

# Narrow Bracketing and Risk in Games\*

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## Abstract

We study finite normal-form games under a narrow bracketing assumption: when players play several games simultaneously, they consider each one separately. We show that under mild additional assumptions, players must play either Nash equilibria, logit quantal response equilibria, or their generalizations, which capture players with various risk attitudes.

**Keywords:** Narrow Bracketing, Strategic Risk, Solution Concepts, Nash Equilibrium, Logit Quantal Response Equilibrium, Statistic Response Equilibrium.

## 1 Introduction

Consider a player who plays both poker and chess at a game night. Since these are separate games, a natural modeling choice for describing her decision-making is to assume that she chooses her strategies independently in the two games. In this paper, we explore the general assumption that when players face unrelated strategic choices, these choices are made independently.

We refer to this assumption as *narrow bracketing* and study its implications for the choice of solution concepts in games. Despite its self-evident nature, narrow bracketing has

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far-reaching implications. Using narrow bracketing and an additional axiom that implies risk neutrality, we characterize Nash equilibrium and logit quantal response equilibrium, providing a novel unifying perspective on these established solution concepts. Relaxing risk neutrality, we use narrow bracketing to suggest new solution concepts that incorporate non-trivial risk attitudes in equilibrium behavior, and yet retain much of the conceptual appeal of Nash equilibria and logit quantal response equilibria. Our paper thus belongs to the literature of axiomatization in game theory (see, e.g., a recent paper by [Brandl and Brandt, 2024](#)) and the literature on risk attitudes in game (e.g., [Goeree, Holt, and Palfrey, 2003](#); [Yekkehkhany, Murray, and Nagi, 2020](#)).

As the game night example hints, our narrow bracketing axiom is a relatively mild assumption: we only suppose that people treat games separately when they are unrelated. Rather than capturing a behavioral bias, narrow bracketing is compatible with rationality since choosing best responses separately in two unrelated games trivially gives a best response when playing the two games simultaneously.<sup>1</sup>

Narrow bracketing is implicitly used by experimentalists when they take models to lab-generated data without considering extraneous information about subjects' lives outside of the lab environment. If narrow bracketing holds, subjects' behavior inside the lab need not be affected by external considerations unknown to the experimentalist, such as their wealth or strategic interactions outside the lab. Narrow bracketing is thus a critical assumption in experimental economics, asserting that economic interactions can be simulated in isolation.

Formally, we take an axiomatic approach to characterizing solution concepts for finite normal form games. A solution concept assigns to each game a set of mixed strategy profiles. We interpret these strategy profiles as potential predictions for how players might behave in the game, referring to them as the game's solutions. To formalize our main axiom, narrow bracketing, we need to clarify what it means for a game to be composed of two unrelated games. Given two games, which we call the *factor games*, involving the same players, we define the *product game* by requiring the players to choose one action in each factor game. To capture the idea that the factor games are unrelated, we think of payoffs as monetary, and define the players' payoffs in the product game to be the sum of their payoffs in the factor games.

We say that a solution concept satisfies narrow bracketing if, given solutions for the factor games, one of the solutions for the product game is for players to choose their actions independently in each factor game, according to its respective solution. More formally, if mixed strategy profile  $(p_i)_i$  is a solution for the first factor game, and  $(q_i)_i$  is a solution for

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<sup>1</sup>This is in contrast with the well-documented (but unrelated) phenomenon that people often treat even related decisions separately, which leads to suboptimal outcomes, and so is incompatible with rationality; e.g., see a survey by [Read, Loewenstein, Rabin, Keren, and Laibson \(2000\)](#).

the second, then one of the solutions for the product of these games is for each player  $i$  to choose a pair of actions independently from  $p_i$  and  $q_i$ . Importantly, we do not require every solution of the product game to be of this form. Rather, thinking of solutions as predictions, we require that combining predictions for the factor game provides one of the valid predictions for the product game. Accordingly, narrow bracketing plays the role of a consistency requirement on predictions across games.

We augment narrow bracketing with various rationality assumptions capturing the idea that players have rational expectations about the strategies of the other players. Moreover, we assume that they prefer higher payoffs to lower payoffs, either in expectation or in distribution, and obtain two corresponding groups of results.

In the first part of the paper, we consider solution concepts that satisfy *expectation-monotonicity*. It requires that choice probabilities are monotone with respect to expected payoffs. Specifically, a player chooses an action more frequently than another action if the expected payoff of the former action is higher, given the strategies of others. This axiom implicitly includes the assumptions that players have rational expectations and respond to expected payoffs. This embodies risk-neutral behavior, since payoffs are monetary.

Our first main result is that the only solution concepts satisfying narrow bracketing and expectation-monotonicity are Nash equilibrium, logit quantal response equilibrium (LQRE), and some of their refinements (Theorem 1). In this way, we provide a common rationality foundation for LQRE and Nash equilibria, highlighting a deep connection between the two concepts.

This characterization offers a new perspective on LQRE, revealing its fundamentally rational nature despite its traditional justification through bounded rationality and errors. Importantly, the usual error-based motivation leads to the broader family of quantal response equilibria (QRE) criticized for being overly permissive (Haile, Hortaçsu, and Kosenok, 2008). Our axioms single out the one-parametric logit subfamily, offering a rationality justification for the use of LQRE and elevating it beyond its typical role of a computationally convenient choice.

In the second part of the paper, we replace expectation-monotonicity with *distribution-monotonicity*, which only requires choice probabilities to be monotone with respect to first-order stochastic dominance of the different actions' payoff distributions. This axiom retains the rational expectations assumption, but relaxes the expected utility and risk neutrality aspects of expectation-monotonicity.

When distribution-monotonicity is combined with narrow bracketing and an additional assumption of strategic invariance, it implies expectation-monotonicity (Theorem 2). Strategic invariance means that players behave identically in games where a player's payoffs differ by a quantity that does not depend on their action. Such games are called

strategically equivalent, and our result uncovers a connection between the seemingly innocent assumption of strategic equivalence and a relatively strong assumption of risk neutrality. It also implies that under narrow bracketing, distribution-monotonicity and strategic invariance, players play either Nash or LQRE.

Dropping strategic invariance, we study solution concepts that satisfy narrow bracketing and distribution-monotonicity. These axioms characterize what we call *statistic response equilibria*, a novel generalization of Nash and LQRE that accommodates various risk attitudes (Theorem 3). Statistic response equilibria allow players to respond not only to the expectation but to any monotone additive statistic (Mu, Pomatto, Strack, and Tamuz, 2021), which can capture risk aversion, risk seeking, and mixed risk attitudes.

The class of statistic response equilibria is broad, and we characterize two parametric subclasses that maintain flexibility in risk attitudes. First, by strengthening distribution-monotonicity to an axiom that captures rational expectations and expected utilities—but not risk neutrality—we obtain a one-parameter family where players respond to the CARA certainty equivalents of the actions (Theorem 4). Second, using a scale-invariance axiom, we characterize a three-parameter family of statistic response equilibria, which we anticipate to be useful in estimating empirical models of games (Theorem 5). In these equilibria, players logit best respond to a convex combination of the minimum, maximum, and expectation of the payoff distributions.

## 1.1 Related literature

We contribute to a large body of literature on the axiomatic approach in economic theory. This methodology has been used extensively to characterize solutions with desirable properties for cooperative games, bargaining, and mechanism design problems; see surveys by Moulin (1995), Roth (2012), and Thomson (2023). In non-cooperative game theory, the axiomatic approach has been applied primarily toward choosing equilibrium refinements (Harsanyi and Selten, 1988; Norde, Potters, Reijnierse, and Vermeulen, 1996; Govindan and Wilson, 2009).

Our paper focuses on characterizing solution concepts in non-cooperative games. The most closely related work is by Brandl and Brandt (2024), who provide an axiomatic characterization of mixed Nash equilibria in general normal form games. The two approaches highlight different aspects of strategic behavior. Our framework emphasizes how players frame multiple unrelated decisions and evaluate risk. By contrast, Brandl and Brandt’s axioms focus on how players’ behavior is affected by negligible changes in the strategic environment. We discuss in depth the connection between the two approaches in Appendix H. Other axiomatic results include Brandl and Brandt (2019), who characterize maximin strategies in zero-sum games, and Voorneveld (2019), where pure Nash equilibria

are derived using a strategic invariance axiom similar to the one in our Theorem 2.

The key axiom in our approach is narrow bracketing. This term is often used in a broader sense than in our paper and includes neglect of interactions between related choices. It has been extensively studied in the context of individual decisions, where much of the literature treats narrow bracketing as a behavioral bias (see, e.g., [Read, Loewenstein, Rabin, Keren, and Laibson, 2000](#); [Barberis, Huang, and Thaler, 2006](#)). We are not aware of previous studies applying narrow bracketing to games.

Our paper belongs to a recent literature offering a rational perspective on narrow bracketing. [Kőszegi and Matějka \(2020\)](#) justify narrow bracketing as rational behavior in a model with costly attention, and [Camara \(2022\)](#) offers a computational complexity justification. [Sandomirskiy and Tamuz \(2023\)](#) use a version of our narrow bracketing assumption as an axiom for single-agent decisions, in order to characterize multinomial logit. We utilize some of their techniques, but ultimately the multi-agent setting is significantly different, particularly because of the endogeneity of randomness. The dissimilarity of games and decision problems is further explored in Appendix I, where we show that the classical independence axiom used by [Luce \(1959\)](#) to characterize logit is inapplicable to LQRE.

Our first set of results provides a rationality-based justification for the logit subfamily of quantal response equilibria (QRE). Introduced by [McKelvey and Palfrey \(1995\)](#), QRE has been empirically successful at explaining deviations from Nash equilibrium predictions across a wide range of experiments ([Goeree, Holt, and Palfrey, 2016, 2020](#)). The axiomatic approach has been applied to define non-parametric subclasses of QRE by imposing axioms directly on the quantal response functions ([Goeree, Holt, and Palfrey, 2005](#); [Friedman and Mauersberger, 2022](#)). By contrast, we do not take QRE as a starting point, and instead axiomatize solution concepts rather than response functions. Furthermore, our results differ in that they pin down the one-parameter class of logit QRE in particular, providing a novel justification for a quantal response function that has been widely used to analyze empirical data (see, e.g., [Goeree, Holt, and Palfrey, 2016](#); [Wright and Leyton-Brown, 2017](#); [Goeree, Holt, and Palfrey, 2020](#)).

Our second set of results gives rise to a novel solution concept—statistic response equilibria—where players respond to a monotone additive statistic of payoffs, a class of statistics characterized by [Mu, Pomatto, Strack, and Tamuz \(2021\)](#). These equilibria capture various risk attitudes toward the uncertainty induced by other players’ mixed strategies. Importantly, these attitudes emerge from our axioms rather than being assumed a priori. This contrasts with the literature on equilibrium concepts that incorporate specified risk attitudes by transforming game payoffs according to some utility function reflecting such an attitude ([Goeree, Holt, and Palfrey, 2003](#); [Yekkehkhany, Murray, and Nagi, 2020](#)). In fact, the risk attitudes resulting from our characterization cannot be

replicated by any such transformation of payoffs. In this way, we contribute to the literature on games with non-expected utility preferences (Shalev, 2000; Metzger and Rieger, 2019). An equilibrium concept related to statistic response equilibria has appeared in the computer science literature, where it has been shown to be efficiently computable under certain conditions (Mazumdar, Panaganti, and Shi, 2024).

## 2 Solution Concepts and Narrow Bracketing

We consider finite normal form games played between a fixed set of players  $\{1, \dots, N\}$ . Such a game  $G = (A, u)$  is given by its finite set of action profiles  $A = \prod_i A_i$  and its payoff function  $u: A \rightarrow \mathbb{R}^N$ , where  $A_i$  is the set of actions of player  $i$  and  $u_i(a)$  is the payoff to player  $i$  when an action profile  $a$  is played.

A mixed strategy  $p_i \in \Delta A_i$  of player  $i$  is a probability distribution over  $A_i$ . A mixed strategy profile  $p = (p_1, \dots, p_n)$  induces a product distribution over  $A$ , also denoted by  $p$ , with its marginal on  $A_{-i} = \prod_{j \neq i} A_j$  denoted  $p_{-i}$ . Given a mixed strategy profile  $p$  and an action  $a_i$  for player  $i$ , we denote by

$$\mathbb{E}[u_i(a_i, p_{-i})] = \sum_{a_{-i}} p_{-i}(a_{-i}) u_i(a_i, a_{-i})$$

the expected payoff of player  $i$  from taking action  $a_i$ , while others play according to  $p_{-i}$ .

A *solution concept*  $S$  assigns to each game  $G$  a nonempty set  $S(G) \subset \prod_i \Delta A_i$  of mixed strategy profiles, referred to as *solutions*.<sup>2</sup> One can think of these solutions as potential predictions for player behavior in  $G$ . We require that every game is assigned at least one solution. For instance, a map  $S(G) = \prod_i \Delta A_i$  assigning to each  $G$  the set of all mixed strategy profiles is a valid—albeit uninformative—solution concept. To compare solution concepts by how specific they are, we say that a solution concept  $S$  is a *refinement* of  $S'$  if  $S(G) \subseteq S'(G)$  for all games  $G$ .

Our main axiom, narrow bracketing, asserts that players who are engaged in multiple unrelated games may consider each game independently. Formally, for games  $G = (A, u)$  and  $H = (B, v)$ , we define the product game  $G \otimes H = (C, w)$  by

$$C_i = A_i \times B_i \quad \text{and} \quad w_i((a_1, b_1), \dots, (a_n, b_n)) = u_i(a) + v_i(b).$$

In  $G \otimes H$ , each player  $i$  chooses an action  $a_i$  from  $A_i$  and an action  $b_i$  from  $B_i$ —effectively playing both games simultaneously—and earns the sum of payoffs from the two games. We

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<sup>2</sup>There is a technical nuance that can be safely ignored without missing the gist of the paper: the collection of all finite sets is not a set, and neither is the collection of all finite games. Hence, for a solution concept to be a well-defined correspondence, we assume that all actions available to any player in any game belong to a universal, non-empty set of actions  $\mathcal{A}$ . We also suppose that  $\mathcal{A}$  is closed under pairing, so that  $\mathcal{A} \times \mathcal{A} \subset \mathcal{A}$ .

call  $G$  and  $H$  *factor games* of the product game  $G \otimes H$ . The additive payoff structure of the product game captures the unrelated nature of the factor games, i.e., the action in one does not affect the payoff in the other. To make sense of the summation of payoffs across different games, we need to think of payoffs as being quoted in the same units across all games. For simplicity, we treat payoffs as monetary.

Given mixed strategy profiles  $p$  and  $q$  for games  $G$  and  $H$ , we define the mixed strategy profile  $p \times q$  for the game  $G \otimes H$  by

$$[p \times q]_i(a_i, b_i) = p_i(a_i) \cdot q_i(b_i).$$

So, when players are playing  $p \times q$  in  $G \otimes H$ , they independently choose strategies in  $G$  from  $p$  and in  $H$  from  $q$ .

**Definition 1.** A solution concept  $S$  satisfies narrow bracketing if, for all games  $G$  and  $H$ ,

$$p \in S(G) \quad \text{and} \quad q \in S(H) \quad \text{implies} \quad p \times q \in S(G \otimes H).$$

When a solution concept satisfies narrow bracketing, the solutions of  $G$  and  $H$  can be combined into a solution of the product game  $G \otimes H$  by having players choose their actions independently in the two factor games. Viewing solutions as predictions, our narrow bracketing assumption requires that independently following the predictions in the factor games constitutes a valid prediction for the product game. Importantly, this assumption does not rule out the existence of other predictions for the product game.

Narrow bracketing is satisfied by the Nash correspondence  $\text{Nash}(G)$ , which assigns to a game  $G$  the set of all its mixed Nash equilibria. Note that not all mixed Nash equilibria in product games are products of equilibria in the factor games, but these products do appear in the solution of the product game, as required by narrow bracketing.

Many refinements of Nash also satisfy narrow bracketing. These include maximal-entropy Nash equilibria, trembling hand perfect equilibria, and welfare-maximizing equilibria. However, not all refinements are guaranteed to satisfy narrow bracketing, e.g., minimal-entropy Nash equilibria—a natural extension of pure Nash equilibria to a non-empty correspondence—violate narrow bracketing as entropy can be reduced by correlating unrelated choices. Likewise, a refinement obtained by eliminating Nash equilibria that are Pareto dominated by other Nash equilibria does not satisfy narrow bracketing.<sup>3</sup>

Given  $\lambda \geq 0$  and a game  $G = (A, u)$ , the logit quantal response equilibrium correspon-

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<sup>3</sup>A standard intuition applies: Pareto optimal allocation in sub-markets may not give rise to a Pareto optimal allocation in the market itself due to beneficial trades across sub-markets.

dence is given by<sup>4</sup>

$$\text{LQRE}_\lambda(G) = \left\{ p \in \Delta(A) \mid p_i(a_i) \propto \exp(\lambda \mathbb{E}[u_i(a_i, p_{-i})]) \right\}.$$

This correspondence also satisfies narrow bracketing. In fact, this correspondence satisfies a stronger property: every solution of a product game is a product of solutions of the factor games.

Other solution concepts that satisfy narrow bracketing include the rationalizable mixed strategies, welfare maximizing (or minimizing) mixed strategy profiles, level  $k$  models in which level 0 players choose uniformly, cognitive hierarchy models with the same base choices, as well as the uniform distribution ( $\text{LQRE}_0$ ). Probit QRE does not satisfy narrow bracketing, and more generally, neither does any QRE that is not logit. The set of all Pareto optimal mixed strategy profiles also violates narrow bracketing.

We also consider anonymity, a simplifying assumption which does not affect the essence of our results. Anonymity requires that permuting players' names results in the corresponding permutation of the solution concept's predictions—that is, the solution concept treats all the players in the same way. Given a game  $G = (A, u)$  and a permutation of players  $\pi: N \rightarrow N$ , define the permuted game  $G_\pi = (B, v)$  by  $B_i = A_{\pi(i)}$ , and  $v_i(a_{\pi(1)}, \dots, a_{\pi(n)}) = u_{\pi(i)}(a_1, \dots, a_n)$  for all  $i \in N$ , and  $a \in A$ . Each mixed strategy profile  $p$  in  $G$  yields a permuted profile  $p_\pi$  in  $G_\pi$  by  $(p_\pi)_i = p_{\pi(i)}$ .

**Definition 2.** *A solution concept  $S$  satisfies anonymity if for any permutation  $\pi$  of players, game  $G$ , and its solution  $p \in S(G)$ , we have  $p_\pi \in S(G_\pi)$ .*

Note that anonymity does not require that every solution to a symmetric game is symmetric, but instead that set of solutions is symmetric. For example, if  $G$  is symmetric then anonymity means that if  $(p_1, p_2) \in S(G)$ , then so is  $(p_2, p_1)$ .

