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Extenstion to Non-Affine Problems

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Empirical Interpolation Method

Context

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We are now interested in the case non affine parametric dependence: The an offline/online decomposition relies on that assumption.

Solution: we recover an approximate affine expansion by means of the empirical interpolation method (EIM).

We shall discuss an alternative called **discrete empirical interpolation method (DEIM)**.

We consider linear problems but it will apply as well to non-linear problems.

EIM i

Figu

Affine decomposition is crucial for the offline/online decomposition for the RB method.

$$a(u,v;\mu) = \sum_{q=1}^{Q} \underbrace{\theta^q(\mu)}_{\text{parameter dependent coefficients}} \underbrace{a^q(u,v)}_{\text{parameter independent matrices}}$$

$$\ell(v;\mu) = \sum_{q=1}^{Q^f} \theta^q(\mu)\ell^q(v)$$

if it is not possible to write the problem in this form, eg. $\ell(v;\mu) = \int_\Omega g(\mathrm{x};\boldsymbol{\mu})v$, we can recover an approximate affine expansion by

EIM ii

means of the empirical interpolation method (EIM), e.g.

Figu

$$g(\mathrm{x};\boldsymbol{\mu}) = g_M(\mathrm{x};\boldsymbol{\mu}) + e_{EIM}(\mathrm{x};\boldsymbol{\mu}) = \sum_{m=1}^{M} \gamma_m(\boldsymbol{\mu})\rho_m(\mathrm{x}) + e_{EIM}(\mathrm{x};\boldsymbol{\mu}) \tag{1}$$

and we require then for a given tolerance

$$\varepsilon_M(\boldsymbol{\mu}) = \|e_{EIM}(\cdot;\boldsymbol{\mu})\|_{L^\infty(\Omega)} \leq \varepsilon \quad \forall \boldsymbol{\mu} \in \mathcal{D} \tag{2}$$

$$\ell_M(v;\boldsymbol{\mu}) = \int_\Omega g_M(\mathrm{x};\boldsymbol{\mu}) v d\Omega = \sum_{m=1}^{M} \gamma_m(\boldsymbol{\mu}) \int_\Omega \rho_m(\mathrm{x}) v d\Omega. \tag{3}$$

We recover the affine expansion provided we set $Q_f = M$ and

$$\Theta_f^m(\boldsymbol{\mu}) = \gamma_m(\boldsymbol{\mu}), \quad \ell^m(v) = \int_\Omega \rho_m(\mathrm{x}) v d\Omega, \quad 1 \leq m \leq M.$$

EIM iii

We now turn to an efficient interpolation technique for μ-dependent functions like $g(\mathrm{x}; \mu)$ and analyzing the impact of this further approximation on the RB methods

EIM v.s. Polynomial Interpolation i

Figu

We denote $\mathcal{G} = \{g(\cdot;\boldsymbol{\mu}), \boldsymbol{\mu} \in \mathcal{D}\} \subset C^0(\bar{\Omega})$

The empirical interpolation method (EIM) relies on the use of basis functions built by sampling g at a suitably selected set of points in $\mathcal{D}$ instead of using predefined basis functions (not exploiting the coupling of $\boldsymbol{x}$ and $\boldsymbol{\mu}$) defined on a fixed domain such as Gauss-Lobatto interpolation.

EIM v.s. Polynomial Interpolation ii

- introduced in Barrault, M., Maday, Y., Nguyen, N.C., Patera, A.T.: An empirical interpolation' method: application to efficient reduced-basis discretization of partial differential equations. C. R. Math. Acad. Sci. Paris 339(9), 667-672 (2004) to treat non-affine problems, eg Grepl, M.A., Maday, Y., Nguyen, N.C., Patera, A.T.: Efficient reduced-basis treatment of nonaffine and nonlinear partial differential equations. ESAIM Math. Modelling Numer. Anal. 41(3), 575-605 (2007)
- more general scope see Maday, Y., Nguyen, N.C., Patera, A.T., Pau, G.S.H.: A general multipurpose interpolation procedure: the magic points. Commun. Pure Appl. Anal. 8(1), 383-404 (2009)
- no predefined basis functions
- adaptive and hierachical interpolation, no need to reconstrut the basis functions when adding a new point
- exponential convergence rate

Figu

Empirical Interpolation i

The purpose of EIM is to find approximations to elements of $\mathcal{G}$ through an operator $\mathcal{I}_M^\times$ that interpolates the function $g(\cdot;\boldsymbol{\mu})$ at some carefully selected points in Ω.

Figu

Given an interpolatory system defined by a set of basis functions $\{\rho_1,\ldots,\rho_M\}$ (linear combination of particular snapshots $g\left(\cdot;\boldsymbol{\mu}^1_{EIM}\right),\ldots,g\left(\cdot;\boldsymbol{\mu}^M_{EIM}\right)$) and interpolation points $T_M=\left\{\mathrm{t}^1,\ldots,\mathrm{t}^M\right\}\subset\bar{\Omega}-$ commonly referred to as magic points $-$ the interpolant $\mathcal{I}_M^\times g(\cdot;\boldsymbol{\mu})$ of $g(\cdot;\boldsymbol{\mu})$ with $\boldsymbol{\mu}\in\mathcal{D}$ admits the separable expansion

$$\mathcal{I}_M^\times g(\mathrm{x};\boldsymbol{\mu})=\sum_{j=1}^{M}\gamma_j(\boldsymbol{\mu})\rho_j(\mathrm{x}),\quad \mathrm{x}\in\Omega \tag{4}$$

and satisfies the M interpolation constraints

$$\mathcal{I}_M^\times g\left(\mathrm{t}^i;\boldsymbol{\mu}\right)=g\left(\mathrm{t}^i;\boldsymbol{\mu}\right),\quad i=1,\ldots,M \tag{5}$$

Empirical Interpolation ii

Indeed, (5) yields the following linear system to solve

$$\sum_{j=1}^{M} \rho_j \left(\mathrm{t}^i\right) \gamma_j(\boldsymbol{\mu}) = g\left(\mathrm{t}^i; \boldsymbol{\mu}\right), \quad i = 1, \dots, M \tag{6}$$

that is, in matrix form

$$\mathbb{B}_M \boldsymbol{\gamma}(\boldsymbol{\mu}) = \mathrm{g}_M(\boldsymbol{\mu}) \quad \forall \boldsymbol{\mu} \in \mathcal{D} \tag{7}$$

where

$$(\mathbb{B}_M)_{ij} = \rho_j\left(\mathrm{t}^i\right), \quad (\boldsymbol{\gamma}(\boldsymbol{\mu}))_j = \gamma_j(\boldsymbol{\mu}), \quad (\mathrm{g}_M(\boldsymbol{\mu}))_i = g\left(\mathrm{t}^i; \boldsymbol{\mu}\right), \quad i,j = 1, \dots, M. \tag{8}$$

We need conditions ensuring that $\mathbb{B}_M$ is invertible.

