Modeling the transient behavior of high field magnets

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Introduction

Formulation
Differential form of Maxwell's equations

$H(r,t)$	the magnetic field intensity,
$E(r,t)$	the electric field intensity,
$B(r,t)$	the magnetic flux density,
$D(r,t)$	the electric flux density,
$J(r,t)$	the electric current density,
$\rho(r,t)$	the electric charge density,
$M(r,t)]$	the magnetization,
$E_i(r,t)$	the impressed electric field,
$P(r,t)$	the polarization,
$\mu_0(r,t)$	the permeability of vaccum,
$\sigma(r,t)$	the conductivity,
$\epsilon_0(r,t)$	the permittivity of vacuum.

Formulation

Differential form of Maxwell's equations

$$
\begin{aligned}
\nabla \times H(r,t) &= J(r,t) + \frac{\partial D(r,t)}{\partial t} \\
\nabla \times E(r,t) &= -\frac{\partial B(r,t)}{\partial t} \\
\nabla \cdot B(r,t) &= 0 \\
\nabla \cdot D(r,t) &= \rho(r,t) \\
B(r,t) &= \mu_0[H(r,t) + M(r,t)] \\
J(r,t) &= \sigma[E(r,t) + E_i(r,t)] \\
D(r,t) &= \epsilon_0 E(r,t) + P(r,t)
\end{aligned}
$$

Formulation

The magnetic vector and electric scalar potentials formulation

The MQS approximation consists in neglecting the so-called displacement current, $\frac{\partial D}{\partial t}$. In this context, the equations to solve are:

$$\nabla \times H(r,t) = J(r,t), \tag{1}$$

$$\nabla \times E(r,t) = -\frac{\partial B(r,t)}{\partial t}, \tag{2}$$

$$B(r,t) = \mu_0 H(r,t) \tag{3}$$

$$J(r,t) = \sigma E(r,t) \tag{4}$$

Formulation
The magnetic vector and electric scalar potentials formulation

A classical way to solve these equations is to introduce a magnetic potential A and a scalar electric potential V. As B is a divergence free field, we can define A as:

$$B = \nabla \times A.$$

To ensure A unicity we will need to add a gauge condition. Most commonly:

$$\nabla \cdot A = 0$$

The Faraday equation may, then, be rewritten as:

$$\nabla \times (E + \frac{\partial A}{\partial t}) = 0.$$

We can define the electric scalar potential V as:

$$E + \frac{\partial A}{\partial t} = -\nabla V.$$

Formulation
The magnetic vector and electric scalar potentials formulation

It follows that:

$$J = \sigma(-\nabla V - \frac{\partial A}{\partial t}).$$

From this expression of the current density, we may rewrite the Ampere equation as:

$$\nabla \times (\frac{1}{\mu} \nabla \times A) + \sigma \frac{\partial A}{\partial t} = -\sigma \nabla V. \quad (1)$$

because :

$$B = \nabla \times A = \mu H$$

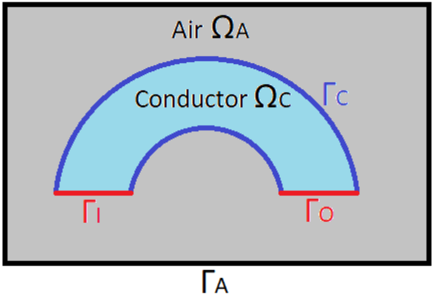
so :

$$\nabla \times H = \nabla \times (\frac{1}{\mu} \nabla \times A)$$

To this equation, we add the conservation of the current density:

$$\nabla \cdot (J) = \nabla \cdot (\sigma(-\nabla V - \frac{\partial A}{\partial t})) = 0 \quad (2)$$

Let us note Ω the domain, comprising the conductor Ω_C and the air Ω_A, and let us note Γ the edge of this domain, comprising the edge of the air Γ_A, the inlet Γ_I and the outlet Γ_O. To simplify, let us note Γ_D the edges with Dirichlet boundary condition, and Γ_N the edges with Neumann bondary condition, such that $\Gamma = \Gamma_D \cup \Gamma_N$.



