Improvable Equilibria

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Introduction

Correlated Equilibria (CE) (Aumann, 1974) generalize Nash equilibria to allow correlation

- capture pre-play communication, intermediation, and joint randomization
  Bárány (1992), Ben-Porath (1998), Gerardi (2004), Lehrer and Sorin (1997)
- result from natural learning dynamics
  Foster and Vohra (1997), Fudenberg and Levine (1999), Hart and Mas-Colell (2000)
- efficiently computable
  Papadimitriou and Roughgarden (2008)

Broad question: When is there potential value in correlation?

Question in context

CE $\simeq$ adding a recommendation system on top of the existing interaction

- $\implies$ What interactions can be improved by a recommendation system?

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CE $\simeq$ outcomes of arbitrary pre-play communication protocols

- $\implies$ What strategic interactions are susceptible to communication / collusion?

Normal-form game

$$\Gamma = \left(N,\ (A_i)_{i \in N},\ (u_i : A \to \mathbb{R})_{i \in N}\right)$$

- $N = \{1, \ldots, n\}$ is finite set of players
- A_i is a finite set of actions of player i
- $A = \prod_{i \in N} A_i$ is the set of action profiles
- $u_i : A \to \mathbb{R}$ is utility of player i

Definition (Aumann, 1974, 1987)

A distribution $\mu \in \Delta(A)$ is a CE if

$$\sum_{a_{-i} \in A_{-i}} \mu(a_i, a_{-i})\, u_i(a_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \mu(a_i, a_{-i})\, u_i(a_i', a_{-i})$$

for all $i \in N$ and all $a_i, a_i' \in A_i$

Interpretation: μ generated by a mediator and players best respond by adhering

Remark: Nash Equilibria (NE) are CE of the form $\mu = \mu_1 \times \ldots \times \mu_n$

Formalizing the Question

- The set of correlated equilibria is a convex polytope
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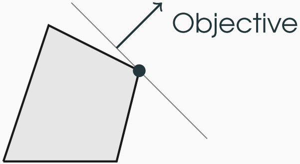
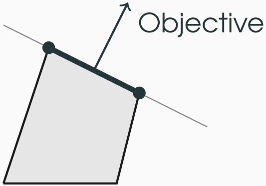
Our Question: When is a Nash equilibrium extreme?

Linear objectives and extreme points

Maximization of a linear objective over a polytope:





- If the optimum is unique, it is an extreme point
- In knife-edge cases, the whole face can be optimal

Definition

Objectives with unique optima are **non-degenerate**

- Tiny perturbations can make degenerate non-degenerate

Improvability of non-extreme equilibria

Observation

NE is non-extreme $\iff$ any non-degenerate linear objective can be improved

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- Linear in probabilities, not in actions $\Rightarrow$ a broad class of objectives
  - e.g., utilitarian welfare, revenue, maximizing/minimizing the probability of a particular action profile

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- Linear in probabilities, not in actions $\Rightarrow$ a broad class of objectives
  - e.g., utilitarian welfare, revenue, maximizing/minimizing the probability of a particular action profile
- The conclusion extends to convex objectives (**Bauer's maximum principle**)
  - including quasi-convex (Ball, 2023)

Improvability of non-extreme equilibria 2

- Non-extreme equilibria are improvable **no matter** the objective
- A conservative notion, agnostic to the designer's objective
- Usually, assess outcomes for a **given** objective

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Main Insight

Despite restrictiveness of improvability notion, **many** equilibria are improvable, i.e., **non-extreme**

Literature

- **Value of correlation in** 2**-player games**: Cripps (1995), Evangelista and Raghavan (1996), Canovas, Hansen, and Jaumard (1999), Nau, Canovas, and Hansen (2004), Peeters and Potters (1999), Calvó-Armengol (2006), Ashlagi, Monderer, and Tennenholtz (2008)
- **Extreme-point approach in info & mech. design:** Manelli and Vincent (2007), Kleiner, Moldovanu, and Strack (2021), Arieli, Babichenko, Smorodinsky, and Yamashita (2023), Yang and Zentefis (2024), Nikzad (2022), Kleiner, Moldovanu, Strack, and Whitmeyer (2024), Lahr and Niemeyer (2024)

Rough Outline

- **Conditions for extremality:**
  in the space of action distributions and payoff space