Beyond narrow bracketing and anonymity, we will also need rationality assumptions reflecting that players tend to choose actions with higher payoffs more often. These assumptions are formalized as monotonicity axioms discussed in the corresponding sections below.

### 3 Expectation-Monotonicity and Logit Quantal Response Equilibrium

In this section, we augment narrow bracketing with the following monotonicity axiom.

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<sup>4</sup>We use  $p_i(a_i) \propto \exp(\lambda \mathbb{E}[u_i(a_i, p_{-i})])$  to indicate equality up to normalization of the probabilities, i.e.,

$$p_i(a_i) = \frac{\exp(\lambda \mathbb{E}[u_i(a_i, p_{-i})])}{\sum_{b \in A_i} \exp(\lambda \mathbb{E}[u_i(b, p_{-i})])}.$$



**Definition 3.** A solution concept  $S$  satisfies expectation-monotonicity if

$$\mathbb{E}[u_i(a_i, p_{-i})] > \mathbb{E}[u_i(b_i, p_{-i})] \quad \text{implies} \quad p_i(a_i) \geq p_i(b_i)$$

for every game  $G = (A, u)$ , solution  $p \in S(G)$ , player  $i$ , and actions  $a_i, b_i \in A_i$ .

This axiom includes a number of conceptual assumptions. First, it introduces a notion of rational expectations into a solution concept, in the sense that players anticipate the others' actions sufficiently well to rank expected payoffs. Second, it implies that players are primarily concerned with the expectation of their payoffs, and hence is, in some sense, an expected utility axiom. Since we think of payoffs as monetary, this axiom furthermore implies that players are risk neutral. Finally, it captures a notion of rationality, because players prefer actions with higher expected payoffs.

Expectation-monotonicity is satisfied by Nash,  $\text{LQRE}_\lambda$ , probit QRE, and, more generally, any regular QRE (Goeree, Holt, and Palfrey, 2005), and M-equilibrium (Goeree and Louis, 2021). It is closed under refinements. The level  $k$  and cognitive hierarchy model solution concept do not satisfy this axiom, and neither does the set of rationalizable mixed strategies.

The three properties—narrow bracketing, anonymity, and expectation-monotonicity—do not seem overly restrictive, as each of them is satisfied by many well-known solution concepts. Among the examples mentioned above, Nash and  $\text{LQRE}_\lambda$  satisfy all three, as do trembling hand Nash and maximum entropy Nash. The following theorem shows that any solution concept satisfying all of the three properties must include only Nash equilibria or only logit quantal response equilibria.

**Theorem 1.** If  $S$  satisfies expectation-monotonicity, narrow bracketing, and anonymity, then  $S$  is either a refinement of Nash or of  $\text{LQRE}_\lambda$  for some  $\lambda \geq 0$ .

Theorem 1 gives yet another piece of evidence for the importance of Nash equilibria. It also provides a novel justification for the particular logit form of QRE, beyond its tractability. This is a simple and important one-parameter family that has been useful for predicting outcomes of games in the lab (Goeree, Holt, and Palfrey, 2016; Wright and Leyton-Brown, 2017; Goeree, Holt, and Palfrey, 2020).

Furthermore, this result establishes a connection between Nash and  $\text{LQRE}_\lambda$ . Notice that Nash is not merely a limiting case of  $\text{LQRE}_\lambda$  as  $\lambda \rightarrow \infty$ ; McKelvey and Palfrey (1995) show that limit points of  $\text{LQRE}_\lambda$  are Nash equilibria, but not every Nash equilibrium can be obtained as a limit point of  $\text{LQRE}_\lambda$ . Indeed, there is an interesting distinction between logit quantal response equilibria (and their limit points) and Nash equilibria. While all logit quantal response equilibria of a product game are products of equilibria of its factor games, there exist Nash equilibria of product games that do not satisfy this property. In

fact, Nash equilibria exhibit a very rich correlation structure: any strategy profile of a product game that marginalizes to Nash equilibria of its factor games constitutes a Nash equilibrium.

The theorem is proved in Appendix A. We illustrate some of the ideas here. Suppose  $S$  satisfies expectation-monotonicity, narrow bracketing, and anonymity. Consider a very simple game  $G$ , where all players but player 1 are dummies facing no strategic choice, and player 1 chooses between two actions, one of which is dominant. Action sets are  $A_1 = \{h, \ell\}$  and  $A_i = \{c\}$  for  $i \neq 1$ , and the payoff for the first player is 1 when playing  $h$  and 0 when playing  $\ell$ . Consider a solution  $p \in S(G)$ . By expectation-monotonicity,  $p_1(h) \geq p_1(\ell)$ .

We consider two cases depending on whether  $p_1(\ell) = 0$  or not. The case of  $p_1(\ell) = 0$  means that player 1 never plays the dominated action and is treated in the following claim:

**Claim 1.** *Suppose  $p_1(\ell) = 0$  in the game  $G$ . Then  $S$  is a refinement of Nash equilibrium.*

*Proof.* Suppose, for the sake of contradiction, that there is a game  $H = (B, v)$  with  $q \in S(H)$  such that  $\mathbb{E}[v_i(b, q_{-i})] > \mathbb{E}[v_i(a, q_{-i})]$  while  $q_i(a) > 0$  for some player  $i$  and some actions  $a, b \in B_i$ . By anonymity, we can assume  $i = 1$ . Choose

$$n > \frac{1}{\mathbb{E}[v_1(b, q_{-1})] - \mathbb{E}[v_1(a, q_{-1})]},$$

and consider the product game  $H^n \otimes G$ , where  $H^n$  denotes the  $n$ -fold product  $H \otimes \cdots \otimes H$ . By narrow bracketing,  $q^n \times p$  (where  $q^n = q \times \cdots \times q$ ) is a solution for this product game. In the product game, the expected payoff to player 1 for playing  $b$  in every copy of  $H$  and playing  $\ell$  in  $G$  is strictly higher than playing  $a$  in every copy of  $H$  and playing  $h$  in  $G$ . This violates expectation-monotonicity since  $q^n \times p$  places a positive probability on the former action and zero probability on the latter, since  $p_1(\ell) = 0$ . Hence players never play strategies that yield lower expected payoffs, i.e., they play a Nash equilibrium.  $\square$

If  $p_1(\ell) > 0$ , meaning that player 1 sometimes plays the dominated strategy, then it turns out that  $S$  is a refinement of  $\text{LQRE}_\lambda$ . This is demonstrated in the appendix by showing that narrow bracketing implies a multiplicative functional equation on players' choice probabilities leading to the exponential function in  $\text{LQRE}_\lambda$ .

Anonymity is not a crucial assumption for Theorem 1, but rather ensures that all players behave alike. Without anonymity, we get a version of  $\text{LQRE}_\lambda$  with player-specific  $\lambda_i$  instead of a common  $\lambda$ , as well as chimeric rules where some agents use logit best responses and others best-respond as in Nash equilibrium.<sup>5</sup> As a corollary, we do not require anonymity to characterize refinements of Nash equilibrium (even though it is an anonymous solution concept).

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<sup>5</sup>McKelvey, Palfrey, and Weber (2000) extended the QRE framework to allow for such  $\lambda$ -heterogeneity.

**Corollary 1.** *If  $S$  satisfies expectation-monotonicity and narrow bracketing, and players never play strictly dominated actions, then  $S$  is a refinement of Nash.*

The assumption on strictly dominated actions serves to distinguish Nash from  $\text{LQRE}_\lambda$ . In fact, under the assumptions of expectation-monotonicity and narrow bracketing, *any* feature of Nash equilibrium that does not apply to  $\text{LQRE}_\lambda$  will lead to a characterization of only Nash equilibrium and vice versa. For example, we can weaken the assumption on strictly dominated actions by supposing that every player plays some action with zero probability in some game.

In Appendix H, we compare this characterization of Nash equilibrium with that of Brandl and Brandt (2024). Their axioms rule out non-trivial refinements of Nash equilibrium, whereas our axioms allow for some refinements, such as trembling hand perfect equilibrium and maximum welfare Nash. Recall that not all refinements of Nash satisfy narrow bracketing, which provides a justification for selecting some refinements of Nash over others.

As the next corollary shows, in the setting of Theorem 1, Nash equilibria can be ruled out in one of two ways: either by adding the following axiom, which is commonly assumed in the QRE literature, or by strengthening expectation-monotonicity. The first approach requires that players play every action with positive probability in all games. Formally, a solution concept  $S$  satisfies *interiority* if for every game  $G = (A, u)$ , solution  $p \in S(G)$ , and player  $i \in N$ , we have  $p_i(a_i) > 0$  for each  $a_i \in A_i$ .

The second approach requires strengthening expectation-monotonicity. Note that expectation-monotonicity only requires actions yielding strictly higher expected payoffs to be played weakly more often. A slight strengthening of expectation-monotonicity would extend this requirement to actions with weakly higher payoffs by assuming that actions with the same expected payoffs are played with the same probabilities. Formally,  $S$  satisfies *expectation-neutrality* if  $\mathbb{E}[u_i(a_i, p_{-i})] = \mathbb{E}[u_i(b_i, p_{-i})]$  implies  $p_i(a_i) = p_i(b_i)$  for every game  $G$ , solution  $p \in S(G)$ , and player  $i \in N$ .

**Corollary 2.** *Suppose  $S$  satisfies expectation-monotonicity, narrow bracketing, and anonymity. If  $S$  also satisfies either interiority or expectation-neutrality, then  $S$  is a refinement of  $\text{LQRE}_\lambda$  for some  $\lambda \geq 0$ .*

The first part follows from the fact that there exist games in which every Nash equilibrium assigns probability zero to some action. This holds, for instance, if there is a strictly dominated action. The second part follows from the fact that there exist games such that every Nash equilibrium assigns different probabilities to two actions with the same expected payoff. This holds, for instance, for the game in Table 4.

Expectation-monotonicity requires that players pay attention even to small differences in payoffs, and a lower-payoff action cannot be played more often. However, real-world decision-making can be influenced by factors beyond pure payoff comparisons. For instance, the way actions are labeled or presented might impact choices, with certain actions drawing more attention regardless of their associated payoffs. Interestingly, while our framework does not explicitly preclude such label effects, the solution concepts we derived, Nash and  $\text{LQRE}_\lambda$ , do not exhibit them. One might argue that the absence of label effects is hardwired in expectation-monotonicity, which prioritizes payoff differences over other factors. Indeed, choice probabilities are required to respond even to negligibly small differences in payoffs.

We now consider a relaxed notion of approximate expectation-monotonicity, which only requires that substantial differences in payoffs matter. I.e., an action with a lower payoff may be chosen more often, but not by too much. In particular, this notion leaves enough room for action labels to play a role.

**Definition 4.** *A solution concept  $S$  satisfies approximate expectation-monotonicity if there exist constants  $M \geq 0$  and  $\varepsilon > 0$  such that*

$$\mathbb{E}[u_i(a_i, p_{-i})] > \mathbb{E}[u_i(b_i, p_{-i})] + M \quad \text{implies} \quad p_i(a_i) \geq \varepsilon \cdot p_i(b_i)$$

*for every game  $G = (A, u)$ , solution  $p \in S(G)$ , player  $i$ , and actions  $a_i, b_i \in A_i$ .*

In contrast to expectation-monotonicity, choice probabilities do not have to be higher but only at least  $\varepsilon$  as high, and for this to hold, expected payoffs need to be higher by at least  $M$ . Approximate expectation-monotonicity is satisfied by, for example,  $\varepsilon$ -Nash equilibrium (where no player can gain more than  $\varepsilon$  by deviating), and the  $\varepsilon$ -proper equilibrium of [Myerson \(1978\)](#) (where any action is played at most  $\varepsilon$  times as often as a better one).

Perhaps surprisingly, when combined with narrow bracketing, approximate expectation-monotonicity implies (exact) expectation-monotonicity, as demonstrated by the following lemma.

**Lemma 1.** *Every solution concept that satisfies narrow bracketing and approximate expectation-monotonicity also satisfies expectation-monotonicity.*

*Proof.* Let  $S$  satisfy narrow bracketing and approximate expectation-monotonicity. Suppose, for the sake of contradiction, that there is a game  $G = (A, u)$ , solution  $p \in S(G)$ , and actions  $a_i, b_i \in A_i$  such that  $\mathbb{E}[u_i(a_i, p_{-i})] > \mathbb{E}[u_i(b_i, p_{-i})]$  and  $\frac{p_i(a_i)}{p_i(b_i)} < 1$ . Then there is a number  $m \in \mathbb{N}$  such that

$$m \cdot \left( \mathbb{E}[u_i(a_i, p_{-i})] - \mathbb{E}[u_i(b_i, p_{-i})] \right) > M \quad \text{and} \quad \left( \frac{p_i(a_i)}{p_i(b_i)} \right)^m < \varepsilon.$$

Let  $G^m$  denote the  $m$ -fold product game  $G \otimes \cdots \otimes G$  and  $p^m$  denote the  $m$ -fold product  $p \times \cdots \times p$ . Define  $v$  as the payoff map of  $G^m$ , i.e.,  $G^m = (A^m, v)$ . Then by narrow bracketing,

we have  $p^m \in S(G^m)$  with  $\mathbb{E}[v_i^m((a_i, \dots, a_i), p_{-i}^m)] > M + \mathbb{E}[v_i^m((b_i, \dots, b_i), p_{-i}^m)]$  and  $\frac{p_i^m(a_i, \dots, a_i)}{p_i^m(b_i, \dots, b_i)} < \varepsilon$ , violating approximate expectation-monotonicity.  $\square$

As a consequence, we can relax expectation monotonicity to approximate expectation monotonicity in Theorem 1, demonstrating the robustness of the result.

**Corollary 3.** *If  $S$  satisfies approximate expectation-monotonicity, narrow bracketing, and anonymity, then  $S$  is either a refinement of Nash or of  $\text{LQRE}_\lambda$  for some  $\lambda \geq 0$ .*

While approximate expectation-monotonicity relaxes the sensitivity to payoff differences, both expectation-monotonicity and its approximate version maintain a central role for expected utility and thus risk neutrality. In the next section, we will relax the risk-neutrality aspect.

## 4 Distribution-Monotonicity and Statistic Response Equilibria

In this section, we relax the assumption of expectation-monotonicity, which demands that each player's choice probabilities are driven by actions' expected payoffs, and that this dependence is monotone. Instead, we introduce a weaker axiom, *distribution-monotonicity*, which allows for a more general dependence on actions' payoff distributions while only requiring monotonicity with respect to first-order stochastic dominance (FOSD).

Given a distribution over actions  $p$ , we use  $u_i(a_i, p_{-i})$  to denote the lottery faced by player  $i$  when playing action  $a_i$ , while other players use mixed strategies  $p_{-i}$ . We say that an action  $a_i \in A_i$  *first-order dominates*  $b_i \in A_i$  if the lottery  $u_i(a_i, p_{-i})$  first-order stochastically dominates  $u_i(b_i, p_{-i})$ , which we denote by  $u_i(a_i, p_{-i}) \geq_{\text{FOSD}} u_i(b_i, p_{-i})$ . If moreover the distributions of  $u_i(a_i, p_{-i})$  and  $u_i(b_i, p_{-i})$  are not identical, we say that  $a_i$  *strictly first-order dominates*  $b_i$  and write  $u_i(a_i, p_{-i}) >_{\text{FOSD}} u_i(b_i, p_{-i})$ .<sup>6</sup>

**Definition 5.** *A solution concept  $S$  satisfies distribution-monotonicity if*

$$u_i(a_i, p_{-i}) >_{\text{FOSD}} u_i(b_i, p_{-i}) \quad \text{implies} \quad p_i(a_i) \geq p_i(b_i)$$

*for every game  $G = (A, u)$ , solution  $p \in S(G)$ , player  $i$ , and actions  $a_i, b_i \in A_i$ .*

In other words, distribution-monotonicity means that if one action strictly first-order dominates the other, the dominated action cannot be played with a higher probability. This axiom is implied by expectation-monotonicity, but is much weaker: it relaxes both the

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<sup>6</sup>First-order dominance means that for any threshold  $t$ , the probability that  $u_i(a_i, p_{-i})$  yields a payoff greater than  $t$  is at least as high as the probability that  $u_i(b_i, p_{-i})$  yields a payoff greater than  $t$ . Equivalently,  $\mathbb{E}[f(u_i(a_i)) | a_i] \geq \mathbb{E}[f(u_i(b_i)) | b_i]$  for any non-decreasing payoff transformation  $f: \mathbb{R} \rightarrow \mathbb{R}$ . Strict dominance corresponds to strict inequality for any strictly increasing  $f$ .

expected utility and the risk neutrality components of expectation-monotonicity, keeping only rational expectations and a preference for higher payoffs. When there is only one player, distribution-monotonicity coincides with expectation-monotonicity. Henceforth, we suppose that there are at least two players.

Unlike expectation-monotonicity, distribution-monotonicity is invariant to monotone transformations of payoffs. For example, Nash equilibrium with monotone-reparameterized payoffs (Weinstein, 2016) and risk-adjusted QRE under CRRA reparameterized payoffs (Goree, Holt, and Palfrey, 2003) satisfy distribution-monotonicity. Distribution-monotonicity is also satisfied by  $S(K)$  equilibria (Osborne and Rubinstein, 1998), where players respond to independent draws from the payoff distribution induced by each action.

#### 4.1 Strategic Invariance and the Emergence of Expected Utility

Relaxing expectation-monotonicity to distribution-monotonicity leads to new families of solution concepts, which we explore in §4.2 below. This demonstrates that distribution-monotonicity is indeed much weaker than expectation-monotonicity, even when coupled with narrow bracketing. In this section, we show that the gap between these axioms can be bridged with the additional assumption of *strategic invariance*, which restricts a solution concept's predictions across *strategically equivalent* games.

**Definition 6.** Games  $(A, v)$  and  $(A, u)$  are strategically equivalent if for each player  $i$  there exists a function  $w_i: A_{-i} \rightarrow \mathbb{R}$  such that  $v_i(a) = u_i(a) + w_i(a_{-i})$ .

That is, strategically equivalent games share the same sets of actions, and while payoffs may differ, they must satisfy  $v_i(a_i, a_{-i}) - v_i(b_i, a_{-i}) = u_i(a_i, a_{-i}) - u_i(b_i, a_{-i})$  for all  $a_i, b_i \in A_i$  and  $a_{-i} \in A_{-i}$ . In other words, player  $i$ 's marginal payoff of switching from an action  $a_i$  to another action  $b_i$  is the same in the two games.

The notion of strategic equivalence is fundamental to the study of solution concepts. For example, strategically equivalent games have identical sets of Nash and correlated equilibria. In mechanism design, strategic equivalence is an important tool. It provides the designer with the flexibility to modify a player  $i$ 's transfers without altering their incentives, simply by adding a quantity that is independent of  $i$ 's report. This flexibility is crucial in mechanisms like VCG, where it helps to achieve the desired normalization of transfers and, in some environments, budget-balancedness.

