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DEMOSTRACIÓN CONSTRUCTIVA DEL TEOREMA DE BARR

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El presente trabajo fué desarrollado en inglés. Aún así nos corresponde escribir una introducción en castellano. La tabla de contenidos indexará la version en inglés.

Introducción

El teorema que demostramos en ésta tesis yace entre medio de dos tipos de teorema comunmente conocidos como *Teoremas de Embebimiento* y *Teoremas de Suficientes Puntos*. Los teoremas de embebimiento son aquellos que buscan representar plenamente un objeto matemático abstracto dentro de uno concreto de la misma naturaleza. Estos teoremas suelen esclarecer la naturaleza de la abstraccion misma. Los siguientes son algunos ejemplos de embebimientos donde la representación se da en algún sentido de forma *canónica*:

1. *Cayley*: $G \hookrightarrow S!$, $S = Conjunto\ Subyacente\,(G)$.
(representación de un grupo como un subgrupo del grupo simétrico).

2. *Stone*: $B \hookrightarrow Sub(S)$, $S = Ideales\ Primos\,(B)$.
(representación de un algebra booleana como una subalgebra de partes de un conjunto)

3. *Gelfand*: $A \hookrightarrow \mathcal{S}et(S, \mathbb{C})$, $S = Ideales\ Maximales\ Cerrados\,(C)$
(representación de una álgebra C^* como una subalgebra del algebra de funciones a valores complejos)

4. *Yoneda*: $\mathcal{C} \hookrightarrow \mathcal{E}ns^{\mathcal{D}}$, $\mathcal{D} = \mathcal{C}^{op}$.
(representación de una categoría pequeña $\mathcal{C}$ como una subcategoria plena de una categoría de funtores a valores en los conjuntos)

El siguiente ejemplo, el *Teorema de Representación de Barr*, es el teorema que nos concierne en éste trabajo.

Theorem 0.1.1. *Para toda categoría pequeña regular $\mathcal{C}$ existe un funtor regular plenamente fiel $\mathcal{C} \overset{h}{\hookrightarrow} \mathcal{E}ns^{\mathcal{D}}$ a una categoría de funtores a valores en los conjuntos donde el exponente tiene como objetos el conjunto $Sub(1)$ de subobjetos de 1.*

Tanto la demostración original de Barr como otras conocidas de éste teorema son altamente no constructivas, pues usan tanto el axioma de elección como inducción transfinita. El propósito de éste trabajo es demostrar una versión débil del teorema de Barr, cuyo enunciado es que el funtor h es *conservativo* (o sea refleja isos), de forma *constructiva*. Ésto es de hecho un teorema de *Suficientes Puntos*.

Posterior a la tésis doctoral de William Lawvere los *teoremas de completud* de teorías lógicas fueron formuladas en términos categóricos como teoremas de *Suficientes Puntos*. Informalmente, dado un modelo $A \in \mathcal{C}$ de una teoría $\mathcal{T}$ en una categoría apropiada $\mathcal{C}$, toda fórmula $\varphi(x)$ tiene una extensión en A que no es más que un subobjeto $[\![x \in A \mid \varphi(x)\ vale\,]\!] \hookrightarrow A$. La idea está en asociar a una teoria $\mathcal{T}$ una categoria *apropiada* $\mathcal{C}_{\mathcal{T}}$ provista de un modelo (*el modelo genérico*) $G_{\mathcal{T}}$ de $\mathcal{T}$ el cual es genérico en los siguientes sentidos:

1. Provee (de forma tautológica por la construcción misma de $\mathcal{C}_{\mathcal{T}}$) un teorema de completud para la teoría $\mathcal{T}$. O sea una fórmula $\varphi(x)$ es demostrable en $\mathcal{T}$ si y solo si $[\![x \in G_{\mathcal{T}} \mid \varphi(x)\ vale\,]\!] = G_{\mathcal{T}}$.

2. Para todo modelo $A \in \mathcal{C}$ de $\mathcal{T}$ en una categoría apropiada, existe un único funtor *apropiado* $\mathcal{C}_{\mathcal{T}} \xrightarrow{F} \mathcal{C}$ tal que $F(G_{\mathcal{T}}) = A$.

Claramente una familia *conservativa* (y por ende *mono conservativa*, o sea que si un monomorfismo va a parar por todo funtor de la familia a un isomorfismo, el monomorfismo mismo es un isomorfismo) (see 0.2.3) de funtores apropiados a valores en los conjuntos $\mathcal{C}_{\mathcal{T}} \xrightarrow{F} \mathcal{E}ns$ (en algunos contextos llamados *puntos*) nos provee un teorema de completud.

Para lógicas geométricas intuicionistas de primer orden (eso es admitiendo el cálculo proposicional intuicionista y solamente el cuantificador existencial) las categorías apropiadas son exactamente las *categorías regulares* y los funtores apropiados son exactamente los *funtores regulares*. De ésta forma la version débil del teorema de Barr nos provee de una demostración del teorema de completud para éstas lógicas mediante el siguiente argumento. Dada una categoría regular $\mathcal{C}$ debemos ver que la familia de funtores regulares $\mathcal{C} \longrightarrow \mathcal{E}ns$ es conservativa. El teorema débil de Barr nos garantiza que hay un funtor regular conservativo $\mathcal{C} \xrightarrow{h} \mathcal{E}ns^{Sub(1)}$. Las evaluaciones $\mathcal{E}ns^{Sub(1)} \longrightarrow \mathcal{E}ns$ son regulares y la familia de evaluaciones es conservativa. Queda entonces demostrado.

Independientemente de éstas consideraciones el propósito de éste trabajo consiste en dar una demostración *constructiva* de la version débil del Teorema de Barr.

Guía de la Construcción

Para la construcción usamos una guía establecida por André Joyal en unas charlas no publicadas a principio de los años setenta en Montreal. Su demostración está inspirada en reinterpretar la demostración del *Teorema de Completud de Gödel* dada por Leon Henkin que consiste en "agregar constantes".

Para llevar a cabo la contrucción de forma constructiva precisamos las hipótesis adicionales de que la categoria regular posea objeto terminal distinguido y a su vez representantes distinguidos de cada clase de subobjetos en $\mathcal{C}$. Ésto no afecta las aplicaciones a la lógica que tenemos en mente. Por ésto quiero decir que las categorías $\mathcal{C}_{\mathcal{T}}$ asociadas a las lógicas geométricas intuicionistas de primer orden verifican éstas hipótesis.

Empezamos por construir para toda categoría regular $\mathcal{A}$, que posee un objeto terminal 1 distinguido, un funtor regular $\mathcal{A} \longrightarrow \mathcal{E}ns$ que es conservativo sobre los monomorfismos cuyo codominio tiene soporte global (Section 0.5). Ésto lo logramos construyendo un funtor $\mathcal{A} \xrightarrow{J_0} \mathcal{A}^\infty$ donde $1 \in \mathcal{A}^\infty$ es proyectivo débil (y por ende el funtor $\mathcal{A}^\infty \xrightarrow{[1,-]} \mathcal{E}ns$ es regular) tal que ambos J_0 y $[1,-]$ son conservativos sobre monomorfismos cuyo codominio tiene soporte global. La construcción del funtor $\mathcal{A} \xrightarrow{J_0} \mathcal{A}^\infty$ se realiza en dos etapas.

En la primer etapa construimos un funtor $\mathcal{A} \xrightarrow{j} \mathcal{A}'$ en donde "agregamos" una *sección global genérica* para cada objeto de $\mathcal{A}$ con soporte global (subsección 0.5.1). La idea de base en ésta construcción yace en la siguiente construcción de una una sección global genérica para un objeto de soporte global $B \overset{e}{\twoheadrightarrow} 1 \in \mathcal{A}$. Si tuvieramos una *elección* de pullbacks a lo largo de e obtendríamos un funtor regular fiel $\mathcal{A} \xrightarrow{e^*} \mathcal{A}/B$ y una sección global $e(B)$ descripta en el siguiente diagrama.

$$\begin{tikzcd}
e^*(1) = B \arrow[rr, dashed, "\Delta_B"] \arrow[dr, "1_B"'] & \equiv & B \times B = e^*(B) \arrow[dl, "\pi_2"] \\
& B &
\end{tikzcd}$$

La sección Δ_B es genérica en el siguiente sentido:

Si $A \overset{m}{\rightarrowtail} B \in \mathcal{A}$ *es tal que* Δ_B *se levanta a lo largo de* m,

$$\begin{tikzcd}
A \times B \arrow[r, tail, "e^*(m)"] & B \times B \\
\equiv & B \arrow[u, "\Delta_B"'] \arrow[ul, dotted, "\exists"]
\end{tikzcd}$$

se sigue que m es un isomorphismo.

Ésto significa que *si Δ_B se levanta a A, entonces* ***toda*** *sección global se levanta a A* (Nota aparte, en nuestra demostración no supondremos que tal elección de pullbacks pueda hacerse). Agregar éstas secciones a todo objeto de soporte global de $\mathcal{A}$ nos lleva a calcular el colimite del siguiente seudo diagram en $\mathcal{C}at$ (es quiere decir que el diagrama conmuta salvo único isomorfismo).

$$\begin{tikzcd} & \mathcal{A}/B \arrow[dr] & \\ \mathcal{A} \arrow[ur] \arrow[rr] \arrow[dr] & & \mathcal{A}/B \times B' = (\mathcal{A}/B)/B' = (\mathcal{A}/B')/B \\ & \mathcal{A}/B' \arrow[ur] & \end{tikzcd} \tag{0.1.2}$$

Construimos una fibración cuyas fibras son isomorfas la las categorías slice $\mathcal{A}/B$ (de hecho como no asumimos que haya una elección de productos la fibras no pueden ser una categoría slice sino que deben ser *multislice* 0.2.2) y cuya base cofiltrante "contiene" de algún modo la categoría que indexa el seudo diagrama de arriba. La inclusión de la fibra $\mathcal{A}/1$ en el colimite de la fibración es lo que tomamos como $\mathcal{A} \xrightarrow{j} \mathcal{A}'$.

La segunda estapa consiste en iterar la construcción previa y así quedarnos con un diagrama filtrante de funtores regulares al cual le calculamos su colimite filtrante $\mathcal{A}^\infty$ (de hecho no calculamos el colímite fitrante en $\mathcal{C}at$ sino que tomamos el colímite de la fibración asociada a éste diagrama que resulta ser equivalente categóricamente)(subsection 0.5.2).

$$\mathcal{A} \xrightarrow{j} \mathcal{A}' \xrightarrow{j'} (\mathcal{A}')' \longrightarrow \cdots \longrightarrow \mathcal{A}^\infty$$

Para nuestra categoría inicial $\mathcal{C}$ obtenemos así un funtor, el cual denotamos $\mathcal{C} \xrightarrow{\Gamma_1} \mathcal{E}ns$, que es conservativo sobre monomorfismos cuyo codominio tiene soporte global. En la subsección 0.6.1 construimos un funtor regular $\mathcal{C} \xrightarrow{h_S} \mathcal{E}ns$ para cada subobjeto distinguido $S \hookrightarrow 1$ que es conservativo sobre monomorfismos cuyo codominio tiene soporte en S (o sea cualquier monomorfismo $A \xrightarrow{m} B$ cuya factorización estricta de $B \longrightarrow 1$ se logra a travéz de S). Con lo cual la familia de funtores regulares $\{h_S\}$ indexada por el conjunto $Sub(1)$ de subobjetos de 1 nos provee de una familia de funtores regulares en los conjuntos mono conservativa (y por ende conservativa (section 0.2.3)).

