Improvable Equilibria

Kirill Rudov - UC Berkeley
Fedor Sandomirskiy - Princeton
Leeat Yariv - Princeton

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Introduction

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This project: When is there potential value in correlation?

Normal-form game

$$\Gamma = \Big(N,\ (A_i)_{i\in N},\ (u_i : A \to \mathbb{R})_{i\in N}\Big)$$

- $N = \{1, \ldots, n\}$ is finite set of players
- A_i is a finite set of actions of player i
- $A = \prod_{i\in N} A_i$ is the set of action profiles
- $u_i : A \to \mathbb{R}$ is utility of player i

Correlated Equilibria (CE)

Definition

A distribution $\mu \in \Delta(A)$ is a correlated equilibrium if

$$\sum_{a_{-i} \in A_{-i}} \mu(a_i, a_{-i})\, u_i(a_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \mu(a_i, a_{-i})\, u_i(a_i', a_{-i})$$

for all $i \in N$ and all $a_i, a_i' \in A_i$

Interpretation: μ as generated by a mediator, where agents best respond by adhering

Approach

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- **Bauer's Maximum Principle:** Any linear or convex objective attains its maximum at an extreme point (uniquely with strict convexity)

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- Improvable $\Leftrightarrow$ **any** strictly convex objective can be strictly improved
- Agnostic about designer's objective

Example: Improving Welfare of Nash

The Game of Chicken by Aumann (1974):

	Dare	Chicken
Dare	0, 0	4, 1
Chicken	1, 4	3, 3

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- Mixed Nash: (1/2, 1/2) for both players
- Correlated equilibrium can increase welfare by shifting weight from (0,0)

- But the mixed Nash is **not improvable**
  - It is a unique optimum for the objective corresponding to weight on diagonal entries, (Dare, Dare) and (Chicken, Chicken)

Improvability: Summary

- Welfare maximization is just one objective
- An objective can maximize likelihood of certain profiles, minimize entropy of the joint distribution over profiles, etc.
- An improvable NE can be improved **no matter the objective**
- Improvability is very demanding

Outline

- Conditions for improvability
- Linear objectives
- Applications
- Symmetric games

Conditions for Improvability

Improvability of Nash Equilibria

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In a generic n-player game, a mixed NE is improvable $\iff \geq 3$ agents randomize

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  $\Rightarrow$ 2-agent games not representative

Non-generic Games

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- In a generic game, any NE is regular (Harsanyi, 1973)
- Hence, Theorem $1' \Rightarrow$ Theorem 1

Why Theorems Hold

Rough intuition: When many agents randomize, there are too many ways to correlate their actions, one must be beneficial

Focus on a particular example to illustrate

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- $2n$ constraints
- Winkler (1988): if k linear constraints are imposed on the set of all distributions $\Delta(A)$, extreme distributions have support $\leq 1 + k$
- $\Rightarrow$ support of an extreme CE μ is bounded by $1 + 2n$

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- At most $\log_2(1 + 2n)$ out of n agents can randomize
- In fact, only 2 agents can randomize

details

Linear Objectives

Linear Objectives and Utilitarian Welfare

Utilitarian welfare is a common objective, which is linear:

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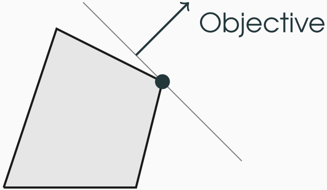
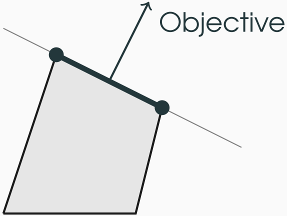
Question: When is a NE improvable in terms of utilitarian welfare, or other linear objectives?