EIM Algorithm i

The construction of the basis functions yielding the approximation space $X_M = \text{span}\{\rho_1, \ldots, \rho_M\}$ and interpolation points $T_M = \left\{\mathrm{t}^1, \ldots, \mathrm{t}^M\right\}$ is based on a greedy algorithm.

This procedure provides also a sample of parameter points $S_M = \left\{\boldsymbol{\mu}^1_{EIM}, \ldots, \boldsymbol{\mu}^M_{EIM}\right\}$, needed to construct the basis functions $\rho_i(\mathrm{x}), i = 1, \ldots, M$.

To start, let us choose our first sample point as

$$\boldsymbol{\mu}^1_{EIM} = \arg\max_{\boldsymbol{\mu} \in \mathcal{D}} \|g(\cdot;\boldsymbol{\mu})\|_{L^\infty(\Omega)},$$

define $S_1 = \left\{\boldsymbol{\mu}^1_{EIM}\right\}$ and the first generating function as

$$\xi_1(\mathrm{x}) = g\left(\mathrm{x};\boldsymbol{\mu}^1_{EIM}\right).$$

EIM Algorithm ii

Concerning the interpolation nodes, we first set

$$\mathrm{t}^1 = \arg\max_{\mathrm{x}\in\bar{\Omega}} |\xi_1(\mathrm{x})|, \quad T_1 = \left\{\mathrm{t}^1\right\};$$

Figu

then, we define the first basis function as

$$\rho_1(\mathrm{x}) = \xi_1(\mathrm{x})/\xi_1\left(\mathrm{t}^1\right),$$

and set $X_1 = \mathrm{span}\{\rho_1\}$.

Finally, we set the initial interpolation matrix

$$(\mathbb{B}_M)_{11} = \rho_1\left(\mathrm{t}^1\right) = 1.$$

At this stage, the available information allows to define the interpolant as the only function colinear with ρ_1 that coincides with g at t^1, that is $\mathcal{I}_1^{\mathrm{x}} g(\mathrm{x};\boldsymbol{\mu}) = g\left(\mathrm{t}^1;\boldsymbol{\mu}\right)\rho_1(\mathrm{x})$. Note that the first interpolation point t^1 is the point where the first basis function attains its maximum.

EIM Algorithm iii

At the m-th step, $m = 1, \dots, M-1$, given the (nested) set $T_m = \left\{\mathrm{t}^1, \dots, \mathrm{t}^m\right\}$ of interpolation points and the set $\{\rho_1, \dots, \rho_m\}$ of basis functions, we select as $(m+1)$ th generating function the snapshot which is the worst approximated by the current interpolant.

In other words, we select the snapshot which maximizes the error between g and $\mathcal{I}_m^{\mathrm{x}} g$

$$\begin{aligned} \boldsymbol{\mu}_{EIM}^{m+1} &= \arg\max_{\boldsymbol{\mu}\in\mathcal{D}} \left\| g(\cdot;\boldsymbol{\mu}) - \mathcal{I}_m^{\mathrm{x}} g(\cdot;\boldsymbol{\mu}) \right\|_{L^\infty(\Omega)}, \\ \xi_{m+1}(\mathrm{x}) &= g\left(\mathrm{x};\boldsymbol{\mu}_{EIM}^{m+1}\right). \end{aligned} \tag{9}$$

We then set $S_{m+1} = S_m \cup \left\{\boldsymbol{\mu}_{EIM}^{m+1}\right\}$.

EIM Algorithm iv

To choose the $(m+1)$-th interpolation point, we first evaluate the residual

$$r_{m+1}(\mathrm{x}) = \xi_{m+1}(\mathrm{x}) - \mathcal{I}_m^{\mathrm{x}} \xi_{m+1}(\mathrm{x})$$

by solving the linear system

$$\sum_{j=1}^{m} \rho_j \left(\mathrm{t}^i\right) \gamma_j = \xi_{m+1}\left(\mathrm{t}^i\right), \quad i = 1, \ldots, m$$

to characterize the interpolant $\mathcal{I}_m^{\mathrm{x}} \xi_{m+1}$; then, we set

$$\mathrm{t}^{M+1} = \arg\max_{\mathrm{x} \in \bar{\Omega}} |r_{m+1}(\mathrm{x})| \tag{10}$$

that is, that point of Ω where ξ_{m+1} is worst approximated. Finally, we define the new basis function as

$$\rho_{m+1}(\mathrm{x}) = \frac{\xi_{m+1}(\mathrm{x}) - \mathcal{I}_m^{\mathrm{x}} \xi_{m+1}(\mathrm{x})}{\xi_{m+1}\left(\mathrm{t}^{m+1}\right) - \mathcal{I}_m^{\mathrm{x}} \xi_{m+1}\left(\mathrm{t}^{m+1}\right)} = \frac{r_{m+1}(\mathrm{x})}{r_{m+1}\left(\mathrm{t}^{m+1}\right)}$$

Figu

EIM Algorithm v

Figu

and we set $X_{m+1} = \text{span}\{\rho_i, i = 1, \ldots, m+1\}$.

EIM algorithm i

Figu

The whole procedure is performed until a given tolerance $\varepsilon_{\mathrm{EIM}}$ is reached, or a given number $M_{\max}$ of terms is computed.

Input: max number of iterations $M_{\max}$, tolerance ε
Output: basis functions $\{\rho_1(\mathbf{x}),\dots,\rho_M(\mathbf{x})\}$, interpolation points $\{\mathbf{t}^1,\dots,\mathbf{t}^M\}$

1: $M=0,\ e_0=\varepsilon+1,\ \mathscr{I}_0^{\mathbf{x}} g(\mathbf{x};\boldsymbol{\mu})=0,$
2: $\boldsymbol{\mu}^1=\arg\max_{\boldsymbol{\mu}\in\mathscr{P}} \|g(\cdot,\boldsymbol{\mu})\|_{L^\infty(\Omega)}$
3: **while** $M<M_{\max}$ and $e_M>\varepsilon$
4: $\quad M\leftarrow M+1$
5: $\quad r(\mathbf{x})=g(\mathbf{x},\boldsymbol{\mu}^M)-\mathscr{I}_{M-1}^{\mathbf{x}} g(\mathbf{x},\boldsymbol{\mu}^M)$
6: $\quad \mathbf{t}^M=\arg\max_{\mathbf{x}\in\overline{\Omega}} |r(\mathbf{x})|$
7: $\quad \rho_M(\mathbf{x})=r(\mathbf{x})/r(\mathbf{t}^M)$
8: $\quad [e_M,\boldsymbol{\mu}^{M+1}]=\arg\max_{\boldsymbol{\mu}\in\mathscr{P}} \|g(\cdot,\boldsymbol{\mu})-\mathscr{I}_M^{\mathbf{x}} g(\cdot,\boldsymbol{\mu})\|_{L^\infty(\Omega)}$
9: **end while**

EIM: properties i

Remark EIM yields a sequence of hierarchical spaces $X_1 \subset X_2 \subset \dots X_M$, such that the interpolation is exact for any $v \in X_M$ - that is,

$$\mathcal{I}_M^x v = v \quad \forall v \in X_M$$

provided that $\dim(X_M) = M$ and that the matrix $\mathbb{B}_M \in \mathbb{R}^{M \times M}$ is invertible.

