Reduced Basis methods: an introduction

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January 30, 2023

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❶ Non Compliant Output and/or Non-Symmetric Elliptic Problems

Reduced Basis Approximation: Primal Only Formulation

Non Compliant/Non Symmetric

"Truth" FEM Approximation

Let $\mu \in \mathcal{D}^\mu$, evaluate

$$s^{\mathcal{N}}(\mu) = \ell(u^{\mathcal{N}}(\mu)) \ ,$$

where $u^{\mathcal{N}}(\mu) \in X^{\mathcal{N}} \subset X$ satisfies

$$a(u^{\mathcal{N}}(\mu), v; \mu) = f(v), \quad \forall\, v \in X^{\mathcal{N}} \ .$$

and we suppose that

- $a(\cdot, \cdot; \mu)$ is bilinear, $f(\cdot; \mu$ and $\ell(\cdot; \mu)$ are linear
- f and ℓ are bounded
- $\ell \neq f$ (non-compliance) and/or a is non-symmetric

"Truth" FEM Approximation: Hypothesis

We assume that $a : X^{\mathcal{N}} \times X^{\mathcal{N}} \to \mathbb{R}$ is

- coercive

$$(0 <)\alpha(\mu) \equiv \inf_{v \in X} \frac{a(v,v;\mu)}{||v||_X^2}$$

- Continuous

$$\gamma(\mu) \equiv \sup_{w \in X} \sup_{v \in X} \frac{a(w,v;\mu)}{\|w\|_X \|v\|_X} (< \infty)$$

- and enjoys affine parametric dependence

$$a(u,v;\mu) = \sum_{q=1}^{Q_a} \Theta_a^q(\mu)\ a^q(u,v), \quad \forall u, v \in X$$

"Truth" FEM Approximation: Inner products and Norms

We next define the

- energy inner product and associated norm (parameter dependent)

$$(((w,v)))_\mu = a_s(w,v;\mu) \qquad \forall u,v \in X$$
$$|||v|||_\mu = \sqrt{a_S(v,v;\mu)} \qquad \forall v \in X$$

- X-inner product and associated norm (parameter independent)

$$(w,v)_X = (((w,v)))_{\bar\mu} \ (\equiv a_S(w,v;\bar\mu)) \qquad \forall u,v \in X$$
$$||v||_X = |||v|||_{\bar\mu} \ (\equiv \sqrt{a_S(v,v;\bar\mu)}) \qquad \forall v \in X$$

where a_s denotes the symmetric part of a.

Ingredients: Noncompliant problems

- Prominent example: convection-diffusion equation
- Reduced basis approximation
  - Galerkin optimality
- A posteriori error estimation
  - Lower bound for coercivity constant
  - Energy-norm and X-norm error bound
  - Output error bound
    Rightarrow Primal-dual formulation
- Offline-online decomposition
  - Additional cost due to dual problem
- Greedy sampling procedure

Spaces

Parameter Samples:

$$\text{Sample}: \quad S_N = \{\mu_1 \in \mathcal{D}^\mu, \ldots, \mu_N \in \mathcal{D}^\mu\} \quad 1 \le N \le N_{\max},$$

with

$$S_1 \subset S_2 \ldots S_{N_{\max}} \subset \mathcal{D}^\mu$$

Lagrangian Hierarchical Space

$$W_N = \operatorname{span}\{\zeta_n \equiv \underbrace{u(\mu^n)}_{u^{\mathcal{N}}(\mu^n)}, n = 1, \ldots, N\}.$$

with

$$W_1 \subset W_2 \ldots W_{N_{\max}} \subset X^{\mathcal{N}} \subset X$$

Formulation: Galerkin method

Galerkin Projection

Given $\mu \in \mathcal{D}^\mu$ evaluate

$$s_N(\mu) = \ell(u_N(\mu); \mu)$$

where $u_N(\mu) \in X_N$ satisfies

$$a(u_N(\mu), v; \mu) = f(v; \mu), \ \forall\, v \in X_N \ .$$

- RB Space $X_N = \mathrm{GramSchmidt}(W_N)$
- Well posed problem (there exists a unique solution: coercivity, continuity, linear independence)

Formulation: a priori convergence

Proposition

For any $\mu \in \mathcal{D}^\mu$, we have the following optimality(thanks to Galerkin) results

$$||u(\mu) - u_N(\mu)||_X \leq \left(1 + \frac{\gamma(\mu)}{\alpha(\mu)}\right) \inf_{v_N \in X_N} ||u(\mu) - v_N(\mu)||_X,$$

and

$$|s(\mu) - s_N(\mu)| \leq C||u(\mu) - v_N(\mu)||_X$$

Remember that

- symmetry : $\left(1 + \frac{\gamma(\mu)}{\alpha(\mu)}\right) \Rightarrow \sqrt{\frac{\gamma(\mu)}{\alpha(\mu)}}$;
- compliance : "quadratic convergence" in the output (not the case anymore)

Formulation: A Priori Convergence Theory

Proof.