**Definition 7.** A solution concept  $S$  satisfies strategic invariance if  $S(A, u) = S(A, v)$  for strategically equivalent games  $(A, u)$  and  $(A, v)$ .

Strategic equivalence is respected by Nash and LQRE $_\lambda$ , as well as many other concepts that do not have a rational expectations component, such as rationalizability and level- $k$  reasoning.

Our next theorem shows that strategic invariance—when coupled with narrow bracketing—becomes a powerful assumption that elevates distribution-monotonicity to expectation-monotonicity.

**Theorem 2.** *Suppose  $S$  satisfies narrow bracketing, strategic invariance, distribution-monotonicity, and anonymity. Then it satisfies expectation-monotonicity.*

Recall that distribution-monotonicity is a rational expectations and monotonicity axiom, and does not have an expected utility or risk neutrality component. In contrast, expectation-monotonicity is a stronger axiom that implies distribution-monotonicity, and furthermore also has expected utility and risk neutrality components. Theorem 2 thus shows that strategic invariance is a potent assumption that highly constrains behavior to resemble risk neutrality.

Combining Theorems 1 and 2, we obtain the following corollary, which offers a foundation for Nash and  $\text{LQRE}_\lambda$  without directly assuming risk neutrality.

**Corollary 4.** *Suppose  $S$  satisfies narrow bracketing, strategic invariance, distribution-monotonicity, and anonymity, then  $S$  is either a refinement of Nash or of  $\text{LQRE}_\lambda$  for some  $\lambda \geq 0$ .*

The proof of Theorem 2 is contained in Appendix B. To develop intuition for this result, note that strategic invariance clearly implies that players display no wealth effects, since adding a constant to all payoffs does not change their behavior. To see why strategic invariance furthermore rules out any non-trivial risk attitudes, consider the following example of a two-player game. Player 2 has two actions,  $a_2$  and  $b_2$ , and gets payoff 0 regardless of the action profile. For now, assume that this player mixes evenly between these two actions. Player 1 has two actions,  $a_1$  and  $b_1$ , and gets the payoffs presented on the left side of Table 1.

	$a_2$	$b_2$		$a_2$	$b_2$
$a_1$	0	2	$a_1$	0	1
$b_1$	1	1	$b_1$	1	0

Table 1: Player 1's payoffs in two strategically equivalent games

In this game, both actions yield the same expected payoff, but action  $a_1$  has variance 1, whereas action  $b_1$  has variance 0, and so would be preferred by any risk-averse player. Consider now the strategically equivalent game described on the right side of Table 1. Here, both actions yield the same distribution of payoffs to player 1, and hence risk attitudes should not influence the choice between  $a_1$  and  $b_1$ . Since this game is strategically



equivalent to the previous, we conclude that under strategic invariance, players would be indifferent between the two actions in the previous game and so are effectively risk-neutral.

The assumption that player 2 mixes evenly between the two actions is crucial for this argument and turns out to be non-trivial: if we cannot guarantee mixing by player 2, we cannot conclude anything about the risk attitudes of player 1. But in these games, player 2 has no particular reason to mix, and so the actual proof of Theorem 2 relies on the following more elaborate construction of a game in which player 2 must mix. Given  $x = (x_1, \dots, x_m) \in \mathbb{R}^m$ ,  $r \in \mathbb{R}$  and  $\varepsilon > 0$ , we define a game  $G_{r,x,\varepsilon}$  with sets of actions  $A_2 = \{1, \dots, m\}$  and

$$A_1 = \{a_x, a_r\} \times \{f: A_2 \rightarrow A_2 \mid f \text{ is a bijection}\}.$$

The payoffs are as follows. For player 1,

$$u_1((a_x, f), a_2) = x_{f(a_2)}, \quad u_1((a_r, f), a_2) = r + \varepsilon x_{f(a_2)}. \quad (1)$$

For player 2,

$$u_2((a_x, f), a_2) = -x_{f(a_2)}.$$

This game admits the following interpretation. Player 1 has  $m$  cards, each showing some amount of money  $x_i$ . Player 1 chooses a re-ordering  $f$  of the cards. Player 2 chooses one of these cards, and pays the amount of money the card shows. Depending on whether player 1 chose  $a_x$  or  $a_r$ , player 1 either receives the money paid by player 2 or else player 1 gains  $r$ , up to an error of  $\varepsilon$ .

We show that, in this game, player 2 must mix uniformly, under the mild assumption that players never play strictly first-order dominated strategies. This leaves player 1 with a choice between the uniform lottery over  $x$  and the (approximately) sure payoff  $r$ . By applying strategic invariance and the idea behind the simple games in Table 1, we show that player 1 will choose the sure thing if it is higher than the average of  $x$ , i.e., the player is driven by expected payoffs in  $G_{r,x,\varepsilon}$ . To prove Theorem 2, we need to extend this conclusion to all games. This step relies on Theorem 3 formulated below.

## 4.2 Statistic Response Equilibria and Risk Attitudes

We now relax the strategic invariance assumption and explore the joint implications of narrow bracketing and distribution-monotonicity, without any assumptions that lead to risk neutrality or indeed expected utility. We show that the only solution concepts satisfying anonymity, narrow bracketing, and distribution-monotonicity are *statistic response equilibria*, where players respond to a statistic of each action's payoff distribution. This



class of equilibria generalizes Nash and  $\text{LQRE}_\lambda$ —in which players evaluate actions by their expected payoffs—and accommodates various risk attitudes.

We use the term *statistic* to refer to a function  $\Phi$  that assigns a real number to every lottery with finitely many outcomes and satisfies  $\Phi[c] = c$  for deterministic lotteries yielding a constant amount  $c$ . Here, a lottery is simply a distribution over monetary payoffs. Lotteries arise in our setting as the payoffs a player anticipates when choosing an action, given the mixed strategies of the other players. We denote lotteries by  $X, Y$ , and  $X + Y$  denotes the lottery corresponding to the sum of outcomes drawn independently from  $X$  and  $Y$ ; that is, the distribution of  $X + Y$  is the convolution of the distributions of  $X$  and  $Y$ .

We now define a class of statistics that are monotone with respect to first-order stochastic dominance and additive for independent lotteries. Below, we demonstrate that players must respond to this class of statistics.

**Definition 8.** *A statistic  $\Phi$  is a monotone additive statistic if*

$$\Phi[X + Y] = \Phi[X] + \Phi[Y] \quad \text{and} \quad \Phi[Z] \leq \Phi[W] \quad \text{for} \quad Z \leq_{\text{FOSD}} W.$$

A canonical example of monotone additive statistics is given by the normalized cumulant-generating function of  $X$ , defined as  $K_a[X] = \frac{1}{a} \log \mathbb{E}[e^{aX}]$  for  $a \in \mathbb{R}$ . Taking limits as  $a$  approaches  $\pm\infty$  and 0, we obtain  $K_{-\infty}[X]$ ,  $K_0[X]$ , and  $K_\infty[X]$ , which correspond to the minimum, expectation, and maximum of  $X$ , respectively. The statistic  $K_a[X]$  has an important economic interpretation: each  $K_a[X]$  represents the certainty equivalent of lottery  $X$  for an agent with constant absolute risk aversion (CARA) utilities, where  $-a$  is the coefficient of absolute risk aversion; negative  $a$  values correspond to risk aversion, and positive values to risk-loving preferences.

Mu, Pomatto, Strack, and Tamuz (2021) show that this example is, in fact, universal: any monotone additive statistics can be represented as  $\Phi[X] = \int_{\overline{\mathbb{R}}} K_a[X] d\mu(a)$  for some probability measure  $\mu$  on the extended real line  $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$ . Hence, any monotone additive statistic is a weighted average of CARA certainty equivalents across risk coefficients, which may reflect both risk-averse and risk-loving preferences, as  $\mu$  can place mass on both negative and positive risk coefficients.<sup>7</sup>

In Nash equilibrium and in  $\text{LQRE}_\lambda$  players respond to the expectation of each action's payoff distribution. In a statistic response equilibrium, players evaluate actions using a monotone additive statistic of the payoff distribution. For example, players

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<sup>7</sup>Mu, Pomatto, Strack, and Tamuz (2021) provide a characterization of monotone additive statistics on the domain of all compactly supported (rather than finitely supported) lotteries. Their characterization also applies to the domain of finitely supported lotteries, since any monotone additive statistic on this restricted domain can be extended to the compactly supported lotteries. For a proof, see Appendix B.1.

may respond to the minimum of each action's payoff distribution,  $\min[u_i(a_i, p_{-i})] = \min_{a_{-i} \in \text{supp}(p_{-i})} u_i(a_i, a_{-i})$ . Below we define two classes of statistic response equilibria.

**Definition 9.** *Given a monotone additive statistic  $\Phi$ , a mixed strategy profile  $p$  is a Nash $_{\Phi}$  equilibrium of the game  $G = (A, u)$  if for all players  $i$*

$$\text{supp } p_i \subseteq \arg \max_a \Phi[u_i(a, p_{-i})].$$

In a Nash $_{\Phi}$  equilibrium, players best respond to the other players' mixed strategies according to  $\Phi$  by randomizing over actions whose payoff distributions maximize  $\Phi$ . Since  $\Phi$  is a monotone additive statistic, players never play an action that is first-order stochastically dominated. Thus any solution concept that only returns Nash $_{\Phi}$  equilibria will satisfy distribution-monotonicity.

The next definition introduces a class of statistic response equilibria in which players “better respond” to a monotone additive statistic of each distribution.

**Definition 10.** *Given a monotone additive statistic  $\Phi$  and  $\lambda \geq 0$ , a mixed strategy profile  $p$  is a LQRE $_{\lambda\Phi}$  equilibrium of the game  $G = (A, u)$  if for all players  $i$  and actions  $a_i \in A_i$*

$$p_i(a_i) \propto \exp(\lambda \Phi[u_i(a_i, p_{-i})]).$$

Nash $_{\Phi}$  and LQRE $_{\lambda\Phi}$  generalize Nash and logit quantal response equilibria, in which players respond to the expectation, i.e.,  $\Phi = \mathbb{E}$ . While every game has a Nash equilibrium, the existence of a Nash $_{\Phi}$  equilibrium is not guaranteed for all  $\Phi$  unless there is only one player.<sup>8</sup> For example, Nash $_{\Phi}$  equilibria may not exist when  $\Phi$  is the minimum or maximum of a distribution, i.e., players are extremely risk-averse or risk-loving. This issue does not arise for LQRE $_{\lambda\Phi}$ , which do exist for every game. As the next result shows, the existence of Nash $_{\Phi}$  equilibria is also guaranteed for a large family of monotone additive statistics, namely those in which the maximum and minimum do not play a role.

**Proposition 1.** *Let  $\Phi = \int_{\mathbb{R}} K_a d\mu(a)$  be a monotone additive statistic. Then*

- *There is an LQRE $_{\lambda\Phi}$  equilibrium for every game and  $\lambda \geq 0$ .*
- *There is a Nash $_{\Phi}$  equilibrium for every game if and only if  $\mu(\{-\infty\}) = \mu(\{+\infty\}) = 0$ .*

Proposition 1 is proved in Appendix C. The existence of an LQRE $_{\lambda\Phi}$  equilibrium follows from Brouwer's fixed-point theorem, applied to a version of the logit response function; the latter must be appropriately modified to ensure continuity when  $\mu$  places positive weight to  $\pm\infty$ . The proof that Nash $_{\Phi}$  equilibria exist when  $\mu$  places no mass on the minimum or

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<sup>8</sup>When there is one player, every payoff is deterministic, so Nash $_{\Phi}$  equilibria coincide with Nash equilibria for all  $\Phi$ .

maximum follows from a standard fixed point argument. This argument does not apply when  $\mu$  places positive mass on the minimum or maximum, since  $\Phi$  may be discontinuous at the limit point. Indeed,  $\text{Nash}_\Phi$  equilibria may fail to exist for such  $\mu$ , as we show using a variant of matching pennies.

We call  $\text{Nash}_\Phi$  and  $\text{LQRE}_{\lambda\Phi}$  *statistic response equilibria* as players best or better respond to the statistic  $\Phi$  of distributions induced by each available action. Formally, a statistic response equilibrium (SRE) is a solution concept that returns all  $\text{Nash}_\Phi$  or all  $\text{LQRE}_{\lambda\Phi}$  equilibria for some  $\Phi$  and  $\lambda$ .<sup>9</sup>

It is easy to verify that the SRE solution concepts satisfy our axioms. Narrow bracketing follows from the additivity of  $\Phi$ , distribution-monotonicity is a consequence of the monotonicity of  $\Phi$ , and anonymity holds since all players use the same  $\Phi$ . The next result shows that these axioms, in fact, characterize SRE.

**Theorem 3.** *Suppose  $S$  satisfies distribution-monotonicity, narrow bracketing, and anonymity. Then  $S$  is a refinement of some SRE.*

In an SRE, the statistic  $\Phi$  assigns a certainty equivalent to each payoff distribution, which players respond to. As a monotone additive statistic,  $\Phi$  is a weighted average of CARA certainty equivalents across different values of coefficients  $a$ . Hence statistic response equilibria incorporate flexible risk attitudes which allow for risk-averse, risk-loving, or mixed risk attitudes. Indeed, consider a statistic  $\Phi[X] = \int_{\mathbb{R}} K_a[X] d\mu(a)$ . If  $\mu$  places mass only on negative values of  $a$ ,  $\Phi[X] \leq \mathbb{E}[X]$  for any lottery  $X$ , i.e.,  $\Phi$  reflects risk-aversion. Conversely, if  $\mu$  places mass only on positive values of  $a$ , then  $\Phi[X] \geq \mathbb{E}[X]$  and  $\Phi$  reflects a risk-loving attitude. If  $\mu$  places mass on both negative and positive values of  $a$ , then  $\Phi$  reflects a mixed risk attitude, i.e. there are lotteries  $X$  and  $Y$  with  $\Phi[X] < \mathbb{E}[X]$  and  $\Phi[Y] > \mathbb{E}[Y]$ ; see Proposition 5 of [Mu, Pomatto, Strack, and Tamuz \(2021\)](#).

While distribution-monotonicity ensures that players' mixing probabilities are monotone with respect to first-order stochastic dominance, it does not provide a way to compare any pair of distributions. Interestingly, Theorem 3 shows that, with narrow bracketing and anonymity, there is a total order, defined by a statistic  $\Phi$ , that dictates how players rank *every* payoff distribution.

Anonymity in Theorem 3 can be removed as in Theorem 1, with the conclusion appropriately altered to allow different players to best or better respond to different statistics.

The proof of Theorem 3 begins with the observation (formalized in Proposition 2 below) that behavior under a solution concept depends crucially on whether it excludes dominated

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<sup>9</sup>By Proposition 1,  $\text{Nash}_\Phi$  is a well-defined solution concept only for  $\Phi$  that puts no mass on the maximum or minimum.

actions. Under distribution-monotonicity (with narrow bracketing and anonymity) there is a dichotomy: every solution concept must either assign zero probability to every first-order dominated action, or else it must assign positive probability to every action. In the latter case, it also satisfies what we call *distribution-neutrality*, which requires that if two actions induce identical payoff distributions, then they must be played with equal probability. Formally,  $S$  satisfies *distribution-neutrality* if  $u_i(a, p_{-i}) = u_i(b, p_{-i})$  implies  $p_i(a) = p_i(b)$  for every game  $G$ , solution  $p \in S(G)$ , and player  $i$ .

**Proposition 2.** *Suppose  $S$  satisfies distribution-monotonicity, narrow bracketing, and anonymity. Then exactly one of the following two statements holds:*

1. *Players never play strictly first-order dominated actions;*
2.  *$S$  satisfies interiority and distribution-neutrality.*

Given Proposition 2, the proof of Theorem 3 breaks into two cases, corresponding to the dichotomy illustrated in the proposition. The first case turns out to be a refinement of  $\text{Nash}_\Phi$ , while the second is a refinement of  $\text{LQRE}_{\lambda\Phi}$ . The proof of each case is similar to the proof in Theorem 1, and hence the proof of Proposition 2 is the main novel component in the proof of Theorem 3.

Proposition 2 is proved in Appendix E. To convey the main idea, recall the classical notion of dominated actions. An action  $a_i$  *strictly dominates*  $b_i$  if  $u_i(a_i, a_{-i}) > u_i(b_i, a_{-i})$  for all  $a_{-i}$ . Note that  $a_i$  strictly dominates  $b_i$  if and only if  $b_i$  is strictly first-order dominated for every  $p$ . However,  $a_i$  can strictly first-order dominate  $b_i$  under some profiles  $p_{-i}$  but not others, in which case there is no strict dominance.

The proof consists of two parts. We first show that if players never play strictly dominated actions, they also never play strictly first-order dominated ones. To demonstrate this, we construct a game  $H$  with the property that in every mixed strategy profile  $p$  there is at least one player  $i$  and actions  $a_i, b_i$  such that  $i$  chooses  $a_i$  with positive probability, yet the lottery resulting from playing  $a_i$  has both a lower minimum outcome and a lower maximum outcome than the lottery corresponding to  $b_i$ .

Using the constructed game  $H$ , we demonstrate that in any game  $G$  with an action  $c_i$  strictly first-order dominated by  $d_i$ , the action  $c_i$  is played with probability zero. For this purpose, we apply a novel lemma about stochastic dominance of i.i.d. sums (Lemma 11) to show that in the product game  $G^n \otimes H^m$  (for carefully chosen  $n, m$ ), the action  $(c_i, \dots, c_i, a_i, \dots, a_i)$  is strictly first-order dominated by  $(d_i, \dots, d_i, b_i, \dots, b_i)$ . The latter action profile is never played since  $b_i$  is assigned probability zero in  $H$ . Distribution-monotonicity then implies that the former profile cannot be played either. Finally, by narrow bracketing,  $c_i$  is never played in  $G$ .

The second part of the proof is to show that if players ever play strictly dominated actions in some game  $G_D$ , the solution concept satisfies interiority and distribution-neutrality. This part of the proof is more straightforward and involves taking the product of a given game  $G$  with powers of  $G_D$  and invoking narrow bracketing and distribution-monotonicity.

## 5 Parametric Families of Statistic Response Equilibria

In statistic response equilibria, players respond to a statistic  $\Phi$ , which is parameterized by a probability measure  $\mu$  on  $\overline{\mathbb{R}}$ , an infinite-dimensional parameter. Having such a large parameter space can be challenging for empirical studies, especially on small datasets. In this section, we introduce additional axioms that single out two tractable parametric families of SRE that are rich enough to capture various risk attitudes but are also empirically viable.

