Beckmann's approach to multi-item multi-bidder auctions

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Bayesian setting: independent private values, seller knows distribution

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Strong duality (informal)

For $n \geq 1$ bidders with additive utilities over $m \geq 1$ items

$$\max_{\text{BIC IR mechanisms}} \text{Revenue} = \min_{\text{transport flows}} \text{Cost}$$

- formal statement later
- left-hand side is intuitive $\Rightarrow$ discuss the right-hand side

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For given geographical distribution of production $\pi_+ \in \Delta(X)$ and consumption $\pi_- \in \Delta(X)$, $X \subset \mathbb{R}^d$, find least costly way of transportation

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[a] M.Beckmann (1952) A continuous model of transportation Econometrica

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flow $f : X \to \mathbb{R}^n$ solving

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Related literature

- **Econ applications of optimal transport**
  - Monge-Kantorovich: Daskalakis et al. (2017), Kleiner, Manelli (2019), Boerma et al.(2021), Chiapporiet et al. (2010), Galichon (2021), Steinerberger, Tsyvinski (2019), Gensbittel (2015), Arieli et al.,(2022), Guo, Shmaya (2021)
  - Beckmann: Fajgelbaum, Schaal (2020), Allenand, Arkolakis (2014), Santambrogio (2015)
- **Non-transport duality in auction design** Giannakopoulos, Koutsoupias (2018), Cai et al. (2019), Bergemann et al. (2016)
- **Simple mechanisms with good revenue guaratees** Hart, Reny (2019), Haghpanah, Hartline (2021), Babaioff et al. (2020,2021), Hart, Nisan (2017), Jehiel et al. (2007), Yao (2017), and many more...
- **Majorization in economics** Hart and Reny (2015), Kleiner et al. (2021), Arieli et al. (2019), Candogan, Strack (2021), Nikzad (2022)

Outline

- Known results: monopolist's problem and its dual
- The case of $n \geq 2$ bidders
  - Similarities and differences
  - Formal statement of duality theorem
- Applications and simulations

$m \geq 2$ **goods,** $n = 1$ **agent**

- agent's values $v = (v_1, \ldots, v_m) \sim \rho(v)\,\mathrm{d}v$
- **Goal:** maximize revenue over BIC IR $(x(v), t(v))$ where $x \in [0,1]^m$
- **Rochet-Chone approach:** mechanisms $\Leftrightarrow$ interim utility functions
  - utility obtained by the agent with values v:

$$u(v) = \langle x(v), v \rangle - t(v) = \max_{v'} \left[\langle x(v'), v \rangle - t(v') \right]$$

  - u is convex and

$$x(v) = \partial u(v), \quad t(v) = \langle x(v), v \rangle - u(v)$$

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Theorem (Rochet and Chone (1998))

$(x(v), t(v)) \leftrightarrow u$ is a bijection between BIC IR mechanisms and convex u with $u(0) = 0$ and $\partial u \in [0,1]^m$

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Revenue maximization becomes:

$$R_m(\rho) = \max_{\substack{\text{convex } u \\ u(0)=0,\ \partial u \in [0,1]^m}} \int_{\mathbb{R}_+^m} \Big(\big\langle \partial u(v), v \big\rangle - u(v) \Big) \rho(v)\,\mathrm{d}v.$$

$m \geq 2$ items, $n = 1$ agent: optimal menus and transportation

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$$= \max_{\substack{\text{convex } u \\ u(0)=0,\ \partial u \in [0,1]^m}} \int_{\mathbb{R}_+^m} u(v)\, \mathrm{d}\psi,$$

where $\mathrm{d}\psi = \big((m+1)\rho(v) + \sum_{j=1}^m v_i \partial_{v_i} \rho \big)\, \mathrm{d}v$ (not necessary positive!)

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What is the dual?

Definition: 2nd-order stochastic dominance aka majorization

$$\mu \succeq \nu \Longleftrightarrow \int g\,\mathrm{d}\mu \geq \int g\,\mathrm{d}\nu \quad \text{for any convex monotone g}$$

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$$R_m(\rho) = \min_{\substack{\text{positive measures } \gamma \\ \text{on } \mathbb{R}_+^m \times \mathbb{R}_+^m \\ \gamma_1 - \gamma_2 \succeq \psi}} \int_{\mathbb{R}_+^m \times \mathbb{R}_+^m} \|v - v'\|_1\,\mathrm{d}\gamma(v, v')$$

This is Monge-Kantorovich problem with majorization

$m \geq 2$ items, $n \geq 2$ agents: reduction to a single agent

Goal: maximize revenue over BIC, IR, symmetric n-agent mechanisms

Can we use the same approach?