- Use Galerkin orthogonality

$$a(u(\mu) - u_N, v_N; \mu) = 0 \quad \forall v_N \in X_N$$

- Note that for $v_N \in X_N$ we have

$$\|u(\mu) - u_N(\mu)\|_X \leq \|u(\mu) - v_N\|_X + \|v_N - u_N(\mu)\|_X$$

- Finally it holds that

$$\begin{aligned}\alpha(\mu)\|v_N - u_N(\mu)\|_X^2 &\leq a(v_N - u_N(\mu), v_N - u_N(\mu); \mu)\\ &= a(v_N - u(\mu), v_N - u_N(\mu); \mu)\\ &\leq \gamma(\mu)\|v_N - u(\mu)\|_X\|v_N - u_N(\mu)\|_X\end{aligned}$$

□

Formulation : a posteriori error estimation

We wish to develop rigorous, sharp and efficient online a posteriori error estimation $\Delta_N(\mu)$, $\Delta_N^s(\mu)$ such that $\forall \mu \in \mathcal{D}^\mu$

$$\|u(\mu) - u_N(\mu)\|_X \leq \Delta_N(\mu)$$
$$|s(\mu) - s_N(\mu)\| \leq \Delta_N^s(\mu)$$

Coercivity Lower Bound	OK
Error Bounds	OK (using a_s)

However two issues remain.

Formulation: coercivity lower bound

For a non-symmetric we introduce

$$a_s(u, v; \mu) = \sum_{q=1}^{Q_{a_s}} \Theta^q_{a_s}(\mu)\ a^q_s(u, v), \quad \forall u, v \in X$$

where

$$a_s(u, v; \mu) = \frac{1}{2}(a(u, v; \mu) + a(v, u; \mu))$$

We then apply either

- the "min Θ" approach if a_s is parametrically coercive

$$\alpha_{\mathrm{LB}}(\mu) \equiv \Theta^{\min,\bar{\mu}}_{a_s} = \min_{q \in \{1 \ldots Q_{a_s}} \frac{\Theta^q_{a_s}(\mu)}{\Theta^q_{a_s}(\bar{\mu})}$$

- or the SCM (a_s)

Formulation: a posteriori error bounds

Given our RB approximation $u_N(\mu)$, we have

$$e(\mu) \equiv u(\mu) - u_N(\mu)$$

that satisfies

$$a(e(\mu), v; \mu) \;=\; r(u_N(\mu), v; \mu), \forall v \in X$$

where $r(u_N(\mu), v; \mu) = f(v) - a(u_N(\mu), v; \mu)$ is the residual. We have then from coercivity and the definitions above that

$$||e(\mu)||_X \;\leq\; \frac{||r(u_N(\mu), v; \mu)||_{X'}}{\alpha(\mu)} \;=\; \frac{\varepsilon_N(\mu)}{\alpha(\mu)}$$

A Posteriori error estimation: Dual norm of the residual

Proposition

Given $\mu \in \mathcal{D}^\mu$, the dual norm of $r(u_N(\mu), \cdot; \mu)$ is defined as follows

$$||r(u_N(\mu), \cdot; \mu)||_{X'} \equiv \sup_{v \in X} \frac{r(u_N(\mu), v; \mu)}{||v||_X} = ||\hat{e}(\mu)||_X$$

where $\hat{e}(\mu) \in X$ satisfies

$$(\hat{e}(\mu), v)_X = r(u_N(\mu), v; \mu)$$

The error residual equation can then be rewritten

$$a(e(\mu), v; \mu) = (\hat{e}(\mu), v)_X, \quad \forall v \in X$$

A Posteriori error estimation: Dual norm of the residual

Then we can define

Definition: Energy error bound

$$\Delta_N(\mu) \equiv \frac{\|\hat{e}(\mu)\|_X}{\alpha_{\mathrm{LB}}(\mu)}$$

Definition: Effectivity

$$\eta_N(\mu) \equiv \frac{\Delta_N(\mu)}{||e(\mu)||_X}$$

Proposition

for $N = 1 \ldots N_{\max}$, the effectivity $\eta_N(\mu)$ verifies

$$1 \le \eta_N(\mu) \le \frac{\gamma_{\mathrm{UB}}(\mu)}{\alpha_{\mathrm{LB}}(\mu)}, \quad \forall \mu \in \mathcal{D}^{\mu}.$$

A Posteriori error estimation: Output error bound

Then we can define

Definition: Output error bound

$$\Delta_N^s(\mu) \equiv \|\ell(\cdot,\mu)\|_{X'} \Delta_N(\mu)$$

Definition: Output Effectivity

$$\eta_N^s(\mu) \equiv \frac{\Delta_N^s(\mu)}{|s(\mu) - s_N(\mu)|}$$

Proposition

for $N = 1 \ldots N_{\max}$, *the error* $|s(\mu) - s_N(\mu)|$ *verifies*

$$|s(\mu) - s_N(\mu)| \leq \Delta_N^s(\mu), \quad \forall \mu \in \mathcal{D}^\mu$$

A Posteriori error estimation: Remarks and Motivations for a Primal/Dual formulation

- Very similar to the compliant case : need only $\|\ell(\cdot,\mu)\|_{X'}$ find $\hat{e}_\ell \in X$ (Riesz representation) such that

$$(\hat{e}_\ell, v)_X = \ell(v,\mu), \quad \forall v \in X$$

  and apply offline-online decomposition similarly to other terms
- Rigorous error bounds
- Best approach if many outputs (little overhead), however in case of few outputs a primal-dual formulation is preferable
  1. Loss of quadratic convergence
  2. Effectivities possibly unbounded,

$$\eta_N^s(\mu) \geq \frac{\|\ell(\cdot,\mu)\|_{X'}}{\gamma(\mu)\|u(\mu)-u_N(\mu)\|_X}$$

  from output error bound (taking $\ell = f$) and energy error bound.

