Reduced Basis methods: an introduction

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Université de Strasbourg

1. Course Syllabus

2. Model Order Reduction

3. Some Examples

4. Reduced Basis Method

Course Syllabus

Date & Time

- Lecture & Seminar
  - Monday 13:30-15:30 Thurdays, 13.30-15.30,
  - Start: 16/01/2023 (total 15 sessions)
  - Homework requires programming in Python
- Website : Moodle
- Assessment
- Final exam
  - Date to be determined

Instructor

- Christophe Prud'homme
  - Room P-210, UFR Math Info (currently located at URFIST, thrid floor)
  - Email: `prudhomme@unistra.fr`
  - Office hours: by appointment

Course Outline

Topics to be discussed: Reduced Basis Methods

- Origin: structural mechanics; Parametrized PDEs
- Stationary and time-dependent problems: a priori theory, certified error bounds, coercivity/inf-sup lower bounds, non intrusive reduced basis
- Non-Affine, Non-Linear problems :
- Data Assimilation for Reduced basis methods

Keywords/Methods: Galerkin, POD, Greedy, RB, Intrusive RB, Non Intrusive RB, CRB, EIM, DEIM, GEIM, PBDW, 3DVAR, 4DVAR, EnKF, RB3DVar, RB4DVar, RBEnKF, heat transfert, transport, Stokes/Navier-Stokes

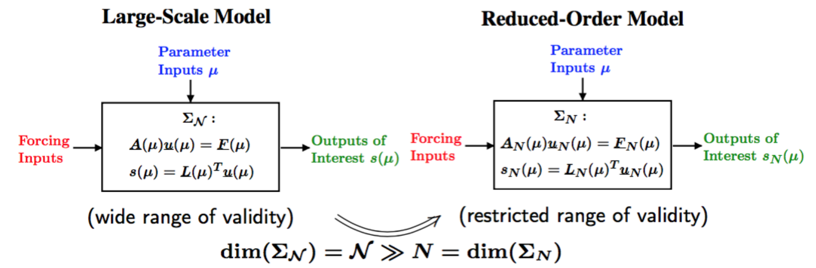
Model Order Reduction

Problem statement

Goal

Replicate input-output behavior of large-scale system Σ over a certain (restricted) range of

- forcing inputs and
- parameter inputs



Problem statement

Given large-scale system $\Sigma_{\mathcal{N}}$ of dimension $\mathcal{N}$, find a reduced order model Σ_N of dimension $N << \mathcal{N}$ such that: The approximation error is small, i.e., there exists a global error bound such that

- $\|u(\mu) - u_N(\mu)\| \leq \varepsilon_{\mathrm{des}}$, and $|s(\mu) - s_N(\mu)| \leq \varepsilon^s_{\mathrm{des}}, \forall \mu \in D^\mu$.
- Stability (and passivity) is preserved.
- The procedure is computationally stable and efficient.

Generalized Inverse Problem

- Given PDE(μ) constraints, find value(s) of parameter μ which:
  - (OPT) minimizes (or maximizes) some functional;
  - (EST) agrees with measurements;
  - (CON) makes the system behave in a desired manner;
  - or some combination of the above
- Full solution computationally very expensive due to a repeated evaluation for many different values of μ
- Goal: Low average cost or real-time online response

Methodologies

- Reduced Basis Methods
- Proper Orthogonal Decomposition
- Balanced Truncation
- Krylov Subspace Methods
- Proper Generalized Decomposition
- Modal Decomposition
- Physical model reduction
- ...

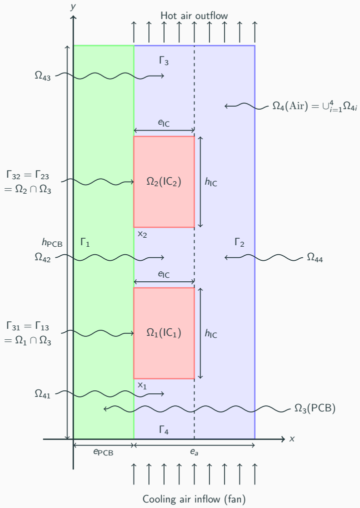
Disclaimer

Model Order Reduction Techniques

- DO NOT replace your favorite discretization scheme (e.g. FE, FV, FD), but instead are build upon and supplement these schemes.
- ARE NOT useful if you are interested in a single high-fidelity solution of your high-dimensional problem, but instead if you are interested in the many-query or real-time context.

