Lecture 2
Root Finding and Iterative Method

Root Finding

Example 1 To what elevation should the cannon be raised to hit the target?

Parameters:

- g = gravitational acceleration (ms^{-2}): known;
- V_0 = initial speed (ms^{-1}): known;
- R = distance to target (m): known;
- θ^* = required elevation (radians): unknown.

Determine elevation θ^* needed to hit target using known values of parameters V_0, R, and g.

- Coordinates of cannonball at time t are $(x(t), y(t))$.
- Motion of cannonball determined by Newton's 2nd law:

$$\begin{cases} x''(t) = 0, & x(0) = 0, x'(0) = V_0 \cos\theta^* \\ y''(t) = -g & y(0) = 0, y'(0) = V_0 \sin\theta^* - \frac{1}{2}gt^2 \end{cases}$$

- Want to find θ^* such that $x(T) = R$ and $y(T) = 0$, where T is the time of flight.
- Can eliminate T using $y(T) = 0$ (i.e. object is on the ground).
- If $y(T) = 0$ then $T = 0$ or $T = \frac{2V_0 \sin\theta^*}{g}$.
- Reject $T = 0$, so to have $x(T) = R$, we must have

$$x(T) = (V_0 \cos\theta^*)(\frac{2V_0 \sin\theta^*}{g}) = R$$

Zero-finding problem: find elevation θ^* such that $f(\theta^*) = 0$, where

$$f(\theta) := 2\sin\theta\cos\theta - \frac{Rg}{V_0^2}$$

Example 2 Suppose you pay an amount of money to a bank every year and they promise to pay you a lump sum when you retire. Over one year, you would pay an amount

$$\nu P = \frac{1}{1+i} P$$

for this service to compensate for the loss of interest, where i is the percentage interest rate.

Over n years you will pay

$$\nu P + \nu^2 P + ... + \nu^n P = \nu P \frac{1-\nu^n}{1-\nu} = T$$

Iterative Methods

- Algorithms for solving $f(x^*) = 0$ are usually iterative.
- Starting from $x^{(0)}$, make sequence of iterates

$$\begin{aligned} x^{(1)} &= \phi(x^{(0)}), \\ x^{(2)} &= \phi(x^{(1)}), \\ &\vdots \\ x^{(k+1)} &= \phi(x^{(k)}), \\ &\vdots \end{aligned}$$

- ϕ function/rule generating successive iterates
- Rule $x^{(k+1)} = \phi(x^{(k)})$ for $k \geq 0$ is a recurrance relation.

Example 3 $x^{(0)} := 1$ and $\phi : \mathbb{R} \to \mathbb{R}$ is the function given by

$$(\forall t \in \mathbb{R}) \quad \phi(t) = 2t$$

Example 4 Given $a > 0$ and $x^{(0)} > 0$, consider sequence

$$x^{(k+1)} = \phi(x^{(k)} = \frac{1}{2}(x^{(k)} + \frac{a}{x^{(k)}}) \quad (k = 0, 1, ...)$$

with $a = 5$ and $x^{(0)} = 3$.

What does this sequence converge to?

Remark 1 Any iteration $x^{(k+1)} = \phi(x^{(k)}$ generates a sequence

$$\{x^{(k)}\}_{k=0}^{\infty} = \{x^{(0)}, x^{(1)}, x^{(2)}, ..., x^{(k-1)}, x^{(k)}, x^{(k+1)}, ...\}$$

Remark 2 Recall that the limit of sequence $\{x^{(k)}\}_{k=0}^{\infty}$ is $x^* \in \mathbb{R}$ iff

$$(\forall \epsilon > 0)(\exists K \in \mathbb{N}) \quad [(k \geq K) \implies |x^{(k)} - x^*| \leq \epsilon]$$

Theorem 1 The sequence $\{x^{(k)}\}_{k=0}^{\infty}$ converges to $x^* \in \mathbb{R}$ or

$$\boxed{\lim_{k\to\infty} x^{(k)} = x^*}$$