Improvable Equilibria

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Penn State, September 27, 2024

Introduction

Communication or intermediation

- precede many interactions: voting, matching, product adoption, etc.
- a possible channel for collusion by auction bidders, market competitors, and the like

Broad question: What strategic interactions are susceptible to communication influences or collusion?

Refined Introduction

Correlated equilibria (Aumann, 1974) generalize Nash equilibria to allow correlation

- Can be implemented via communication, as well as mediation or joint randomization
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This project: When is there potential value in correlation?

Normal-form game

$$\Gamma = \Big(N,\ (A_i)_{i\in N},\ (u_i : A \to \mathbb{R})_{i\in N}\Big)$$

- $N = \{1, \ldots, n\}$ is finite set of players
- A_i is a finite set of actions of player i
- $A = \prod_{i\in N} A_i$ is the set of action profiles
- $u_i : A \to \mathbb{R}$ is utility of player i

Correlated Equilibria (CE)

Definition

A distribution $\mu \in \Delta(A)$ is a correlated equilibrium if

$$\sum_{a_{-i} \in A_{-i}} \mu(a_i, a_{-i})\, u_i(a_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \mu(a_i, a_{-i})\, u_i(a_i', a_{-i})$$

for all $i \in N$ and all $a_i, a_i' \in A_i$

Interpretation: μ generated by a mediator and players best respond by adhering

Remark: Nash Equilibria (NE) are CE of the form $\mu = \mu_1 \times \ldots \times \mu_n$

Approach

- The set of correlated equilibria is a convex polytope
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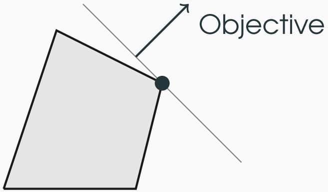
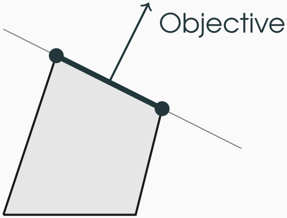
Our Question: When is a Nash equilibrium extreme?

Improvability of non-extreme equilibria

Maximization of a linear objective—e.g., utilitarian welfare—over a polytope P:





Two cases:

- If the optimum is unique, it is an extreme point
  - We call objectives with a unique optimum **non-degenerate**
  - Utilitarian welfare is non-degenerate, as we will see
- In knife-edge cases, the whole face of P can be optimal

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Remark: linear in probabilities, not in actions $\Rightarrow$ a broad class of objectives

Improvability of non-extreme equilibria 2

Bauer's Maximum Principle

Any non-degenerate linear or (quasi-)convex objective attains its maximum at an extreme point

- $\Rightarrow$ Non-extreme equilibria are improvable **no matter** the objective

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Main Insight

Despite restrictiveness of improvability notion, **many** equilibria are improvable, i.e., **non-extreme**

Literature Review

- **Value of correlation in 2-player games**: Cripps (1995), Evangelista and Raghavan (1996), Canovas et al. (1999), Nau et al. (2004), Peeters and Potters (1999), Calvó-Armengol (2006), Ashlagi et al. (2008)
- **Communication $\Leftrightarrow$ correlation:** Forges (2020), Bárány (1992), Ben-Porath (1998), Gerardi (2004), Lehrer and Sorin (1997)
- **Communication & collusion in specific contexts**:
  - Bargaining: Crawford (1990), Agranov and Tergiman (2014), Baranski and Kagel (2015)
  - Auctions: McAfee and McMillan (1992), Lopomo et al. (2011), Feldman et al. (2016), Agranov and Yariv (2018), Pavlov (2023)
  - Voting: Gerardi and Yariv (2007), Goeree and Yariv (2011)
  - Matching: Beyhaghi and Tardos (2018), Echenique et al. (2022)
- **Extreme-point approach in info & mech. design:** Kleiner et al. (2021), Arieli et al. (2023), Yang and Zentefis (2024), Kleiner et al. (2024)

Outline

- **Part 1**
  - Conditions for extremality/improvability
  - Translation to payoffs
  - Applications
- **Part 2**
  - Proof idea
  - Simple description of extreme CE

Conditions for Extremality

Extremality of Nash Equilibria

Theorem 1

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  $\Rightarrow$ 2-player games not representative

Non-generic Games

Genericity can be dropped in any game, by considering **regular** NE only

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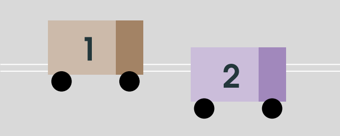
- In a generic game, any NE is regular (Harsanyi, 1973)
- Hence, Theorem $1' \Rightarrow$ Theorem 1