### 5.1 CARA Equilibria

Our first parametric family corresponds to players with constant absolute risk aversion (CARA) preferences. Consider  $\text{Nash}_\Phi$  equilibria with the statistic  $\Phi$  given by the cumulant-generating function  $K_a$  or, equivalently,  $\mu$  equal to the point mass at some  $a \in \mathbb{R}$ . In these equilibria, players randomize over the actions whose payoff distributions have maximal CARA certainty equivalents under risk-coefficient  $-a$ . Equivalently, players randomize over the actions whose payoff distributions maximize CARA expected utility. These equilibria thus coincide with the Nash equilibria of a game whose payoff function is reparameterized according to the CARA utility function

$$c_{-a}(x) = \begin{cases} \frac{e^{ax}-1}{a} & a \neq 0 \\ x & a = 0. \end{cases}$$

We consider two families of CARA-based solution concepts— $\text{Nash}_{K_a}$  and  $\text{LQRE}_{\lambda K_a}$ —and characterize them by strengthening the distribution-monotonicity axiom in a way that adds an expected utility aspect, without imposing risk neutrality. Specifically, we assume that choice probabilities are monotone with respect to the expectation of transformed payoffs  $f \circ u_i$  for some transformation  $f$ .

**Definition 11.** *A solution concept  $S$  satisfies  $f$ -monotonicity if there exist a strictly increasing function  $f: \mathbb{R} \rightarrow \mathbb{R}$  such that*

$$\mathbb{E}[f(u_i(a_i, p_{-i}))] > \mathbb{E}[f(u_i(b_i, p_{-i}))] \quad \text{implies} \quad p_i(a_i) \geq p_i(b_i)$$

*for every game  $G = (A, u)$ , solution  $p \in S(G)$ , player  $i$ , and actions  $a_i, b_i \in A_i$ .*

The function  $f$  plays the role of a utility for monetary payoffs, generalizing the linear utility implicit in the expectation-monotonicity axiom.

**Theorem 4.** *Suppose that  $S$  satisfies narrow bracketing, anonymity, and  $f$ -monotonicity. Then  $S$  is a refinement of either  $\text{Nash}_{K_a}$  or  $\text{LQRE}_{\lambda K_a}$  for some  $a \in \mathbb{R}$  and  $\lambda \geq 0$ .*

The theorem is proved in Appendix F by combining Theorem 3 with the observation that the only monotone additive statistics compatible with  $f$ -monotonicity are given by  $K_a$ .

Theorem 4 shows that CARA preferences emerge endogenously. It singles out the one-parameter family of  $\text{Nash}_{K_a}$  and the two-parameter family of  $\text{LQRE}_{\lambda K_a}$  equilibria. The parameter  $a$  captures the degree and direction of risk attitudes. Each of the two families separately can be pinned down by specifying whether the solution concept ever excludes a certain action profile, as in Corollaries 1 and 2.

Note that  $\text{Nash}_{K_a}$  and  $\text{LQRE}_{\lambda K_a}$  are  $f$ -monotone with respect to the CARA utility function  $c_{-a}$ . Accordingly, any  $f$  compatible with the rest of the axioms equals  $c_{-a}$  up to an affine transformation. Moreover,  $\text{Nash}_{K_a}$  are the only  $\text{Nash}_\Phi$  equilibria that coincide with Nash equilibria under a monotone transformation of payoffs. Indeed, any such solution concept would satisfy the assumptions of the theorem and thus the transformation must follow  $c_{-a}$ .

By contrast, there is no  $\Phi$  other than the expectation such that  $\text{LQRE}_{\lambda \Phi}$  equilibria coincide with  $\text{LQRE}_\lambda$  under transformed payoffs. To see this, observe that in  $\text{LQRE}_{\lambda K_a}$  players logit respond to certainty equivalents of actions' payoff distributions

$$p_i(a_i) \propto \left( \mathbb{E}[\exp(a \cdot u_i(a_i, p_{-i}))] \right)^{\lambda/a},$$

rather than logit responding to the transformed payoffs, as in Goeree, Holt, and Palfrey (2003). We conclude that introducing risk aversion or risk seeking behavior to  $\text{LQRE}_\lambda$  cannot be achieved via payoff transformations without giving up on narrow bracketing, while  $\text{LQRE}_{\lambda K_a}$  can incorporate risk attitudes while maintaining narrow bracketing.

Since  $\text{Nash}_{K_a}$  coincide with Nash equilibria under a CARA transformation of payoffs, these solution concepts satisfy an analogue of the sure-thing principle (Savage, 1954). To formalize the sure-thing principle in games, we introduce a notion of equivalence between games that is stronger than strategic equivalence (see Definition 6). For two games to be *strongly strategically equivalent*, we require that a player's payoffs across the games only differ over actions where they have no agency to affect payoffs. Formally, we say that players receive a *sure thing* under  $a_{-i}$  if  $u(a_i, a_{-i}) = u(b_i, a_{-i})$  for all  $a_i, b_i \in A_i$ . Let  $E_i^u \subset A_{-i}$  denote the set of opponent profiles for which players receive a sure thing.

**Definition 12.** *Games  $(A, u)$  and  $(A, v)$  are strongly strategically equivalent if there exists a function  $w_i: A_{-i} \rightarrow \mathbb{R}$  with  $\text{supp } w_i \subset E_i^u$  such that  $v_i(a) = u_i(a) + w_i(a_{-i})$ .*

	$a_2$	$b_2$	$c_2$		$a_2$	$b_2$	$c_2$
$a_1$	(0, 1)	(2, -1)	(1, 1)	$a_1$	(0, 1)	(2, -1)	(-1, 1)
$b_1$	(1, 1)	(1, 2)	(1, 1)	$b_1$	(1, 1)	(1, 2)	(-1, 1)

Table 2: Player 1 and 2's payoffs in two strongly strategically equivalent games.

The payoffs in the two games in Table 2 differ only over player 1's payoffs when player 2 plays  $c_2$ . Since the players receive a sure thing when player 2 plays  $c_2$ , these games are strongly strategically equivalent.

A player  $i$  faces the same incentives for strongly strategically equivalent games, since the only difference between such games pertains to payoffs under  $E_i^u$ , in which  $i$ 's actions do not matter. The sure-thing principle requires solutions to be invariant to such differences.

**Definition 13.** *We say that  $S$  satisfies the sure-thing principle if  $S(A, u) = S(A, v)$  for strongly strategically equivalent games  $(A, u)$  and  $(A, v)$ .*

The logic behind the sure-thing principle is similar to the idea of conditioning on pivotality in the strategic voting literature (see, for example, Feddersen and Pesendorfer, 1996, 1998): when a player decides between actions, they condition on the event that choosing different actions will lead to different outcomes, since otherwise it makes no difference which action is chosen. In our setting, this means that a player will condition on the event that  $a_{-i}$  is not in  $E_i^u$ . Hence a difference in payoffs on  $E_i^u$  should not change a player's strategy.

It is easy to see that  $\text{Nash}_{K_a}$  and  $\text{LQRE}_\lambda$  satisfy the sure-thing principle. We conjecture that these solution concepts are the only SREs that satisfy the sure-thing principle.

## 5.2 Min-Max-Mean Response Equilibria

While CARA equilibria offer a parsimonious way to introduce risk aversion or risk seeking behavior into equilibria, this family cannot capture more complex patterns such as mixed risk attitudes. Our second parametric family allows for mixed risk attitudes by combining three fundamental statistics of payoff distributions: the minimum (worst-case), maximum (best-case), and expectation (average-case). To recover this family we consider the following assumption on behavior across games.

**Definition 14.**  *$S$  satisfies scale-invariance if whenever  $p_i$  is the uniform distribution on  $A_i$  for all  $i$ ,  $p \in S(A, u)$  implies  $p \in S(A, \alpha \cdot u)$  for all  $\alpha \in (0, 1)$ .*

Scale-invariance follows from the consistency and consequentialism axioms of Brandl and Brandt (2024); see Appendix H. This assumption is weak, along two dimensions. First, it only applies when all players mix uniformly. In a way, this means that it only restricts

players' behavior when all players are indifferent among all action.<sup>10</sup> Second, the restriction is imposed only for scales  $\alpha$  less than unity. Intuitively, it seems plausible that if a player is indifferent between all actions, then they would still be indifferent when the stakes are made lower.

To keep the statement of the next theorem simple, we rule out Nash equilibria by assuming that players play every action with positive probability; due to Proposition 1, removing this assumption would only add back the Nash solution concept and some of its refinements.

**Theorem 5.** *Suppose  $S$  satisfies distribution-monotonicity, narrow bracketing, scale-invariance, anonymity, and interiority. Then there is  $\lambda \in \mathbb{R}_{\geq 0}^3$  such that*

$$p_i(a_i) \propto \exp \left( \lambda_1 \min_{a_{-i}} u_i(a_i, a_{-i}) + \lambda_2 \mathbb{E}[u_i(a_i, p_{-i})] + \lambda_3 \max_{a_{-i}} u_i(a_i, a_{-i}) \right)$$

for all games  $G = (A, u)$ , players  $i$ , and actions  $a_i \in A_i$ .

Perhaps surprisingly, scale-invariance rules out all risk attitudes except for extreme risk aversion, extreme risk-seeking, and risk neutrality (and their convex combinations). From a behavioral point of view, taking the minimum and maximum into account is consistent with these values being more salient than intermediate values, especially when players face multiple games simultaneously (Avoyan and Schotter, 2020). This family is simple enough to be represented by only three parameters, but is rich enough to capture mixed risk attitudes, allowing players to be risk-averse in one game but risk-seeking in another. This combination of parsimony and flexibility suggests these equilibria as a promising tool for analyzing experimental data.

## 6 Conclusion

We have developed an axiomatic theory of how players frame and evaluate strategic decisions, characterizing both classical solution concepts and new ones that accommodate mixed risk attitudes. Our analysis of narrow bracketing reveals it as a unifying principle underlying Nash equilibrium and  $\text{LQRE}_\lambda$ , while also suggesting natural generalizations of these concepts. Our work raises three questions for future research discussed below.

In the definition of our solution concepts, it is hardwired that players randomize their actions independently. A natural next step would be to consider solution concepts that allow for correlated actions, such as the set of correlated equilibria (CE). Since CE satisfies a version of the narrow-bracketing axiom, extending our approach to correlated actions would potentially result in a new perspective on correlated equilibria and suggest their

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<sup>10</sup>A longer but more cumbersome name such as indifference-scale-invariance might be more appropriate.



generalizations. The main challenge is finding the right analogue of the monotonicity axioms.

Our results show that Nash equilibria,  $\text{LQRE}_\lambda$ , and their generalizations arise under narrow bracketing, as do some of their refinements. A complete characterization of which refinements satisfy narrow bracketing remains open. This question is particularly interesting for  $\text{LQRE}_\lambda$ , which might admit only a small class of such refinements.

Narrow bracketing may have different implications if restricted to particular classes of games, e.g., symmetric or zero-sum games. Solution concepts that fail our axioms when applied to all games might satisfy them within these restricted domains, potentially yielding novel equilibrium notions tailored to particular strategic environments.

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## A Proof of Theorem 1

We begin by introducing a number of definitions and establishing a key lemma. Say that  $S$  satisfies *expectation-neutrality* if for any  $G = (A, u)$  and  $p \in S(G)$ , if  $\mathbb{E}[u_i(a, p_{-i})] = \mathbb{E}[u_i(b, p_{-i})]$ , then  $p_i(a) = p_i(b)$ .

**Lemma 2.** *If  $S$  satisfies expectation-neutrality, narrow bracketing, interiority, expectation-monotonicity, and anonymity, then  $S$  is a refinement of  $\text{LQRE}_\lambda$  for some  $\lambda \geq 0$ .*

*Proof of Lemma 2.* We first show that  $S$  coincides with  $\text{LQRE}_\lambda$  on a particular class of games where all but one player have a single action. Fix a player  $i \in N$  and consider for each  $x \in \mathbb{R}$  the game  $G_x = (A, u)$ , where  $A_i = \{a_0, a_x\}$ , and payoffs of the player  $i$  are given by  $u_i(a_0, a_{-i}) = 0$  and  $u_i(a_x, a_{-i}) = x$ , while all the other players have a single action and receive a payoff of zero for all action profiles. Let  $p_x \in S(G_x)$  and define

$$f(x) = \ln \frac{p_{x_i}(a_x)}{1 - p_{x_i}(a_x)},$$

which is well-defined by interiority. We aim to demonstrate that  $f(x) = \lambda \cdot x$  and so  $S(G_x) = \text{LQRE}_\lambda(G_x)$ . For this purpose, we derive a functional equation on  $f$  using an argument similar to the one used by [Sandomirskiy and Tamuz \(2023\)](#) in the case of decision problems. Let  $x, y \in \mathbb{R}$ ,  $p_x \in S(G_x)$ ,  $p_y \in S(G_y)$ , and  $p_{x+y} \in S(G_{x+y})$ . By narrow bracketing,  $p_x \times p_y \times p_{x+y} \in S(G_x \otimes G_y \otimes G_{x+y})$ . By expectation-neutrality,  $p_{x_i}(a_x)p_{y_i}(a_y)(1 - p_{x+y_i}(a_{x+y})) = (1 - p_{x_i}(a_x))(1 - p_{y_i}(a_y))p_{x+y_i}(a_{x+y})$ . Rearranging and taking logs, we get  $f(x + y) = f(x) + f(y)$ , Cauchy's functional equation. Since  $S$  satisfies narrow bracketing and expectation-monotonicity,  $p_{x_i}(a_x)$  is increasing in  $x$ , as  $p_{x_i}(a_x)p_{x'_i}(a_0) \geq p_{x'_i}(a'_x)p_{x_i}(a_0)$  for  $x > x'$ . Thus  $f$  is nondecreasing, and  $f(x) = \lambda x$ , for some  $\lambda \geq 0$ , because all monotone solutions to the Cauchy equation are linear.

Fix any  $G = (B, v)$ ,  $q \in S(G)$ , with  $b, c \in B_i$ . Let  $x = \mathbb{E}[v_i(b, q_{-i})]$  and  $y = \mathbb{E}[v_i(c, q_{-i})]$ . By narrow bracketing,  $r := q \times p_x \times p_y \in S(G \otimes G_x \otimes G_y)$ . Let  $w$  denote the payoff map for  $G \otimes G_x \otimes G_y$ . Note that  $\mathbb{E}[w_i((b, a_0, a_y), r_{-i})] = \mathbb{E}[w_i((c, a_x, a_0), r_{-i})]$ . By expectation-neutrality,

$$q_i(b)(1 - p_{x_i}(a_x))p_{y_i}(a_y) = q_i(c)p_{x_i}(a_x)(1 - p_{y_i}(a_y)).$$

Rearranging, we have

$$\frac{q_i(b)}{q_i(c)} = \frac{p_{x_i}(a_x)/(1 - p_{x_i}(a_x))}{p_{y_i}(a_y)/(1 - p_{y_i}(a_y))} = \frac{\exp(f(x))}{\exp(f(y))}.$$

Since  $c$  was arbitrary,

$$q_i(b) \propto \exp(f(x)) = \exp(\lambda x) = \exp(\lambda \mathbb{E}[v_i(b, q_{-i})]),$$

for some  $\lambda \geq 0$ . By anonymity, this holds for all  $i \in N$ . □

**Remark 6.** If the payoffs in  $v_i(\cdot, q_{-i})$  are deterministic,  $w_i((b, a_0, a_y), r_{-i}) = w_i((c, a_x, a_0), r_{-i})$  as distributions. Thus, even under distribution-neutrality we have the result  $q_i(b) \propto \exp(\lambda \mathbb{E}[v_i(b, q_{-i})])$ .

We are now ready to prove Theorem 1.

*Proof of Theorem 1.* Consider the game  $G$  with action sets  $A_1 = \{h, \ell\}$  and  $A_i = \{c\}$  for  $i \neq 1$ , and where the payoff for player 1 is 1 when playing  $h$  and 0 when playing  $\ell$ . Let  $p \in S(G)$ . By Claim 1, if  $p_1(h) = 1$ , then  $S$  is a refinement of Nash.

For the remainder of this proof, suppose that  $p_1(h) < 1$ . We will show  $S$  is a refinement of  $\text{LQRE}_\lambda$  for some  $\lambda \geq 0$ . First, we show that  $S$  satisfies interiority. By anonymity, it is without loss of generality to suppose, toward a contradiction, that there is a game  $H = (B, v)$  with  $q \in S(H)$  and  $a, b \in B_1$  such that  $q_1(a) = 0 < q_1(b)$ . Let  $n > \mathbb{E}[v_1(b, q_{-1})] - \mathbb{E}[v_1(a, q_{-1})]$ , and, as previously, consider that  $r := q \times p^n \in S(H \otimes G^n)$ . However,

$$\mathbb{E}[u_1((h, \dots, h, a), r_{-1})] = n + \mathbb{E}[v_1(a, q_{-1})] > \mathbb{E}[v_1(b, q_{-1})] = \mathbb{E}[u_1((\ell, \dots, \ell, b), r_{-1})],$$

while

$$r_1(h, \dots, h, a) = (p_1(h))^n q_1(a) = 0 < r_1(\ell, \dots, \ell, b) = (p_1(\ell))^n q_1(b),$$

violating expectation-monotonicity.

We now show that  $S$  also satisfies expectation-neutrality. By anonymity, it is without loss of generality to suppose, toward a contradiction, that there is a game  $H = (B, v)$  with  $q \in S(H)$  and  $a, b \in B_1$  such that  $\mathbb{E}[v_1(a, q_{-1})] = \mathbb{E}[v_1(b, q_{-1})]$ , while  $q_1(a) < q_1(b)$ . By interiority we may let  $n$  such that  $\left(\frac{q_1(b)}{q_1(a)}\right)^n > \frac{p_1(h)}{p_1(\ell)}$ . By narrow bracketing,  $r := q^n \times p \in S(H^n \otimes G)$ . However,  $\mathbb{E}[u_1((h, a, \dots, a), r_{-1})] > \mathbb{E}[u_1((\ell, b, \dots, b), r_{-1})]$ , while

$$r_1(h, a, \dots, a) = (q_1(a))^n p_1(h) < (q_1(b))^n p_1(\ell) = r_1(\ell, b, \dots, b),$$

violating expectation-monotonicity.

Since  $S$  satisfies the hypotheses of Lemma 2, there is a  $\lambda \geq 0$ , such that for any  $i \in N$ ,  $G = (A, u)$ , and  $p \in S(G)$ ,

$$p_i(a) \propto \exp(\lambda \mathbb{E}[u_i(a, p_{-i})]).$$

□

## B Proof of Theorem 2

We first need some observations about the set of lotteries that can arise as an action's payoff distribution in finite normal form games. We use  $\Delta_{\text{finite}}$  to denote the set of all

lotteries with finitely many real-valued outcomes and note that all lotteries arising from finite normal form games belong to  $\Delta_{\text{finite}}$ . An important subset of  $\Delta_{\text{finite}}$  will be the set of lotteries with rational-valued CDFs. We denote this set by  $\Delta_{\mathbb{Q}}$ . The next lemma suggests that  $\Delta_{\mathbb{Q}}$  is rich enough to approximate any compactly supported lottery and in particular, any lottery in  $\Delta_{\text{finite}}$ .

