}^{m} \varphi_i^*(|f_i|) \quad \text{where } \varphi_i^*(y) = \sup_x \langle x, y \rangle - \varphi_i(x)$$

$m \geq 2$ items, $n \geq 2$ agents: dual problem

- **Beckmann's problem:**

$$\mathrm{Beck}_\rho(\pi, \Phi) = \min_{f:\ \mathrm{div}[\rho \cdot f] + \pi = 0} \int_{\mathbb{R}_+^m} \Phi(f(v)) \cdot \rho(v)\, \mathrm{d}v.$$

- **The choice of costs:**
  for convex monotone φ_i on $\mathbb{R}_+$ with $\varphi_i(0) = 0$ define

$$\Phi(f) = \sum_{i=1}^{m} \varphi_i^*(|f_i|) \quad \text{where } \varphi_i^*(y) = \sup_x \langle x, y \rangle - \varphi_i(x)$$

Strong duality

$$R_{n,m}(\rho) = n \cdot \min_{\substack{\pi \succeq \psi \\ \varphi_i \text{ on } \mathbb{R}_+ \text{ s.t.} \\ \text{convex, monotone, } \varphi_i(0) = 0}} \left[\mathrm{Beck}_\rho\Big(\pi, \Phi\Big) + \sum_{i=1}^{m} \int_0^1 \varphi_i\left(z^{n-1}\right)\, \mathrm{d}z \right]$$

$m \geq 2$ items, $n \geq 2$ agents: dual problem

- $\mathrm{Beck}_\rho(\pi, \Phi) = \min_{f\,:\,\mathrm{div}[\rho \cdot f] + \pi = 0} \int_{\mathbb{R}_+^m} \Phi(f(v)) \cdot \rho(v)\, \mathrm{d}v.$
- $\Phi(f) = \sum_{i=1}^m \varphi_i^*(|f_i|)$ where $\varphi_i^*(y) = \sup_x \langle x, y\rangle - \varphi_i(x)$

Strong duality

$$R_{n,m}(\rho) = n \cdot \min_{\substack{\pi \succeq \psi \\ \varphi_i \text{ on } \mathbb{R}_+ \text{ s.t.} \\ \text{convex, monotone, } \varphi_i(0)=0}} \left[\mathrm{Beck}_\rho\Big(\pi, \Phi\Big) + \sum_{i=1}^m \int_0^1 \varphi_i\left(z^{n-1}\right) \mathrm{d}z \right]$$

- **weak duality ($\leq$) is easy:** see the blackboard
  - gives complementary slackness conditions link
- **ingredients for strong duality:** careful choice of topology, a priori estimates on (u, φ_i), functional minimax theorem
- **the existence of optimal vector field f:** allow for generalized fields (vector measures) and use weak derivatives

$m \geq 2$ items, $n \geq 2$ agents: dual problem

- $\mathrm{Beck}_\rho(\pi, \Phi) = \min_{f:\, \mathrm{div}[\rho \cdot f] + \pi = 0} \int_{\mathbb{R}_+^m} \Phi(f(v)) \cdot \rho(v) \, \mathrm{d}v.$
- $\Phi(f) = \sum_{i=1}^m \varphi_i^*(|f_i|)$ where $\varphi_i^*(y) = \sup_x \langle x, y \rangle - \varphi_i(x)$

Strong duality

$$R_{n,m}(\rho) = n \cdot \min_{\substack{\pi \succeq \psi \\ \varphi_i \text{ on } \mathbb{R}_+ \text{ s.t.} \\ \text{convex, monotone, } \varphi_i(0) = 0}} \left[\mathrm{Beck}_\rho\Big(\pi, \Phi\Big) + \sum_{i=1}^m \int_0^1 \varphi_i\left(z^{n-1}\right) \mathrm{d}z \right]$$

- **weak duality ($\leq$) is easy:** see the blackboard
  - gives complementary slackness conditions link
- **ingredients for strong duality:** careful choice of topology, a priori estimates on (u, φ_i), functional minimax theorem
- **the existence of optimal vector field f:** allow for generalized fields (vector measures) and use weak derivatives

$m \geq 2$ items, $n \geq 2$ agents: dual problem

- $\mathrm{Beck}_\rho(\pi, \Phi) = \min_{f\,:\,\mathrm{div}[\rho \cdot f] + \pi = 0} \int_{\mathbb{R}_+^m} \Phi(f(v)) \cdot \rho(v)\, \mathrm{d}v$.
- $\Phi(f) = \sum_{i=1}^m \varphi_i^*(|f_i|)$ where $\varphi_i^*(y) = \sup_x \langle x, y \rangle - \varphi_i(x)$

Strong duality

$$R_{n,m}(\rho) = n \cdot \min_{\substack{\pi \succeq \psi \\ \varphi_i \text{ on } \mathbb{R}_+ \text{ s.t.} \\ \text{convex, monotone, } \varphi_i(0) = 0}} \left[\mathrm{Beck}_\rho\Big(\pi, \Phi\Big) + \sum_{i=1}^m \int_0^1 \varphi_i\left(z^{n-1}\right) \mathrm{d}z \right]$$

- **weak duality ($\leq$) is easy:** see the blackboard
  - gives complementary slackness conditions link
- **ingredients for strong duality:** careful choice of topology, a priori estimates on (u, φ_i), functional minimax theorem
- **the existence of optimal vector field f:** allow for generalized fields (vector measures) and use weak derivatives

The case for $n = 1$ agent

Question: How can it be that seller's problem admits two duals: Monge-Kantorovich and Beckmann?