Motivation I

The Primal only approach has to major deficiencies:

1. We lose the "quadratic convergence" effect for our output: Recall for compliance we had

$$|s(\mu) - s_N(\mu)| \leq \gamma^e(\mu)\|u(\mu) - u_N(\mu)\|_X^2$$

and

$$\Delta_N^s(\mu) = \alpha_{LB}(\mu)(\Delta_N(\mu))2$$

But now we obtain

$$|s(\mu) - s_N(\mu)| \leq C\|u(\mu) - u_N(\mu)\|_X$$

and

$$\Delta_N^s(\mu) = \|l(\cdot;\mu)\|_X' \Delta_N(\mu).$$

Motivation II

2. The output effectivities, $\eta_N^s(\mu)$, can be unbounded: If $\ell = f$, we know that

$$|s(\mu) - s_N(\mu)| \leq \gamma^e(\mu)\|u(\mu) - u_N(\mu)\|_X^2,$$

and from the X-norm bound, we have

$$\|u(\mu) - u_N(\mu)\|_X \leq \Delta_N(\mu).$$

It thus follows that

$$\frac{\Delta s_N(\mu)}{|s(\mu) - s_N(\mu)|} \geq \frac{\|l(\cdot;\mu)\|'_X}{\gamma^e(\mu)\|u(\mu) - u_N(\mu)\|_X},$$

and therefore

$$\eta_N^s(\mu) \to \infty \text{ as } (N \to \infty \text{ and}) u_N(\mu) \to u(\mu).$$

Dual Problem

Given $\mu \in D \subset \mathbb{R}^P$, we define $(H^1_0(\Omega) \subset X^e \subset H^1(\Omega))$

- Exact Statement: $\psi(x;\mu) \in X^e$ satisfies

$$a(v, \psi(\mu); \mu) = -l(v; \mu), \forall v \in X^e(\Omega).$$

- Truth Approximation: $\psi_N(x;\mu) \in X_{\mathcal{N}} \subset X^e$ satisfies

$$a(v, \psi_N(\mu); \mu) = -l(v; \mu), \forall v \in X_{\mathcal{N}}(\Omega).$$

Note

- Problem is well-posed due to hypotheses on a and ℓ.
- We still assume affine $\mu-$dependence of a and ℓ (and f).

Formulation (Linear Case)

Sample : $S_N = \{\mu_1 \in \mathcal{D}^\mu, \ldots, \mu_N \in \mathcal{D}^\mu\}$.

Sample : $S^{\mathrm{du}}_{N^{\mathrm{du}}} = \{\mu^{\mathrm{du}} \in \mathcal{D}^\mu, \ldots, \mu^{\mathrm{du}}_{N^{\mathrm{du}}} \in \mathcal{D}^\mu\}$.

Space : $W_N = \mathrm{span}\,\{\zeta_n \equiv \underbrace{u(\mu^n)}_{u^{\mathcal{N}}(\mu^n)}, n = 1, \ldots, N\}$.

Space : $W^{\mathrm{du}}_{N^{\mathrm{du}}} = \mathrm{span}\,\{\zeta^{\mathrm{du}}_n \equiv \underbrace{\Psi(\mu^{\mathrm{du}}_n)}_{\Psi^{\mathcal{N}}(\mu^{\mathrm{du}}_n)}, n = 1, \ldots, N^{\mathrm{du}}\}$.

Sampling strategies?

- Equidistributed points in $\mathcal{D}^\mu$ (curse of dimensionality)
- Log-random distributed points in $\mathcal{D}^\mu$
- See later for more efficient, adaptive strategies

Reduced Basis Sample and Space

Note that the primal and dual parameter samples and associated reduced basis spaces are fundamentally different, i.e.,

$$S^{pr}_{N_{pr}} \neq S^{du}_{N_{du}} \text{ and } W^{pr}_{N_{pr}} \neq W^{du}_{N_{du}},$$

and in general we also have $N^{pr} \neq N^{du}$. For notational convenience, we drop the sub-/superscript for the primal problem:

$$N = N_{pr}, S_N = S^{pr}_{N_{pr}}, \text{ and } W_N = W^{pr}_{N_{pr}}$$

Formulation (Linear Case): a Galerkin method

Galerkin Projection

Given $\mu \in \mathcal{D}^\mu$ evaluate

$$s_N(\mu) = \ell(u_N(\mu)) - r(u_N(\mu), \Psi_{N^{\mathrm{du}}}(\mu); \mu) \ ;$$

where $u_N(\mu) \in X_N$ and $\Psi_{N^{\mathrm{du}}}(\mu) \in X^{\mathrm{du}}_{N^{\mathrm{du}}}$ satisfy

$$a(u_N(\mu), v; \mu) = f(v), \ \forall\, v \in X_N \ .$$

and

$$a(v, \Psi_{N^{\mathrm{du}}}(\mu)\ ; \mu) = -\ell(v; \mu), \ \forall\, v \in X^{\mathrm{du}}_{N^{\mathrm{du}}} \ .$$

- Note that RB Space
  $X_N = \mathrm{GramSchmidt}(W_N), X_N^{\mathrm{du}} = \mathrm{GramSchmidt}(W_N^{\mathrm{du}})$
- In general $N \neq N^{\mathrm{du}}$ (primal and dual are different problems)

Formulation (Linear Case): back to the compliant case

Recall that in compliance

- a is symmetric
- $\ell = f$

such that $\Psi(\mu) = -u(\mu)$.