Por último en la subsección 0.6.2 describimos una manera genérica de construir un funtor $\mathcal{C} \longrightarrow \mathcal{E}ns^{\mathcal{I}}$ a partir de una familia de funtores $\{h_i\}_{i \in I}$ que en nuestro caso nos da el resultado deseado.

Resultados

Para demostrar la version débil del teorema de Barr constructivamente desarrollamos el concepto de *Fibración Regular*, el cual no aparece en la literatura, y demostramos *constructivamente* el resultado fundamental que es que el colímite de una fibración regular con base cofiltrante es una categoria *regular*. Ésto demuestra en particular que los colímites filtrantes de categorias regulares es una categoría regular.

INTRODUCTION

The theorem we prove in this work lies in between the realms of *Embedding Theorems* and *Sufficient Points Theorems*. Embedding theorems are those that fully represent an abstract mathematical object within a concrete one of the same nature. These theorems shed light on the nature of the abstraction itself. Some embeddings where the representation is in a way *canonical* are the following:

1. *Cayley*: $G \hookrightarrow S!$, $S = Underlying\ set\,(G)$.
(representation of a group as a subgroup of the symmetric group).

2. *Stone*: $B \hookrightarrow Sub(S)$, $S = Prime\ Ideals\,(B)$.
(representation of a boolean algebra as a subalgebra of the algebra of subsets)

3. *Gelfand*: $A \hookrightarrow \mathcal{S}et(S, \mathbb{C})$, $S = Closed\ Maximal\ Ideals\,(C)$
(representation of a C^*-algebra as a subalgebra of the algebra complex valued functions)

4. *Yoneda*: $\mathcal{C} \hookrightarrow \mathcal{S}et^{\mathcal{D}}$, $\mathcal{D} = \mathcal{C}^{op}$.
(representation of a small category $\mathcal{C}$ as a full subcategory of a category of $\mathcal{S}et$-valued functors.

The next example, *Barr's Representation Theorem*, is the theorem that concerns us in this work.

Theorem *For any small regular category $\mathcal{C}$ there exists a fully faithful regular functor $\mathcal{C} \overset{h}{\hookrightarrow} \mathcal{E}ns^{\mathcal{D}}$ into a set valued functor category, where the exponent has as objects the set $Sub(1)$ of subobjects of* 1.

Barr's original proof as well as all known proofs of this theorem are highly not constructive, using transfinite induction and the axiom of choice. The purpose of this work is to develop a constructive proof of a weaker form of Barr's theorem, namely that the functor h is *conservative* (reflects isomorphisms). This is in fact a *Sufficient Points Theorem*.

After the leading work of William Lawvere *completeness theorems* of logical theories were formulated in categorical terms as *Sufficient Points Theorems*. Informally, given a model $A \in \mathcal{C}$ of a theory $\mathcal{T}$ in an *appropiate* category $\mathcal{C}$, any formula $\varphi(x)$ has an extension in A which is a subobject $[\![x \in A \mid \varphi(x)\ holds\,]\!] \hookrightarrow A$. The idea is to associate to a theory $\mathcal{T}$ an *appropiate* category $\mathcal{C}_{\mathcal{T}}$ equipped with a model (*the generic model*) $G_{\mathcal{T}}$ of $\mathcal{T}$ that is generic in two senses:

1. It furnishes a (tautological by the very construction of $\mathcal{C}_{\mathcal{T}}$) completeness theorem for the theory $\mathcal{T}$. That is, a formula $\varphi(x)$ is provable in $\mathcal{T}$ if and only if $[\![x \in G_{\mathcal{T}} \mid \varphi(x)\ holds\,]\!] = G_{\mathcal{T}}$.

2. For any model $A \in \mathcal{C}$ of $\mathcal{T}$ in an appropiate category, there exist a unique *appropiate* functor $\mathcal{C}_{\mathcal{T}} \xrightarrow{F} \mathcal{C}$ such that $F(G_{\mathcal{T}}) = A$.

Clearly a *conservative* (thus a *monic-conservative*, i.e. if a monomorphism is sent by every functor in the family to an isomorphism it is itself an isomorphism) (see 0.2.3) family of appropriate set valued functors $\mathcal{C}_{\mathcal{T}} \xrightarrow{F} \mathcal{E}ns$ (in some contexts called *points*) yields a completeness theorem.

For first order intuicionistic geometric logic (that is admitting the intuicionistic propositional calculus and only the existencial quantifier), the appropriate categories are exactly the *regular categories* and the appropriate functors are *regular functors*. In this way, the weak version of Barr's theorem yields a completeness theorem for these logics using the following argument. Given any regular category $\mathcal{C}$ we wish to see that the family of regular functors $\mathcal{C} \longrightarrow \mathcal{E}ns$ is conservative. Barr's weak theorem guarantees us that there is a conservative regular functor $\mathcal{C} \xrightarrow{h} \mathcal{E}ns^{Sub(1)}$. Evaluations $\mathcal{E}ns^{Sub(1)} \longrightarrow \mathcal{E}ns$ are regular and the family of evaluations is conservative. The claim follows.

Independently of these considerations the purpose of this work is to develop a *constructive* proof of the weaker form of Barr's Theorem.

Outline of the Construction

For the construction we followed a guideline set by André Joyal in some unpublished talks given in Montreal in the early seventies. His proof was inspired in reinterpreting Leon Henkin's proof by adding constants of the *Gödel Completeness Theorem* of first order logic.

To carry out the whole proof constructively we need the additional hypothesis that the regular category $\mathcal{C}$ possesses a distinguished terminal object 1 and that distinguished subobject representatives for every subobject class exist in $\mathcal{C}$. These hypothesis will not affect our desired range of applications to logic. That is to say, for first order intuicionistic geometric logic theories the categories $\mathcal{C}_{\mathcal{T}}$ verify these additional hypothesis.

We start by constructing for any regular category $\mathcal{A}$ that possesses a distinguished terminal object 1 a regular functor $\mathcal{A} \longrightarrow \mathcal{E}ns$ that is conservative over monics with globally supported codomain (Section 0.5). This is achieved by constructing a regular functor $\mathcal{A} \xrightarrow{J_0} \mathcal{A}^\infty$ where $1 \in \mathcal{A}^\infty$ is weakly projective (thus the functor $\mathcal{A}^\infty \xrightarrow{[1,-]} \mathcal{E}ns$ is regular) and such that both J_0 and $[1,-]$ are conservative over monics with globally supported codomain. The construction of the functor $\mathcal{A} \xrightarrow{J_0} \mathcal{A}^\infty$ is carried out in two stages.

In the first stage we construct a functor $\mathcal{A} \xrightarrow{j} \mathcal{A}'$ in which we "add" a *generic global section* for each globally supported object in $\mathcal{A}$ (subsection 0.5.1). The basis of this construction lies on the following *idea* of how to construct a generic global section for a chosen globally supported object $B \overset{e}{\twoheadrightarrow} 1 \ \in \mathcal{A}$. If we had a *choice* of pullbacks along e we would have a faithful regular functor $\mathcal{A} \xrightarrow{e^*} \mathcal{A}/B$ and a global section of $e(B)$ described in the following diagram.

$$\begin{tikzcd} e^*(1) = B \arrow[rr, dashed, "\Delta_B"] \arrow[dr, "1_B"'] \arrow[rr, phantom, shift right=4ex, "\equiv"] & & B \times B = e^*(B) \arrow[dl, "\pi_2"] \\ & B & \end{tikzcd}$$

The section Δ_B is generic in the following sense:

If $A \overset{m}{\rightarrowtail} B \in \mathcal{A}$ *is such that* Δ_B *lifts along* m,

$$\begin{tikzcd} A \times B \arrow[r, tail, "e^*(m)"] & B \times B \\ & B \arrow[u, "\Delta_B"'] \arrow[ul, dotted, "\exists"] \arrow[ul, phantom, shift right=2ex, "\equiv"] \end{tikzcd}$$

it follows that m *is an isomorphism.*

That is to say *if* Δ_B *lifts to* A*, then* ***every*** *global section will lift to* A (As a note, in our proof we will not suppose that a choice of pullbacks can be made). In order to add these sections for every globally supported object of $\mathcal{A}$ leads to the calculation of the colimit of the following pseudo diagram in $\mathcal{C}at$ (that is the diagram commutes up to a unique isomorphism).

$$\begin{tikzcd} & \mathcal{A}/B \arrow[dr] & \\ \mathcal{A} \arrow[ur] \arrow[r] \arrow[dr] & \mathcal{A}/B \times B' & = (\mathcal{A}/B)/B' = (\mathcal{A}/B')/B \\ & \mathcal{A}/B' \arrow[ur] & \end{tikzcd} \tag{0.1.3}$$

We construct a fibration whose fibres are isomorphic to the slice categories $\mathcal{A}/B$ (in fact, since we do not assume there is a choice of products the fibres cannot be single sliced categories but must be *multislice* categories 0.2.2) and whose cofiltered base category contains in *some* way the indexing category of the pseudo diagram above. The inclusion of the fibre $\mathcal{A}/1$ in the colimit of the fibration is what we take as $\mathcal{A} \xrightarrow{j} \mathcal{A}'$.

The second stage consists of iterating the previous construction, yielding a filtered diagram of regular functors and calculating the filtered colimit $\mathcal{A}^{\infty}$ of this diagram (In fact we do not take the filtered colimit $\mathcal{C}at$ but take the colimit of the fibration associated to this diagram which is in fact *equivalent* to it) (subsection 0.5.2).

$$\mathcal{A} \xrightarrow{j} \mathcal{A}' \xrightarrow{j'} (\mathcal{A}')' \longrightarrow \cdots \longrightarrow \mathcal{A}^{\infty}$$

For our initial category $\mathcal{C}$ we have thus obtained a regular functor, which we label $\mathcal{C} \xrightarrow{\Gamma_1} \mathcal{E}ns$, that is conservative over monics with globally supported codomain. In subsection 0.6.1 we construct a regular functor $\mathcal{C} \longrightarrow \mathcal{E}ns$ for each distinguished subobject $S \hookrightarrow 1$ the is conservative over monics whose codomain is supported in S(This is a monic $A \xrightarrow{m} B$ for which the strict factorization of $B \longrightarrow 1$ is through S). Thus that family of regular functors indexed by the the set $Sub(1)$ of subobjects of 1 yields a monic conservative family of functors(thus a conservative family (section 0.2.3)).

