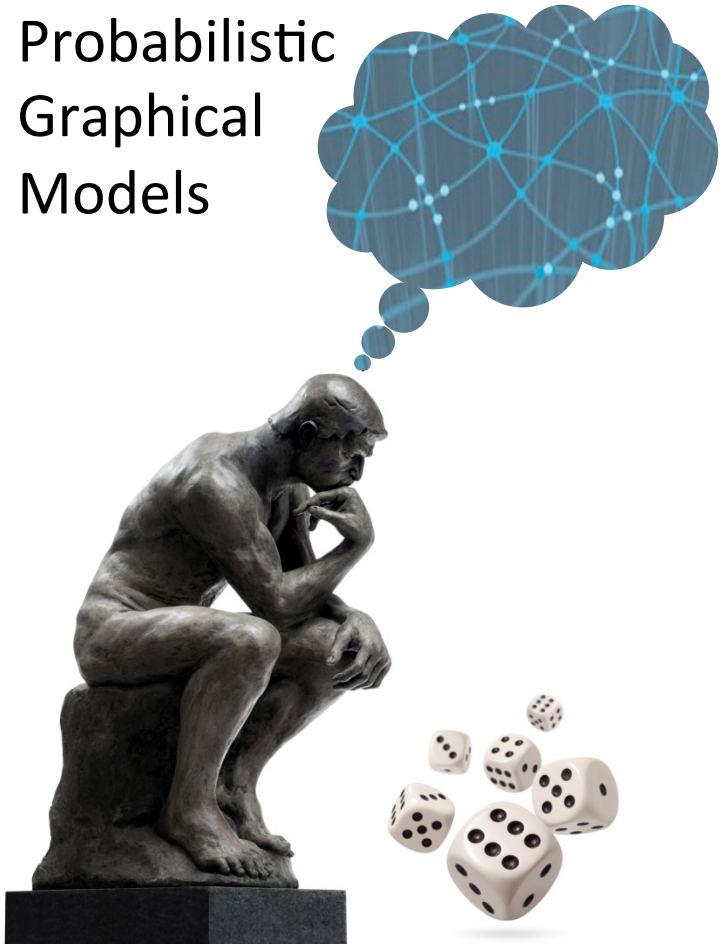
Probabilistic Graphical Models

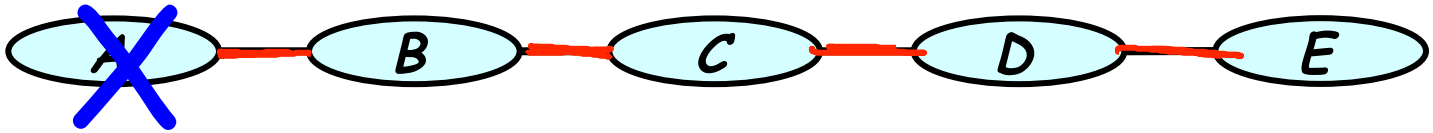


Inference

Variable Elimination

Variable Elimination Algorithm

Elimination in Chains



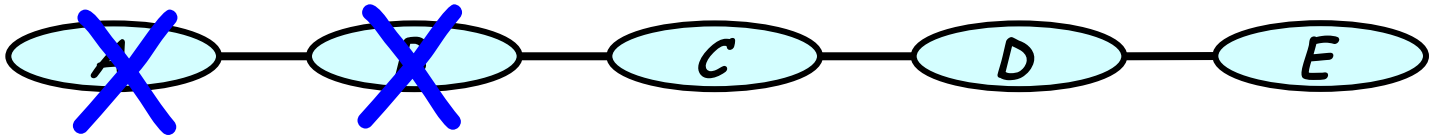
$$P(E) \propto \sum_D \sum_C \sum_B \sum_A \tilde{P}(A,B,C,D,E)$$

$$= \sum_D \sum_C \sum_B \sum_A \phi_1(A,B)\phi_2(B,C)\phi_3(C,D)\phi_4(D,E)$$

$$= \sum_D \sum_C \sum_B \phi_2(B,C)\phi_3(C,D)\phi_4(D,E) \sum_A \phi_1(A,B) \quad \tau_1(B)$$

$$= \sum_D \sum_C \sum_B \phi_2(B,C)\phi_3(C,D)\phi_4(D,E)\tau_1(B)$$

Elimination in Chains



$$P(E) \propto \sum_D \sum_C \sum_B \phi_2(B,C)\phi_3(C,D)\phi_4(D,E)\tau_1(B)$$

$$= \sum_D \sum_C \phi_3(C,D)\phi_4(D,E)\left(\sum_B \phi_2(B,C)\tau_1(B)\right)$$

$$= \sum_D \sum_C \phi_3(C,D)\phi_4(D,E)\tau_2(C)$$

Variable Elimination

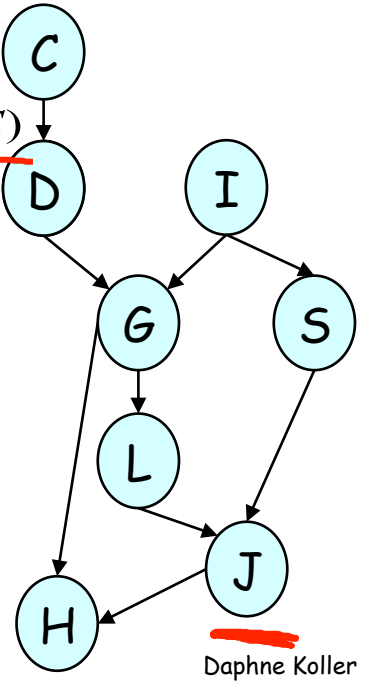
- Goal: P(J)
- Eliminate: C,D,I,H,G,S,L

$$\sum_{L,S,G,H,I,D,C} \phi_J(J,L,S)\phi_L(L,G)\phi_S(S,I)\phi_G(G,I,D)\phi_H(H,G,J)\phi_I(I)\phi_D(C,D)\phi_C(C)$$

$$\sum_{L,S,G,H,I,D} \phi_J(J,L,S)\phi_L(L,G)\phi_S(S,I)\phi_G(G,I,D)\phi_H(H,G,J)\phi_I(I)\sum_C \phi_D(C,D)\phi_C(C)$$

Compute $\tau_1(D) = \sum_C \phi_C(C)\phi_D(C,D)$

$$= \sum_{L,S,G,H,I,D} \phi_J(J,L,S)\phi_L(L,G)\phi_S(S,I)\phi_G(G,I,D)\phi_H(H,G,J)\phi_I(I)\tau_1(D)$$



Variable Elimination

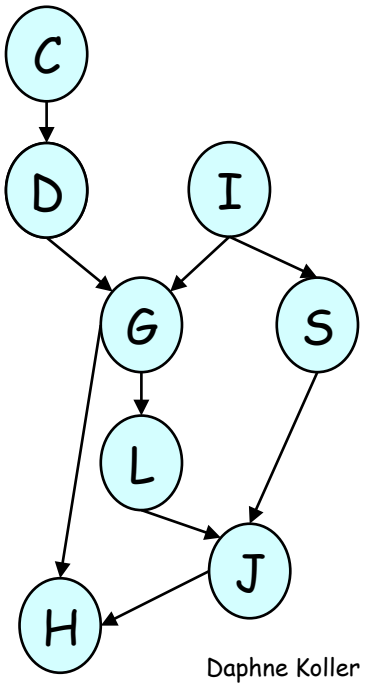
- Goal: P(J)
- Eliminate: D,I,H,G,S,L

$$\sum_{L,S,G,H,I,D} \phi_J(J,L,S)\phi_L(L,G)\phi_S(S,I)\phi_G(G,I,D)\phi_H(H,G,J)\phi_I(I)\tau_1(D)$$

$$= \sum_{L,S,G,H,I} \phi_J(J,L,S)\phi_L(L,G)\phi_S(S,I)\phi_H(H,G,J)\phi_I(I)\sum_D \phi_G(G,I,D)\tau_1(D)$$

Compute $\tau_2(G,I) = \sum_D \phi_G(G,I,D)\tau_1(D)$

$$= \sum_{L,S,G,H,I} \phi_J(J,L,S)\phi_L(L,G)\phi_S(S,I)\phi_H(H,G,J)\phi_I(I)\tau_2(G,I)$$



