# Reduced Basis methods: an introduction

Christophe Prud'homme
prudhomme@unistra.fr

September 19, 2023

CeMosis - http://www.cemosis.fr
IRMA
Université de Strasbourg

1. Reduced Basis Approximation

2. Geometric Transformations and Reduced Basis Methods

# RB Approx.

# Reduced Basis Objectives

For any given accuracy $\epsilon$, evaluate — Accuracy

$$\mu \in \mathcal{D}^{\mu} \to s_N(\mu)(\approx s^{\mathcal{N}}(\mu)) \text{ and } \Delta_N^s(\mu)$$

that provably achieves the desired accuracy — Reliability

$$|s^{\mathcal{N}}(\mu) - s_N(\mu)| \leq \Delta_N^s(\mu) \leq \epsilon$$

for a very low cost $t_{\text{comp}}$ — Efficiency

Independent of $\mathcal{N}$ as $\mathcal{N} \to \infty$

where $t_{\text{comp}}$ is the time to perform the input-output relationship

$$\mu \to (s_N(\mu), \Delta_N^s(\mu))$$

# Reduced Basis Objective : Rapid Convergence

Build a rapidly convergent approximation of

$$s_N(\mu) \in \mathbb{R} \text{ and } u_N(\mu) \in X^N \subset X^{\mathcal{N}} \subset X$$

such that for all $\mu$, we have

$$s_N(\mu) \to s^{\mathcal{N}}(\mu) \text{ and } u_N(\mu) \to u^{\mathcal{N}}(\mu)$$

rapidly as $N = \dim X_N \to \infty (= 10 - 200)$ (and independently of $\mathcal{N}$)

# Reduced Basis Objective : Reliability and Sharpness

Provide a posteriori error bound $\Delta_N(\mu)$ and $\Delta_N^s(\mu)$ :

$$1(\text{rigor}) \leq \frac{\Delta_N(\mu)}{\|u^{\mathcal{N}}(\mu) - u_N(\mu)\|_X} \leq \; E(\text{sharpness})$$

and

$$1(\text{rigor}) \leq \frac{\Delta_N^s(\mu)}{|s^{\mathcal{N}}(\mu) - s_N(\mu)|} \leq \; E(\text{sharpness})$$

for all $N = 1 \ldots N_{\max}$ and $\mu \in \mathcal{D}^\mu$.

# Reduced Basis Objective : Efficiency

Develop a two stage strategy : Offline/Online

**Offline:** very expensive pre-processing, we have typically that for a given $\mu \in \mathcal{D}^\mu$

$$t_{\text{comp}}^{\text{offline}} >> t_{\text{comp}}^{\mu \to s^{\mathcal{N}}(\mu)}$$

**Online:** very rapid convergent certified reduced basis input-output relationship

$$t_{\text{comp}}^{\text{online}} \text{ independent of } \mathcal{N}$$

**Remark**

*$\mathcal{N}$ may/should be chosen conservatively*

# Parametric Manifold $\mathcal{M}^{\mathcal{N}}$

We assume

- the form $a$ is continuous and coercive (or inf-sup stable); and
- affine $\mu - dependence; and the \theta^q(\mu), 1 \leq q \leq Q$, are smooth (i.e., $\theta^q \in C^\infty(\mathcal{D})$ ;

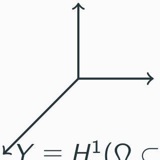
then

$$\mathcal{M}^{\mathcal{N}} = \{u^{\mathcal{N}}(\mu),\ \mu \in \mathcal{D}\} \tag{12}$$

is a smooth $P$-dimensional manifold in $X^{\mathcal{N}}$, since

$$\|D_\sigma y^{\mathcal{N}}(\mu)\| \leq C_\sigma \forall \mu \in \mathcal{D},\ \text{for any order } |\sigma| \in \mathbb{N}_{+0} \tag{13}$$

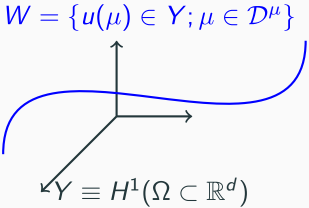
# Approximation opportunities: Low-Dimension Manifold



$Y \equiv H^1(\Omega \subset \mathbb{R}^d)$

To approximate $u(\mu)$, and thus $s(\mu)$, we need not represent all functions in $Y$

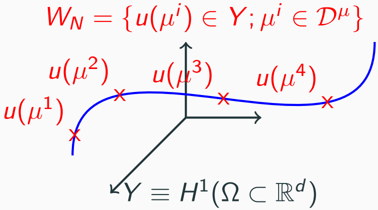
# Approximation opportunities: Low-Dimension Manifold




