ORDERED LIST REASONING using SEMANTIC LOSS

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Theoretical Foundations of Semantic Loss

THEORETICAL MOTIVATION

PROBLEM

Standard Neural Networks **ignore logical structure**

Output may violate **hard constraints** (e.g., x_i > x_i+1 in a sorted list)

SOLUTION

Semantic Loss (*L_sem*)

- Acts as a regularizer for **logical consistency**
- Bridges **propositional logic** and **differentiable learning**

GOAL

Inject **symbolic knowledge** into Deep Learning

Enforce **valid configurations** during training (semi-supervised signal).

MATHEMATICAL FORMULATION

CONCEPT: Treat the constraint α as a probabilistic event

FORMULA:

$$\mathcal{L}_{sem}(\mathbf{p}) \propto -\log \sum_{\mathbf{x} \models \alpha} P(\mathbf{x} \mid \mathbf{p})$$

COMPONENTS:

- **p:** Output probabilities from the Neural Network
- **x:** A possible state (variable assignment)
- **x ⊨ α:** The set of all states where the constraint is α True

INTERPRETATION: Minimize the negative log-probability of generating a state consistent with α

INTEGRATION & TRAINING

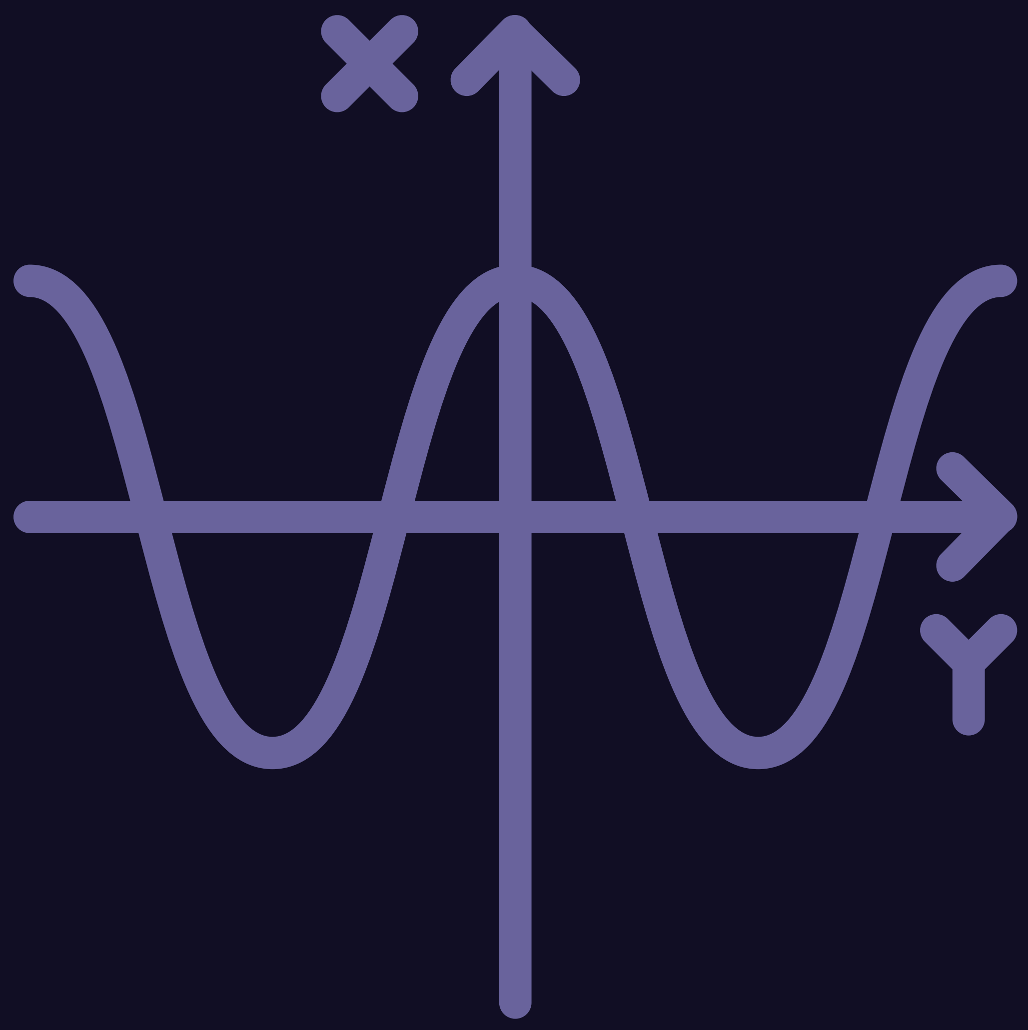
LOSS FUNCTION: $L_{total} = (1 - \lambda) L_{sup} + \lambda L_{sem}$