Let us consider the equation (1) : By making the scalar product with $\phi \in H^{curl}_{A_D}(\Omega)$ and by integrating on Ω we get :

$$\int_{\Omega} \phi \cdot (\nabla \times (\frac{1}{\mu} \nabla \times A) + \sigma \frac{\partial A}{\partial t}) = \int_{\Omega_C} \phi \cdot (-\sigma \nabla V)$$

Using the relationship:

$$\nabla \cdot (u \times v) = v \cdot (\nabla \times u) - u \cdot (\nabla \times v)$$

we deduce that:

$$\int_{\Omega} \frac{1}{\mu} (\nabla \times \phi) \cdot (\nabla \times A) + \int_{\Omega} \frac{1}{\mu} \nabla \cdot (\phi \times (\nabla \times A)) = - \int_{\Omega_C} \sigma \phi \cdot (\nabla V + \frac{\partial A}{\partial t})$$

Formulation
Weak formulation

Using the divergence theorem we get:

$$\int_{\Omega} \frac{1}{\mu} (\nabla \times \phi) \cdot (\nabla \times A) + \int_{\Gamma_D} \frac{1}{\mu} (\phi \times (\nabla \times A)) \cdot n$$
$$+ \int_{\Gamma_N} \frac{1}{\mu} (\phi \times (\nabla \times A)) \cdot n = - \int_{\Omega_C} \sigma \phi \cdot (\nabla V + \frac{\partial A}{\partial t})$$

By performing a circular permutation on Γ_N and Γ_D we have :

$$\begin{aligned}\int_{\Omega} \frac{1}{\mu} (\nabla \times \phi) \cdot (\nabla \times A) - \int_{\Gamma_D} \frac{1}{\mu} (\phi \times n) \cdot (\nabla \times A) \\ + \int_{\Gamma_N} \frac{1}{\mu} ((\nabla \times A) \times n) \cdot \phi \\ = - \int_{\Omega_C} \sigma \phi \cdot (\nabla V + \frac{\partial A}{\partial t})\end{aligned}$$

So we have :

$$\int_{\Omega} \frac{1}{\mu} (\nabla \times \phi) \cdot (\nabla \times A) - \int_{\Gamma_D} \frac{1}{\mu} A_D \cdot (\nabla \times A) = - \int_{\Omega_C} \sigma \phi \cdot (\nabla V + \frac{\partial A}{\partial t})$$

Formulation
Weak formulation

The border of Ω_C is considered to be splitted into Γ_I, Γ_O respectively the input and output of current and the rest will be noted Γ_C. On Γ_I and Γ_O we consider Dirichlet Boundary condition for the electrical potential. Thus we will take $\psi \in H^1(\Omega_C)$.
Let's consider the equation (2) : By making the scalar product with ψ and integrating over Ω_C we get :

$$\int_{\Omega_C} \psi \cdot \nabla \cdot (\sigma(-\nabla V - \frac{\partial A}{\partial t})) = 0 \tag{5}$$

Using the relationship:

$$\nabla \cdot (u \cdot v) = v \cdot \nabla u + u \nabla \cdot v$$

we get :

$$\int_{\Omega_C} \nabla \cdot (\sigma \psi \cdot (-\nabla V - \frac{\partial A}{\partial t})) - \int_{\Omega_C} \sigma(-\nabla V - \frac{\partial A}{\partial t}) \cdot \nabla \psi = 0$$

By using the formula of divergence we get:

$$\int_{\Gamma_C} \sigma\psi \cdot (-\nabla V - \frac{\partial A}{\partial t}) \cdot n - \int_{\Omega_C} \sigma(-\nabla V - \frac{\partial A}{\partial t}) \cdot \nabla\psi = 0$$

Or we know that $j \cdot n = 0$ on Γ_C due to the current density conservation law. Or $j = \sigma E = \sigma(\nabla V + \frac{\partial A}{\partial t})$ so we get finally :

$$-\int_{\Omega_C} \sigma(-\nabla V - \frac{\partial A}{\partial t}) \cdot \nabla\psi = 0$$

Discretization

To solve these two differential equations, we can first discretize the time derivative by finite differences. If u is a functions. Let us note u^n the quantity designating u at time n. Let us note $\Delta t > 0$ the step time, such that $t_n = n\Delta t$. Let us note $A^n(x) := A(t_n, x)$.

