Reduced Basis methods: an introduction

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1 A Priori Reduced Basis Theory

A Priori Reduced Basis Theory

So far we assumed $\boldsymbol{S_N}$ (and hence $\boldsymbol{W_N}$) to be known. Problem: How do we (optimally) choose a sample set $S_{\boldsymbol{N}}$, that provides rapid convergence of RB approximation?

Generic a priori recipes to generate $S_N\,(W_N)$ are available only for one-dimensional parameter problems ($\boldsymbol{P} = 1$).

- For higher parameter dimensions we will use "adaptive" sampling procedures. We consider

1. A priori convergence theory ($\boldsymbol{P} = 1$), and
2. A priori convergence theory ($\boldsymbol{P} > 1$), and
3. Adaptive sampling strategies.

Parameter domain and grids I

We recall

- parameter $\boldsymbol{\mu}$ and
- closed, bounded and suitably regular parameter domain $\mathcal{D} \subset \mathbb{R}^P$ such that

$$\mu = (\mu_1, \ldots, \mu_P) \in \mathcal{D} \subset \mathbb{R}^P$$

We also define

$$\mu_p^{\text{min}} = \min_{\mu \in \mathcal{D}} \mu_p, \quad , \mu_p^{\text{max}} = \max_{\mu \in \mathcal{D}} \mu_p, \quad 1 \leq p \leq P$$

and the smallest parallel-" P "ped, $\mathcal{D}_{\text{box}} \subset \mathbb{R}^P$, such that

$$\mathcal{D} \subset \mathcal{D}_{\text{box}} \equiv \left[\mu_1^{\text{min}}, \mu_2^{\text{max}}\right] \times \ldots \times \left[\mu_P^{\text{min}}, \mu_P^{\text{max}}\right]$$

Parameter domain and grids II

We introduce one-dimensional deterministic grids (for $z_2 \in \mathbb{R} > z_1 \in \mathbb{R}$, and $m \in \mathbb{N}$):

$$G^{\text{lin}}_{[z_1,z_2;m]} = \left\{ z_1 + \frac{i-1}{m-1}\left(z_2 - z_1\right), 1 \le i \le m \right\},$$
$$G^{\ln}_{[z_1,z_2;m]} = \left\{ z_1 \exp\left\{ \frac{i-1}{m-1} \ln\left(\frac{z_2}{z_1}\right)\right\}, 1 \le i \le m \right\};$$

and note that $\ln(z_i)$ is equi-distributed for $G^{\ln}_{[z_1,z_2;m]}$. For $\boldsymbol{P} \ge 2$: Grid of $\boldsymbol{m}^P$ points in $\mathcal{D}_{\text{box}}$

$$G^{\text{lin}}_{[\mu_1^{\min},\mu_2^{\max};m]} \times \ldots \times G^{\text{lin}}_{[\mu_P^{\min},\mu_P^{\max};m]}$$

Parameter domain and grids III

We also introduce the Monte-Carlo samples (of size $\boldsymbol{m} \in \mathbb{N}$):

- $G^{\mathrm{lin}}_{[\mathrm{MC};m]}$ with elements

$$\mu_p = \mu_p^{\mathrm{min}} + \mathrm{rand} \times \left(\mu_p^{\mathrm{max}} - \mu_p^{\mathrm{min}}\right), 1 \leq p \leq P$$

- $G^{\mathrm{ln}}_{[\mathrm{MC};m]}$ with elements

$$\mu_p = \mu_p^{\mathrm{min}} \exp\left\{\mathrm{rand} \times \ln\left(\frac{\mu_p^{\mathrm{max}}}{\mu_p^{\mathrm{min}}}\right)\right\}, 1 \leq p \leq P;$$

where rand is a random variable uniformly distributed over $[0, 1]$ and we reject any $\mu = (\mu_1, \ldots, \mu_P) \notin \mathcal{D}$.