The case for $n = 1$ agent

Question: How can it be that seller's problem admits two duals: Monge-Kantorovich and Beckmann?

$$R_{n=1,m}(\rho) = n \;\cdot \min_{\substack{\pi \succeq \psi \\ \varphi_i \text{ on } \mathbb{R}_+ \text{ s.t.} \\ \text{convex, monotone, } \varphi_i(0)=0}} \left[\mathrm{Beck}_\rho\Big(\pi,\, \Phi\Big) + \sum_{i=1}^{m} \int_0^1 \varphi_i\left(z^{n-1}\right)\,\mathrm{d}z \right]$$

The case for $n = 1$ agent

Question: How can it be that seller's problem admits two duals: Monge-Kantorovich and Beckmann?

$$R_{n=1,m}(\rho) = n \cdot \min_{\substack{\pi \succeq \psi \\ \varphi_i \text{ on } \mathbb{R}_+ \text{ s.t.} \\ \text{convex, monotone, } \varphi_i(0)=0}} \left[\mathrm{Beck}_\rho\Big(\pi, \Phi\Big) + \sum_{i=1}^{m} \int_0^1 \varphi_i\left(z^{n-1}\right) \mathrm{d}z \right]$$

The case for $n = 1$ agent

Question: How can it be that seller's problem admits two duals: Monge-Kantorovich and Beckmann?

$$R_{n=1,m}(\rho) = \min_{\substack{\pi \succeq \psi \\ \varphi_i \text{ on } \mathbb{R}_+ \text{ s.t.} \\ \text{convex, monotone, } \varphi_i(0)=0}} \left[\mathrm{Beck}_\rho\Big(\pi, \Phi\Big) + \sum_{i=1}^{m} \int_0^1 \varphi_i\,(1)\,\mathrm{d}z \right]$$

The case for $n = 1$ agent

Question: How can it be that seller's problem admits two duals: Monge-Kantorovich and Beckmann?

$$R_{n=1,m}(\rho) = \min_{\substack{\pi \succeq \psi \\ \varphi_i \text{ on } \mathbb{R}_+ \text{ s.t.} \\ \text{convex, monotone, } \varphi_i(0)=0}} \left[\mathrm{Beck}_\rho\Big(\pi, \Phi\Big) + \sum_{i=1}^{m} \varphi_i(1) \right]$$

The case for $n = 1$ agent

Question: How can it be that seller's problem admits two duals: Monge-Kantorovich and Beckmann?

$$R_{n=1,m}(\rho) = \min_{\substack{\pi \succeq \psi \\ \varphi_i \text{ on } \mathbb{R}_+ \text{ s.t.} \\ \text{convex, monotone, } \varphi_i(0)=0}} \left[\mathrm{Beck}_\rho\Big(\pi,\, \Phi\Big) + \sum_{i=1}^m \varphi_i(1) \right]$$

- if we increase $\varphi_i(x)$, then $\varphi_i^*(y) = \sup_x \langle x, y\rangle - \varphi_i(x)$ decreases
- thus $\varphi_i(x) = x \cdot \varphi_i(1)$ in $[0,1]$ and $\varphi_i(x) = +\infty$ for $x \geq 1$
- optimization over $\varphi_i(1)$ gives $\varphi_i(1) = 0$
- $\varphi_i^*(y) = y$ and so $\Phi = \|\cdot\|_1$

The case for $n = 1$ agent

Question: How can it be that seller's problem admits two duals: Monge-Kantorovich and Beckmann?

$$R_{n=1,m}(\rho) = \min_{\substack{\pi \succeq \psi \\ \varphi_i \text{ on } \mathbb{R}_+ \text{ s.t.} \\ \text{convex, monotone, } \varphi_i(0)=0}} \left[\mathrm{Beck}_\rho\Big(\pi, \Phi\Big) + \sum_{i=1}^m \varphi_i(1) \right]$$

- if we increase $\varphi_i(x)$, then $\varphi_i^*(y) = \sup_x \langle x, y \rangle - \varphi_i(x)$ decreases
- thus $\varphi_i(x) = x \cdot \varphi_i(1)$ in $[0, 1]$ and $\varphi_i(x) = +\infty$ for $x \geq 1$
- optimization over $\varphi_i(1)$ gives $\varphi_i(1) = 0$
- $\varphi_i^*(y) = y$ and so $\Phi = \| \cdot \|_1$

The case for $n = 1$ agent

Question: How can it be that seller's problem admits two duals: Monge-Kantorovich and Beckmann?

$$R_{n=1,m}(\rho) = \min_{\substack{\pi \succeq \psi \\ \varphi_i \text{ on } \mathbb{R}_+ \text{ s.t.} \\ \text{convex, monotone, } \varphi_i(0)=0}} \left[\mathrm{Beck}_\rho\Big(\pi,\, \Phi\Big) + \sum_{i=1}^{m} \varphi_i(1) \right]$$

- if we increase $\varphi_i(x)$, then $\varphi_i^*(y) = \sup_x \langle x, y \rangle - \varphi_i(x)$ decreases
- thus $\varphi_i(x) = x \cdot \varphi_i(1)$ in $[0, 1]$ and $\varphi_i(x) = +\infty$ for $x \geq 1$
- optimization over $\varphi_i(1)$ gives $\varphi_i(1) = 0$
- $\varphi_i^*(y) = y$ and so $\Phi = \| \cdot \|_1$

The case for $n = 1$ agent

Question: How can it be that seller's problem admits two duals: Monge-Kantorovich and Beckmann?

$$R_{n=1,m}(\rho) = \min_{\substack{\pi \succeq \psi \\ \varphi_i \text{ on } \mathbb{R}_+ \text{ s.t.} \\ \text{convex, monotone, } \varphi_i(0)=0}} \left[\mathrm{Beck}_\rho\Big(\pi,\, \Phi\Big) + \sum_{i=1}^{m} \varphi_i(1) \right]$$

- if we increase $\varphi_i(x)$, then $\varphi_i^*(y) = \sup_x \langle x, y \rangle - \varphi_i(x)$ decreases
- thus $\varphi_i(x) = x \cdot \varphi_i(1)$ in $[0, 1]$ and $\varphi_i(x) = +\infty$ for $x \geq 1$
- optimization over $\varphi_i(1)$ gives $\varphi_i(1) = 0$
- $\varphi_i^*(y) = y$ and so $\Phi = \| \cdot \|_1$

The case for $n = 1$ agent

Question: How can it be that seller's problem admits two duals: Monge-Kantorovich and Beckmann?

$$R_{n=1,m}(\rho) = \min_{\pi \succeq \psi} \left[\mathrm{Beck}_\rho\Big(\pi, \Phi\Big)\right]$$

