

Improvable Equilibria

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ASSA 2025, Advances in Mechanism and Information Design, January 3, 2025

Correlated Equilibria (CE) (Aumann, 1974) generalize Nash equilibria to allow correlation

- capture communication, intermediation, natural learning dynamics

Broad question: When is there potential value in correlation?

CE \simeq adding a recommendation system on top of the existing interaction

- \implies What interactions can be improved by a recommendation system?

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CE \simeq outcomes of arbitrary pre-play communication protocols

Bárány (1992), Ben-Porath (1998), Gerardi (2004), Lehrer and Sorin (1997)

- \implies What strategic interactions are susceptible to communication / collusion?

Normal-form game

$$\Gamma = \left(N, (A_i)_{i \in N}, (u_i: A \rightarrow \mathbb{R})_{i \in N} \right)$$

- $N = \{1, \dots, n\}$ is finite set of players
- A_i is a finite set of actions of player i
- $A = \prod_{i \in N} A_i$ is the set of action profiles
- $u_i: A \rightarrow \mathbb{R}$ is utility of player i

Definition

A distribution $\mu \in \Delta(A)$ is a CE if

$$\sum_{a_{-i} \in A_{-i}} \mu(a_i, a_{-i}) u_i(a_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \mu(a_i, a_{-i}) u_i(a'_i, a_{-i})$$

for all $i \in N$ and all $a_i, a'_i \in A_i$

Interpretation: μ generated by a mediator and players best respond by adhering

Remark: Nash Equilibria (NE) are CE of the form $\mu = \mu_1 \times \dots \times \mu_n$

Formalizing the Question

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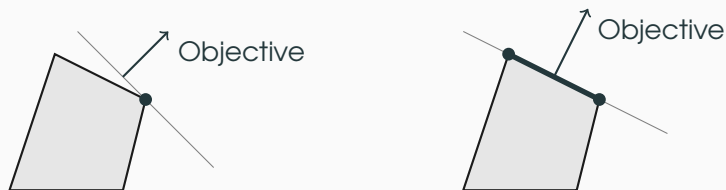
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Our Question: When is a Nash equilibrium extreme?

Linear objectives and extreme points

Maximization of a linear objective over a polytope:



- If the optimum is unique, it is an extreme point
- In knife-edge cases, the whole face can be optimal

Definition

We call objectives with a unique optimum **non-degenerate**

Observation

NE is non-extreme \iff any non-degenerate linear objective can be improved

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Remark: Linear in probabilities, not in actions \Rightarrow a broad class of objectives

- e.g., utilitarian welfare, revenue, maximizing/minimizing the probability of a particular action profile

Improvability of non-extreme equilibria 2

- Non-extreme equilibria are improvable **no matter** the objective
- A conservative notion, agnostic to the designer's objective
- Usually, assess outcomes for a **given** objective

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Main Insight

Despite restrictiveness of improvability notion, **many** equilibria are improvable, i.e., **non-extreme**

- **Value of correlation in 2-player games:** Cripps (1995), Evangelista and Raghavan (1996), Canovas, Hansen, and Jaumard (1999), Nau, Canovas, and Hansen (2004), Peeters and Potters (1999), Calvó-Armengol (2006), Ashlagi, Monderer, and Tennenholtz (2008)
- **Extreme-point approach in info & mech. design:** Kleiner, Moldovanu, and Strack (2021), Arieli, Babichenko, Smorodinsky, and Yamashita (2023), Yang and Zentefis (2024), Kleiner, Moldovanu, Strack, and Whitmeyer (2024)

Conditions for Extremality

Theorem

In a generic n -player game, a mixed NE is extreme $\iff \leq 2$ players randomize

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A version of the theorem applies to **all games** and **regular** NE

Proof Intuition

High-level idea: When many players randomize, there are too many ways to correlate their actions \implies one must be beneficial

Focus on a particular example to illustrate

- Game with n players, each with 2 actions

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- $2n$ constraints
- **Winkler (1988)**: if k linear constraints are imposed on the set of all distributions $\Delta(A)$, extreme distributions have support $\leq k + 1$
- \Rightarrow support of an extreme CE μ is bounded by $2n + 1$

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- The same argument applies to equilibria where players mix over the same number of pure strategies
- The main difficulty is handling very asymmetric equilibria [▶ details](#)

Extreme Points in Payoff Space

- The set of CE $\subset \Delta(A)$ subset of a space of dimension $|A_1| \cdot \dots \cdot |A_n|$
- Equilibria are often represented via payoffs in \mathbb{R}^n

Definition

A Nash equilibrium is **payoff-extreme** if its payoff vector is an extreme point of the set of CE payoffs

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Question: What can we say about payoff-extreme equilibria?