We may take $N^{\mathrm{du}} = N$, $S_N^{\mathrm{du}} = S_N$ and $X_N^{\mathrm{du}} = X_N$ and get

$$\Psi_N(\mu) = -u_N(\mu)$$

Compliant case

- The dual problem is never formed/solved
- We simply identify $\Psi_N(\mu) = -u_N(\mu)$
- We get a 50% cost reduction

Formulation : A priory convergence

Proposition

For any $\mu \in \mathcal{D}^\mu$, we have

$$|s(\mu) - s_N(\mu)| \leq C \left(\inf_{v_N \in X_N} ||u(\mu) - v_N(\mu)||_X \right) \times \left(\inf_{v_N^{\mathrm{du}} \in X_N^{\mathrm{du}}} ||\psi(\mu) - \psi_N^{\mathrm{du}}(\mu)||_X \right)$$

- Recovery of quadratic convergence for the output !
- Alternative: build RB space comprising both primal and dual basis functions (output dual correction not needed however more costly and conditioning issues)

Formulation (Linear Case): offline-online decomposition

Expand our RB approximations:

$$u_N(\mu) = \sum_{j=1}^{N} u_{Nj}(\mu)\ \zeta_j$$

$$\Psi_{N^{\mathrm{du}}}(\mu) = \sum_{j=1}^{N^{\mathrm{du}}} \Psi_{Nj}(\mu)\ \zeta_j^{\mathrm{du}}$$

Express $s_N(\mu)$

$$s_N(\mu) = \sum_{j=1}^{N} u_{Nj}(\mu)\ \ell(\zeta_j) - \sum_{j=1}^{N^{\mathrm{du}}} \Psi_{Nj}(\mu)\ f(\zeta_j^{\mathrm{du}})$$
$$+ \sum_{j=1}^{N}\sum_{j'=1}^{N^{\mathrm{du}}}\sum_{q=1}^{Q} u_{Nj}(\mu)\ \Psi_{Nj'}(\mu)\ \Theta^q(\mu)\ a^q(\zeta_j, \zeta_{j'}^{\mathrm{du}})$$

Formulation (Linear Case): offline-online decomposition

$u_{N\,i}(\mu), 1 \leq i \leq N$ and $\Psi_{N\,i}(\mu), 1 \leq i \leq N^{\mathrm{du}}$ satisfy

$$\sum_{j=1}^{N} \left\{ \sum_{q=1}^{Q} \Theta^q(\mu)\; a^q(\zeta_i, \zeta_j) \right\} u_{N\,j}(\mu) = f(\zeta_i), \quad 1 \leq i \leq N$$

$$\sum_{j=1}^{N^{\mathrm{du}}} \left\{ \sum_{q=1}^{Q} \Theta^q(\mu)\; a^q(\zeta_i^{\mathrm{du}}, \zeta_j^{\mathrm{du}}) \right\} \Psi_{N^{\mathrm{du}}\,j}(\mu) = -\ell(\zeta_i^{\mathrm{du}}), \quad 1 \leq i \leq N^{\mathrm{du}}$$

Formulation (Linear Case): matrix form

Solve

$$\underline{A}_N(\mu)\,\underline{u}_N(\mu) = \underline{F}_N$$

and

$$\underline{A}^{\mathrm{du}}_{N^{\mathrm{du}}}(\mu)\,\underline{\Psi}_{N^{\mathrm{du}}}(\mu) = -\underline{L}_N$$

where

$$\underset{1 \le i,j \le N}{(A_N)_{i\,j}(\mu) = \textstyle\sum_{q=1}^{Q}\ \Theta^q(\mu)\ a^q(\zeta_i,\zeta_j)}, \quad \underset{1 \le i \le N}{F_{N\,i} = f(\zeta_i)}\ .$$

and

$$\underset{1 \le i,j \le N^{\mathrm{du}}}{(A^{\mathrm{du}}_{N^{\mathrm{du}}})_{i\,j}(\mu) = \textstyle\sum_{q=1}^{Q}\ \Theta^q(\mu)\ a^q(\zeta^{\mathrm{du}}_i,\zeta^{\mathrm{du}}_j)}, \quad \underset{1 \le i \le N^{\mathrm{du}}}{L_{N\,i} = \ell(\zeta^{\mathrm{du}}_i)}\ .$$