- if we increase $\varphi_i(x)$, then $\varphi_i^*(y) = \sup_x \langle x, y \rangle - \varphi_i(x)$ decreases
- thus $\varphi_i(x) = x \cdot \varphi_i(1)$ in $[0, 1]$ and $\varphi_i(x) = +\infty$ for $x \geq 1$
- optimization over $\varphi_i(1)$ gives $\varphi_i(1) = 0$
- $\varphi_i^*(y) = y$ and so $\Phi = \| \cdot \|_1$

The case for $n = 1$ agent

Question: How can it be that seller's problem admits two duals: Monge-Kantorovich and Beckmann?

$$R_{n=1,m}(\rho) = \min_{\pi \succeq \psi} \quad \left[\mathrm{Beck}_\rho\Big(\pi,\, \Phi\Big)\right]$$

- if we increase $\varphi_i(x)$, then $\varphi_i^*(y) = \sup_x \langle x, y \rangle - \varphi_i(x)$ decreases
- thus $\varphi_i(x) = x \cdot \varphi_i(1)$ in $[0, 1]$ and $\varphi_i(x) = +\infty$ for $x \geq 1$
- optimization over $\varphi_i(1)$ gives $\varphi_i(1) = 0$
- $\varphi_i^*(y) = y$ and so $\Phi = \| \cdot \|_1$

The case for $n = 1$ agent

Question: How can it be that seller's problem admits two duals: Monge-Kantorovich and Beckmann?

$$R_{n=1,m}(\rho) = \min_{\pi \succeq \psi} \left[\mathrm{Beck}_\rho\Big(\pi, \Phi\Big)\right]$$

- if we increase $\varphi_i(x)$, then $\varphi_i^*(y) = \sup_x \langle x, y\rangle - \varphi_i(x)$ decreases
- thus $\varphi_i(x) = x \cdot \varphi_i(1)$ in $[0, 1]$ and $\varphi_i(x) = +\infty$ for $x \geq 1$
- optimization over $\varphi_i(1)$ gives $\varphi_i(1) = 0$
- $\varphi_i^*(y) = y$ and so $\Phi = \| \cdot \|_1$

The case for $n = 1$ agent

Question: How can it be that seller's problem admits two duals: Monge-Kantorovich and Beckmann?

$$R_{n=1,m}(\rho) = \min_{\pi \succeq \psi} \left[\mathrm{Beck}_\rho\Big(\pi, \, \| \cdot \|_1\Big) \right]$$

- if we increase $\varphi_i(x)$, then $\varphi_i^*(y) = \sup_x \langle x, y \rangle - \varphi_i(x)$ decreases
- thus $\varphi_i(x) = x \cdot \varphi_i(1)$ in $[0, 1]$ and $\varphi_i(x) = +\infty$ for $x \geq 1$
- optimization over $\varphi_i(1)$ gives $\varphi_i(1) = 0$
- $\varphi_i^*(y) = y$ and so $\Phi = \| \cdot \|_1$

The case for $n = 1$ agent

Question: How can it be that seller's problem admits two duals: Monge-Kantorovich and Beckmann?

$$R_{n=1,m}(\rho) = \min_{\pi \succeq \psi} \quad \left[\mathrm{Beck}_\rho\Big(\pi, \; \|\cdot\|_1\Big)\right]$$

- if we increase $\varphi_i(x)$, then $\varphi_i^*(y) = \sup_x \langle x, y\rangle - \varphi_i(x)$ decreases
- thus $\varphi_i(x) = x \cdot \varphi_i(1)$ in $[0,1]$ and $\varphi_i(x) = +\infty$ for $x \geq 1$
- optimization over $\varphi_i(1)$ gives $\varphi_i(1) = 0$
- $\varphi_i^*(y) = y$ and so $\Phi = \|\cdot\|_1$

The case for $n = 1$ agent

Question: How can it be that seller's problem admits two duals: Monge-Kantorovich and Beckmann?

$$R_{n=1,m}(\rho) = \min_{\pi \succeq \psi} \quad \left[\mathrm{Beck}_\rho\Big(\pi, \|\cdot\|_1\Big)\right]$$

- if we increase $\varphi_i(x)$, then $\varphi_i^*(y) = \sup_x \langle x, y\rangle - \varphi_i(x)$ decreases
- thus $\varphi_i(x) = x \cdot \varphi_i(1)$ in $[0, 1]$ and $\varphi_i(x) = +\infty$ for $x \geq 1$
- optimization over $\varphi_i(1)$ gives $\varphi_i(1) = 0$
- $\varphi_i^*(y) = y$ and so $\Phi = \|\cdot\|_1$

Theorem (Santambrogio (2015))

$$\mathrm{Beck}_\rho\Big(\pi, \|\cdot\|_1\Big) = \min_{\substack{\text{positive measures } \gamma \\ \text{with marginals } \pi_+, \pi_-}} \int \|v - v'\|_1 \, \mathrm{d}\gamma(v, v')$$

The case for $n = 1$ agent

Question: How can it be that seller's problem admits two duals: Monge-Kantorovich and Beckmann?

$$R_{n=1,m}(\rho) = \min_{\pi \succeq \psi} \quad \left[\mathrm{Beck}_\rho\Big(\pi,\ \|\cdot\|_1\Big)\right]$$

- if we increase $\varphi_i(x)$, then $\varphi_i^*(y) = \sup_x \langle x, y\rangle - \varphi_i(x)$ decreases
- thus $\varphi_i(x) = x \cdot \varphi_i(1)$ in $[0, 1]$ and $\varphi_i(x) = +\infty$ for $x \geq 1$
- optimization over $\varphi_i(1)$ gives $\varphi_i(1) = 0$
- $\varphi_i^*(y) = y$ and so $\Phi = \|\cdot\|_1$

Theorem (Santambrogio (2015))

$$\mathrm{Beck}_\rho\Big(\pi,\ \|\cdot\|_1\Big) = \min_{\substack{\text{positive measures } \gamma \\ \text{with marginals } \pi_+, \pi_-}} \int \|v - v'\|_1 \,\mathrm{d}\gamma(v, v')$$

The case for $n = 1$ agent

Question: How can it be that seller's problem admits two duals: Monge-Kantorovich and Beckmann?