Lastly in subsection 0.6.2 we describe generic method of constructing a functor $\mathcal{C} \longrightarrow \mathcal{E}ns^{\mathcal{I}}$ from a given family of functors $\{h_i\}_{i \in I}$ which in our case will yield the desired result.

Results

In order to prove the weaker form of Barr's theorem constructively we developed the concept of *Regular Fibration* which does not apear in the literature and proved *constructively* the fundamental result that the colimit of a regular fibration over a cofiltered category yields a *regular* category. This in fact has as particular case that a filtered colimit of regular categories is a regular category.

Contents

0.2 CATEGORICAL PRELIMINARIES

0.2.1 Regular Categories

In this section we will introduce the basic concepts we need to define what a regular category is and prove some basic properties these categories satisfy. We will denote with $\mathcal{C}at$ the category of small categories whose morphisms are functors and $\mathcal{E}ns$ the category of small sets.

Definition

Let $\mathcal{C}$ be a small category. In what follows all the diagrams are in $\mathcal{C}$.

Definition 0.2.1. *$\mathcal{C}$ is* ***finitely complete*** *if finite limits exist.*

Remark 0.2.2. $\mathcal{C}$ is finitely complete if and only if pullbacks exist and a terminal object exists.

We will denote $\mathcal{C}at_{fl}$ the subcategory of $\mathcal{C}at$ whose objects are finitely complete categories and whose morphisms are limit-preserving functors.

Observation. A morphism $X \xrightarrow{f} Y$ is monic if and only if the following diagram is a pullback.

$$\begin{tikzcd}
X \arrow[r, "id_X"] \arrow[d, "id_X"'] \arrow[dr, phantom, "p.b."] & X \arrow[d, "f"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Proposition 0.2.3. *If $\mathcal{C} \xrightarrow{F} \mathcal{D} \in \mathcal{C}at_{fl}$, then F preserves monics.*

Proof. It is an immediate consequence from the previous observation. □

For what follows it is useful to have the following result present.

Lemma 0.2.4. *If a monic $X \xrightarrow{f} Y$ admits a section, that is there exists a morphism $Y \xrightarrow{h} X$ such that $fh = id_Y$, then f is an isomorphisms.*

Proof. We have that $f \cdot hf = fh \cdot f = id_Y \cdot f = f \cdot id_X$. Since f is monic it follows that $hf = id_X$. □

Definition 0.2.5. *For morphisms $X \xrightarrow{f} Y$ and $X \xrightarrow{g} Z$ we say that g is* ***compatible*** *with f if for every pair $W \underset{y}{\overset{x}{\rightrightarrows}} X \in \mathcal{C}$ such that $fx = fy$ it follows that $gx = gy$.*

We will also express this by saying that g is f-compatible.

Definition 0.2.6. *A morphism $X \xrightarrow{f} Y$ is a* ***strict epimorphism*** *if for every f-compatible morphism $X \xrightarrow{g} Z$ there exists a unique morphism $Y \xrightarrow{h} Z$ such that $hf = g$.*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\forall\ f-compatible\ g"'] \arrow[dr, phantom, "\equiv"] & Y \arrow[dl, dashed, "\exists!"] \\
Z &
\end{tikzcd}$$

We will use the symbol $\twoheadrightarrow$ to label strict epimorphisms.

Observation. Strict epimorphism are epimorphism.

Observation. A morphism $X \xrightarrow{f} Y$ is monic if and only if id_X is f-compatible.

Proposition 0.2.7. *If $X \xrightarrow{f} Y$ is monic and a strict epimorphism , then f is an isomorphism.*

Proof. We will prove that f admits a section. From the previous observation we know there is a unique morphism $Y \xrightarrow{h} X$ such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "id_X"'] \arrow[dr, phantom, "\equiv"] & Y \arrow[dl, "h"] \\ X & \end{tikzcd}$$

Thus we have that $fh \cdot f = f \cdot hf = f \cdot id_X = id_Y \cdot f$. Since f is epic we conclude that $fh = id_Y$. The result follows.

$\square$

We will use the symbol $\rightarrowtail$ to label monomorphisms. A monomorphism $A \rightarrowtail Y$ will be called a *subobject* of Y. Consider the following commutative diagram.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "e"'] & \equiv & Y \\ & A \arrow[ur, tail, "m"'] & \end{tikzcd}$$

We will say the pair $X \xrightarrow{e} A \xrightarrowtail{m} Y$ is a *factorization of f through a subobject of Y*. If e is a strict epimorphism we call the pair $X \overset{e}{\twoheadrightarrow} A \xrightarrowtail{m} Y$ a *strict* factorization of f.

Remark 0.2.8. A strict factorization $X \overset{e}{\twoheadrightarrow} A \xrightarrowtail{m} Y$ of f is universal with respect to every factorization of f throught subobjects of Y. That is if $X \xrightarrow{e'} A' \xrightarrowtail{m'} Y$ is a factorization of f through a subobject of Y, then there exists a unique $A \xrightarrow{s} A'$ such that $se = e'$ and $m's = m$.

Consider the following pullback diagram.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "p.b."] & B \arrow[d, "f"] \\ D \arrow[r, "g"'] & C \end{tikzcd}$$

We will call f' a *pullback of f along g* or simply a *pullback of f*.

Definition 0.2.9. *Strict epimorphisms are* ***stable*** *if a pullback of a strict epimorphism is a strict epimorphism.*

Definition 0.2.10. *$\mathcal{C}$ is* ***regular*** *if finite limits exist, strict factorizations exist for every morphism and strict epimorphisms are stable.*

We will denote $\mathcal{R}eg$ the subcategory of $\mathcal{C}at$ whose objects are regular categories and whose morphisms are limit-preserving functors that preserve strict factorizations. We call the morphisms in $\mathcal{R}eg$ *regular* functors. $\mathcal{R}eg$ is in fact a subcategory of $\mathcal{C}at_{fl}$.

A few facts

In what follows $\mathcal{C}$ represents a regular category.

Proposition 0.2.11. *Strict epimorphisms are closed under composition.*

Proof. Take composable strict epimorphisms $X \overset{f}{\twoheadrightarrow} Y \overset{g}{\twoheadrightarrow} Z$ and take $X \overset{e}{\twoheadrightarrow} A \overset{m}{\rightarrowtail} Z$ a strict factorization of gf. We will prove m is an isomorphism. It suffices to prove that m admits a section. We have that e is f-compatible because m is monic. Thus there exists a unique h such that $hf = e$.

$$\begin{tikzcd}
X \arrow[r, two heads, "f"] \arrow[dr, two heads, "e"'] & Y \arrow[r, two heads, "g"] \arrow[d, dashed, "\exists! h"'] \arrow[dl, phantom, "\equiv"] \arrow[dr, phantom, "\equiv"] & Z \\
& A \arrow[ur, tail, "m"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "P"] \arrow[dr, "t"] & Y \arrow[r, "u"] \arrow[d] & Z \\
& A \arrow[ur, "a"] &
\end{tikzcd}$$

Since f is epic we also have that $mh = g$. Because m is monic h is g-compatible. Thus there exists a unique $Z \xrightarrow{h'} A$ such that $h'g = h$. Composing with m we obtain that $mh' \cdot g = id_Y \cdot g$ and since g is epic we have that $mh' = id_Y$. The result follows.

□

Definition 0.2.12. *An object X is* ***globally supported*** *if for every terminal object 1 the unique morphism $X \longrightarrow 1$ is a strict epimorphism.*

Observation. If for any terminal object 1 the unique morphism $X \longrightarrow 1$ is a strict epimorphism, then X is globally supported.

Let $[n] = \{0, 1, 2, ..., n-1\}$ denote the finite ordinal. Take a product $\{P \xrightarrow{\pi_i} B_i\}_{i \in [n]}$ of the family $\{B_i\}_{i \in [n]}$.

Observation. For $B_n \in \mathcal{C}$ we can calculate a product of the family $\{B_i\}_{i \in [n+1]}$ taking the following pullback.

$$\begin{tikzcd} P' \arrow[r] \arrow[d, "\hat{\pi}_n"'] \arrow[dr, phantom, "p.b."] & B_n \arrow[d] \\ P \arrow[r] & 1 \end{tikzcd}$$

The morphism $\hat{\pi}_n$ can be interpreted as *truncating the* n^{th} *coordinate.*

Lemma 0.2.13. *If B_n is globally supported, then truncating the n^{th} coordinate $P' \xrightarrow{\hat{\pi}_n} P$ is a strict epimorphism.*

Proof. It is immediate from the previous diagram. □

Corollary 0.2.14. *If the objects of the family $\{B_i\}_{i\in[n]}$ are globally supported, then every projection $P \xrightarrow{\hat{\pi}_i} B_i$ is a strict epimorphism. Consequently P is globally supported.*

Proof. The result follows from that fact that strict epimorphisms are closed under composition and that truncating one coordinate at a time we reach to the desired projection. □

The following result will be useful throughout the thesis.

Proposition 0.2.15. *If $\mathcal{D}$ is a regular category and $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is a limit-preserving functor, the the following are equivalent:*

- *F is a regular functor.*
- *F preserves strict epimorphisms.*

Proof. It follows from Proposition 0.2.3. □

0.2.2 Multislice Categories of Regular Categories

In this section we will define what a multislice category is and prove that a multislice category of a finitely complete category is finitely complete. More so a multislice category of a regular category is regular.

Definition 0.2.16. *For a category $\mathcal{C}$ and a family $\{B_i\}_{i\in[n]}$ of objects of $\mathcal{C}$ we define the **multi-slice category** $\mathcal{C}_{/\{B_i\}_{i\in[n]}}$ whose objects are families $\{X \xrightarrow{x_i} B_i\}_{i\in[n]}$ of arrows of $\mathcal{C}$ and morphisms $\{X \xrightarrow{x_i} B_i\}_{i\in[n]} \xrightarrow{f} \{Y \xrightarrow{y_i} B_i\}_{i\in[n]}$ are arrows $X \xrightarrow{f} Y$ in $\mathcal{C}$ such that for every $i \in [n]$*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "x_i"'] & \equiv & Y \arrow[dl, "y_i"] \\ & B_i & \end{tikzcd}$$

Remark 0.2.17. The domain functor $\mathcal{C}_{/\{B_i\}_{i\in I}} \xrightarrow{\Sigma} \mathcal{C}$ is faithful, conservative and preserves pullbacks. It follows that Σ reflects monomorphisms and pullbacks (0.2.30).

