

# Improvable Equilibria

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Correlated Equilibria (CE) (Aumann, 1974) generalize Nash equilibria to allow correlation

- capture communication, intermediation, natural learning dynamics

**Broad question:** When is there potential value in correlation?

CE  $\simeq$  adding a recommendation system on top of the existing interaction

- $\implies$  What interactions can be improved by a recommendation system?

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CE  $\simeq$  outcomes of arbitrary pre-play communication protocols

Bárány (1992), Ben-Porath (1998), Gerardi (2004), Lehrer and Sorin (1997)

- $\implies$  What strategic interactions are susceptible to communication / collusion?

## Normal-form game

$$\Gamma = \left( N, (A_i)_{i \in N}, (u_i: A \rightarrow \mathbb{R})_{i \in N} \right)$$

- $N = \{1, \dots, n\}$  is finite set of players
- $A_i$  is a finite set of actions of player  $i$
- $A = \prod_{i \in N} A_i$  is the set of action profiles
- $u_i: A \rightarrow \mathbb{R}$  is utility of player  $i$

## Definition

A distribution  $\mu \in \Delta(A)$  is a CE if

$$\sum_{a_{-i} \in A_{-i}} \mu(a_i, a_{-i}) u_i(a_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \mu(a_i, a_{-i}) u_i(a'_i, a_{-i})$$

for all  $i \in N$  and all  $a_i, a'_i \in A_i$

**Interpretation:**  $\mu$  generated by a mediator and players best respond by adhering

**Remark:** Nash Equilibria (NE) are CE of the form  $\mu = \mu_1 \times \dots \times \mu_n$

## Formalizing the Question

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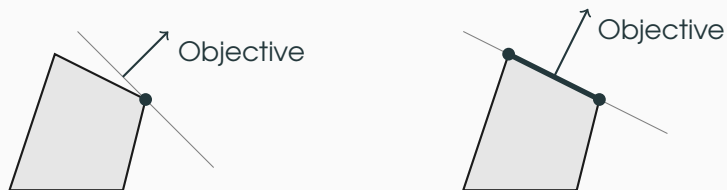
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**Our Question:** When is a Nash equilibrium extreme?

# Linear objectives and extreme points

Maximization of a linear objective over a polytope:



- If the optimum is unique, it is an extreme point
- In knife-edge cases, the whole face can be optimal

## Definition

We call objectives with a unique optimum **non-degenerate**

## Observation

NE is non-extreme  $\iff$  any non-degenerate linear objective can be improved

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**Remark:** Linear in probabilities, not in actions  $\Rightarrow$  a broad class of objectives

- e.g., utilitarian welfare, revenue, maximizing/minimizing the probability of a particular action profile

## Improvability of non-extreme equilibria 2

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### Main Insight

Despite restrictiveness of improvability notion, **many** equilibria are improvable, i.e., **non-extreme**

- **Value of correlation in 2-player games:** Cripps (1995), Evangelista and Raghavan (1996), Canovas, Hansen, and Jaumard (1999), Nau, Canovas, and Hansen (2004), Peeters and Potters (1999), Calvó-Armengol (2006), Ashlagi, Monderer, and Tennenholtz (2008)
- **Extreme-point approach in info & mech. design:** Kleiner, Moldovanu, and Strack (2021), Arieli, Babichenko, Smorodinsky, and Yamashita (2023), Yang and Zentefis (2024), Kleiner, Moldovanu, Strack, and Whitmeyer (2024)

## Conditions for Extremality

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## Theorem

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A version of the theorem applies to **all games** and **regular** NE

## Proof Intuition

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**High-level idea:** When many players randomize, there are too many ways to correlate their actions  $\implies$  one must be beneficial

Focus on a particular example to illustrate

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- **Winkler (1988)**: if  $k$  linear constraints are imposed on the set of all distributions  $\Delta(A)$ , extreme distributions have support  $\leq k + 1$
- $\Rightarrow$  support of an extreme CE  $\mu$  is bounded by  $2n + 1$

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- The same argument applies to equilibria where players mix over the same number of pure strategies
- The main difficulty is handling very asymmetric equilibria [▶ details](#)

## Extreme Points in Payoff Space

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- The set of CE  $\subset \Delta(A)$  subset of a space of dimension  $|A_1| \cdot \dots \cdot |A_n|$
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**Question:** What can we say about payoff-extreme equilibria?