To approximate $u(\mu)$, and thus $s(\mu)$, we need only approximate functions in low-dimensional manifold

$$W = \{u(\mu) \in Y; \mu \in \mathcal{D}^\mu\}$$

# Approximation opportunities: Low-Dimension Manifold




To approximate $u(\mu)$, and thus $s(\mu)$, we construct the approximation space

$$W_N = \{u(\mu^i) \in Y; \\ (\mu^i)_{i=1..N} \in \mathcal{D}^\mu\}$$

# Spaces & Bases

We define the RB approximation space

$$X_N = \text{span}\{\xi^n, 1 \leq n \leq N\},\ 1 \leq N \leq N_{max} \tag{14}$$

with linearly independent basis functions

$$\xi^n \in X,\ 1 \leq n \leq N_{max} \tag{15}$$

We thus obtain

$$X_N \subset X,\ \dim(X_N) = N,\ 1 \leq N \leq N_{max} \tag{16}$$

and nested (hierarchical) spaces

$$X_1 \subset X_2 \subset \ldots X_{N_{max}} (\subset X) \tag{17}$$

We denote non-hierarchical RB spaces as $X_N^{nh}, 1 \leq N \leq Nmax$,

$$X_N^{nh} \subset X,\ \dim(X_N^{nh}) = N,\ 1 \leq N \leq N_{max} \tag{18}$$

# Spaces & Bases - Lagrangian

Parameter Samples:

$$\text{Sample}: \quad S_N = \{\mu_1 \in \mathcal{D}^\mu, \dots, \mu_N \in \mathcal{D}^\mu\} \quad 1 \le N \le N_{\max},$$

with

$$S_1 \subset S_2 \dots S_{N_{\max}-1} \subset S_{N_{\max}} \subset \mathcal{D}^\mu$$

Lagrangian Hierarchical Space

$$W_N = \operatorname{span}\{\xi^n \equiv \underbrace{u(\mu^n)}_{u^{\mathcal{N}}(\mu^n)}, n = 1, \dots, N\}.$$

with

$$W_1 \subset W_2 \dots \subset W_{N_{\max}} \subset X^{\mathcal{N}} \subset X$$

## Sampling strategies?

- Equidistributed points in $\mathcal{D}^\mu$ (curse of dimensionality)
- Log-random distributed points in $\mathcal{D}^\mu$
- See later for more efficient, adaptive strategies

# Space & Bases - Taylor & Hermite

- Taylor reduced basis spaces: hierarchical

$$W_N^{Taylor} = \text{span}\{D_\sigma u(\mu), \forall \sigma \in I^{P,N-1}\}, 1 \leq N \leq N_{max}, \qquad (19)$$

field variable and sensitivity derivatives at one point in D.

- Hermite reduced basis spaces: hierarchical

$$W_N^{Hermite} \text{``}=\text{''}\ W_N^{Lagrangian} \cup W_N^{Taylor} \qquad (20)$$

field variable and sensitivity derivatives at several points in $\mathcal{D}$

Note: We will exclusively use Lagrangian RB spaces in this course.

# Space & Bases - Orthogonal Basis

Given $\xi^n = u(\mu^n), 1 \leq n \leq N_{max}$ (Lagrange case) we construct the basis set $\{\zeta^n, 1 \leq n \leq Nmax\}$, from

**Definition (Gram-Schmidt Orthogonalisation)**

$\zeta^1 = \xi^1 / \|\xi^1\|_X;$
for n = 2 : Nmax
  $z^n = \xi^n - \sum_{m=1}^{n-1} (\xi^n, \zeta^m)_X \zeta^m;$
  $\zeta^n = z^n / \|z^n\|_X;$
end.

Note: $(\zeta^n, \zeta^m)_X = \delta_{nm}, 1 \leq n, m \leq Nmax$

# Space & Bases - Orthogonal Basis

Given reduced basis space

$$X_N = \mathrm{span}\,\{\zeta^n, n = 1, \dots, N\}, 1 \leq N \leq N_{max} \tag{21}$$

we can express any $w_N \in X_N$ as

$$w_N = \sum_{k=1}^{N} w_{N\,n} \zeta^n \tag{22}$$

for unique $w_{N\,n} \in \mathbb{R}, 1 \leq n \leq N$

Reduced basis "matrices" $Z_N \in \mathbb{R}^{\mathcal{N} \times N}, 1 \leq N \leq N_{max}$ :

$$Z_N = [\zeta^1, \zeta^2, ..., \zeta^N], 1 \leq N \leq N_{max} \tag{23}$$

where, from orthogonality, $Z^T_{N_{max}} X Z^T_{N_{max}} = I_{N_{max}}$, and $I_M$ is the Identity matrix in $\mathbb{R}^{M \times M}$.

# Formulation (Linear Compliant Case): a Galerkin method

## Galerkin Projection

Given $\mu \in \mathcal{D}^\mu$ evaluate

$$s_N(\mu) = f(u_N(\mu); \mu) \tag{24}$$

where $u_N(\mu) \in X_N$ satisfies

$$a(u_N(\mu), v; \mu) = f(v; \mu), \ \forall\, v \in X_N \ .$$

# Formulation (Linear Compliant Case): Optimality

For any $\mu \in \mathcal{D}^\mu$, we have the following optimality results (thanks to Galerkin)

$$|||u(\mu) - u_N(\mu)|||_\mu = \inf_{v_N \in X_N} |||u(\mu) - v_N(\mu)|||_\mu,$$

$$||u(\mu) - u_N(\mu)||_X \leq \sqrt{\frac{\gamma(\mu)}{\alpha(\mu)}} \inf_{v_N \in X_N} ||u(\mu) - v_N(\mu)||_X,$$

and

$$\begin{aligned} s(\mu) - s_N(\mu) &= |||u(\mu) - u_N(\mu)|||^2_\mu, \\ &= \inf_{v_N \in X_N} |||u(\mu) - v_N(\mu)|||^2_\mu, \end{aligned}$$

and finally

$$0 \leq s(\mu) - s_N(\mu) \leq \gamma(\mu) \inf_{v_N \in X_N} ||u(\mu) - v_N(\mu)||^2_X$$