$$R_{n=1,m}(\rho) = \min_{\pi \succeq \psi} \left[\min_{\substack{\text{positive measures } \gamma \\ \text{with marginals } \pi_+, \pi_-}} \int \|v - v'\|_1 \, \mathrm{d}\gamma(v, v') \right]$$

- if we increase $\varphi_i(x)$, then $\varphi_i^*(y) = \sup_x \langle x, y \rangle - \varphi_i(x)$ decreases
- thus $\varphi_i(x) = x \cdot \varphi_i(1)$ in $[0, 1]$ and $\varphi_i(x) = +\infty$ for $x \geq 1$
- optimization over $\varphi_i(1)$ gives $\varphi_i(1) = 0$
- $\varphi_i^*(y) = y$ and so $\Phi = \| \cdot \|_1$

Theorem (Santambrogio (2015))

$$\mathrm{Beck}_\rho\Big(\pi, \| \cdot \|_1\Big) = \min_{\substack{\text{positive measures } \gamma \\ \text{with marginals } \pi_+, \pi_-}} \int \|v - v'\|_1 \, \mathrm{d}\gamma(v, v')$$

The case for $n = 1$ agent

Question: How can it be that seller's problem admits two duals: Monge-Kantorovich and Beckmann?

$$R_{n=1,m}(\rho) = \min_{\pi \succeq \psi} \left[\min_{\substack{\text{positive measures } \gamma \\ \text{with marginals } \pi_+, \pi_-}} \int \|v - v'\|_1 \, \mathrm{d}\gamma(v, v') \right]$$

- if we increase $\varphi_i(x)$, then $\varphi_i^*(y) = \sup_x \langle x, y \rangle - \varphi_i(x)$ decreases
- thus $\varphi_i(x) = x \cdot \varphi_i(1)$ in $[0, 1]$ and $\varphi_i(x) = +\infty$ for $x \geq 1$
- optimization over $\varphi_i(1)$ gives $\varphi_i(1) = 0$
- $\varphi_i^*(y) = y$ and so $\Phi = \| \cdot \|_1$

Theorem (Santambrogio (2015))

$$\mathrm{Beck}_\rho\Big(\pi, \, \| \cdot \|_1\Big) = \min_{\substack{\text{positive measures } \gamma \\ \text{with marginals } \pi_+, \pi_-}} \int \|v - v'\|_1 \, \mathrm{d}\gamma(v, v')$$

The case for $n = 1$ agent

Question: How can it be that seller's problem admits two duals: Monge-Kantorovich and Beckmann?

$$R_{n=1,m}(\rho) = \min_{\substack{\text{positive measures } \gamma \\ \gamma_1 - \gamma_2 \succeq \psi}} \left[\int \|v - v'\|_1 \, \mathrm{d}\gamma(v, v') \right]$$

- if we increase $\varphi_i(x)$, then $\varphi_i^*(y) = \sup_x \langle x, y \rangle - \varphi_i(x)$ decreases
- thus $\varphi_i(x) = x \cdot \varphi_i(1)$ in $[0, 1]$ and $\varphi_i(x) = +\infty$ for $x \geq 1$
- optimization over $\varphi_i(1)$ gives $\varphi_i(1) = 0$
- $\varphi_i^*(y) = y$ and so $\Phi = \| \cdot \|_1$

Theorem (Santambrogio (2015))

$$\mathrm{Beck}_\rho\Big(\pi, \| \cdot \|_1\Big) = \min_{\substack{\text{positive measures } \gamma \\ \text{with marginals } \pi_+, \pi_-}} \int \|v - v'\|_1 \, \mathrm{d}\gamma(v, v')$$

The case for $n = 1$ agent

Question: How can it be that seller's problem admits two duals: Monge-Kantorovich and Beckmann?

$$R_{n=1,m}(\rho) = \min_{\substack{\text{positive measures } \gamma \\ \gamma_1 - \gamma_2 \succeq \psi}} \left[\int \|v - v'\|_1 \, \mathrm{d}\gamma(v, v') \right]$$

- if we increase $\varphi_i(x)$, then $\varphi_i^*(y) = \sup_x \langle x, y \rangle - \varphi_i(x)$ decreases
- thus $\varphi_i(x) = x \cdot \varphi_i(1)$ in $[0, 1]$ and $\varphi_i(x) = +\infty$ for $x \geq 1$
- optimization over $\varphi_i(1)$ gives $\varphi_i(1) = 0$
- $\varphi_i^*(y) = y$ and so $\Phi = \| \cdot \|_1$

Theorem (Santambrogio (2015))

$$\mathrm{Beck}_\rho\Big(\pi, \, \| \cdot \|_1\Big) = \min_{\substack{\text{positive measures } \gamma \\ \text{with marginals } \pi_+, \pi_-}} \int \|v - v'\|_1 \, \mathrm{d}\gamma(v, v')$$

Corollary: duality by Daskalakis et al. (2017)

Why strong duality is useful?

- **Upper bound on revenue**
- **Controlling how far a given mechanism is from the optimum:** numerical methods with provable approximation guarantees
- **Complementary slackness conditions**
  - can be used to show that a mechanism is not optimal if the conditions are infeasible.
    Example: For $\rho(v) = \rho_1(v_1) \cdot \ldots \cdot \rho_m(v_m)$, selling separately is never optimal.[3]
  - help to guess/construct an explicit solution and to prove its optimality (dual solution is a certificate)
    Example: For $n = 1$ and $m = 2$ i.i.d. uniform items, selling each for $\frac{2}{3}$ or both for $\frac{4-\sqrt{2}}{3}$ is optimal.[4]

[3] Obtained by P. Jehiel, M.Meyer-Ter-Vehn, B.Moldovanu (2007) Mixed bundling auctions JET

[4] Obtained by A.Manelli, D.Vincent (2007) Multidimensional Mechanism Design JET

Why strong duality is useful?