Parameter domain and grids IV

We define

- Train sample

$$\Xi_{\text{train}} \equiv \{\mu^1_{\text{train}}, \ldots, \mu^{n_{\text{train}}}_{\text{train}}\} \subset \mathcal{D},$$

  with cardinality (size) $|\Xi_{\text{train}}| = n_{\text{train}}$;
- Test sample

$$\Xi_{\text{test}} \equiv \{\mu^1_{\text{test}}, \ldots, \mu^{n_{\text{test}}}_{\text{test}}\} \subset \mathcal{D},$$

  with cardinality (size) $|\Xi_{\text{test}}| = \boldsymbol{n}_{\text{test}}$. $\Rightarrow \Xi_{\text{train}}$ and Ξ_{test} serve as our (finite) surrogates for $\mathcal{D}$.

Example: $\Xi_{\text{train}} = G^{\text{lin}}_{[0.1,10;10^4]}$ – equi-distributed deterministic grid on $[0.1, 10]$ of size 10^4.

Definitions

- Given $y : \mathcal{D} \to \mathbb{R}$, we define

$$\|y\|_{L^\infty(\Xi)} \equiv \max_{\mu \in \Xi} |y(\mu)|,$$

$$\|y\|_{L^p(\Xi)} \equiv \left(|\Xi|^{-1} \sum_{\mu \in \Xi} |y(\mu)|^p \right)^{1/p} .$$

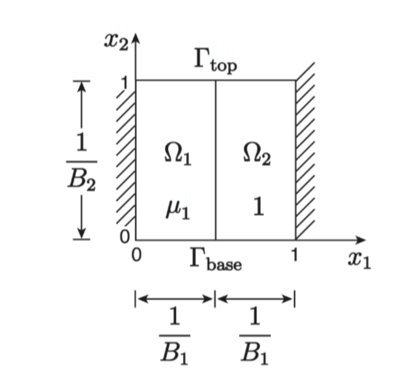
- Given $z : \mathcal{D} \to X^{\mathcal{N}}$, we define

$$\|z\|_{L^\infty(\Xi;X)} \equiv \max_{\mu \in \Xi} \|z(\mu)\|_X,$$

$$\|z\|_{L^p(\Xi;X)} \equiv \left(|\Xi|^{-1} \sum_{\mu \in \Xi} \|z(\mu)\|_X^p \right)^{1/p} .$$

Also: $\Pi_{\boldsymbol{X}_N} : \boldsymbol{X} \to \boldsymbol{X}_N$ projection in the $\boldsymbol{X}$-inner product

A Priori Theory: thermal block $P = 1$ I



We use $\mathcal{N} = 1024$

We consider the case

$$B_1 = 2, \quad B_2 = 1,$$

and $\mu_r = 100$, thus and

$$\mu_{\text{min}} = 1/\sqrt{\mu_r} = 0.1$$
$$\mu_{\text{max}} = \sqrt{\mu_r} = 10.$$

We thus have

$$\mu \in \mathcal{D} = [0.1, 10] \subset \mathbb{R}^{P=1}.$$

For our inner product we choose

$$\bar{\mu} = 1.$$

A Priori Theory: thermal block $P = 1$ II

Given $\mu \in \mathcal{D}$, evaluate

$$s(\boldsymbol{\mu}) = f(u(\boldsymbol{\mu}))$$

where $u(\mu) \in X \equiv \left\{ v \in H^1(\Omega) | v|_{\Gamma_{\mathbf{top}}} = 0 \right\}$ satisfies

$$\mu_1 a^1(u(\mu), v) + a^2(u(\mu), v) = f(v), \quad \forall v \in X.$$

Here,

$$a^q(w, v) = \int_{\Omega_q} \nabla w \cdot \nabla v, \quad \forall w, v \in X, 1 \leq q \leq 2,$$

and

$$f(v) \equiv \int_{\Gamma_{\mathbf{base}}} v, \quad \forall v \in X.$$

We obtain the inner product, $\forall w, v \in X$,

$$(w, v)_X = a^1(u(\mu), v) + a^2(u(\mu), v) \equiv \int_0 \nabla w \cdot \nabla v$$

A Priori Theory: thermal block $P = 1$ III

We next introduce the (in general) non-nested samples

$$S_N^{\mathrm{nh},\mathsf{ln}} = G^{\mathrm{ln}}_{[\mu_{\min},\mu_{\max};N]}, 2 \leq N \leq N_{\max}$$

and associated Lagrange Space

$$W_N^{\mathrm{nh},\mathsf{ln}} = \mathrm{span}\left\{u\left(\mu_N^n\right), 1 \leq n \leq N\right\}, 2 \leq N \leq N_{\max}$$