We can show that the construction discussed so far yields indeed a set $\{\rho_1, \dots, \rho_M\}$ of linearly independent basis functions.

EIM: properties ii

Figu

Theorem (EIM Properties)
The construction of the interpolation points is well-defined and, for any $M \leq M_{\max} < \dim(\mathrm{span}\{\mathcal{G}\}), X_M = \mathrm{span}\{\rho_1, \ldots, \rho_M\} = \mathrm{span}\{\xi_1, \ldots, \xi_M\}$ *is of dimension* M*. In addition,* $\mathbb{B}_M$ *is lower triangular with* $(\mathbb{B}_M)_{ii} = 1, i = 1, \ldots, M$*.*

EIM: properties iii

Figu

Proof.

The property span $\{\rho_1, \ldots, \rho_M\} = \text{span}\{\xi_1, \ldots, \xi_M\} = X_M$ directly follows from the construction of the normalized ρ_i 's with respect to the ξ_j 's. Then, we proceed by induction. Clearly, $X_1 = \text{span}\{\rho_1\}$ has dimension 1 and $\mathbb{B}_1 = 1$ is invertible. Next, lets us assume that $X_{M-1} = \text{span}\{\rho_1, \ldots, \rho_{M-1}\}$ is of dimension $M - 1$. If

- $\mathbb{B}_{M-1}$ is invertible
- $\left|r_M\left(\mathrm{t}^M\right)\right| > 0$

EIM: properties iv

we may form $X_M = \text{span}\,\{\rho_1, \ldots, \rho_M\}$.

To prove that $\mathbb{B}_M$ is invertible, it is enough to observe that

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$$(\mathbb{B}_{M-1})_{ij} = \rho_j\left(\mathrm{t}^i\right) = r_j\left(\mathrm{t}^i\right)/r_j\left(\mathrm{x}^j\right), \quad i,j = 1,\ldots,M-1. \qquad (11)$$

Then $(\mathbb{B}_{M-1})_{ij} = 0$ if $i < j$, $(\mathbb{B}_{M-1})_{ij} = 1$ if $i = j$, whereas $\left|(\mathbb{B}_M)_{ij}\right| \leq 1$ if $i > j$ since $\mathrm{x}^j = \arg\max_{\mathrm{x}\in\bar{\Omega}} |r_j(\mathrm{x})|\,, j = 1,\ldots,M$, according to (10). The matrix $\mathbb{B}_M$ is lower triangular and it is such that $(\mathbb{B}_M)_{ii} = 1, i = 1,\ldots,M$, hence it is invertible.

To prove that $\dim(X_M) = M$ (whence $\mathrm{t}^1,\ldots,\mathrm{t}^M$ are distinct) we observe that

$$\begin{aligned}\|r_M\|_{L^\infty(\Omega)} &= \|g\left(\cdot;\boldsymbol{\mu}^M_{EIM}\right) - \mathcal{I}^{\mathrm{x}}_{M-1} g\left(\cdot;\boldsymbol{\mu}^M_{EIM}\right)\|_{L^\infty(\Omega)} \\ &\geq \max_{\boldsymbol{\mu}\in\mathcal{D}} \left\|g(\cdot;\boldsymbol{\mu}) - \mathcal{I}^{\mathrm{x}}_{M-1} g(\cdot;\boldsymbol{\mu})\right\|_{L^\infty(\Omega)} \geq d_{M_{\max}}(\mathcal{G};X),\end{aligned}$$

EIM: properties v

with

$$d_{M_{\max}}(\mathcal{G}; X) = \inf_{\substack{\hat{X} \subset X \\ \dim(\tilde{X}) = M_{\max}}} \sup_{\boldsymbol{\mu} \in \mathcal{D}} \inf_{z \in \hat{X}} \|g(\cdot; \boldsymbol{\mu}) - z\|_{L^\infty(\Omega)} > 0$$

since $M_{\max} < \dim(\mathrm{span}\{\mathcal{G}\})$. If $\dim(X_M) \neq M$, we have that $\xi_M \in X_{M-1}$ and thus $\|r_M\|_{L^\infty(\Omega)} = 0$, which provides the contradiction and yields $\dim(X_M) = M$. $\square$

EIM: properties vi

Note: if $\dim(\mathrm{span}\{\mathcal{G}\}) = M^*$, the algorithm stops after $M = M^*$ iterations. As long as $M \leq M^*$, the previous theorem ensures that the basis functions $\{\rho_1, \ldots, \rho_M\}$ and the snapshots $\{\xi_1, \ldots, \xi_M\}$ span the same space X_M. In particular, it is better to deal with the former since, as resulting from 11,

$$\rho_i\left(\mathrm{t}^i\right) = 1, \quad i = 1, \ldots, M, \quad \rho_j\left(\mathrm{t}^i\right) = 0, \quad 1 \leq i < j \leq M$$

EIM: Error Analysis i

We carry out an error analysis of the EIM based on the results found in previous references.

Let us first introduce a set of characteristic (Lagrangian) functions $\{l_i^M \in X_M\}$ to construct the interpolation operator $\mathcal{I}_M^{\mathrm{x}}$ in X_M over the set of magic points T_M. For any given M, we can express

$$\mathcal{I}_M^{\mathrm{x}} g(\mathrm{x};\boldsymbol{\mu}) = \sum_{i=1}^{M} g\left(\mathrm{t}^i;\boldsymbol{\mu}\right) l_i^M(\mathrm{x}), \quad l_i^M(\mathrm{x}) = \sum_{j=1}^{M} \rho_j(\mathrm{x}) \left(\mathbb{B}_M^{-1}\right)_{ji}$$

by definition, $l_i^M\left(\mathrm{x}^j\right) = \delta_{ij}, i, j = 1, \ldots, M$. The existence of characteristic functions directly follows from the nonsingularity of the matrix $\mathbb{B}_M$.

EIM: Error Analysis ii

The a priori error analysis of the EIM involves the Lebesgue constant

$$\Lambda_M = \sup_{x \in \Omega} \sum_{i=1}^{M} \left| l_i^M(x) \right| ;$$

note that Λ_M depends on X_M and the magic points T_M, but is μ-independent.

An upper bound (indeed quite pessimistic) for the Lebesgue constant is $\Lambda_M \leq 2^M - 1$.

EIM: Error Analysis iii

Proposition
For any $g \in \mathcal{G}$, the interpolation error satisfies

$$\varepsilon_M(\boldsymbol{\mu}) := \|g(\cdot;\boldsymbol{\mu}) - \mathcal{I}_M^{\times} g(\cdot;\boldsymbol{\mu})\|_{L^\infty(\Omega)} \leq (1+\Lambda_M) \inf_{g_M \in X_M} \|g(\cdot;\boldsymbol{\mu}) - g_M\|_{L^\infty(\Omega)} \tag{12}$$

The estimate provides a theoretical basis for the stability of the empirical interpolation method. The last term in the right-hand side of (12) is referred to as the best approximation of g by elements in X_M in the L^∞-norm.