KEY COMPONENTS:

- ***L_{sup}*:** Standard Cross-Entropy (matches labels)
- ***L_{sem}*:** Penalizes logically invalid predictions
- **λ:** Hyperparameter controlling symbolic influence

PROJECT CONTEXT:

- **Constraint:** Monotonicity ($pos_i < pos_{i+1}$)
- **Outcome:** Improved robustness with limited data



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Connection between Propositional Logic and Differentiable Loss

THE BRIDGE (WMC)

PROBLEM

- **Logic is discrete** (True/False)
- **Loss functions** must **be continuous**

SOLUTION

- Map **logical satisfaction** to **Weighted Model Counting (WMC)**
- **Calculates** the **total probability mass** of **all valid worlds**

MECHANISM

- **Interpret NN outputs** as **variable probabilities**
- **Sum probabilities** of all **possible worlds satisfying constraint** α
- **Result** is a **differentiable scalar value** (probability mass of valid states)

COMPUTATIONAL TRACTABILITY

CHALLENGE

Naive summation over truth assignments **is #P-hard (exponential)**

METHOD

KNOWLEDGE COMPILATION: Compile **Propositional Logic -> Arithmetic Circuit** (AC) (e.g., d-DNNF)

ADVANTAGE

- AC allows **computing WMC** in **linear time** w.r.t. circuit size
- **AC operations** (Sum/Product) are **fully differentiable**
- Enables **efficient backpropagation** of **logical errors**

FROM LOGIC TO GRADIENT

PIPELINE:

DEFINE:
Logic Constraint α
(Hard Constraint)

COMPILE:
Generate Arithmetic
Circuit (Offline Step)

FORWARD:
Pass NN probabilities
into Circuit → Get
Probability of
Satisfaction

BACKWARD:
Compute Gradient of
-log(*P_sat*) to update
NN weights

OUTCOME:

The **logic constraint** drives the **gradient descent** towards **valid regions**

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Our Architecture and Training Methodology

SEMANTIC LOSS I (SYMBOLIC CNF)

METHOD: Explicit construction of Propositional Logic trees (via SymPy)

THE LOGIC: For adjacent positions j, j+1:

- $(P_{i,j} \wedge P_{k,j+1}) \Rightarrow Val(i) \leq Val(k)$

CONFLICT CLAUSE: If Val(i) > Val(k), the configuration is invalid

- CNF Formula: $\neg P_{i,j} \vee \neg P_{k,j+1}$
- Probabilistic Eval: $1 - (P(i@j) \cdot P(k@j+1))$

LIMITATION: Requires recursive tree traversal on CPU

SEMANTIC LOSS II (PAIRWISE OPTIM.)

METHOD: Matrix operations on GPU (Tensor-based)

THE MECHANISM: Compute **Joint Probability** for all pairs **simultaneously** using Batch Matrix Multiplication

FORMULA: $\mathscr{L}$_local $\propto -\log \sum_{i,k} (P(i@j) \cdot P(k@j+1) \cdot \mathbf{M}_{i,k})$

- $\mathbf{M}_{i,k}$: Validity Mask (1 if Val(i) ≤ Val(k), else 0)

KEY ADVANTAGE:

- **Transitivity:** Local checks (j < j+1) enforce global order
- **Speed:** Parallelizes the logic evaluation (O(N^2) matrix op)
- **Result:** Enables scalable training with symbolic guarantees

MODEL & STRUCTURAL CONSTRAINTS

ARCHITECTURE: SortNet (MLP)

- **Input:** 5 Integers → **Hidden:** 64-32 (ReLU) → **Output:** 5x5 Matrix
- **Activation:** Softmax over positions

CONSTRAINT 1: Exactly - One (Structural)

- **Logic:** Every item must be in exactly one position; every position must hold exactly one item

IMPLEMENTATION: Differentiable penalty on probability mass

- **Rows:** $\sum_j P_{i,j} = 1$ (Item i placed once)
- **Cols:** $\sum_i P_{i,j} = 1$ (Position j filled once)

ROLE: Prevents the network from "cheating" (e.g., duplicating easy numbers)