- **Upper bound on revenue**
- **Controlling how far a given mechanism is from the optimum:** numerical methods with provable approximation guarantees
- **Complementary slackness conditions**
  - can be used to show that a mechanism is <u>not optimal</u> if the conditions are infeasible.
    Example: For $\rho(v) = \rho_1(v_1) \cdot \ldots \cdot \rho_m(v_m)$, selling separately is never optimal.[3]
  - help to guess/construct an explicit solution and to prove its optimality (dual solution is a certificate)
    Example: For $n = 1$ and $m = 2$ i.i.d. uniform items, selling each for $\frac{2}{3}$ or both for $\frac{4-\sqrt{2}}{3}$ is optimal.[4]

---

[3] Obtained by P. Jehiel, M.Meyer-Ter-Vehn, B.Moldovanu (2007) Mixed bundling auctions JET

[4] Obtained by A.Manelli, D.Vincent (2007) Multidimensional Mechanism Design JET

Why strong duality is useful?

- **Upper bound on revenue**
- **Controlling how far a given mechanism is from the optimum:** numerical methods with provable approximation guarantees
- **Complementary slackness conditions**
  - can be used to show that a mechanism is not optimal if the conditions are infeasible.
    Example: For $\rho(v) = \rho_1(v_1) \cdot \ldots \cdot \rho_m(v_m)$, selling separately is never optimal.[3]
  - help to guess/construct an explicit solution and to prove its optimality (dual solution is a certificate)
    Example: For $n = 1$ and $m = 2$ i.i.d. uniform items, selling each for $\frac{2}{3}$ or both for $\frac{4-\sqrt{2}}{3}$ is optimal.[4]

[3] Obtained by P. Jehiel, M.Meyer-Ter-Vehn, B.Moldovanu (2007) Mixed bundling auctions JET

[4] Obtained by A.Manelli, D.Vincent (2007) Multidimensional Mechanism Design JET

Why strong duality is useful?

- **Upper bound on revenue**
- **Controlling how far a given mechanism is from the optimum:** numerical methods with provable approximation guarantees
- **Complementary slackness conditions**
    - can be used to show that a mechanism is not optimal if the conditions are infeasible.
      Example: For $\rho(v) = \rho_1(v_1) \cdot \ldots \cdot \rho_m(v_m)$, selling separately is never optimal.[3]
    - help to guess/construct an explicit solution and to prove its optimality (dual solution is a certificate)
      Example: For $n = 1$ and $m = 2$ i.i.d. uniform items, selling each for $\frac{2}{3}$ or both for $\frac{4-\sqrt{2}}{3}$ is optimal.[4]

[3] Obtained by P. Jehiel, M.Meyer-Ter-Vehn, B.Moldovanu (2007) Mixed bundling auctions JET

[4] Obtained by A.Manelli, D.Vincent (2007) Multidimensional Mechanism Design JET

Why strong duality is useful?

- **Upper bound on revenue**
- **Controlling how far a given mechanism is from the optimum:** numerical methods with provable approximation guarantees
- **Complementary slackness conditions**
  - can be used to show that a mechanism is not optimal if the conditions are infeasible.
    Example: For $\rho(v) = \rho_1(v_1) \cdot \ldots \cdot \rho_m(v_m)$, selling separately is never optimal.[3]
  - help to guess/construct an explicit solution and to prove its optimality (dual solution is a certificate)
    Example: For $n = 1$ and $m = 2$ i.i.d. uniform items, selling each for $\frac{2}{3}$ or both for $\frac{4-\sqrt{2}}{3}$ is optimal.[4]

[3] Obtained by P. Jehiel, M.Meyer-Ter-Vehn, B.Moldovanu (2007) Mixed bundling auctions JET

[4] Obtained by A.Manelli, D.Vincent (2007) Multidimensional Mechanism Design JET

Numerical simulations

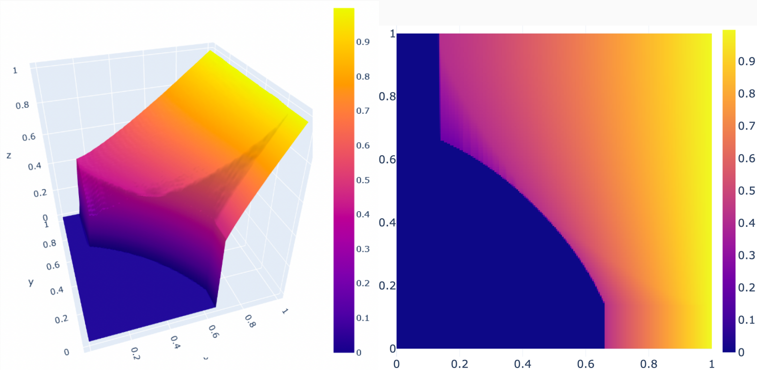
Question: Any hope for an explicit solution with $n \geq 2$ and $m = 2$ i.i.d. uniform items?

Numerical simulations

Question: Any hope for an explicit solution with $n \geq 2$ and $m = 2$ i.i.d. uniform items? **Perhaps, not.**

Numerical simulations

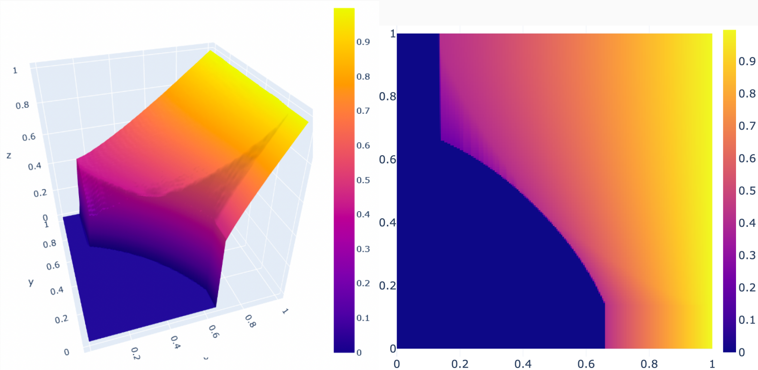
Question: Any hope for an explicit solution with $n \geq 2$ and $m = 2$ i.i.d. uniform items? **Perhaps, not.**



Probability to receive the first item as a function of bidder's values (v_1, v_2) in the optimal auction

Numerical simulations

Question: Any hope for an explicit solution with $n \geq 2$ and $m = 2$ i.i.d. uniform items? **Perhaps, not.**



Probability to receive the first item as a function of bidder's values (v_1, v_2) in the optimal auction

Remark: computing the optimum numerically is a non-trivial task requiring extra optimal transportation insights about algorithm

The end

- Another justification that mechanism design $\simeq$ optimal transport
- Surprising appearance of Beckmann's problem, its first non-transportation application
- **Remaining question to unleash full power:** Guess a good candidate for an explicit solution? (for some particular ρ)

The end

- Another justification that mechanism design $\simeq$ optimal transport
- Surprising appearance of Beckmann's problem, its first non-transportation application
- **Remaining question to unleash full power:** Guess a good candidate for an explicit solution? (for some particular ρ)

Thank you!

