Reduced Basis methods: an introduction

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1 Notations, Definitions, Problem Statement, Example

Linear Compliant Elliptic Problems

Notations, Definitions, Problem Statement, Example

Definitions

Definition (Linear Space)

A space Z is a linear or vector space if, for any $\alpha \in \mathbb{R}$, $w, v \in Z$, $\alpha w + v \in Z$

Note: $\mathbb{R}$ denotes the real numbers, and $\mathbb{N}$ and $\mathbb{C}$ shall denote the natural and complex numbers, respectively.

Definition (Inner Product Space)

An inner product space (or Hilbert space) Z is a linear space equipped with

- an inner product $(w, v)_Z, \forall w, v \in Z$,and
- induced norm $\|w\|_Z = (w, w)_Z, \forall w \in Z$.

Inner Product

Definition (Inner Product)

An inner product $w, v \in Z \rightarrow (w, v)_Z \in \mathbb{R}$ has to satisfy

- Bilinearity
$$(\alpha w + v, z)_Z = \alpha (w, z)_Z + (v, z)_Z \forall \alpha \in R, w, v, z \in Z$$
$$(z, \alpha w + v)_Z = \alpha (z, w)_Z + (z, v)_Z, \forall \alpha \in R, w, v, z \in Z$$
- Symmetry
$$(w, v)_Z = (v, w)_Z, \forall w, v \in Z$$
- Positivity
$$(w, w)_Z > 0, \forall w \in Z, w \neq 0$$
$$(w, w)_Z = 0, \text{ only if } w = 0$$

Cauchy-Schwarz inequality:

$$(w, v)_Z \leq \|w\|_Z \|v\|_Z, \forall w, v \in Z.$$

Norm

Definition (Norm)

A norm is a map $\|\cdot\| : Z \to \mathbb{R}$ such that

- $\|w\|_Z > 0 \quad \forall w \in Z, w \neq 0,$
- $\|\alpha w\|_Z = |\alpha| \|w\|_Z \quad \forall \alpha \in \mathbb{R},\ \forall w \in Z,$
- $\|w + v\|_Z \leq \|w\|_Z + \|v\|_Z \quad \forall w \in Z,\ \forall v \in Z.$

Equivalence of norms $\|\cdot\|_Z$ and $\|\cdot\|_Y$: there exist positive constants C_1, C_2 such that

$$C_1 \|v\|_Z \leq \|v\|_Y \leq C_2 \|v\|_Z.$$

Cartesian Product Space

Given two inner product spaces Z_1 and Z_2, we define

$$Z = Z_1 \times Z_2 \equiv \{(w_1, w_2) \mid w_1 \in Z_1,\ w_2 \in Z_2\}$$

and given $w = (w_1, w_2) \in Z, v = (v_1, v_2) \in Z$, we define

$$w + v \equiv (w_1 + v_1, w_2 + v_2).$$

We also equip Z with the inner product

$$(w, v)_Z = (w_1, v_1)_{Z_1} + (w_2, v_2)_{Z_2}$$

and induced norm

$$\|w\|_Z = (w, w)_Z.$$

Linear Forms

Definition (Linear functional)

A functional $g : Z \to \mathbb{R}$ is a linear functional if, for any $\alpha \in \mathbb{R}, w, v \in Z$

$$g(\alpha w + v) = \alpha g(w) + g(v)$$

A linear form is bounded, or continuous, over Z if

$$|g(v)| \leq C\|v\|_Z, \forall v \in Z,$$

for some finite real constant C.