Variable Elimination

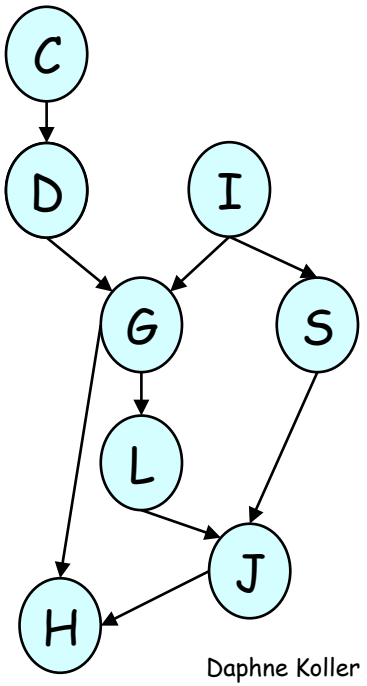
- Goal: P(J)
- Eliminate: I,H,G,S,L

$$\sum_{L,S,G,H,I} \phi_J(J,L,S)\phi_L(L,G)\phi_S(S,I)\phi_H(H,G,J)\phi_I(I)\tau_2(G,I)$$

$$= \sum_{L,S,G,H} \phi_J(J,L,S)\phi_L(L,G)\phi_H(H,G,J)\sum_I \phi_S(S,I)\phi_I(I)\tau_2(G,I)$$

Compute $\tau_3(S,G) = \sum_I \phi_S(S,I)\phi_I(I)\tau_2(G,I)$

$$= \sum_{L,S,G,H} \phi_J(J,L,S)\phi_L(L,G)\phi_H(H,G,J)\tau_3(S,G)$$



Daphne Koller

Variable Elimination

- Goal: P(J)
- Eliminate: H,G,S,L

$$\sum_{L,S,G,H} \phi_J(J,L,S)\phi_L(L,G)\phi_H(H,G,J)\tau_3(S,G)$$

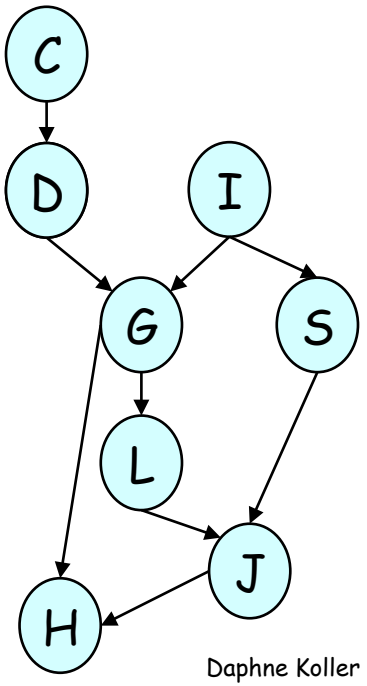
$$\sum_{L,S,G} \phi_J(J,L,S)\phi_L(L,G)\tau_3(S,G)\sum_H \phi_H(H,G,J)$$

$\sum_H P(H|G,J) = 1$

$\tau_4(G,J)$

Compute $\tau_4(G,J) = \sum_H \phi_H(H,G,J)$

$$\sum_{L,S,G} \phi_J(J,L,S)\phi_L(L,G)\tau_3(S,G)\tau_4(G,J)$$



Variable Elimination

- Goal: P(J)
- Eliminate: G,S,L

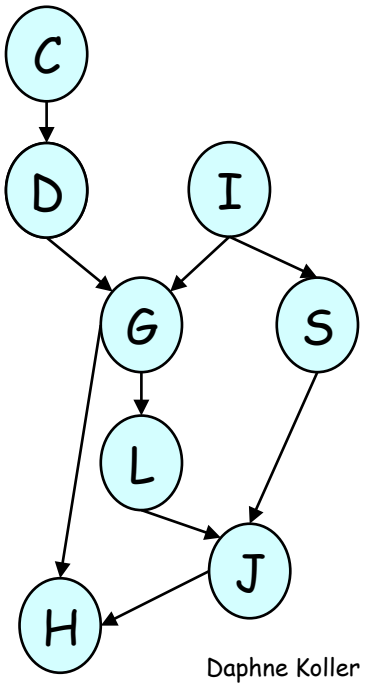
$$\sum_{L,S,G} \phi_J(J,L,S)\phi_L(L,G)\tau_3(S,G)\tau_4(G,J)$$

L, G, S, J

$$\sum_{L,S} \phi_J(J,L,S) \sum_G \phi_L(L,G)\tau_4(G,J)\tau_3(S,G)$$

Compute $\tau_5(L,J,S) = \sum_G \phi_L(L,G)\tau_3(S,G)\tau_4(G,J)$

$$\sum_{L,S} \phi_J(J,L,S)\tau_5(L,J,S)$$

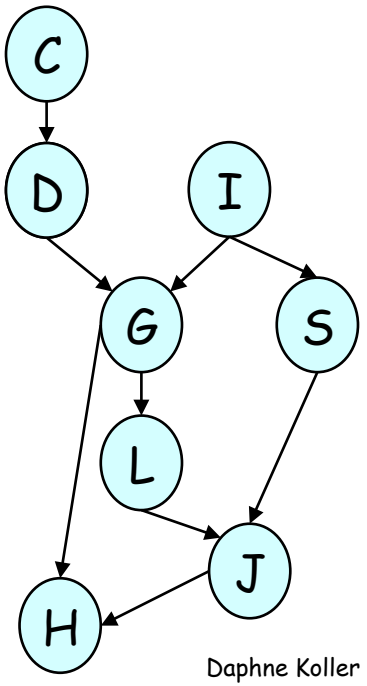


Daphne Koller

Variable Elimination

- Goal: P(J)
- Eliminate: S,L

$$\sum_{L,S} \phi_J(J,L,S)\tau_5(L,J)$$



Daphne Koller

Variable Elimination with evidence

- Goal: P(J,I=i,H=h)
- Eliminate: C,D,G,S,L (no H, I)

$$\sum_{L,S,G,,D,C} \phi_J(J,L,S)\phi_L(L,G)\phi_S'(S)\phi_G'(G,D)\phi_H'(G,J)\phi_I'()\phi_D(C,D)\phi_C(C)$$

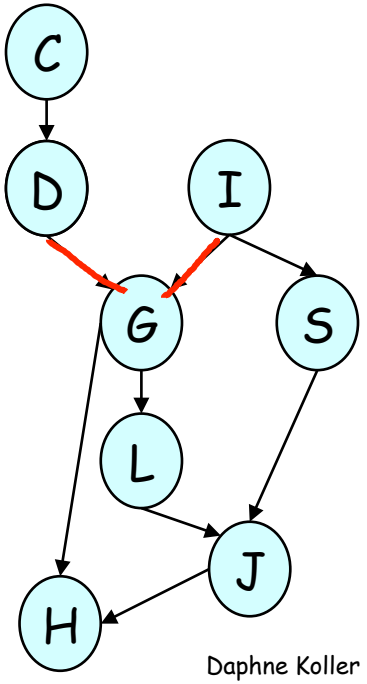
$\phi_G'(G,D)$: P(G|i,D) $\quad$ $\phi_I'()$: $\phi_I(i)$

elimination as before

How do we get P(J | I=i,H=h)?