Example: 2 Players vs 3 Players

Example: 2-Player Games

A version of the Game of Chicken by Aumann (1974):



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Risky	6, 6	10, 7
Safe	7, 10	9, 9

Example: 2-Player Games

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	p	$1-p$

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  Solves linear equation: $6p + 10(1-p) = 7p + 9(1-p) \implies p = 1/2$

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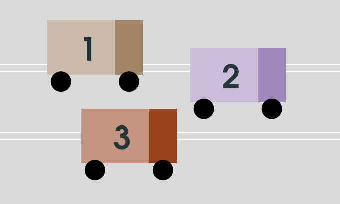
- Indeed, it is the optimum for a non-degenerate objective

$$\text{weight of (Risky, Risky) \& (Safe, Safe)} \to \max$$

Example: 3-Player Games



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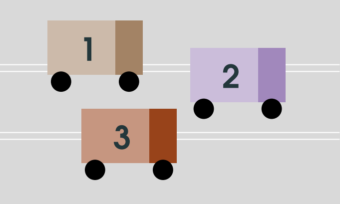

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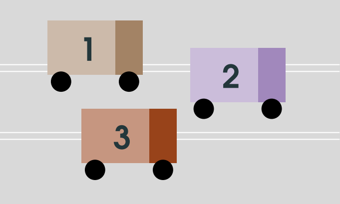

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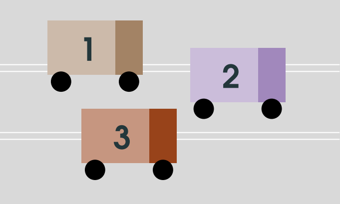

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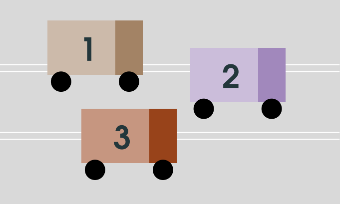

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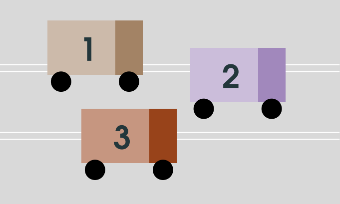

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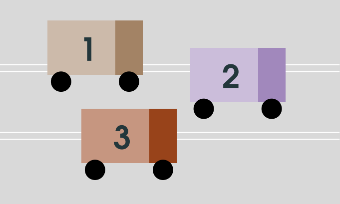

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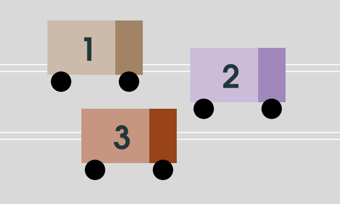

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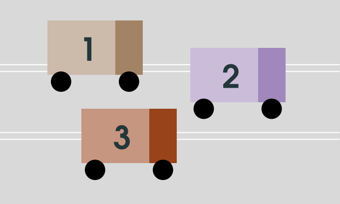

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More than 2 players mixing makes a difference...

Extreme Points in Payoff Space

Payoff-extreme Equilibria

- The set of CE $\subset \Delta(A)$ subset of a space of dimension $|A_1| \cdot \ldots \cdot |A_n|$
- Equilibria are often represented via payoffs in $\mathbb{R}^n$

Definition

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Question: What can we say about payoff-extreme equilibria?

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- CE payoffs $=$ projection of CE to a lower-dimensional space
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- Projection of an extreme point **need not** be an extreme point of a projection
- $\Rightarrow$ pure NE and NE with 2 mixers **need not** be payoff-extreme
    - e.g, the mixed NE in the Game of Chicken

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Proposition

In a generic game, utilitarian welfare is non-degenerate

Applications to Particular Classes of Games

Costly Voting

Costly voting model of Palfrey and Rosenthal (1983):

- Two finite groups of voters: D and R, $|R| > |D|$
- Voters in D get utility of 1 if d-candidate wins and 0 otherwise
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Other Applications: games where players want to mismatch actions of others

- e.g., network games (with substitutes), congestion games, all-pay auctions, Boston matching mechanism

Symmetric Games

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Take-away: caution when focusing on symmetric mixed equilibria in symmetric games

Games with Unique Correlated Equilibrium

- Games with a unique CE form an open set (Viossat, 2010)
- NE=CE $\Rightarrow$ robustness to incomplete information about payoffs (Einy et al., 2022)