Dual Spaces

Definition (Dual Space)

Given Z, we define the dual space Z' as the space of all bounded linear functionals over Z. We associate to Z' the dual norm

$$\|g\|_{Z'} = \sup_{v \in Z} \frac{g(v)}{\|v\|_Z}, \forall g \in Z'.$$

Theorem (Riesz representation)

For any $g \in Z'$, there exists a unique $w_g \in Z$ such that

$$(w_g, v)_Z = g(v), \forall v \in Z.$$

It directly follows that

$$\|g\|_{Z'} = \|w_g\|_Z.$$

Bilinear Forms

Definition (Bilinear Form)

A form $b : Z_1 \times Z_2 \to \mathbb{R}$ is bilinear if, for any $\alpha \in R$,

- $b(\alpha w + v, z) = \alpha b(w, z) + b(v, z), \forall w, v \in Z_1, z \in Z_2$
- $b(z, \alpha w + v) = \alpha b(z, w) + b(z, v), \forall z \in Z_1, w, v \in Z_2$

The bilinear form $b : Z \times Z \to \mathbb{R}$ is

- symmetric, if
$$b(w, v) = b(v, w),$$
- skew-symmetric, if
$$b(w, v) = -b(v, w),$$
- positive definite, if
$$b(v, v) \geq 0 \text{ , with equality only for } v = 0.$$

Bilinear Forms

The bilinear form $b : Z \times Z \to \mathbb{R}$ is positive semidefinite, if

$$b(v, v) \geq 0, \forall v \in Z.$$

We also define, for a general bilinear form $b : Z \times Z \to \mathbb{R}$, the

- symmetric part as

$$b_S(w, v) = 1/2(b(w, v) + b(v, w)), \forall w, v \in Z;$$

- the skew-symmetric part as

$$b_{SS}(w, v) = 1/2(b(w, v) - b(v, w)), \forall w, v \in Z.$$

Bilinear Forms

The bilinear form $b : Z \times Z \to \mathbb{R}$ is

- coercive over Z if

$$\alpha \equiv \inf_{w \in Z} \frac{b(w,w)}{\|w\|_Z^2}$$

  is positive;

- continuous over Z if

$$\gamma \equiv \sup_{w \in Z} \sup_{v \in Z} \frac{b(w,v)}{\|w\|_Z \|v\|_Z}$$

  is finite.

Parametric Linear and Bilinear Forms

We introduce

- $D \in \mathbb{R}^P$: closed bounded parameter domain;
- $\mu = (\mu_1, \ldots, \mu_P) \in D$: parameter vector.

We shall say that

- $g : Z \times D \to \mathbb{R}$ is a parametric linear form if, for all $\mu \in D, g(\cdot;\mu) : Z \to \mathbb{R}$ is a linear form;
- $b : Z \times Z \times D \to \mathbb{R}$ is a parametric bilinear form if,for all $\mu \in D, b(\cdot,\cdot;\mu) : Z \times Z \to \mathbb{R}$ is a bilinear form.

Concepts of symmetry,. . . directly extend to the parametric case.

Parametric Linear and Bilinear Forms

The parametric bilinear form $b : Z \times Z \times D \to \mathbb{R}$ is

- coercive over Z if
$$\alpha(\mu) \equiv \inf_{w \in Z} \frac{b(w, w; \mu)}{\|w\|_Z^2}$$
is positive for all $\mu \in D$;
- continuous over Z if
$$\gamma(\mu) \equiv \sup_{w \in Z} \sup_{v \in Z} \frac{b(w, v; \mu)}{\|w\|_Z \|v\|_Z}$$
is finite for all $\mu \in D$.

We also define
$$(0 <)\alpha_0 \equiv \min_{\mu \in D} \alpha(\mu)$$
$$\gamma_0 \equiv \max_{\mu \in D} \gamma(\mu)(< \infty).$$

Coercivity EigenProblem

We have

$$\alpha(\mu) \equiv \inf_{w \in Z} \frac{b_S(w, w; \mu)}{\|w\|_Z^2}$$

Associated generalized eigenproblem:

Given $\mu \in D$, find $(\chi^{co}, \nu^{co})_i(\mu) \in Z \times \mathbb{R}, 1 \leq i \leq \dim(Z)$, such that

$$b_S(\chi_i^{co}(\mu), v; \mu) = \nu_i^{co}(\mu)(\chi_i^{co}(\mu), v)_Z$$

and

$$\|\chi_i^{co}(\mu)\|_Z = 1$$

Let $\nu_1^{co}(\mu) \leq \nu_2^{co}(\mu) \leq \ldots \leq \nu_{\dim Z}^{co}(\mu)$ and b coercive, then

$$\alpha(\mu) = \nu_1^{co}(\mu) > 0.$$

Parameter affine Dependence

We assume

$$g(v;\mu) = \sum_{q=1}^{Q_g} \theta_g^q(\mu) g^q(v), \forall v \in Z,$$

where, for $1 \leq q \leq Q_g$ (finite),

- parameter-dependent functions $\theta_g^q : D \to \mathbb{R}$,
- parameter-independent forms $g^q : Z \to \mathbb{R}$;

and

$$b(w,v;\mu) = \sum_{q=1}^{Q_b} \theta_b^q(\mu) b^q(w,v), \quad \forall w, v \in Z,$$

where, for $1 \leq q \leq Q_b$ (finite),

- parameter-dependent functions $\theta_b^q : D \to \mathbb{R}$,
- parameter-independent forms $b^q : Z \times Z \to \mathbb{R}$.

Parametric Coercivity

Definition (Parametric coercivity)

The coercive bilinear form $b : Z \times Z \times D \to \mathbb{R}$

$$b(w, v; \mu) = \sum_{q=1}^{Q_b} \theta_b^q(\mu) b^q(w, v), \quad \forall w, v \in Z,$$

is parametrically coercive if $c \equiv b_S$ is affine

$$c(w, v; \mu) = \sum_{q=1}^{Q_c} \theta_c^q(\mu) c^q(w, v), \quad \forall w, v \in Z,$$

and satisfies and

- $\theta_c^q(\mu) > 0, \forall \mu \in D, 1 \le q \le Q_c,$
- $c^q(v, v) \ge 0, \forall v \in Z, 1 \le q \le Q_c.$

Scalar and Vector Fields

We consider (real)

- scalar-valued field variables (e.g., temperature, pressure) $w : \Omega \to \mathbb{R}^{d=1}$
- vector-valued field variables (e.g., displacement, velocity) $\mathrm{w} : \Omega \to \mathbb{R}^{d}$, where $\mathrm{w}(x) = (w_1(x), \ldots, w_d(x))$;

and

- $\Omega \in \mathbb{R}^d, d = 1, 2, \text{or} 3$ is an open bounded domain
- $x = (x_1, ..., x_d) \in \Omega$;
- Ω has Lipschitz continuous boundary $\partial\Omega$; and
- we define the canonical basis vectors as $e_i, 1 \leq i \leq d$.

Multi-Index Derivative

Given a scalar (or one component of a vector)

- field $w : \Omega \to \mathbb{R}$ SPATIAL DERIVATIVE

$$(D^\sigma w)(x) = \frac{\partial^\sigma w}{\partial x_1^{\sigma_1} \ldots \partial x_d^{\sigma_d}}$$

- parametric field $w : \Omega \times D \to \mathbb{R}$ SENSITIVITY DERIVATIVE

$$(D_\sigma w)(x) = \frac{\partial^\sigma w}{\partial \mu_1^{\sigma_1} \ldots \partial \mu_d^{\sigma_d}}$$

where

- $\sigma = (\sigma_1, \ldots, \sigma_d)$, $\sigma_i, 1 \leq i \leq d$, non-negative integers;
- $|\sigma| = \sum_{j=1}^d \sigma_j$ is the order of the derivative; and
- $I^{d,n}$ is set of all index vectors $\sigma \in N_0^d$ such that $|\sigma| \leq n$.

Function Spaces

Definition (Spaces of Continuous Functions)

Let $m \in N_0$, the space $C^m(\Omega)$ is defined as

$$C^m(\Omega) \equiv \{w | D^\sigma w \in C^0(\Omega), \forall \sigma \in I^{d,m}\},$$

and $C^0(\Omega)$ is the space of continuous functions over $\Omega \in \mathbb{R}^d$.

We denote by $C^\infty(\Omega)$ the space of functions w for which D^σ exists and is continuous for any order $|\sigma|$.