Complementary slackness conditions

back to theorem

Optimal u^{opt}, functions φ_i^{opt}, measure π^{opt}, and vector field f^{opt} satisfy:

$$\int u^{\mathrm{opt}}(v)\,\mathrm{d}\psi(v) = \int u^{\mathrm{opt}}(v)\,\mathrm{d}\pi^{\mathrm{opt}}(v)$$

$$f_i^{\mathrm{opt}}(v) \in \partial\varphi_i^{\mathrm{opt}}\left(\frac{\partial u^{\mathrm{opt}}}{\partial v_i}(v)\right)$$

$$\int \varphi_i^{\mathrm{opt}}\left(\frac{\partial u^{\mathrm{opt}}}{\partial v_i}(v)\right)\rho(v)\,\mathrm{d}v = \int_0^1 \varphi_i^{\mathrm{opt}}\left(z^{n-1}\right)\,\mathrm{d}z$$

Algorithmic ideas

back to simulations

- **Automated mechanism design:** revenue maximization is an LP, let's feed it to an LP solver; Sandholm (2003)
- **Curse of dimensionality:** If each of n agents can have q different values for each of m items $\Rightarrow$ the dimension $\sim \left(q^n\right)^m$
  - intractable for $(m = 2,\ q = 100\ n = 2)$ or for $(m = 2\ q = 10\ n = 4)$
  - deep neural networks improve the bounds; Dutting et al. (2019)
- **How to avoid:**

$$R_{n,m}(\rho) = \max_{\substack{\text{convex monotone } u \\ u(0)=0,\ \partial_{v_i} u(v) \preceq z^{n-1}}} n \cdot \int_{\mathbb{R}_+^m} u(v)\, \mathrm{d}\psi(v)$$

  - Pros: dependence on n is killed; Cai et al.(2012), Alaei et al. (2019)
  - Cons: non-linear program
- **Linearization via transport:**
  - μ on $[0,1]$ majorizes ν if and only if there is γ on $[0,1]^2$ with marginals μ on y and ν on x and such that $\int y\, \mathrm{d}\gamma(y \mid x) \geq x$ for γ-almost all x
  - solve for (u, γ)
- **Performance:** algorithm handles $(m = 2\ q = 100\ n = 10)$ revenue curve

Algorithmic ideas

back to simulations

- **Automated mechanism design:** revenue maximization is an LP, let's feed it to an LP solver; Sandholm (2003)
- **Curse of dimensionality:** If each of n agents can have q different values for each of m items $\Rightarrow$ the dimension $\sim \left(q^n\right)^m$
  - intractable for $(m = 2,\ q = 100\ n = 2)$ or for $(m = 2\ q = 10\ n = 4)$
  - deep neural networks improve the bounds; Dutting et al. (2019)
- **How to avoid:**

$$R_{n,m}(\rho) = \max_{\substack{\text{convex monotone } u \\ u(0)=0,\ \partial_{v_i} u(v) \preceq z^{n-1}}} n \cdot \int_{\mathbb{R}_+^m} u(v)\, \mathrm{d}\psi(v)$$

  - Pros: dependence on n is killed; Cai et al.(2012), Alaei et al. (2019)
  - Cons: non-linear program
- **Linearization via transport:**
  - μ on $[0,1]$ majorizes ν if and only if there is γ on $[0,1]^2$ with marginals μ on y and ν on x and such that $\int y\, \mathrm{d}\gamma(y \mid x) \geq x$ for γ-almost all x
  - solve for (u, γ)
- **Performance:** algorithm handles $(m = 2\ q = 100\ n = 10)$ revenue curve

Algorithmic ideas back to simulations

- **Automated mechanism design:** revenue maximization is an LP, let's feed it to an LP solver; Sandholm (2003)
- **Curse of dimensionality:** If each of n agents can have q different values for each of m items $\Rightarrow$ the dimension $\sim \left(q^n\right)^m$
  - intractable for ($m = 2$, $q = 100$ $n = 2$) or for ($m = 2$ $q = 10$ $n = 4$)
  - deep neural networks improve the bounds; Dutting et al. (2019)
- **How to avoid:**

$$R_{n,m}(\rho) = \max_{\substack{\text{convex monotone } u \\ u(0)=0,\ \partial_{v_i} u(v) \preceq z^{n-1}}} n \cdot \int_{\mathbb{R}_+^m} u(v)\, \mathrm{d}\psi(v)$$

  - Pros: dependence on n is killed; Cai et al.(2012), Alaei et al. (2019)
  - Cons: non-linear program
- **Linearization via transport:**
  - μ on $[0,1]$ majorizes ν if and only if there is γ on $[0,1]^2$ with marginals μ on y and ν on x and such that $\int y \, \mathrm{d}\gamma(y \mid x) \geq x$ for γ-almost all x
  - solve for (u, γ)
- **Performance:** algorithm handles ($m = 2$ $q = 100$ $n = 10$) revenue curve

Algorithmic ideas

back to simulations

- **Automated mechanism design:** revenue maximization is an LP, let's feed it to an LP solver; Sandholm (2003)
- **Curse of dimensionality:** If each of n agents can have q different values for each of m items $\Rightarrow$ the dimension $\sim \left(q^n\right)^m$
  - intractable for ($m = 2,\ q = 100\ n = 2$) or for ($m = 2\ q = 10\ n = 4$)
  - deep neural networks improve the bounds; Dutting et al. (2019)
- **How to avoid:**

$$R_{n,m}(\rho) = \max_{\substack{\text{convex monotone } u \\ u(0)=0,\ \partial_{v_i} u(v) \preceq z^{n-1}}} n \cdot \int_{\mathbb{R}_+^m} u(v)\, \mathrm{d}\psi(v)$$