Finite completeness

Proposition 0.2.18. *If pullbacks exist in $\mathcal{C}$, then pullbacks exist in $\mathcal{C}_{/\{B_i\}_{i\in[n]}}$.*

Proof. For each i consider the following commutative diagram in $\mathcal{C}$

$$\begin{tikzcd}
P \arrow[rr, "\pi_1"] \arrow[dr, "\pi_2"'] & & X \arrow[dr, "f"] \arrow[dd, "x_i"'] & \\
& Y \arrow[rr, "g"] \arrow[dr, bend right, "y_i"'] \arrow[ur, phantom, "p.b."] & & Z \arrow[dl, bend left, "z_i"] \\
& & B_i &
\end{tikzcd}$$

This diagram determines a cone for f and g in $\mathcal{C}_{/\{B_i\}_{i\in[n]}}$. It follows from Remark 0.2.17 that this cone is a pullback. □

Observation. If distinguished pullbacks exist in $\mathcal{C}$, then distinguished pullbacks exist in $\mathcal{C}_{/\{B_i\}_{i\in[n]}}$ and Σ preserves them.

Remark 0.2.19. If a product of the family $\{B_i\}_{i\in[n]}$ exists, then the family of its projections $\{P \xrightarrow{\pi_i} B_i\}_{i\in[n]}$ form a terminal object of the multislice category.

Corollary 0.2.20. *If $\mathcal{C}$ is finitely complete, then $\mathcal{C}_{/\{B_i\}_{i\in[n]}}$ is finitely complete.*

Proof. It follows from Proposition 0.2.18 and Remark 0.2.19. □

Regularity

Remark 0.2.21. Σ preserves compatibility. In other words if we consider the following commutative diagrams in $\mathcal{C}$

$$\begin{tikzcd}
Z \arrow[r, shift left, "x"] \arrow[r, shift right, "y"'] & X \arrow[rr, "f"] \arrow[dr, "g"] \arrow[ddr, bend right, "x_i"'] & & Y \arrow[ddl, bend left, "y_i"] \\
& & Z \arrow[d, "z_i"] & \\
& & B_i &
\end{tikzcd}$$

then g is f-compatible in $\mathcal{C}_{/\{B_i\}_{i\in[n]}}$ if and only if g is f-compatible in $\mathcal{C}$. Moreover Σ reflects and preserves strict epimorphisms.

Proposition 0.2.22. *If every morphism in $\mathcal{C}$ admits a strict factorization, then so does every morphism in $\mathcal{C}_{/\{B_i\}_{i\in[n]}}$.*

Proof. For $\{X \xrightarrow{x_i} B_i\}_{i\in[n]} \xrightarrow{f} \{Y \xrightarrow{y_i} B_i\}_{i\in[n]}$ take a strict factorization of f in $\mathcal{C}$.

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "e"] \arrow[ddr, bend right, "x_i"'] & {} & Y \arrow[ddl, bend left, "y_i"] \\
& A \arrow[ur, tail, "m"'] \arrow[d, dashed, "t_i"] \arrow[u, phantom, "\equiv"] & \\
& B_i &
\end{tikzcd}$$

Since x_i is e-compatible, there is a unique arrow $T \xrightarrow{t_i} B_i$ such that $t_i e = x_i$. Since e is epic it follows that $my_i = t_i$. We conclude from Remarks 0.2.17 and 0.2.21 that this is a strict factorization. ☐

Observation. If every morphism in $\mathcal{C}$ has a distinguished strict factorization, then so does every morphism in $\mathcal{C}_{/\{B_i\}_{i\in[n]}}$ and Σ preserves them.

Theorem 0.2.23. *If $\mathcal{C}$ is regular, then $\mathcal{C}_{/\{B_i\}_{i\in[n]}}$ is regular.*

Proof. The only thing left to verify is that strict epimorphisms are stable. But this follows from the fact that the domain functor Σ preserves and reflects pullbacks and strict epimorphisms. ☐

0.2.3 Families of Functors With Common Domain

In this section we will establish some generalities on the collective behaviour that a family of functors may have. We will prove that for a regular category $\mathcal{C}$, a *monic-conservative* family of regular functors is faithful and conservative.

Let $\mathcal{F}$ be a family of functors with common domain $\mathcal{C}$.

Definition 0.2.24. *We will say that pullbacks (pushouts,...) are* ***preserved*** *by $\mathcal{F}$ if for every $F \in \mathcal{F}$ the functor F preserves pullbacks (pushouts,...).*

Definition 0.2.25. *We will say that monomorphisms (epimorphisms,...) are* ***reflected*** *by $\mathcal{F}$ if for every arrow $X \xrightarrow{u} Y \in \mathcal{C}$ such that for every $F \in \mathcal{F}$ its image Fu is a monomorphism (epimorphism,...), it follows that u is a monomorphism (epimorphism,...).*

Definition 0.2.26. *$\mathcal{F}$ is* ***conservative*** *if $\mathcal{F}$ reflects isomorphisms.*

Definition 0.2.27. *$\mathcal{F}$ is* ***monic (epic)-conservative*** *if for every monic (epic) $X \xrightarrow{u} Y \in \mathcal{C}$ such that for every $F \in \mathcal{F}$ its image Fu is an isomorphism, it follows that u is an isomorphism.*

Observation. A conservative family is monic (epic)-conservative.

Definition 0.2.28. *$\mathcal{F}$ is* ***faithful*** *if for every pair $X \underset{v}{\overset{u}{\rightrightarrows}} Y \in \mathcal{C}$ such that for every $F \in \mathcal{F}$ their images are equal ($Fu = Fv$), it follows that $u = v$.*

Lemma 0.2.29. *If $\mathcal{F}$ is faithful, then $\mathcal{F}$ reflects monics(epics).*

Proof. Let $X \xrightarrow{u} Y$ in $\mathcal{C}$ be such that for every $F \in \mathcal{F}$, Fu is monic. Suppose we have $A \underset{y}{\overset{x}{\rightrightarrows}} X$ such that $ux = uy$. Then for every $F \in \mathcal{F}$, $Fu \cdot Fx = Fu \cdot Fy$ and since Fu is monic, it follows that for every $F \in \mathcal{F}$, $Fx = Fy$. Thus $x = y$. The dual proposition follows. ☐

Lemma 0.2.30. *If $\mathcal{C}$ has pullbacks (pushouts), $\mathcal{F}$ preserves them and $\mathcal{F}$ is monic (epic)-conservative, then $\mathcal{F}$ reflects monics (epics).*

Proof. Let $X \xrightarrow{u} Y$ be such that for every $F \in \mathcal{F}$, Fu is monic. We will prove that the following diagram is a pullback

$$\begin{tikzcd} X \arrow[r, "id_X"] \arrow[d, "id_X"'] & X \arrow[d, "u"] \\ X \arrow[r, "u"'] & Y \end{tikzcd}$$

Take a pullback in $\mathcal{C}$ and δ as follows:

$$\begin{tikzcd} X \arrow[dr, dashed, "\exists!\delta"'] \arrow[drr, bend left, "id_X"] \arrow[ddr, bend right, "id_X"'] \arrow[drr, phantom, "\equiv", near start] \arrow[ddr, phantom, "\equiv", near start] & & \\ & P \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "p.b."] & X \arrow[d, "u"] \\ & X \arrow[r, "u"'] & Y \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[dr, dashed, "\exists!\delta"'] \arrow[drr, bend left, "id_X"] \arrow[ddr, bend right, "id_X"'] \arrow[drr, phantom, "\equiv", near start] \arrow[ddr, phantom, "\equiv", near start] & & \\ & P \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "p.b."] & X \arrow[d, "u"] \\ & X \arrow[r, "u"'] & Y \end{tikzcd}$$

From this diagram we see δ is monic and for every $F \in \mathcal{F}$, $F\delta$ is the isomorphism between the corresponding pullback diagrams. Therefore δ is an isomorphism. The dual proposition follows. □

Remark 0.2.31. The proof of Lemma 0.2.30 gives us a technique to prove that under the additional hypothesis of preserving limits (colimits) of a certain type, conservative families reflect limits (colimits) of that type.

Proposition 0.2.32. *If $\mathcal{C}$ has pullbacks (pushouts), $\mathcal{F}$ preserves them and $\mathcal{F}$ is monic (epic)-conservative, then $\mathcal{F}$ is conservative.*

Proof. Observe that monic (epic)-conservative families that reflect monics (epics) are conservative. □

Proposition 0.2.33. *If $\mathcal{C}$ has equalizers (coequalizers), $\mathcal{F}$ preserves them and $\mathcal{F}$ is monic/strict-monic (epic/strict-epic)-conservative, then $\mathcal{F}$ is faithful.*

Proof. Take $X \underset{v}{\overset{u}{\rightrightarrows}} Y$ in $\mathcal{C}$ such that for every $F \in \mathcal{F}$ their images are equal ($Fu = Fv$). An equalizer $E \xrightarrow{e} X$ of u and v is monic and its image Fe is an equalizer of Fu and Fv which are equal, so Fe is an isomorphism. Thus e is an isomorphism and consequently $u = v$. For the strict case we need only note that equalizers are strict monics. The dual propositions follow. □

Proposition 0.2.34. *If $\mathcal{C}$ has equalizers (coequalizers), $\mathcal{F}$ preserves them and $\mathcal{F}$ is monic (epic)-conservative, then $\mathcal{F}$ is conservative.*

Proof. It follows from Lemma 0.2.29 and the observation made in Proposition 0.2.32. □

Corollary 0.2.35. *If $\mathcal{C}$ has equalizers, $\mathcal{F}$ preserves them and $\mathcal{F}$ is monic (epic)-conservative, then $\mathcal{F}$ is conservative and faithfull.*

Remark 0.2.36. If $\mathcal{C}$ is regular and $\mathcal{F}$ the set of all regular functors $\mathcal{C} \longrightarrow \mathcal{E}ns$ is monic conservative, it follows that it is conservative and faithful.

Remark 0.2.37. Even under the strictest limit-preserving conditions we will not be able to guarantee that a faithful family is conservative in any sense. Take the following counterexample: Let $\mathcal{C} = \{0 \xrightarrow{u} 1\}$ and take the family whose only member is the functor $\mathcal{C} \xrightarrow{F} \{*\}$. $\mathcal{C}$ is a regular category that in fact has all limits and colimits, F is regular and preserves all limits and colimits , F is faithful but nevertheless does not reflect the isomorphism Fu.

Proposition 0.2.38. *If in $\mathcal{C}$ every bimorphism (a morphism that is both epic and monic) is an isomorphism and $\mathcal{F}$ is faithful, then $\mathcal{F}$ is conservative.*

0.2.4 Weakly Projective Objects

Here we give a characterization of which hom-functors of a regular category $\mathcal{A}$ are regular functors.

Observation. A hom-functor $hom_{\mathcal{A}}(A, -)$ always preserves limits but doesn't necessarily preserve strict epimorphisms.