- **Particular classes of games:**
  symmetric, having unique CE

- **Extensions:**
  Bayesian CE and Coarse CE

Conditions for Extremality

Extremality of Nash Equilibria

Theorem

In a generic n-player game, a mixed NE is extreme $\iff$ ≤ 2 players randomize

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  $\Rightarrow$ 2-player games not representative

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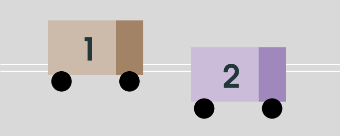
- In a generic game, any NE is regular (Harsanyi, 1973)
- Hence, Theorem $1' \Rightarrow$ Theorem 1

Example: 2 Players vs 3 Players

Example: 2-Player Games

A version of the Game of Chicken by Aumann (1974):



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Risky	6, 6	10, 7
Safe	7, 10	9, 9

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  Solves linear equation: $6p + 10(1-p) = 7p + 9(1-p) \quad \Longrightarrow \quad p = 1/2$

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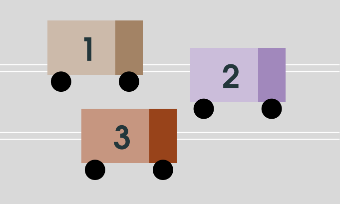
- Aumann (1974): CE can increase utilitarian welfare by shifting weight from (6,6)
- However, the mixed NE is an **extreme point**
- Indeed, it is the optimum for a non-degenerate objective

$$\text{weight of (Risky, Risky) \& (Safe, Safe)} \to \max$$

Example: 3-Player Games



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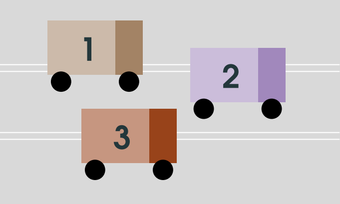

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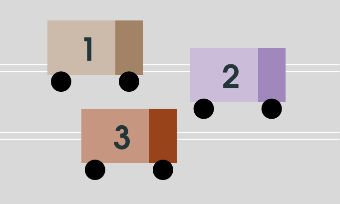

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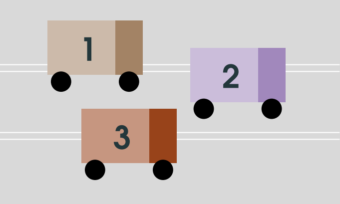

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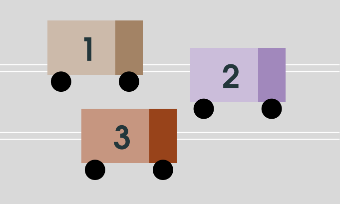

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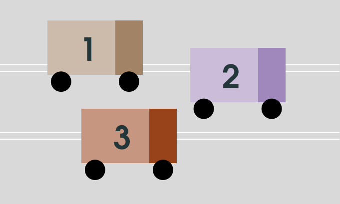

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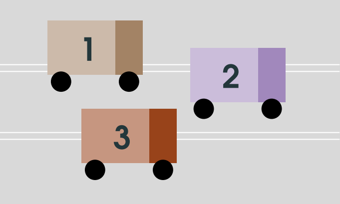

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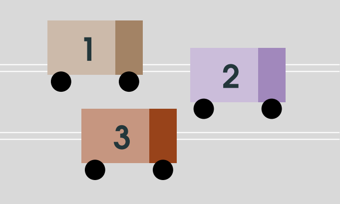

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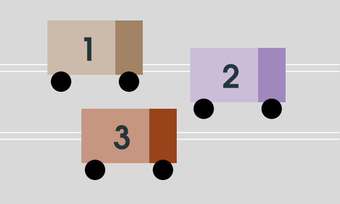

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More than 2 players mixing makes a difference...

General Proof Intuition

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High-level idea: When many players randomize, there are too many ways to correlate their actions $\implies$ one must be beneficial

Focus on a particular example to illustrate

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- Game with n players, each with 2 actions

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- If μ is a CE, must satisfy incentive constraints

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- $\Rightarrow$ support of an extreme CE μ is bounded by $2n+1$

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- The main difficulty is handling very asymmetric equilibria ▸ details

Extreme Points in Payoff Space

Payoff-extreme Equilibria

- The set of CE $\subset \Delta(A)$ subset of a space of dimension $|A_1| \cdot \ldots \cdot |A_n|$
- Equilibria are often represented via payoffs in $\mathbb{R}^n$

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Question: What can we say about payoff-extreme equilibria?