Formulation (Linear Case): complexity analysis

Offline: independent of μ

- Solve: $N + N^{\mathrm{du}}$ FEM system depending on $\mathcal{N}$
- Form and store: $f(\zeta_i)$, $\ell(\zeta_i)$, $f(\zeta_i^{\mathrm{du}})$, $\ell(\zeta_i^{\mathrm{du}})$
- Form and store: $a^q(\zeta_i, \zeta_j)$, $a^q(\zeta_i^{\mathrm{du}}, \zeta_j^{\mathrm{du}})$, $a^q(\zeta_i, \zeta_j^{\mathrm{du}})$

Online: independent of $\mathcal{N}$

- Given a new $\mu \in \mathcal{D}^\mu$
- Form and solve $A_N(\mu)$: $O(QN^2)$ and $O(N^3)$
- Form and solve $A_{N^{\mathrm{du}}}{}^{\mathrm{du}}(\mu)$: $O(Q{N^{\mathrm{du}}}^2)$ and $O({N^{\mathrm{du}}}^3)$
- Compute $s_N(\mu)$

Online: $N, N^{\mathrm{du}} << \mathcal{N}$

Online we realize often orders of magnitude computational economies relative to FEM in the context of many μ-queries

$u_N(\mu)$: Error equation and residual dual norm

Given our RB approximation $u_N(\mu)$, we have

$$e(\mu) \equiv u(\mu) - u_N(\mu)$$

that satisfies

$$a(e(\mu), v; \mu) \;=\; r(u_N(\mu), v; \mu), \forall v \in X$$

where $r(u_N(\mu), v; \mu) = f(v) - a(u_N(\mu), v; \mu)$ in the linear case is the residual. We have then

$$||e(\mu)||_X \;\leq\; \frac{||r(u_N(\mu), v; \mu)||_{X'}}{\alpha(\mu)} \;=\; \frac{\varepsilon_N(\mu)}{\alpha(\mu)}$$

$u_N(\mu)$: Defintions of energy error bounds and effectivity

Given $\alpha_{\mathrm{LB}}(\mu)$ a nonnegative lower bound of $\alpha(\mu)$:

$$\alpha(\mu) \geq \alpha_{\mathrm{LB}}(\mu) \geq \epsilon_\alpha \alpha(\mu),\ \ \epsilon_\alpha \ \in]0,1[,\ \forall \mu \in \mathcal{D}^\mu$$

Definition: Energy error bound

$$\Delta_N(\mu) \ \equiv \ \frac{\varepsilon_N(\mu)}{\alpha_{\mathrm{LB}}(\mu)}$$

Definition: Effectivity

$$\eta_N(\mu) \ \equiv \ \frac{\Delta_N(\mu)}{||e(\mu)||_X}$$

$u_N(\mu)$: Rigorous sharp error bounds

One can prove that

$$1 \;\le\; \eta_N(\mu) \;\le \frac{\gamma(\mu)}{\alpha_{\mathrm{LB}}(\mu)}, \quad 1 \le N \le N_{\max}, \quad \forall \mu \;\in\; \mathcal{D}^\mu$$

Remarks

- Rigorous: Left inequality ensures rigorous upper bound measured in $||\cdot||_X$, i.e. $||e(\mu)||_X \le \Delta_N(\mu),\ \forall \mu \in \mathcal{D}^\mu$
- Sharp: Right inequality states that $\Delta_N(\mu)$ overestimates the "true" error by at most $\gamma(\mu)/\alpha_{\mathrm{LB}}(\mu)$

$\Psi_N(\mu)$: error bounds

We have a similar result for the dual problem

$$||\Psi(\mu) - \Psi_{N^{\mathrm{du}}}||_X \;\leq\; \Delta_N^{\mathrm{du}}(\mu), \quad 1 \;\leq\; N^{\mathrm{du}} \;\leq\; N^{\mathrm{du}}_{\max}, \quad \forall \mu \;\in\; \mathcal{D}^\mu$$

where

$$\Delta_N^{\mathrm{du}}(\mu) \;\equiv\; \frac{\varepsilon_N^{\mathrm{du}}(\mu)}{\alpha_{\mathrm{LB}}(\mu)} \;\equiv\; \frac{|| -\ell(\cdot) - a(\cdot, \Psi_{N^{\mathrm{du}}}(\mu); \mu)||_{X'}}{\alpha_{\mathrm{LB}}(\mu)}$$

$\varepsilon_N^{\mathrm{du}}(\mu)$ is the dual norm of the residual.

$s_N(\mu)$: error bounds

From primal and dual energy error bounds we have

$$|s(\mu) - s_N(\mu)| \leq \Delta_N^s(\mu), \quad \mu \in \mathcal{D}^\mu$$

where

$$\Delta_N^s(\mu) \equiv \varepsilon_N(\mu)\Delta_N^{\mathrm{du}}(\mu)(\mu) = \alpha_{\mathrm{LB}}(\mu)\Delta_N(\mu)\Delta_N^{\mathrm{du}}(\mu)$$

Rapid convergence of the error in the output

Note that the error in the output vanishes as the product of the error in the primal and dual error

Back to compliance: a symmetric and $\ell = f$

We obtain

$$\Delta_N^s(\mu) \equiv \frac{\varepsilon_N^2(\mu)}{\alpha_{\mathrm{LB}}(\mu)}, \quad \forall \mu \in \mathcal{D}^\mu$$

Offline-Online decomposition (Primal problem)