Definition 0.2.39. *An object A in a regular category $\mathcal{A}$ is **weakly projective** if the functor $\mathcal{A} \xrightarrow{hom_{\mathcal{A}}(A,-)} \mathcal{E}ns$ preserves strict epimorphisms.*

Remark 0.2.40. A is weakly projective if and only if given any strict epimorphism $X \overset{u}{\twoheadrightarrow} Y$ it follows that

$$\begin{tikzcd} X \arrow[r, two heads, "u"] & Y \\ & A \arrow[u, "\forall t"'] \arrow[ul, dotted, bend left, "\exists"] \arrow[ul, phantom, "\equiv", shift left=3] \end{tikzcd}$$

We call such a dotted arrow a *lift* of t along u.

Lemma 0.2.41. *A is weakly projective if and only if every strict epimorphism $X \overset{u}{\twoheadrightarrow} A$ admits a section.*

Proof. If A is weakly projective, a lift of 1_A along the given strict epimorphism yields a section. For the converse let $X \overset{u}{\twoheadrightarrow} Y$ be a strict epimorphism. For a given $A \xrightarrow{t} Y$ take a pullback of u along t.

$$\begin{tikzcd} X \arrow[r, two heads, "u"] & Y \\ P \arrow[u, dotted, "p_1"] \arrow[r, dotted, two heads, "p_2"'] \arrow[ur, phantom, "p.b."] & A \arrow[u, "t"'] \end{tikzcd}$$

Since p_2 is a strict epimorphism it admits a section v. The morphism $p_1 v$ is a lift of t along u. □

0.3 PREFIBRED CATEGORIES

0.3.1 Basic Notions

Let $\mathcal{G}$ be a category. We denote $\mathcal{C}at/\mathcal{G}$ the category whose objects are functors $\mathcal{E} \xrightarrow{F} \mathcal{G}$ and whose morphisms from $\mathcal{E} \xrightarrow{F} \mathcal{G}$ to $\mathcal{E}' \xrightarrow{F'} \mathcal{G}$ are functors $\mathcal{E} \xrightarrow{f} \mathcal{E}'$ such that $F'f = F$.

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "f"] \arrow[dr, "F"'] & \arrow[d, phantom, "\equiv"] & \mathcal{E}' \arrow[dl, "F'"] \\
& \mathcal{G} &
\end{tikzcd}$$

We will call f a $\mathcal{G}$-functor and with an abuse of language we will denote with $hom_{\mathcal{G}}(\mathcal{E}, \mathcal{E}')$ the set of $\mathcal{G}$-functors from F to F'.

Definition 0.3.1. *For a functor $\mathcal{E} \xrightarrow{F} \mathcal{G}$ and $\alpha \in \mathcal{G}$, the **fibre** over α is the subcategory of $\mathcal{E}$ given by the preimage of F of the punctual category defined by α. That is the subcategory of $\mathcal{E}$ whose objects are $X \in \mathcal{E}$ such that $F(X) = \alpha$ and morphisms $s \in \mathcal{E}$ such that $F(s) = id_\alpha$. We will use $\mathcal{E}_\alpha$ to label this category.*

Observation. A $\mathcal{G}$-functor induces functors between fibres. That is to say if $f \in hom_{\mathcal{G}}(\mathcal{E}, \mathcal{E}')$ and $\alpha \in \mathcal{G}$, then f sends $\mathcal{E}_\alpha$ into $\mathcal{E}'_\alpha$. We will use $\mathcal{E}_\alpha \xrightarrow{f_\alpha} \mathcal{E}'_\alpha$ to denote these restriction.

For $\alpha \xrightarrow{\varphi} \beta$ in $\mathcal{G}$, $A \in \mathcal{E}_\alpha$ and $B \in \mathcal{E}_\beta$ we will denote with $hom_\varphi(A, B)$ the set of morphisms $s \in hom_{\mathcal{E}}(A, B)$ such that $F(s) = \varphi$. We will represent an element of this set with a double diagram.

$$\begin{tikzcd}
A \arrow[r, "s"] & B \\
\\
\alpha \arrow[r, "\varphi"] & \beta
\end{tikzcd}$$

We will refer to arrows in a fibre as *vertical arrows*, and if an arrow is in $\mathcal{E}_\alpha$, we draw the arrow vertically above α. We write hom_α instead of hom_{id_α}.

For $s \in hom_\varphi(A, B)$, $\gamma \xrightarrow{\psi} \alpha \xrightarrow{\varphi} \beta \in \mathcal{G}$ and $X \in \mathcal{E}_\gamma$ we will use the notation $hom_\psi(X, A) \xrightarrow{s_*} hom_{\varphi\psi}(X, B)$ to indicate the function defined by postcomposing with s.

$$\begin{tikzcd}
X \arrow[dr, "g"'] \arrow[drr, dashed, "s_*(g)"] & & \\
& A \arrow[r, "s"'] \arrow[u, phantom, "\equiv"] & B \\
\\
\gamma \arrow[r, "\psi"'] & \alpha \arrow[r, "\varphi"'] & \beta
\end{tikzcd}$$

Definition 0.3.2. *For $s \in hom_\varphi(A, B)$ we say that s is **cartesian** (or **cartesian over** φ) if for every $X \in \mathcal{E}_\alpha$ the function*

$$hom_\alpha(X, A) \xrightarrow{s_*} hom_\varphi(X, B)$$

is a bijection.

$$
\begin{tikzcd}
X \arrow[d, dashed, "\exists!"'] \arrow[dr, bend left, "\forall"] \arrow[dr, phantom, "\equiv"] & \\
A \arrow[r, "s"'] & B
\end{tikzcd}
$$

$$
\begin{tikzcd}
\alpha \arrow[r, "\varphi"'] & \beta
\end{tikzcd}
$$

Definition 0.3.3. *For $s \in hom_\varphi(A, B)$ we say that s is **strong cartesian** (or **strong cartesian over** φ) if for every $\gamma \xrightarrow{\psi} \alpha \xrightarrow{\varphi} \beta \in \mathcal{G}$ and for every $X \in \mathcal{E}_\gamma$ the function*

$$hom_\psi(X, A) \xrightarrow{s_*} hom_{\varphi\psi}(X, B)$$

is a bijection.

$$
\begin{tikzcd}
X \arrow[dr, dashed, "\exists!"'] \arrow[drr, bend left, "\forall"] \arrow[drr, phantom, "\equiv"] & & \\
& A \arrow[r, "s"'] & B
\end{tikzcd}
$$

$$
\begin{tikzcd}
\gamma \arrow[r, "\psi"'] & \alpha \arrow[r, "\varphi"'] & \beta
\end{tikzcd}
$$

Observation. If s is strong cartesian, then s is cartesian.

Definition 0.3.4. *A functor $\mathcal{E} \xrightarrow{F} \mathcal{G}$ is **prefibred** if for every $\alpha \xrightarrow{\varphi} \beta \in \mathcal{G}$ and for every $B \in \mathcal{E}_\beta$ there exists a cartesian morphism over φ with target B. A prefibred functor is **fibred** if the set of cartesian morphisms is closed under composition.*

Remark 0.3.5. Strong cartesian morphisms are closed under composition. In a fibration every cartesian morphism is strong cartesian.

Observation. We have the dual definitions of cocartesian morphism and precofibration. We will freely use these notions and the dual theorems.

Definition 0.3.6. *A functor $\mathcal{E} \xrightarrow{F} \mathcal{G}$ is said to be **cleaved** if we are provided with a set K of cartesian morphisms that verifies that for each pair $\alpha \xrightarrow{\varphi} \beta \in \mathcal{G}$ and $X \in \mathcal{E}_\beta$ there is a unique morphism $s \in K$ over φ with target X.*

Observation. Every cleaved functor is a prefibration. Using choice we have that every prefibration admits a clivage. We will not assume such a choice has been made.

Such a set K is called a *clivage* of the functor.

Definition 0.3.7. *A cleaved prefibration $\mathcal{E} \xrightarrow{F} \mathcal{G}$ with clivage K is **split** if the morphisms in K are closed under composition.*

Observation. Every functor that admits a split clivage is a fibration. Not every fibration is split. Take as example a surjective group homomorphism $G \xrightarrow{f} H$. Interpreting the groups as punctual categories f is a fibration. In fact every morphism is cartesian. Nevertheless this fibration is split if and only if f admits a section [5].

The category of prefibrations over $\mathcal{G}$ as the subcategory of $\mathcal{C}at/\mathcal{G}$ whose objects are prefibrations and whose morphisms are $\mathcal{G}$-functors that transform cartesian morfisms into cartesian morphisms. We call these morphisms *cartesian* $\mathcal{G}$-functors and we will denote this category $Prefib(\mathcal{G})$. The category of fibrations over $\mathcal{G}$ is the full subcategory of $Prefib(\mathcal{G})$ whose objects are fibrations. We denote this category $Fib(\mathcal{G})$. We define the category of cleaved (split) prefibrations as the category whose objects are pairs $(\mathcal{E} \xrightarrow{F} \mathcal{G}, K)$ where K is a clivage for F and whose morphisms are cartesian $\mathcal{G}$-functors that preserve the clivages. We will denote this category $Cprefib(\mathcal{G})$ ($Sprefib(\mathcal{G})$). Similar notations will be used for the categories of cleaved and split fibrations.

Remark 0.3.8. In a cleaved prefibration $(\mathcal{E} \xrightarrow{F} \mathcal{G}, K)$ there is associated to $\alpha \xrightarrow{\varphi} \beta \in \mathcal{G}$ a functor $\mathcal{E}_\beta \xrightarrow{\varphi^*} \mathcal{E}_\alpha$ called the *pullback functor* along φ of the prefibration determined by the diagram below where $X \xrightarrow{m} Y \in \mathcal{E}_\beta$ and $s, t \in K$.

$$\begin{tikzcd}
\varphi^* X \arrow[r, "s"] \arrow[d, dashed, "\varphi^*(m)"'] \arrow[dr, phantom, "\equiv"] & X \arrow[d, "m"] \\
\varphi^* Y \arrow[r, "t"'] & Y
\end{tikzcd}$$

$$\alpha \xrightarrow{\varphi} \beta$$

For a general prefibration we will use a simpler version of this diagram notation to indicate we are labelling a cartesian morphism over φ with target X.

$$X^* \longrightarrow X$$

$$\alpha \xrightarrow{\varphi} \beta$$

This variation in notation is done to remind us we are making a momentary choice of a single cartesian arrow and that we do not assume to have a clivage. We may also indicate this by saying we have a cartesian morphism $X^* \longrightarrow X$ over φ.