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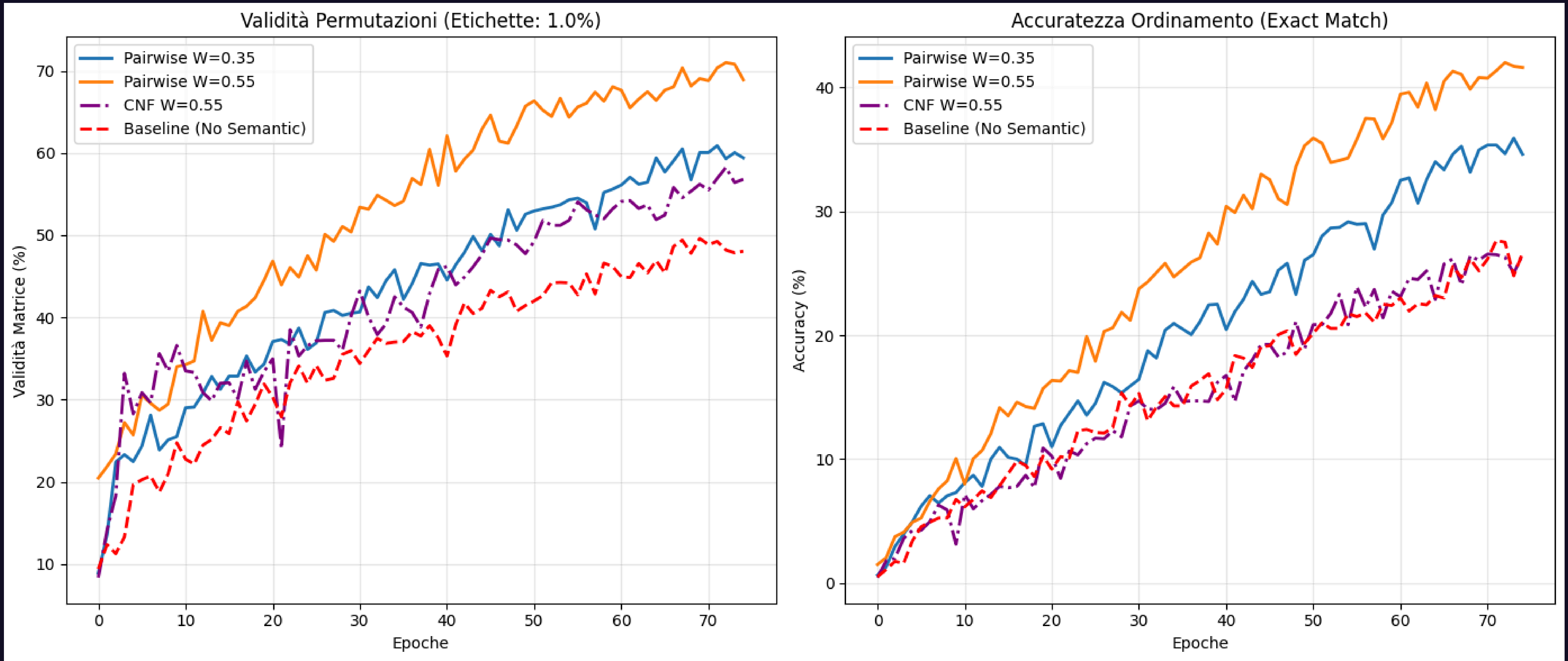
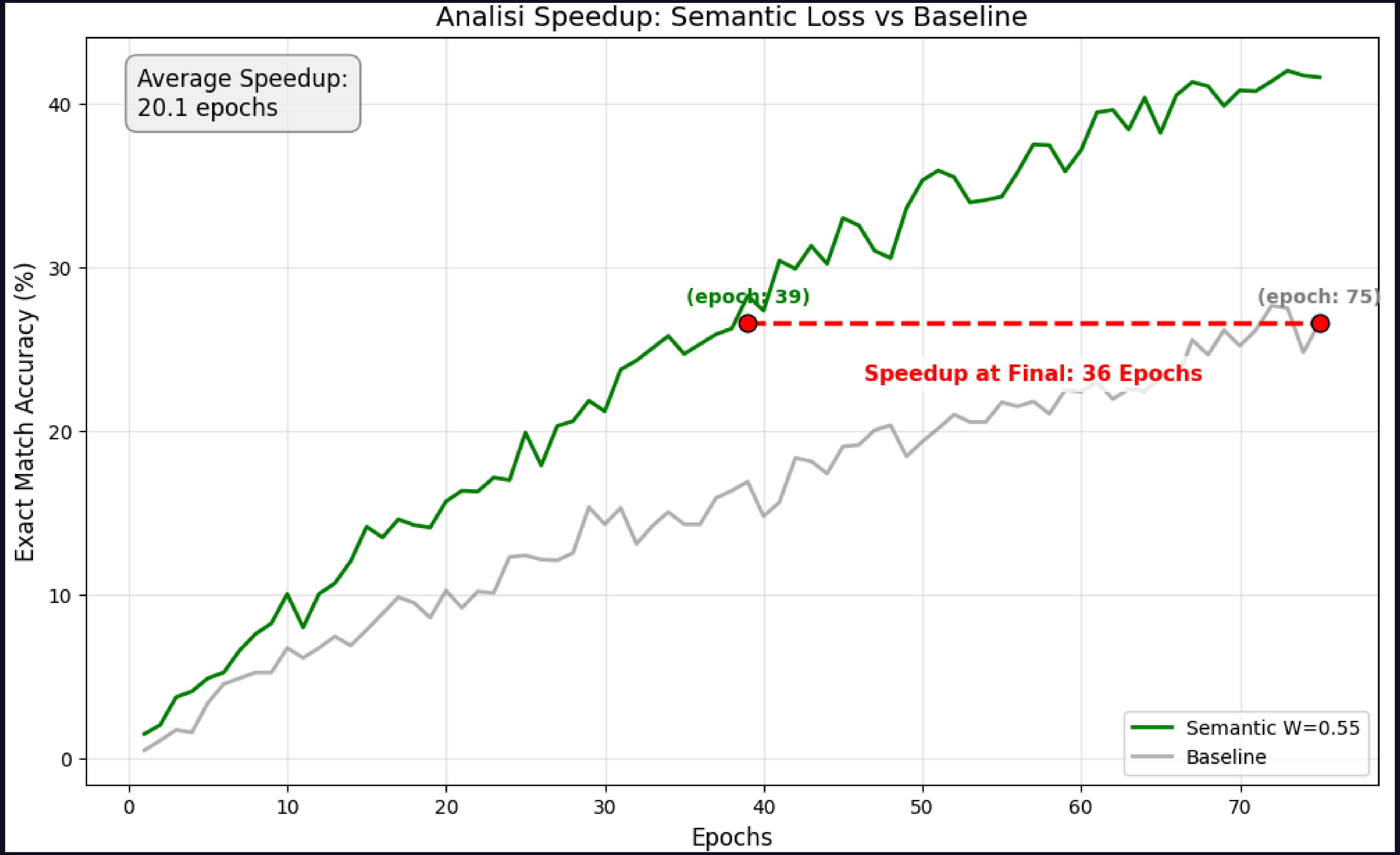
Experimental Results and Comparisons