renormalize

P(I=i,H=h) is the renormalizing constant



Variable Elimination in MNs

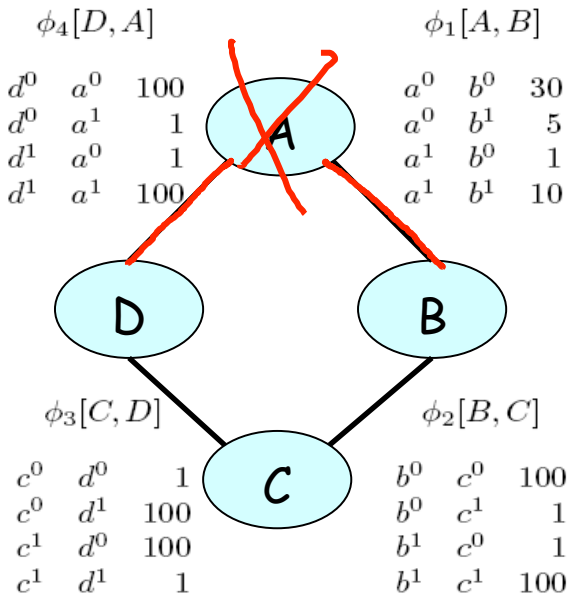
- Goal: P(D)
- Eliminate: A,B,C

$$\sum_{A,B,C} \phi_1(A,B)\phi_2(B,C)\phi_3(C,D)\phi_4(A,D)$$

$$\sum_{B,C} \phi_2(B,C)\phi_3(C,D) \sum_{A} \phi_1(A,B)\phi_4(A,D)$$

$$\sum_{B,C} \phi_2(B,C)\phi_3(C,D)\tau_1(B,D)$$

At the end of elimination get $\tau_3(D) \propto P(D)$



Eliminate-Var Z from Φ

all factors that involve z

$$\Phi' = \{\phi_i \in \Phi \ : \ Z \in Scope[\phi_i]\}$$

$$\psi = \prod_{\phi_i \in \Phi'} \phi_i$$

multiply them

$$\tau = \sum_{Z} \psi$$

sum out z

$$\Phi := \Phi - \Phi' \cup \{\tau\}$$

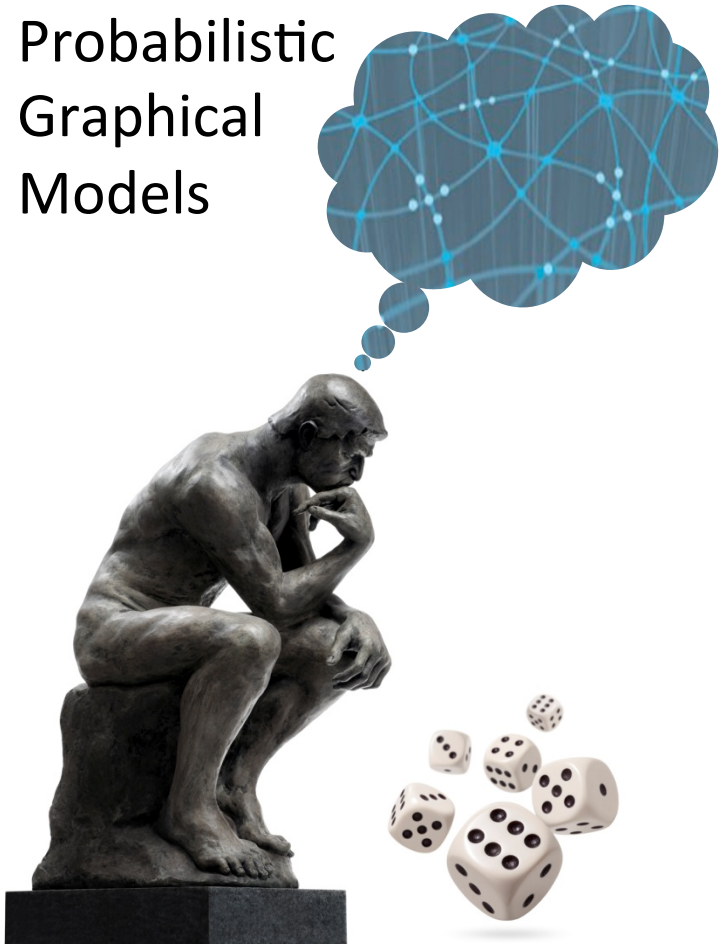
VE Algorithm Summary

- Reduce all factors by evidence
  - Get a set of factors Φ
- For each non-query variable Z
  - Eliminate-Var Z from Φ adds removes Φ' adds τ
- Multiply all remaining factors
- Renormalize to get distribution

Summary

- Simple algorithm
- Works for both BNs and MNs
- Factor product and summation steps can be done in any order, subject to:
  - when Z is eliminated, all factors involving Z have been multiplied in

Probabilistic Graphical Models



Inference

Variable Elimination

Complexity Analysis

Eliminating Z

$$\psi_k(\boldsymbol{X}_k) = \prod_{i=1}^{m_k} \phi_i$$

factor product

$$\tau_k(\boldsymbol{X}_k - \{Z\}) = \sum_{Z} \psi_k(\boldsymbol{X}_k)$$

marginalization

Reminder: Factor Product

$$\psi_k(\boldsymbol{X}_k) = \prod_{i=1}^{m_k} \phi_i$$

$N_k = |Val(\boldsymbol{X}_k)|$

a^1	b^1	0.5
a^1	b^2	0.8
a^2	b^1	0.1
a^2	b^2	0
a^3	b^1	0.3
a^3	b^2	0.9

b^1	c^1	0.5
b^1	c^2	0.7
b^2	c^1	0.1
b^2	c^2	0.2

a^1	b^1	c^1	0.5·0.5 = 0.25
a^1	b^1	c^2	0.5·0.7 = 0.35
a^1	b^2	c^1	0.8·0.1 = 0.08
a^1	b^2	c^2	0.8·0.2 = 0.16
a^2	b^1	c^1	0.1·0.5 = 0.05
a^2	b^1	c^2	0.1·0.7 = 0.07
a^2	b^2	c^1	0·0.1 = 0
a^2	b^2	c^2	0·0.2 = 0
a^3	b^1	c^1	0.3·0.5 = 0.15
a^3	b^1	c^2	0.3·0.7 = 0.21
a^3	b^2	c^1	0.9·0.1 = 0.09
a^3	b^2	c^2	0.9·0.2 = 0.18

Cost: $(m_k-1)N_k$ multiplications

Reminder: Factor Marginalization

$$\tau_k(\boldsymbol{X}_k - \{Z\}) = \sum_Z \psi_k(\boldsymbol{X}_k)$$

$N_k = |Val(\mathbf{X}_k)|$

Cost: ~N_k additions

each number used exactly once marg B

a^1	b^1	c^1	0.25
a^1	b^1	c^2	0.35
a^1	b^2	c^1	0.08
a^1	b^2	c^2	0.16
a^2	b^1	c^1	0.05
a^2	b^1	c^2	0.07
a^2	b^2	c^1	0
a^2	b^2	c^2	0
a^3	b^1	c^1	0.15
a^3	b^1	c^2	0.21
a^3	b^2	c^1	0.09
a^3	b^2	c^2	0.18

a^1	c^1	0.33
a^1	c^2	0.51
a^2	c^1	0.05
a^2	c^2	0.07
a^3	c^1	0.24
a^3	c^2	0.39

Complexity of Variable Elimination

- Start with m factors
  - m ≤ n for Bayesian networks (one for every variable)
  - can be larger for Markov networks
- At each elimination step generate 1 factor
- At most n elimination steps
- Total number of factors: m* ≤ m + n

Complexity of Variable Elimination

- $N = \max(N_k)$ = size of the largest factor
- Product operations: $\sum_k (m_k - 1) N_k \leq N \sum_k (m_k - 1) \leq Nm^*$
  each factor multiply in at most once
  $N m^*$
- Sum operations: $\leq \sum_k N_k \leq N \cdot \#\text{eliminationsteps} \leq N \cdot n$
- Total work is linear in N and m*

Complexity of Variable Elimination

- Total work is linear in N and m — exponential blowup
- $N_k = |Val(\mathbf{X}_k)| = O(d^{r_k})$ where — # variables in k^{th} factor
  - $d = \max(|Val(X_i)|)$ — d values in their scope
  - $r_k = |\mathbf{X}_k|$ = cardinality of the scope of the k^{th} factor

Complexity Example

$$\tau_1(D) = \sum_C \phi_C(C)\phi_D(C,D)$$ 2

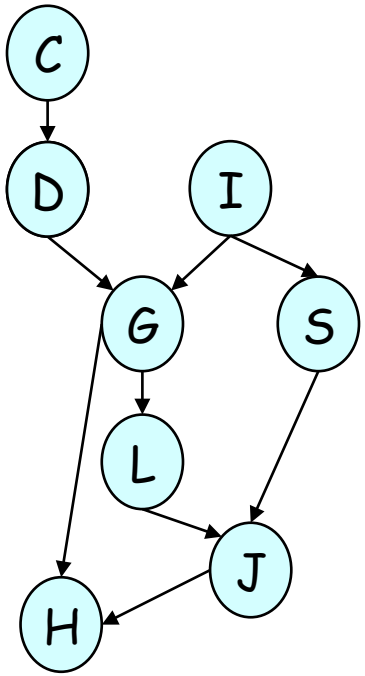
$$\tau_2(G,I) = \sum_D \phi_G(G,I,D)\tau_1(D)$$ 3

$$\tau_3(S,G) = \sum_I \phi_S(S,I)\phi_I(I)\tau_2(G,I)$$ 3

$$\tau_4(G,J) = \sum_H \phi_H(H,G,J)$$ 3

$$\tau_5(J,L,S) = \sum_G \phi_L(L,G)\tau_3(S,G)\tau_4(G,J)$$ 4 ←

$$\tau_6(J) = \sum_{L,S} \phi_J(J,L,S)\tau_5(J,L,S)$$ 3

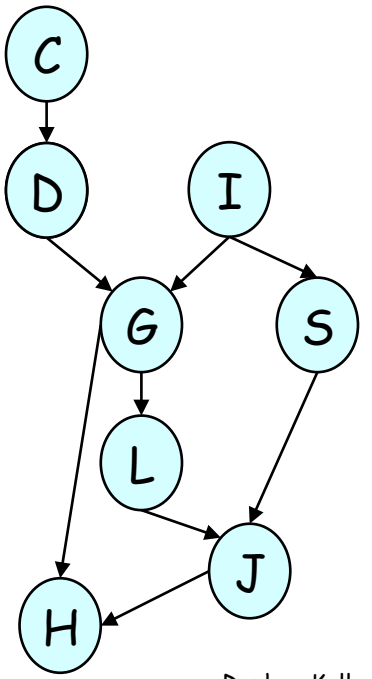


Complexity and Elimination Order

$$\sum_{L,S,G,H,I,D,C} \phi_J(J,L,S)\phi_L(L,G)\phi_S(S,I)\phi_G(G,I,D)\phi_H(H,G,J)\phi_I(I)\phi_D(C,D)\phi_C(C)$$