For a compactly supported lottery  $X$  with CDF  $F$ , we denote by  $\underline{X}_n$  the lottery in  $\Delta_{\mathbb{Q}}$  with CDF  $\underline{F}_n$  defined by  $\underline{F}_n(t) = \frac{1}{2^n} \lceil 2^n \cdot F(t) \rceil$ . Likewise, denote by  $\bar{X}_n$  the lottery with CDF  $\bar{F}_n$  defined by  $\bar{F}_n(t) = \frac{1}{2^n} \lfloor 2^n \cdot F(t) \rfloor$ . Note that for all  $n$ ,

$$\underline{X}_n \leq_{\text{FOSD}} \underline{X}_{n+1} \leq_{\text{FOSD}} X \leq_{\text{FOSD}} \bar{X}_{n+1} \leq_{\text{FOSD}} \bar{X}_n, \quad (2)$$

since first-order dominance is equivalent to having a lower CDF (pointwise).

**Lemma 3.** *Let  $\Phi: \Delta_{\mathbb{Q}} \rightarrow \mathbb{R}$  be a monotone additive statistic and  $X \in \Delta_{\text{finite}}$ . Then*

$$\lim_n \Phi[\underline{X}_n] = \lim_n \Phi[\bar{X}_n]. \quad (3)$$

See §4.2 for a discussion of monotone additive statistics and related notation.

*Proof.* Let  $F$  denote the CDF of  $X$ . Since  $X \in \Delta_{\text{finite}}$ , there is a closed interval  $I \subset \mathbb{R}$  containing all the values  $X$  takes with positive probability. Denote by  $\Delta_I$  the set of all lotteries supported on  $I$ . Lotteries  $X$ ,  $\bar{X}_n$ , and  $\underline{X}_n$  belong to  $\Delta_I$ .

Define a sequence of functions  $(g_n)_n$  with  $g_n: \bar{\mathbb{R}} \rightarrow \mathbb{R}$  by  $g_n(a) = K_a[\bar{X}_n] - K_a[\underline{X}_n]$ . For any  $a \in \mathbb{R}$ , we have  $\lim_n g_n(a) = 0$ , since  $\bar{X}_n$  and  $\underline{X}_n$  converge to  $X$  weakly, and  $K_a$ , considered as a mapping  $\Delta_I \rightarrow \mathbb{R}$ , is continuous in the weak topology. Consider now  $a = \pm\infty$ . Recall that  $K_{-\infty}$  and  $K_{\infty}$  are the leftmost and the rightmost points of the support, respectively. Therefore,  $\lim_n g_n(a) = 0$  also for  $a = \pm\infty$ . Indeed,  $K_{-\infty}[\underline{X}_n] = K_{-\infty}[X]$  for all  $n$ , and  $K_{-\infty}[\bar{X}_n]$  converges to  $K_{-\infty}[X]$ , since for any  $\varepsilon > 0$  there exists  $n \geq 1$  with  $2^n \cdot F(K_{-\infty}[X] + \varepsilon) \geq 1$  and  $\bar{F}_n(K_{-\infty}[X] + \varepsilon) > 0$ . Thus  $\lim_n g_n(-\infty) = 0$ , and an analogous argument shows that  $\lim_n g_n(\infty) = 0$ .

As each  $K_a$  is monotone with respect to first-order dominance,  $g_{n+1}(a) \leq g_n(a)$  for any  $a$  and  $n$ . Thus,  $(g_n)_n$  is a decreasing sequence of continuous functions such that  $g_n(a) \rightarrow 0$  for each  $a$  from the compact set  $\bar{\mathbb{R}}$ . By Dini's theorem, monotone pointwise convergence to a continuous function on a compact set implies uniform convergence. Thus, for each  $\varepsilon > 0$ , there exists  $M \in \mathbb{N}$  such that for  $n \geq M$ ,

$$K_a[\underline{X}_n + \varepsilon] = K_a[\underline{X}_n] + \varepsilon \geq K_a[\bar{X}_n],$$

for all  $a \in \bar{\mathbb{R}}$ . By Lemma 1 of [Mu et al. \(2021\)](#),  $\Phi[\underline{X}_n] + \varepsilon \geq \Phi[\bar{X}_n]$ . Taking the limit as  $\varepsilon$  goes to zero, we conclude that  $\lim_n \Phi[\underline{X}_n] = \lim_n \Phi[\bar{X}_n]$ . □

The next remark establishes the relationship between a lottery and a random variable, which we use to construct games.

**Remark 7.** Any  $X \in \Delta_{\mathbb{Q}}$  can be represented as a random variable defined on the probability space  $(\Omega = \{1, \dots, m\}, 2^{\Omega}, \nu)$ , where  $\nu$  is the uniform distribution,  $m \in \mathbb{N}$ , and  $X: \Omega \rightarrow \mathbb{R}$ .

The following lemma shows that under the assumption that players never play first-order dominated actions, there are games where players must evaluate a rich set of lotteries against (almost) sure things. It is without loss of generality to construct a game played between player 1 and player 2, where it is implicit that other players' actions do not affect the payoffs of player 1 and player 2.

We consider a version of the game described in (1). Let  $X \in \Delta_{\mathbb{Q}}$  be a nonconstant lottery,  $r \in \mathbb{R}$ , and  $\varepsilon > 0$ . As in Remark 7, we represent  $X$  as a random variable  $X: \Omega \rightarrow \mathbb{R}$  where  $\Omega = \{1, \dots, m\}$  belongs to the probability space  $(\Omega, 2^{\Omega}, \nu)$ , and  $\nu$  is the uniform distribution on  $\Omega$ . A two-player game  $G_{r,X,\varepsilon} = (A, u)$  has action sets

$$A_2 = \Omega, \quad \text{and} \quad A_1 = \{a_r, a_X\} \times \{f: A_2 \rightarrow A_2 \mid f \text{ is a bijection}\}$$

and the payoffs are given by

$$\begin{aligned} u_1((a_X, f), a_2) &= X(f(a_2)), \\ u_1((a_r, f), a_2) &= r + \varepsilon \cdot X(f(a_2)), \\ u_2((a_r, f), a_2) &= u_2((a_X, f), a_2) = -X(f(a_2)). \end{aligned}$$

**Lemma 4.** Suppose that players never play first-order dominated actions according to a solution concept  $S$ . Then for each  $r \in \mathbb{R}$ , every nonconstant  $X \in \Delta_{\mathbb{Q}}$ , and  $\varepsilon > 0$ , the game  $G_{r,X,\varepsilon}$  has the following properties:

- $|u_1((a_r, f), a_2) - r| \leq \varepsilon \max_{\omega} |X(\omega)|$  for all  $(a_r, f) \in A_1$  and  $a_2 \in A_2$ ;
- Any  $p \in S(G_{r,X,\varepsilon})$  satisfies the following:
  - $p_2$  is the uniform distribution over  $A_2$ ;
  - The lottery  $u_1((a_X, f), p_2)$  is distributed as  $X$  for all  $(a_X, f) \in A_1$ .

*Proof of Lemma 4.* Fix a nonconstant  $X \in \Delta_{\mathbb{Q}}$ ,  $r \in \mathbb{R}$ , and  $\varepsilon > 0$ , and consider the game  $G_{r,X,\varepsilon}$ . In this game player 1 makes two choices: the first choice is between a guaranteed payoff very close to  $r$  and a payoff that depends on player 2's action.

The first bullet point is immediate. Moreover, if  $p_2$  is uniform over  $A_2$  then, for any  $f: A_2 \rightarrow A_2$ , the lottery  $u_1((a_X, f), p_2)$  is distributed as  $X$ . We thus only need to show that for any  $p \in S(G_{r,X,\varepsilon})$ ,  $p_2$  is uniform over  $A_2$ .

First note that for all  $f$  chosen by player 1,  $X \circ f$  must be weakly increasing in  $p_2$ , i.e.,  $p_2(s) > p_2(t) \implies X(f(s)) \geq X(f(t))$ .<sup>11</sup> Indeed, if, for some  $s, t \in A_2$ ,  $p_2(s) > p_2(t)$  while  $X(f(s)) < X(f(t))$ , then  $f$  is first-order dominated by  $f'$  which coincides with  $f$  except on  $\{s, t\}$ , where  $f'(s) = f(t)$  and  $f'(t) = f(s)$ .

Fix any  $f$  played with positive probability by player 1, and suppose, for the sake of contradiction, that  $p_2$  is not uniform. Since  $X$  is nonconstant, there must be  $s, t \in A_2$  with  $p_2(s) > p_2(t)$  and  $X(f(s)) > X(f(t))$ .<sup>12</sup> Moreover, for any  $f'$  played with positive probability,  $X(f'(s)) \geq X(f'(t))$ , by the weakly increasing property of  $X \circ f'$ . Thus  $s$  is first-order dominated by  $t$  for player 2, a contradiction.  $\square$

The following lemma shows that under the assumptions of distribution-neutrality, there are games where players must evaluate a rich set of lotteries against sure things.

**Lemma 5.** *Suppose  $S$  satisfies distribution-neutrality. Then for each,  $r \in \mathbb{R}$ , and  $X \in \Delta_{\mathbb{Q}}$ , there is a game  $H_{r,X} = (B, v)$  with  $B_1 = \{b_r, b_X\}$  such that for all  $p \in S(H_{r,X})$  the following properties hold:*

- *the action  $b_r$  results in deterministic payoff of  $r$  to player 1, i.e.,  $u_1(b_r, p_{-1})$  is a degenerate lottery which yields  $r$ ;*
- *$u_1(b_X, p_{-1})$  is distributed as  $X$ . Moreover, if  $S$  also satisfies narrow bracketing, then  $S(H_{r,X})$  is a singleton.*

*Proof of Lemma 5.* Fix a player  $i$  and lottery  $X \in \Delta_{\mathbb{Q}}$ . As in Remark 7, we represent  $X$  as a random variable  $X: \Omega \rightarrow \mathbb{R}$  where  $\Omega = \{1, \dots, m\}$  belongs to the probability space  $(\Omega, 2^\Omega, \nu)$ , and  $\nu$  is the uniform distribution on  $\Omega$ . We construct  $H_{r,X} = (B, v)$  by  $B_1 = \{b_r, b_X\}$ ,  $B_2 = \Omega$ ,

$$\begin{aligned} u_1(b_r, \cdot) &= r && \text{always,} \\ u_1(b_X, b_2) &= X(b_2) && \text{and, for } i \neq 1, \\ u_i &= 0 && \text{always.} \end{aligned} \tag{4}$$

Clearly  $u_1(a_r, p_{-1}) = r$ , and by distribution-neutrality,  $p_2$  is uniform, so  $u_1(a_X, p_{-1})$  is distributed as  $X$ .

<sup>11</sup>In the card game interpretation of this game, this statement means that if player 2 chooses the first card with probability strictly higher than the second card, then player 1 will order the cards so that the first card shows a higher payoff than the second.

<sup>12</sup>Indeed, fix any  $s, t \in A_2$  such that  $p_2(s) > p_2(t)$ . If  $X(f(s)) > X(f(t))$ , we are done. Suppose then that  $X(f(s)) = X(f(t)) = c$ . Since  $X \circ f$  is nonconstant, there is an  $h \in A_2$  such that  $X(f(h)) \neq c$ . Consider the case where  $X(f(h)) > c$ . Then  $p_2(h) \geq p_2(s)$ , since  $X \circ f$  is weakly increasing in  $p_2$ . We thus have  $p_2(h) > p_2(t)$  and  $X(f(h)) > X(f(t))$ , as desired. An identical argument works for the case where  $X(f(h)) < c$ .



Finally, we show that  $S(H_{r,X})$  is a singleton under the additional assumption of narrow bracketing. Indeed, let  $p, q \in S(H_{r,X})$ , so  $p \times q \in S(H_{r,X} \otimes H_{r,X})$  by narrow bracketing. Since

$$u_1(b_X, p_{-1}) + u_1(b_r, q_{-1}) = X + r = u_1(b_r, p_{-1}) + u_1(b_X, q_{-1}),$$

it must be that  $p(b_X) \cdot q(b_r) = p(b_r) \cdot q(b_X)$ . Equivalently,  $p(b_X) \cdot (1 - q(b_X)) = (1 - p(b_X)) \cdot q(b_X)$ , so  $p_1 = q_1$ . By distribution neutrality,  $p_i = q_i$  for  $i \neq 1$ , since they must both be the uniform distribution over  $B_i$ . Hence  $p = q$ , and  $S(H_{r,X})$  is a singleton.  $\square$

### B.1 Representation of Monotone Additive Statistics

Mu, Pomatto, Strack, and Tamuz (2021) provide a characterization of monotone additive statistics on the domain of all compactly supported lotteries. The following lemma shows that we can apply their characterization to monotone additive statistics on the restricted domain of lotteries that arise from finite normal form games.

**Lemma 6.** *Let  $\Phi: \Delta_{\mathbb{Q}} \rightarrow \mathbb{R}$  be a monotone additive statistic. Then*

$$\Phi[X] = \int_{\mathbb{R}} K_a[X] d\mu(a)$$

for some Borel probability measure  $\mu$  on  $\overline{\mathbb{R}}$ .

*Proof of Lemma 6.* Let  $\Phi: \Delta_{\mathbb{Q}} \rightarrow \mathbb{R}$  be a monotone additive statistic and fix a lottery  $X$  that is compactly supported. As above, we denote by  $\underline{X}_n$  the lottery in  $\Delta_{\mathbb{Q}}$  with CDF  $F_n(t) = \frac{1}{2^n} \lceil 2^n \cdot F(t) \rceil$ . Define the real-valued function  $\Psi$  on the set of compactly supported lotteries by  $\Psi[X] = \lim_{n \rightarrow \infty} \Phi[\underline{X}_n]$ . By (2),  $(\underline{X}_n)_n$  is an increasing sequence in terms of first-order dominance, so  $\Psi[X] \geq \Phi[\underline{X}_n]$  for all  $n$ . By Lemma 3, for  $X \in \Delta_{\mathbb{Q}}$ ,  $\Psi[X] = \lim_n \Phi[\underline{X}_n] = \Phi[X]$ , i.e.,  $\Psi$  extends  $\Phi$ .

Let  $X, Y$  be compactly supported lotteries. If  $X \geq_{\text{FOSD}} Y$ , then  $\underline{X}_n \geq_{\text{FOSD}} \underline{Y}_n$  for all  $n$ , so  $\Psi[X] \geq \Psi[Y]$ , and  $\Psi$  is monotone. Moreover, if  $X$  and  $Y$  are independent, then the minima and maxima of  $\underline{X}_n + \underline{Y}_n$  and  $(\underline{X} + \underline{Y})_n$  converge to those of  $X + Y$  by the same argument as in the proof of Lemma 3. Additionally, they both converge in distribution to  $X + Y$ , so  $\lim_n \Phi[\underline{X}_n + \underline{Y}_n] = \lim_n \Phi[(\underline{X} + \underline{Y})_n]$ . It thus holds that

$$\begin{aligned} \Psi[X + Y] &= \lim_{n \rightarrow \infty} \Phi[(\underline{X} + \underline{Y})_n] = \lim_{n \rightarrow \infty} \Phi[\underline{X}_n + \underline{Y}_n] \\ &= \lim_{n \rightarrow \infty} \Phi[\underline{X}_n] + \Phi[\underline{Y}_n] = \Psi[X] + \Psi[Y]. \end{aligned}$$

Hence, by the characterization of Mu et al. (2021),  $\Phi[X] = \Psi[X] = \int_{\mathbb{R}} K_a[X] d\mu(a)$  for some Borel probability measure  $\mu$  on  $\overline{\mathbb{R}}$ .  $\square$

## B.2 Additivity for all Lotteries

**Lemma 7.** *Let  $\Phi: \Delta_{\mathbb{Q}} \rightarrow \mathbb{R}$  be a monotone statistic. If  $\Phi$  is additive for all lotteries, then  $\Phi$  is the expectation.*

*Proof of Lemma 7.* We consider each finite probability space  $(\Omega = \{1, \dots, m\}, 2^{\Omega}, \nu)$ , where  $\nu$  is the uniform distribution on  $\Omega$ . Each  $X \in \Delta_{\mathbb{Q}}$  can be represented as a random variable  $X: \Omega \rightarrow \mathbb{R}$  for a large enough  $m$ . We require that  $\Phi[X + Y] = \Phi[X] + \Phi[Y]$  for any random variables  $X, Y \in \mathbb{R}^{\Omega}$ .

For  $X: \Omega \rightarrow \mathbb{R}$ , we may write  $X = \sum_{\omega} I(\omega)X(\omega)$ , where  $I$  denotes the indicator function. For each  $\omega \in \Omega$ , define  $f_{\omega}: \mathbb{R} \rightarrow \mathbb{R}$  by  $f_{\omega}(x) = \Phi[I(\omega) \cdot x]$ , so  $\Phi[X] = \sum_{\omega} f_{\omega}(X(\omega))$ . It follows that each  $f_{\omega}$  is a monotone additive function and is therefore linear. Thus there is  $Z \in \mathbb{R}^{\Omega}$  such that  $\Phi[X] = Z \cdot X$  for all  $X$ . Since  $\Phi$  only depends on the distribution of  $X$  and  $\mu$  is uniform,  $\Phi[X] = \Phi[X \circ \pi]$  for any permutation  $\pi: \Omega \rightarrow \Omega$ , which is only possible for constant  $Z$ . Finally, since  $\Phi$  is a statistic, it maps any constant random variable to its value, so  $Z(\omega) = \frac{1}{m}$  for all  $\omega$ . We have thus shown that for any  $X \in \mathbb{R}^{\Omega}$ ,  $\Phi[X] = \frac{1}{|\Omega|} \sum_{\omega} X(\omega) = \mathbb{E}[X]$ .  $\square$

The remainder of the proof of Theorem 2 relies on the existence of a monotone additive statistic  $\Phi$  that players respond to (Theorem 3). Theorem 3 is proved in the subsequent section.

*Proof of Theorem 2.* Let  $X, Y \in \Delta_{\mathbb{Q}}$ . As previously, we represent  $X$  and  $Y$  as random variables on  $(\Omega = \{1, \dots, m\}, 2^{\Omega}, \nu)$ , where  $\nu$  is uniform. We consider the two families of SREs characterized by Theorem 3. First, we consider  $S$  that is a refinement of a Nash $_{\Phi}$  equilibrium.