LEARNING CURVE - 1% LABELS



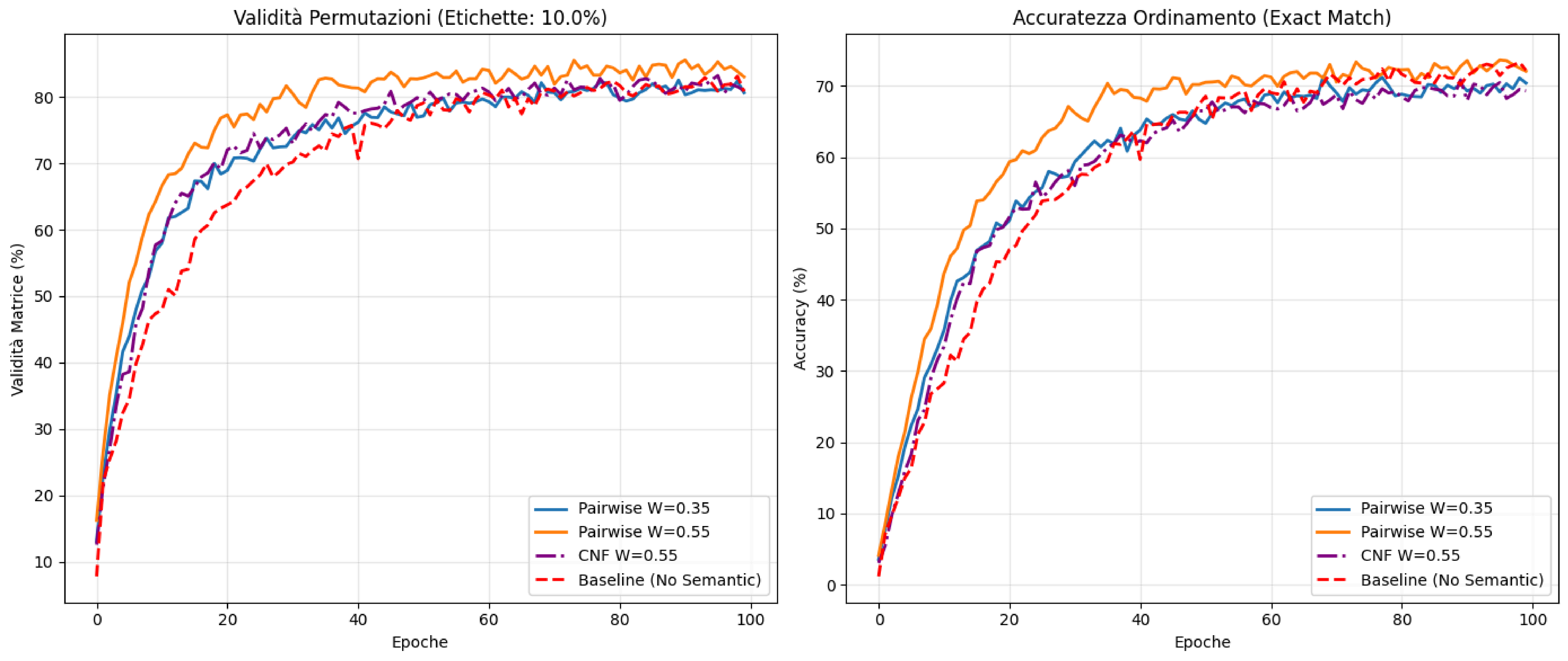
INSIGHT: Semantic Loss **drastically outperforms** the baseline in data-scarce scenarios, effectively **preventing overfitting** where supervised signals are insufficient