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- Projection of an extreme point **need not** be an extreme point of a projection
- \Rightarrow pure NE and NE with 2 mixers **need not** be payoff-extreme

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Proposition

In a generic game, utilitarian welfare is non-degenerate

Applications to Particular Classes of Games

Costly voting model of Palfrey and Rosenthal (1983):

- Two finite groups of voters: D and R , $|R| > |D|$
- Voters in D get utility of 1 if d -candidate wins and 0 otherwise
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- Majority voting (among those who participate), ties broken randomly
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Other Applications: games where players want to mismatch actions of others

- e.g., network games (with substitutes), congestion games, all-pay auctions, Boston matching mechanism

Games with Unique Correlated Equilibrium

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- No genericity assumption since games with a unique CE form an open set (Viossat, 2010)

Conclusions

Several papers effectively show extremality of NE in specific contexts:

- Tullock contests, Cournot and Bertrand, patent races, location games (Einy, Haimanko, and Lagziel, 2022)
- First-price auctions (Feldman, Lucier, and Nisan, 2016)
- Convex potential games (Neyman, 1997; Ui, 2008)
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Our paper:

- a tension between equilibrium randomness and extremality
- detail-free criterion for extremality in various settings

Main takeaway: inherent suboptimality of equilibria with a lot of mixing

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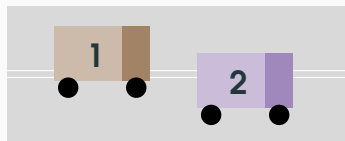
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Thank you!

Example: 2 Players vs 3 Players

Example: 2-Player Games

A version of the Game of Chicken by **Aumann (1974)**:



	Risky	Safe
Risky	6, 6	10, 7
Safe	7, 10	9, 9

Example: 2-Player Games

	Risky	Safe
Risky	6, 6	10, 7
Safe	7, 10	9, 9
	p	$1 - p$

- Mixed NE: $(1/2, 1/2)$ for both players

Solves linear equation: $6p + 10(1 - p) = 7p + 9(1 - p) \implies p = 1/2$

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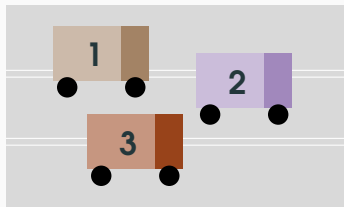
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- Indeed, it is the optimum for a non-degenerate objective

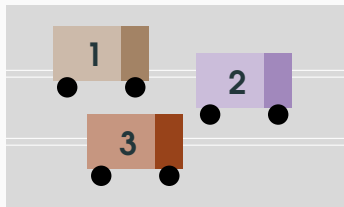
weight of (Risky, Risky) & (Safe, Safe) $\rightarrow \max$

Example: 3-Player Games



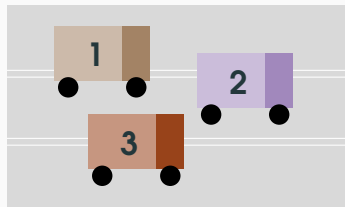
		Safe		Risky	
		Risky	Safe	Risky	Safe
Risky	Risky	6, 6, 5	10, 7, 7	0, 0, 0	6, 5, 6
	Safe	7, 10, 7	9, 9, 9	5, 6, 6	7, 7, 10

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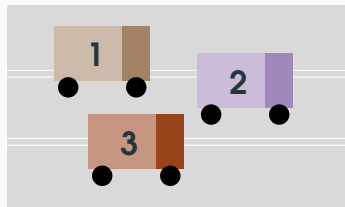
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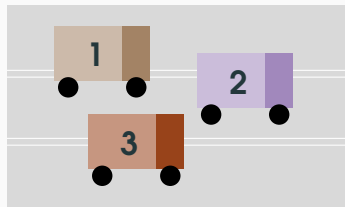
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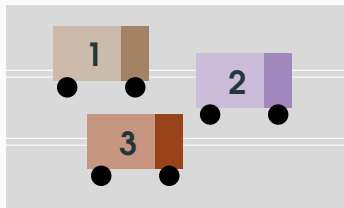
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- Symmetric Mixed NE: $(\sqrt{3/2} - 1, 2 - \sqrt{3/2})$ for each player