Let  $r < \Phi[X]$  and  $\varepsilon \in \left(0, \frac{\Phi[X] - r}{\max |X| + 1}\right)$ , and let  $G_{r,X,\varepsilon} = (A, u)$  be the game defined in (1). We will consider the probability that player 1 chooses the (almost) sure things  $(a_r, \cdot)$ . Note that Lemma 4 applies since players never play first-order dominated actions in any Nash $_{\Phi}$  equilibrium. We will show that  $\Phi$  is additive for all lotteries (rather than only independent ones).

We first show super-additivity. Let  $p \in S(G_{r,X,\varepsilon})$ . Since  $p$  is a Nash $_{\Phi}$  equilibrium and

$$\Phi[r + \varepsilon X] \leq r + \Phi[\varepsilon \max |X|] \leq r + \Phi\left[(\Phi[X] - r) \frac{\max |X|}{\max |X| + 1}\right] < r + \Phi[X] - r = \Phi[X],$$

$p_1(a_r, f) = 0$  for all  $f$ , and  $p_1(a_X, g) > 0$  for some  $g$ . Consider the game  $(A, v)$ , where  $v_1(a_1, a_2) = u_1(a_1, a_2) + Y(g(a_2))$  for each  $(a_1, a_2) \in A_1 \times A_2$ , and  $v_i = u_i$  for  $i \neq 1$ . Let  $p \in S(A, v)$ . By strategic invariance  $p \in S(G_{r,X,\varepsilon})$ , and by Lemma 4,  $p_2$  is the uniform distribution.

Note that  $v_1((a_X, g), p_2)$  is distributed as  $X + Y$ , and  $v_1((a_r, g), p_2)$  is distributed as  $r + Y + \varepsilon X$ . Since  $p_1((a_r, g)) = 0$  and  $p_1((a_X, g)) > 0$ , it must be that  $\Phi[r + Y + \varepsilon X] = r + \Phi[Y + \varepsilon X] \leq \Phi[X + Y]$ . Taking  $\varepsilon \rightarrow 0$ , we see that  $r + \Phi[Y] \leq \Phi[X + Y]$ .<sup>13</sup> Since  $r < \Phi[X]$  was arbitrary, it follows that  $\Phi[X] + \Phi[Y] \leq \Phi[X + Y]$ .

We next show sub-additivity. Let  $r > \Phi[X]$  and  $\varepsilon \in \left(0, \frac{r - \Phi[X]}{\max |X| + 1}\right)$ , and let  $G_{r, X, \varepsilon} = (A, u)$  be the game defined in (1). Let  $p \in S(G_{r, X, \varepsilon})$ . Since  $p$  is a Nash $_{\Phi}$  equilibrium and

$$\begin{aligned} \Phi[r + \varepsilon X] &= r + \Phi[\varepsilon X] \geq r + \Phi[-\varepsilon \max |X|] \\ &\geq r + \Phi\left[-(r - \Phi[X]) \frac{\max |X|}{\max |X| + 1}\right] > r - r + \Phi[X] = \Phi[X], \end{aligned}$$

$p_1(a_X, f) = 0$  for all  $f$ , and  $p_1(a_r, g) > 0$  for some  $g$ . Consider the game  $(A, v)$ , where  $v_1(a_1, a_2) = u_1(a_1, a_2) + Y(g(a_2))$  for each  $(a_1, a_2) \in A_1 \times A_2$ , and  $v_i = u_i$  for  $i \neq 1$ . By strategic invariance,  $p \in S(A, v)$  and by Lemma 4,  $p_2$  is the uniform distribution.

Note that  $v_1((a_X, g), p_2)$  is distributed as  $X + Y$ , and  $v_1((a_r, g), p_2)$  is distributed as  $r + Y + \varepsilon X$ . Since  $p_1((a_X, g)) = 0$  and  $p_1(a_r, g) > 0$ , it must be that  $\Phi[r + Y + \varepsilon X] = r + \Phi[Y + \varepsilon X] \geq \Phi[X + Y]$ . Then by an analogous argument for sub-additivity, we have  $\Phi[X] + \Phi[Y] \geq \Phi[X + Y]$ . Thus  $\Phi[X + Y] = \Phi[X] + \Phi[Y]$ . By Lemma 7,  $\Phi$  is the expectation on  $\Delta_{\mathbb{Q}}$ , so by Lemma 6,  $\Phi$  is the expectation.

We next consider the case where  $S$  is a refinement of some  $\text{LQRE}_{\lambda\Phi}$  equilibrium. If  $\lambda = 0$ , the result holds trivially. Consider then  $\lambda > 0$ . Note we can apply Lemma 5 since  $\text{LQRE}_{\lambda\Phi}$  equilibrium satisfies distribution-neutrality. Given the game  $H_{r, X} = (B, v)$  defined in (4) with  $r = \Phi[X]$ , we define a new game  $(B, w)$  by  $w_1(b_1, b_2) = v_1(b_1, b_2) + Y(a_2)$ , and  $w_i = v_i$  for  $i \neq 1$ . Let  $p \in S(B, w)$ . By strategic invariance  $p_2 \in S(B, v)$ , so  $p_2$  is uniform by Lemma 5 and  $w_1(b_r, p_2)$  is distributed as  $r + Y$ , while  $w_1(b_X, p_2)$  is distributed as  $X + Y$ .

Since  $r = \Phi[X]$  and  $p$  is an  $\text{LQRE}_{\lambda\Phi}$  equilibrium, we have

$$p_1(b_r) = p_1(b_X) \Rightarrow \Phi[X + Y] = \Phi[r + Y] = \Phi[Y] + r = \Phi[X] + \Phi[Y].$$

Thus  $\Phi$  is additive for all lotteries and is therefore the expectation by Lemma 7.  $\square$

## C Proof of Proposition 1

We show the two parts of the proposition in two separate claims: Claim 2 and Claim 3.

Given  $\lambda \geq 0$  and  $\Phi = \int K_t d\mu(t)$ , we prove in Claim 2 that every game  $G = (A, u)$  has an  $\text{LQRE}_{\lambda\Phi}$  equilibrium. A natural approach would be to follow the proof of the existence of

<sup>13</sup>Note that  $\Phi[Y] + \varepsilon \min X = \Phi[Y + \varepsilon \min X] \leq \Phi[Y + \varepsilon X] \leq \Phi[Y + \varepsilon \max X] = \Phi[Y] + \varepsilon \max X$ , so  $\lim_{\varepsilon \rightarrow 0} \Phi[Y + \varepsilon X] = \Phi[Y]$ .

quantal response equilibria by defining the quantal response operator  $T: \prod_i \Delta A_i \rightarrow \prod_i \Delta A_i$  by  $T_i(p)(a_i) \propto \exp(\lambda \Phi[u_i(a_i, p_{-i})])$  and applying a fixed point theorem. The issue is that when  $\mu$  has positive mass at  $t = \infty$  or  $t = -\infty$ , then  $T$  is not continuous, and so we cannot apply Brouwer's fixed point theorem.

To overcome this issue, we define a closely related, continuous operator  $T'$ , apply the fixed point theorem to it, and then show that this is also a fixed point of  $T$ , and hence an equilibrium. To define  $T'$ , fix  $i \in N$ ,  $a_i \in A_i$  and  $p \in \prod_i \Delta A_i$ . Let  $L: \mathbb{R} \times A_i \times \prod_{j \neq i} \Delta(A_j) \rightarrow \mathbb{R}$  be given by

$$L(t, a_i, p_{-i}) = \begin{cases} K_t[u_i(a_i, p_{-i})] & \text{if } t \in \mathbb{R} \\ \min_{a_{-i}} u_i(a_i, a_{-i}) & \text{if } t = -\infty \\ \max_{a_{-i}} u_i(a_i, a_{-i}) & \text{if } t = +\infty. \end{cases} \quad (5)$$

That is, when  $t \in \mathbb{R}$ ,  $L(t, a_i, p_{-i})$  is equal to the monotone additive statistic  $K_t$ , evaluated on the lottery that player  $i$  gets when playing  $a_i$  and when the rest of the players play  $p_{-i}$ . For  $t \in \{-\infty, +\infty\}$ ,  $L(t, a_i, p_{-i})$  is independent of  $p_{-i}$ , and returns the minimum or maximum payoff that the action  $a_i$  can yield. Let  $\Phi'[a_i, p_{-i}] = \int_{\mathbb{R}} L(t, a_i, p_{-i}) d\mu(t)$ . Note that if  $p_{-i}$  is totally mixed then  $\Phi'[a_i, p_{-i}] = \Phi[u_i(a_i, p_{-i})]$ . Note also that if  $\mu(\{-\infty, +\infty\}) = 0$  then  $\Phi'[a_i, p_{-i}] = \Phi[u_i(a_i, p_{-i})]$  for all  $p_{-i}$ .

Define  $T': \prod_i \Delta A_i \rightarrow \prod_i \Delta A_i$  by

$$T'_i(p)(a_i) \propto \exp(\lambda \Phi'[a_i, p_{-i}]). \quad (6)$$

To prove the existence of  $\text{LQRE}_{\lambda\Phi}$  equilibria, we show that  $T'$  is continuous, and that its fixed points coincide with those of  $T$ , and hence are equilibria.

**Claim 2.** *Let  $\Phi = \int K_a d\mu(a)$  be a monotone additive statistic. Then there is an  $\text{LQRE}_{\lambda\Phi}$  equilibrium for every game, for every  $\lambda \geq 0$ .*

*Proof.* Fix  $\lambda \geq 0$ ,  $\Phi = \int K_a d\mu(a)$ , and let  $G = (A, u)$ . Define  $L$  and  $T'$  as in (5) and (6).

For  $t \in \mathbb{R}$ , the map  $p_{-i} \mapsto L(t, a_i, p_{-i})$  varies continuously in  $p_{-i}$ . It is also (trivially) continuous when  $t \in \{-\infty, +\infty\}$ , since then it does not depend on  $p_{-i}$ . It follows that  $L(t, a_i, p_{-i})$  is continuous in  $p_{-i}$  for all  $t \in \mathbb{R}$  and  $a_i \in A_i$ .

We show that  $T'$  has a fixed-point. Since  $L(t, a_i, p_{-i})$  is continuous in  $p$  for all  $t \in \mathbb{R}$  and

$$|L(t, a_i, p_{-i})| \leq \max_{a_{-i}} |u_i(a_i, a_{-i})|,$$

by the dominated convergence theorem and the continuity of  $L$ ,

$$\begin{aligned} \lim_{p_n \rightarrow p} \Phi'[a_i, (p_n)_{-i}] &= \lim_{p_n \rightarrow p} \int L(t, a_i, (p_n)_{-i}) d\mu(t) \\ &= \int \lim_{p_n \rightarrow p} L(t, a_i, (p_n)_{-i}) d\mu(t) = \int L(t, a_i, p_{-i}) d\mu(t) = \Phi'[a_i, p_{-i}], \end{aligned}$$

hence,  $T'$  is continuous in  $p$ . Since  $\prod_i \Delta A_i$  is convex and compact,  $T'$  has a fixed-point  $q^*$  by Brouwer's fixed-point theorem. Since  $T'$  maps every mixed strategy profile to a totally mixed strategy profile,  $q^*$  must be totally mixed. We thus have

$$q_i^*(a_i) \propto \exp(\lambda \Phi'[a_i, q_{-i}^*]) = \exp(\lambda \Phi[u_i(a_i, q_{-i}^*)]),$$

as  $\Phi$  and  $\Phi'$  agree when  $i$ 's opponents play totally mixed strategy profiles. Hence we have shown that an  $\text{LQRE}_{\lambda\Phi}$  equilibrium exists.  $\square$

Next, we show the second part of the proposition:

**Claim 3.** *Let  $\Phi = \int K_a d\mu(a)$  be a monotone additive statistic. Then there is a  $\text{Nash}_\Phi$  equilibrium for every game if and only if  $\mu(\{-\infty, +\infty\}) = 0$ .*

*Proof.* The existence of  $\text{Nash}_\Phi$  equilibria for  $\mu(\{-\infty, +\infty\}) = 0$  follows from Kakutani's fixed-point theorem since the best response correspondence is upper hemicontinuous for such  $\mu$ . Alternatively, we can show the existence of  $\text{Nash}_\Phi$  as a limit point of  $\text{LQRE}_{\lambda\Phi}$ .

Next, we demonstrate how to construct a game with no  $\text{Nash}_\Phi$  equilibrium when  $\mu$  places a positive weight on the minimum or maximum. For such a  $\Phi$ , let  $\varepsilon = \mu(-\infty) + \mu(+\infty)$  and consider the game in table 3.

	$a_2$	$b_2$
$a_1$	$(1 + \frac{1}{\varepsilon}, 0)$	$(0, 1)$
$b_1$	$(-\frac{1}{\varepsilon}, 1)$	$(1, 0)$

Table 3: Variant of matching pennies for which extremal  $\text{Nash}_\Phi$  equilibria do not exist.

Since pure  $\text{Nash}_\Phi$  equilibria coincide with pure Nash equilibria for all  $\Phi$ , it is easy to see that the game has no pure equilibria. Likewise, there are no equilibria where either player plays a pure strategy, since the best responses to pure strategies in this game are pure for all  $\Phi$ . In particular, any supposed  $\text{Nash}_\Phi$  equilibrium  $q$  would have player 2 playing a totally mixed strategy. We thus have

$$\begin{aligned} \Phi[u_1(a_1, q_2)] - \Phi[u_1(b_1, q_2)] &= (\mu(-\infty) + \mu(+\infty)) \cdot \frac{1}{\varepsilon} \\ &+ \underbrace{\int_{\mathbb{R}} K_t[u_1(a_1, q_2)] d\mu(t)}_{\text{nonnegative}} - \underbrace{\int_{\mathbb{R}} K_t[u_1(b_1, q_2)] d\mu(t)}_{\leq 1} \geq 1 - \mu(\mathbb{R}) \cdot 1 = \varepsilon > 0. \end{aligned}$$

This contradicts the assumption that  $q$  is a totally mixed  $\text{Nash}_\Phi$  equilibrium, which would require that  $\Phi[u_1(a_1, q_2)] = \Phi[u_1(b_1, q_2)]$ .  $\square$

## D Proof of Theorem 3

Recall the definition of the game  $G_{r,X,\varepsilon}$  from (1). The following lemma shows how certainty equivalents may be deduced from player 1's mixing probabilities in this game under the assumption that players never play first-order dominated actions. We will refer to  $X \in \Delta_{\mathbb{Q}}$  with the understanding that  $X$  is a random variable as in Remark 7, so that  $G_{r,X,\varepsilon}$  is a well-defined game.

**Lemma 8.** *Suppose  $S$  satisfies narrow bracketing, distribution-monotonicity, and players never play first-order dominated actions according to  $S$ . Define  $\Phi_{\varepsilon}: \Delta_{\mathbb{Q}} \rightarrow \mathbb{R}$  by*

$$\Phi_{\varepsilon}[X] = \sup\{r \in \mathbb{R} \mid \exists p \in S(G_{r,X,\varepsilon}), \exists f: A_2 \rightarrow A_2 \text{ with } p_1(a_X, f) > 0\}. \quad (7)$$

*Then the limit  $\Phi[X] = \lim_{\varepsilon \rightarrow 0} \Phi_{\varepsilon}[X]$  exists and is a monotone additive statistic.*

*Proof.* We first show that the limit exists. Suppose, for the sake of contradiction that  $\liminf_{\varepsilon \rightarrow 0} \Phi_{\varepsilon}[X] < \limsup_{\varepsilon \rightarrow 0} \Phi_{\varepsilon}[X]$  for some  $X \in \Delta_{\mathbb{Q}}$ . There then exist  $\delta > 0$  and  $c > 0$  such that for any  $\varepsilon < \delta$ , there are  $\varepsilon_1, \varepsilon_2 < \varepsilon$  with  $\Phi_{\varepsilon_1}[X] + c < \Phi_{\varepsilon_2}[X]$ . Thus, there exist  $(A, u) = G_{r_1, X, \varepsilon_1}$  and  $(B, v) = G_{r_2, X, \varepsilon_2}$  with  $r_1 + c < r_2$ ,  $p \in S(A, u)$ ,  $q \in S(B, v)$ , such that  $p_1(a_X, f) = 0$  for all  $f$  and  $q_1(b_X, g) > 0$  for some  $g$ . Since  $u_1((a_{r_1}, f), p_{-1}) + v_1((b_X, g), q_{-1}) = r_1 + \varepsilon_1 X + X$  and  $u_1((a_X, f), p_{-1}) + v_1((b_{r_2}, g), q_{-1}) = X + r_2 + \varepsilon_2 X$ , for  $\varepsilon$  small enough,

$$u_1((a_{r_1}, f), p_{-1}) + v_1((b_X, g), q_{-1}) <_{\text{FOSD}} u_1((a_X, f), p_{-1}) + v_1((b_{r_2}, g), q_{-1}).$$

However, since  $p_1(a_X, f) = 0$  for all  $f$ , there must be  $f_0$  so that  $p_1(a_{r_1}, f_0) > 0$ . By narrow bracketing,  $p \times q \in S((A, u) \otimes (B, v))$ . This violates distribution-monotonicity, since  $p_1(a_{r_1}, f_0) \cdot q_1(b_X, g) > 0$ , while  $p_1(a_X, f_0) \cdot q_1(b_{r_2}, g) = 0$ . Hence, the limit  $\Phi[X]$  exists.

We next show  $\Phi$  is a monotone additive statistic. Since players never play first-order dominated actions, it is immediate that, for  $c \in \mathbb{R}$ ,  $\Phi_{\varepsilon}[c] = (1 - \varepsilon)c$ , so  $\Phi[c] = c$ ; i.e.,  $\Phi$  is a statistic.

We need to show that  $\Phi$  is additive for independent variables. We will show subadditivity; superadditivity follows an identical argument. Let  $X, Y \in \Delta_{\mathbb{Q}}$  be independent, and let  $r > \Phi[X], s > \Phi[Y]$ . Fix  $t > r + s$  and let  $(A, u) = G_{r, X, \varepsilon}$ ,  $(B, v) = G_{s, Y, \varepsilon}$ , and  $(C, w) = G_{t, X+Y, \varepsilon}$ . Fix  $o \in S(A, u)$ ,  $p \in S(B, v)$ , and  $q \in S(C, w)$ . By narrow bracketing,  $o \times p \times q \in S((A, u) \otimes (B, v) \otimes (C, w))$ . Note that for any  $f, g$  and  $h$ ,

$$\begin{aligned} u_1((a_r, f), o_{-1}) + u_1((a_s, g), p_{-1}) + u_1((a_{X+Y}, h), q_{-1}) &= X + Y + \varepsilon(X + Y) + r + s \\ u_1((a_X, f), o_{-1}) + u_1((a_Y, g), p_{-1}) + u_1((a_t, h), q_{-1}) &= X + Y + \varepsilon(X + Y) + t, \end{aligned}$$

where the latter expression first-order dominates the former one. By distribution-monotonicity, we must have

$$o_1(a_r, f) \cdot p_1(a_s, g) \cdot q_1(a_{X+Y}, h) \leq o_1(a_X, f) \cdot p_1(a_Y, g) \cdot q_1(a_t, h).$$

From the definition of  $\Phi$ , for  $\varepsilon$  small enough  $o_1(a_X, f) = p_1(a_Y, g) = 0$ . Since the above inequality must hold for all  $f$  and  $g$ , we see that  $q_1(a_{X+Y}, h) = 0$ . Since we can choose  $r, s$  and  $t$  so that  $t$  is arbitrarily close to  $\Phi[X] + \Phi[Y]$ , it follows that  $\Phi[X + Y] \leq \Phi[X] + \Phi[Y]$ .