RB Approximation: Given $\boldsymbol{\mu} \in \mathcal{D}$, evaluate

$$s_N^{\mathrm{nh},\mathsf{ln}}(\mu) = f\left(u_N^{\mathrm{nh},\mathsf{ln}}(\mu)\right)$$

where $\boldsymbol{u_N}(\boldsymbol{\mu}) \in \boldsymbol{W}_{\boldsymbol{N}}^{\mathrm{nh},\mathsf{ln}}$ satisfies

$$\mu a^1\left(u_N^{\mathrm{nh},\mathsf{ln}}(\mu), v\right) + a^2\left(u_N^{\mathrm{nh},\mathsf{ln}}(\mu), v\right) = f(v), \forall v \in W_N^{\mathrm{nh},\mathsf{ln}}.$$

A Priori Theory: thermal block $P = 1$ IV

Proposition (Patera Rozza 2007)

For any $f \in X'$, and for all $\mu \in \mathcal{D}$,

$$\frac{\left|\left|\left| u(\mu) - u_N^{\mathrm{nh},\ln}(\mu) \right|\right|\right|_\mu}{|||u(\mu)|||_\mu} \leq \exp\left\{ -\frac{N-1}{N_{crit}-1} \right\},$$

and

$$\frac{s(\mu) - s_N^{\mathrm{nh},\ln}(\mu)}{s(\mu)} \leq \exp\left\{ -\frac{2(N-1)}{N_{\mathrm{crit}}-1} \right\},$$

for $N \geq N_{crit} = 1 + [2e \ln \mu_r]_+$.

Here $[\arg]_+$returns the smallest integer greater than or equal to its real argument $\arg \in \mathbb{R}$.

A Priori Theory: thermal block $P = 1$ V

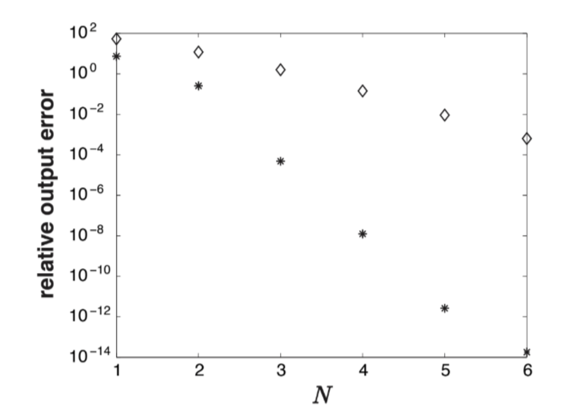


Figure 2: ThermalBlock $P = 1$: Relative error in the output over $\Xi^{0}_{\text{train}} = G^{\ln}_{[\mu_{\min},\mu_{\max};10^4]}$ for $s_N^{\mathrm{nh},\ln}$: actual error ($\star$) and "pseudo" *a* priori bound ($\diamond$). Source: PR (2007).

A Priori Theory: thermal block $P = 1$ VI

Remarks

- Convergence of reduced basis approximation relies on smoothness in parameter but not on spatial regularity $-a$ priori result is valid for any $f \in \boldsymbol{X}'$.
- Reduced basis convergence rate does not depend on $\mathcal{N}$.
- Reduced basis convergence rate depends quite weakly on the extent of $\mathcal{D}$ (logarithmic in $\mu_{\boldsymbol{r}}$) - global approximation.
- Reduced basis approximation can converge very quickly.

Problems:

- Spaces are non-hierachical - and thus not very practical.
- For $\boldsymbol{P} > 1$ a priori theory requires more general framework.

Next Steps

- Kolmogorov N-Width
- A priori theory result
- A POD Approach Method of Snapshots
- A Greedy Approach
  - $\boldsymbol{X}$-norm bound
  - Energy norm bound
- Comparison
  - Greedy vs. log-Sample Greedy vs. POD

Kolmogorov N-Width I

Concerning the approximability of the solution set, we start by asking how well $\mathcal{M}$ can be approximated (uniformly with respect to $\boldsymbol{\mu}$) by a finite-dimensional subspace of prescribed dimension. To answer this question, we recall the important notion of Kolmogorov n-width[5].