- **Reduced-forms mechanism:** expected allocation and payment of an agent as a function of her vector of values
- As before, one-agent mechanism $\leftrightarrow$ convex u
- $m = 1$ proved by Hart and Reny[1]; equivalent to earlier result by K.Border.

[1] S.Hart, P.Reny (2015) Implementation of reduced form mechanisms ET Bulletin

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New feasibility constraint

u corresponds to a symmetric n-agent mechanism if and only if

$$\partial_{v_i} u(v) \preceq z^{n-1} \quad \forall i = 1, \ldots m,$$

where $v \sim \rho$ and $z \sim \mathrm{Uniform}([0,1])$.

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$m \geq 2$ items, $n \geq 2$ agents: primal problem

multi-bidder version of Rochet-Chone theorem

$$R_{n,m}(\rho) = \max_{\substack{\text{convex monotone } u \\ u(0)=0,\ \partial_{v_i} u(v) \preceq z^{n-1}\ \forall i}} n \cdot \int_{\mathbb{R}_+^m} u(v)\, \mathrm{d}\psi(v),$$

where $\mathrm{d}\psi = \left((m+1)\rho(v) + \sum_{j=1}^m v_i \partial_{v_i} \rho\right) \mathrm{d}v$.

- Looks similar to one-agent case
- **Major obstacle:** Local feasibility constraint $\partial u \in [0,1]^m$ is replaced by a non-local non-linear majorization constraint on gradient's distribution.

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What is the dual?

$m \geq 2$ items, $n \geq 2$ agents: dual problem

- **Beckmann's problem:**

$$\mathrm{Beck}_\rho(\pi, \Phi) = \min_{f:\ \mathrm{div}[\rho \cdot f] + \pi = 0} \int_{\mathbb{R}_+^m} \Phi(f(v)) \cdot \rho(v)\, \mathrm{d}v.$$

- **The choice of costs:**
for convex monotone φ_i on $\mathbb{R}_+$ with $\varphi_i(0) = 0$ define

$$\Phi(f) = \sum_{i=1}^m \varphi_i^*(|f_i|) \quad \text{where } \varphi_i^*(y) = \sup_x \langle x, y \rangle - \varphi_i(x)$$

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Corollary: duality by Daskalakis et al. (2017)

Why strong duality is useful?

- **Upper bound on revenue**
- **Controlling how far a given mechanism is from the optimum:** numerical methods with provable approximation guarantees
- **Complementary slackness conditions**
  - can be used to show that a mechanism is not optimal if the conditions are infeasible.
    Example: For $\rho(v) = \rho_1(v_1) \cdot \ldots \cdot \rho_m(v_m)$, selling separately is never optimal.[2]
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    Example: For $n = 1$ and $m = 2$ i.i.d. uniform items, selling each for $\frac{2}{3}$ or both for $\frac{4-\sqrt{2}}{3}$ is optimal.[3]

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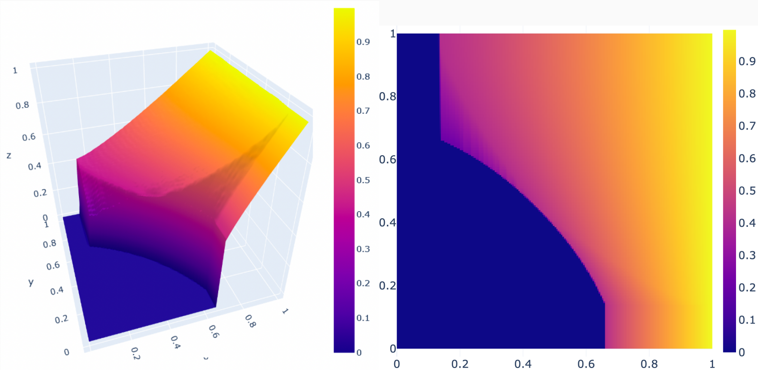

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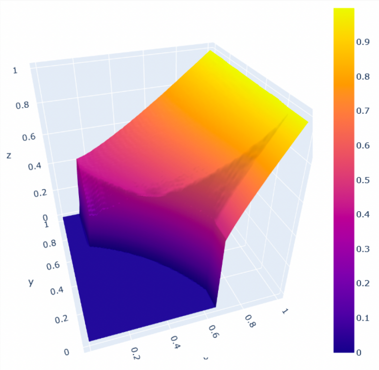