## Observations:

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- Projection of an extreme point **need not** be an extreme point of a projection
- $\Rightarrow$  pure NE and NE with 2 mixers **need not** be payoff-extreme

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## Proposition

In a generic game, utilitarian welfare is non-degenerate

## **Applications to Particular Classes of Games**

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## Costly voting model of Palfrey and Rosenthal (1983):

- Two finite groups of voters:  $D$  and  $R$ ,  $|R| > |D|$
- Voters in  $D$  get utility of 1 if  $d$ -candidate wins and 0 otherwise
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**Other Applications:** games where players want to mismatch actions of others

- e.g., network games (with substitutes), congestion games, all-pay auctions, Boston matching mechanism

# Games with Unique Correlated Equilibrium

- Unique CE  $\implies$  CE=NE
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- No genericity assumption since games with a unique CE form an open set (Viossat, 2010)

# Conclusions

Several papers effectively show extremality of NE in specific contexts:

- Tullock contests, Cournot and Bertrand, patent races, location games (Einy, Haimanko, and Lagziel, 2022)
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Our paper:

- a tension between equilibrium randomness and extremality
- detail-free criterion for extremality in various settings

Main takeaway: inherent suboptimality of equilibria with a lot of mixing

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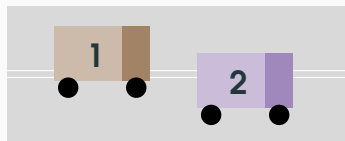
# Thank you!

## **Example: 2 Players vs 3 Players**

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## Example: 2-Player Games

A version of the Game of Chicken by **Aumann (1974)**:



	Risky	Safe
Risky	6, 6	10, 7
Safe	7, 10	9, 9

## Example: 2-Player Games

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Safe	7, 10	9, 9
	$p$	$1 - p$

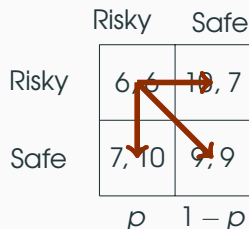
- Mixed NE:  $(1/2, 1/2)$  for both players

Solves linear equation:  $6p + 10(1 - p) = 7p + 9(1 - p) \implies p = 1/2$



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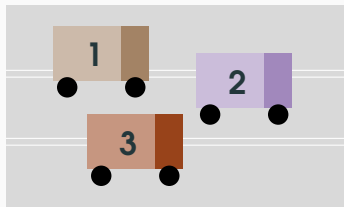
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- Indeed, it is the optimum for a non-degenerate objective

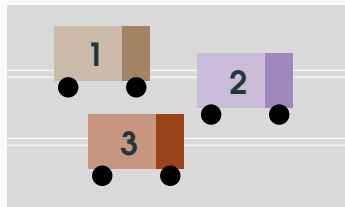
weight of (Risky, Risky) & (Safe, Safe)  $\rightarrow \max$

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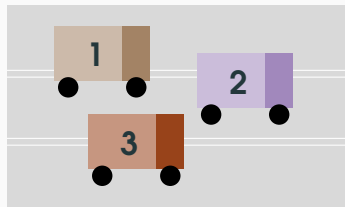
		Safe		Risky	
		Risky	Safe	Risky	Safe
Risky	Risky	6, 6, 5	10, 7, 7	0, 0, 0	6, 5, 6
	Safe	7, 10, 7	9, 9, 9	5, 6, 6	7, 7, 10

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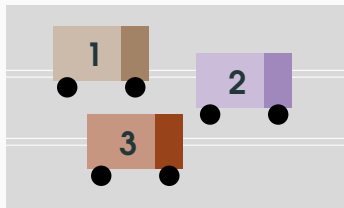
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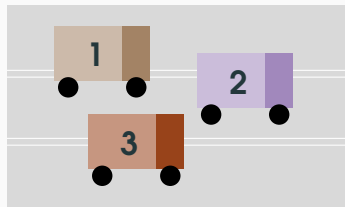
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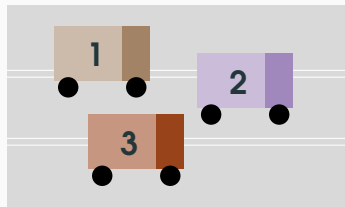


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- Symmetric Mixed NE:  $(\sqrt{3/2} - 1, 2 - \sqrt{3/2})$  for each player



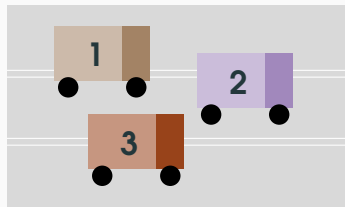
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- Non-linear equation in  $p \Rightarrow$  irrational weights (Nash, 1950)