Let K be a compact set of a generic Hilbert space X, and consider a generic n dimensional subspace $X_n \subset X$. If we define the distance between an element $x \in X$ and X_n as

$$d\left(x; X_n\right) = \inf_{x_n \in X_n} \|x - x_n\|_X \tag{6}$$

any element $\hat{x}_n \in X_n$ which realizes the infimum, that is

$$\|x - \hat{x}_n\|_X = d\left(x, X_n\right), \tag{7}$$

is called the best approximation of x in X_n.

Kolmogorov N-Width II

A very natural question is whether the n-dimensional subspace is suitable to approximate all the elements $x \in K$.

To be precise, we quantify the worst possible best approximation as the angle between the subspace X_n and the set K, defined[6] by

$$d(K; X_n) = \sup_{x \in K} d(x; X_n). \tag{8}$$

Kolmogorov N-Width III

The distance between a subspace X_n and K is determined by the worst-case scenario.

Finding the best n-dimensional subspace of X for approximating K determines the minimum, over all possible n-dimensional subspaces of X, of the deviation 8, that is,

$$d_n(K;X) = \inf_{\substack{X_n \subset X \\ \dim(X_n)=n}} d\left(K;X_n\right) = \inf_{\substack{X_n \subset X \\ \dim(X_n)=n}} \sup_{x \in K} \inf_{x_n \in X_n} \left\| x - x_n \right\|_V .$$

(9)

The number $d_n(K;X)$ is called the Kolmogorov n-width of K, first

Kolmogorov N-Width IV

introduced by Kolmogorov[7]. It represents the best achievable accuracy in the X-norm when all possible elements of K are approximated by elements belonging to a linear n dimensional subspace $X_n \subset X$. A subspace $\hat{X}_n$ of dimension at most n such that

$$d\left(K;\hat{X}_n\right) = d_n(K;X)$$

is called an optimal n-dimensional subspace for $d_n(K;X)$.

Kolmogorov N-Width V

Replacing X by V_h and K by $\mathcal{M}$, we can now define the Kolmogorov n-width of the solution set $\mathcal{M}$ as

$$d_n\left(\mathcal{M}; V_h\right) = \inf_{\substack{V_n \subset V_h \\ \dim(V_n)=n}} d\left(\mathcal{M}; V_n\right) = \inf_{\substack{V_n \subset V_h \\ \dim(V_n)=n}} \sup_{\boldsymbol{\mu} \in \mathcal{D}} \inf_{v_n \in V_n} \left\| u_h(\boldsymbol{\mu}) - v_n \right\|_V \tag{10}$$

Kolmogorov N-Width VI

Since V_h is a Hilbert space, there exists an orthogonal projection operator $\Pi_{V_n} : V \to V_n$ such that

$$\|v - \Pi_{V_n} v\|_V = \min_{v_n \in V_n} \|v - v_n\|_V \quad \forall v \in V_h.$$

The Kolmogorov n-width of $\mathscr{M}_h$ can thus be expressed as

$$d_n(\mathcal{M}; V_h) = \inf_{\substack{V_n \subset V_h \\ \dim(V_n) = n}} \|u_h - \Pi_{V_n} u_h\|_{L^\infty(\mathcal{D}; V)} . \tag{11}$$

For $n = N$, 11 corresponds to the best achievable error in a uniform sense when approximating the solution manifold $\mathcal{M}$ by elements of the RB space V_N. In this regard, the Kolmogorov n-width is relevant for deciding whether or not a given parametrized problem can be efficiently reduced. Evaluating this quantity is a hard task from a theoretical standpoint.

Kolmogorov N-Width VII

In some cases, the n-width of the solution manifold can be directly deduced from that of the space of the parametric coefficients (say, $\theta_q^a(\boldsymbol{\mu}), q = 1, \ldots, Q_a$ and $\theta_q^f(\boldsymbol{\mu}), q = 1, \ldots, Q_f\big)$ of the affine expansions. See [Cohen-Devore-2015][8] for further details.

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[5]Melenk, J.M.: On n-widths for elliptic problems. J. Math. Anal. Appl. 247, 272-289 (2000)
Pinkus, A.: n-Widths in Approximation Theory. Springer-Verlag, Ergebnisse (1985)

[6]We can refer to $d\,(K; X_n)$ as to the discrepancy or deviation between X_n and K as well.