# Formulation (Linear Compliant Case): offline-online decomposition

Expand our RB approximations:

$$u_N(\mu) \ = \sum_{j=1}^{N} u_{N\,j}(\mu)\ \zeta_j \tag{25}$$

Express $s_N(\mu)$

$$s_N(\mu) = \sum_{j=1}^{N} u_{N\,j}(\mu)\ \left\{ \sum_{q=1}^{Q_f} \Theta_f^q(\mu)\ f^q(\zeta_j) \right\} \tag{26}$$

where $u_{N\,i}(\mu), 1 \leq i \leq N$ satisfies

$$\sum_{j=1}^{N} \left\{ \sum_{q=1}^{Q_a} \Theta_a^q(\mu)\ a^q(\zeta_i, \zeta_j) \right\} u_{N\,j}(\mu) = \ \ \textstyle\sum_{q=1}^{Q_f} \ \Theta_f^q(\mu)\ f^q(\zeta_i),$$

$$1 \leq i \leq N \tag{27}$$

$$\tag{28}$$

# Formulation (Linear Compliant Case): matrix form

Solve

$$\underline{A}_N(\mu)\,\underline{u}_N(\mu) = \underline{F}_N$$

where

$$(A_N)_{i\,j}(\mu) = \sum_{q=1}^{Q_a} \Theta_a^q(\mu)\; a^q(\zeta_i, \zeta_j),$$

$$F_{N\,i} = \sum_{q=1}^{Q_f} \Theta_f^q(\mu) f^q(\zeta_i)\ .$$

$$1 \le i, j \le N, \quad 1 \le i \le N$$

# Formulation (Linear Compliant Case): complexity analysis

**Offline: independent of $\mu$**

- Solve: $N$ FEM system depending on $\mathcal{N}$
- Form and store: $f^q(\zeta_i)$
- Form and store: $a^q(\zeta_i, \zeta_j)$

**Online: independent of $\mathcal{N}$**

- Given a new $\mu \in \mathcal{D}^\mu$
- Form and solve $A_N(\mu)$ : $O(QN^2)$ and $O(N^3)$
- Compute $s_N(\mu)$

**Online: $N << \mathcal{N}$**

Online we realize often orders of magnitude computational economies relative to FEM in the context of many $\mu$-queries

# Formulation (Linear Compliant Case): Condition number

**Proposition**

*Thanks to the orthonormalization of the basis function, we have that the condition number of $A_N(\mu)$ is bounded by the ratio $\gamma(\mu)/\alpha(\mu)$.*

**Proof.**

- Write the Rayleigh Quotient

$$\frac{v_N^T A_N(\mu) v_N}{v_N^T v_N}, \quad \forall v_N \in \mathbb{R}^N$$

- Express

$$v_N = \sum_{n=1}^{N} v_{N_n} \zeta^n$$

- Use coercivity, continuity and orthonormality.

□

# Geometric Transformations and Reduced Basis Methods

# Hypothesis: Reference Geometry

Note $\boldsymbol{\Omega}$ is parameter-independent: the reduced basis requires a common spatial configuration, i.e., a reference domain $\Omega_{\text{ref}}$

Introduce a piecewise affine mapping $\mathcal{T}(\cdot;\mu) : \Omega \to \widetilde{\Omega}(\mu)$

$$\widetilde{a}\,(\widetilde{w}, \widetilde{v}; \mu) \text{ over } \widetilde{\Omega}(\mu)$$
$$\Downarrow$$
$$\mathcal{T}(\cdot;\mu)^{-1} : \widetilde{\Omega}(\boldsymbol{\mu}) \to \Omega_{\text{ref}} \equiv \Omega \quad \left(\Omega_{\text{ref}} = \widetilde{\Omega}\,(\mu_{\text{ref}})\right)$$
$$\Downarrow$$
$$a(w, v; \mu) \text{ over } \Omega$$

where

$$a(w, v; \mu) = \widetilde{a}\,(\widetilde{w} \circ \mathcal{T}_\mu, \widetilde{v} \circ \mathcal{T}_\mu; \mu)$$

We will discuss this issue in detail later on.

## Problem statement

Find $u(\mu) \in X$ which satisfies

$$a(u(\mu), v; \mu) = f(v; \mu) \quad \forall v \in X$$

where; for $\mu \in D \subset \mathbb{R}^P$

- a $(; ; \mu)$ is continuous and coercive
- $f(\cdot, \mu$ is bounded
- $a$ and $f$ depend affinely on parameter

$$a(u, v; \mu) = \sum_{q=1}^{Q_a} \theta_a^q(\mu) a^q(u, v)$$

$$f(v; \mu) = \sum_{q=1}^{Q_f} \theta_f^q(\mu) f^q(v)$$

# Example 1

PDE

$$-\nabla^2 u + \rho u = 1$$
$$u = 0 \text{ on } \Gamma$$

The weak form is then (1) where

$$a(u, v; \mu) = \underbrace{1}_{\theta_a^1(\mu)} \underbrace{\int_\Omega \frac{\partial v}{\partial x_i} \frac{\partial u}{\partial x_i}}_{a^1(u,v)} + \underbrace{\rho}_{\theta_a^2(\mu)} \underbrace{\int_\Omega vu}_{a^2(u,v)}$$

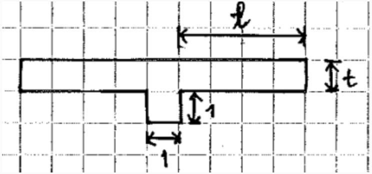
$$f(v; \mu) = 1 \int_\Omega v$$

Here, $\mu = (p)$

$$u = u(\mu) = u(x, y; \mu), \quad (x, y) \in \Omega$$

But what if the domain depends on the parameter?