EIM: Error Analysis iv

Similarly to the convergence result for the greedy RB algorithm, also in the case of the empirical interpolation method it is possible to link the convergence rate of EIM approximations to the Kolmogorov M-width of the manifold $\mathcal{G}$.

Figu

Theorem
Assume that $\mathcal{G} \subset X = C^0(\Omega)$, and that there exists a sequence of nested finite-dimensional spaces $\mathcal{Z}_1 \subset \mathcal{Z}_2 \ldots, \dim(\mathcal{Z}_M) = M$, and $\mathcal{Z}_M \subset \operatorname{span}\{\mathcal{G}\}$ such that there exists $c > 0$ and $\alpha > \log 4$ with

$$d(\mathcal{G}, \mathcal{Z}_M) = \sup_{\boldsymbol{\mu} \in \mathcal{D} \inf_{M \in \mathcal{Z}_M}} \|g(\cdot; \boldsymbol{\mu}) - v_M\|_X \leq c e^{-\alpha M}.$$

Then

$$\sup_{\boldsymbol{\mu} \in \mathcal{D}} \|g(\cdot; \boldsymbol{\mu}) - \mathcal{I}_M^x g(\cdot; \boldsymbol{\mu})\|_{L^\infty(\Omega)} \leq c e^{-(\alpha - \log 4) M}.$$

EIM: Error Analysis v

Figu

Remark the condition implies in particular that $d_M(\mathcal{G}; X) \leq ce^{-\alpha M}$, i.e. $\mathcal{G}$ has an exponentially small Kolmogorov M-width.

A Posteriori error estimate i

Figu

We now derive an a posteriori error estimate for the EIM error we proceed as follows.

Given an approximation $g_M(\cdot;\boldsymbol{\mu})$, for $M \leq M_{\max} - 1$, let us define

$$E_M(\mathrm{x};\boldsymbol{\mu}) = \hat{\varepsilon}_M(\boldsymbol{\mu})\rho_{M+1}(\mathrm{x}), \quad \hat{\varepsilon}_M(\boldsymbol{\mu}) = \left|g\left(\mathrm{t}^{M+1};\boldsymbol{\mu}\right) - \mathcal{I}_M^{\mathrm{x}} g\left(\mathrm{t}^{M+1};\boldsymbol{\mu}\right)\right|.$$

A Posteriori error estimate ii

Figu

In general, $\varepsilon_M(\boldsymbol{\mu}) \geq \hat{\varepsilon}_M(\boldsymbol{\mu})$.

However, it is possible to show that

Proposition
If $g(\cdot;\boldsymbol{\mu}) \in X_{M+1}$, *then*

1. $g(\mathrm{x};\boldsymbol{\mu}) - \mathcal{I}_M^{\mathrm{x}} g(\mathrm{x};\boldsymbol{\mu}) = \pm E_M(\mathrm{x};\boldsymbol{\mu})$;
2. $\varepsilon_M(\boldsymbol{\mu}) = \hat{\varepsilon}_M(\boldsymbol{\mu})$.

A Posteriori error estimate iii

Proof.

If $g(\cdot;\boldsymbol{\mu}) \in X_{M+1}$ there exists $\kappa(\boldsymbol{\mu}) \in \mathbb{R}^{M+1}$ such that $g(\mathrm{x};\boldsymbol{\mu}) - \mathcal{I}_M^{\mathrm{x}} g(\mathrm{x};\boldsymbol{\mu}) = \sum_{i=1}^{M+1} \kappa_i(\boldsymbol{\mu})\rho_i(\mathrm{x})$. Taking $\mathrm{x} = \mathrm{t}^j, j = 1, \ldots, M+1$, we get

$$\sum_{i=1}^{M+1} \kappa_i(\boldsymbol{\mu})\rho_i\left(\mathrm{t}^i\right) = g\left(\mathrm{t}^j;\boldsymbol{\mu}\right) - \mathcal{I}_M^{\mathrm{x}} g\left(\mathrm{t}^j;\boldsymbol{\mu}\right),$$

from which we obtain that $(i)\kappa_i(\boldsymbol{\mu}) = 0, i = 1, \ldots, M$ since $g\left(\mathrm{t}^i;\boldsymbol{\mu}\right) - \mathcal{I}_M^{\mathrm{x}} g\left(\mathrm{t}^i;\boldsymbol{\mu}\right) = 0, i = 1, \ldots, M$ and $\mathbb{B}_M$ is lower triangular, i.e. $\rho_i\left(\mathrm{x}^j\right) = 0$ if $i < j$, and that (ii) $\kappa_{M+1}(\boldsymbol{\mu}) = g\left(\mathrm{t}^{M+1};\boldsymbol{\mu}\right) - \mathcal{I}_M^{\mathrm{x}} g\left(\mathrm{t}^{M+1};\boldsymbol{\mu}\right)$ since $\rho_{M+1}\left(\mathrm{t}^{M+1}\right) = 1$. This concludes the proof of point 1 . Point 2 directly follows since $\|\rho_{M+1}\|_{L^\infty(\Omega)} = 1$. □

Figu

A Posteriori error estimate iv

Figu

Note that in general $g(\cdot;\boldsymbol{\mu}) \notin X_{M+1}$, so we only have that $\varepsilon_M(\boldsymbol{\mu}) \geq \hat{\varepsilon}_M(\boldsymbol{\mu})$ for any $\boldsymbol{\mu} \in \mathcal{D}$, i.e., $\hat{\varepsilon}_M(\boldsymbol{\mu})$ is a lower bound of the interpolation error in L^∞-norm.

Nevertheless, if $\varepsilon_M(\boldsymbol{\mu}) \to 0$ very fast, we expect the effectivity

$$\eta_M(\boldsymbol{\mu}) = \frac{\hat{\varepsilon}(\boldsymbol{\mu})}{\|g(\cdot;\boldsymbol{\mu}) - \mathcal{I}_M^{\times} g(\cdot;\boldsymbol{\mu})\|_{L^\infty(\Omega)}}$$

to be close to 1 .

A Posteriori error estimate v

In any case, evaluating the estimator only requires an additional evaluation of $g(\cdot;\boldsymbol{\mu})$ at a single point in Ω. For this reason, we define the one point error estimator as

$$\Delta_M(\boldsymbol{\mu}) = \hat{\varepsilon}_M(\boldsymbol{\mu})$$

corresponding to the interpolation error at the $(M+1)$-th magic point, the one where the residual $r_M(\mathrm{x})$ attains its maximum.

Practical Implementation i

Similarly to the (weak) greedy RB algorithm , finding the supremum in (9) and (10) is not computationally feasible unless an approximation of both Ω and $\mathcal{D}$ is considered.