Proposition 0.3.9. *If F is a prefibred and cofibred, then F is a fibration.*

Proof. We will prove that cartesian morphisms are strong cartesian (see Remark 0.3.5). Take $Y^* \xrightarrow{f} Y$ a cartesian morphisms over $\alpha \xrightarrow{\varphi} \beta$. Take $\gamma \xrightarrow{\psi} \alpha \xrightarrow{\varphi} \beta \in \mathcal{G}$ and $X \in \mathcal{E}_\gamma$. We will prove that $hom_\psi(X, Y^*) \xrightarrow{f_*} hom_{\varphi\psi}(X, Y)$ is a bijection. Take $X \xrightarrow{r} X_*$ a strong cocartesian morphism over ψ. The situation can be described as follows.

$$\begin{tikzcd}
X \arrow[r, "r"] \arrow[drr, dashed, "\exists! c"'] \arrow[rrrd, bend left, "\forall"] & X_* \arrow[r, phantom, "\equiv"] \arrow[dr, dashed, "\exists! a"] \arrow[d, dashed, "\exists! b"] & {} & \\
& & Y^* \arrow[r, "f"'] & Y
\end{tikzcd}$$

$$\gamma \xrightarrow{\psi} \alpha \xrightarrow{\varphi} \beta$$

The arrow c shows that f is strong cartesian.

For an alternative proof take the following commutative diagram.

$$\begin{tikzcd}
hom_{\psi}(X,Y^*) \arrow[r, "f_*"] & hom_{\varphi\psi}(X,Y) \\
hom_{\alpha}(X_*,Y^*) \arrow[u, "r^*"] \arrow[r, "f_*"'] \arrow[ur, phantom, "\equiv"] & hom_{\varphi}(X_*,Y) \arrow[u, "r^*"']
\end{tikzcd}$$

The bottom arrow is bijective because f is cartesian and the vertical arrows are bijections because r is strong cocartesian. The result follows.

□

0.3.2 Stability in a Prefibration

The following concepts are inspired in defining properties of the pullback functors of a cleaved prefibration without using clivages. Let $\mathcal{E} \xrightarrow{F} \mathcal{G}$ be a prefibration.

Pulling back objects and arrows

Definition 0.3.10. *A subset $\mathcal{A} \subset Ob(\mathcal{E})$ is **stable** if for every $X \in \mathcal{A}$ and any cartesian morphism $X^* \longrightarrow X$ it follows that $X^* \in \mathcal{A}$.*

If $X^* \longrightarrow X$ is cartesian over $\alpha \xrightarrow{\varphi} \beta$ we think of X being *pulled back* to X^* along φ. We will call X^* a *pullback* of X along φ. This means that $\mathcal{A}$ is stable when its objects are pulled back to objects of $\mathcal{A}$ exclusively.

Remark 0.3.11. Stable subsets are closed under isomorphisms.

Definition 0.3.12. *We will say that **terminal objects are stable** if the set of terminal objects of the fibres is a stable set.*

That is to say if 1_β is a terminal object of $\mathcal{E}_\beta$ and $(1_\beta)^* \longrightarrow 1_\beta$ is cartesian over $\alpha \xrightarrow{\varphi} \beta \in \mathcal{G}$, it follows that $(1_\beta)^*$ is terminal in $\mathcal{E}_\alpha$.

Observation. Vertical arrows can be pulled back as follows. For $X \xrightarrow{m} Y \in \mathcal{E}_\beta$ choose two cartesian morphisms $X^* \longrightarrow X$ and $Y^* \xrightarrow{s} Y$ over $\alpha \xrightarrow{\varphi} \beta \in \mathcal{G}$. These determine an arrow in $X^* \xrightarrow{m^*} Y^* \in \mathcal{E}_\alpha$ given by m^* in the following diagram.

$$\begin{tikzcd} X^* \arrow[r] \arrow[d, dashed, "\exists! m^*"'] \arrow[dr, phantom, "\equiv"] & X \arrow[d, "m"] \\ Y^* \arrow[r, "s"'] & Y \end{tikzcd}$$

$$\alpha \xrightarrow{\varphi} \beta$$

We call m^* a *pullback* of m along φ. In a fibration the diagram on top is in fact a pullback in $\mathcal{E}$ and thus m^* is a pullback of m along s.

Arrows in $\mathcal{E}_\alpha$ that are isomorphisms (epimorphisms,...) *in* $\mathcal{E}_\alpha$ will be refered to as *vertical* isomorphisms (epimorphisms,...).

Definition 0.3.13. *We will say that* ***isomorphisms (epimorphisms,...) are stable*** *if for every vertical isomorphism (epimorphism,...) m it follows that every pullback m^* of it is a vertical isomorphism (epimorphism,...).*

Observation. Isomorphisms are stable in any prefibration or precofibration.

Pulling back finite diagrams

Objects in $\mathcal{E}$ and vertical arrows are examples of a more general type of object that we can *pullback* in a prefibration. It is well known that a functor preserves finite limits if and only if the functor preserves terminal objects and pullbacks. We have already developed a notion of stability for terminal object and classes of morphisms in the previous section. In this section we define stability of pullbacks taken in a fibre. To encompass the three types of objects we will take a fixed finite category $\mathcal{D}$ with a terminal object t and consider the set of functors $\mathcal{D} \xrightarrow{G} \mathcal{E}$ that factor through a fibre.

$$\begin{tikzcd} & \mathcal{D} \arrow[dl, "G_\alpha"'] \arrow[dr, "G"] \arrow[d, phantom, "\equiv"] & \\ \mathcal{E}_\alpha \arrow[rr, "j_\alpha"'] & {} & \mathcal{E} \end{tikzcd}$$

We will allow the abuse of notation $\mathcal{D} \xrightarrow{G} \mathcal{E}_\alpha$ to indicate through which fibre such a functor factors.

These functors form the objects of a category $\mathcal{E}^{(\mathcal{D})}$ whose morphisms from $\mathcal{D} \xrightarrow{G} \mathcal{E}_\alpha$ to $\mathcal{D} \xrightarrow{H} \mathcal{E}_\beta$ are natural transformations $G \overset{\eta}{\Longrightarrow} H$ of functors $\mathcal{D} \longrightarrow \mathcal{E}$ that are projected onto a single arrow in $\mathcal{G}$. That is for every $d \in \mathcal{D}$, it follows that $F(\eta_d) = F(\eta_t)$.

$$\begin{tikzcd} & \mathcal{D} \arrow[dl, "G_\alpha"'] \arrow[dr, "H_\beta"] & \\ \mathcal{E}_\alpha \arrow[rr, phantom, "\overset{\eta}{\Longrightarrow}"] \arrow[dr, "j_\alpha"'] & & \mathcal{E}_\beta \arrow[dl, "j_\beta"] \\ & \mathcal{E} & \end{tikzcd} \qquad \begin{array}{c} Gd \xrightarrow{\eta_d} Hd \\ \\ \alpha \xrightarrow{F(\eta_t)} \beta \end{array}$$

The category $\mathcal{E}^{(\mathcal{D})}$ is a subcategory of the functor category $\mathcal{E}^{\mathcal{D}}$ and we denote the inclusion $\mathcal{E}^{(\mathcal{D})} \xrightarrow{i} \mathcal{E}^{\mathcal{D}}$. The assignments $G \longmapsto F(G_t)$ and $\eta \longmapsto F(\eta_t)$ yield a functor $\mathcal{E}^{(\mathcal{D})} \xrightarrow{F^{(\mathcal{D})}} \mathcal{G}$. For this functor there is a natural identification between the categories $(\mathcal{E}^{(\mathcal{D})})_\alpha$ and ${\mathcal{E}_\alpha}^{\mathcal{D}}$ and we will allow the abuse of language of sometimes using the latter as if it were the fibre itself.

Remark 0.3.14. The functor $F^{(\mathcal{D})}$ is prefibred. A morphism $G \xrightarrow{\eta} H \in \mathcal{E}^{(\mathcal{D})}$ is cartesian if and only if for every $d \in \mathcal{D}$ the morphisms η_d are cartesian. Consequently if F is a fibration, then $F^{(\mathcal{D})}$ is a fibration. Similarly if F is cleaved (split), then so is $F^{(\mathcal{D})}$.

Observation. For any category $\mathcal{A}$ and $X \in \mathcal{A}$ we will use $\mathcal{D} \xrightarrow{\Delta X} \mathcal{A}$ to denote the functor defined for every $d \xrightarrow{a} d' \in \mathcal{D}$ as $(\Delta X)(d \xrightarrow{a} d') = X \xrightarrow{1_X} X$. We call this the *constant* functor X. For $X \xrightarrow{f} Y \in \mathcal{A}$ we associate a natural transformation between the constant functors $\Delta X \xrightarrow{\Delta f} \Delta Y$ defined as the constant family f. This yields a functor $\mathcal{A} \xrightarrow{\Delta} \mathcal{A}^{\mathcal{D}}$.

Remark 0.3.15. Take $\mathcal{E}^{\mathcal{D}} \xrightarrow{F^{\mathcal{D}}} \mathcal{G}^{\mathcal{D}}$ the functor defined as postcomposing with F. The functor $F^{(\mathcal{D})}$ is a pullback of $F^{\mathcal{D}}$ along $\mathcal{G} \xrightarrow{\Delta} \mathcal{G}^{\mathcal{D}}$ in $\mathcal{C}at$ and we have the following diagram.

$$\begin{tikzcd}
\mathcal{E} \arrow[dr, "\delta"] \arrow[drr, bend left, "\Delta"] \arrow[ddr, bend right, "F"'] & & \\
\equiv & \mathcal{E}^{(\mathcal{D})} \arrow[r, "i"] \arrow[d, "F^{(\mathcal{D})}"'] \arrow[u, phantom, "\equiv"] \arrow[dr, phantom, "p.b."] & \mathcal{E}^{\mathcal{D}} \arrow[d, "F^{\mathcal{D}}"] \\
& \mathcal{G} \arrow[r, "\Delta"'] & \mathcal{G}^{\mathcal{D}}
\end{tikzcd}$$

The functor δ is cartesian. If F is cleaved, thenr δ is a morphism between cleaved functors.

Remark 0.3.16. If F is a prefibration, $F^{\mathcal{D}}$ will not necessarily be a prefibration. This does not happen with fibrations. We show now that when F is a fibration, so is $F^{\mathcal{D}}$.