# Example 2




$$\mu = (\ell, t) \in D \subset \mathbb{R}^2$$
$$\widetilde{\Omega}(\mu) \subset \mathbb{R}^2$$

## Back to Reduced Basis

Recall that the RB method computes

$$u_N(\mu) = \sum_{n=1}^{N} c_n(\mu)\xi_n$$

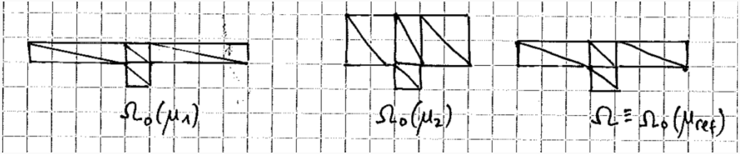
where the $\xi_n = u(\mu_n)$, ie $\operatorname{span}\{\xi_n, n = 1 \dots N\} = \operatorname{span}(u(\mu_n))\}$ and the $\xi_n$ are orthonormalized basis functions

$\Rightarrow$ We need to be able to add the basis functions (or, more precisely, the FE vectors)

$\Rightarrow$ the RB recipe requires that $\Omega$ be parameter-independent

# Solution: Apply Divide and Conquer, domain decomposition

We need to distinguish between the PHYSICAL/ACTUAL/ORIGINAL Domain $\widetilde{\Omega}(\mu)$ (and associated Fe Mesh) and a REFERENCE DOMAIN $\Omega \equiv \widetilde{\Omega}\,(\mu_{\text{ref}}\,)$ (and REFERENCE mesh)




$\Omega$ will be the pre-image of $\widetilde{\Omega}(\mu)$ that is $\widetilde{\Omega}(\mu)$ will be obtained using a geometric transformation or mapping of $\Omega$.

## Solution: domain decomposition

Introduce for any $\mu \in \mathcal{D}$ a domain decomposition of $\widetilde{\Omega}(\mu)$ expressed as

$$\bar{\widetilde{\Omega}}(\mu) = \bigcup_{k=1}^{K} \bar{\widetilde{\Omega}}^k(\mu)$$

where the $\Omega^k(\mu), 1 \leq k \leq K$ are mutually non-overlapping regions

$$\widetilde{\Omega}^k(\mu) \cap \widetilde{\Omega}^l(\mu) = \emptyset \quad 1 \leq k < l \leq k$$

Note that

- regions may correspond to different materials or PDE coefficients
- regions may also be introduced for algorithmic purposes

# Solution: domain decomposition

We now choose a reference parameter value $\mu_{\text{ref}} \in \mathcal{D}$ and define $\Omega = \widetilde{\Omega}(\mu_{\text{ref}})$

It follows that

$$\bar{\Omega} = \bigcup_{k=1}^{K} \bar{\Omega}^k \quad \text{where } \bar{\Omega}^k = \bar{\widetilde{\Omega}}^k(\mu_{\text{ref}})$$

$$\Omega^k \cap \Omega^l = \emptyset \quad 1 \leq k < l \leq K$$

Note: $\mu_{\text{ref}}$ only affects the accuracy of the underlying FE approx. Typically $\mu_{\text{ref}}$ is chosen at the "center" of $D$ to mimize distortion and reduce $\mathcal{N}$ (to satisfy acceptable FE error in $\mathcal{D}$.

# Geometric transformation

we assume $\tilde{x} = g^k\left(x^k;\mu\right)$ or $\bar{\tilde{\Omega}}^k(\mu) = g^k\left(\Omega^k;\mu\right) \quad 1 \leq k \leq K$ where the $g_k(\cdot;\mu) : \Omega^k \rightarrow \widetilde{\Omega}^k(\mu)$ are

1. bijective: $\left(g^k\right)^{-1}$ exists: $x = \left(g^k\right)^{-1}(\tilde{x};\mu)$
2. Continuous across internal foundaries

$$g^k(x,\mu) = g^l(x;\mu) \quad \forall x \in \bar{\Omega}^k \cap \Omega^l$$

3. affine (linear in x) for $x \in \Omega^k$,

$$\tilde{x} = g^k(x;\mu) = \underline{c}^k(\mu) + \underbrace{\underline{G}^k(\mu)}_{\in\mathbb{R}^{d\times d}} x$$

or $\tilde{x}_i = c_i^k(\mu) + \sum_{j=1}^d G_{ij}^k(\mu) x_j$

# Geometric transformation

We now define $g(x;\mu)$ to be piecewise affine

$$g(\mathrm{x};\mu) = g^k(\mathrm{x};\mu) \quad \mathrm{x} \in \Omega^k$$

The inverse mapping is then

$$\begin{aligned} \mathrm{x} &= \left(\underline{G}^k(\mu)\right)^{-1}\left[\tilde{\mathrm{x}} - \underline{c}^k(\mu)\right] \\ \text{or } \mathrm{x} &= \underline{H}^k(\mu)\mathrm{x}_0 + \underline{d}^k(\mu) \\ \text{where } \underline{H}^k(\mu) &= \left(\underline{G}^k(\mu)\right)^{-1} \\ \underline{d}^k(\mu) &= -\left(\underline{G}^k(\mu)\right)^{-1}\underline{c}^k(\mu) \end{aligned}$$

Note: In 2 dimensions, straight lines map to straight limes, parallel lines to parallel lines.