- Dual problem : similar treatment
- Denote $\hat{e}(\mu) \in Y$

$$||\hat{e}(\mu)||_Y = \varepsilon_N(\mu) = ||g(u_N(\mu), \cdot; \mu)||_Y \tag{31}$$

such that

$$(\hat{e}(\mu), v)_Y = -g(u_N(\mu), v; \mu), \quad \forall v \in Y \tag{32}$$

- Recall that

$$-g(u_N(\mu), v; \mu) = f(v) - \sum_{q=1}^{Q} \sum_{n=1}^{N} \Theta^q(\mu)\, u_{N\,n}(\mu)\, a^q(\zeta_n, v), \quad \forall v \in X \tag{33}$$

Offline-Online decomposition (Primal problem)

- It follows next that $\hat{e}(\mu) \in Y$ satisfies

$$(\hat{e}(\mu), v)_Y \;=\; f(v) - \sum_{q=1}^{Q} \sum_{n=1}^{N} \Theta^q(\mu)\; u_{N\,n}(\mu)\; a^q(\zeta_n, v), \quad \forall v \;\in\; X \tag{34}$$

- Observe then that the rhs is the *sum* of products of parameter dependent functions and parameter independent linear functionals, thus invoking linear superposition

$$\hat{e}(\mu) \;=\; \mathcal{C} - \sum_{q=1}^{Q} \sum_{n=1}^{N} \Theta^q(\mu)\; u_{N\,n}(\mu)\; \mathcal{L}^q_n \tag{35}$$

  - $\mathcal{C} \in Y$ satisfies

$$(\mathcal{C}, v) = f(v), \forall v \in Y \tag{36}$$

  - $\mathcal{L} \in Y$ satisfies

$$(\mathcal{L}^q_n, v)_Y = -a^q(\zeta_n, v), \forall v \in Y,\; 1 \leq n \leq N, 1 \leq q \leq Q \tag{37}$$

(37) are parameter independe

Offline-Online decomposition: Error bounds

From (35) we get

$$||\hat{e}(\mu)||_Y^2 = (\mathcal{C},\mathcal{C})_Y + \sum_{q=1}^{Q} \sum_{n=1}^{N} \Theta^q(\mu)\, u_{N\,n}(\mu) \Big\{ 2(\mathcal{C}, \mathcal{L}_n^q)_Y + \sum_{q'=1}^{Q'} \sum_{n'=1}^{N'} \Theta^{q'}(\mu)\, u_{N\,n'}(\mu)\, (\mathcal{L}_n^q, \mathcal{L}_{n'}^{q'})_Y \Big\} \quad (38)$$

Remark

In (38), $||\hat{e}(\mu)||_Y^2$ is the sum of products of

- parameter dependent (simple/known) functions and
- parameter independent inner-product,

the offline-online for the error bounds is now clear.

Offline-Online decomposition: steps and complexity

Offline:

- Solve for $\mathcal{C}$ and $\mathcal{L}_n^q$, $1 \le n \le N$, $1 \le q \le Q$
- Form and save $(\mathcal{C}, \mathcal{C})_Y$, $(\mathcal{C}, \mathcal{L}_n^q)_Y$ and $(\mathcal{L}_n^q, \mathcal{L}_{n'}^{q'})_Y$, $1 \le n, n' \le N$, $1 \le q, q' \le Q$

Online

- Given a new $\mu \in \mathcal{D}^\mu$
- Evaluate the sum (38) in terms of $\Theta^q(\mu)$ and $u_{N\,n}(\mu)$
- Complexity in $O(Q^2 N^2)$ independent of $\mathcal{N}$

The linear symmetric coercive case

- We require a lower bound $\beta_{\mathrm{LB}}(\mu)$ for $\beta(\mu) = \alpha_c(\mu),\ \forall \mu \in \mathcal{D}^\mu$
- If
- Primal-Dual problem : similar treatment as in Primal-only formulation
- New ingredient introduced is the correction term for the output $r(u_N(\mu), \Psi_{N^{\mathrm{du}}}(\mu); \mu) = f(\psi_N^{\mathrm{du}}(\mu); \mu) - a(u_N(\mu), \psi_N^{\mathrm{du}}(\mu); \mu)$ which requires cross terms between primal and dual problems
  - $f^q(\zeta_n^{\mathrm{du}}), \quad 1 \le n \le N^{\mathrm{du}}, 1 \le q \le Q_f$
  - $a^q(\zeta_n, \zeta_n^{\mathrm{du}}), \quad 1 \le n \le N,\ 1 \le n \le N^{\mathrm{du}},\ 1 \le q \le Q_a$

Greedy Sampling Procedure

Note that we can write

$$|s(\mu) - s_N^{pr,du}(\mu)| \leq \left(\frac{\|r^{du}(\cdot;\mu)\|_{X'}}{\alpha_{LB}(\mu)} \right) \left(\frac{\|r(\cdot;\mu)\|_{X'}}{\alpha_{LB}(\mu)} \right)$$

Given a desired output error tolerance $\varepsilon^s_{tol,min}$, we perform

- a primal greedy sampling procedure until ($\Rightarrow N^{pr,max}$)

$$\frac{\|r(\cdot;\mu)\|_{X'}}{\sqrt{\alpha_{LB}(\mu)}} \leq \sqrt{\varepsilon_{tol,min}} \text{ over } \Xi^{pr}_{\mathrm{train}}$$

- a dual greedy sampling procedure until ($\Rightarrow N^{du,max}$) :

$$\frac{\|r^{du}(\cdot;\mu)\|_{X'}}{\sqrt{\alpha_{LB}(\mu)}} \leq \sqrt{\varepsilon_{tol,min}} \text{ over } \Xi^{pr}_{\mathrm{train}}$$

Note

"best" (most efficient) approach strongly problem dependent.