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- No genericity assumption needed thanks to the open-set property

PART II

How to Prove Theorem 1

Proof Idea

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- We've seen intuition based on the possibility of having irrational NE for $n \geq 3$ mixers

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In any game, a regular mixed NE is extreme $\iff$ ≤ 2 players randomize

- We've seen intuition based on the possibility of having irrational NE for $n \geq 3$ mixers
- This is not how the actual proof goes since not all NE with $n = 3$ mixers are irrational

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- This is not how the actual proof goes since not all NE with $n = 3$ mixers are irrational

Idea: When many players randomize, there are too many ways to correlate their actions, one must be beneficial

Proof Idea

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Focus on a particular example to illustrate

How the Proof Goes: Example

- Game with n players, each with 2 actions

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- $2n$ constraints
- Winkler (1988): if k linear constraints are imposed on the set of all distributions $\Delta(A)$, extreme distributions have support $\leq k + 1$
- $\Rightarrow$ support of an extreme CE μ is bounded by $2n + 1$

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- The same argument applies to equilibria, where players mix over the **same number of pure strategies**
- The main difficulty is handling very asymmetric equilibria

Key Lemmas for the General Proof

Support Size of Extreme Correlated Equilibria (follows from Winkler (1988))

If μ is an extreme correlated equilibrium, then

$$\text{supp}(\mu) \leq 1 + \sum_{i \in N} |A_i| \cdot (|A_i| - 1)$$

Support Size of Regular Nash Equilibria (McKelvey and McLennan, 1997)

For a regular Nash equilibrium, $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$:

$$\text{supp}(\nu_i) - 1 \leq \sum_{j \neq i} (\text{supp}(\nu_j) - 1), \quad \text{for any player } i$$

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Let's combine these two observations

How the Proof Goes

Consider a game $\Gamma = (A, u)$ and a non-pure **extreme** regular Nash equilibrium ν

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By the lemmas from the previous slide:

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- $\Rightarrow$ ν with $n \geq 3$ mixers cannot be extreme
- The proposition is proved via majorization & Schur convexity

What Extreme CE Look Like

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Question: What is the structure of extreme CE?

- For general games, we only know that extreme CE have small support
- For symmetric games and symmetric CE, we can say more

Symmetric CE and Exchangability

Observation:

- For a symmetric CE, the random variables $a_1, \ldots, a_n$ are exchangeable

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Theorem (de Finetti)

Any infinite exchangeable sequence $a_1, a_2, a_3 \dots$ is a mixture of i.i.d. distributions

Extreme Symmetric CE with Many Players

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- A version of Proposition 2 holds: sampling without replacement instead of i.i.d.

details

Conclusions

Several papers effectively show extremality of NE in specific contexts:

- Tullock contests, Cournot and Bertrand, patent races, location games (Einy, Haimanko, and Lagziel, 2022)
- First-price auctions (Feldman, Lucier, and Nisan, 2016)
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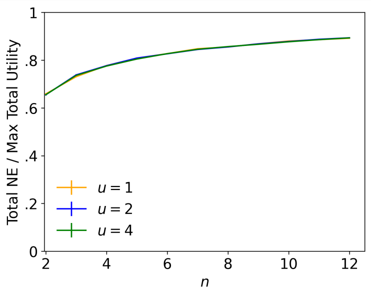
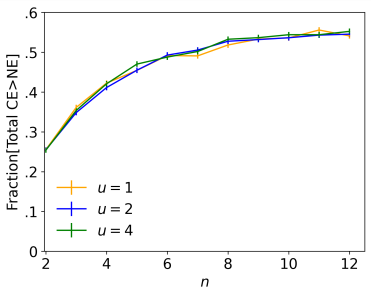
Thank you!

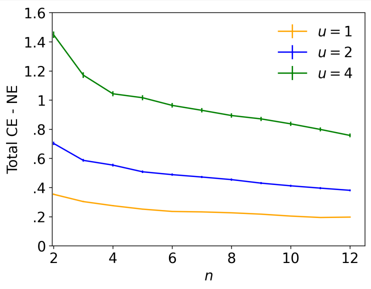
Simulations

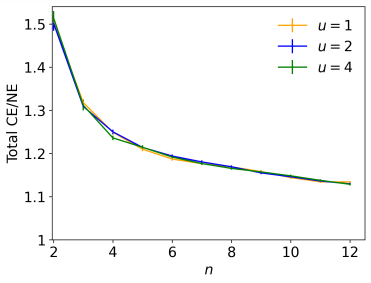


(a) NE/2u



(b) Fraction of CE > NE*





General linear objectives

- Consider a NE ν
- For simplicity, ν has full support
- By Farkas lemma, a linear objective L can be improved for $\nu \iff L$ **cannot** be expressed as