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- Projection of an extreme point **need not** be an extreme point of a projection
- $\Rightarrow$ pure NE and NE with 2 mixers **need not** be payoff-extreme
  - e.g, the mixed NE in the Game of Chicken

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Proposition

In a generic game, utilitarian welfare is non-degenerate

Applications to Particular Classes of Games

Costly Voting

Costly voting model of Palfrey and Rosenthal (1983):

- Two finite groups of voters: D and R, $|R| > |D|$
- Voters in D get utility of 1 if d-candidate wins and 0 otherwise
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Palfrey and Rosenthal (1983): For intermediate values of c, all equilibria involve at least one group all mixing

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- Two finite groups of voters: D and R, $|R| > |D|$
- Voters in D get utility of 1 if d-candidate wins and 0 otherwise
- Voters in R get utility of 1 if r-candidate wins and 0 otherwise
- Majority voting (among those who participate), ties broken randomly
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Other Applications: games where players want to mismatch actions of others

- e.g., network games (with substitutes), congestion games, all-pay auctions, Boston matching mechanism

Symmetric Games

- In many applications, strategic interactions are symmetric
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Take-away: caution when focusing on symmetric mixed equilibria in symmetric games

What Extreme CE Look Like

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Question: What is the structure of extreme CE?

- For general games, we only know that extreme CE have small support
- For symmetric games and symmetric CE, we can say more

Symmetric CE and Exchangability

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Theorem (de Finetti)

Any infinite exchangeable sequence $a_1, a_2, a_3 \ldots$ is a mixture of i.i.d. distributions

Extreme Symmetric CE with Many Players

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- A version of Proposition 2 holds: sampling without replacement instead of i.i.d.

details

Games with Unique Correlated Equilibrium

skip

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- Unique CE $\implies$ CE=NE
- Such NE is robust to communication/collusion, incomplete information about payoffs, and can be computed without fixed points (Einy et al., 2022)
- **Examples:** games with dominant strategies, some congestion games, Cournot competition

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Corollary

If a game has a unique CE, then it is either:

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- No genericity assumption since games with a unique CE form an open set (Viossat, 2010)

Bayesian Correlated Equilibria

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General games with incomplete information (Bergemann and Morris, 2019):

- Common payoff uncertainty: a finite set of states Θ
- Private information: finite sets of types T_i
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Definition

A distribution $\psi \in \Delta(A \times \Theta \times T)$ is a BCE if

- its marginal on $\Theta \times T$ coincides with π
- no agent can gain by deviating from a recommended action a_i to another action a_i', given her private type t_i

$$\sum_{a_{-i}\in A_{-i},\ \theta\in\Theta,\ t_{-i}\in T_{-i}} \psi(a,\theta,t)\, u_i(a,\theta,t) \geq \sum_{a_{-i}\in A_{-i},\ \theta\in\Theta,\ t_{-i}\in T_{-i}} \psi(a,\theta,t)\, u_i(a_i', a_{-i},\theta,t)$$

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A Bayesian Nash equilibrium (BNE) is a BCE where $\theta, a_1, \ldots, a_n$ are independent, conditional on $t_1, \ldots, t_n$

Bayesian Correlated Equilibria: Extremality

Theorem

For a generic game with either:

- non-trivial common payoff uncertainty ($|\Theta| \geq 2$), or
- non-trivial private information ($|T_i| \geq 2$ for at least 3 agents),

a BNE is an extreme point of BCE $\Leftrightarrow$ it is pure

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Intuition: Randomness in the state and private types provides more degrees of freedom for correlation to be beneficial

Coarse Correlated Equilibria

skip

Coarse Correlated Equilibria

Consider situations where agents commit **ex-ante** to a correlating device, before receiving recommendations. For example,

- firms entering binding collusive agreements (McAfee and McMillan, 1992)
- users opting in to algorithmic recommendation systems

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Definition (Hannan, 1957; Moulin and Vial, 1978)

A distribution $\mu \in \Delta(A)$ is a coarse correlated equilibrium (CCE) if, for all $i \in N$,

$$\sum_{a \in A} \mu(a) u_i(a) \geq \max_{a_i' \in A_i} \sum_{a \in A} u_i(a_i', a_{-i}) \mu(a),$$

i.e., the expected payoff from following the correlated strategy is at least as high as the best deterministic deviation