Improving Linear Objectives

Maximization of a linear objective over a polytope P:





Two cases:

- For a generic objective, the unique optimum is at an extreme point
- In the knife-edge case, the whole face of P can be optimal

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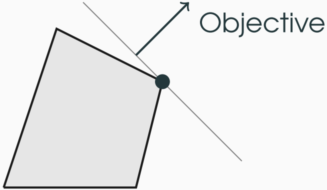

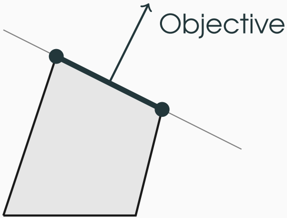

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Corollary

NE is improvable $\Longrightarrow$ a generic linear objective can be strictly improved

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Question: When can the welfare objective

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Intuition: Perturb utilities $u_i'(s) = u_i(s) + \delta_i(s_{-i})$. This

- does not affect the incentive constraints
- affects W

$\implies$ the hyperplane $W =$ const cannot be parallel to a face of $CE(\Gamma)$ in a generic Γ

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Other linear objectives: conditions for strict improvability from Farkas lemma details

Sample Applications

Costly voting model of Palfrey and Rosenthal (1983):

- Two finite groups of voters: D and R, $|R| > |D|$
- Voters in D get utility of 1 if d-candidate wins and 0 otherwise
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Other Applications: games where agents want to mismatch actions of others

- e.g., network games (with substitutes), congestion games, all-pay auctions, Boston matching mechanism

Symmetric Games

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- No genericity or regularity assumptions
- Any pure strategy must be played with a positive probability

Symmetric Games: Highlights

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  For $n \geq 3$, completely mixed symmetric NE cannot be extreme within the (smaller!) set of *symmetric CE*

- $\Rightarrow$ a generic *symmetric* linear objective can be strictly improved, e.g., utilitarian welfare

Take-away: caution when focusing on symmetric equilibria in symmetric games

What Improvements Look Like

For an improvable NE, any strictly convex objective can be strictly improved by switching to an extreme CE

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Question: What is the structure of extreme CE?

- For general games, we only know that extreme CE have small support
- For symmetric games and symmetric CE, we can say more

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Theorem (de Finetti)

Any infinite exchangeable sequence $a_1, a_2, a_3 \ldots$ is a mixture of i.i.d.

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- Consider a symmetric game with m actions per player
- Assume the number of players n is large

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- A version of Proposition 2 holds: sampling without replacement instead of i.i.d.

details

Conclusions

Several papers effectively show non-improvability in specific contexts:

- Tullock contests, Cournot and Bertrand, patent races, location games (Einy, Haimanko, and Lagziel, 2022)
- First-price auctions (Feldman, Lucier, and Nisan, 2016)
- Convex potential games (Neyman, 1997; Ui, 2008)
- Two-player normal-form games (Cripps, 1995)

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Thank you!

Implications: Games with Unique Correlated Equilibrium

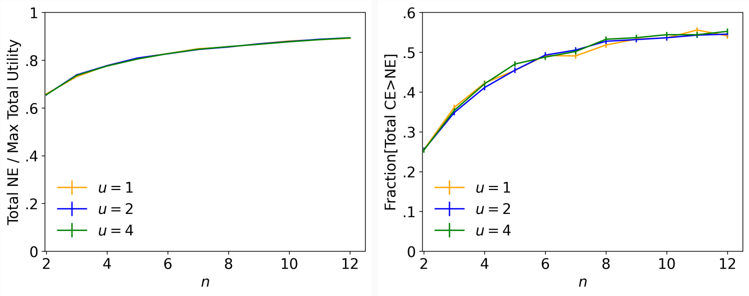
Corollary

If a generic game has a unique correlated equilibrium ν, then ν is either:

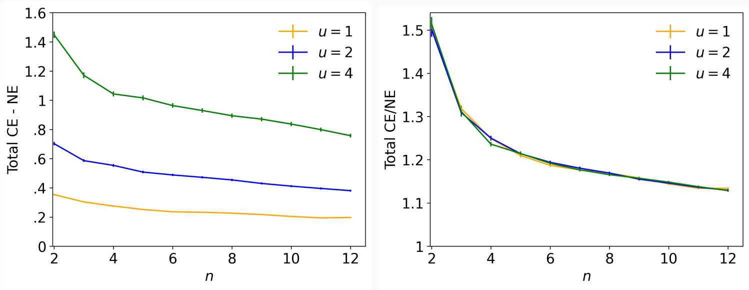
- A pure Nash equilibrium, or
- A Nash equilibrium where exactly two agents randomize

- Examples: some congestion games, Cournot competition

Simulations



(a) NE/2u **(b)** Fraction of CE > NE*



Key Lemmas

Support Size of Extreme Correlated Equilibria (follows from Winkler (1988))

If μ is an extreme correlated equilibrium, then

$$\text{supp}(\mu) \leq 1 + \sum_{i \in N} |S_i| \cdot (|S_i| - 1)$$

Support Size of Regular Nash Equilibria (McKelvey and McLennan, 1997)

For a regular Nash equilibrium, $\nu = (\nu_1, \nu_2, \dots, \nu_n)$:

$$\text{supp}(\nu_i) - 1 \leq \sum_{j \neq i} (\text{supp}(\nu_j) - 1), \quad \text{for any player } i$$

Regularity of Generic games (Harsanyi, 1973)

In a generic game, any Nash equilibrium is regular

back

General linear objectives

- Consider a NE ν
- For simplicity, ν has full support
- By Farkas lemma, a linear objective L can be improved for $\nu \iff L$ **cannot** be expressed as

$$L(\mu) = C + \sum_{i,a_i,a_i',a_{-i}} \mu(a) \cdot \lambda_i(a_i, a_i') \cdot \left(u_i(a_i, a_{-i}) - u_i(a_i', a_{-i})\right)$$

  for some $\lambda_i(a_i, a_i') \geq 0$.
- For improvable NE ν, "bad" L form a lower-dimensional subspace

back

Extreme Symmetric CE with Any Number of Players

Consider n players with m actions each

Proposition

Any extreme symmetric CE can be obtained as follows:

- there are M urns, each with n balls labeled by actions

$$1 \leq M \leq m(m-1)+1$$

- an urn is selected at random according to $p \in \Delta_M$, secretly from players
- players draw balls sequentially without replacement
- i's action = her ball's label, no incentive to deviate

Remark: If n is large, sampling without replacement can be approximated by i.i.d.

back

Bayesian games

Bayesian game

$$\mathcal{B} = \Big(N,\ (A_i)_{i\in N},\ (T_i)_{i\in N},\ \tau \in \Delta(T),\ (u_i : A \times T_i \to \mathbb{R})_{i\in N}\Big)$$

- Each player $i \in N$ has a type $t_i \in T_i$
- Profile of types $(t_1, \ldots, t_n) \in T$ sampled from τ
- Each player i observes her realized type
- Utility $u_i : A \times T_i \to \mathbb{R}$ depends on the action profile and i's type

Technical assumption: sets of types T_i are finite

Bayesian Correlated Equilibria (BCE)

Definition

A joint distribution $\mu \in \Delta(A \times T)$ is a Bayesian correlated equilibrium if

- The marginal on T coincides with τ
- For each player i, type t_i, recommended action a_i, and deviation a_i',

$$\sum_{(a_{-i},t_{-i})} \mu((a_i, t_i), (a_{-i}, t_{-i}))\, u_i(a_i, t_i, a_{-i}) \geq \sum_{(a_{-i},t_{-i})} \mu((a_i, t_i), (a_{-i}, t_{-i}))\, u_i(a_i', t_i, a_{-i})$$

Interpretation: a mediator having access to realized types recommends actions to each player. Two aspects:

1. *Ex-ante coordination:* a source of correlated randomness (as in CE)
2. *Information sharing:* providing i more info about t_{-i} than contained in t_i