CONVERGENCE SPEED - 1% LABELS



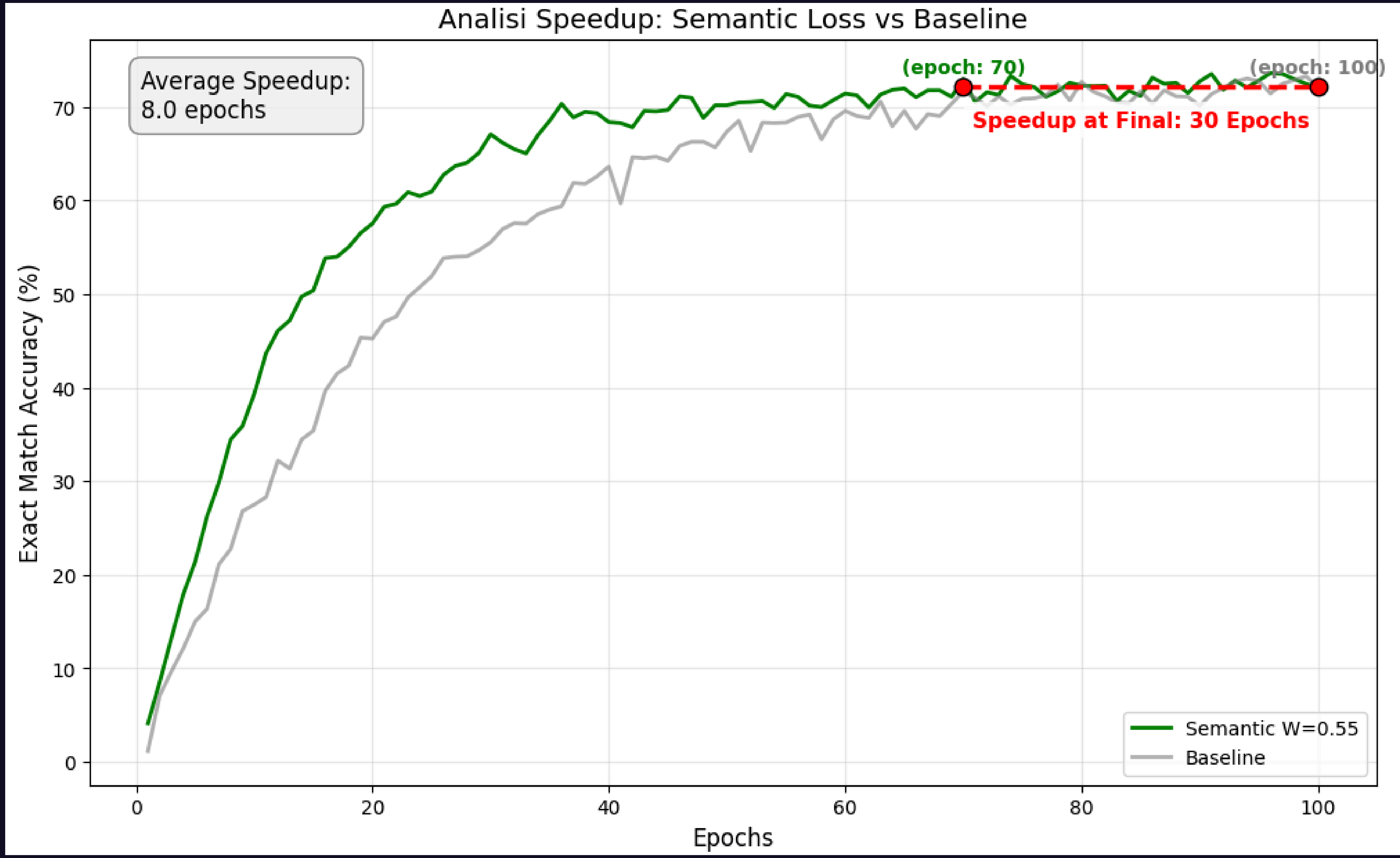
INSIGHT: The injection of logic acts as a **strong** "cold start" **accelerator**, allowing the model to reach **optimal performance significantly earlier** than the baseline

