Constructive Blackwell Theorem

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Introduction

Blackwell's theorem: key tool in information economics

- Tells which belief distributions can be induced by a signal
- But not how...

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Our project: a simple economically relevant construction for signals in Blackwell's theorem

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Theorem (Blackwell, 1951; Strassen, 1965)

F can be induced by a signal

$$\iff$$

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Downward Uniform Signals

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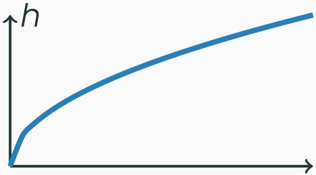
Given monotone $h : [0, 1] \to [0, 1]$,
sample $s \sim \text{Uniform}\big(\,[0,\ h(\theta)]\,\big)$



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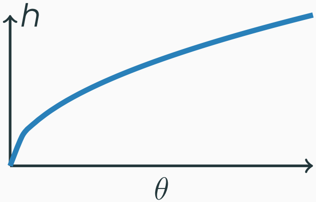

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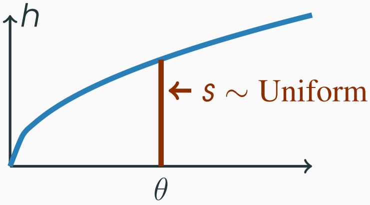

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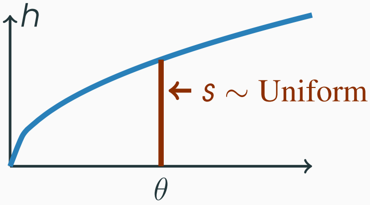

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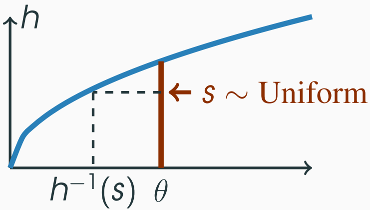

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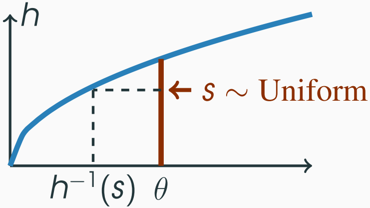

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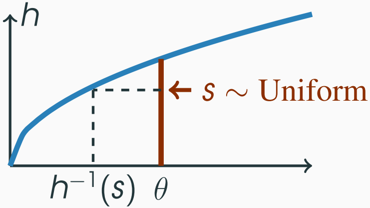

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- θ: total number of voters for "A", s: exit poll number
- θ true unreported income, s: amount discovered during an audit

Downward Uniform Signals are W.L.O.G.

Constructive Blackwell Theorem

$$F \text{ can be induced by a downward uniform signal}$$
$$\Longleftrightarrow$$
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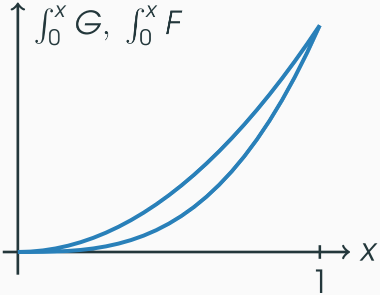

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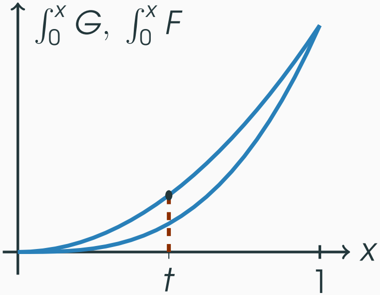

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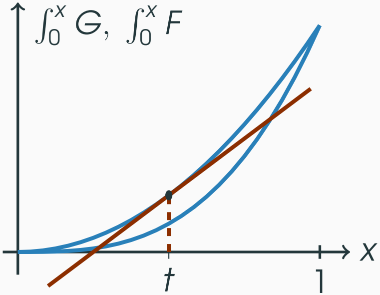

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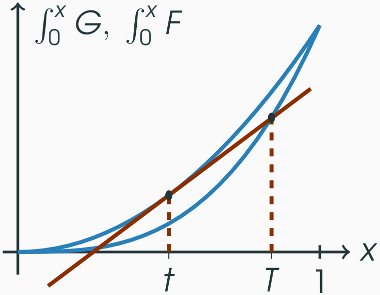