For this reason, we introduce:

1. a fine sample $\Xi_{\text{train}}^{EIM} \subset \mathcal{D}$ of cardinality $\left|\Xi_{\text{train}}^{EIM}\right| = n_{\text{train}}^{EIM}$ to train the EIM algorithm ;
2. a discrete approximation $\Omega_h = \left\{x^k\right\}_{k=1}^{N_q}$ of Ω of dimension N_q. In the finite element context, the points x^i can be for instance the vertices of the computational mesh, or the quadrature points.

Problems (9) and (10) are now turned into simpler enumeration problems.

Practical Implementation ii

In this setting, we can also provide an algebraic version of the EIM.

Figu

function $[\mathbb{Q}, \mathfrak{I}] = \text{EIM\_OFFLINE}(\Xi_{\text{train}}^{EIM}, \Omega_h, M_{\max}, \varepsilon_{\text{EIM}})$
    $M = 0,\ e_0 = \varepsilon_{\text{EIM}} + 1$
    $\boldsymbol{\mu}^1 = \arg\max_{\boldsymbol{\mu} \in \Xi_{\text{train}}^{EIM}} \|\mathbf{g}(\boldsymbol{\mu})\|_\infty$
    $\mathbf{r} = \mathbf{g}(\boldsymbol{\mu}^1),\ \ \mathbb{Q} = [\,]$
    while $M < M_{\max}$ and $e_M > \varepsilon_{\text{EIM}}$
        $M \leftarrow M + 1$
        $i_M = \arg\max_{i=1,\dots,N_q} |\mathbf{r}_i|$
        $\boldsymbol{\rho}_M = \mathbf{r}/\mathbf{r}_{i_M}$
        $\mathbb{Q} \leftarrow \mathbb{Q} \cup \boldsymbol{\rho}_M,\ \ \mathfrak{I} \leftarrow \mathfrak{I} \cup i_M,$
        $[e_M, \boldsymbol{\mu}^{M+1}] = \arg\max_{\boldsymbol{\mu} \in \Xi_{\text{train}}^{EIM}} \|\mathbf{g}(\boldsymbol{\mu}) - \mathbb{Q}\mathbb{Q}_{\mathfrak{I}}^{-1}\mathbf{g}_{\mathfrak{I}}(\boldsymbol{\mu})\|_\infty$
        $\mathbf{r} = \mathbf{g}(\boldsymbol{\mu}^{M+1}) - \mathbb{Q}\mathbb{Q}_{\mathfrak{I}}^{-1}\mathbf{g}_{\mathfrak{I}}(\boldsymbol{\mu}^{M+1})$
    end while
end function

function $\boldsymbol{\gamma}(\boldsymbol{\mu}) = \text{EIM\_ONLINE}(\mathbb{Q}_{\mathfrak{I}}, \boldsymbol{\mu}, \{\mathbf{t}^1, \dots, \mathbf{t}^M\})$
    form $\mathbf{g}_{\mathfrak{I}}(\boldsymbol{\mu})$ by evaluating $g(\cdot, \boldsymbol{\mu})$ in the interpolation points $\{\mathbf{t}^1, \dots, \mathbf{t}^M\}$
    solve $\mathbb{Q}_{\mathfrak{I}}\boldsymbol{\gamma}(\boldsymbol{\mu}) = \mathbf{g}_{\mathfrak{I}}(\boldsymbol{\mu})$
end function

Practical Implementation iii

We first introduce the vector representation $\mathrm{g} : \mathcal{D} \to \mathbb{R}^{N_q}$ of $g : \Omega_h \times \mathcal{D} \to \mathbb{R}$, defined as

$$(\mathrm{g}(\boldsymbol{\mu}))_k = g\left(\mathrm{x}^k; \boldsymbol{\mu}\right), \quad k = 1, \ldots, N_q$$

obtained by evaluating the function g in Ω_h, for any $\boldsymbol{\mu} \in \mathcal{D}$. Then, we denote by $\mathbb{Q} \in \mathbb{R}^{N_q \times M}$ the matrix

$$\mathbb{Q} = [\boldsymbol{\rho}_1 | \ldots | \boldsymbol{\rho}_M]$$

whose columns are the discrete representation of the basis functions $\{\rho_1, \ldots, \rho_M\}$, i.e. $(\mathbb{Q})_{kj} = \rho_j\left(\mathrm{x}^k\right)$. Moreover we denote by $\mathfrak{I} = \{i_1, \ldots, i_M\}$ a set of interpolation indices such that

Figu

Practical Implementation iv

$\{t^1, \ldots, t^M\} = \{x^{i_1}, \ldots, x^{i_M}\}$. The discrete representation $g_M : \mathcal{D} \to \mathbb{R}^{N_q}$ of the interpolation operator $\mathcal{I}_M^x$ is given by

$$g_M(\boldsymbol{\mu}) = \mathbb{Q}\boldsymbol{\gamma}(\boldsymbol{\mu}) \in \mathbb{R}^{N_q},$$

where $\gamma(\boldsymbol{\mu}) \in \mathbb{R}^M$ is the solution of the following linear system

$$(g_M(\boldsymbol{\mu}))_{i_m} = \sum_{j=1}^{M} \gamma_j(\boldsymbol{\mu}) \left(\boldsymbol{\rho}_j\right)_{i_m} = (g(\boldsymbol{\mu}))_{i_m}, \quad m = 1, \ldots, M. \tag{13}$$

Denoting by $g_{\mathfrak{I}}(\boldsymbol{\mu}) \in \mathbb{R}^M$ the vector whose components are $(g_{\mathfrak{I}}(\boldsymbol{\mu}))_m = (g(\boldsymbol{\mu}))_{i_m}$ for $m = 1, \ldots, M$, and noting that the $M \times M$ matrix $\mathbb{B}_M$ can be easily formed by restricting the $N_q \times M$ matrix $\mathbb{Q}$ to the rows $\mathfrak{I}$, i.e. $\mathbb{B}_M = \mathbb{Q}_{\mathfrak{I}}$,

Figu

Practical Implementation v

Figu

(13) can be written in compact form as

$$\mathbb{Q}_{\mathfrak{I}}\boldsymbol{\gamma}(\boldsymbol{\mu}) = \mathrm{g}_{\mathfrak{I}}(\boldsymbol{\mu}). \tag{14}$$

We finally obtain the following expression for the EIM approximation

$$\mathrm{g}_M(\boldsymbol{\mu}) = \mathbb{Q}_{\mathfrak{I}}^{-1}\mathrm{g}_{\mathfrak{I}}(\boldsymbol{\mu}) \quad \forall \boldsymbol{\mu} \in \mathcal{D}.$$

Practical Implementation vi

Note: that the solution of the dense linear system (14) has complexity $O\left(M^2\right)$, since the matrix $\mathbb{Q}_{\mathfrak{J}}$ is lower triangular.