# Computing the derivatives and the integrals I

Consider the Jacobian matrix $\mathcal{J}_{\boldsymbol{g}}(\mathrm{x};\boldsymbol{\mu}) \in \mathbb{R}^{d\times d}$ of the map $\boldsymbol{g}(\cdot;\boldsymbol{\mu})$ introduced in (8.1),

$$(\mathbb{J}_{\boldsymbol{g}}(\mathrm{x};\boldsymbol{\mu}))_{kl} = \frac{\partial \tilde{\mathrm{x}}_k}{\partial \mathrm{x}_l}(\mathrm{x}) = \frac{\partial g_k(\mathrm{x};\boldsymbol{\mu})}{\partial \mathrm{x}_l}(\mathrm{x}), \quad k,l = 1,\dots,d$$

We have that its determinant $|\mathbb{J}_{\boldsymbol{g}}(\mathrm{x};\boldsymbol{\mu})| \neq 0$

For any integrable function $\phi : \Omega \to \mathbb{R}$ the following formula

$$\int_{\widetilde{\Omega}(\boldsymbol{\mu})} \widetilde{\phi}(\tilde{\mathrm{x}}) d\widetilde{\Omega} = \int_{\Omega} \phi(\mathrm{x}) \left|\mathbb{J}_{\boldsymbol{g}}(\mathrm{x};\boldsymbol{\mu})\right| d\Omega$$

provides the change of variable, where $\phi = \widetilde{\phi} \circ \boldsymbol{g}$.

The Jacobian matrix and its determinant depend a priori on both the spatial coordinates x and the parameter vector $\boldsymbol{\mu}$.

# Computing the derivatives and the integrals II

In case of integrals involving derivatives, we need to introduce some extra transformations. Let us denote by $\boldsymbol{g}^{-1}(\cdot;\boldsymbol{\mu})$ the inverse of $\boldsymbol{g}(\cdot;\boldsymbol{\mu})$, such that $\Omega = \boldsymbol{g}^{-1}(\widetilde{\Omega}(\boldsymbol{\mu});\boldsymbol{\mu})$, and by

$$\left(\mathbb{J}_{\boldsymbol{g}^{-1}}(\tilde{\mathrm{x}};\boldsymbol{\mu})\right)_{kl} = \frac{\partial x_k}{\partial \tilde{x}_l}(\tilde{\mathrm{x}}) = \frac{\partial g_k^{-1}(\tilde{\mathrm{x}};\boldsymbol{\mu})}{\partial \tilde{x}_l}(\tilde{\mathrm{x}}), \quad k,l = 1,\dots,d$$

its Jacobian matrix. Then

$$\mathbb{J}_{\boldsymbol{g}}(\tilde{\mathrm{x}};\boldsymbol{\mu}) = \left(\mathbb{J}_{\boldsymbol{g}}(\mathrm{x};\boldsymbol{\mu})\right)^{-1}, \quad \tilde{\mathrm{x}} = \boldsymbol{g}(\mathrm{x};\boldsymbol{\mu})$$

so that

$$\left|\mathbb{J}_{\boldsymbol{g}^{-1}}(\tilde{\mathrm{x}};\boldsymbol{\mu})\right| = \frac{1}{\left|\, \mathbb{J}_{\boldsymbol{g}}(\mathrm{x};\boldsymbol{\mu}))\,\right|}$$

# Computing the derivatives and the integrals III

Thanks to the chain rule,

$$\frac{\partial \widetilde{\phi}(\tilde{x})}{\partial \tilde{x}_i} = \sum_{j=1}^{d} \frac{\partial \phi(x)}{\partial x_j} \frac{\partial x_j}{\partial \tilde{x}_i}, \quad i = 1, \dots, d$$

we obtain the compact expression

$$\nabla_{\tilde{x}} \widetilde{\phi}(\tilde{x}) = \left[ \mathbb{J}_{g^{-1}}(\tilde{x}) \right]^T \nabla_x \phi(x) = \left( \mathbb{J}_{\boldsymbol{g}}(x; \boldsymbol{\mu}) \right)^{-T} \nabla_x \phi(x),$$

where $\nabla_{\tilde{x}}$ ( resp. $\nabla_x$) is the gradient with respect to the coordinates of the original (resp. reference) domain.

# Computing the derivatives and the integrals IV

We obtain the following relations for any $\widetilde{\phi}, \widetilde{\chi} \in H^1(\widetilde{\Omega})$ :

$$\int_{\tilde{\Omega}(\boldsymbol{\mu})} \nabla_{\tilde{x}} \widetilde{\boldsymbol{\phi}} \cdot \nabla_{\tilde{x}} \widetilde{\chi} d\widetilde{\Omega} = \int_{\Omega} \left(\mathbb{J}_{\boldsymbol{g}}^{-T}(x; \boldsymbol{\mu}) \nabla_{x} \phi\right) \cdot \left(\mathbb{J}_{\boldsymbol{g}}^{-T}(\tilde{x}; \boldsymbol{\mu}) \nabla_{x} \chi\right) \left|\mathbb{J}_{\boldsymbol{g}}(x; \boldsymbol{\mu})\right| d\Omega$$