How do we choose N_{pr} vs N_{du}?

Suppose that

$$\|u(\mu) - uN(\mu)\|_X = \|u(\mu)\|_X g_{err}(N^{pr})$$
$$\|\psi(\mu) - \psi_N(\mu)\|_X = \|\psi(\mu)\|_X g_{err}(N^{du})$$

where $g_{err} : \mathbb{N}_0 \rightarrow \mathbb{R}$ is a monotonically decreasing “convergence” function with inverse g^1 (such that $g^1_{err}(g_{err}(N)) = N$) and $g_{err}(0) = 1$.

From our a priori convergence result, we suppose that

$$|s(\mu) - s_N^{pr,du}(\mu)| = C_s \|u(\mu)\|_X \|\psi(\mu)\|_X g_{err}(N^{pr}) g_{err}(N^{du}).$$

At fixed output error

$$|s(\mu) - s_N^{pr,du}(\mu)| = C_s \|u(\mu)\|_X \|\psi(\mu)\|_X \varepsilon,$$

we then obtain . . .

How do we choose N_{pr} vs N_{du}?

$$\frac{\text{Online Cost of } s_{N_{pr,du}}(\mu) \text{ WITH DUAL}}{\text{Online Cost of } s_N(\mu) \text{ WITHOUT DUAL}} = 2\left(\frac{g_{err}^{-1}(\sqrt{\varepsilon})}{g_{err}^{-1}(\varepsilon)}\right)^3.$$

where

- WITH DUAL: $N^{pr} = N^{du}$
- WITHOUT DUAL: $N^{pr} \neq= 0, N^{du} = 0$

We obtain for

- exponential convergence, $g_{err}(N) = e^{-\omega N}(\omega \in \mathbb{R}^+)$:
  Cost Ratio $= \frac{1}{4} \Rightarrow 75\%$ reduction in online cost;
- algebraic convergence, $g_{err}(N) = (N+1)^{-\omega}$:
  Cost Ratio $\approx 2\varepsilon^{3/(2\omega)} \Rightarrow$ reduction even more significant.

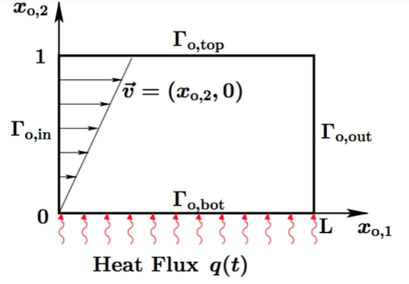
How do we choose N_{pr} vs N_{du}?

Remarks:

- In general, g_{err} not equal for primal and dual problem, i.e., we need to replace g_{err} by g_{err}^{pr} and g_{err}^{du}.
- Optimal choice of N_{pr} vs. N_{du} depends on ratio between g_{err}^{pr} and g_{err}^{du} , e.g.:
  - if $g^{pr} \ll g^{du}$,choose $N^{pr} \neq 0, N^{du} = 0$.
  - if $g^{pr} \gg g^{du}$,choose $N^{pr} = 0, N^{du} \neq 0$.
- Choice of N^{pr} vs. N^{du} thus strongly problem dependent.
- For many outputs, dual formulation becomes expensive (one dual for each output).

Graetz Flow

Scalar advection-diffusion in $\Omega_o(\mu) =]0, L \times]0, 1[$



Boundary conditions

- Neumann on $\Gamma_{o,bot}$ (flux) and $\Gamma_{o,out}$ (homogenous);
- Homogeneous Dirichlet on Γ_{in} and $\Gamma_{o,top}$.

Graetz Flow — Problem Statement

We consider $P = 2$

- $\mu_1 = L$ length of the channel
- $\mu_2 = \mathrm{Pe}$ Peclet number

Problem statement

Given $\mu = (\mu_1, \mu_2) \in \mathcal{D} \equiv [1, 10] \times [0.1, 100]$, evaluate

$$s^e(\mu) = \ell(u^e(\mu))$$

where $u^e(\mu) \in X^e \equiv \{v \in H^1(\Omega(\mu)) | v|_{\Gamma_{\mathrm{top}} \cup \Gamma_{\mathrm{in}}}\}$ satisfies

$$a(u^e(\mu), v; \mu) = f(v), \quad v \in X^e$$

Note: affine geometric mapping required:

$$\Rightarrow (x_1, x_2) = (x_{o,1}/L, x_{o,2})$$

Graetz Flow — Problem Statement

We have $\forall v \in X^e$,

$$f(v;\mu) = \ell(v;\mu) \equiv \mu_1 \int_{\Gamma_{\mathrm{root}}} v,$$

and, $\forall w, v \in X^e$

$$a(w,v;\mu) = \int_\Omega x_2 \frac{\partial w}{\partial x_1} v + \frac{1}{\mu_1 \ \mu_2} \int_\Omega \frac{\partial w}{\partial x_1}\frac{\partial v}{\partial x_1} + \frac{\mu_1}{\mu_2} \int_\Omega \frac{\partial w}{\partial x_2}\frac{\partial v}{\partial x_2}$$

Note

Problem is non-compliant since bilinear form a is non-symmetric (although $f = \ell$).