Some Examples

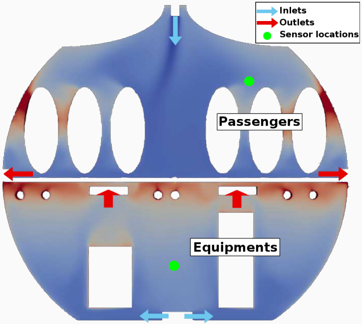
Thermal Testcase Description



Overview

- Heat-Transfer with conduction and convection possibly coupled with Navier-Stokes
- Simple but complex enough to contain all difficulties to test the certified reduced basis
  - non symmetric, non compliant
  - steady/unsteady
  - physical and geometrical parameters
  - coupled models
- Testcase can be easily complexified

Airbus Use-Case



Objective

Apply reduced basis methods on an aerothermal simulation in an avionic bay

Model

- Steady Navier-Stokes/Heat transfert
- Incompressible Newtonian Fluid
- Boussinesq Approximation
- Turbulent Flow

Some Scientific Issues

- Turbulence
- Mixed forced and natural convection
- Boundary conditions coupled to an ECS (Environment Control System)
- Error prediction (Reduced Basis)

HiFiMagnet project

High Field Magnet Modeling

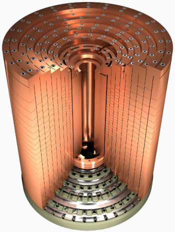
Laboratoire National des Champs Magnétiques Intenses

Large scale user facility in France

- High magnetic field : from 24 T
- Grenoble : continuous magnetic field (36 T)
- Toulouse : pulsed magnetic field (90 T)

Application domains

- Magnetoscience
- Solide state physic
- Chemistry
- Biochemistry



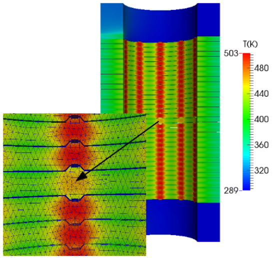
Magnetic Field

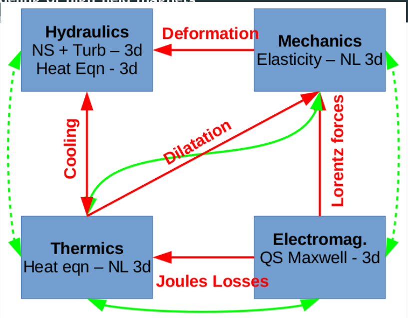
- Earth : $5.8 \cdot 10^{-4}\,T$
- Supraconductors : $24\,T$
- Continuous field : $36\,T$
- Pulsed field : $90\,T$

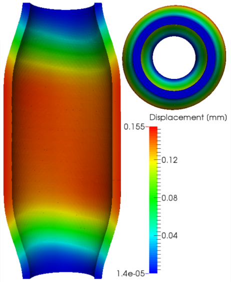
Access

- Call for Magnet Time : 2 × per year
- ≈ 140 projects per year

High Field Magnet Modeling







Why use Reduced Basis Methods ?

Challenges

- Modeling : multi-physics non-linear models, complex geometries, genericity
- Account for uncertainties : uncertainty quantification, sensitivity analysis
- Optimization : shape of magnets, robustness of design

Objective 1 : Fast

- Complex geometries
  - Large number of dofs
- Uncertainty quantification
  - Large number of runs

Objective 2 : Reliable

- Field quality
- Design optimization
  - Certified bounds
  - Reach material limits

Summary

Many problems in computational engineering require

many or real-time evaluations of PDE(μ)-induced
input-output relationships.

Model order reduction techniques enable

certified, real-time calculation
of outputs of PDE(μ)
for parameter estimation, optimization, and control.

Reduced Basis Method

The Reduced Basis Method

Reduced basis method
A model order reduction technique that allows efficient and reliable reduced order approximations for a large class of parametrized partial differential equations (PDEs) in real-time or in the limit of many queries.

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- Comparison to other model reduction techniques:
  - Parametrized problems(material, constants, geometry,...)
  - A posteriori error estimation
  - Offline-online decomposition
  - Greedy algorithm (to construct reduced basis space)

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- Comparison to other model reduction techniques:
  - Parametrized problems(material, constants, geometry,...)
  - A posteriori error estimation
  - Offline-online decomposition
  - Greedy algorithm (to construct reduced basis space)
- Motivation:
  - Efficient solution of optimization and optimal control problems governed by parametrized PDEs.