$$\int_{\tilde{\Omega}(\boldsymbol{\mu})} \widetilde{\chi} \widetilde{\boldsymbol{b}} \cdot \nabla_{\tilde{x}} \widetilde{\phi} d\widetilde{\Omega} = \int_{\Omega} \chi \boldsymbol{b} \cdot \left(\mathbb{J}_{\boldsymbol{g}}^{-T}(x; \boldsymbol{\mu}) \nabla_{x} \phi\right) \left|\mathbb{J}_{\boldsymbol{g}}(x; \boldsymbol{\mu})\right| d\Omega,$$

where $\phi = \widetilde{\phi} \circ \boldsymbol{g}, \chi = \widetilde{\chi} \circ \boldsymbol{g}$, and $\boldsymbol{b} = \widetilde{\boldsymbol{b}} \circ \boldsymbol{g}$.

# Computing the derivatives and the integrals V

In a more compact form, we can write

$$\int_{\tilde{\Omega}(\boldsymbol{\mu})} \nabla_{\tilde{x}} \widetilde{\phi} \cdot \nabla_{\tilde{x}} \widetilde{\chi} d\widetilde{\Omega} = \sum_{k,l=1}^{d} \int_{\Omega} \frac{\partial \phi}{\partial x_k} v_{kl} \frac{\partial \chi}{\partial x_l} d\Omega$$

where for any $\boldsymbol{\mu} \in \mathcal{D}$, $\boldsymbol{v} : \mathbb{R}^d \times \mathcal{D} \to \mathbb{R}^{d \times d}$ is given by

$$\boldsymbol{v}(\mathrm{x}; \boldsymbol{\mu}) = \mathbb{J}_{\boldsymbol{g}}^{-1}(\mathrm{x}; \boldsymbol{\mu}) \mathbb{J}_{\boldsymbol{g}}^{-T}(\mathrm{x}; \boldsymbol{\mu}) \left| \mathbb{J}_{\boldsymbol{g}}(\mathrm{x}; \boldsymbol{\mu}) \right| . \tag{29}$$

# Computing the derivatives and the integrals VI

In the same way,

$$\int_{\tilde{\Omega}(\boldsymbol{\mu})} \widetilde{\boldsymbol{b}} \cdot \nabla_{\tilde{x}} \widetilde{\phi} \widetilde{\chi} d\widetilde{\Omega} = \sum_{k,l=1}^{d} \int_{\Omega} b_k \eta_{kl} \frac{\partial \phi}{\partial x_l} \chi d\Omega$$

where $\boldsymbol{\eta} : \mathbb{R}^d \times \mathcal{D} \to \mathbb{R}^{d \times d}$ is given by

$$\boldsymbol{\eta}(\mathrm{x}; \boldsymbol{\mu}) = \mathbb{J}_{\boldsymbol{g}}^{-T}(\mathrm{x}; \boldsymbol{\mu}) \left| \mathbb{J}_{\boldsymbol{g}}(\mathrm{x}; \boldsymbol{\mu}) \right| . \tag{30}$$

The parametrized tensors $\boldsymbol{v}(\mathrm{x}; \boldsymbol{\mu}), \boldsymbol{\eta}(\mathrm{x}; \boldsymbol{\mu})$ encode all the information concerning the parameters, and allow to derive the weak formulation of a parametrized PDE, which stands at the basis of the implementation of a RB method.

# The Vector case I

The vector counterpart of formula for the change of variables under the sign of integral is given by

$$\int_{\widetilde{\Omega}(\boldsymbol{\mu})} \widetilde{\phi} d\widetilde{\Omega} = \int_{\Omega} \phi \left|\mathbb{J}_{\boldsymbol{g}}\right| d\Omega$$

for any integrable function $\widetilde{\boldsymbol{\phi}} : \Omega \to \mathbb{R}^d$, where $\boldsymbol{\phi} = \widetilde{\boldsymbol{\phi}} \circ \boldsymbol{g}$ and $|\mathbb{J}_{\boldsymbol{g}}(\mathrm{x}; \boldsymbol{\mu})|$ denotes the determinant of the Jacobian matrix, defined as in the scalar case.

## The Vector case II

In case of integrals involving derivatives, we have

$$\int_{\widetilde{\Omega}(\boldsymbol{\mu})} \nabla_{\tilde{x}} \widetilde{\boldsymbol{\phi}} : \nabla_{\tilde{x}} \widetilde{\boldsymbol{\chi}} d\widetilde{\Omega} = \sum_{i,k,l=1}^{d} \int_{\Omega} \frac{\partial \phi_i}{\partial x_k} v_{kl} \frac{\partial \chi_i}{\partial x_l} d\Omega$$

where $\boldsymbol{v} : \mathbb{R}^d \times \mathcal{D} \to \mathbb{R}^{d\times d}$ is the parametrized tensor defined in 29.