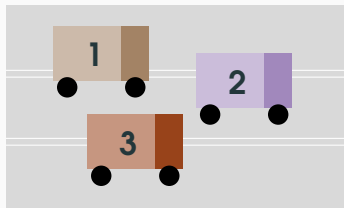
## Example: 3-Player Games



		Safe		Risky		
		Risky	Safe	Risky	Safe	
Risky		6, 6, 5	10, 7, 7	0, 0, 0	6, 5, 6	$p$
Safe		7, 10, 7	9, 9, 9	5, 6, 6	7, 7, 10	$1 - p$
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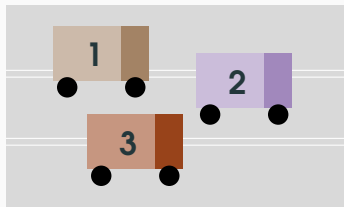
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More than 2 players mixing makes a difference...

## More Applications

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# Symmetric Games

- In many applications, strategic interactions are symmetric
- When are symmetric equilibria extreme?

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**Take-away:** symmetric mixed equilibria are inherently suboptimal

## What Extreme CE Look Like

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**Question:** What is the structure of extreme CE?

- For general games, we only know that extreme CE have small support
- For symmetric games and symmetric CE, we can say more

**Observation:**

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## Theorem (de Finetti)

Any infinite exchangeable sequence  $a_1, a_2, a_3 \dots$  is a mixture of i.i.d. distributions

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- A version of Proposition 2 holds: sampling without replacement instead of i.i.d.

## Key Lemmas

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## Support Size of Extreme Correlated Equilibria (follows from **Winkler (1988)**)

If  $\mu$  is an extreme correlated equilibrium, then

$$\text{supp}(\mu) \leq 1 + \sum_{i \in N} |S_i| \cdot (|S_i| - 1)$$

## Support Size of Regular Nash Equilibria (**McKelvey and McLennan, 1997**)

For a regular Nash equilibrium,  $\nu = (\nu_1, \nu_2, \dots, \nu_n)$ :

$$\text{supp}(\nu_i) - 1 \leq \sum_{j \neq i} (\text{supp}(\nu_j) - 1), \quad \text{for any player } i$$

## Regularity of Generic games (**Harsanyi, 1973**)

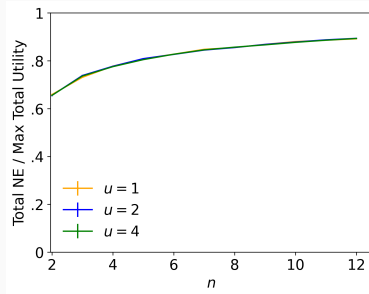
In a generic game, any Nash equilibrium is regular

# Simulations

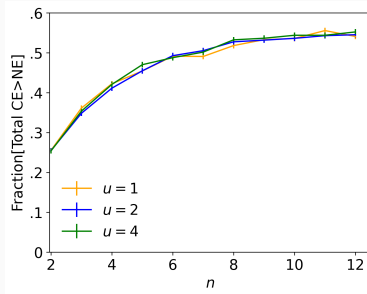
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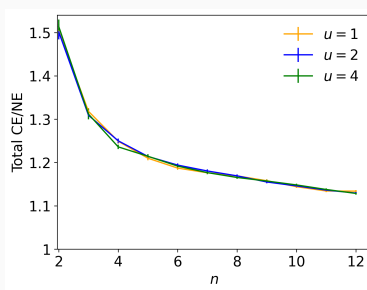
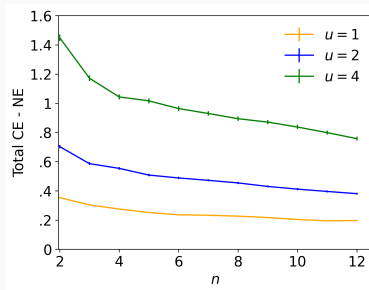
# Simulations



(a)  $NE/2u$



(b) Fraction of CE  $>$  NE\*



# General linear objectives

- Consider a NE  $\nu$
- For simplicity,  $\nu$  has full support
- By Farkas lemma, a linear objective  $L$  can be improved for  $\nu \iff L$  **cannot** be expressed as

$$L(\mu) = C + \sum_{i, a_i, a'_i, a_{-i}} \mu(a) \cdot \lambda_i(a_i, a'_i) \cdot (u_i(a_i, a_{-i}) - u_i(a'_i, a_{-i}))$$

for some  $\lambda_i(a_i, a'_i) \geq 0$ .