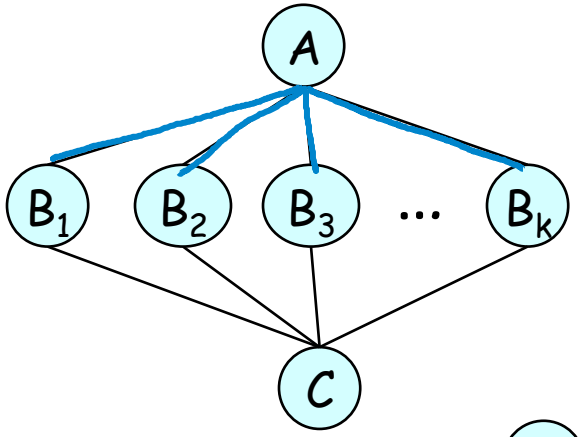
- Eliminate: G

$$\sum_{G} \phi_L(L,G)\phi_G(G,I,D)\phi_H(H,G,J)$$



Complexity and Elimination Order

Eliminate A first:

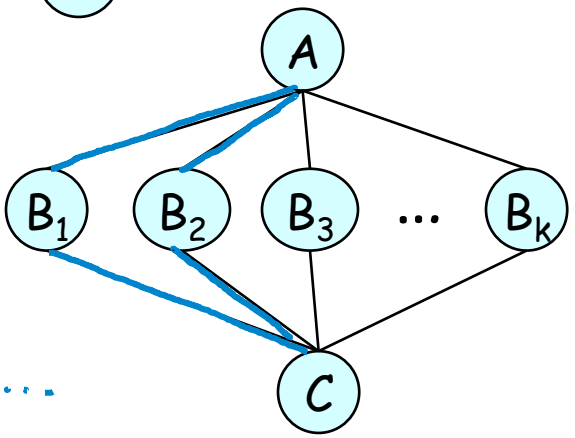


$\{A, B_1, \ldots, B_k\}$

size of factor is exp in k

$\prod_i \tau_i(A,C)$

Eliminate B_i's first:



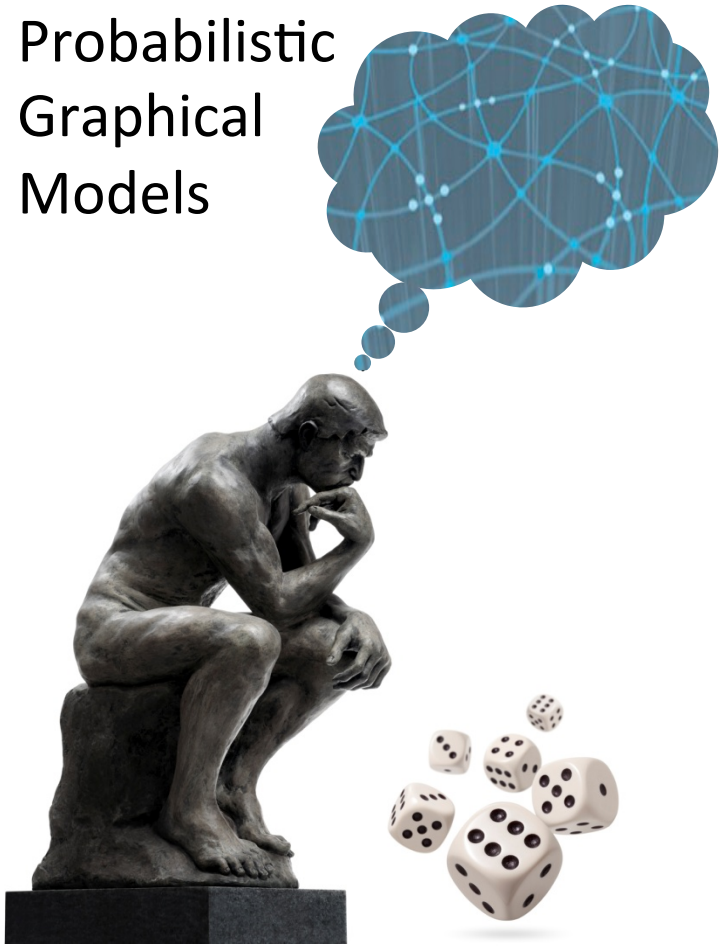
$\underbrace{\phi_1(A,B_1) \cdot \phi_2(C,B_2)}_{\text{scope } A,B_1,C} \Rightarrow \tau_1(A,C)$

$\tau_2(A,C) \ldots$

Summary

- Complexity of variable elimination linear in
  - size of the model (# factors, # variables)
  - size of the largest factor generated
- Size of factor is exponential in its scope
- Complexity of algorithm depends heavily on elimination ordering

Probabilistic Graphical Models



Inference

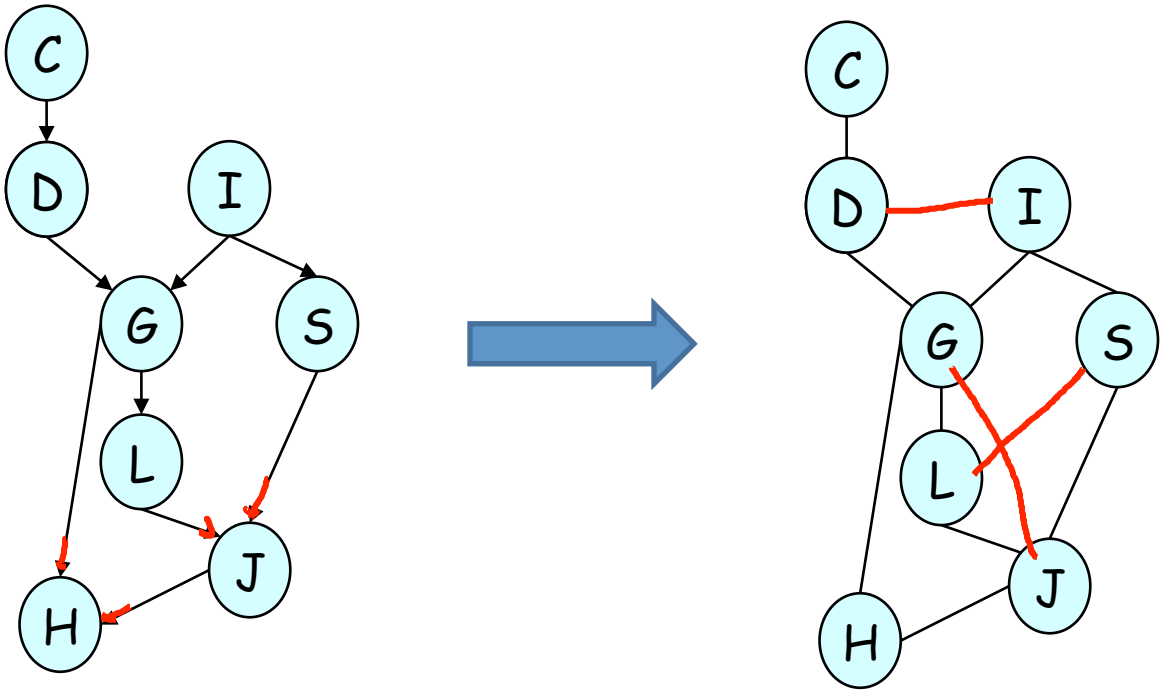
Variable Elimination

Graph-Based Perspective

Initial Graph

$$\phi_J(J,L,S)\phi_L(L,G)\phi_S(S,I)\phi_G(G,I,D)\phi_H(H,G,J)\phi_I(I)\phi_D(C,D)\phi_C(C)$$

moralization

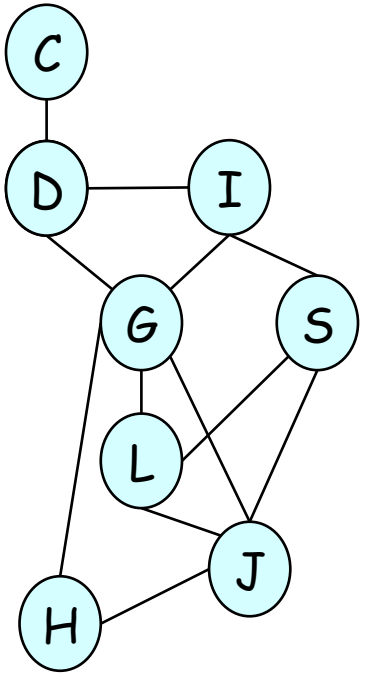


Elimination as Graph Operation

$$\phi_J(J,L,S)\phi_L(L,G)\phi_S(S,I)\phi_G(G,I,D)\phi_H(H,G,J)\phi_I(I)\phi_D(C,D)\phi_C(C)$$

