# Model Order Reduction Techniques
# Problem Set 1: RB for Linear Affine Elliptic Problems
# Design of a Thermal Fin

C. Prud'homme

January 26, 2023

We consider the problem of designing a thermal fin to effectively remove heat from a surface. The two-dimensional fin, shown in Figure 1, consists of a vertical central "post" and four horizontal "subfins"; the fin conducts heat from a prescribed uniform flux "source" at the root, $\Gamma_{\mathrm{root}}$ , through the large-surface-area subfins to surrounding flowing air. The fin is characterized by a five-component parameter vector, or "input," $\mu=(\mu_1, \mu_2, \ldots, \mu_5)$,where $\mu_i = k^i, i = 1, \ldots, 4$, and $\mu_5 = \mathrm{Bi}$; $\mu$ may take on any value in a specified design set $D \subset \mathbb{R}^5$.

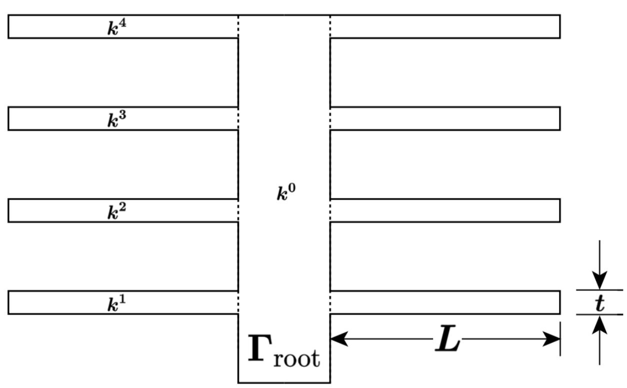


Figure 1: Thermal fin

Here $k^i$ is the thermal conductivity of the ith subfin (normalized relative to the post conductivity $k^0 \equiv 1$); and Bi is the Biot number, a nondimensional heat transfer coefficient reflecting convective transport to the air at the fin surfaces (larger Bi means better heat transfer). For example, suppose we choose a thermal fin with $k^1 = 0.4, k^2 = 0.6, k^3 = 0.8, k^4 = 1.2$, and $\mathrm{Bi} = 0.1$; for this particular configuration $\mu = \{0.4, 0.6, 0.8, 1.2, 0.1\}$, which corresponds to a single point in the set of all possible configurations D (the parameter or design set). The post is of width unity and height four; the subfins are of fixed thickness $t = 0.25$ and length $L = 2.5$.

We are interested in the design of this thermal fin, and we thus need to look at certain outputs or cost-functionals of the temperature as a function of $\mu$. We choose for our output $T_{\text{root}}$, the average steady-state temperature of the fin root normalized by the prescribed heat flux into the fin root. The particular output chosen relates directly to the cooling efficiency of the fin — lower values of $T_{\text{root}}$ imply better thermal performance. The steady–state temperature distribution within the fin, $u(\mu)$, is governed by the elliptic partial differential equation

$$-k^i \Delta u^i = 0 \text{ in } \Omega^i, i = 0, \ldots, 4, \tag{1}$$

where $\Delta$ is the Laplacian operator, and $u_i$ refers to the restriction of $u$ to $\Omega^i$ . Here $\Omega^i$ is the region of the fin with conductivity $k^i, i = 0, \ldots, 4 : \Omega^0$ is thus the central post, and $\Omega^i, i = 1, \ldots, 4$, corresponds to the four subfins. The entire fin domain is denoted $\Omega(\bar{\Omega} = \cup_{i=0}^4 \bar{\Omega}^i)$; the boundary $\Omega$ is denoted $\Gamma$. We must also ensure continuity of temperature and heat flux at the conductivity–discontinuity interfaces $\Gamma^i_{int} \equiv \partial\Omega^0 \cap \partial\Omega^i, i = 1, \ldots, 4$, where $\partial\Omega^i$ denotes the boundary of $\Omega^i$, we have on $\Gamma^i_{int} i = 1, \ldots, 4$ :