LEARNING CURVE - 10% LABELS



INSIGHT: As labeled data **increases the performance gap** narrows, yet the Neuro-Symbolic model **maintains superior stability** and **lower variance** during training

CONVERGENCE SPEED - 10% LABELS



INSIGHT: Even with sufficient data, symbolic regularization **enforces faster gradient descent, reducing the overall computational budget** required for convergence

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Advantages and Limitations of Semantic Loss

▶ PERFORMANCE MATRIX (1% LABELS)

METRIC	SEMANTIC CNF	BASELINE	NEUROSYMBOLIC	Δ IMPACT(Baseline → Nesy)
EXACTLY-ONE (VALIDITY)	~ 57%	~ 48%	~ 69%	+21%
EXACT MATCH (ACCURACY)	~ 26%	~ 27%	~ 42%	+15%
EPOCHS-TO-27%-ACCURACY	/	75 Epochs	39 Epochs	-36

PERFORMANCE MATRIX (10% LABELS)

METRIC	SEMANTIC CNF	BASELINE	NEUROSYMBOLIC	Δ IMPACT(Baseline → Nesy)
EXACTLY-ONE (VALIDITY)	~ 81%	~ 81%	~ 83%	+2%
EXACT MATCH (ACCURACY)	~ 69%	~ 72%	~ 72%	+0%
EPOCHS-TO-72%-ACCURACY	/	100 Epochs	70 Epochs	-30

ANALYSIS & KEY ADVANTAGES

DATA EFFICIENCY (The "1% Label" Case):

- **Observation: Massive accuracy gap** (+15%) when supervision is **scarce**
- **Why:** Semantic Loss acts as a **structural regularizer**, injecting "rules of the game" where labels are missing

GRADIENT GUIDANCE:

- **Observation: 2x faster convergence** despite **higher** per-epoch time
- **Why:** Logic provides **informative gradients** even for **unlabeled data**, actively steering the model away from invalid states

GUARANTEED CONSISTENCY:

- **Advantage:** Unlike standard MLPs, the NeuroSymbolic model **enforces strict constraints** (e.g., Exactly-One, Monotonicity), **ensuring output validity** by design

LIMITATIONS & TECHNICAL ISSUES

COMPUTATIONAL BOTTLENECK:

- **Issue:** Exact logical inference (WMC) is **#P-Hard** and **scales exponentially**
- **Impact:** Pure symbolic implementation (CNF) is **slow**; **requires approximations** or **matrix-based optimizations** (Pairwise) to be **feasible on GPU**

TRAINING OVERHEAD:

- **Metric:** Logic calculation **increases training time per epoch** (**30x slower**)
- **Trade-off: Higher computational cost** per step vs. fewer total steps to converge

HYPERPARAMETER SENSITIVITY:

- **Risk:** If logic weight λ is **too high,** the model may **collapse to "Trivial Validity"** (satisfying rules but ignoring input features)

THANKS!

Do you have any questions?