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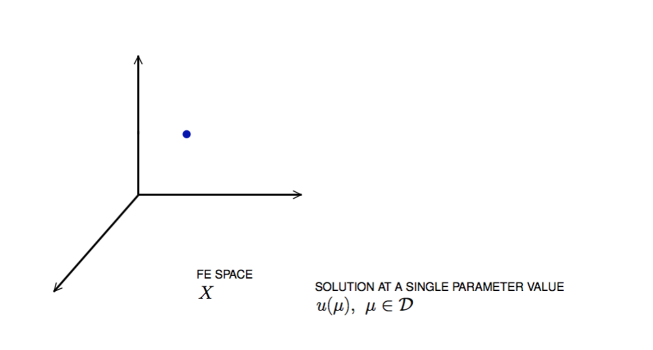
Difficulties:

- Need to solve $PDE_{FE}(\mu)$ numerous times at different values of μ
- Finite element space X has large dimension $\mathcal{N}$

The Main Idea - Key Observation



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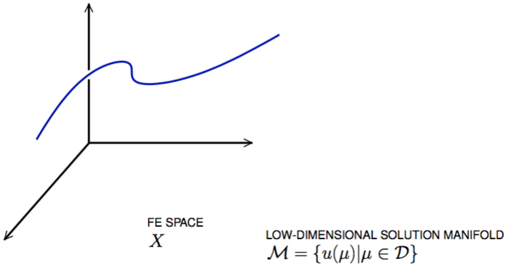

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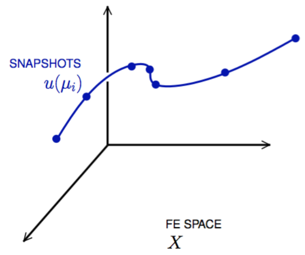

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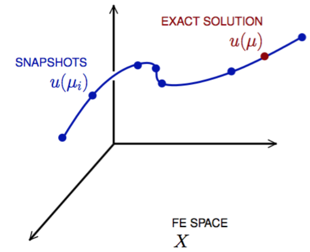

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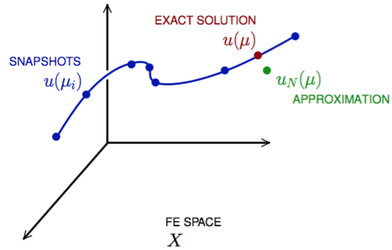

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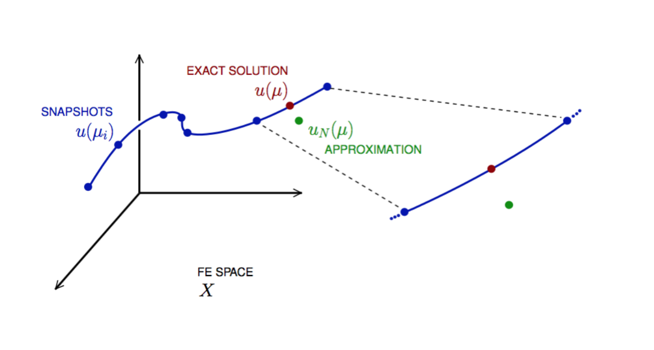

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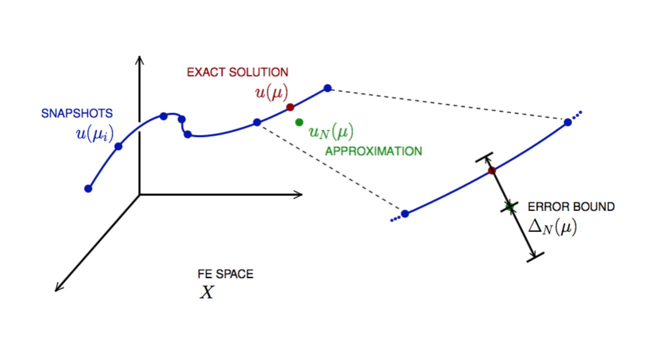

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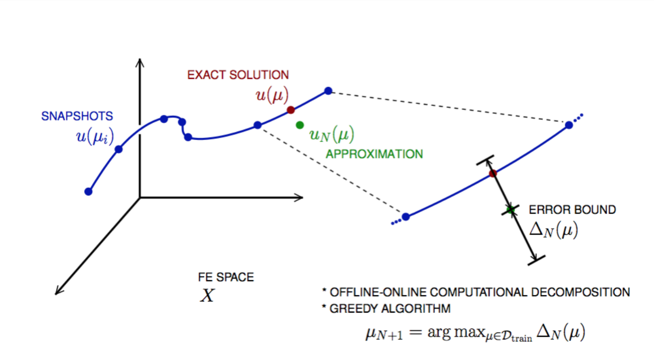