[7]Kolmogorov, A.N.: Ber die beste annaherung von funktionen einer gegebenen funktionenklasse. Ann. of Math. 37, 107-110 (1936)

[8]Cohen, A., DeVore, R.: Approximation of high-dimensional parametric PDEs. ArXiv eprints 1502.06797 (2015)

A priori convergence $P > 1$ I

Theorem (Buffa, Maday, Patera, Prud'homme[9])

Assume that the set of all solutions $\mathcal{M} = \{u(\boldsymbol{\mu}), \boldsymbol{\mu} \in \mathcal{D}\}$ *to has an exponentially small Kolmogorov n-width* $d_k(\mathcal{M}, X) \leq ce^{-\alpha k}$ *with* $\alpha > \log\left(1 + \sqrt{\frac{\gamma}{\alpha_{\mathbf{coer}}}}\right)$*; then the reduced basis method converges exponentially in the sense that there exists* $\beta > 0$ *such that*

$$\forall \boldsymbol{\mu} \in \mathcal{D}, \quad \|u(\mu) - u_N(\mu)\|_X \leq Ce^{-\beta N}$$

A priori convergence $P > 1$ II

Note that in practice the sup over $\mathcal{D}$ is replaced with a sup over a very fine sample in $\mathcal{D}$, requiring nevertheless many expensive evaluations. In order to construct a computable algorithm, we need in addition to replace the theoretical error bound with a relatively cheap procedure that maintain the performance stated in the estimate. We thus use

$$\boldsymbol{\mu}_i = \arg\sup_{\boldsymbol{\mu} \in D} \Delta_{i-1}(\boldsymbol{\mu})$$

in the practical greedy procedure

[9] A priori convergence of the Greedy algorithm for the parametrized reduced basis method, ESAIM: M2AN Volume 46, Number 3, May-June 2012

Back to POD

Proper Orthogonal Decomposition (POD) (or Karhunen-Loève expansion) approach:

- POD Spaces

$$X_N^{\mathrm{POD}} = \arg \operatorname{spaces}_{X_N \subset \operatorname{span}\{u(\mu) | \mu \in \Xi_{\mathbf{train}}\}} \left\| u - \Pi_{X_N} u \right\|_{L^2(\Xi_{\mathbf{train}};X)}$$

- "Best" approximation error

$$\bar{\bar{\varepsilon}}_N^{\mathrm{POD}} \equiv \left\| u - \Pi_{X_N^{\mathbf{POD}}} u \right\|_{L^2(\Xi_{\mathbf{train}};X)}$$

Note:

- $\boldsymbol{X}_{\boldsymbol{N}}^{\mathrm{POD}}$ are hierarchical,
- Weaker norm over $\boldsymbol{\Xi}_{\mathrm{train}}$, ,
- Optimization can be solved using "method of snapshosts."

Greedy approach: W_N I

Given:

- desired error tolerance $\varepsilon_{\mathrm{tol,min}}$
- initial sample $S_1 = \mu_1^*$ (random or $\mu^{\mathrm{min,max}}$), and
- space $\boldsymbol{X}_1 = \mathrm{span}\,\{u\,(\mu_1^*)\}$

Greedy approach: W_N II

Greedy Algorithm

while $\Delta_{N-1}^{\max} \geq \varepsilon_{\text{tol,min}}$

$N = N + 1;$

$\mu_N^* = \arg \max_{\mu \in \Xi_{\text{train}}} \Delta_{N-1}(\mu);$

$\Delta_{N-1}^{\max} = \Delta_{N-1}\left(\mu_N^*\right);$

$S_N = S_{N-1} \cup \mu_N^*;$

$\boldsymbol{X}_N = \boldsymbol{X}_{N-1} + \operatorname{span}\left\{u^{\mathcal{N}}\left(\mu_N^*\right)\right\};$

end

Greedy approach: W_N III

Comments

- $\boldsymbol{X_N} = \boldsymbol{X_N}^{\text{Greedy}}$ are hierarchical.
- Sub-optimal solution to $\boldsymbol{L}^\infty(\Xi_{\text{train}})$ optimization problem.
- Define the "true" error, $1 \leq N \leq N_{\max}$,

$$\bar{\varepsilon}_N^* = \arg \max_{\mu \in \Xi_{\textbf{train}}} \|u(\mu) - u_N(\mu)\|_X \,,$$

then $\bar{\varepsilon}_N^*$ is bounded by

$$\bar{\varepsilon}_N^* \leq \Delta_N(\mu) \leq \varepsilon_{\text{tol,min}} \,, \quad \forall \mu \in \Xi_{\text{train}}$$

- Condition on $\boldsymbol{N}_{\max}$ possible (hp-Reduced Basis)[10].
- Perform Gram-Schmidt orthogonalization on $\boldsymbol{X_N}$.