$$u^0 = u^i \tag{2}$$

$$-(\nabla u^0 \cdot n^i) = -k^i(\nabla u^i \cdot n^i) \tag{3}$$

here $n^i$ is the outward normal on $\partial\Omega^i$ . Finally, we introduce a Neumann flux boundary condition on the fin root

$$-(\nabla u^0 \cdot n^0) = -1 \text{ on } \Gamma_{\text{root}}, \tag{4}$$

which models the heat source; and a Robin boundary condition

$$-k^i(\nabla u^i \cdot n^i) = \text{Bi}u^i \text{ on } \Gamma^i_{ext}, i = 0, \ldots, 4, \tag{5}$$

which models the convective heat losses. Here $\Gamma^i_{ext}$ is that part of the boundary of $\Omega^i$ exposed to the flowing fluid; note that $\cup_{i=0}^4 \Gamma^i_{ext} = \Gamma \backslash \Gamma_{\text{root}}$. The average temperature at the root, $T_{\text{root}}(\mu)$, can then be expressed as $\ell^O(u(\mu))$, where

$$\ell^O(v) = \int_{\Gamma_{\text{root}}} v \tag{6}$$

(recall $\Gamma_{\text{root}}$ is of length unity). Note that $\ell(v) = \ell^O(v)$ for this problem.

# 1 Part 1 - Finite Element Approximation

We saw in class that the reduced basis approximation is based on a "truth" finite element approximation of the exact (or analytic) problem statement. To

begin, we have to show that the exact problem described above does indeed satisfy the affine parameter dependence and thus fits into the framework shown in class.
a) Show that $u^e(\mu) \in X^e \equiv H^1(\Omega)$ satisfies the weak form

$$a(u^e(\mu), v; \mu) = \ell(v), \forall v \in X^e, \tag{7}$$

with

$$\begin{aligned} a(w, v; \mu) &= \sum_{i=0}^{4} k^i \int_{\Omega^i} \nabla w \cdot \nabla v dA + \mathrm{Bi} \int_{\Gamma \backslash \Gamma_{\mathrm{root}}} wv dS, \\ \ell(v) &= \int_{\Gamma_{\mathrm{root}}} v \end{aligned}$$

b) Show that $u^e(\mu)$ is the argument that minimizes

$$J(w) = \frac{1}{2} \sum_{i=0}^{4} k^i \int_{\Omega_i} \nabla w \cdot \nabla w dA + \frac{\mathrm{Bi}}{2} \int_{\Gamma \backslash \Gamma_{\mathrm{root}}} w^2 dS - \int_{\Gamma_{\mathrm{root}}} w dS \tag{8}$$

over all functions $w$ in $X^e$ .

We now consider the linear finite element space

$$X^{\mathcal{N}} = \{v \in H^1(\Omega) | v|_{T_h} \in \mathbb{P}^1(\mathcal{T}_h), \forall T_h \in \mathcal{T}_h\},$$

and look for $u^{\mathcal{N}}(\mu) \in X^{\mathcal{N}}$ such that

$$a(u^{\mathcal{N}}(\mu), v; \mu) = \ell(v), \forall v \in X^{\mathcal{N}}; \tag{9}$$

our output of interest is then given by

$$T^{\mathcal{N}}_{\mathrm{root}}(\mu) = \ell^O(u^{\mathcal{N}}(\mu)). \tag{10}$$

Applying our usual nodal basis, we arrive at the matrix equations

$$\underline{A}^{\mathcal{N}}(\mu) \underline{u}^{\mathcal{N}}(\mu) = \underline{F}^{\mathcal{N}}, \tag{11}$$

$$T^{\mathcal{N}}_{\mathrm{root}}(\mu) = (\underline{L}^{\mathcal{N}})^T \underline{u}^{\mathcal{N}}(\mu), \tag{12}$$

where $\underline{A}^{\mathcal{N}} \in \mathbb{R}^{\mathcal{N} \times \mathcal{N}}, \underline{u}^{\mathcal{N}} \in \mathbb{R}^{\mathcal{N}}, \underline{F}^{\mathcal{N}} \in \mathbb{R}^{\mathcal{N}}, and \underline{L}^{\mathcal{N}} \in \mathbb{R}^{\mathcal{N}}$; here $\mathcal{N}$ is the dimension of the finite element space $X^{\mathcal{N}}$, which (given our natural boundary conditions) is equal to the number of nodes in $\mathcal{T}_h$.