- For non-extreme NE  $\nu$ , “bad”  $L$  form a lower-dimensional subspace

# Extreme Symmetric CE with Any Number of Players

Consider  $n$  players with  $m$  actions each

## Proposition

Any extreme symmetric CE can be obtained as follows:

- there are  $M$  urns, each with  $n$  balls labeled by actions

$$1 \leq M \leq m(m-1) + 1$$

- an urn is selected at random according to  $p \in \Delta_M$ , secretly from players
- players draw balls sequentially without replacement
- $i$ 's action = her ball's label, no incentive to deviate

**Remark:** If  $n$  is large, sampling without replacement can be approximated by i.i.d.

## Bayesian games

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## Bayesian game

$$\mathcal{B} = \left( N, (A_i)_{i \in N}, (T_i)_{i \in N}, \tau \in \Delta(T), (u_i : A \times T_i \rightarrow \mathbb{R})_{i \in N} \right)$$

- Each player  $i \in N$  has a type  $t_i \in T_i$
- Profile of types  $(t_1, \dots, t_n) \in T$  sampled from  $\tau$
- Each player  $i$  observes her realized type
- Utility  $u_i : A \times T_i \rightarrow \mathbb{R}$  depends on the action profile and  $i$ 's type

**Technical assumption:** sets of types  $T_i$  are finite

# Bayesian Correlated Equilibria (BCE)

## Definition

A joint distribution  $\mu \in \Delta(A \times T)$  is a Bayesian correlated equilibrium if

- The marginal on  $T$  coincides with  $\tau$
- For each player  $i$ , type  $t_i$ , recommended action  $a_i$ , and deviation  $a'_i$ ,

$$\sum_{(a_{-i}, t_{-i})} \mu((a_i, t_i), (a_{-i}, t_{-i})) u_i(a_i, t_i, a_{-i}) \geq \sum_{(a_{-i}, t_{-i})} \mu((a_i, t_i), (a_{-i}, t_{-i})) u_i(a'_i, t_i, a_{-i})$$

**Interpretation:** a mediator having access to realized types recommends actions to each player. Two aspects:

1. **Ex-ante coordination:** a source of correlated randomness (as in CE)
2. **Information sharing:** providing  $i$  more info about  $t_{-i}$  than contained in  $t_i$

**Remark:** Bergemann and Morris (2016) allow for a broader class of BCE, where player  $i$  observes a noisy signal about her type

# Induced Complete Information Game

We can associate a complete information normal form game  $\Gamma_{\mathcal{B}}$  with  $\mathcal{B}$ :

- Replace  $A_i$  with set of functions  $\sigma_i : T_i \rightarrow A_i$
- $\Sigma_i$  is the set of all such  $\sigma_i$
- Utility  $v_i : \Sigma \rightarrow \mathbb{R}$  is given by

$$v_i(\sigma) = \sum_{t \in T} \tau(t) \cdot u_i((\sigma_1(t_1), \dots, \sigma_n(t_n)), t_i)$$

## Induced Complete Information Game

$$\Gamma_{\mathcal{B}} = (N, (\Sigma_i)_{i \in N}, (v_i)_{i \in N})$$

**Question:** What is a relation between CE of  $\Gamma_{\mathcal{B}}$  and BCE of  $\mathcal{B}$ ?

# Induced complete information game

## Relationship between equilibria in $\Gamma_{\mathcal{B}}$ and $\mathcal{B}$

CE in  $\Gamma_{\mathcal{B}} \Leftrightarrow$  ex-ante coordination in  $\mathcal{B}$  with no information sharing

- i.e., BCE such that  $a_i$  is independent of  $t_{-i}$  conditionally on  $t_i$

Nash in  $\Gamma_{\mathcal{B}} \Leftrightarrow$  Bayes-Nash in  $\mathcal{B}$

**Observation:** Generic  $\mathcal{B}$  leads to generic  $\Gamma_{\mathcal{B}}$

- $\Rightarrow$  we can apply our theorem to  $\Gamma_{\mathcal{B}}$  to learn about generic  $\mathcal{B}$

## Corollary

For a generic Bayesian game, a Bayes-Nash equilibrium is improvable via ex-ante coordination  $\Leftrightarrow$  at least 3 players randomize

Applies to Bayesian games where players randomize in equilibrium, e.g., costly voting with private types (Feddersen and Pesendorfer, 1997) and contests (Baranski and Goel, 2024)



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