Greedy approach: W_N IV

Greedy, $\boldsymbol{L}^\infty(\boldsymbol{\Xi}_{\text{train}}, \boldsymbol{X})$, space "economization"

$$\begin{array}{llll} \boldsymbol{n}_{\text{train}} & \text{contestants} \Rightarrow & \boldsymbol{N}_{\max} \ll \boldsymbol{n}_{\text{train}} & \text{winners} \\ & \in \Xi_{\text{train}} & & \boldsymbol{\mu}_1^*, \ldots, \boldsymbol{\mu}_{N_{\max}}^* \end{array}$$

in which we never form/calculate most snapshots:

$$\begin{array}{ccc} \|u(\mu) - u_N(\mu)\| X & \text{replaced} & \Delta_N(\mu) \\ n_{\text{train}} \cdot O(\mathcal{N}^\bullet) & \text{by} & n_{\text{train}} \cdot O\left(Q^2 N^2\right) \dagger \end{array}$$

note good effectivity of estimator is crucial.

[10] Eftang, Jens L.; Patera, Anthony T.; Ronquist, Einar M. "An "hp" Certified Reduced Basis Method for Parametrized Elliptic Partial Differential Equations." SIAM Journal on Scientific Computing, 32, pp. 3170-3200

Greedy Approach : W_N^{out} I

Given:

- desired error tolerance $\varepsilon_{\text{tol,min}}$
- initial sample $S_1^{\text{out}} = \mu_1^{\text{out},*}$ (random or $\mu^{\text{min,max}}$), and
- space $X_1^{\text{out}} = \text{span}\left\{u\left(\mu_1^{\text{out},*}\right)\right\}$

Greedy Algorithm

$$
\begin{aligned}
&\text{while } \Delta_{N-1}^{\text{out,max}} \geq \varepsilon_{\text{tol,min}}\\
&\qquad N = N+1;\\
&\qquad \mu_N^{\text{out,}} = \arg\max_{\mu\in\Xi_{\text{train}}} \left((\omega(\mu))^{-1}\Delta_{N-1}^{\text{en}}(\mu)\right);\\
&\qquad \Delta_{N-1}^{\text{out,max}} = \left(\left(\omega\left(\mu_N^{\text{out},*}\right)\right)^{-1}\Delta_{N-1}^{\text{en}}\left(\mu_N^{\text{out},*}\right)\right);\\
&\qquad S_N^{\text{out}} = S_{N-1}^{\text{out}} \cup \mu_N^{\text{out},*};\\
&\qquad X_N^{\text{out}} = X_{N-1}^{\text{out}} + \text{span}\left\{u^{\mathcal{N}}\left(\mu_N^{\text{out},*}\right)\right\}\\
&\text{end}
\end{aligned}
$$

Greedy Approach : W_N^{out} II

Comments

- $\boldsymbol{X}_N^{\text{out}} = \boldsymbol{X}_N^{\text{out,Greedy}}$ are hierarchical.
- Sub-optimal solution to $\boldsymbol{L}^\infty(\boldsymbol{\Xi}_{\text{train}})$ optimization problem.
- Computational cost equivalent to W_{N^*}-Greedy
- Define the relative "true" error, $1 \leq \boldsymbol{N} \leq \boldsymbol{N}_{\max}$,

$$\bar{\varepsilon}_N^{\text{out},} = \arg \max_{\mu \in \Xi_{\textbf{train}}} (\omega(\mu))^{-1} |||u(\mu) - u_N(\mu)|||_\mu,$$

then $\varepsilon_{\boldsymbol{N}}$ out,* is bounded by

$$\varepsilon_N^{\text{out},}, \leq (\omega(\mu))^{-1} \Delta_N^{\text{en}}(\mu) \leq \varepsilon_{\text{tol,min}}\ , \quad \forall \mu \in \Xi_{\text{train}}$$