General Problem Statement

Given a system $\Sigma_{\mathcal{N}}$ of large dimension N,

$$\begin{array}{c} \text{PARAM INPUT } \boldsymbol{\mu} \longrightarrow \\ \text{FORCING INPUT } \boldsymbol{g} \longrightarrow \end{array} \boxed{\begin{array}{rcl} \boldsymbol{M}(\boldsymbol{\mu})\dot{\boldsymbol{u}} & = & \boldsymbol{A}(\boldsymbol{\mu})\boldsymbol{u} + \boldsymbol{B}(\boldsymbol{\mu})\boldsymbol{g} \\ \boldsymbol{s} & = & \boldsymbol{L}^{\boldsymbol{T}}(\boldsymbol{\mu})\boldsymbol{u} \end{array}} \longrightarrow \text{OUTPUT } \boldsymbol{s}$$

where

- $u(\mu, t) \in \mathbb{R}^{\mathcal{N}}$, the state
- $s(\mu, t)$, the outputs of interest
- $g(t)$, the forcing or control inputs

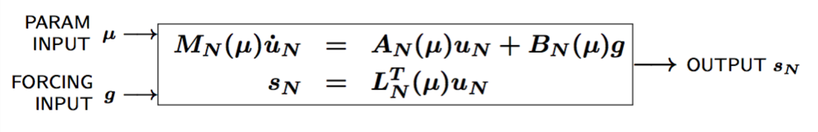
are functions of

- $\mu \in D$, the parameter inputs
- t, time

and the matrices M, A, B, and L also depend on μ . . .

General Problem Statement

. . . construct a reduced order system Σ_N of dimension $N << \mathcal{N}$,



where $u_N(\mu) \in \mathbb{R}^N$ is the reduced state.

Special case

We start by considering $\dot{u} = 0$

Full Model	**Reduced Model**
$A(\mu)u(\mu) = F(\mu)$	$A_N(\mu)u_N(\mu) = F_N(\mu)$
$s(\mu) = L^T(\mu)u(\mu)$	$s_N(\mu) = L_N^T(\mu)u_N(\mu)$

Approximation

- Take "snapshots" at different μ-values: $u(\mu_i), i = 1 \dots N$, and let

$$Z_N = [\xi_1, \dots, \xi_N] \in \mathbb{R}^{\mathcal{N} \times N}$$

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- For any new μ, approximate u by a linear combination of the ξ_i

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$$\underbrace{Z_N^T A(\mu) Z_N}_{\equiv A_N(\mu)} u_N(\mu) = \underbrace{Z_N^T F(\mu)}_{\equiv F_N(\mu)}$$

$$s_N(\mu) = \underbrace{L^T(\mu) Z_N}_{\equiv L_N^T(\mu)} u_N(\mu)$$

A posteriori error estimation

- Assume well-posedness; $A(\mu)$ pos.def. with min eigenvalue $\alpha_a := \lambda_1 > 0$, where $A\xi = \lambda X\xi$ and X corresponds to the X-inner product, $(v, v)_X = \|v\|_X^2$

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- Let $\underbrace{e_N = u - Z_N\ u_N}_{\text{error}}$, and $\underbrace{r = F - A\ Z_N\ u_N}_{\text{residual}}, \forall \mu \in D$, so that

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- Then for any $\mu \in D$, LAX-MILGRAM

$$\|u(\mu) - Z_N u_N(\mu)\|_X \leq \frac{\|r(\mu)\|_{X'}}{\alpha_{LB}(\mu)} =: \Delta_N(\mu)$$

$$|s(\mu) - s_N(\mu)| \leq \|L\|_{X'} \Delta_N(\mu) =: \Delta_N^s(\mu)$$

where $\alpha_{LB}(\mu)$ is a lower bound to $\alpha_a(\mu)$, and $\|r\|_{X'} = r^T X^{-1} r$.

Offline-Online decomposition

How do we compute u_N, s_N, Δ_N^s for any μ efficiently online?

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Since all required quantities can be decomposed in this manner, we can

- **OFFLINE**: Form and store μ-independent quantities at cost $O(\mathcal{N}^*)$
- ONLINE: For any $\mu \in D$, compute approx and error bounds at cost $O(QN^2 + N^3)$ and $O(Q^2N^2)$.

Greedy Algorithm

How do we choose the sample points μ_i (snapshots) optimally?