If $G \xrightarrow{f} H \in \mathcal{E}^{\mathcal{D}}$ satisfies that for every $d \in \mathcal{D}$ the morphisms f_d are strong cartesian, it follows that f is cartesian. If F is a fibration, given $A \xrightarrow{\eta} B \in \mathcal{G}^{\mathcal{D}}$ and $X \in (\mathcal{E}^{\mathcal{D}})_B$, a choice of strong cartesian morphisms ${X_d}^* \xrightarrow{f_d} X_d$ over η_d determines a functor $\mathcal{D} \xrightarrow{X^*} \mathcal{E}$ over A and a cartesian morphism $X^* \xrightarrow{f} X$ over η (only finite choices are being made).

$$\begin{tikzcd}
{X_d}^* \arrow[rr, "f_d"] \arrow[dr, dashed, "\exists! X^*(a)"'] & \equiv & X_d \arrow[dr, "X(a)"] & \\
& {X_{d'}}^* \arrow[rr, "f_{d'}"] & & X_{d'}
\end{tikzcd}$$

$$\begin{tikzcd}
A_d \arrow[rr, "\eta_d"] \arrow[dr, "G(a)"'] & \equiv & B_d \arrow[dr, "H(a)"] & \\
& A_{d'} \arrow[rr, "\eta_{d'}"] & & B_{d'}
\end{tikzcd}$$

Thus the existence of the necessary cartesian morphisms for $F^{\mathcal{D}}$ to be a fibration follows without using choice. In such a case we have that i in Remark 0.3.15 preserves cartesian morphisms. If F is a cleaved (split) fibration, then $F^{\mathcal{D}}$ is cleaved (split) fibration and i preserves transport morphisms.

Remark 0.3.17. The conclusions in Remarks 0.3.16 and 0.3.15 are true if we replace the words *fibration* and *cartesian* for *cofibration* and *cocartesian* respectively.

Observation. Let $\mathbf{2}$ denote the category $\{0 \longrightarrow 1\}$. Vertical arrows in a fibration F are naturally identified with the objects of $\mathcal{E}^{(\mathbf{2})}$ and pulling back vertical arrows in F is the same as pulling back objects in $F^{(\mathbf{2})}$. This allows us to generalize the notion of pulling back diagrams of type $\mathcal{D}$ a prefibration.

Pulling back cones

Observation. For any category $\mathcal{A}$ and a functor $\mathcal{D} \xrightarrow{G} \mathcal{A}$, a cone $\{C \xrightarrow{c_d} G_d\}_{d\in\mathcal{D}}$ of G in $\mathcal{A}$ is nothing but an arrow $\Delta C \xrightarrow{c} G$ in $\mathcal{A}^{\mathcal{D}}$.

We will call an arrow $\delta C \xrightarrow{c} G$ in ${\mathcal{E}_\alpha}^{\mathcal{D}}$ a *vertical cone* in F.

Definition 0.3.18. *We will say a vertical cone $\delta C \xrightarrow{c} G \in {\mathcal{E}_\alpha}^{\mathcal{D}}$ is **universal** if $\{C \xrightarrow{c_d} G_d\}_{d\in\mathcal{D}}$ is a limit cone of G in $\mathcal{E}_\alpha$.*

Observation. A vertical cone $\delta C \xrightarrow{c} G$ in ${\mathcal{E}_\alpha}^{\mathcal{D}}$ is universal if and only if for every $X \in \mathcal{E}_\alpha$ and for every vertical cone $\delta X \xrightarrow{x} G$ in ${\mathcal{E}_\alpha}^{\mathcal{D}}$ there exists a unique $X \xrightarrow{f} C \in \mathcal{E}_\alpha$ such that $x = c \cdot \delta(f)$. That is for every $X \in \mathcal{E}_\alpha$ we have the following universal property.

$$\begin{tikzcd}
X \arrow[d, dashed, "\exists! f"'] & \delta X \arrow[d, "\delta f"'] \arrow[dr, "\forall"] \arrow[dr, phantom, "\equiv", shift right=3] & \\
C & \delta C \arrow[r, "c"'] & G \\
\alpha & \alpha \arrow[r, "id_\alpha"] & \alpha
\end{tikzcd} \tag{0.3.19}$$

Observation. We can pullback a cone $\delta C \xrightarrow{c} G \in {\mathcal{E}_\beta}^{\mathcal{D}}$ along $\alpha \xrightarrow{\varphi} \beta \in \mathcal{G}$ to a vertical cone in ${\mathcal{E}_\alpha}^{\mathcal{D}}$ choosing a cartesian morphism $C^* \xrightarrow{s} C$ over φ in F and a cartesian morphism $G^* \xrightarrow{t} G$ over φ in $F^{(\mathcal{D})}$.

$$\begin{tikzcd}
\delta(C^*) \arrow[r, "\delta s"] \arrow[d, dashed, "\exists!"'] \arrow[dr, phantom, "\equiv"] & \delta C \arrow[d, "c"] \\
G^* \arrow[r, "t"'] & G \\
\alpha \arrow[r, "\varphi"] & \beta
\end{tikzcd} \tag{0.3.20}$$

Definition 0.3.21. *We will say that **limits of type $\mathcal{D}$ are stable** if vertical universal cones are stable for pullbacks such as 0.3.20.*

Observation. Pullbacks are limits of functors whose domain is the following category $\mathcal{P}$.

$$\begin{tikzcd}
 & 0 \arrow[d] \\
2 \arrow[r] & 1
\end{tikzcd}$$

Definition 0.3.22. *We will say that **pullbacks are stable** if limits of type $\mathcal{P}$ are stable.*

A property equivalent to the stability of pullbacks

We will give a more comprehensive characterization of the stability of pullbacks in a fibration. Nevertheless we will develop it for the general type of finite category $\mathcal{D}$ with terminal object t.

Lemma 0.3.23. *For every category $\mathcal{A}$ the functor $\mathcal{A} \xrightarrow{\Delta} \mathcal{A}^{\mathcal{D}}$ is fully faithful (this actually holds for any connected category $\mathcal{D}$).*

Proof. For $X, Y \in \mathcal{A}$ and $\Delta X \xrightarrow{\eta} \Delta Y \in \mathcal{A}^{\mathcal{D}}$ we have $\eta = \Delta(\eta_t)$. This is because for every $d \in \mathcal{D}$ the unique arrow $d \longrightarrow t \in \mathcal{D}$ yields the following diagram.

$$\begin{tikzcd} d \arrow[d] & X \arrow[r, "\eta_d"] \arrow[d, "1_X"'] \arrow[dr, phantom, "\equiv"] & Y \arrow[d, "1_Y"] \\ t & X \arrow[r, "\eta_t"'] & Y \end{tikzcd}$$

□

Corollary 0.3.24. *$\mathcal{E} \xrightarrow{\delta} \mathcal{E}^{(\mathcal{D})}$ and $\mathcal{E}^{(\mathcal{D})} \xrightarrow{i} \mathcal{E}^{\mathcal{D}}$ are fully faithful functors.*

Proof. Since fully faithful functors are stable in $\mathcal{C}at$ [5, page 128] it follows from Remark 0.3.15 that i and consequently δ are fully faithful. □

Remark 0.3.25. In this context we can merge the sets $hom_{\mathcal{E}}(X, Y)$ and $hom_{\mathcal{E}^{(\mathcal{D})}}(G, H)$ with $hom_{\mathcal{E}^{(\mathcal{D})}}(\delta X, \delta Y)$ and $hom_{\mathcal{E}^{\mathcal{D}}}(iG, iH)$ respectively. Thus we will adopt the abuse of notation of suppressing δ, Δ and i in our expressions. Looking at diagram 0.3.19 we have that a vertical cone $C \xrightarrow{c} G$ in ${\mathcal{E}_\alpha}^{\mathcal{D}}$ is universal if and only if it verifies the following universal property for every $X \in \mathcal{E}_\alpha$.

$$\begin{tikzcd} X \arrow[d, dashed, "\exists! f"'] \arrow[dr, "\forall"] \arrow[dr, phantom, "\equiv", shift right=3] & \\ C \arrow[r, "c"'] & G \end{tikzcd}$$

$$\alpha \xrightarrow{id_\alpha} \alpha$$

That is for every $X \in \mathcal{E}_\alpha$ the function

$$hom_\alpha(X, C) \xrightarrow{c_*} hom_\alpha(X, G)$$

is a bijection.

Proposition 0.3.26. *Limits of type $\mathcal{D}$ are stable if and only if for every $\alpha \xrightarrow{\varphi} \beta \in \mathcal{G}$, every $\mathcal{D} \xrightarrow{G} \mathcal{E}_\beta$ and every vertical universal cone $C \xrightarrow{c} G \in {\mathcal{E}_\beta}^{\mathcal{D}}$ we have that for every $X \in \mathcal{E}_\alpha$ the function*

$$hom_\varphi(X, C) \xrightarrow{c_*} hom_\varphi(X, G)$$

is a bijection.

Proof. Take cartesian morphisms $C^* \xrightarrow{s} C$ and $G^* \xrightarrow{t} G$ over φ. Diagram 0.3.20 can be written as follows.

$$
\begin{tikzcd}
C^* \arrow[r, "s"] \arrow[d, "c^*"'] \arrow[dr, phantom, "\equiv"] & C \arrow[d, "c"] \\
G^* \arrow[r, "t"] & G
\end{tikzcd}
$$

$$
\alpha \xrightarrow{\varphi} \beta
$$

Thus we have the following commutative diagram.

$$
\begin{tikzcd}
hom_\alpha(X, C^*) \arrow[r, "s_*"] \arrow[d, "(c^*)_*"'] \arrow[dr, phantom, "\equiv"] & hom_\varphi(X, C) \arrow[d, "c_*"] \\
hom_\alpha(X, G^*) \arrow[r, "t_*"'] & hom_\varphi(X, G)
\end{tikzcd}
$$

The result follows from this diagram and Remark 0.3.25. □

Remark 0.3.27. For a functor $\mathcal{D} \xrightarrow{G} \mathcal{E}$ a cone $C \xrightarrow{c} G \in \mathcal{E}^{\mathcal{D}}$ of G in $\mathcal{E}$ is a limit cone if and only if for every $X \in \mathcal{E}$ the following function is a bijection.

$$
hom_{\mathcal{E}^{\mathcal{D}}}(X, C) \xrightarrow{c_*} hom_{\mathcal{E}^{\mathcal{D}}}(X, G)
$$

Proposition 0.3.28. *Limits of type $\mathcal{D}$ are stable if and only if the functors $\mathcal{E}_\alpha \xrightarrow{j_\alpha} \mathcal{E}$ preserves limits of type $\mathcal{D}$.*

Proof. Take $\mathcal{D} \xrightarrow{G} \mathcal{E}_\alpha$ a functor and $C \xrightarrow{c} G$ a universal cone of G in $\mathcal{E}_\alpha$. The result follows from Proposition 0.3.26, Remark 0.3.25, Remark 0.3.27 and the following sequence.

$$
hom_{\mathcal{E}(\mathcal{D})}(X, C) = \coprod_\varphi hom_\varphi(X, C) \xrightarrow{c_*} \coprod_\varphi hom_\varphi(X, G) = hom_{\mathcal{E}(\mathcal{D})}(X, G)
$$

□

Theorem 0.3.29. *If F is precofibred, then terminal objects and limits of type $\mathcal{D}$ are stable.*

Proof. For $\alpha \xrightarrow{\varphi} \beta \in \mathcal{G}$, 1_β a terminal object of $\mathcal{E}_\beta$ and $(1_\beta)^* \xrightarrow{s} 1_\beta$ cartesian over φ we will prove that $(1_\beta)^*$ is a terminal object in $\mathcal{E}_\alpha$. Take $X \in \mathcal{E}_\alpha$ and $X \xrightarrow{r} X_*$ cocartesian over φ. we have the following diagram.

$$
hom_\alpha(X, (1_\beta)^*) \xrightarrow{s_*} hom_\varphi(X, 1_\beta) \xleftarrow{r^*} hom_\beta(X_*, 1_\beta) \tag{0.3.30}
$$

Both arrows are bijections and the set on the right is a singleton. The result follows.