Greedy Approach : W_N^{out} III

Comments

- Direct control of (relative) RB error (Galerkin optimality)

$$\omega(\mu) \equiv |||u_N(\mu)|||_\mu, \quad \forall \mu \in \mathcal{D}.$$

- Compliant case: direct control of (relative) error in the RB output prediction

$$\omega(\mu) \equiv |s_N(\mu)|, \quad \forall \mu \in \mathcal{D}.$$

- Use of sharper bound since $\boldsymbol{\eta}^{\mathrm{en}}_{\mathsf{max},\mathrm{UB}} \leq \boldsymbol{\eta}_{\mathsf{max},\mathrm{UB}}$.
- Perform Gram-Schmidt orthogonalization on $\boldsymbol{X_N}$.
- $\Rightarrow$ In this course we use $\omega(\mu) \equiv |||u_N(\mu)|||_\mu$

Greedy Approach vs $W_N^{nh,ln}$

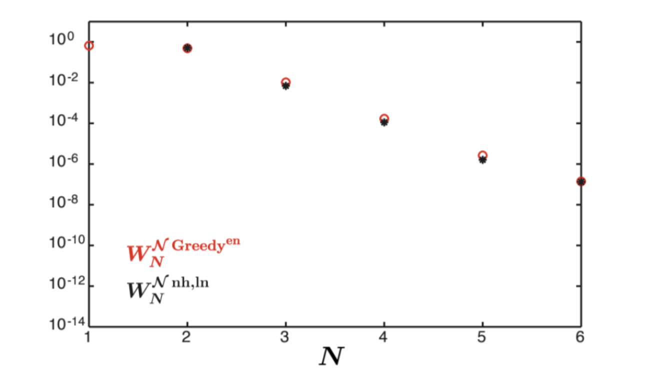

Figure 3: ThermalBlock $P = 1$: $L^\infty(\Xi)$ relative energy error as a function of N for the spaces $W_N^{\mathrm{nh,ln}}(\star)$ and $W_N^{\mathrm{Greedy,en}}$ (o).

Greedy Approach vs POD I

- Greedy
  - heuristically minimizes RB error bound in $L^{\infty}(\Xi_{\text{train}}, X)$, at
  - cost

$$\begin{array}{ll} \text{RB formation} & \boldsymbol{N}_{\max} \underline{A}^{\mathcal{N}}\text{-solve} + N_{\max}^2 Q X^{\mathcal{N}\mathcal{N}}\text{-inprod} \\ \Delta_N \text{ formation} & + N_{\max} Q \mathbb{X}^{\mathcal{N}}\text{-solve} + N_{\max}^2 Q^2 X^{\mathcal{N}}\text{-inprod} \\ \text{Greedy} & + n_{\text{train}}\, O\left(N_{\max}^4 + N_{\max}^3 Q^2\right) \end{array}$$

- POD
  - truly minimizes projection error in $L^2(\Xi_{\text{train}}, X)$, at
  - cost RB formation $n_{\text{train}}\ \underline{A}^{\mathcal{N}}$-solves $+ n_{\text{train}}^2\ X^{\mathcal{N}}$-inprod $+\underline{\boldsymbol{C}}$-eigenvalue problem

$\Rightarrow$ Cost(Greedy) $\ll$ Cost(POD) for large $\boldsymbol{n}_{\text{train}}$

Greedy Approach vs POD II

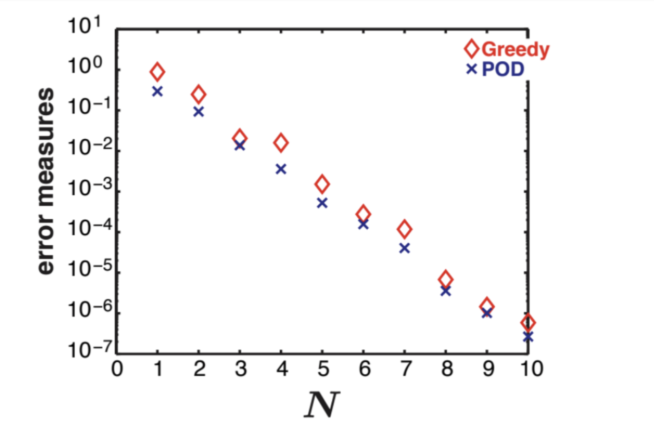

Figure 4: ThermalBlock $P = 1$: RB error $\|u(\mu) - u_N(\mu)\|_{L^2(\Xi;X)}$ as a function of N for POD and Greedy sampling.