$$L(\mu) = C + \sum_{i, a_i, a_i', a_{-i}} \mu(a) \cdot \lambda_i(a_i, a_i') \cdot \left(u_i(a_i, a_{-i}) - u_i(a_i', a_{-i})\right)$$

  for some $\lambda_i(a_i, a_i') \geq 0$.
- For non-extreme NE ν, "bad" L form a lower-dimensional subspace

back

Extreme Symmetric CE with Any Number of Players

Consider n players with m actions each

Proposition

Any extreme symmetric CE can be obtained as follows:

- there are M urns, each with n balls labeled by actions

$$1 \leq M \leq m(m-1)+1$$

- an urn is selected at random according to $p \in \Delta_M$, secretly from players
- players draw balls sequentially without replacement
- i's action = her ball's label, no incentive to deviate

Remark: If n is large, sampling without replacement can be approximated by i.i.d.

back

Bayesian games

Bayesian game

$$\mathcal{B} = \Big(N,\ (A_i)_{i\in N},\ (T_i)_{i\in N},\ \tau \in \Delta(T),\ (u_i : A \times T_i \to \mathbb{R})_{i\in N}\Big)$$

- Each player $i \in N$ has a type $t_i \in T_i$
- Profile of types $(t_1, \dots, t_n) \in T$ sampled from τ
- Each player i observes her realized type
- Utility $u_i : A \times T_i \to \mathbb{R}$ depends on the action profile and i's type

Technical assumption: sets of types T_i are finite

Bayesian Correlated Equilibria (BCE)

Definition

A joint distribution $\mu \in \Delta(A \times T)$ is a Bayesian correlated equilibrium if

- The marginal on T coincides with τ
- For each player i, type t_i, recommended action a_i, and deviation a_i',

$$\sum_{(a_{-i},t_{-i})} \mu((a_i,t_i),(a_{-i},t_{-i}))\, u_i(a_i,t_i,a_{-i}) \geq \sum_{(a_{-i},t_{-i})} \mu((a_i,t_i),(a_{-i},t_{-i}))\, u_i(a_i',t_i,a_{-i})$$

Interpretation: a mediator having access to realized types recommends actions to each player. Two aspects:

1. **Ex-ante coordination:** a source of correlated randomness (as in CE)
2. **Information sharing:** providing i more info about t_{-i} than contained in t_i

Remark: Bergemann and Morris (2016) allow for a broader class of BCE, where player i observes a noisy signal about her type

Induced Complete Information Game

We can associate a complete information normal form game $\Gamma_{\mathcal{B}}$ with $\mathcal{B}$:

- Replace A_i with set of functions $\sigma_i : T_i \to A_i$
- Σ_i is the set of all such σ_i
- Utility $v_i : \Sigma \to \mathbb{R}$ is given by

$$v_i(\sigma) = \sum_{t \in T} \tau(t) \cdot u_i\big((\sigma_1(t_1), \ldots, \sigma_n(t_n)),\ t_i\big)$$

Induced Complete Information Game

$$\Gamma_{\mathcal{B}} = \big(N,\ (\Sigma_i)_{i \in N},\ (v_i)_{i \in N}\big)$$

Question: What is a relation between CE of $\Gamma_{\mathcal{B}}$ and BCE of $\mathcal{B}$?

Induced complete information game

Relationship between equilibria in $\Gamma_{\mathcal{B}}$ and $\mathcal{B}$

CE in $\Gamma_{\mathcal{B}}$ $\Leftrightarrow$ ex-ante coordination in $\mathcal{B}$ with no information sharing

- i.e., BCE such that a_i is independent of t_{-i} conditionally on t_i

Nash in $\Gamma_{\mathcal{B}}$ $\Leftrightarrow$ Bayes-Nash in $\mathcal{B}$

Observation: Generic $\mathcal{B}$ leads to generic $\Gamma_{\mathcal{B}}$

- $\Rightarrow$ we can apply our theorem to $\Gamma_{\mathcal{B}}$ to learn about generic $\mathcal{B}$

Corollary

For a generic Bayesian game, a Bayes-Nash equilibrium is improvable via ex-ante coordination $\Longleftrightarrow$ at least 3 players randomize

Applies to Bayesian games where players randomize in equilibrium, e.g., costly voting with private types (Feddersen and Pesendorfer, 1997) and contests (Baranski and Goel, 2024)

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