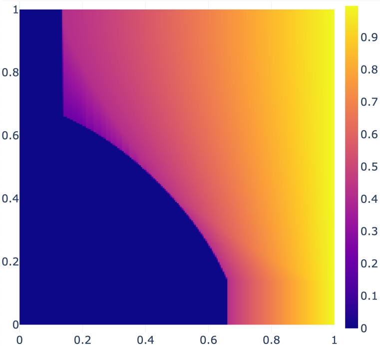

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Remark: computing the optimum numerically is a non-trivial task requiring extra optimal transportation insights about algorithm

The end

- Another justification that mechanism design $\simeq$ optimal transport
- Surprising appearance of Beckmann's problem, its first non-transportation application
- **Remaining question to unleash full power:** Guess a good candidate for an explicit solution? (for some particular ρ)

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Thank you!

Complementary slackness conditions

back to theorem

Optimal u^{opt}, functions φ_i^{opt}, measure π^{opt}, and vector field f^{opt} satisfy:

$$\int u^{\mathrm{opt}}(v)\,\mathrm{d}\psi(v) = \int u^{\mathrm{opt}}(v)\,\mathrm{d}\pi^{\mathrm{opt}}(v)$$

$$f_i^{\mathrm{opt}}(v) \in \partial\varphi_i^{\mathrm{opt}}\left(\frac{\partial u^{\mathrm{opt}}}{\partial v_i}(v)\right)$$

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  - μ on $[0,1]$ majorizes ν if and only if there is γ on $[0,1]^2$ with marginals μ on y and ν on x and such that $\int y\,\mathrm{d}\gamma(y \mid x) \geq x$ for γ-almost all x
  - solve for (u, γ)
- **Performance:** algorithm handles $(m=2\ q=100\ n=10)$ revenue curve

Algorithmic ideas

back to simulations

- **Automated mechanism design:** revenue maximization is an LP, let's feed it to an LP solver; Sandholm (2003)
- **Curse of dimensionality:** If each of n agents can have q different values for each of m items $\Rightarrow$ the dimension $\sim \left(q^n\right)^m$
  - intractable for $(m = 2,\ q = 100\ n = 2)$ or for $(m = 2\ q = 10\ n = 4)$
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- **How to avoid:**

$$R_{n,m}(\rho) = \max_{\substack{\text{convex monotone } u \\ u(0)=0,\ \partial_{v_i} u(v) \preceq z^{n-1}}} n \cdot \int_{\mathbb{R}_+^m} u(v)\,\mathrm{d}\psi(v)$$

  - Pros: dependence on n is killed; Cai et al.(2012), Alaei et al. (2019)
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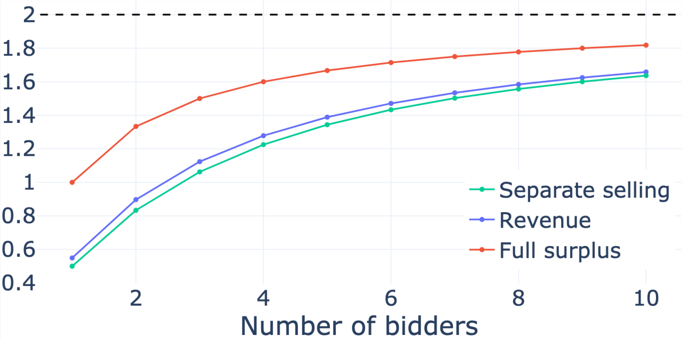
Revenue

back to algorithmic ideas

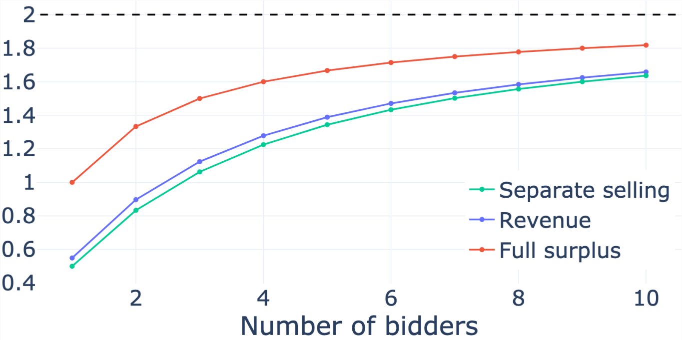


Revenue as a function of the number of bidders n for two items with i.i.d. values uniform on $[0, 1]$. Graphs from bottom to top: selling separately (light-green), selling optimally (blue), full surplus extraction (red), limit for $n \to \infty$ (the dashed line).

Remark: For $n = 2$, selling optimally improves upon selling separately by 5%

Revenue

back to algorithmic ideas



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