- Eliminate: C

$$\tau_1(D) = \sum_C \phi_C(C)\phi_D(C,D)$$



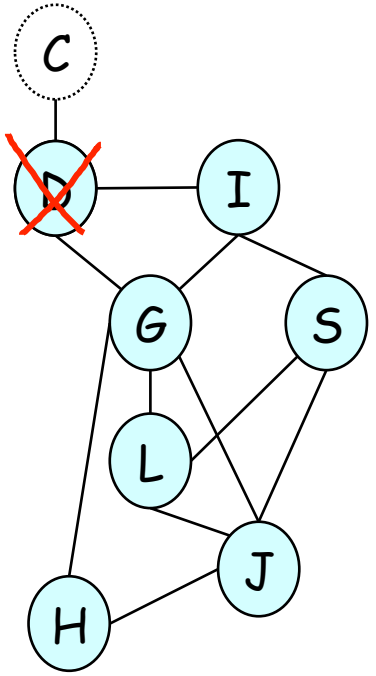
Induced Markov network for the current set of factors

Elimination as Graph Operation

$$\phi_J(J,L,S)\phi_L(L,G)\phi_S(S,I)\phi_G(G,I,D)\phi_H(H,G,J)\phi_I(I)\tau_1(D)$$

- Eliminate: D

$$\tau_2(G,I) = \sum_D \phi_G(G,I,D)\tau_1(D)$$



Induced Markov network for the current set of factors

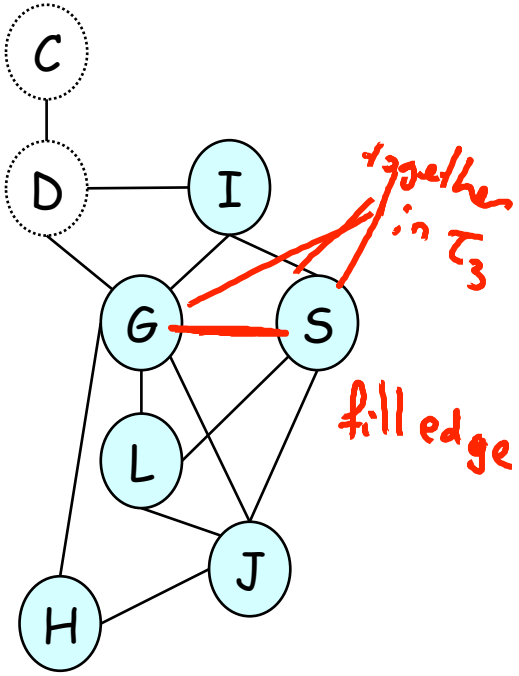
Elimination as Graph Operation

$$\phi_J(J,L,S)\phi_L(L,G)\phi_S(S,I)\phi_I(I)\phi_H(H,G,J)\tau_2(G,I)$$

- Eliminate: I

$$\tau_3(S,G) = \sum_I \phi_S(S,I)\phi_I(I)\tau_2(G,I)$$

all variables connected to I become connected directly



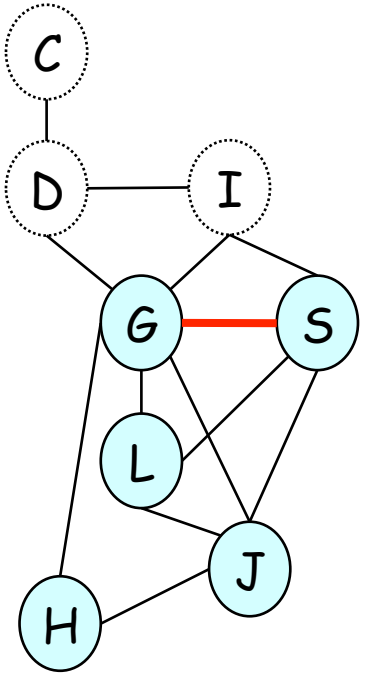
Induced Markov network for the current set of factors

Elimination as Graph Operation

$$\phi_J(J,L,S)\phi_L(L,G)\phi_H(H,G,J)\tau_3(S,G)$$

- Eliminate: H

$$\tau_4(G,J) = \sum_H \phi_H(H,G,J)$$

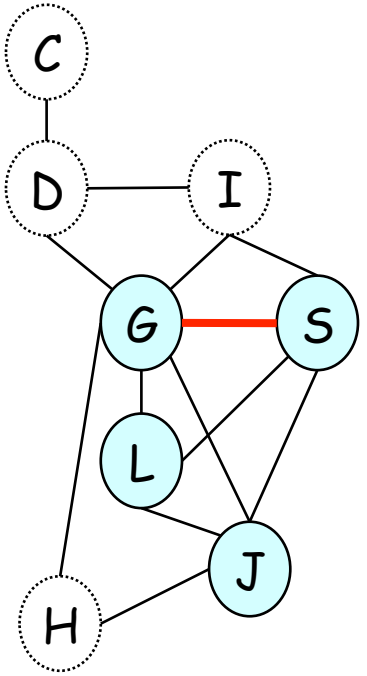


Induced Markov network for the current set of factors

Elimination as Graph Operation

$$\phi_J(J,L,S)\phi_L(L,G)\tau_3(S,G)\tau_4(G,J)$$

- Eliminate: *G*



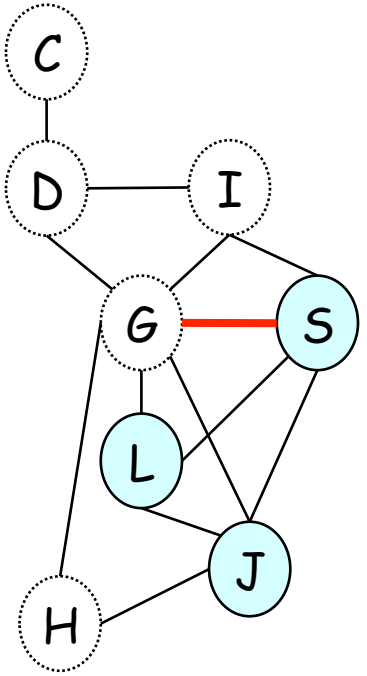
$$\tau_5(L,J) = \sum_G \phi_L(L,G)\tau_3(S,G)\tau_4(G,J)$$

Induced Markov network for the current set of factors

Elimination as Graph Operation

$$\phi_J(J,L,S)\tau_5(L,J)$$

- Eliminate: L,S

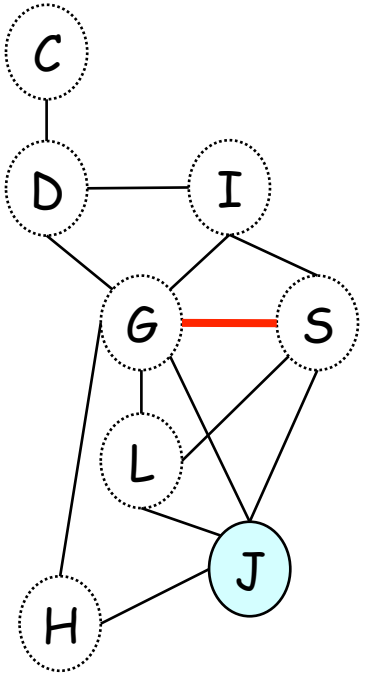


Induced Markov network for the current set of factors

Elimination as Graph Operation

$$\phi_J(J,L,S)\tau_5(L,J)$$

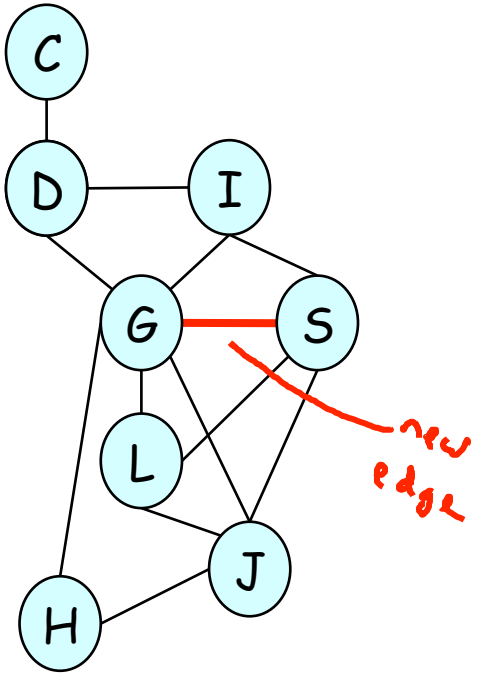
- Eliminate: L,S



Induced Markov network for the current set of factors

Induced Graph

- The induced graph $I_{\Phi,\alpha}$ over factors Φ and ordering α:
  - Undirected graph
  - X_i and X_j are connected if they appeared in the same factor in a run of the VE algorithm using α as the ordering