Lebesgue Spaces

Definition (Lebesgue Spaces)

We define, for $1 \leq p < \infty$, the Lebesgue space $L^p(\Omega)$ as

$$L^p(\Omega) \equiv \{w \text{ measurable} \mid \quad \|w\|_{L^p(\Omega)} < \infty\}$$

where

- $\|w\|_{L^p(\Omega)} \equiv \left(\int_\Omega |w|^p dx\right)^{1/p}, 1 \leq p < \infty,$
- $\|w\|_{L^\infty(\Omega)} \equiv \operatorname{ess\,sup}_{x\in\Omega} |w(x)|, p = \infty.$

Hilbert Space

Definition (Hilbert Spaces)

Let $m \in \mathbb{N}_0$, the space $H^m(\Omega)$ is then defined as

$$H^m(\Omega) \equiv \{w| \quad D^\sigma w \in L^2(\Omega), \forall \sigma \in I^{d,m}\},$$

with associated inner product

$$(w, v)_{H^m(\Omega)} \equiv \sum_{\sigma \in I^{d,m}} \int_\Omega D^\sigma w D^\sigma v dx,$$

and induced norm

$$\|w\|_{H^m(\Omega)} \equiv \sqrt{(w, w)_{H^m(\Omega)}}.$$

Special (most important) cases

Since we only consider second-order PDEs, we require mostly

- $L^2(\Omega) = H^0(\Omega)$: Lebesgue Space $p = 2$

$$(w, v)_{L^2(\Omega)} = \int_\Omega wv \quad \forall w, v \in L^2(\Omega)$$

$$\|w\|_{L^2(\Omega)} = \sqrt{(w, w)_{L^2(\Omega)}} \forall w \in L^2(\Omega),$$

$\Rightarrow$ Space of all functions $w : \Omega \to \mathbb{R}$ square-integrable over Ω .

Special (most important) cases

Since we only consider second-order PDEs, we require mostly

- $H^1(\Omega)$

$$H^1(\Omega) \equiv \{w \in L^2(\Omega) | \frac{\partial w}{\partial xi} \in L^2(\Omega), 1 \leq i \leq d\}$$

with inner product and induced norm

$$(w, v)_{H^1(\Omega)} \equiv \int_\Omega \nabla w \cdot \nabla v + wv \quad \forall w, v \in H^1(\Omega),$$

,

$$\|w\|_{H^1(\Omega)} \equiv \sqrt{(w, w)_{H^1(\Omega)}} \quad \forall w \in H^1(\Omega),$$

and seminorm

$$|w|_{H^1(\Omega)} \equiv \int_\Omega \nabla w \cdot \nabla w, \quad \forall w \in H^1(\Omega).$$

Special (most important) cases

Since we only consider second-order PDEs, we require mostly

- the space $H^1_0(\Omega)$

$$H^1_0(\Omega) \equiv \{v \in H^1(\Omega) | v_{|\partial\Omega} = 0\}$$

where $v = 0$ on the boundary $\partial\Omega$.
Note that, for any $v \in H^1_0(\Omega)$, we have

$$C_{PF}\|v\|_{H^1(\Omega)} \le |v|_{H^1(\Omega)} \le \|v\|_{H^1(\Omega)},$$

and thus

$$\|v\|_{H^1(\Omega)} = 0 \Rightarrow v = 0$$

$\Rightarrow |v|_{H^1(\Omega)}$ constitutes a norm for $v \in H^1_0(\Omega)$.