  - Pros: dependence on n is killed; Cai et al.(2012), Alaei et al. (2019)
  - Cons: non-linear program
- **Linearization via transport:**
  - μ on $[0,1]$ majorizes ν if and only if there is γ on $[0,1]^2$ with marginals μ on y and ν on x and such that $\int y\, \mathrm{d}\gamma(y \mid x) \geq x$ for γ-almost all x
  - solve for (u, γ)
- **Performance:** algorithm handles ($m = 2\ q = 100\ n = 10$) revenue curve

Algorithmic ideas back to simulations

- **Automated mechanism design:** revenue maximization is an LP, let's feed it to an LP solver; Sandholm (2003)
- **Curse of dimensionality:** If each of n agents can have q different values for each of m items $\Rightarrow$ the dimension $\sim \left(q^n\right)^m$
  - intractable for $(m = 2,\ q = 100\ n = 2)$ or for $(m = 2\ q = 10\ n = 4)$
  - deep neural networks improve the bounds; Dutting et al. (2019)
- **How to avoid:**

$$R_{n,m}(\rho) = \max_{\substack{\text{convex monotone } u \\ u(0)=0,\ \partial_{v_i} u(v) \preceq z^{n-1}}} n \cdot \int_{\mathbb{R}_+^m} u(v)\, \mathrm{d}\psi(v)$$

  - Pros: dependence on n is killed; Cai et al.(2012), Alaei et al. (2019)
  - Cons: non-linear program
- **Linearization via transport:**
  - μ on $[0,1]$ majorizes ν if and only if there is γ on $[0,1]^2$ with marginals μ on y and ν on x and such that $\int y\, \mathrm{d}\gamma(y \mid x) \geq x$ for γ-almost all x
  - solve for (u, γ)
- **Performance:** algorithm handles $(m = 2\ q = 100\ n = 10)$ revenue curve

Algorithmic ideas

back to simulations

- **Automated mechanism design:** revenue maximization is an LP, let's feed it to an LP solver; Sandholm (2003)
- **Curse of dimensionality:** If each of n agents can have q different values for each of m items $\Rightarrow$ the dimension $\sim \left(q^n\right)^m$
  - intractable for $(m = 2,\ q = 100\ n = 2)$ or for $(m = 2\ q = 10\ n = 4)$
  - deep neural networks improve the bounds; Dutting et al. (2019)
- **How to avoid:**

$$R_{n,m}(\rho) = \max_{\substack{\text{convex monotone } u \\ u(0)=0,\ \partial_{v_i} u(v) \preceq z^{n-1}}} n \cdot \int_{\mathbb{R}_+^m} u(v)\,\mathrm{d}\psi(v)$$

  - Pros: dependence on n is killed; Cai et al.(2012), Alaei et al. (2019)
  - Cons: non-linear program
- **Linearization via transport:**
  - μ on $[0,1]$ majorizes ν if and only if there is γ on $[0,1]^2$ with marginals μ on y and ν on x and such that $\int y\,\mathrm{d}\gamma(y \mid x) \geq x$ for γ-almost all x
  - solve for (u, γ)
- **Performance:** algorithm handles $(m = 2\ q = 100\ n = 10)$ revenue curve

Algorithmic ideas

back to simulations

- **Automated mechanism design:** revenue maximization is an LP, let's feed it to an LP solver; Sandholm (2003)
- **Curse of dimensionality:** If each of n agents can have q different values for each of m items $\Rightarrow$ the dimension $\sim \left(q^n\right)^m$
    - intractable for $(m = 2,\ q = 100\ n = 2)$ or for $(m = 2\ q = 10\ n = 4)$
    - deep neural networks improve the bounds; Dutting et al. (2019)
- **How to avoid:**

$$R_{n,m}(\rho) = \max_{\substack{\text{convex monotone } u \\ u(0)=0,\ \partial_{v_i} u(v) \preceq z^{n-1}}} n \cdot \int_{\mathbb{R}_+^m} u(v)\, \mathrm{d}\psi(v)$$

    - Pros: dependence on n is killed; Cai et al.(2012), Alaei et al. (2019)
    - Cons: non-linear program
- **Linearization via transport:**
    - μ on $[0,1]$ majorizes ν if and only if there is γ on $[0,1]^2$ with marginals μ on y and ν on x and such that $\int y\, \mathrm{d}\gamma(y \mid x) \geq x$ for γ-almost all x
    - solve for (u, γ)
- **Performance:** algorithm handles $(m = 2\ q = 100\ n = 10)$ revenue curve

Algorithmic ideas

back to simulations

- **Automated mechanism design:** revenue maximization is an LP, let's feed it to an LP solver; Sandholm (2003)
- **Curse of dimensionality:** If each of n agents can have q different values for each of m items $\Rightarrow$ the dimension $\sim \left(q^n\right)^m$
  - intractable for $(m = 2,\ q = 100\ n = 2)$ or for $(m = 2\ q = 10\ n = 4)$
  - deep neural networks improve the bounds; Dutting et al. (2019)
- **How to avoid:**

$$R_{n,m}(\rho) = \max_{\substack{\text{convex monotone } u \\ u(0)=0,\ \partial_{v_i} u(v) \preceq z^{n-1}}} n \cdot \int_{\mathbb{R}_+^m} u(v)\, \mathrm{d}\psi(v)$$

  - Pros: dependence on n is killed; Cai et al.(2012), Alaei et al. (2019)
  - Cons: non-linear program
- **Linearization via transport:**
  - μ on $[0,1]$ majorizes ν if and only if there is γ on $[0,1]^2$ with marginals μ on y and ν on x and such that $\int y\, \mathrm{d}\gamma(y \mid x) \geq x$ for γ-almost all x
  - solve for (u, γ)
- **Performance:** algorithm handles $(m = 2\ q = 100\ n = 10)$ revenue curve

