The geometry of consumer preference aggregation

Fedor Sandomirskiy (Caltech) **Philip Ushchev** (ECARES, U.libre de Bruxelles)

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- **Our paper is a middle ground:** a rich enough tractable setting

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Information economic tools for demand aggregation with application to robust welfare analysis and economic design

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- complexity of pseudo-market allocation mechanisms

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- **Economic applications of extreme points, Choquet theory, and convexification**
  - Kleiner et al. (2021), Arieli et al. (2020), Manelli & Vincent (2010), Aumann et al. (1995), Kamenica & Gentzkow (2011)

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- Demand as a function of prices $\mathbf{p}$

$$D(\mathbf{p}, b) = \underset{\mathbf{x} \in \mathbb{R}_+^n \,:\, \langle \mathbf{p}, \mathbf{x} \rangle \le b}{\arg\max} u(\mathbf{x})$$

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  - aggregate consumers' utility $\Leftrightarrow$ the Nash product maximization:

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- Challenging problem, no structural insights

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For a population $(\succsim_k, b_k)_{k=1,\ldots,m}$, the LEF of the aggregate is the average of individual LEFs

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- A simple result with numerous implications

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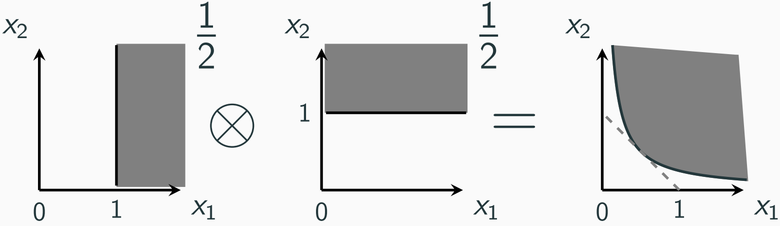
Corollary

The upper contour set of the aggregate consumer is the geometric mean of individual upper contour sets

$$\left\{u_{\mathrm{aggr}}(\mathbf{x}) \geq 1\right\} = \left\{u_1(\mathbf{x}) \geq 1\right\}^{\beta_1} \otimes \left\{u_2(\mathbf{x}) \geq 1\right\}^{\beta_2} \otimes \ldots \otimes \left\{u_m(\mathbf{x}) \geq 1\right\}^{\beta_k}.$$

Example: single-minded consumers



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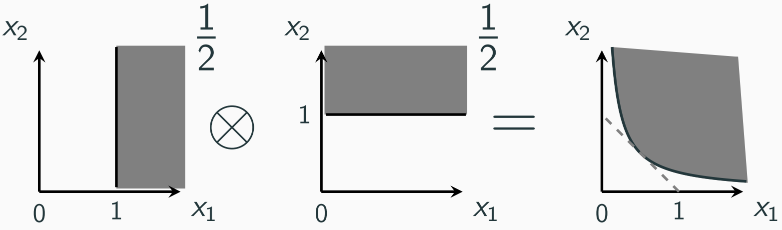

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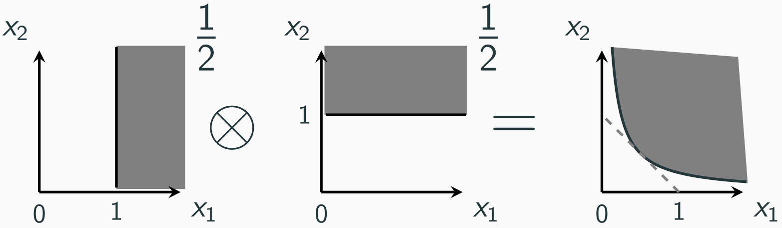

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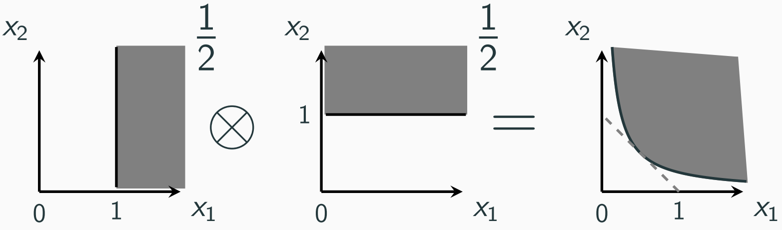

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- **Economics:** two single-minded consumers generate the same demand as one Cobb-Douglas consumer

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- Compatible with a range of welfare levels $[\underline{W}, \overline{W}]$

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- a population behaves like a Cobb-Douglas consumer

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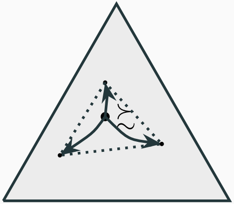

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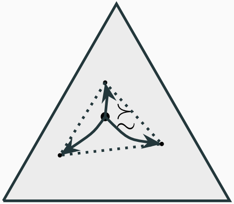

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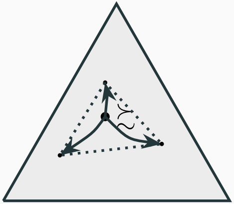

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Theorem 2

$$[\underline{W}, \overline{W}] = \sum_k b_k \cdot \Big[\mathrm{vex}[w](\succsim_{\mathrm{aggr}}), \;\; \mathrm{cav}[w](\succsim_{\mathrm{aggr}})\Big]$$

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The same aggregate behavior is compatible with a range $W \in [\underline{W}, \overline{W}]$

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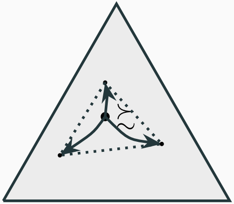

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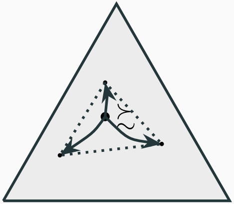

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Corollary of Theorem 1 (finitely-generated domains): If $\mathcal{D} = \{\succsim_1, \ldots, \succsim_m\}$, the completion = all preferences with E s.t.

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  - **Geometric meaning:** the domain of substitutes is a "simplex" and linear preferences are extreme points

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Our goal: find preference domains where CEEI is easy to compute

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- For large-scale applications, use bidding languages based on finitely-generated domains

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Thank you!

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Example: linear preferences for $n \geq 3$ goods

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