Graetz Flow — Problem Statement

Inner product, $w, v \in X^e$,

$$(w,v)_{X^e} = \frac{1}{\bar{\mu}_1\ \bar{\mu}_2} \int_\Omega \frac{\partial w}{\partial x_1}\frac{\partial v}{\partial x_1} + \frac{\bar{\mu}_1}{\bar{\mu}_2} \int_\Omega \frac{\partial w}{\partial x_2}\frac{\partial v}{\partial x_2}$$

with reference parameter $\bar{\mu} = (1, 1)$

Note

- Linear form $f = \ell$ is bounded
- Bilinear form a is coercive and continuous
- Linea form $f = \ell$ and bilinear form a satisfy affine parameter dependence
- symmetric part of a is parametrically coercive

Graetz Flow — Problem Statement

We obtain

$$a(w, v; \mu) = \sum_{q=1}^{3} \Theta^q(\mu) a^q(w, v)$$

for

$$\Theta^1 = 1$$
$$\Theta^2 = \frac{1}{\mu_1 \ \mu_2}$$
$$\Theta^3 = \frac{\mu_1}{\mu_2}$$

$$a^1(w, v) = \int_\Omega x_2 \frac{\partial w}{\partial x_1} v$$
$$a^2(w, v) = \int_\Omega \frac{\partial w}{\partial x_1} \frac{\partial v}{\partial x_1}$$
$$a^3(w, v) = \int_\Omega \frac{\partial w}{\partial x_2} \frac{\partial v}{\partial x_2}$$

Affine assumption is satisfied

Truth approximation

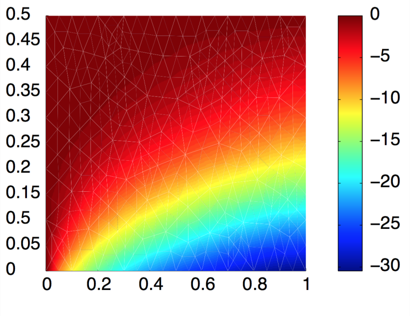


Figure 1: Example de solution

Convergence Results

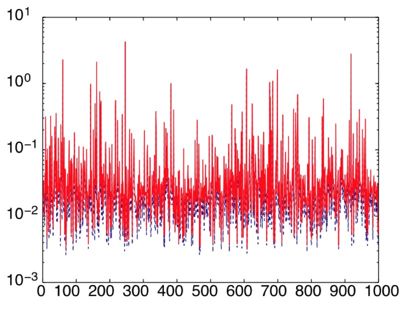


Figure 2: Graetz flow: $\alpha_{\rm LB}(\mu)$ (upper cuve, red) and $\alpha_{\rm LB}(\mu)$ (lower bound, dotted, blue) as a function $\mu' \in \Xi_{\rm train,SCM}$ after $J = 4$ iterations of SCM greedy; abscissa represents index of $\mu^k in \Xi_{\rm train,scm}$, source [HRP (2008)].

Convergence Results

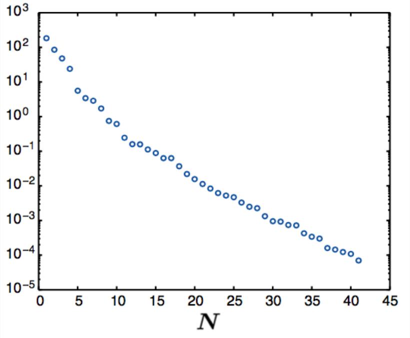


Figure 3: Graetz flow: $\max_{\mu} \in \Xi_{\text{train}} \Delta_N^s(\mu)$ as a function of N Source [RHP 2008]

Convergence Results

N	$\Delta^s_{N,\max}(\mu)$	$\eta^s_{N,\mathrm{ave}}$	$\eta^s_{N,\max}$
10	1.9E-01	7.9	63.1
15	5.3E-02	9.3	46.8
25	4.0E-03	5.9	48.5
33	1.0E-03	8.2	94.3
40	2.5E-04	17.8	81.4

Table 2: Output error bounds and effectivities [RHP2008]

Note that: $\eta^s_{N,\max}(\mu_1) \leq \eta^s_{\max,\mathrm{UB}} \equiv \sqrt{\mu_r} = 10$

- Maximum output error bound: $\Delta^s_{N,\max} = \max_{\mu\in\Xi_{\mathrm{train}}} \Delta^s_N(\mu)$
- Average output effectivity: $\eta^s_{N,\mathrm{ave}} = \frac{1}{\Xi_{\mathrm{train}}} \sum_{\mu\in\Xi_{\mathrm{train}}} \eta^s_N(\mu)$
- Maximum output effectivity: $\eta^s_{N,\max} = \max_{\mu\in\Xi_{\mathrm{train}}} \eta^s_N(\mu)$