Remark At each iteration, the algorithm requires to evaluate $\mathrm{g}(\boldsymbol{\mu})$ for $\boldsymbol{\mu} \in \Xi_{\text{train}}^{EIM}$. Should this operation be expensive, one may form and store the (possibly dense) matrix

$$\mathbb{S} = \left[\mathrm{g}\left(\boldsymbol{\mu}^1\right) | \ldots | \mathrm{g}\left(\boldsymbol{\mu}^{n_{\textbf{train}}^{EIM}}\right)\right] \in \mathbb{R}^{N_q \times n_{\textbf{train}}^{EIM}}$$

once and for all before entering the while loop. However, already for moderately large N_q and n_{train}^{EIM} , storing the matrix $\mathbb{S}$ can be quite challenging. For instance, approximating a function defined over a set of $N_q = 4 \cdot 10^5$ points (corresponding to 4 quadrature points for each tetrahedron) using a training set of dimension $n_{\text{train}}^{EIM} = 10^3$, requires to store as much as about 7 GB of data.

DEIM: an alternative

Discrete Empirical Interpolation Method i

Figu

An alternative to the EIM for approximating a nonaffinely parametrized function is the so-called discrete empirical interpolation method (DEIM), originally introduced in Chaturantabut, S., Sorensen, D.C.: Nonlinear model reduction via discrete empirical interpolation. SIAM J. Sci. Comput. 32(5), 2737-2764 (2010) .

Similarly to EIM, DEIM approximates a nonlinear function $\mathrm{g} : \boldsymbol{\mu} \in \mathcal{D} \subset \mathbb{R}^P \to \mathrm{g}(\boldsymbol{\mu}) \in \mathbb{R}^{N_q}$ by projection onto a low-dimensional subspace spanned by a basis $\mathbb{Q}$,

$$\mathrm{g}(\boldsymbol{\mu}) \approx \mathrm{g}_M(\boldsymbol{\mu}) = \mathbb{Q}\gamma(\boldsymbol{\mu}), \tag{15}$$

where $\mathbb{Q} = [\boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_M] \in \mathbb{R}^{N_q \times M}$ and $\gamma(\boldsymbol{\mu}) \in \mathbb{R}^M$ is the corresponding vector of coefficients, with $M \ll N_q$.

Discrete Empirical Interpolation Method ii

Figu

The difference is on the construction of the basis $\mathbb{Q}$, that is obtained operating a POD on a set of snapshots

$$\mathbb{S} = \left[\mathrm{g}\left(\boldsymbol{\mu}_{DEIM}^{1}\right) | \dots | \mathrm{g}\left(\boldsymbol{\mu}_{DEIM}^{n_s}\right) \right], \quad n_s^{DEIM} > M$$

instead than being embedded in the EIM greedy algorithm.

Note that for both EIM and DEIM the interpolation points are iteratively selected with the same greedy algorithm.

Discrete Empirical Interpolation Method iii

DEIM thus requires to:

Figu

1. construct a set of snapshots obtained by sampling $g(\boldsymbol{\mu})$ at values $\boldsymbol{\mu}^i_{DEIM}, i = 1, \ldots, n_s$ and apply POD to extract the basis

$$[\boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_M] = \mathsf{POD}\left(\left[\mathrm{g}\left(\boldsymbol{\mu}^1_{DEIM}\right), \ldots, \mathrm{g}\left(\boldsymbol{\mu}^{n_s}_{DEIM}\right)\right], \varepsilon_{\mathrm{POD}}\right),$$

   where $\varepsilon_{\mathsf{DEIM}}$ is a prescribed tolerance;
2. select iteratively M indices $\mathfrak{I} \subset \{1, \cdots, N_q\}, |\mathfrak{I}| = M$ from the basis $\mathbb{Q}$ using a greedy procedure, which minimizes at each step the interpolation error over the snapshots set measured in the maximum norm. This operation is indeed the same as for the selection of the EIM magic points;

Discrete Empirical Interpolation Method iv

Figu

3. given a new $\boldsymbol{\mu}$, in order to compute the coefficients vector $\gamma(\boldsymbol{\mu})$, interpolation constraints are imposed at the M points corresponding to the selected indices, thus requiring the solution of the following linear system

$$\mathbb{Q}_{\mathfrak{I}}\boldsymbol{\gamma}(\boldsymbol{\mu}) = \mathrm{g}_{\mathfrak{I}}(\boldsymbol{\mu}), \tag{16}$$

where $\mathbb{Q}_{\mathfrak{I}} \in \mathbb{R}^{M\times M}$ is the matrix formed by the $\mathfrak{I}$ rows of $\mathbb{Q}$. As a result,

$$\mathrm{g}_M(\boldsymbol{\mu}) = \mathbb{Q}_{\mathfrak{I}}^{-1}\mathrm{g}_{\mathfrak{I}}(\boldsymbol{\mu})$$

Discrete Empirical Interpolation Method v

Figu

Here we only point out that the error between g and its DEIM approximation g_M can be bounded as

$$\|g(\boldsymbol{\mu}) - g_M(\boldsymbol{\mu})\|_2 \leq \left\|\mathbb{Q}_{\mathfrak{J}}^{-1}\right\|_2 \left\|\left(\mathbb{I} - \mathbb{Q}^T\right) g(\boldsymbol{\mu})\right\|_2, \tag{17}$$

with

$$\left\|\left(\mathbb{I} - \mathbb{Q}^T\right) g(\boldsymbol{\mu})\right\|_2 \approx \sigma_{M+1}, \tag{18}$$

being σ_{M+1} the first discarded singular value of the matrix $\mathbb{S}$ when selecting M basis through the POD procedure.

Discrete Empirical Interpolation Method vi

Figu

This approximation holds for any $\boldsymbol{\mu} \in \mathcal{D}$ provided a suitable sampling in the parameter space has been carried out to build the snapshot matrix $\mathbb{S}$.

- the predictive projection error (18) is comparable to the training projection error σ_{M+1}
- Similar to the one point error estimator of EIM, (17) exploits the information related to the first discarded term and can be seen as a heuristic measure of the DEIM error.

Discrete Empirical Interpolation Method vii

THE DEIM procedure reads

function $[\mathbb{Q}, \mathfrak{I}] = \text{DEIM\_OFFLINE}(\mathbb{S}, \varepsilon_{\text{DEIM}})$
  $[\boldsymbol{\rho}_1 \mid \dots, \mid \boldsymbol{\rho}_M] = \text{POD}(\mathbb{S}, \varepsilon_{\text{DEIM}})$
  $i_m = \arg\max_{i=1,\dots,N_q} |(\boldsymbol{\rho}_1)_i|$
  $\mathbb{Q} = \boldsymbol{\rho}_1, \ \mathfrak{I} = \{i_1\},$
  for $m = 2 : M$
    $\mathbf{r} = \boldsymbol{\rho}_m - \mathbb{Q}\mathbb{Q}_{\mathfrak{I}}^{-1}(\boldsymbol{\rho}_m)_{\mathfrak{I}}$
    $i_m = \arg\max_{i=1,\dots,N_q} |\mathbf{r}_i|$
    $\mathbb{Q} \leftarrow [\mathbb{Q} \ \ \boldsymbol{\rho}_m], \ \mathfrak{I} \leftarrow \mathfrak{I} \cup i_m$
  end for
end function

function $\boldsymbol{\gamma}(\boldsymbol{\mu}) = \text{DEIM\_ONLINE}(\mathbb{Q}_{\mathfrak{I}}, \boldsymbol{\mu}, \{\mathbf{t}^1, \dots, \mathbf{t}^M\})$
  form $\mathbf{g}_{\mathfrak{I}}(\boldsymbol{\mu})$ by evaluating $g(\cdot, \boldsymbol{\mu})$ in the interpolation points $\{\mathbf{t}^1, \dots, \mathbf{t}^M\}$
  solve $\mathbb{Q}_{\mathfrak{I}}\boldsymbol{\gamma}(\boldsymbol{\mu}) = \mathbf{g}_{\mathfrak{I}}(\boldsymbol{\mu})$
end function

Discrete Empirical Interpolation Method viii

Remark(Gappy POD) The interpolation condition (16) can be generalized to the case where more sample indices ($|\mathfrak{J}| > M$) than basis functions are considered.