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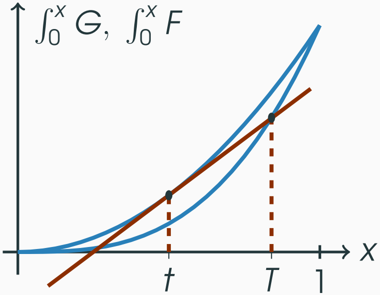

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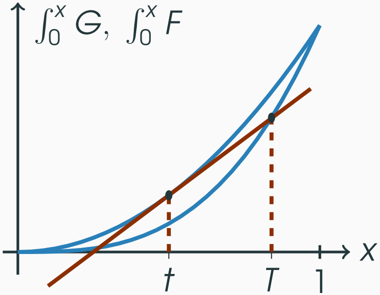

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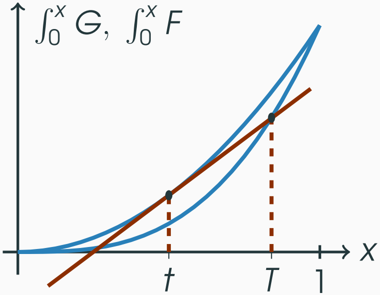

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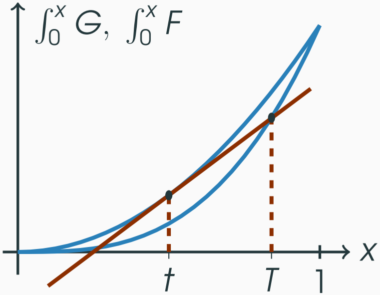

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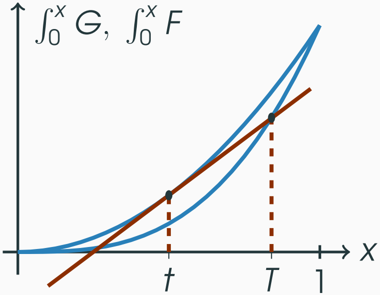

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- Receiver is partially informed with prior $\sim F$

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Information design:

- real-valued state and posterior-mean driven agents
  e.g., Dworczak and Martini (2019)
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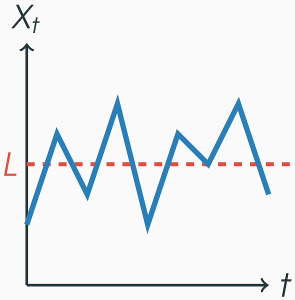

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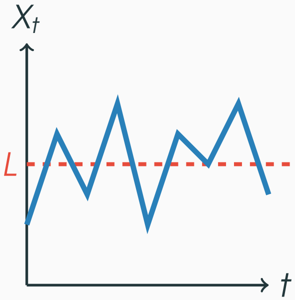

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Question:
How overoptimistic can a rational learner get for given F and G?

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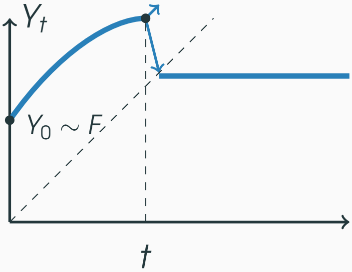

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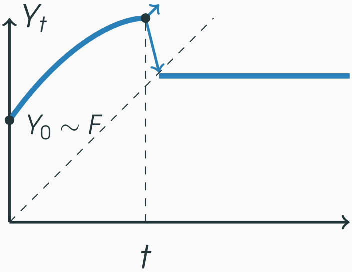

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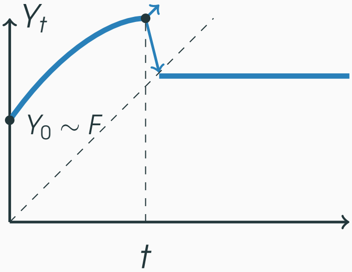

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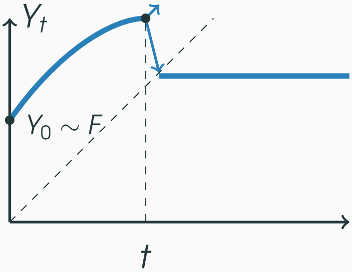

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- Y is the most optimistic martingale

Conclusion

- A simple construction for signals in Blackwell's theorem
- Applications in information design, mechanism design, and learning
- **Future:** More applications? Other explicit constructions?

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Thank you!

Related Math Literature

- Martingale optimal transport and left-curtain coupling: Beiglböck, Cox, and Huesmann (2017); Hobson and Norgilas (2022)
- Maximal maximum martingales: Dubins and Gilat (1978); Hobson (2012, 1998)

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