Cliques in the Induced Graph

- **Theorem**: Every factor produced during VE is a clique in the induced graph

$$\tau_1(D) = \sum_C \phi_C(C)\phi_D(C,D)$$

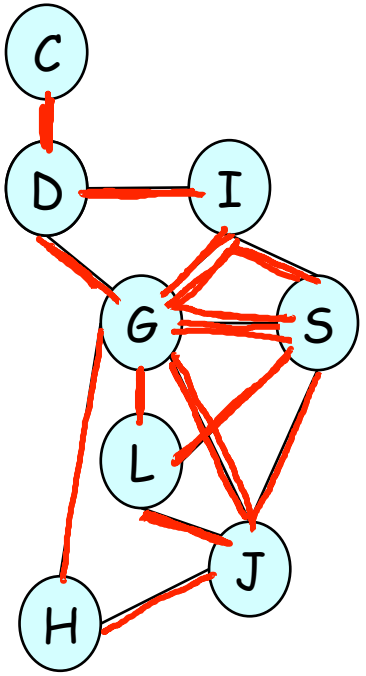
$$\tau_2(G,I) = \sum_D \phi_G(G,I,D)\tau_1(D)$$

$$\tau_3(S,G) = \sum_I \phi_S(S,I)\phi_I(I)\tau_2(G,I)$$

$$\tau_4(G,J) = \sum_H \phi_H(H,G,J)$$

$$\tau_5(L,J) = \sum_G \phi_L(L,G)\tau_3(S,G)\tau_4(G,J)$$

$$\tau_6 = \sum_{L,S} \phi_J(J,L,S)\tau_5(L,J)$$



Cliques in the Induced Graph

- **Theorem:** Every (maximal) clique in the induced graph is a factor produced during VE

$$\tau_1(D) = \sum_C \phi_C(C)\phi_D(C,D)$$

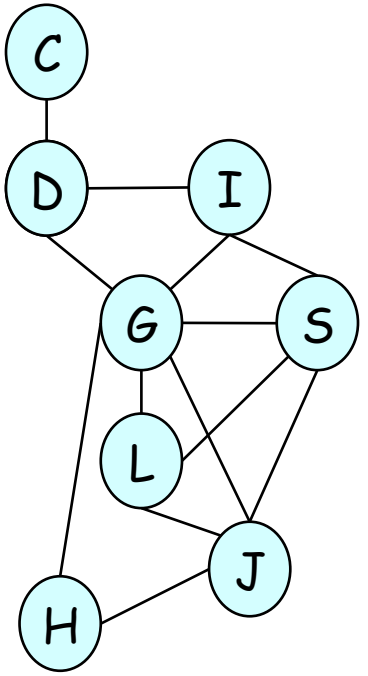
$$\tau_2(G,I) = \sum_D \phi_G(G,I,D)\tau_1(D)$$

$$\tau_3(S,G) = \sum_I \phi_S(S,I)\phi_I(I)\tau_2(G,I)$$

$$\tau_4(G,J) = \sum_H \phi_H(H,G,J)$$

$$\tau_5(L,J) = \sum_G \phi_L(L,G)\tau_3(S,G)\tau_4(G,J)$$

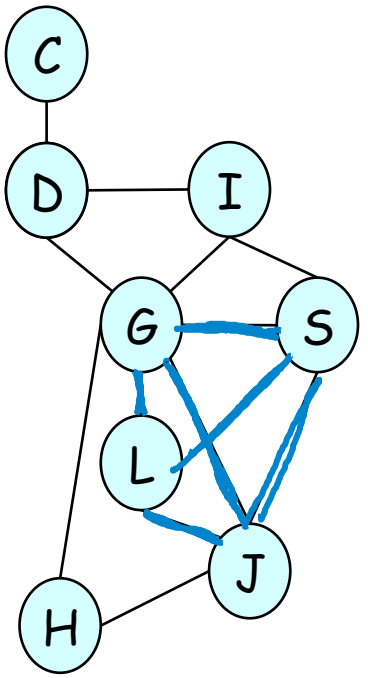
$$\tau_6 = \sum_{L,S} \phi_J(J,L,S)\tau_5(L,J)$$



Cliques in the Induced Graph

- **Theorem:** Every (maximal) clique in the induced graph is a factor produced during VE

Consider a max clique –
some variable is first to be eliminated
once a variable is eliminated - no new neighbors are added to it
⇒ when eliminated it already had all the clique members as neighbors
⇒ participated in factor with all these other variables
⇒ when multiplied together, we have a factor over all of them



Induced Width

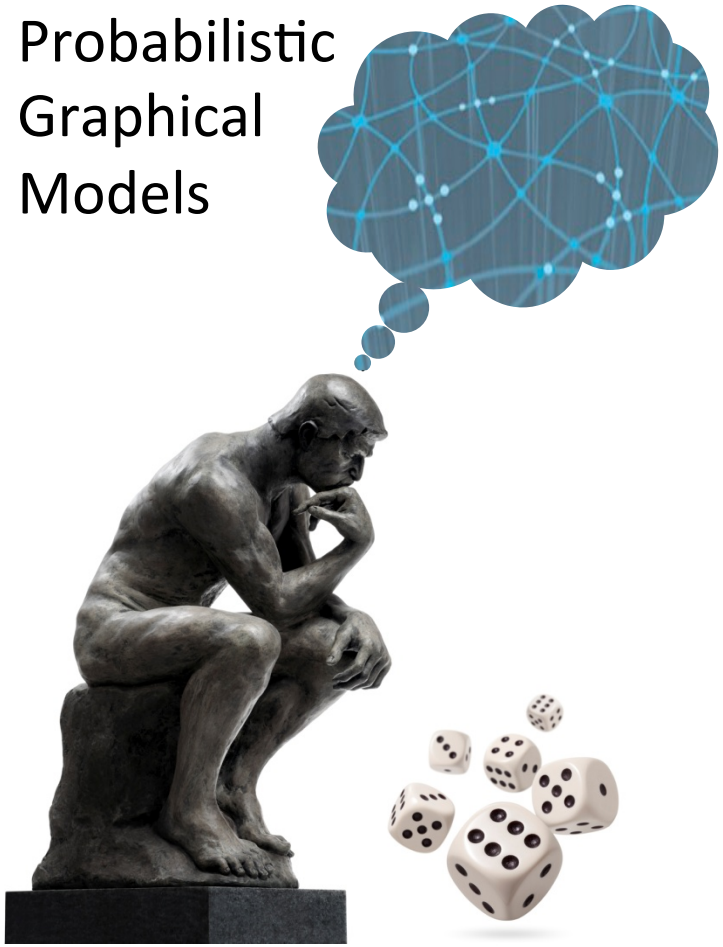
- The width of an induced graph is the number of nodes in the largest clique in the graph minus 1
- Minimal induced width of a graph K is $\min_{\alpha}(\text{width}(I_{K,\alpha}))$
- Provides a lower bound on best performance of VE to a model factorizing over K

Summary

- Variable elimination can be viewed as transformations on undirected graph
  - Elimination connects all node's current neighbors

- Cliques in resulting induced graph directly correspond to algorithm's complexity

Probabilistic Graphical Models



Inference

Variable Elimination

Finding Elimination Orderings

Finding Elimination Orderings

- Theorem: For a graph H, determining whether there exists an elimination ordering for H with induced width ≤ K is NP-complete

