Improvable Equilibria

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Introduction

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This project: When is there potential value in correlation?

Normal-form game is given by $\left(N, (A_i)_{i \in N}, (u_i)_{i \in N}\right)$, where

- $N = \{1, \ldots, n\}$ is finite set of players
- A_i is a finite set of actions of agent i
- $A = \prod_{i \in N} A_i$ is the set of action profiles
- $u_i : A \to \mathbb{R}$ is utility of agent i

Correlated Equilibria

Definition

A distribution $\mu \in \Delta(A)$ is a **correlated equilibrium** if

$$\sum_{a_{-i} \in A_{-i}} \mu(a_i, a_{-i}) u_i(a_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \mu(a_i, a_{-i}) u_i(a_i', a_{-i})$$

for all $i \in N$ and all $a_i, a_i' \in A_i$

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Interpretation: μ as generated by a mediator, where agents best respond by adhering

Approach

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- **Bauer's Maximum Principle:** Any linear or convex objective attains its maximum at an extreme point (uniquely with strict convexity)

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Definition

A Nash equilibrium is **improvable** if it is not an extreme point of the set of correlated equilibria

Improvability of Nash Equilibria

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In a generic n-player game, a mixed **Nash equilibrium is improvable** $\Longleftrightarrow$ **three or more agents randomize**

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The genericity assumption can be dropped

- in games with 2 actions per player
- in any game, by considering *regular* Nash equilibria only

Intuition through Example

Rough intuition: When many agents randomize, there are too many ways to correlate their actions, one must be beneficial

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Focus on particular example to illustrate

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- $2n$ constraints
- Winkler (1988) $\Rightarrow$ support of any extreme μ is bounded by $1 + 2n$

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- At most $\log_2(1+2n)$ out of n agents can randomize
- In fact, only 2 agents can randomize (requires more careful analysis ▸ details)

Sample Applications

Costly voting model of Palfrey and Rosenthal (1983):

- Two finite groups of voters: D and R, $|R| > |D|$
- Voters in D get utility of 1 if d-candidate wins and 0, otherwise
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- Majority voting (among those who participate), ties broken randomly
- Costly participation: $c > 0$

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Other Applications: games where agents want to mismatch actions of others

- e.g., network games (with substitutes), congestion games, all-pay auctions, Boston matching mechanism

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Take-away: caution when focusing on symmetric equilibria in symmetric games

Conclusions

Several papers effectively show non-improvability in specific contexts:

- Tullock contests, Cournot and Bertrand, patent races, location games (Einy, Haimanko, and Lagziel, 2022)
- First-price auctions (Feldman, Lucier, and Nisan, 2016)
- Convex potential games (Neyman, 1997; Ui, 2008)
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Ongoing:

- Incomplete information
- "Correlated implementation" in mechanism design

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Thank you!

Implications: Games with Unique Correlated Equilibrium

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If a generic game has a unique correlated equilibrium ν, then ν is either:

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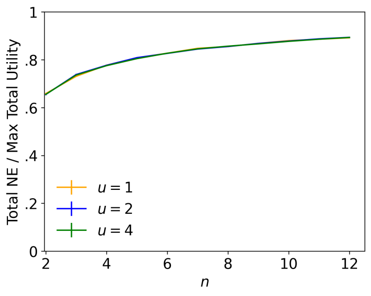
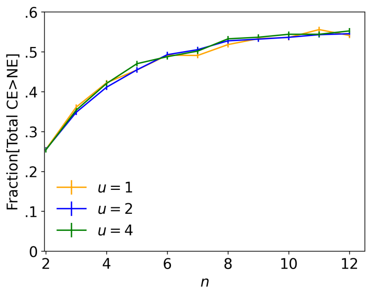
- Examples: some congestion games, Cournot competition

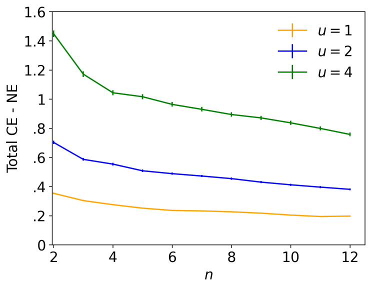
Simulations

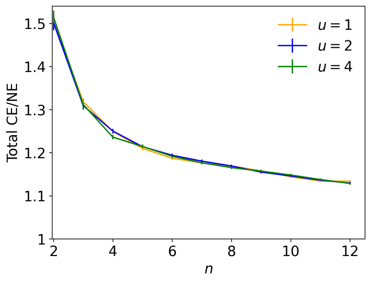


(a) NE/$2u$



(b) Fraction of CE > NE*





Key Lemmas

Support Size of Extreme Correlated Equilibria (follows from Winkler (1988))

If μ is an extreme correlated equilibrium, then

$$\text{supp}(\mu) \leq 1 + \sum_{i \in N} |S_i| \cdot (|S_i| - 1)$$

Support Size of Regular Nash Equilibria (McKelvey and McLennan, 1997)

For a regular Nash equilibrium, $\nu = (\nu_1, \nu_2, \dots, \nu_n)$:

$$\text{supp}(\nu_i) - 1 \leq \sum_{j \neq i} (\text{supp}(\nu_j) - 1), \quad \text{for any player } i$$

Regularity of Generic games (Harsanyi, 1973)

In a generic game, any Nash equilibrium is regular

▸ back

References

Aumann, R. J. (1974). Subjectivity and correlation in randomized strategies. *Journal of mathematical Economics 1*(1), 67–96.

Cripps, M. (1995). Extreme correlated and nash equilibria in two-person games.

Einy, E., O. Haimanko, and D. Lagziel (2022). Strong robustness to incomplete information and the uniqueness of a correlated equilibrium. *Economic Theory 73*(1), 91–119.

Feldman, M., B. Lucier, and N. Nisan (2016). Correlated and coarse equilibria of single-item auctions. In *Web and Internet Economics: 12th International Conference, WINE 2016, Montreal, Canada, December 11-14, 2016, Proceedings 12*, pp. 131–144. Springer.

Harsanyi, J. C. (1973). Oddness of the number of equilibrium points: a new proof. *International Journal of Game Theory 2*, 235–250.

McKelvey, R. D. and A. McLennan (1997). The maximal number of regular totally mixed nash equilibria. *Journal of Economic Theory 72*(2), 411–425.

Neyman, A. (1997). Correlated equilibrium and potential games. *International Journal of Game Theory 26*, 223–227.

Palfrey, T. R. and H. Rosenthal (1983). A strategic calculus of voting. *Public Choice 41*(1), 7–53.

Ui, T. (2008). Correlated equilibrium and concave games. *International Journal of Game Theory 37*, 1–13.

Winkler, G. (1988). Extreme points of moment sets. *Mathematics of Operations Research 13*(4), 581–587.