Greedy Approach vs POD III

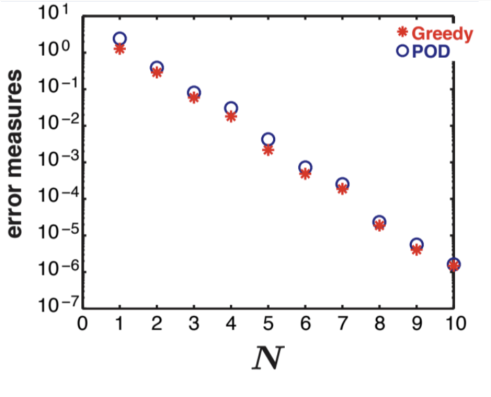

Figure 5: ThermalBlock $P = 1$: RB error $\|u(\mu) - u_N(\mu)\|_{L^\infty(\Xi;X)}$ as a function of N for POD and Greedy sampling.

Thermal Block $P = 8$ I

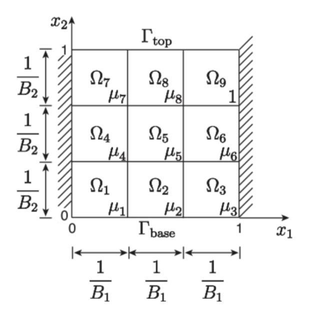

Figure 6: ThermalBlock $\boldsymbol{P} = 8$.

We consider

$$B_1 = B_2 = 3$$

and thus

$$P = 8.$$

We choose

$$\begin{aligned} \Xi_{\text{train}} &= G^{\text{ln}}_{[\text{MC};5000]} \\ \text{and } \Xi_{\text{test}} &= \Xi_{\text{train}} \end{aligned}$$

Thermal Block $P = 8$ II

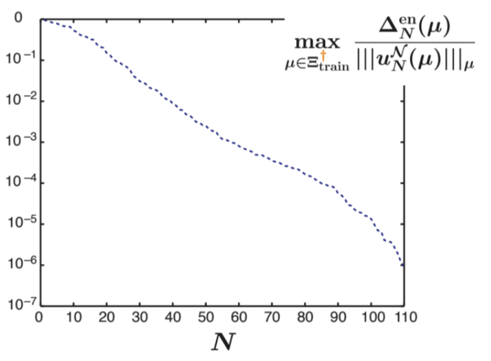

Figure 7: ThermalBlock $P = 8$: L^∞ (Ξ_{test}) relative energy error as a function of N.

Thermal Block $P = 8$ III

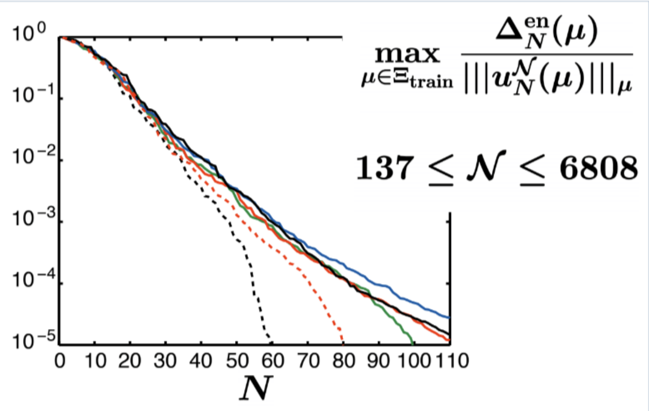

Figure 8: ThermalBlock $\boldsymbol{P} = 8$: $\boldsymbol{L}^\infty$ ($\boldsymbol{\Xi}_{\text{test}}$) relative energy error as a function of N for $\mathcal{N} = 137, \mathcal{N} = 137\mathcal{N} = 453$, and $\mathcal{N} = 661, 1737, 2545, 6808$.