Projection

Definition (Projection)

Given Hilbert Spaces Y and $Z \subset Y$, the projection, $\Pi : Y \to Z$, of $y \in Y$ onto Z is defined as

$$(\Pi y, v)_Y = (y, v)_Y, \forall v \in Z$$

Properties:

- Orthogonality: $(y - \Pi y, v)_Y = 0$
- Idempotence: $\Pi(\Pi y) = \Pi y$
- Best Approximation $\|y - \Pi y\|_Y^2 = \inf_{v \in Z} \|y - v\|_Y^2$,

Given an orthonormal basis $\{\varphi i\}_{i=1,N=\dim(Z)}$, then

$$\Pi y = \sum_{i=1}^{\dim(Z)} (\varphi i, y)_Y \varphi_i, \forall y \in Y$$

Notations and Definitions

Notations

- $(\cdot)^{\mathcal{N}}$ finite element approximation
- $(\cdot)_N$ reduced basis approximation
- μ input parameter (physical, geometrical,...)
- $s(t;\mu) \approx s^{\mathcal{N}}(t;\mu) \approx s_N(t;\mu)$ output approximations
- $\mu \to s(t;\mu)$ input-output relationship

Definitions

- $\Omega \subset \mathbb{R}^d$ spatial domain
- μ P-uplet
- $\mathcal{D}^\mu \subset \mathbb{R}^P$ parameter space
- s output, ℓ, f functionals
- u field variable
- X function space $H^1_0(\Omega)^\nu \subset X \subset H^1(\Omega)^\nu$ ($\nu = 1$ for simplicity) $(\cdot,\cdot)_X$ scalar product and $\|\cdot\|_X$ norm associated to X

Problem Statement

The formal problem statement reads: Given $\mu \in \mathcal{D}^\mu$, evaluate

$$s(\mu) = \ell(u(\mu); \mu)$$

where $u(x; \mu) \in X$ satisfies

$$a(u(\mu), v; \mu) = f(v; \mu), \quad \forall v \in X$$

Remark

We consider first the case of linear affine compliant elliptic problem and then complexify

Hypothesis: Reference Geometry

Note $\boldsymbol{\Omega}$ is parameter-independent: the reduced basis requires a common spatial configuration, i.e., a reference domain Ω_{ref}

Introduce a piecewise affine mapping $\mathcal{T}(\cdot;\mu) : \Omega \to \Omega_o(\mu)$

$$a_o\left(w_o, v_o; \mu\right) \text{ over } \Omega_o(\mu)$$
$$\Downarrow$$
$$\mathcal{T}(\cdot;\mu)^{-1} : \Omega_o(\boldsymbol{\mu}) \to \Omega_{\text{ref}} \equiv \Omega \quad \left(\Omega_{\text{ref}} = \Omega_o\left(\mu_{\text{ref}}\right)\right)$$
$$\Downarrow$$
$$a(w, v; \mu) \text{ over } \Omega$$

where

$$a(w, v; \mu) = a_o\left(w_o \circ \mathcal{T}_\mu, v_o \circ \mathcal{T}_\mu; \mu\right)$$

We will discuss this issue in detail later on.

Hypothesis: Continuity, stability, compliance

We consider the following μ−PDE

$$
\begin{array}{rl}
a(\cdot,\cdot;\mu) & \text{bilinear} \\
 & \text{symmetric} \\
 & \text{continuous} \\
 & \text{coercive } (\forall\mu \in \mathcal{D}^\mu) \\
 & \\
f(\cdot;\mu), \ell(\cdot;\mu) & \text{linear} \\
 & \text{bounded } (\forall\mu \in \mathcal{D}^\mu)
\end{array}
$$

and in particular, to start, the compliant case

- a symmetric
- $f(\cdot;\mu) = \ell(\cdot;\mu) \quad \forall\mu \in \mathcal{D}^\mu$

Hypothesis: Affine dependence in the parameter

We require for the RB methodology

$$a(u, v; \mu) = \sum_{q=1}^{Q_a} \Theta_a^q(\mu)\ a^q(u, v),$$

where for $q = 1, \dots, Q_a$

$$\begin{aligned} \Theta_a^q :&\quad \mathcal{D}^\mu \to \mathbb{R} & \mu - \text{dependent functions} \\ a^q :&\quad X \times X \to \mathbb{R} & \mu - \text{independent bilinear forms} \end{aligned}$$