We have, using an implicit euler's schema : $\frac{\partial A}{\partial t} = \frac{A^n - A^{n-1}}{\Delta t}$.

Discretization

So our two equations become :

$$\int_\Omega \frac{\Delta t}{\mu}(\nabla \times \phi) \cdot (\nabla \times A^n) - \int_{\Gamma_D} \frac{1}{\mu} A_D \cdot (\nabla \times A^n) + \int_{\Omega_C} \sigma\phi \cdot (A^n + \Delta t \nabla V) = \int_{\Omega_C} \sigma\phi \cdot A^{n-1}$$

$$\int_{\Omega_C} \sigma(A^n + \Delta t \nabla V) \cdot \nabla\psi = \int_{\Omega_C} \sigma A^{n-1} \cdot \nabla\psi$$

Implementation
First equation

To solve the first equation only, we will assume that V is known. By slightly transforming the first equation, we obtain :

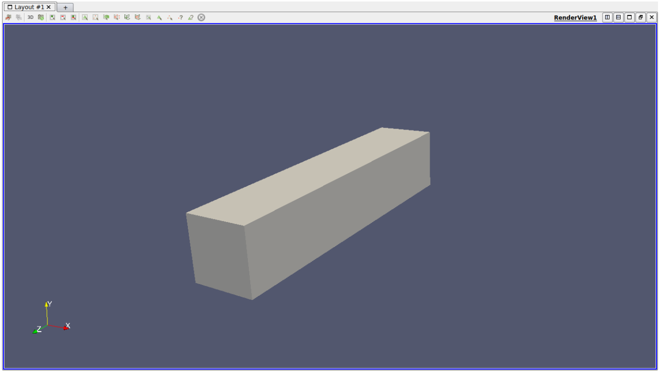
$$\int_\Omega \frac{\Delta t}{\mu} (\nabla \times \phi) \cdot (\nabla \times A^n) - \int_{\Gamma_D} \frac{1}{\mu} A_D \cdot (\nabla \times A^n) + \int_{\Omega_C} \sigma \phi \cdot A^n = \int_{\Omega_C} \sigma \phi \cdot (A^{n-1} - \Delta t \nabla V)$$

We will now implement this equation under feelpp.

Implementation

First equation

Now, we will test this program using the function $gradV = (-xz, 0, -\frac{t}{\sigma})$ with $\sigma = 58000$ and $\mu = 1$. Note that the exact solution A for this V is $A = (xzt, 0, 0)$. We will also take the time $t \in [0, 1]$, with a step time $dt = 0.025$. We'll run the simulation on a bar, representing the conductor.

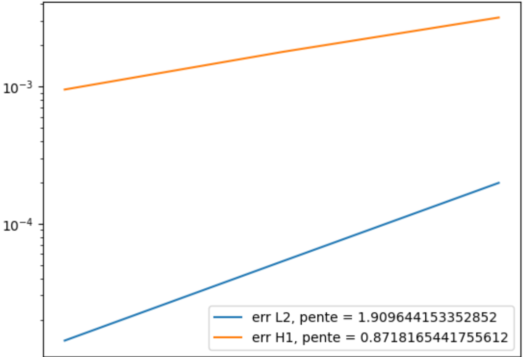


Implementation

First equation

$t = 0.1$

h	0.2	0.1	O.05
L^2 error	0.000198386	5.29439e-05	1.40537e-05
L^2 relative error	0.000532326	0.000142063	3.77101e-05
H^1 error	0.00318232	0.00178333	0.000950294