Similarly, we obtain

$$\int_{\widetilde{\Omega}(\boldsymbol{\mu})} \left(\widetilde{\boldsymbol{b}} \cdot \nabla_{\tilde{x}}\right) \widetilde{\boldsymbol{\phi}} \cdot \widetilde{\boldsymbol{\chi}} d\widetilde{\Omega} = \sum_{i,k,l=1}^{d} \int_{\Omega} b_k \eta_{kl} \frac{\partial \phi_i}{\partial x_l} \chi_i d\Omega$$

$$\int_{\widetilde{\Omega}(\boldsymbol{\mu})} \widetilde{\phi} \operatorname{div} \widetilde{\boldsymbol{\chi}} d\widetilde{\Omega} = \sum_{k,l=1}^{d} \int_{\Omega} \phi \eta_{kl} \frac{\partial \chi_k}{\partial x_l} d\Omega$$

where $\boldsymbol{\eta} : \mathbb{R}^d \times \mathcal{D} \to \mathbb{R}^{d\times d}$ is the parametrized tensor defined in 30.

# Example: heat transfer I

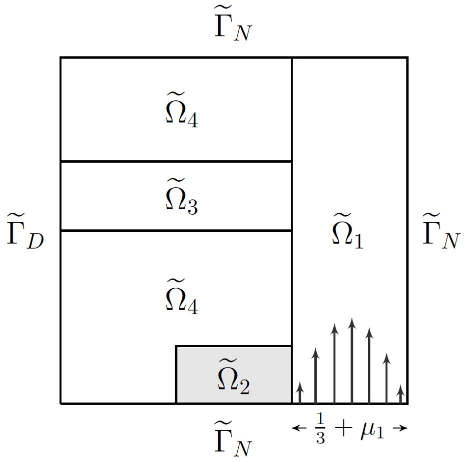
We consider a heat conduction/convection problem occurring, e.g., in electronic devices made by different materials characterized by different thermal conductivities. We suppose that the advection field has a varying intensity, and acts on a portion of the domain whose width is also changing.

We denote by $\widetilde{\Omega}\,(\mu_1) = (0, 1+\mu_1) \times (0,1)$ a domain consisting of four subdomains, see Figure,

$$\widetilde{\Omega}_1\,(\mu_1) = \left(\frac{2}{3}, 1+\mu_1\right) \times (0,1), \quad \widetilde{\Omega}_2 = \left(\frac{1}{3}, \frac{2}{3}\right) \times \left(0, \frac{1}{6}\right),$$

$$\widetilde{\Omega}_3 = \left(0, \frac{2}{3}\right) \times (0.5, 0.7), \quad \widetilde{\Omega}_4 = \left(0, \frac{2}{3}\right) \times (0,1) \backslash \left(\bar{\widetilde{\Omega}}_2 \cup \bar{\widetilde{\Omega}}_3\right).$$

# Example: heat transfer II



# Example: heat transfer III

The governing parametrized PDE problem for the temperature $\tilde{u} = \tilde{u}(\boldsymbol{\mu})$ reads

$$\begin{cases} -\operatorname{div}(\tilde{k}(\tilde{x};\boldsymbol{\mu})\nabla\tilde{u}) + \tilde{\boldsymbol{b}}(\tilde{x};\boldsymbol{\mu})\cdot\nabla\tilde{u} = \tilde{s}(\tilde{x}) & \text{in } \widetilde{\Omega}(\mu_1) \\ \tilde{u} = 0 & \text{on } \widetilde{\Gamma}_D \\ \tilde{k}(\tilde{x};\boldsymbol{\mu})\nabla\tilde{u}\cdot\tilde{\boldsymbol{n}} = 0 & \text{on } \widetilde{\Gamma}_N \end{cases}$$

where $\widetilde{\Gamma}_D = \{0\}\times[0,1]$ and $\widetilde{\Gamma}_N = \partial\widetilde{\Omega}\backslash\widetilde{\Gamma}_D$, while $\tilde{\boldsymbol{b}}$ is a prescribed advection field.

Across all internal interfaces the solution $\tilde{u}$ (the temperature) as well as its flux $\tilde{k}(\tilde{x};\boldsymbol{\mu})\nabla\tilde{u}\cdot\tilde{\boldsymbol{n}}$ are continuous.

The Problem can, e.g., model a cooling device for (an array of) electronic components; here the component - which is a poor conductor - is assumed to occupy the subdomain $\widetilde{\Omega}_4$, a conducting material occupies

# Example: heat transfer IV

the domain $\widetilde{\Omega}_3$, whereas $\widetilde{\Omega}_1$ is occupied by a fluid whose velocity is described by a fully-developed parabolic field;

# Example: heat transfer V

Since these subregions are characterized by different thermal conductivities, $\tilde{k}(\mathrm{x}; \boldsymbol{\mu})$ can be expressed by

$$\tilde{k}\left(\mathrm{x}; \mu_3\right) = \chi_{\widetilde{\Omega}_1 \cup \widetilde{\Omega}_4}(\tilde{\mathrm{x}}) + 100\chi_{\widetilde{\Omega}_2}(\tilde{\mathrm{x}}) + \mu_3 \chi_{\widetilde{\Omega}_3}(\tilde{\mathrm{x}}).$$

Here $\chi_A$ denotes the characteristic function of the subdomain $A \subset \Omega$, whereas $\mu_3$ is the conductivity that varies over $\widetilde{\Omega}_3$. The advection field in $\widetilde{\Omega}_1$ is given by

$$\tilde{\boldsymbol{b}}\left(\mu_2; \tilde{\mathrm{x}}\right) = \left[0, \chi_{\widetilde{\Omega}_1} 162\mu_2 / \left(1 + 3\mu_1\right)^3 \left(\tilde{x}_1 - 2/3\right)\left(1 + \mu_1 - \tilde{x}_1\right)\right]^T.$$