Figu

This leads to the so-called gappy *POD* reconstruction [1],[2], which provides an approximation under the form (15) where instead of the linear system (16) we need to solve

$$\gamma(\boldsymbol{\mu}) = \arg\min_{\mathrm{x}\in\mathbb{R}^M} \|\mathrm{g}_{\mathcal{I}}(\boldsymbol{\mu}) - \mathbb{Q}_{\mathcal{I}}\mathrm{x}\|_2 \,.$$

The solution of this least-squares problem yields $g_M(\boldsymbol{\mu}) = \mathbb{Q}^+_{\mathfrak{J}}\mathrm{g}_{\mathfrak{J}}(\boldsymbol{\mu})$, where $\mathbb{Q}^+_{\mathfrak{J}}$ is the Moore-Penrose pseudoinverse of the matrix $\mathbb{Q}_{\mathfrak{J}}$.

[1] Bui-Thanh, T., Damodaran, M., Willcox, K.: Proper orthogonal decomposition extensions for parametric applications in transonic aerodynamics (AIAA Paper 2003-4213). In: Proceedings of the 15th AIAA Computational Fluid Dynamics Conference (2003)

[2] Carlberg, K., Farhat, C., Cortial, J., Amsallem, D.: The GNAT method for nonlinear model reduction: Effective implementation and application to computational fluid dynamics and turbulent flows. J. Comput. Phys. 242, 623 - 647 (2013)

Non affine problems

EIM for Non Affine Problems i

We now exploit the empirical interpolation method to recover an affine parametric dependence in the originally nonaffine operators.

Although both EIM and DEIM lead to the same computational procedure from a practical standpoint, we exploit the former to develop our analysis in a more straightforward way.

Consider our usual model problem where for the sake of interpolation we assume that both $s(\cdot;\boldsymbol{\mu})$ and $k(\cdot;\boldsymbol{\mu}) \in C^0(\Omega)$ for any $\boldsymbol{\mu} \in \mathcal{D}$.

Note: We assume to deal with a strongly coercive problem, although all the results can be extended to the more general case of weakly coercive problems.

EIM for Non Affine Problems ii

The high-fidelity approximation reads: find $u_h(\boldsymbol{\mu}) \in| V_h$ such that

$$a\left(u_h, v_h; \boldsymbol{\mu}\right) = f\left(v_h; \boldsymbol{\mu}\right) \quad \forall v_h \in V_h,$$

Figu

with

$$a\left(u_h, v_h; \boldsymbol{\mu}\right) = \int_\Omega k(\mathrm{x}; \boldsymbol{\mu}) \nabla u_h \cdot \nabla v_h d\Omega, \quad f\left(v_h; \boldsymbol{\mu}\right) = \int_\Omega s(\mathrm{x}; \boldsymbol{\mu}) v_h d\Omega$$

We assume that the diffusion coefficient $k(\mathrm{x}; \boldsymbol{\mu})$ and the source term $s(\mathrm{x}; \boldsymbol{\mu})$ are nonaffine functions of $\boldsymbol{\mu}$, yielding a nonaffine parametric dependence of the linear and bilinear forms. Using the EIM, we replace them by the corresponding interpolants

$$k_M(\mathrm{x}; \boldsymbol{\mu}) = \mathcal{I}_M^{\mathrm{x}} k(\mathrm{x}; \boldsymbol{\mu}) = \sum_{j=1}^{M_k} \gamma_j^k(\boldsymbol{\mu}) \rho_j^k(\mathrm{x})$$

$$s_M(\mathrm{x}; \boldsymbol{\mu}) = \mathcal{I}_M^{\mathrm{x}} s(\mathrm{x}; \boldsymbol{\mu}) = \sum_{j=1}^{M_s} \gamma_j^s(\boldsymbol{\mu}) \rho_j^s(\mathrm{x})$$

EIM for Non Affine Problems iii

We now write the corresponding high-fidelity problem with EIM approximation (to which we refer to as EIM high-fidelity approximation) becomes:

find $u_h^M(\boldsymbol{\mu}) \in V_h$ such that

$$a_M\left(u_h^M(\boldsymbol{\mu}), v_h; \boldsymbol{\mu}\right) = f_M\left(v_h; \boldsymbol{\mu}\right) \quad \forall v_h \in V_h,$$

where we have set

$$a_M\left(u_h, v_h; \boldsymbol{\mu}\right) = \int_\Omega k_M(\mathrm{x}; \boldsymbol{\mu}) \nabla u_h \cdot \nabla v_h d\Omega, \quad f_M\left(v_h; \boldsymbol{\mu}\right) = \int_\Omega s_M(\mathrm{x}; \boldsymbol{\mu}) v_h d\Omega.$$

The linear and the bilinear forms now depend on the number M_k, M_s of terms appearing in the interpolants; we denote the EIM dependence by the subscript M.

EIM for Non Affine Problems iv

Figu

The associated Galerkin RB problem, thus reads:

find $u_N^M(\boldsymbol{\mu}) \in V_N$ such that

$$a_M\left(u_N^M(\boldsymbol{\mu}), v_N; \boldsymbol{\mu}\right) = f_M\left(v_N; \boldsymbol{\mu}\right) \quad \forall v_N \in V_N \tag{19}$$

EIM for Non Affine Problems v

Figu

Remark: We remark that the bilinear form $a_M(\cdot, \cdot; \boldsymbol{\mu})$ does not necessarily preserve the properties of $a(\cdot, \cdot; \boldsymbol{\mu})$. While the symmetry of $a(\cdot, \cdot; \boldsymbol{\mu})$ is automatically inherited by its approximation, this is not the case for the (strong) coercivity. However, by requiring that the high-fidelity problem is coercive, the EIM RB problem is well-posed too, being a Galerkin projection.

Note: The EIM RB problem can be considered as a generalized Galerkin method.