# 2 Part 2 - Reduced-Basis Approximation

In general, the dimension of the finite element space, $\dim X = \mathcal{N}$, will be quite large (in particular if we were to treat the more realistic three-dimensional fin problem), and thus the solution of $\underline{A}^{\mathcal{N}} \underline{u}^{\mathcal{N}}(\mu) = \underline{F}^{\mathcal{N}}$ can be quite expensive. We thus investigate the reduced-basis methods that allow us to accurately and very rapidly predict $T_{\text{root}}(\mu)$ in the limit of many evaluations — that is, at many different values of $\mu$ — which is precisely the "limit of interest" in design and optimization studies. To derive the reduced-basis approximation we shall exploit the energy principle,

$$u(\mu) = \underset{w \in X}{\operatorname{argmin}}\, J(w), \tag{13}$$

where $J(w)$ is given by (8).

To begin, we introduce a sample in parameter space, $S_N = \{\mu_1, \mu_2, \ldots, \mu_N\}$ with $N \ll \mathcal{N}$. Each $\mu_i, i = 1, \ldots, N$ , belongs in the parameter set $\mathcal{D}$. For our parameter set we choose $\mathcal{D} = [0.1, 10.0]^4 \times [0.01, 1.0]$, that is, $0.1 \leq k^i \leq 10.0, i = 1, \ldots, 4$ for the conductivities, and $0.01 \leq \text{Bi} \leq 1.0$ for the Biot number. We then introduce the reduced-basis space as

$$W_N = \text{span}\{u^{\mathcal{N}}(\mu_1), u^{\mathcal{N}}(\mu_2), \ldots, u^{\mathcal{N}}(\mu_N)\} \tag{14}$$

where $u_N(\mu_i)$ is the finite-element solution for $\mu = \mu_i$.

To simplify the notation, we define $\xi^i \in X$ as $\xi^i = u_N(\mu_i), i = 1, \ldots, N$; we can then write $W_N = span\{\xi^i, i = 1, \ldots, N\}$. Recall that $W_N = \text{span}\{\xi^i, i = 1, \ldots, N\}$ means that $W_N$ consists of all functions in $X$ that can be expressed as a linear combination of the $\xi^i$ ; that is, any member $v_N$ of $W_N$ can be represented as

$$v_N = \sum_{i=1}^{N} \beta^j \xi^j, \tag{15}$$

for some unique choice of $\beta^j \in \mathbb{R}, j = 1, \ldots, N$. (We implicitly assume that the $\xi^i, i = 1, \ldots, N$, are linearly independent; it follows that $W_N$ is an $N$ -dimensional subspace of $X^{\mathcal{N}}$.) In the reduced-basis approach we look for an approximation $u_N(\mu)$ to $u^{\mathcal{N}}(\mu)$ (which for our purposes here we presume is arbitrarily close to $u^e(\mu)$) in $W_N$ ; in particular, we express $u_N(\mu)$ as

$$u_N(\mu) = \sum_{i=1}^{N} u_N^j \xi^j; \tag{16}$$

we denote by $\underline{u}_N(\mu) \in \mathbb{R}^N$ the coefficient vector $(u_N^1, \ldots, u_N^N)^T$. The premise — or hope — is that we should be able to accurately represent the solution at some new point in parameter space, $\mu$, as an appropriate linear combination of solutions previously computed at a small number of points in parameter space (the $\mu_i, i = 1, \ldots, N$). But how do we find this appropriate linear combination? And how good is it? And how do we compute our approximation efficiently? The energy principle is crucial here (though more generally the weak form would suffice). To wit, we apply the classical Rayleigh-Ritz procedure to define

$$u_N(\mu) = \underset{wN \in W_N}{\operatorname{argmin}}\, J(w_N); \tag{17}$$

alternatively we can apply Galerkin projection to obtain the equivalent statement

$$a(u_N(\mu), v; \mu) = \ell(v), \quad \forall v \in W_N. \tag{18}$$

The output can then be calculated from

$$T_{\mathrm{root}\,N}(\mu) = \ell^O(u_N(\mu)). \tag{19}$$

We now study this approximation in more detail.