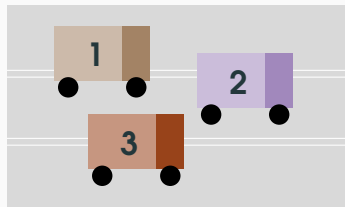
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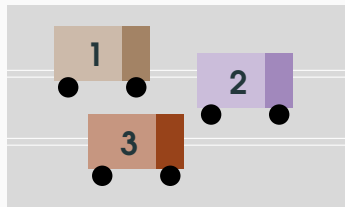
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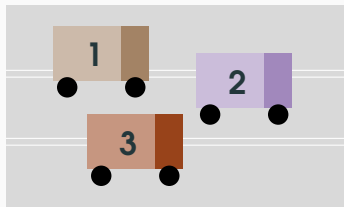
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More than 2 players mixing makes a difference...

More Applications

Symmetric Games

- In many applications, strategic interactions are symmetric
- When are symmetric equilibria extreme?

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Take-away: symmetric mixed equilibria are inherently suboptimal

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- For general games, we only know that extreme CE have small support
- For symmetric games and symmetric CE, we can say more

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Theorem (de Finetti)

Any infinite exchangeable sequence $a_1, a_2, a_3 \dots$ is a mixture of i.i.d. distributions

Extreme Symmetric CE with Many Players

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- A version of Proposition 2 holds: sampling without replacement instead of i.i.d.

Key Lemmas

Support Size of Extreme Correlated Equilibria (follows from **Winkler (1988)**)

If μ is an extreme correlated equilibrium, then

$$\text{supp}(\mu) \leq 1 + \sum_{i \in N} |S_i| \cdot (|S_i| - 1)$$

Support Size of Regular Nash Equilibria (**McKelvey and McLennan, 1997**)

For a regular Nash equilibrium, $\nu = (\nu_1, \nu_2, \dots, \nu_n)$:

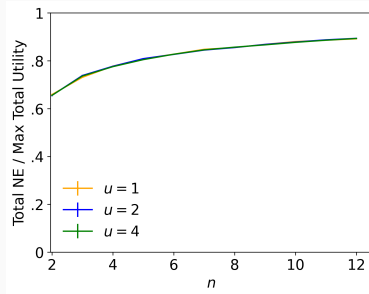
$$\text{supp}(\nu_i) - 1 \leq \sum_{j \neq i} (\text{supp}(\nu_j) - 1), \quad \text{for any player } i$$

Regularity of Generic games (**Harsanyi, 1973**)

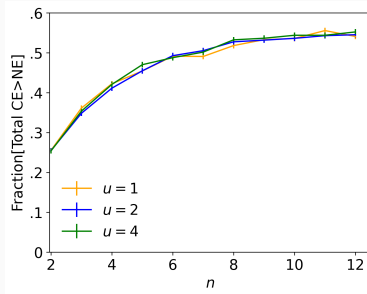
In a generic game, any Nash equilibrium is regular

Simulations

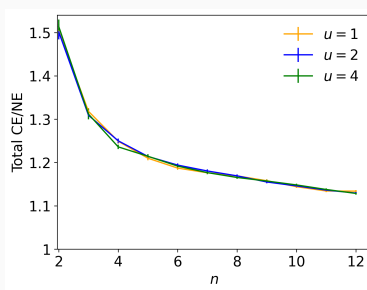
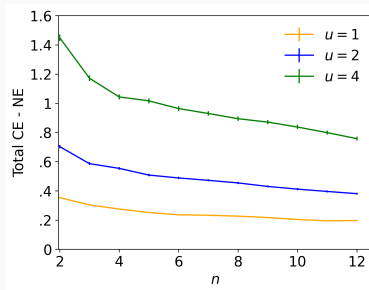
Simulations



(a) $NE/2u$



(b) Fraction of CE $> NE^*$



General linear objectives

- Consider a NE ν
- For simplicity, ν has full support
- By Farkas lemma, a linear objective L can be improved for $\nu \iff L$ **cannot** be expressed as

$$L(\mu) = C + \sum_{i, a_i, a'_i, a_{-i}} \mu(a) \cdot \lambda_i(a_i, a'_i) \cdot (u_i(a_i, a_{-i}) - u_i(a'_i, a_{-i}))$$

for some $\lambda_i(a_i, a'_i) \geq 0$.