Remark

- *similar decomposition is required for $\ell(v; \mu)$ and $f(v; \mu)$, and denote Q_ℓ and Q_f the corresponding number of terms*
- *applicable to a large class of problems including geometric variations*
- *can be relaxed (see non affine/non linear problems)*

Inner Products and Norms

- energy inner product and associated norm (parameter dependent)

$$
\begin{aligned}
(((w,v)))_\mu &= a(w,v;\mu) && \forall u, v \in X\\
|||v|||_\mu &= \sqrt{a(v,v;\mu)} && \forall v \in X
\end{aligned}
$$

- X-inner product and associated norm (parameter independent)

$$
\begin{aligned}
(w,v)_X &= (((w,v)))_{\bar\mu}\ (\equiv a(w,v;\bar\mu)) && \forall u, v \in X\\
||v||_X &= |||v|||_{\bar\mu}\ (\equiv \sqrt{a(v,v;\bar\mu)}) && \forall v \in X
\end{aligned}
$$

Coercivity and Continuity Constants

We assume *a* coercive and continuous

Recall that

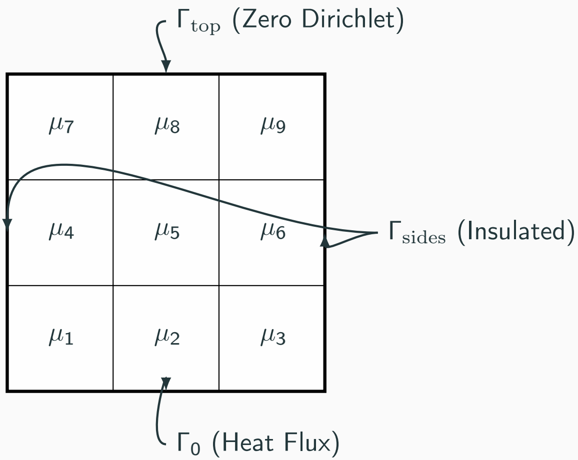
- coercivity constant

$$(0 <)\alpha(\mu) \equiv \inf_{v\in X} \frac{a(v,v;\mu)}{||v||_X^2}$$

- Continuity constant

$$\gamma(\mu) \equiv \sup_{w\in X} \sup_{v\in X} \frac{a(w,v;\mu)}{\|w\|_X \|v\|_X} (< \infty)$$

Example Thermal Block: Heat Transfer



Example Thermal Block: Problem statement

Given $\mu \in (\mu_1, ...\mu_P) \in \mathcal{D}^\mu \equiv [\mu^{\mathsf{min}}, \mu^{\mathsf{max}}]^P$, evaluate (recall that $\ell = f$)

$$s(\mu) = f(u(\mu))$$

where $u(\mu) \in X \equiv \{v \in H^1(\Omega), v|_{\Gamma_{\mathbf{top}}} = 0\}$ satisfies

$$a(u(\mu), v; \mu) = f(v; \mu) \quad \forall v \in X$$

we have $P = 8$ and given $1 < \mu_r < \infty$ we set

$$\mu^{\min} = 1/\sqrt{\mu_r}, \quad \mu^{\max} = \sqrt{\mu_r}$$

such that $\mu^{\max}/\mu^{\min} = \mu_r$.

Example Thermal Block

Recall we are in the compliant case $\ell = f$, we have

$$f(v) = \int_{\Gamma_0} v \quad \forall v \in X$$

and

$$a(u, v; \mu) = \sum_{i=1}^{P} \mu_i \int_{\Omega_i} \nabla u \cdot \nabla v + 1 \int_{\Omega_{P+1}} \nabla u \cdot \nabla v \quad \forall u, \ v \ \in X$$

where $\Omega = \cup_{i=1}^{P+1} \Omega_i$.