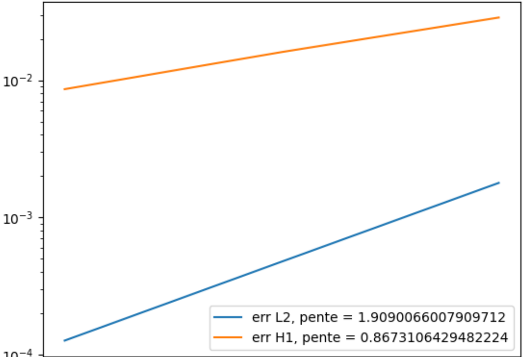


Implementation

First equation

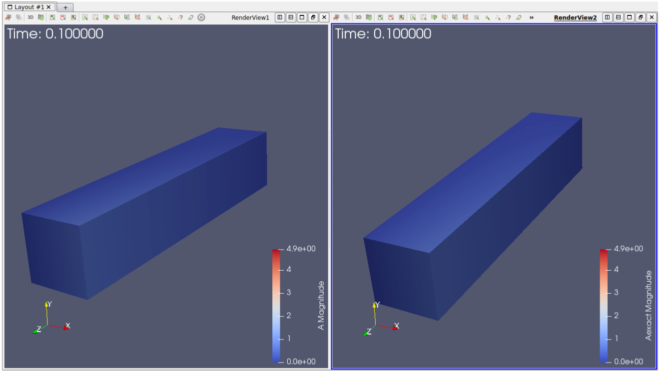
$t = 0.9$

h	0.2	0.1	0.05
L^2 error	0.00178574	0.000476291	0.000126614
L^2 relative error	0.000532405	0.000142003	3.77491e-05
H^1 error	0.0286545	0.0160523	0.00861033

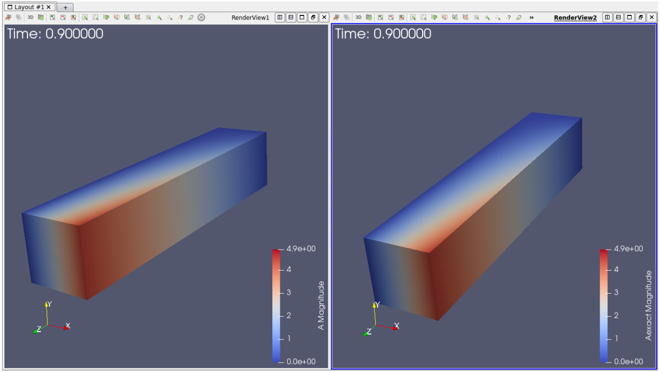


Implementation
First equation

Here is a comparison under paraview between the exact solution and the calculated solution, at different time :



Implementation
First equation



Implementation
Second equation

To solve the second equation only, we will assume that $\frac{\partial A}{\partial t}$ is known. By slightly transforming this equation, we obtain :

$$\int_{\Omega_C} \sigma \nabla V \cdot \nabla \psi = -\int_{\Omega_C} \frac{\partial A}{\partial t} \cdot \nabla \psi$$

We will now implement this equation under feelpp.

Now we will test this program with two different set of function : First will be with the function $A = (-t, 0, 0)$, so $\frac{\partial A}{\partial t} = (-1, 0, 0)$. Note that the exact solution is $V = zt$. We will run the simulation on the same geometry as before, with same time and step time.

hsize $= 0.1$

t	0.1	0.5	0.9
L^2 error	1.40624e-15	3.01653e-15	7.75481e-15
L^2 relative error	2.17853e-15	9.34637e-16	1.33486e-15
H^1 error	4.47471e-15	2.05823e-14	3.85823e-14

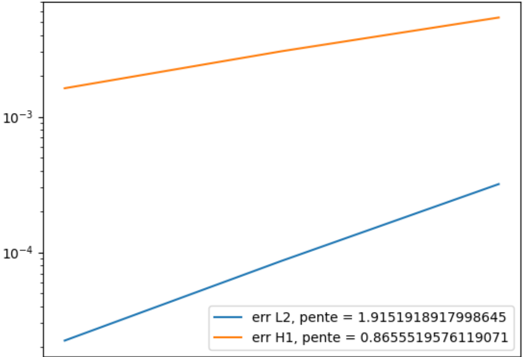
The error is 0 at epsilon machine, which is what is expected because the function is linear in space.

Now we will use the function $A = (-xt, 0, zt)$, so $\frac{\partial A}{\partial t} = (-x, 0, z)$. Note that the exact solution is $V = zxt$. We will run the simulation on the same geometry as before, with same time and step time. Below are the errors we get at different times, with the associated graph in log scale.