EIM for Non Affine Problems vi

Theorem (Convergence)
Let us suppose that $a_M(\cdot,\cdot;\boldsymbol{\mu})$ is continuous on $V_h \times V_h$ and coercive on V_h for any $\boldsymbol{\mu} \in \mathcal{D}$, that is

$$\exists \alpha_h^M(\boldsymbol{\mu}) : a_M(v,v;\boldsymbol{\mu}) \geq \alpha_h^M(\boldsymbol{\mu})\|v\|_V^2 \quad \forall v \in V_h, \forall \boldsymbol{\mu} \in \mathcal{D}.$$

Moreover, let us suppose that $f_M(\cdot;\boldsymbol{\mu})$ is continuous on V_h.

Then problem (19) admits a unique solution $u_N^M(\boldsymbol{\mu}) \in V_N$ which satisfies

$$\left\|u_N^M(\boldsymbol{\mu})\right\|_V \leq \frac{1}{\alpha_N^M(\boldsymbol{\mu})} \left\|f_M(\cdot;\boldsymbol{\mu})\right\|_{V_h'},$$

being

$$\alpha_N^M(\boldsymbol{\mu}) = \inf_{v \in V_N} \frac{a_M(v,v;\boldsymbol{\mu})}{\|v\|_V^2} \geq \alpha_h^M(\boldsymbol{\mu})$$

the stability factor of the EIM RB problem, and fulfills the following a priori error estimate

Figu

EIM RB method A posteriori estimate i

It is also possible to derive an a posteriori error estimate by combining the error estimator obtained in the case of a generic linear elliptic PDE, and a suitable indicator of the empirical interpolation error.

Let us denote by $e_h^M(\boldsymbol{\mu}) = u_h(\boldsymbol{\mu}) - u_N^M(\boldsymbol{\mu})$ the error between the high-fidelity and the EIM RB solutions and by

$$r_M(v;\boldsymbol{\mu}) = f_M(v;\boldsymbol{\mu}) - a_M\left(u_N^M(\boldsymbol{\mu}), v; \boldsymbol{\mu}\right) \quad \forall v \in V$$

the residual of the EIM high-fidelity problem computed on the RB solution. We assume that $a(\cdot,\cdot;\boldsymbol{\mu})$ is strongly coercive over $V_h \times V_h$ for any $\boldsymbol{\mu} \in \mathcal{D}$ and denote by $\alpha_h(\boldsymbol{\mu})$ its stability factor

EIM RB method A posteriori estimate ii

Proposition
The following a posteriori estimates hold

$$\left\|u_h(\boldsymbol{\mu}) - u_N^M(\boldsymbol{\mu})\right\|_V \leq \frac{1}{\alpha_h(\boldsymbol{\mu})}\left(\|r_M(\cdot;\boldsymbol{\mu})\|_{V_h'} + C_f\delta_s(\boldsymbol{\mu}) + C_a\delta_k(\boldsymbol{\mu})\left\|u_N^M(\boldsymbol{\mu})\right\|_V\right)$$

where

$$C_f = \sup_{v\in V_h}\frac{\int_\Omega v d\Omega}{\|v\|_V}, \quad C_a = \sup_{v\in V_h w\in V_h}\sup_{\Omega}\frac{\int_\Omega \nabla v\cdot\nabla w d\Omega}{\|v\|_V\|w\|_V}$$

and

$$\delta_k(\boldsymbol{\mu}) = \|k(\cdot;\boldsymbol{\mu}) - k_M(\cdot;\boldsymbol{\mu})\|_{L^\infty(\Omega)}, \quad \delta_s(\boldsymbol{\mu}) = \|s(\cdot;\boldsymbol{\mu}) - s_M(\cdot;\boldsymbol{\mu})\|_{L^\infty(\Omega)}$$

EIM RB method A posteriori estimate iii

Proof: we have that, for all $v_h \in V_h$,

$$a\left(u_h(\boldsymbol{\mu}), v_h; \boldsymbol{\mu}\right) - a\left(u_N^m(\boldsymbol{\mu}), v_h; \boldsymbol{\mu}\right) = f\left(v_h; \boldsymbol{\mu}\right) - a\left(u_N^m(\boldsymbol{\mu}), v_h; \boldsymbol{\mu}\right)$$
$$= f\left(v_h; \boldsymbol{\mu}\right) - f_M\left(v_h; \boldsymbol{\mu}\right) + f_M\left(v_h; \boldsymbol{\mu}\right)$$
$$- a_M\left(u_N^m(\boldsymbol{\mu}), v_h; \boldsymbol{\mu}\right) + a_M\left(u_N^m(\boldsymbol{\mu}), v_h; \boldsymbol{\mu}\right) - a\left(u_N^m(\boldsymbol{\mu}), v_h; \boldsymbol{\mu}\right)$$

Figu

By taking $v_h = e_h^M(\boldsymbol{\mu})$, exploiting the continuity of the linear and bilinear forms and the coercivity of $a(\cdot, \cdot; \boldsymbol{\mu})$, and dividing by $\left\|e_h^M(\boldsymbol{\mu})\right\|_V$, we find

$$\alpha_h(\boldsymbol{\mu}) \left\|e_h^M(\boldsymbol{\mu})\right\|_V \leq \left\|f(\cdot; \boldsymbol{\mu}) - f_M(\cdot; \boldsymbol{\mu})\right\|_{V_h'}$$
$$+ \left\|a(\cdot, ; ; \boldsymbol{\mu}) - a_M(\cdot, \cdot; \boldsymbol{\mu})\right\|_{\mathcal{L}\left(V_h, V_h'\right)} \left\|u_N^M(\boldsymbol{\mu})\right\|_V + \left\|r_M(\cdot; \boldsymbol{\mu})\right\|_{V_h'}.$$

The first term can be bounded as

$$\left\|f(\cdot; \boldsymbol{\mu}) - f_M(\cdot; \boldsymbol{\mu})\right\|_{V_h'} \leq \sup_{v \in V_h} \frac{\left|\int_\Omega \left(s(\cdot; \boldsymbol{\mu}) - s_M(\cdot; \boldsymbol{\mu})\right) v d\Omega\right|}{\|v\|_V}$$
$$\leq C_f \left\|s(\cdot; \boldsymbol{\mu}) - s_M(\cdot; \boldsymbol{\mu})\right\|_{L^\infty(\Omega)}.$$

EIM RB method A posteriori estimate iv

Figu

Similarly for the second term we find

$$\|a(\cdot,\cdot;\boldsymbol{\mu}) - a_M(\cdot,\cdot;\boldsymbol{\mu})\|_{\mathcal{L}(V_h,V_h')} \leq \sup_{v\in V_h w\in V_h} \frac{\left|\int_\Omega (k(\cdot;\boldsymbol{\mu}) - k_M(\cdot;\boldsymbol{\mu}))\nabla v\cdot\nabla w d\Omega\right|}{\|v\|_V\|w\|_V}$$
$$\leq C_a \|k(\cdot;\boldsymbol{\mu}) - k_M(\cdot;\boldsymbol{\mu})\|_{L^\infty(\Omega)}.$$

$\square$

Remark: Surrogates for $\delta_k(\boldsymbol{\mu})$ and $\delta_s(\boldsymbol{\mu})$ could be either the tolerance of the EIM algorithm or the estimator $\hat{\varepsilon}_M(\boldsymbol{\mu})$.