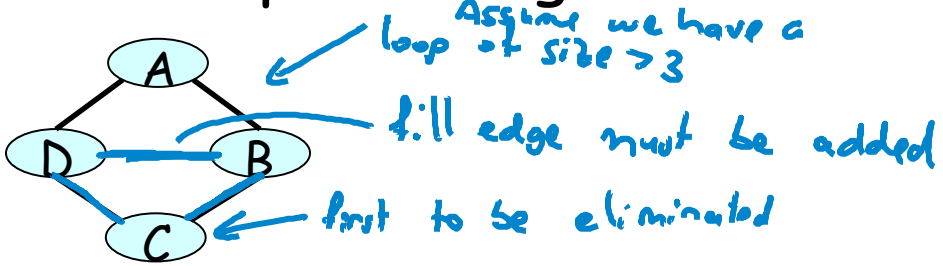
- Note: This NP-hardness result is distinct from the NP-hardness result of inference
  - Even given the optimal ordering, inference may still be exponential

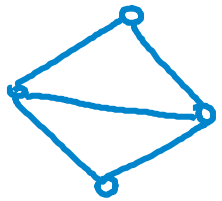
Finding Elimination Orderings

- Greedy search using heuristic cost function
  - At each point, eliminate node with smallest cost
- Possible cost functions:
  - min-neighbors: # neighbors in current graph (smallest factor)
  - min-weight: weight (# values) of factor formed
  - min-fill: number of new fill edges
  - weighted min-fill: total weight of new fill edges (edge weight = product of weights of the 2 nodes)

Finding Elimination Orderings

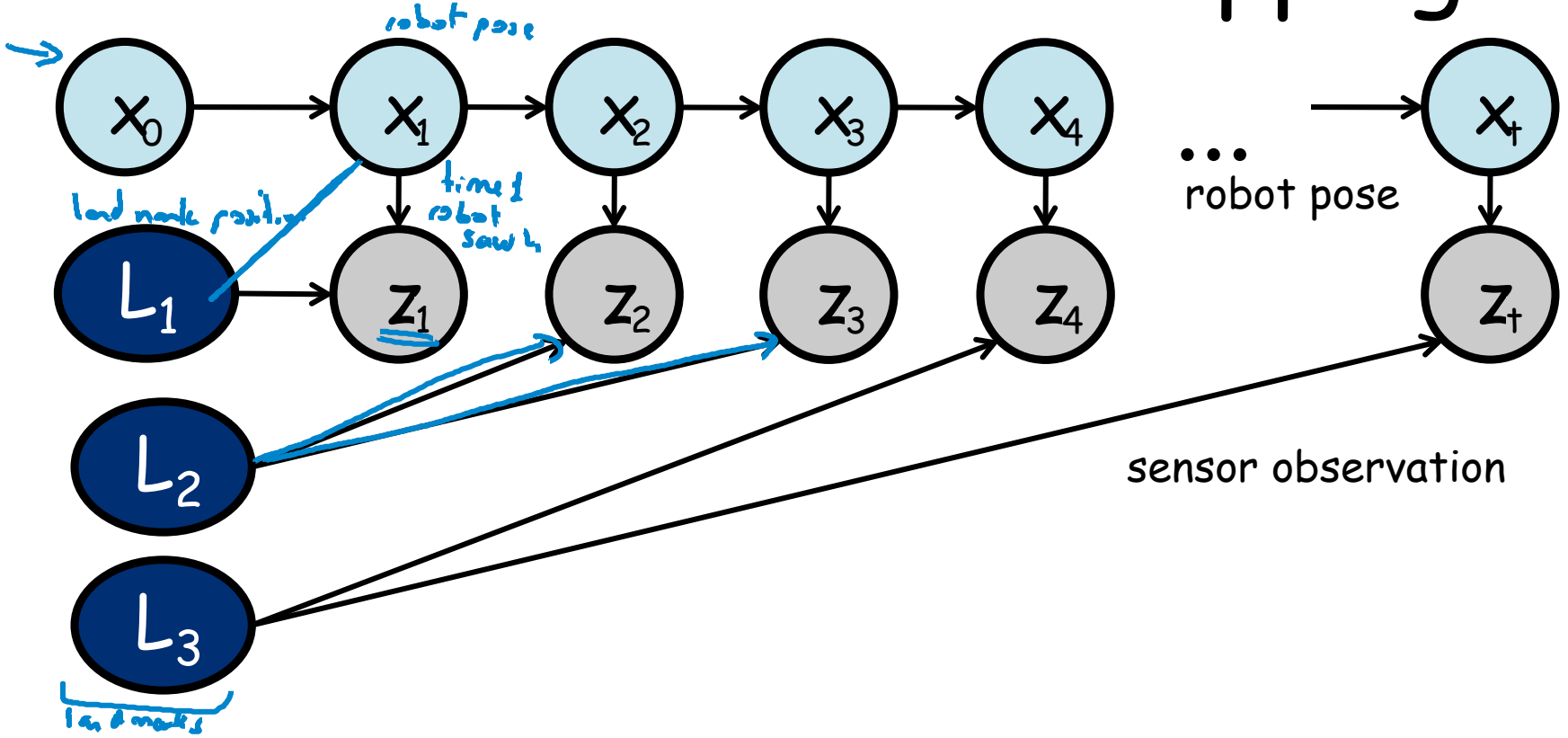
- Theorem: The induced graph is triangulated
  - No loops of length > 3 without a "bridge"





- Can find elimination ordering by finding a low-width triangulation of original graph H_Φ