We next show that  $\Phi$  is monotone with respect to first-order stochastic dominance. Let  $\varepsilon > 0$ ,  $X, Y \in \Delta_{\mathbb{Q}}$  with  $X >_{\text{FOSD}} Y$ , and fix  $r < s$ . Let  $(A, u) = G_{r, X, \varepsilon}$ ,  $(B, v) = G_{s, Y, \varepsilon}$ , and fix  $p \in S(A, u)$  and  $q \in S(B, v)$ . Note that, for all  $f$  and  $g$  and for all  $\varepsilon$  small enough, since  $Y <_{\text{FOSD}} X$  and  $r < s$ ,

$$\begin{aligned} u_1((a_r, f), p_{-1}) + v_1((b_Y, g), q_{-1}) &= Y + \varepsilon X + r \\ &<_{\text{FOSD}} X + \varepsilon Y + s \\ &= u_1((a_X, f), p_{-1}) + v_1((b_s, g), q_{-1}). \end{aligned}$$

Thus, by distribution-monotonicity,  $p_1(a_r, f) \cdot q_1(b_Y, g) \leq p_1(a_X, f) \cdot q_1(b_s, g)$ . Hence, if  $p_1(a_X, f) = 0$  for all  $f$ , it must be that  $q_1(b_Y, g) = 0$  for all  $g$ . Recalling the definition of  $\Phi_\varepsilon$  in (7), it follows that for any  $r < s$ , there is a  $\delta > 0$  such that for all  $\varepsilon < \delta$ , if  $\Phi_\varepsilon[X] < r$  then  $\Phi_\varepsilon[Y] \leq s$ . Hence,  $\Phi[X] \geq \Phi[Y]$ .

Finally, if  $X$  and  $Y$  have the same distribution then, for any  $\varepsilon > 0$ ,  $\Phi[Y] - \varepsilon = \Phi[Y - \varepsilon] \leq \Phi[X] \leq \Phi[Y + \varepsilon] = \Phi[Y] + \varepsilon$ , so  $\Phi[X] = \Phi[Y]$ . □

**Lemma 9.** *Suppose a solution concept  $S$  satisfies distribution-neutrality, interiority, narrow bracketing, distribution-monotonicity, and anonymity. Then there is a monotone additive statistic  $\Phi$  such that for all  $i, G = (A, u), p \in S(G), a_i \in A_i$ ,*

$$p_i(a_i) \propto \exp(\lambda \Phi[u_i(a_i, p_{-i})])$$

for some  $\lambda \geq 0$ .

*Proof of Lemma 9.* For each  $X \in \Delta_{\mathbb{Q}}$ , let  $H_{0, X} = (B, v)$  be the game defined in (4), where  $r = 0$ . Since  $S$  satisfies narrow bracketing, by Lemma 5,  $S(H_{0, X})$  is a singleton. Let  $p \in S(H_{0, X})$ . Define  $f: \Delta_{\mathbb{Q}} \rightarrow \mathbb{R}$  by

$$f(X) := \ln \left( \frac{p_1(b_X)}{1 - p_1(b_X)} \right),$$

which is well-defined by interiority.

Let  $X, Y \in \Delta_{\mathbb{Q}}$  be independent lotteries and let  $o \in S(H_{0,X}), p \in S(H_{0,Y})$ , and  $q \in S(H_{0,X+Y})$ . By narrow bracketing,  $o \times p \times q \in S(H_{0,X} \otimes H_{0,Y} \otimes H_{0,X+Y})$ . By distribution-neutrality,  $(1 - o_1(b_X)) \cdot (1 - p_1(b_Y)) \cdot q_1(b_{X+Y}) = o_1(b_X) \cdot p_1(b_Y) \cdot (1 - q_1(b_{X+Y}))$ . Rearranging gives

$$\frac{q_1(b_{X+Y})}{1 - q_1(b_{X+Y})} = \frac{o_1(b_X)}{1 - o_1(b_X)} \cdot \frac{p_1(b_Y)}{1 - p_1(b_Y)}.$$

Taking logs, we have  $f(X + Y) = f(X) + f(Y)$ ; i.e.,  $f$  is additive for independent lotteries.

Since  $S$  satisfies narrow bracketing and distribution-monotonicity, for  $X >_{\text{FOSD}} Y$ ,  $p \in S(H_{0,X}), q \in S(H_{0,Y})$ , we must have  $p_1(b_X) \cdot q_1(b_0) \geq q_1(b_Y) \cdot p_1(b_0)$ ; i.e.,  $p_1(b_X) \geq p_1(b_Y)$ . Hence,  $f$  is non-decreasing in first-order dominance. Finally, define  $g: \mathbb{R} \rightarrow \mathbb{R}$  by  $g(x) = f(x)$  for deterministic lotteries yielding  $x$  for sure. Then  $g(x + y) = f(x + y) = f(x) + f(y) = g(x) + g(y)$ . Since  $f$  is monotone,  $g$  must be monotone, so there is a  $\lambda \in [0, \infty)$  such that  $f(x) = g(x) = \lambda x$  for all  $x \in \mathbb{R}$ . Hence,  $f$  is a scaled monotone additive statistic, i.e.,  $f(X) = \lambda \Phi[X]$ , for some monotone additive statistic  $\Phi$ .

Fix any  $G = (A, u), o \in S(G)$  with  $a, a' \in A_1$ . Let  $X = u_1(a, o_{-1})$  and  $Y = u_1(a', o_{-1})$ . Fix  $n \geq 1$ , and let  $(B, v) = H_{0, \bar{X}_n}, (C, w) = H_{0, \underline{Y}_n}$ , and let  $p \in S(H_{0, \bar{X}_n})$  and  $q \in S(H_{0, \underline{Y}_n})$ . By narrow bracketing,  $o \times p \times q \in S(G \otimes H_{0, \bar{X}_n} \otimes H_{0, \underline{Y}_n})$ . By (2),

$$\begin{aligned} u_1(a, o_{-1}) + v_1(b_0, p_{-1}) + w_1(c_{\underline{Y}_n}, q_{-1}) &= \\ X + \underline{Y}_n &<_{\text{FOSD}} \bar{X}_n + Y = \\ u_1(a', o_{-1}) + v_1(b_{\bar{X}_n}, p_{-1}) + w_1(c_0, q_{-1}). \end{aligned}$$

Thus, by distribution-monotonicity,

$$o_1(a) \cdot (1 - p_1(b_{\bar{X}_n})) \cdot q_1(c_{\underline{Y}_n}) \leq o_1(a') \cdot p_1(b_{\bar{X}_n}) \cdot (1 - q_1(c_{\underline{Y}_n})).$$

By interiority, we have

$$\frac{o_1(a)}{o_1(a')} \leq \frac{p_1(b_{\bar{X}_n})}{1 - p_1(b_{\bar{X}_n})} \cdot \frac{1 - q_1(c_{\underline{Y}_n})}{q_1(c_{\underline{Y}_n})} = \frac{\exp f(\bar{X}_n)}{\exp f(\underline{Y}_n)} = \frac{\exp(\lambda \Phi[\bar{X}_n])}{\exp(\lambda \Phi[\underline{Y}_n])}.$$

By a symmetric argument,

$$\frac{\exp(\lambda \Phi[\underline{X}_n])}{\exp(\lambda \Phi[\bar{Y}_n])} \leq \frac{o_1(a)}{o_1(a')}.$$

Since these inequalities hold for all  $n \geq 1$ , by Lemma 3,

$$\frac{o_1(a)}{o_1(a')} = \frac{\exp(\lambda \Phi[X])}{\exp(\lambda \Phi[Y])}.$$

Since  $a'$  was an arbitrary element of  $A_1$ ,

$$o_1(a) \propto \exp(\lambda \Phi[u_1(a, o_{-1})]).$$



By anonymity, this holds for all  $i$ .

□

*Proof of Theorem 3.* Since  $S$  satisfies the assumptions of Proposition 2 proved below, either players never play first-order dominated actions, or  $S$  satisfies interiority and distribution-neutrality. In the latter case,  $S$  is a refinement of an  $\text{LQRE}_{\lambda\Phi}$  by Lemma 9.

We show that if players never play first-order dominated actions then  $S$  is a refinement of a  $\text{Nash}_\Phi$  equilibrium, concluding the proof of Theorem 3. Define the monotone additive statistic  $\Phi: \Delta_{\text{finite}} \rightarrow \mathbb{R}$  using Lemma 8 to first define it on  $\Delta_{\mathbb{Q}}$  and then applying the representation of Lemma 6 to obtain a monotone additive statistic on  $\Delta_{\text{finite}}$ .

We will show that in any game only maximizers of  $\Phi$  can be played with positive probability. Let  $G = (A, u)$ , with  $a, b \in A_1, o \in S(G)$ , and  $\Phi[u_1(a, o_{-1})] > \Phi[u_1(b, o_{-1})]$ . Let  $X = u_1(a, o_{-1}), Y = u_1(b, o_{-1})$  so that  $\Phi[Y] < \Phi[X]$ , and denote  $a_X = a$  and  $a_Y = b$ .

By Lemma 3, there is an  $n \geq 1$ , such that  $\Phi[\bar{Y}_n] < \Phi[X_n]$ . Fix  $r, s$  with  $\Phi[\bar{Y}_n] < r < s < \Phi[X_n]$ .

Define

$$\Phi_\varepsilon[X] = \sup\{r \in \mathbb{R} \mid \exists p \in S(G_{r, X, \varepsilon}), \exists f: A_2 \rightarrow A_2 \text{ with } p_1(a_X, f) > 0\},$$

as in Lemma 8. By that lemma, there exists  $\delta > 0$  such that for all  $\varepsilon < \delta$ ,  $\Phi_\varepsilon[\bar{Y}_n] < r < s < \Phi_\varepsilon[X_n]$ . Fix  $\varepsilon < \delta$ , so that by the definition of  $\Phi_\varepsilon$  there is a  $t \in [s, \Phi_\varepsilon[X_n]]$ , a  $p \in S(G_{t, X_n, \varepsilon})$  and an  $f$  such that  $p_1(b_{X_n}, f) > 0$ . Let  $(B, v) = G_{t, X_n, \varepsilon}$  and  $(C, w) = G_{r, \bar{Y}_n, \varepsilon}$ . Let  $q \in S(G_{r, \bar{Y}_n, \varepsilon})$ , and note that for all  $g$ ,  $q_1(c_{\bar{Y}_n}, g) = 0$ . For all  $\varepsilon$  small enough,

$$\begin{aligned} & u_1(a_Y, o_{-1}) + v_1((b_{X_n}, f), p_{-1}) + w_1((c_r, g), q_{-1}) \\ &= Y + X_n + r + \varepsilon \cdot \bar{Y}_n \\ &<_{\text{FOSD}} X + t + \varepsilon \cdot X_n + \bar{Y}_n \\ &= u_1(a_X, o_{-1}) + v_1((b_t, f), p_{-1}) + w_1((c_{\bar{Y}_n}, g), q_{-1}), \end{aligned}$$

by (2) and the fact that  $r < t$ . By narrow bracketing,  $o \times p \times q \in S((A, u) \otimes (B, v) \otimes (C, w))$ . Thus, by distribution-monotonicity,

$$o_1(a_Y) \cdot p_1(b_{X_n}, f) \cdot q_1(c_r, g) \leq o_1(a_X) \cdot p_1(b_s, f) \cdot q_1(c_{\bar{Y}_n}, g).$$

The right-hand side is zero, since  $q_1(c_{\bar{Y}_n}, g) = 0$  for all  $g$ . There is therefore a  $g_0$  with  $q_1(c_r, g_0) > 0$ . Since  $p_1(b_{X_n}, f) > 0$ , it must be that  $o_1(a_Y) = 0$ , demonstrating that only maximizers of  $\Phi$  can be played with positive probability.

Since  $S$  satisfies anonymity, we have now shown that if this case holds, there is a monotone additive statistic  $\Phi$  such that for all games  $G$ ,  $p \in S(G)$ , players  $i$ , and  $a_i \in A_i$ ,

$$\text{supp } p_i \subseteq \arg \max_a \Phi[u_i(a, p_{-i})].$$

□

## E Proof of Proposition 2

Consider a solution concept  $S$  satisfying distribution-monotonicity, narrow bracketing, and anonymity. For such  $S$ , Proposition 2 claims that players never play strictly first-order dominated actions, or the solution concept satisfies interiority and distribution-neutrality. We prove this proposition by showing the following dichotomy:

1. If players never play strictly dominated actions, then they never play strictly first-order dominated actions.
2. If players play a strictly dominated action in some game, then the solution concept satisfies interiority and distribution-neutrality.

We consider each of the two cases separately in Claims 4 and 5 below. The proposition follows directly from them. To prove Claim 4, we will need a technical lemma about stochastic dominance in large numbers. We identify a lottery  $X$  with a distribution over  $\mathbb{R}$ . We write  $X + Y$  for a lottery, corresponding to the sum of the outcomes independently sampled from  $X$  and  $Y$ , and denote by  $X^m$  the sum of  $m$  independent copies of  $X$ .

**Definition 15.** *Let  $X$  and  $Y$  be compactly supported lotteries. Say  $X$  dominates  $Y$  in large numbers, denoted  $X >_L Y$ , if there exists  $M \in \mathbb{N}$  such that for all  $m \geq M$ ,  $X^m >_{\text{FOSD}} Y^m$ .*

The following lemma is due to [Aubrun and Nechita \(2007\)](#):

**Lemma 10.** *Let  $X$  and  $Y$  be compactly supported lotteries with  $K_a[X] > K_a[Y]$  for all  $a \in \mathbb{R}$ . Then  $X >_L Y$ .*

The next lemma provides a condition under which adding independent lotteries to lotteries ranked with respect to first-order stochastic dominance preserves the dominance ranking.

**Lemma 11.** *Let  $X, Y, A$ , and  $B$  be compactly supported lotteries with  $X >_{\text{FOSD}} Y$ ,  $\max(A) > \max(B)$  and  $\min(A) > \min(B)$ . Then there exist  $m, n \in \mathbb{N}$  such that*

$$X^m + A^n >_{\text{FOSD}} Y^m + B^n.$$

Here and below, by writing  $Z + W$  we mean a lottery obtained by summing outcomes of two independent lotteries  $Z$  and  $W$ .

*Proof.* Since  $X >_{\text{FOSD}} Y$ , we have  $K_a[X] > K_a[Y]$  for all  $a \in \mathbb{R}$  and  $K_a[X] \geq K_a[Y]$  for  $a = \pm\infty$ . Moreover,  $K_a[A] > K_a[B]$  for  $a = \pm\infty$ . By continuity of  $K_a$  in  $a$ , there exists  $M > 0$  such that  $K_a[A] > K_a[B]$  for all  $a \in \mathbb{R} \setminus [-M, M]$ . Consider  $t = \min_{a \in [-M, M]} (K_a[X] - K_a[Y]) > 0$  and  $s = \min_{a \in [-M, M]} (K_a[A] - K_a[B])$ . Choose  $d \in \mathbb{N}$  such that  $d \cdot t + s > 0$ . It thus follows from Lemma 10 that  $X^d + A >_L Y^d + B$ . The result follows from the definition of  $>_L$ .  $\square$

To make use of the above lemmata, we construct a variant of matching pennies such that any solution must involve someone playing, with positive probability, an action that generates a lottery with a lower max and min than its alternative.

	$a_2$	$b_2$
$a_1$	$(2, 0)$	$(0, 1)$
$b_1$	$(-1, 1)$	$(1, 0)$

Table 4: Variant of matching pennies

**Lemma 12.** *There exists a game  $H$  for  $n \geq 2$  players, with integral payoffs, and such that for all mixed strategy profiles  $p$ , there is a player  $i$  and actions  $a_i, b_i \in A_i$  such that  $p_i(b_i) > 0$  and  $\min[u_i(b_i, p_{-i})] < \min[u_i(a_i, p_{-i})]$  and  $\max[u_i(b_i, p_{-i})] < \max[u_i(a_i, p_{-i})]$ .*

*Proof.* Let  $H$  be the game in which players 1 and 2 play the game described in Table 4, and the remaining players' actions do not affect the payoffs of players 1 and 2.

Since this game has no pure Nash equilibria, any pure strategy profile has the desired property for whichever of players 1 and 2 has a profitable deviation. It is easy to verify that if  $p$  has the property that one of the first two players totally mixes and the other plays a pure strategy, then  $p$  has the desired property with respect to the player who is mixing.

Finally, if both players 1 and 2 play totally mixed strategies, then

$$\begin{aligned}\min[u_1(b_1, p_{-1})] &= -1 < \min[u_1(a_1, p_{-1})] = 0, \\ \max[u_1(b_1, p_{-1})] &= 1 < \max[u_1(a_1, p_{-1})] = 2.\end{aligned}$$

Thus  $p$  has the desired property with respect to player 1.  $\square$

**Claim 4.** *Suppose  $S$  satisfies distribution-monotonicity and narrow bracketing. Assume also that players never play strictly dominated actions. Then players never play strictly first-order dominated actions.*

*Proof.* Let  $H$  be a game that satisfies the property whose existence is guaranteed by Lemma 12 (e.g., the one described in Table 4). Fix any  $p \in S(H)$ . By the defining property of  $H$  there is a player  $i$  such that  $i$  plays with positive probability an action  $\underline{a}_i$  that yields a lottery with a lower max and min than its alternative  $\bar{a}_i$ .