- For non-extreme NE ν , “bad” L form a lower-dimensional subspace

Extreme Symmetric CE with Any Number of Players

Consider n players with m actions each

Proposition

Any extreme symmetric CE can be obtained as follows:

- there are M urns, each with n balls labeled by actions

$$1 \leq M \leq m(m-1) + 1$$

- an urn is selected at random according to $p \in \Delta_M$, secretly from players
- players draw balls sequentially without replacement
- i 's action = her ball's label, no incentive to deviate

Remark: If n is large, sampling without replacement can be approximated by i.i.d.

Bayesian games

Bayesian game

$$\mathcal{B} = \left(N, (A_i)_{i \in N}, (T_i)_{i \in N}, \tau \in \Delta(T), (u_i: A \times T_i \rightarrow \mathbb{R})_{i \in N} \right)$$

- Each player $i \in N$ has a type $t_i \in T_i$
- Profile of types $(t_1, \dots, t_n) \in T$ sampled from τ
- Each player i observes her realized type
- Utility $u_i: A \times T_i \rightarrow \mathbb{R}$ depends on the action profile and i 's type

Technical assumption: sets of types T_i are finite

Bayesian Correlated Equilibria (BCE)

Definition

A joint distribution $\mu \in \Delta(A \times T)$ is a Bayesian correlated equilibrium if

- The marginal on T coincides with τ
- For each player i , type t_i , recommended action a_i , and deviation a'_i ,

$$\sum_{(a_{-i}, t_{-i})} \mu((a_i, t_i), (a_{-i}, t_{-i})) u_i(a_i, t_i, a_{-i}) \geq \sum_{(a_{-i}, t_{-i})} \mu((a_i, t_i), (a_{-i}, t_{-i})) u_i(a'_i, t_i, a_{-i})$$

Interpretation: a mediator having access to realized types recommends actions to each player. Two aspects:

1. **Ex-ante coordination:** a source of correlated randomness (as in CE)
2. **Information sharing:** providing i more info about t_{-i} than contained in t_i

Remark: Bergemann and Morris (2016) allow for a broader class of BCE, where player i observes a noisy signal about her type

Induced Complete Information Game

We can associate a complete information normal form game $\Gamma_{\mathcal{B}}$ with \mathcal{B} :

- Replace A_i with set of functions $\sigma_i : T_i \rightarrow A_i$
- Σ_i is the set of all such σ_i
- Utility $v_i : \Sigma \rightarrow \mathbb{R}$ is given by

$$v_i(\sigma) = \sum_{t \in T} \tau(t) \cdot u_i((\sigma_1(t_1), \dots, \sigma_n(t_n)), t_i)$$

Induced Complete Information Game

$$\Gamma_{\mathcal{B}} = (N, (\Sigma_i)_{i \in N}, (v_i)_{i \in N})$$

Question: What is a relation between CE of $\Gamma_{\mathcal{B}}$ and BCE of \mathcal{B} ?

Induced complete information game

Relationship between equilibria in $\Gamma_{\mathcal{B}}$ and \mathcal{B}

CE in $\Gamma_{\mathcal{B}} \Leftrightarrow$ ex-ante coordination in \mathcal{B} with no information sharing

- i.e., BCE such that a_i is independent of t_{-i} conditionally on t_i

Nash in $\Gamma_{\mathcal{B}} \Leftrightarrow$ Bayes-Nash in \mathcal{B}

Observation: Generic \mathcal{B} leads to generic $\Gamma_{\mathcal{B}}$

- \Rightarrow we can apply our theorem to $\Gamma_{\mathcal{B}}$ to learn about generic \mathcal{B}

Corollary

For a generic Bayesian game, a Bayes-Nash equilibrium is improvable via ex-ante coordination \Leftrightarrow at least 3 players randomize

Applies to Bayesian games where players randomize in equilibrium, e.g., costly voting with private types (Feddersen and Pesendorfer, 1997) and contests (Baranski and Goel, 2024)

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