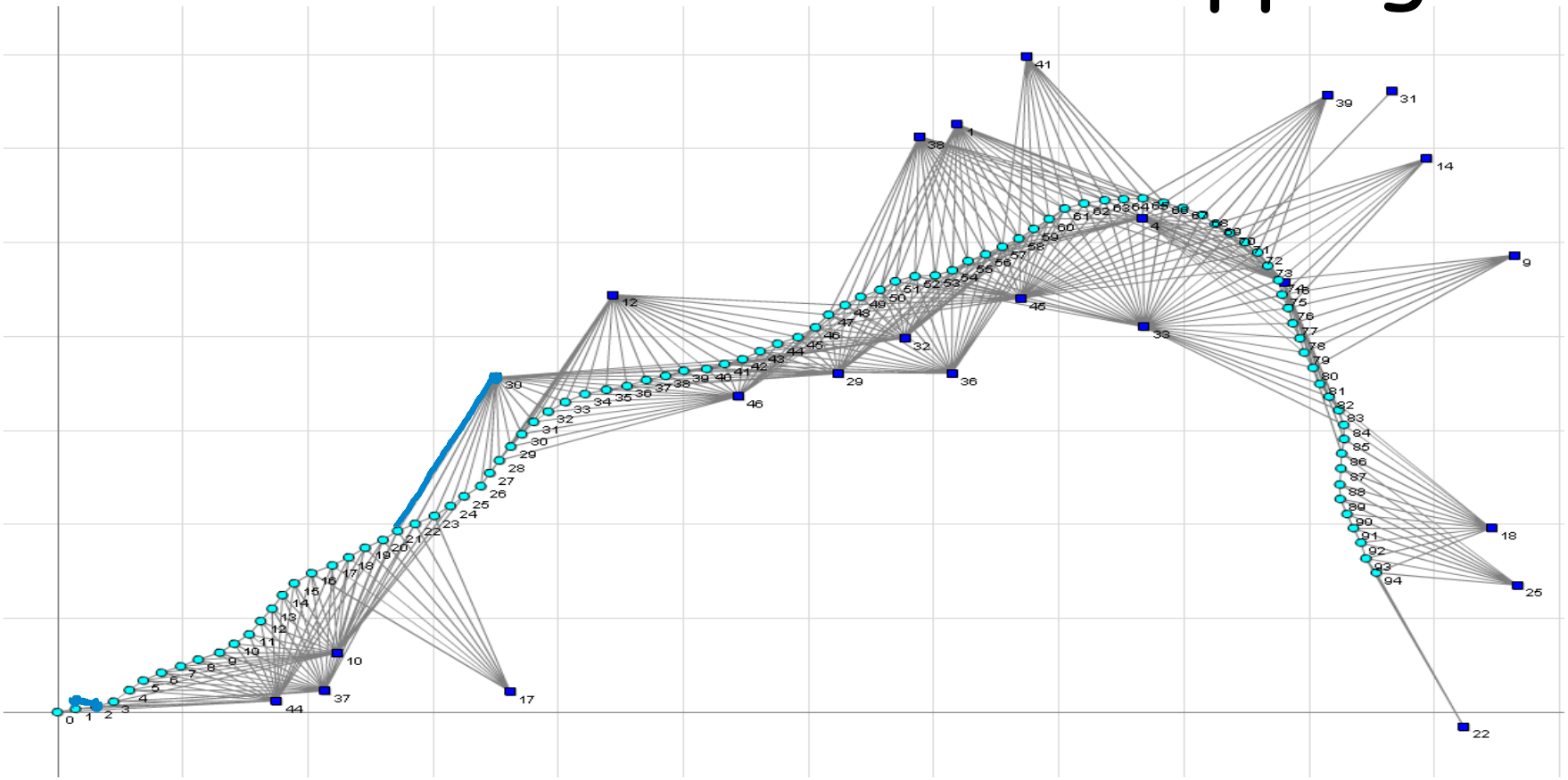
Robot Localization & Mapping



Daphne Koller

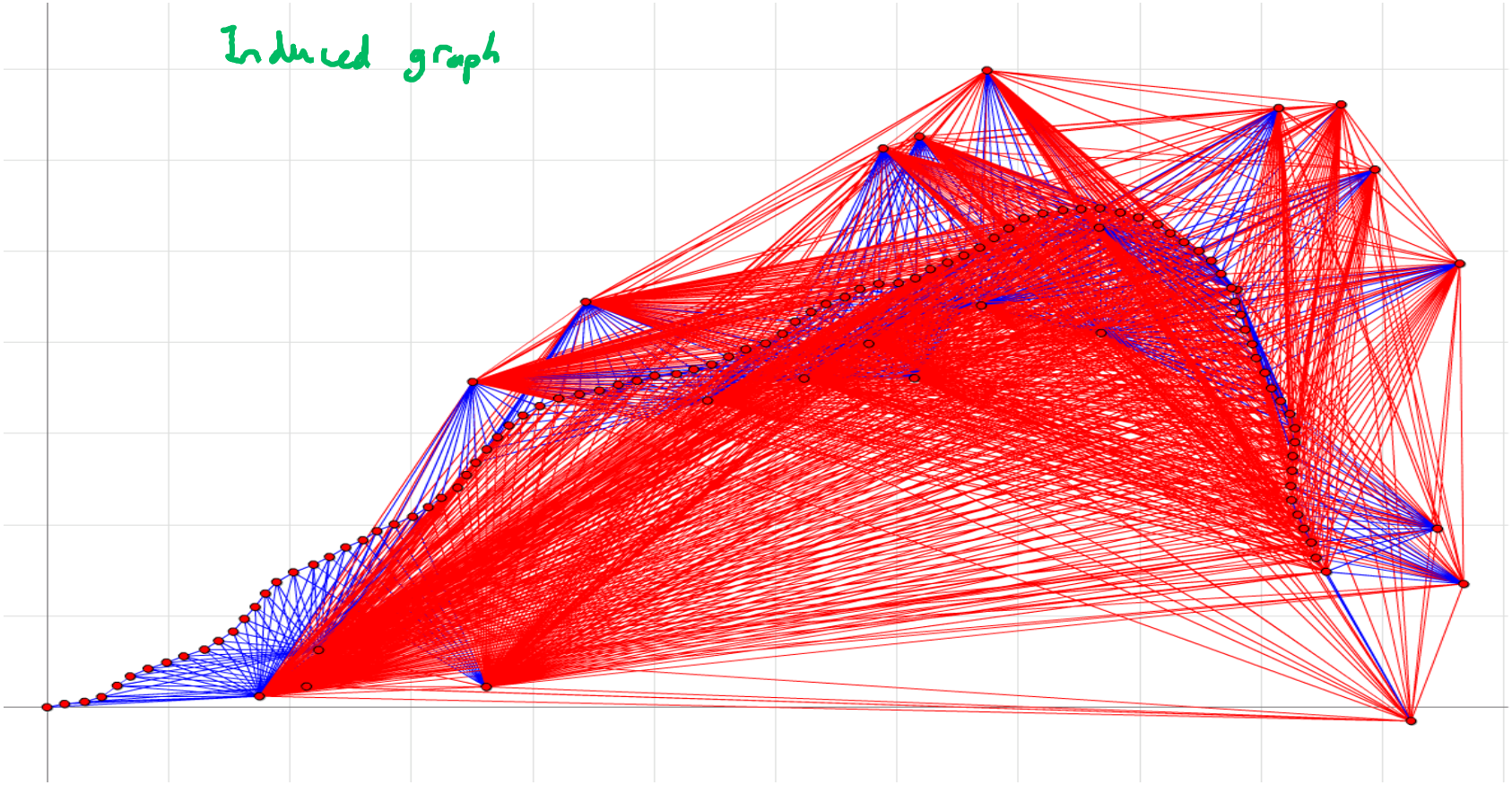
Square Root SAM, F. Dellaert and M. Kaess, IJRR, 2006

Robot Localization & Mapping



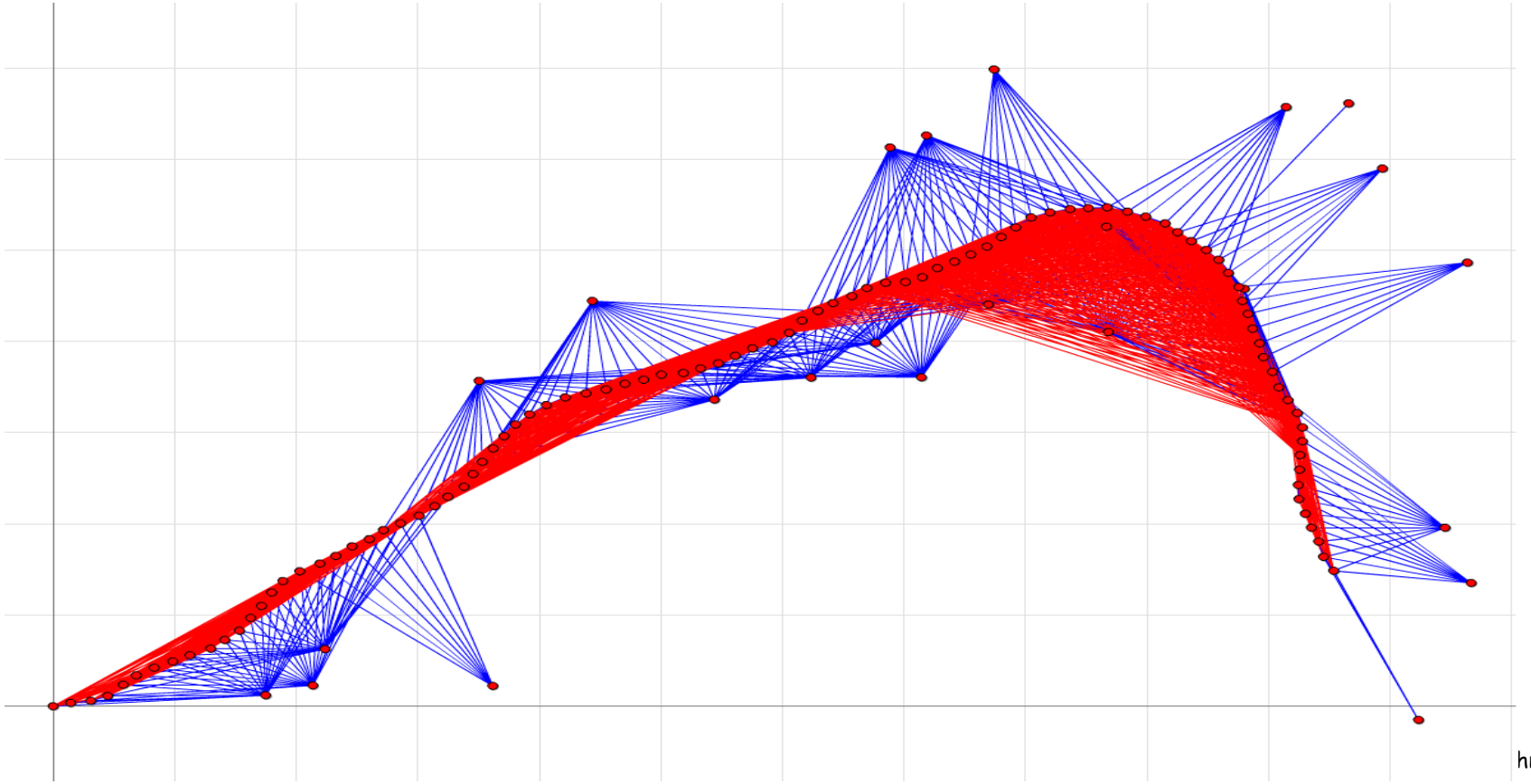
Square Root SAM, F. Dellaert and M. Kaess, IJRR, 2006

Eliminate Poses then Landmarks



Square Root SAM, F. Dellaert and M. Kaess, IJRR, 2006

Eliminate Landmarks then Poses



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Summary

- Finding the optimal elimination ordering is NP-hard
- Simple heuristics that try to keep induced graph small often provide reasonable performance