a) Prove that, in the energy norm $|||\cdot||| \equiv (a(,;\mu))^{1/2}$ ,

$$|||u(\mu) - u_N(\mu)||| \le |||u(\mu) - w_N|||, \forall w_N \in W_N. \tag{20}$$

This inequality indicates that out of all the possible choices of wN in the space $W_N$ , the reduced basis method defined above will choose the "best one" (in the energy norm). Equivalently, we can say that even if we knew the solution $u(\mu)$, we would not be able to find a better approximation to $u(\mu)$ in $W_N$ — in the energy norm — than $u_N(\mu)$.

b) Prove that

$$T_{\mathrm{root}}(\mu) - T_{\mathrm{root}\,N}(\mu) = |||u(\mu) - u_N(\mu)|||^2. \tag{21}$$

c) Show that $u_N(\mu)$ as defined in (17)-(19) satisfies a set of $N \times N$ linear equations,

$$\underline{\mathrm{A}}_N(\mu)\underline{u}_N(\mu) = \underline{F}_N; \tag{22}$$

and that

$$T_{\mathrm{root}\,N}(\mu) = \underline{L}_N^T \underline{u}_N(\mu). \tag{23}$$

Give expressions for $\underline{A}_N(\mu) \in \mathbb{R}^{N\times N}$ in terms of $\underline{A}^{\mathcal{N}}(\mu)$ and $Z$, $\underline{F}^N \in \mathbb{R}^N$ in terms of $\underline{F}^{\mathcal{N}}$ and $Z$, and $\underline{L}^N \in \mathbb{R}^N$ in terms of $\underline{L}^{\mathcal{N}}$ and $Z$; here $Z$ is an $\mathcal{N} \times N$ matrix, the $j$ th column of which is $\underline{u}_N(\mu^j)$ (the nodal values of $\underline{u}^{\mathcal{N}}(\mu^j)$).

d) Show that the bilinear form $a(w, v; \mu)$ can be decomposed as

$$a(w, v; \mu) = \sum_{q=0}^{Q} \theta^q(\mu) a^q(w, v), \forall w, v \in X, \forall \mu \in D, ; \tag{24}$$

for $Q = 6$ and give expressions for the $\theta^q(\mu)$ and the $a^q(w, v)$. Notice that the $aq(w, v)$ are not dependent on $\mu$; the parameter dependence enters only through the functions $\theta^q(\mu), q = 1, \ldots, Q$. Further show that

$$\underline{A}^{\mathcal{N}}(\mu) = \sum_{q=1}^{Q} \theta^q(\mu) \underline{A}^{\mathcal{N}q}, \tag{25}$$

and

$$\underline{A}^N(\mu) = \sum_{q=1}^{Q} \theta^q(\mu) \underline{A}_N^q, \tag{26}$$

Give an expression for the $\underline{A}^{\mathcal{N}q}$ in terms of the nodal basis functions; and develop a formula for the $\underline{A}_N^q$ in terms of the $\underline{A}^{\mathcal{N}q}$ and $Z$.

e) The coercivity and continuity constants of the bilinear form for the continuous problem are denoted by $\alpha^e(\mu)$ and $\gamma^e(\mu)$, respectively. We now assume that the basis function $\xi i, i = 1, \ldots, N$, are orthonormalized. Show that the condition number of $\underline{A}_N(\mu)$ is then bounded from above by $\gamma^e(\mu)/\alpha^e(\mu)$.

f): Take into account the parameters $L$ and $t$ as parameters: the geometric transformation must be taken into account in the affine decomposition procedure.

*acknowledgment:* This problem set is based on a problem set in the class "16.920 Numerical Methods for Partial Differential Equations" at MIT. We would like to thank Prof. A.T. Patera and Debbie Blanchard for providing the necessary material and the permission to use it.