- CCE $\supseteq$ CE $\supseteq$ NE

Coarse Correlated Equilibria: Extremality

Proposition

In a generic game, a NE is an extreme point of the set of CCE $\Leftrightarrow$ it is pure

Coarse Correlated Equilibria: Extremality

Proposition

In a generic game, a NE is an extreme point of the set of CCE $\Leftrightarrow$ it is pure

- The tension between randomness and optimality is even stronger for CCE than for CE: any randomness allows for improvement
- Consistent with the prevalence of examples where CCE improves over NE, even in two-player games (Moulin and Vial, 1978; Gérard-Varet and Moulin, 1978; Moulin, Ray, and Gupta, 2014; Dokka, Moulin, Ray, and SenGupta, 2023)

Conclusions

Several papers effectively show extremality of NE in specific contexts:

- Tullock contests, Cournot and Bertrand, patent races, location games (Einy, Haimanko, and Lagziel, 2022)
- First-price auctions (Feldman, Lucier, and Nisan, 2016)
- Convex potential games (Neyman, 1997; Ui, 2008)
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Our paper:

- a general tension between equilibrium randomness and extremality
- detail-free criterion for extremality in various settings

Main takeaway: inherent suboptimality of equilibria with a lot of mixing

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Thank you!

Key Lemmas

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Support Size of Extreme Correlated Equilibria (follows from Winkler (1988))

If μ is an extreme correlated equilibrium, then

$$\mathrm{supp}(\mu) \leq 1 + \sum_{i \in N} |S_i| \cdot (|S_i| - 1)$$

Support Size of Regular Nash Equilibria (McKelvey and McLennan, 1997)

For a regular Nash equilibrium, $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$:

$$\mathrm{supp}(\nu_i) - 1 \leq \sum_{j \neq i} (\mathrm{supp}(\nu_j) - 1), \quad \text{for any player } i$$

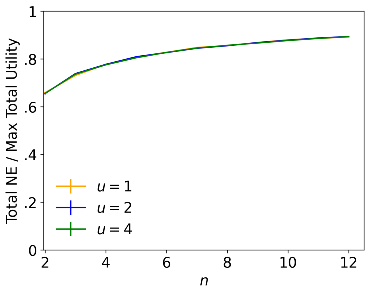
Regularity of Generic games (Harsanyi, 1973)

In a generic game, any Nash equilibrium is regular

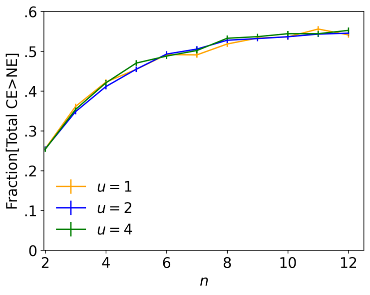
back

Simulations

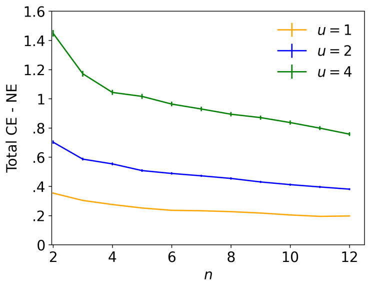
Simulations

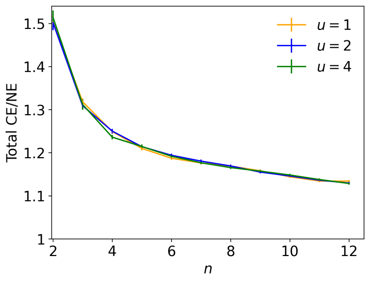


(a) NE/2u



(b) Fraction of CE > NE*





Extreme Symmetric CE with Any Number of Players

Consider n players with m actions each

Proposition

Any extreme symmetric CE can be obtained as follows:

- there are M urns, each with n balls labeled by actions

$$1 \leq M \leq m(m-1)+1$$

- an urn is selected at random according to $p \in \Delta_M$, secretly from players
- players draw balls sequentially without replacement
- i's action = her ball's label, no incentive to deviate

Remark: If n is large, sampling without replacement can be approximated by i.i.d.

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