Algorithmic ideas

back to simulations

- **Automated mechanism design:** revenue maximization is an LP, let's feed it to an LP solver; Sandholm (2003)
- **Curse of dimensionality:** If each of n agents can have q different values for each of m items $\Rightarrow$ the dimension $\sim \left(q^n\right)^m$
  - intractable for $(m=2,\ q=100\ n=2)$ or for $(m=2\ q=10\ n=4)$
  - deep neural networks improve the bounds; Dutting et al. (2019)
- **How to avoid:**

$$R_{n,m}(\rho) = \max_{\substack{\text{convex monotone } u \\ u(0)=0,\ \partial_{v_i} u(v) \preceq z^{n-1}}} n \cdot \int_{\mathbb{R}_+^m} u(v)\,\mathrm{d}\psi(v)$$

  - <u>Pros:</u> dependence on n is killed; Cai et al.(2012), Alaei et al. (2019)
  - <u>Cons:</u> non-linear program
- **Linearization via transport:**
  - μ on $[0,1]$ majorizes ν if and only if there is γ on $[0,1]^2$ with marginals μ on y and ν on x and such that $\int y\,\mathrm{d}\gamma(y \mid x) \geq x$ for γ-almost all x
  - solve for (u, γ)
- **Performance:** algorithm handles $(m=2\ q=100\ n=10)$ revenue curve

Algorithmic ideas back to simulations

- **Automated mechanism design:** revenue maximization is an LP, let's feed it to an LP solver; Sandholm (2003)
- **Curse of dimensionality:** If each of n agents can have q different values for each of m items $\Rightarrow$ the dimension $\sim \left(q^n\right)^m$
  - intractable for $(m = 2,\ q = 100\ n = 2)$ or for $(m = 2\ q = 10\ n = 4)$
  - deep neural networks improve the bounds; Dutting et al. (2019)
- **How to avoid:**

$$R_{n,m}(\rho) = \max_{\substack{\text{convex monotone } u \\ u(0)=0,\ \partial_{v_i} u(v) \preceq z^{n-1}}} n \cdot \int_{\mathbb{R}_+^m} u(v)\,\mathrm{d}\psi(v)$$

  - Pros: dependence on n is killed; Cai et al.(2012), Alaei et al. (2019)
  - Cons: non-linear program
- **Linearization via transport:**
  - μ on $[0,1]$ majorizes ν if and only if there is γ on $[0,1]^2$ with marginals μ on y and ν on x and such that $\int y\,\mathrm{d}\gamma(y \mid x) \geq x$ for γ-almost all x
  - solve for (u, γ)
- **Performance:** algorithm handles $(m = 2\ q = 100\ n = 10)$ revenue curve

Algorithmic ideas

back to simulations

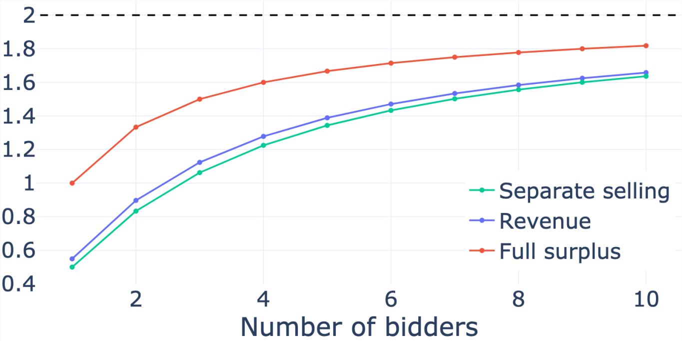
- **Automated mechanism design:** revenue maximization is an LP, let's feed it to an LP solver; Sandholm (2003)
- **Curse of dimensionality:** If each of n agents can have q different values for each of m items $\Rightarrow$ the dimension $\sim \left(q^n\right)^m$
  - intractable for $(m = 2,\ q = 100\ n = 2)$ or for $(m = 2\ q = 10\ n = 4)$
  - deep neural networks improve the bounds; Dutting et al. (2019)
- **How to avoid:**

$$R_{n,m}(\rho) = \max_{\substack{\text{convex monotone } u \\ u(0)=0,\ \partial_{v_i} u(v) \preceq z^{n-1}}} n \cdot \int_{\mathbb{R}_+^m} u(v)\, \mathrm{d}\psi(v)$$

  - Pros: dependence on n is killed; Cai et al.(2012), Alaei et al. (2019)
  - Cons: non-linear program
- **Linearization via transport:**
  - μ on $[0,1]$ majorizes ν if and only if there is γ on $[0,1]^2$ with marginals μ on y and ν on x and such that $\int y\, \mathrm{d}\gamma(y \mid x) \geq x$ for γ-almost all x
  - solve for (u, γ)
- **Performance:** algorithm handles $(m = 2\ q = 100\ n = 10)$ revenue curve

Revenue

back to algorithmic ideas

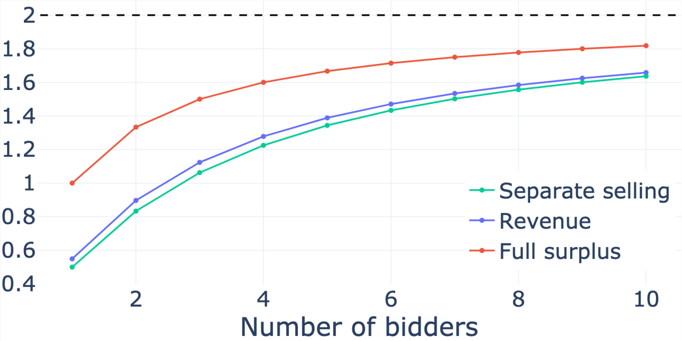


Revenue as a function of the number of bidders n for two items with i.i.d. values uniform on $[0, 1]$. Graphs from bottom to top: selling separately (light-green), selling optimally (blue), full surplus extraction (red), limit for $n \to \infty$ (the dashed line).

Remark: For $n = 2$, selling optimally improves upon selling separately by 5%

Revenue

back to algorithmic ideas



Revenue as a function of the number of bidders n for two items with i.i.d. values uniform on $[0, 1]$. Graphs from bottom to top: selling separately (light-green), selling optimally (blue), full surplus extraction (red), limit for $n \to \infty$ (the dashed line).

Remark: For $n = 2$, selling optimally improves upon selling separately by 5%