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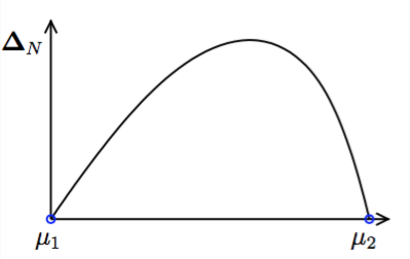
Given $Z_{N=2}$ "=" $[u(\mu_1), u(\mu_2)]$, we choose μ_{N+1} as follows:



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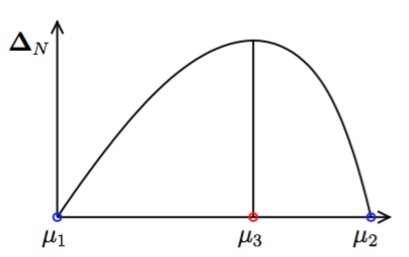

$$\mu_{N+1} = \operatorname{argmax}_{\mu \in D^{\text{train}}} \frac{\Delta_N(\mu)}{\|u_N(\mu)\|_X}$$

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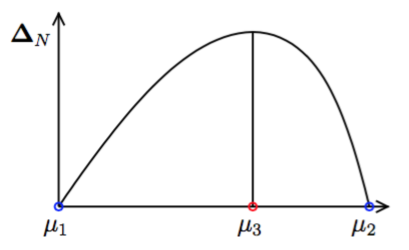

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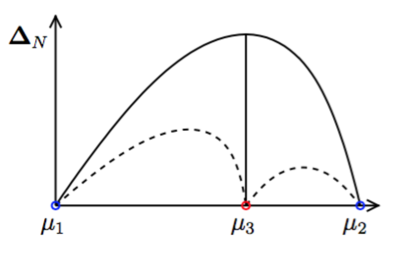

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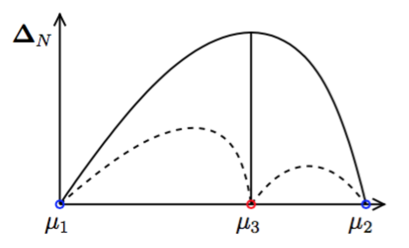

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$$Z_{N+1} = [u(\mu_1), \dots, u(\mu_{N+1})]$$

- Key: $\Delta_N(\mu)$ is *sharp* and *inexpensive* to compute (online)
- Error bound "optimal" samples $\Rightarrow$ good approximation $u_N(\mu)$.

Reduced Basis Opportunities

Computational Opportunities

- We restrict our attention to the typically smooth and low-dimensional manifold induced by the parametric dependence.
  $\Rightarrow$ Dimension reduction
- We accept greatly increased offline cost in exchange for greatly decreased online cost.
  $\Rightarrow$ Real-time and/or many-query context

Reduced Basis Relevance

Real-Time Context (control,...):

$$\begin{array}{ccc} & \mu \rightarrow \quad s_N(\mu), \Delta_N^s(\mu) & \\ t_0(\text{“}input\text{''}) & & t_0 + \delta t_{\mathrm{comp}}(\text{“}response\text{''}) \end{array}$$

Many-Query Context (design,...):

$$\begin{array}{ll} \mu_j \rightarrow & s_N(\mu_j), \Delta_N^s(\mu_j), \quad j = 1 \ldots J \\ t_0 & \qquad t_0 + \delta t_{\mathrm{comp}} J \quad (J \rightarrow \infty) \end{array}$$

$\Rightarrow$ Low marginal (real-time) and/or low average (many-query) cost.

Reduced Basis Challenges

- A Posteriori error estimation
  - Rigorous error bounds for outputs of interest
  - Lower bounds to the stability "constants"
- Offline-online computational procedures
  - Full decoupling of finite element and reduced basis spaces
  - A posteriori error estimation
  - Nonaffine and nonlinear problems
- Effective sampling strategies
  - High parameter dimensions

Reduced Basis Outline

1. Affine Elliptic Problems
    - (non)symmetric, (non)compliant, (non)coercive
    - (Convection)-diffusion, linear elasticity, Helmholtz
2. Affine Parabolic Problems
    - (Convection)-diffusion equation
3. Nonaffine and Nonlinear Problems
    - Nonaffine parameter dependence, nonpolynomial nonlinearities
4. Reduced Basis (RB) Method for Fluid Flow
    - Saddle-Point Problems (Stokes)
    - Navier-Stokes Equations
5. Applications
    - Data Assimilation : 3DVar, 4DVar, EnKF
    - Parameter Optimization and Estimation (Inverse Problems)
    - Optimal Control