$\Omega_2$ is the heated element, and we represent the source term as

$$\tilde{s}(\tilde{\mathrm{x}}) = 10\chi_{\widetilde{\Omega}_2}(\tilde{\mathrm{x}}).$$

# Example: heat transfer VI

The weak formulation of the problem over the original domain $\widetilde{\Omega}(\mu_1)$ reads:

find $\tilde{u}(\boldsymbol{\mu}) \in \tilde{V}(\boldsymbol{\mu}) = H^1_{\tilde{\Gamma}_D}\left(\widetilde{\Omega}(\mu_1)\right)$ such that

$$\tilde{a}(\tilde{u}(\boldsymbol{\mu}), v; \boldsymbol{\mu}) = \tilde{f}(v; \boldsymbol{\mu}) \quad \forall v \in \tilde{V}(\boldsymbol{\mu}),$$

where

$$\begin{aligned}
\tilde{a}(u, v; \boldsymbol{\mu}) = {} & \int_{\tilde{\Omega}_1(\mu_1)} \nabla u \cdot \nabla v d\widetilde{\Omega} + \\
& \frac{162\mu_2}{(1+3\mu_1)^3} \int_{\tilde{\Omega}_1(\mu_1)} \left(x_1 - \frac{2}{3}\right)(1 + \mu_1 - x_1) \frac{\partial u}{\partial x_2} v d\widetilde{\Omega} \\
& + \int_{\tilde{\Omega}_2} 100 \nabla u \cdot \nabla v d\widetilde{\Omega} + \mu_3 \int_{\tilde{\Omega}_3} \nabla u \cdot \nabla v d\widetilde{\Omega} + \int_{\tilde{\Omega}_4} \nabla u \cdot \nabla v d\widetilde{\Omega}, \\
\tilde{f}(v; \boldsymbol{\mu}) = {} & \int_{\tilde{\Omega}_2} 10 v d\widetilde{\Omega}.
\end{aligned}$$

# Reference Domain I

We introduce the following reference domain

$$\Omega = \widetilde{\Omega}\left(\mu_1^{\mathrm{ref}}\right) = (0,1) \times (0,1),$$

corresponding to the choice $\mu_1^{\mathrm{ref}} = 0$

We have

$$\Omega_1 = \widetilde{\Omega}_1\left(\mu_1^{ref}\right) = \left(\frac{2}{3}, 1\right) \times (0,1)$$

# Reference Domain II

We have in $\Omega_1$

$$\boldsymbol{g}_{\Omega_1}^{-1}(\mathrm{x}, \mu_1) = \begin{bmatrix} \frac{1}{1+3\mu_1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{-2\mu_1}{1+3\mu_1} \\ 0 \end{bmatrix}$$

and $\boldsymbol{g}_{\Omega_i}^{-1}(\mathrm{x}) = \mathbb{I}$ for $i = 2, 3, 4$

$$\left(\mathbb{J}_{\boldsymbol{g}_{\Omega_1}}(\mathrm{x}; \boldsymbol{\mu})\right)^{-1} = \begin{bmatrix} \frac{1}{1+3\mu_1} & 0 \\ 0 & 1 \end{bmatrix}$$

## Reference Domain III

We evaluate first the expression of the conductivity coefficient

$$k\,(\mathrm{x};\mu_3) = \chi_{\Omega_1\cup\Omega_4}(\mathrm{x}) + 100\chi_{\Omega_2}(\mathrm{x}) + \mu_3\chi_{\Omega_3}(\mathrm{x}),$$

the advection field $\boldsymbol{b} = \tilde{\boldsymbol{b}} \circ \boldsymbol{g}(\mathrm{x};\boldsymbol{\mu})$ over the reference domain and source term

$$\boldsymbol{b}\,(\mu_2;\mathrm{x}) = \left[0, \chi_{\Omega_1}\frac{162\mu_2}{1+3\mu_1}\,(1-x_1)\left(x_1-\frac{2}{3}\right)\right]^T, \quad s(\mathrm{x}) = 10\chi_{\Omega_2}(\mathrm{x})$$

# Reference Domain IV

The weak formulation of the problem on the reference domain reads:

find $u(\boldsymbol{\mu}) \in V = H^1_{\Gamma_D}(\Omega)$ such that

$$a(u(\boldsymbol{\mu}), v; \boldsymbol{\mu}) = f(v; \boldsymbol{\mu}) \quad \forall v \in V$$

where

$$a(u, v; \boldsymbol{\mu}) = \frac{1}{1+3\mu_1} \int_{\Omega_1} \frac{\partial u}{\partial x_1} \frac{\partial v}{\partial x_1} d\Omega + (1+3\mu_1) \int_{\Omega_1} \frac{\partial u}{\partial x_2} \frac{\partial v}{\partial x_2} d\Omega$$
$$+ 162\mu_2 \int_{\Omega_1} (1-x_1)\left(x_1 - \frac{2}{3}\right) \frac{\partial u}{\partial x_2} v d\Omega$$
$$+ 100 \int_{\Omega_2} \nabla u \cdot \nabla v d\Omega + \mu_3 \int_{\Omega_3} \nabla u \cdot \nabla v d\Omega + \int_{\Omega_4} \nabla u \cdot \nabla v d\Omega$$
$$f(v; \boldsymbol{\mu}) = 10 \int_{\Omega_2} v d\Omega$$

The affine decomposition is now readily obtained for $a$ and $f$.