Consider a game  $F = (C, w)$  where  $C_i = \{a_{0.5}, a_0\}$  and payoffs are given by  $w_i(a_{0.5}, \cdot) = 0.5$  and  $w_i(a_0, \cdot) = 0$ . Note that  $q_i(a_{0.5}) = 1$  for any solution  $q \in S(F)$ , by the assumption that strictly dominated actions are never played. Consider the product game  $H \otimes F$ . By narrow bracketing,  $p \times q$  is one of its solutions. It puts positive weight on  $(\underline{a}_i, a_{0.5})$  since  $p_i(\underline{a}_i) \cdot q_i(a_{0.5}) > 0$ , while  $(\bar{a}_i, a_0)$  has probability zero as  $p_i(\bar{a}_i) \cdot q_i(a_0) = 0$ . Since the

payoffs in  $H$  are integral, a difference in the max or min of lotteries generated by a player's actions is at least 1. Consequently, the payoff distribution of  $(\bar{a}_i, a_0)$  has strictly higher maximum and minimum than those of  $(\underline{a}_i, a_{0.5})$ :

$$\begin{aligned}\max(v_i(\bar{a}_i, p_{-i}) + w_i(a_0, q_{-i})) &> \max(v_i(\underline{a}_i, p_{-i}) + w_i(a_{0.5}, q_{-i})), \\ \min(v_i(\bar{a}_i, p_{-i}) + w_i(a_0, q_{-i})) &> \min(v_i(\underline{a}_i, p_{-i}) + w_i(a_{0.5}, q_{-i})).\end{aligned}$$

Let  $G = (A, u)$  be an arbitrary game with  $a_Y, a_Z \in A_i$  and  $r \in S(G)$  such that  $u_i(a_Y, r_{-i}) >_{\text{FOSD}} u_i(a_Z, r_{-i})$ . We need to show that  $r_i(a_Z) = 0$ . Indeed, by Lemma 11 there exist  $m, n \in \mathbb{N}$  such that

$$u_i(a_Y, r_{-i})^m + (v_i(\bar{a}_i, p_j) + w_i(a_0, q_j))^n >_{\text{FOSD}} u_i(a_Z, r_{-i})^m + (v_i(\underline{a}_i, p_j) + w_i(a_{0.5}, q_j))^n.$$

By narrow bracketing,  $r^m \times p^n \times q^n \in S(G^m \otimes H^n \otimes F^n)$ . Since  $q_i(a_0) = 0$ , and  $p_i(\underline{a}_i), q_i(a_{0.5}) > 0$ , by distribution-monotonicity, it must be that  $r_i(a_Z) = 0$ .  $\square$

**Claim 5.** *Suppose  $S$  satisfies distribution-monotonicity, narrow bracketing, and anonymity. Also, assume that there is a game  $G_D$ , a player  $i$ , and a solution  $p \in S(G_D)$  in which player  $i$  plays a dominated action with positive probability. Then  $S$  satisfies interiority and distribution-neutrality.*

*Proof.* By definition, there are actions  $a_h, a_\ell$  in  $G_D$  and  $D > 0$  such that  $p_i(a_\ell) > 0$  and  $u_i(a_h, a_{-i}) \geq u_i(a_\ell, a_{-i}) + D$  for all  $a_{-i}$ . We show that  $S$  must then satisfy interiority. Let  $G = (A, u)$  be any other game, and consider some  $a \in A_i$  and  $q \in S(G)$ . To prove interiority, we need to show that  $q_i(a) > 0$ . Pick some  $b$  such that  $q_i(b) > 0$ . Consider  $m \in \mathbb{N}$  such that  $mD > \max[u_i(b, q_{-i})] - \min[u_i(a, q_{-i})]$ . By narrow bracketing,  $r = q \times p^m$  is in  $S(G_D^m \otimes G)$ . Denote the payoff map of  $G_D^m \otimes G$  by  $v$ , and note that

$$v_i(a_h, \dots, a_h, a, r_{-i}) >_{\text{FOSD}} v_i(a_\ell, \dots, a_\ell, b, r_{-i}).$$

Therefore, by distribution-monotonicity and narrow bracketing, while

$$r_i(a_h, \dots, a_h, a) > r_i(a_\ell, \dots, a_\ell, b) > 0,$$

and so  $q_i$  is totally mixed. We conclude that  $S$  satisfies interiority.

We next show that such an  $S$  must satisfy distribution-neutrality. Toward a contradiction, suppose that distribution-neutrality is violated in a game  $G = (A, u)$ . By anonymity, we can assume that the violation occurs for the same player  $i$  that played the dominated action  $a_\ell$  in  $G_D$ . Hence, there is  $q \in S(G)$  and  $a, b \in A_i$ , such that  $u_i(a, q_{-i}) = u_i(b, q_{-i})$ , while  $q_i(a) < q_i(b)$ . By assumption  $p \in S(G_D)$  satisfies  $p_i(a_\ell) > 0$ . Pick  $m \in \mathbb{N}$  such that

$\left(\frac{q_i(b)}{q_i(a)}\right)^m > \frac{p_i(a_h)}{p_i(a_\ell)}$ . By narrow bracketing,  $r = q^m \times p \in S(G^m \otimes G_D)$ . Let  $v$  denote the payoff map of  $G^m \otimes G_D$ , and note that  $v_i(a, \dots, a, a_h, r_{-i}) >_{\text{FOSD}} v_i(b, \dots, b, a_\ell, r_{-i})$ , while

$$r_i(a, \dots, a, a_h) < r_i(b, \dots, b, a_\ell),$$

violating distribution-monotonicity. This contradiction implies that  $S$  satisfies distribution-neutrality.  $\square$

Claim 4 and Claim 5 together immediately imply Proposition 2.

## F Proof of Theorem 4

*Proof.* By Theorem 3,  $S$  is a refinement of either  $\text{Nash}_\Phi$  or  $\text{LQRE}_{\lambda\Phi}$  for some monotone additive statistic  $\Phi$  and  $\lambda \geq 0$ . For  $\text{LQRE}_{\lambda\Phi}$  where  $\lambda = 0$ , the result is trivial. For any other SRE with  $\Phi$ , we define the statistic  $\Psi$  by  $\Psi[X] = f^{-1}(\mathbb{E}[f(X)])$ . Suppose  $a_i$  is an action chosen by player  $i$  with maximal probability:  $a_i \in \arg \max_a p_i(a)$ . Then by the definition of  $\Phi$ ,  $a_i \in \arg \max_a \Phi[u_i(a, p_{-i})]$ , i.e.,  $a_i$  maximizes  $\Phi$ . But by the definition of  $\Psi$ , it likewise holds that  $a_i$  maximizes  $\Psi$ . It follows from Lemmas 4 and 5 that  $\Psi$  and  $\Phi$  agree on  $\Delta_{\mathbb{Q}}$ , so by Lemma 6  $\Psi = \Phi$ . Since  $f \circ \Psi$  is an expected utility over lotteries,  $\Psi$  satisfies independence, i.e., for all compactly supported lotteries  $X, Y, Z$  and all  $\beta \in (0, 1)$ ,  $X \succsim Y$  implies  $\beta X + (1 - \beta)Z \succsim \beta Y + (1 - \beta)Z$ . Hence, the result follows from Proposition 8 of [Mu, Pomatto, Strack, and Tamuz \(2021\)](#).  $\square$

## G Proof of Theorem 5

Since  $S$  satisfies distribution-monotonicity, narrow bracketing, anonymity and interiority, by Theorem 3,  $S$  is a refinement of some  $\text{LQRE}_{\lambda\Phi}$ . Scale invariance ensures that  $\Phi$  belongs to the class of positively homogenous monotone additive statistics, which we characterize in the following lemma. We use  $\Delta_{\text{finite}}$  to refer to the set of all lotteries with finite outcomes.

**Lemma 13.** *Suppose that  $\Phi: \Delta \rightarrow \mathbb{R}$  is a monotone additive statistic such that  $\Phi[\beta X] = \beta\Phi[X]$  for all  $X \in \Delta_{\text{finite}}$  and some  $\beta > 0$ ,  $\beta \neq 1$ . Then  $\Phi$  is a convex combination of the minimum, the maximum, and the expectation.*

*Proof of Lemma 13.* Let  $\beta > 0$  and let  $\Phi$  be a monotone additive statistic. By Lemma 6,

$\Phi[X] = \int K_a[X] d\mu(a)$ . Then

$$\begin{aligned}\Phi[\beta X] &= \int \frac{1}{a} \log \mathbb{E}[e^{a\beta X}] d\mu(a) \\ &= \int \frac{\beta}{a\beta} \log \mathbb{E}[e^{a\beta X}] d\mu(a) \\ &= \beta \int K_{a\beta}[X] d\mu(a) \\ &= \beta \int K_a[X] d(\beta_*\mu)(a).\end{aligned}$$

Denote  $\Psi[X] = \int K_a[X] d(\beta_*\mu)(a)$ , and note that this is also a monotone additive statistic. Then  $\Phi[\beta X] = \beta\Psi[X]$ .

Suppose  $\Phi[\beta X] = \beta\Phi[X]$  for all  $X$  and some  $\beta > 0$ . Hence  $\beta\Phi[X] = \beta\Psi[X]$  for all  $X$ , and so  $\Phi = \Psi$ . By Lemma 5 of [Mu, Pomatto, Strack, and Tamuz \(2021\)](#) it follows that  $\mu = \beta_*\mu$ . Since a probability measure on  $\mathbb{R}$  can only be invariant to rescaling by  $\beta \neq 1$  if it is the point mass at 0, it follows that  $\mu(\{-\infty, +\infty, 0\}) = 1$ .  $\square$

It is straightforward to see that if  $\mu$  is supported on  $\{-\infty, +\infty, 0\}$ , then  $\Phi$  satisfies  $\Phi[\beta X] = \beta\Phi[X]$  for all  $\beta > 0$ . We proceed with the proof of Theorem 5.

*Proof of Theorem 5.* Since  $S$  satisfies distribution-monotonicity, narrow bracketing, anonymity and interiority, by Theorem 3,  $S$  is a refinement of some  $\text{LQRE}_{\lambda\Phi}$ . Let  $X \in \Delta_{\mathbb{Q}}$  and consider the game  $H_{r,X} = (B, v)$  where  $r = \Phi[X]$  (see (4)). Any  $p \in S(H_{r,X})$  must satisfy  $p_1(b_r) = p_1(b_X)$ , and so all players play uniform strategies. Hence, by scale invariance, this is also the case for each  $p \in S(B, \frac{1}{2} \cdot v)$ . It follows that  $\Phi[\frac{1}{2}X] = \Phi[\frac{1}{2}r] = \frac{1}{2}r = \frac{1}{2}\Phi[X]$ . By Lemma 13,  $\Phi$  is a convex combination of the minimum, the maximum and the expectation. The desired representation follows by setting  $\lambda_1 = \lambda\mu(\{-\infty\})$ ,  $\lambda_2 = \lambda\mu(\{0\})$  and  $\lambda_3 = \lambda\mu(\{+\infty\})$ .  $\square$

## H Connections to [Brandl and Brandt \(2024\)](#)

[Brandl and Brandt \(2024\)](#) characterize Nash equilibrium as the unique solution concept satisfying consistency, consequentialism, and rationality. Their work focuses on how minor changes in the strategic environment impact behavior, while our approach emphasizes how individuals frame and evaluate risk across multiple independent decisions. To further explore the relationship between the two perspectives, we now outline their axioms.

**Definition 16.** A solution concept  $S$  is consistent if for any  $(A, u)$ ,  $(A, v)$ , and  $\alpha \in [0, 1]$

$$S(A, u) \cap S(A, v) \subset S(A, \alpha u + (1 - \alpha)v).$$

Consistency requires that if a strategy profile is a solution to two games, it must also be a solution to any convex combination of them. Differently stated, given a mixed strategy profile  $p$ , the set of games  $\{G : p \in S(G)\}$  that it solves must be convex for consistent  $S$ .

Among SRE, only Nash equilibrium and  $\text{LQRE}_\lambda$  satisfy consistency. While our Theorem 1 characterizes these two solution concepts, consistency is not implied by the hypotheses of the theorem, as the theorem also allows for refinements. For example, trembling hand perfect equilibrium is a refinement of Nash equilibrium that satisfies narrow bracketing, expectation-monotonicity, and anonymity but violates consistency.<sup>14</sup>

To formulate the next axiom, we say that a game  $G = (A, u)$  is a *blowup* of  $H = (B, v)$  if there exist  $f_i: A_i \rightarrow B_i$  such that  $u_i(a) = v_i(f_i(a))$  for all players  $i$  and  $a \in A$ . Given a mixed strategy profile  $p$  in  $(G)$ , we denote by  $q = f(p)$  the profile in  $H$  given by  $q_i(b_i) = p_i(f_i^{-1}(b_i))$ .

**Definition 17.** *A solution concept  $S$  satisfies consequentialism if for any games  $H$  and  $G$  such that  $G$  is a blowup of  $H$ , it holds that  $p \in S(G)$  if and only if  $f(p) \in S(H)$  for the witnessing  $f$ .*

Equivalently, consequentialism means that if two games  $G, H$  are identical, except that  $G$  contains an additional action  $a'_i$  that is indistinguishable from  $a_i$ , then the solutions of the two games are the same, except that the probabilities assigned to  $a_i$  in  $H$  can be divided in any way between  $a_i$  and  $a'_i$  in  $G$ . In other words, duplicating actions should not affect behavior beyond splitting probabilities. Consequentialism is satisfied by every  $\text{Nash}_\Phi$  but is violated by every  $\text{LQRE}_{\lambda\Phi}$ .

**Definition 18.** *A solution concept  $S$  satisfies rationality if for any game  $G = (A, u)$ , player  $i$ , and a strictly dominant action  $a_i \in A_i$ , it holds that  $p_i(a_i) > 0$  for all  $p \in S(G)$ .*

That is, players play dominant strategies with positive probability. Distribution-monotonicity implies rationality and, in particular, rationality is satisfied by all  $\text{Nash}_\Phi$  and  $\text{LQRE}_{\lambda\Phi}$ .

## H.1 Implications of Consistency and Consequentialism

The rationality assumption of Brandl and Brandt is straightforward, and so we focus on the connection between our axioms and their consequentialism and consistency. In particular, we investigate the connection between these properties and our scale-invariance and narrow bracketing.

First, we point out that consequentialism and consistency imply scale-invariance. In fact, they imply a much stronger property:  $S(A, u) \subset S(A, \alpha \cdot u)$  for any game  $(A, u)$  and

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<sup>14</sup>See §5 of Brandl and Brandt (2024) for an illustration of a consistency violation by trembling hand perfect equilibrium.

all  $\alpha \in (0, 1)$ .<sup>15</sup> Indeed, consequentialism implies that every mixed-strategy profile is a solution to the game  $(A, 0)$  whose payoff map vanishes for every action. The result then follows from consistency, since any scaled down game  $(A, \alpha \cdot u)$  is the convex combination  $(A, \alpha \cdot u + (1 - \alpha) \cdot 0)$ .

Consequentialism and consistency also imply a property that is a weakening of narrow bracketing: for any games  $G$  and  $H$  there exist solutions  $p \in S(G)$  and  $q \in S(H)$  such that  $p \times q \in S(G \otimes H)$ . To see this, we first show that consequentialism and consistency imply that if  $p \in S(G)$  and  $q \in S(H)$  then  $p \times q$  solves the game with the same action space as  $G \otimes H$  but with payoffs halved. We write  $\alpha \cdot W$  for a game with the same action set as  $W$  and payoffs scaled by  $\alpha$ , so our goal is to show  $p \times q \in S\left(\frac{1}{2}(G \otimes H)\right)$ . Let  $G = (A, u)$  and  $H = (B, v)$ . Denote  $C = A \times B$  and define two auxiliary games  $\hat{G} = (C, \hat{u})$  and  $\hat{H} = (C, \hat{v})$  by

$$\hat{u}_i((a_1, b_1), \dots, (a_n, b_n)) = u_i(a) \quad \text{and} \quad \hat{v}_i((a_1, b_1), \dots, (a_n, b_n)) = v_i(b).$$

The game  $\hat{G}$  is a blowup of  $G$  under the map  $f$  that projects  $A \times B$  to  $A$ ; similarly,  $\hat{H}$  is a blowup of  $H$  under the projection to  $B$ . Consequentialism implies that  $p \times q \in S(\hat{G}) \cap S(\hat{H})$ . By consistency,  $p \times q$  is also a solution to the game  $\left(C, \frac{1}{2}\hat{u} + \frac{1}{2}\hat{v}\right)$  which is precisely  $\frac{1}{2}(G \otimes H)$ . Finally, let  $p \in S(2G)$  and  $q \in S(2H)$ , so that by consequentialism and consistency  $p \in S(G)$ ,  $q \in S(H)$ , and  $p \times q \in S(G \otimes H)$ , as  $G \otimes H = \frac{1}{2}((2G) \otimes (2H))$ .

Curiously, narrow bracketing is not implied by consequentialism and consistency: there are solution concepts  $S$  satisfying these properties, with  $p \in S(G)$  and  $q \in S(H)$  such that  $p \times q \notin S(G \otimes H)$ . For example, consider  $\varepsilon$ -Nash that assigns to a game  $G$  all mixed strategy profiles  $p$  such that  $p_i(a_i) > 0$  implies  $\mathbb{E}[u_i(a_i, p_{-i})] \geq \max_{b_i} \mathbb{E}[u_i(b_i, p_{-i})] - \varepsilon$ . It is straightforward to check that  $\varepsilon$ -Nash satisfies consequentialism and consistency. However,  $\varepsilon$ -Nash violates narrow bracketing. Indeed, if player  $i$  plays an action that is  $\varepsilon$ -suboptimal in  $G$  then narrow bracketing would imply that they play a  $2\varepsilon$ -suboptimal action in  $G \otimes G$  (see also Lemma 1). Of course, by the main result of Brandl and Brandt (2024), narrow bracketing is implied if we add their rationality assumption, highlighting another connection between narrow bracketing and rationality.

## I IIA Property for Games

LQRE is, in some sense, a natural extension to games of the single-agent multinomial logit choice rule. In this section, we discuss the IIA property that characterizes the multinomial logit choice rule (Luce, 1959). We show that this property cannot be naively adapted to

<sup>15</sup>This is in contrast with our scale-invariance axiom which only imposes that solutions which are uniform distributions be invariant to scaling down the payoff map.



games to characterize LQRE. In Table 5, we consider two games that differ by  $c_1$ , which is an additional action available to player 1 in the game on the right.

	$a_2$	$b_2$		$a_2$	$b_2$
$a_1$	(0, 0)	(2, 0)	$a_1$	(0, 0)	(2, 0)
$b_1$	(1, 0)	(1, 0)	$b_1$	(1, 0)	(1, 0)
			$c_1$	( $\alpha, \beta$ )	( $\gamma, \delta$ )

Table 5: Additional actions are not irrelevant

In any  $\text{LQRE}_\lambda$ , the unique solution to the left game is uniform mixing by both players. A naive adaptation of the IIA property to games would require any solution  $q$  to the game on the right to satisfy  $q_1(a_1) = q_1(b_1)$ . However, unless  $\beta = \delta$ , player 2 will not randomize uniformly under  $q$  (provided  $\lambda \neq 0$ ). Thus  $q_1(a_1) \neq q_1(b_1)$  for all  $\text{LQRE}_\lambda$ , with  $\lambda \neq 0$ .