Implementation
Second equation

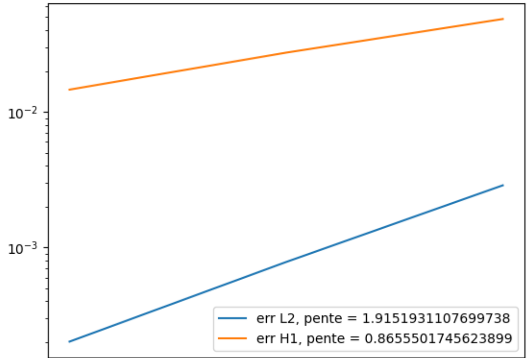
$t = 0.1$

h	0.2	0.1	O.05
L^2 error	0.000318572	8.66208e-05	2.23948e-05
L^2 relative error	0.00085482	0.000232428	6.00916e-05
H^1 error	0.00537588	0.00303521	0.00161933



$t = 0.9$

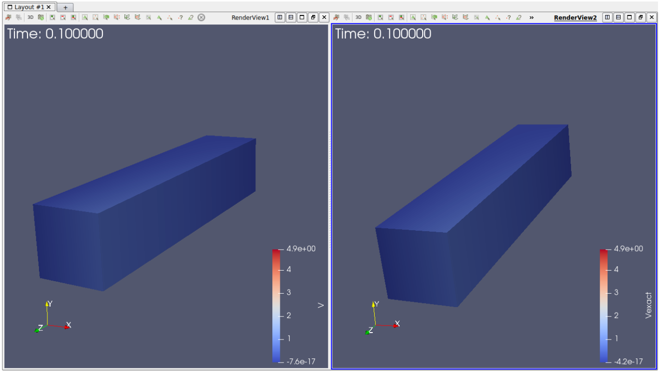
h	0.2	0.1	O.05
L^2 error	0.00286715	0.000779587	0.000201553
L^2 relative error	0.00085482	0.000232428	6.00916e-05
H^1 error	0.0483829	0.0273169	0.014574



Implementation

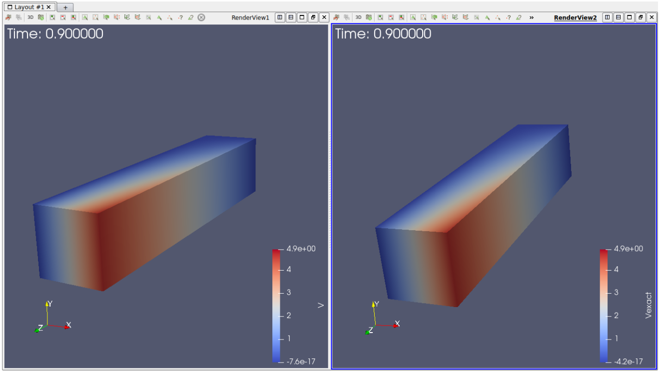
Second equation

Here is a comparison under paraview between the exact solution and the calculated solution, at different time :



Implementation

Second equation



Implementation

Coupled system

$$\int_\Omega \frac{\Delta t}{\mu}(\nabla \times \phi)\cdot(\nabla \times A^n) - \int_{\Gamma_D} \frac{1}{\mu} A_D \cdot (\nabla \times A^n) + \int_{\Omega_C} \sigma\phi \cdot (A^n + \Delta t \nabla V) = \int_{\Omega_C} \sigma\phi \cdot A^{n-1}$$

$$\int_{\Omega_C} \sigma(A^n + \Delta t \nabla V)\cdot \nabla\psi = \int_{\Omega_C} \sigma A^{n-1} \cdot \nabla\psi$$

Implementation
Coupled system

Which can be rewrite :

$$\int_\Omega \frac{\Delta t}{\mu}(\nabla \times \phi)\cdot(\nabla \times A^n) + \int_{\Omega_C} \sigma\phi\cdot A^n - \int_{\Gamma_D} \frac{1}{\mu}A_D\cdot(\nabla \times A^n) + \int_{\Omega_C} \sigma\phi\cdot\Delta t\nabla V = \int_{\Omega_C} \sigma\phi\cdot A^{n-1}$$

$$\int_{\Omega_C} \sigma A^n\cdot\nabla\psi + \int_{\Omega_C} \sigma\Delta t\nabla V\cdot\nabla\psi = \int_{\Omega_C} \sigma A^{n-1}\cdot\nabla\psi$$

Conclusion