Example Thermal Block

The inner product is defined as follows

$$(u,v)_X = \sum_{i=1}^{P} \bar{\mu}_i \int_{\Omega_i} \nabla u \cdot \nabla v + 1 \int_{\Omega_{P+1}} \nabla u \cdot \nabla v$$

where $\bar{\mu}_i$ is a reference parameter. We have readily that a is

- symmetric
- parametrically coercive

$$0 < \frac{1}{\sqrt{\mu_r}} \leq \min(\mu_1/\bar{\mu}_1, \ldots, \mu_P/\bar{\mu}_P, 1) \leq \alpha(\mu)$$

- and continuous

$$\gamma(\mu) \leq \max(\mu_1/\bar{\mu}_1, \ldots, \mu_P/\bar{\mu}_P, 1) \leq \sqrt{\mu_r} < \infty$$

and the linear form f is bounded.

Example Thermal Block: Affine decomposition

We obtain the affine decomposition

$$a(u,v;\mu) = \sum_{q=1}^{P+1} \Theta^q(\mu) a^q(u,v)$$

with

$$\Theta^1(\mu) = \mu_1 \qquad a^1(u,v) = \int_{\Omega_1} \nabla u \cdot \nabla v$$

$$\vdots$$

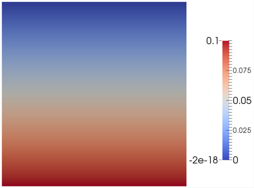
$$\Theta^P(\mu) = \mu_P \qquad a^P(u,v) = \int_{\Omega_P} \nabla u \cdot \nabla v$$

$$\Theta^{P+1}(\mu) = 1 \qquad a^{P+1}(u,v) = \int_{\Omega_{P+1}} \nabla u \cdot \nabla v$$

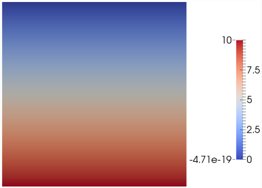
Example Thermal Block

- Homogeneous parameters

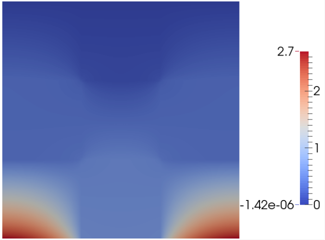
- **Maximum parameters values.**



- **Minimum parameters values.**



- Heterogeneous parameters



"Truth" FEM Approximation

Let $\mu \in \mathcal{D}^\mu$, evaluate

$$s^{\mathcal{N}}(\mu) = \ell(u^{\mathcal{N}}(\mu)) \ ,$$

where $u^{\mathcal{N}}(\mu) \in X^{\mathcal{N}}$ satisfies

$$a(u^{\mathcal{N}}(\mu), v; \mu) = f(v), \quad \forall\, v \in X^{\mathcal{N}} \ .$$

Here $X^{\mathcal{N}} \subset X$ is a Truth finite element approximation of dimension $\boxed{\mathcal{N} \gg 1}$ equiped with an inner product $(\cdot,\cdot)_X$ and induced norm $||\cdot||_X$. Denote also X' and associated norm

$$\ell \in X', \qquad ||\ell||_{X'} \equiv \sup_{v \in X} \frac{\ell(v)}{||v||_X}$$

.

Purpose

- Equate $u(\mu)$ and $u_{\mathcal{N}}(\mu)$ in the sense that

$$||u(\mu) - u_{\mathcal{N}}(\mu)||_X \leq \mathrm{tol} \quad \forall \mu \in \mathcal{D}^{\mu}$$

- Build the reduced basis approximation using the FEM approximation
- Measure the error associated with the reduced basis approximation relative to the FEM approximation

$\Rightarrow u^{\mathcal{N}}(\mu)$ is a calculable surrogate for $u(\mu)$.

$$\|u(\mu) - u^{\mathcal{N}}(\mu)\|_X \leq \underbrace{\|u(\mu) - u^{\mathcal{N}}(\mu)\|_X}_{\leq \varepsilon^{\mathcal{N}}} + \underbrace{\|u^{\mathcal{N}}(\mu) - u^{N}(\mu)\|_X}_{\varepsilon_{\mathrm{tol,min}}}$$

with $\varepsilon^{\mathcal{N}} << \varepsilon_{\mathrm{tol,min}}$