Remark: Bergemann and Morris (2016) allow for a broader class of BCE, where player i observes a noisy signal about her type

Induced Complete Information Game

We can associate a complete information normal form game $\Gamma_{\mathcal{B}}$ with $\mathcal{B}$:

- Replace A_i with set of functions $\sigma_i : T_i \to A_i$
- Σ_i is the set of all such σ_i
- Utility $v_i : \Sigma \to \mathbb{R}$ is given by

$$v_i(\sigma) = \sum_{t \in T} \tau(t) \cdot u_i\big((\sigma_1(t_1), \dots, \sigma_n(t_n)),\ t_i\big)$$

Induced Complete Information Game

$$\Gamma_{\mathcal{B}} = \big(N,\ (\Sigma_i)_{i \in N},\ (v_i)_{i \in N}\big)$$

Question: What is a relation between CE of $\Gamma_{\mathcal{B}}$ and BCE of $\mathcal{B}$?

Induced complete information game

Relationship between equilibria in $\Gamma_{\mathcal{B}}$ and $\mathcal{B}$

CE in $\Gamma_{\mathcal{B}} \Leftrightarrow$ ex-ante coordination in $\mathcal{B}$ with no information sharing

- i.e., BCE such that a_i is independent of t_{-i} conditionally on t_i

Nash in $\Gamma_{\mathcal{B}} \Leftrightarrow$ Bayes-Nash in $\mathcal{B}$

Observation: Generic $\mathcal{B}$ leads to generic $\Gamma_{\mathcal{B}}$

- $\Rightarrow$ we can apply our theorem to $\Gamma_{\mathcal{B}}$ to learn about generic $\mathcal{B}$

Corollary

For a generic Bayesian game, a Bayes-Nash equilibrium is improvable via ex-ante coordination $\Longleftrightarrow$ at least 3 players randomize

Applies to Bayesian games where agents randomize in equilibrium, e.g., costly voting with private types (Feddersen and Pesendorfer, 1997) and contests (Baranski and Goel, 2024)

References

Aumann, R. J. (1974). Subjectivity and correlation in randomized strategies. *Journal of mathematical Economics 1*(1), 67–96.

Baranski, A. and S. Goel (2024). Contest design with a finite type-space. *to appear*.

Bergemann, D. and S. Morris (2016). Bayes correlated equilibrium and the comparison of information structures in games. *Theoretical Economics 11*(2), 487–522.

Cripps, M. (1995). Extreme correlated and nash equilibria in two-person games.

Einy, E., O. Haimanko, and D. Lagziel (2022). Strong robustness to incomplete information and the uniqueness of a correlated equilibrium. *Economic Theory 73*(1), 91–119.

Feddersen, T. and W. Pesendorfer (1997). Voting behavior and information aggregation in elections with private information. *Econometrica: Journal of the Econometric Society*, 1029–1058.

Feldman, M., B. Lucier, and N. Nisan (2016). Correlated and coarse equilibria of single-item auctions. In *Web and Internet Economics: 12th International Conference, WINE 2016, Montreal, Canada, December 11-14, 2016, Proceedings 12*, pp. 131–144. Springer.

Harsanyi, J. C. (1973). Oddness of the number of equilibrium points: a new proof. *International Journal of Game Theory 2*, 235–250.

McKelvey, R. D. and A. McLennan (1997). The maximal number of regular totally mixed nash equilibria. *Journal of Economic Theory 72*(2), 411–425.

Neyman, A. (1997). Correlated equilibrium and potential games. *International Journal of Game Theory 26*, 223–227.

Palfrey, T. R. and H. Rosenthal (1983). A strategic calculus of voting. *Public Choice 41*(1), 7–53.

Ui, T. (2008). Correlated equilibrium and concave games. *International Journal of Game Theory 37*, 1–13.

Winkler, G. (1988). Extreme points of moment sets. *Mathematics of Operations Research 13*(4), 581–587.