Take $C \xrightarrow{c} G$ a universal vertical cone in ${\mathcal{E}_\beta}^{\mathcal{D}}$. We will prove that for every $X \in \mathcal{E}_\alpha$ the function $hom_\varphi(X, C) \xrightarrow{c_*} hom_\varphi(X, G)$ is a bijection. Take $X \xrightarrow{r} X_*$ cocartesian over φ. We have the following diagram.

$$\begin{tikzcd} hom_{\varphi}(X,C) \arrow[r, "c_*"] & hom_{\varphi}(X,G) \\ hom_{\beta}(X_*,C) \arrow[r, "c_*"'] \arrow[u, "r^*"] \arrow[ur, phantom, "\equiv"] & hom_{\beta}(X_*,G) \arrow[u, "r^*"'] \end{tikzcd}$$

The bottom arrow and the vertical arrows are bijections. The result follows.

□

Definition 0.3.31. *We will say that **finite limits are stable** if terminal objects and pullbacks are stable.*

Definition 0.3.32. *A prefibration $\mathcal{E} \xrightarrow{F} \mathcal{G}$ is **finitely complete** if the categories $\mathcal{E}_\alpha$ are finitely complete and finite limits are stable.*

Accordingly we have the category of finitely complete prefibrations whose morphisms are cartesian $\mathcal{G}$-functors $f \in hom_{\mathcal{G}}(\mathcal{E}, \mathcal{E}')$ such that for every $\alpha \in \mathcal{G}$ the restrictions

$$\mathcal{E}_\alpha \xrightarrow{f_\alpha} \mathcal{E}'_\alpha$$

preserve finite limits.

Definition 0.3.33. *A prefibration $\mathcal{E} \xrightarrow{F} \mathcal{G}$ is **regular** if the categories $\mathcal{E}_\alpha$ are regular, finite limits are stable and strict epimorphisms are stable.*

The category of regular prefibrations over $\mathcal{G}$ is the category that has regular prefibrations (fibrations) as objects and whose morphisms are cartesian $\mathcal{G}$-functors $f \in hom_{\mathcal{G}}(\mathcal{E}, \mathcal{E}')$ such that for every $\alpha \in \mathcal{G}$ the restrictions

$$\mathcal{E}_\alpha \xrightarrow{f_\alpha} \mathcal{E}'_\alpha$$

are regular functors. This category is included in the category if finitely complete prefibrations. The restrictions to fibrations are natural and coherent.

Reflection properties in a prefibration

Properties of functors such as reflecting limits and other types of categorical objects can be defined in a prefibration. Here we define without using clivages the property that the pullback functors are conservative over a specific set of morphisms as defined in section 0.2.3.

Consider $\mathcal{A}$ a *stable* set of vertical arrows. We will use the notation $\mathcal{A}_\alpha$ to represent the subset $\mathcal{A} \cap \mathcal{E}_\alpha$. We will work freely identifying vertical arrows in F with objects in $F^{\mathbf{2}}$.

Definition 0.3.34. *The prefibration $\mathcal{E} \xrightarrow{F} \mathcal{G}$ is **conservative over** $\mathcal{A}$ if for every cartesian morphism $f^* \longrightarrow f$ such that $f \in \mathcal{A}$ and f^* is an isomorphism, it follows that f is an isomorphism.*

We will say the prefibration $\mathcal{E} \xrightarrow{F} \mathcal{G}$ is *conservative* if it is conservative over the complete set of arrows of $\mathcal{E}$.

0.3.3 Two facts about epimorphisms in a prefibration

Lemma 0.3.35. *If $A \xrightarrow{f} B$ is an epimorphism in $\mathcal{E}_\alpha$ and $g, h \in hom_\varphi(B, C)$ satisfy $gf = hf$, then $g = h$.*

Proof. Take $C^* \xrightarrow{s} C$ a cartesian arrow over φ. We have the following diagram.

$$\begin{tikzcd}
hom_\varphi(B,C) \arrow[r, "f^*"] & hom_\varphi(A,C) \\
hom_\alpha(B,C^*) \arrow[u, "s_*"] \arrow[r, "f^*"'] \arrow[ur, phantom, "\equiv"] & hom_\alpha(A,C^*) \arrow[u, "s_*"']
\end{tikzcd}$$

Since the vertical arrows are bijections and the bottom is injective, the result follows. □

Lemma 0.3.36. *If $A \xrightarrow{f} B$ is a strict epimorphism in $\mathcal{E}_\alpha$, then every compatible morphism with f in $\mathcal{E}$ factors through f.*

Proof. Let $A \xrightarrow{g} C$ be compatible with f. Take $C^* \xrightarrow{s} C$ cartesian over $F(g) = \alpha \xrightarrow{\varphi} \beta$. It is straightforward that the only factorization of g through s over α is compatible with f. In a diagram:

$$\begin{tikzcd}
& A \arrow[dl, "f"'] \arrow[dd, dashed, "\exists! a"] \arrow[ddr, "g"] & \\
B \arrow[dr, dashed, "\exists! b"'] \arrow[r, phantom, "\equiv"] & \arrow[dr, phantom, "\equiv"] & \\
& C^* \arrow[r, "s"'] & C
\end{tikzcd}$$

$$\alpha \xrightarrow{id_\alpha} \alpha \xrightarrow{\varphi} \beta$$

□

0.4 COLIMIT OF A FIBRATION WITH COFILTERED BASE

In this section we will study the structure of the colimit of a fibration developed in [1] for the particular case where the base category is cofiltered.

Let $\mathcal{E} \xrightarrow{F} \mathcal{G}$ be a fibration where $\mathcal{G}$ is cofiltered. If S denotes the set of cartesian morphisms in $\mathcal{E}$, the *colimit* of the fibration is defined as the category of fractions $\mathcal{E}[S^{-1}]$ characterized by a functor $\mathcal{E} \xrightarrow{Q} \mathcal{E}[S^{-1}]$ that satisfies the following universal property in $\mathcal{C}at$.

Proposition 0.4.1. *For every functor $\mathcal{E} \xrightarrow{G} \mathcal{I}$ such that G transforms cartesian morphisms into isomorphisms, there exists a unique functor $\mathcal{E}[S^{-1}] \xrightarrow{H} \mathcal{I}$ such that $HQ = G$.*

Remark 0.4.2. Because $\mathcal{G}$, is cofiltered S admits a calculus of right fractions [1, 3]. Thus we can describe the category of fractions $\mathcal{E}[S^{-1}]$ as done in [4] as follows. The objects of $\mathcal{E}[S^{-1}]$ are the objects of $\mathcal{E}$. A morphism $X \longrightarrow Y$ in $\mathcal{E}[S^{-1}]$ is an equivalence class of the quotient set of the set of pairs $X \xleftarrow{s} A \xrightarrow{f} Y$ with $s \in S$ where the equivalence relation is given by the relation $X \xleftarrow{s} A \xrightarrow{f} Y \sim X \xleftarrow{s'} A' \xrightarrow{f'} Y$ if and only if there exists $X \xleftarrow{s''} A'' \xrightarrow{f''} Y$ with $s'' \in S$ and arrows $A'' \longrightarrow A$ and $A'' \longrightarrow A'$ in $\mathcal{E}$ such that the following diagram commutes.

$$\begin{tikzcd} & A \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & A'' \arrow[l, "s''"'] \arrow[r, "f''"] \arrow[u] \arrow[d] & Y \\ & A' \arrow[ul, "s'"] \arrow[ur, "f'"'] & \end{tikzcd}$$

We will denote the class of the pair $X \xleftarrow{s} A \xrightarrow{f} Y$ by $X \xrightarrow{f/s} Y$. For $X \xrightarrow{f/s} Y \xrightarrow{g/t} Z$ having a calculus of right fractions guarantees that there is a pair $A \xleftarrow{u} C \xrightarrow{h} B$ with $u \in S$ such that $su \in S$ and the following diagram commutes.

$$\begin{tikzcd} & & C \arrow[dl, "u"'] \arrow[dr, "h"] & & \\ & A \arrow[dl, "s"'] \arrow[dr, "f"] & & B \arrow[dl, "t"'] \arrow[dr, "g"] & \\ X & & Y & & Z \end{tikzcd}$$

The operation $(g/t)(f/s) := (gh)/(su)$ is well defined, which defines the composition. The functor Q is defined as the identity on objects and for $A \xrightarrow{f} Y \in \mathcal{E}$ we have $Q(f) = f/1_A$. For $A \xrightarrow{s} X \in S$, and adopting the abuse of notations $f/1_A = f$ and $1_A/s = 1/s$, we have that $(1/s) \cdot s = 1_A$, $s \cdot (1/s) = 1_X$ and $f/s = f \cdot (1/s)$. For details see [4].

0.4.1 Colimit of a Finitely Complete Fibration

Our objective in this section is to prove the following theorem.

Theorem 0.4.3. *If F is finitely complete, then $\mathcal{E}[S^{-1}]$ is a finitely complete category and the functors*

$$\begin{tikzcd} \mathcal{E}_\alpha \arrow[rr, "J_\alpha"] \arrow[dr, "j_\alpha"'] \arrow[rr, phantom, shift right=4ex, "\equiv"] & & \mathcal{E}[S^{-1}] \\ & \mathcal{E} \arrow[ur, "Q"'] & \end{tikzcd}$$

preserve finite limits. More so if $\mathcal{I} \in \mathcal{C}at_{fl}$ and $\mathcal{E}[S^{-1}] \xrightarrow{H} \mathcal{I}$ is a functor such that for every $\alpha \in \mathcal{G}$ the functors $H \cdot J_\alpha \in \mathcal{C}at_{fl}$, it follows that $H \in \mathcal{C}at_{fl}$.

Corollary 0.4.4. *If* $\mathcal{I} \in \mathcal{C}at_{fl}$ *and* $\mathcal{E} \xrightarrow{G} \mathcal{I}$ *is such that* G *transforms cartesian morphisms into isomorphisms and for every* $\alpha \in \mathcal{G}$ *the functors* $G \cdot j_\alpha \in \mathcal{C}at_{fl}$*, then there exists a unique functor* $\mathcal{E}[S^{-1}] \xrightarrow{H} \mathcal{I} \in \mathcal{C}at_{fl}$ *such that* $HQ = G$.

Proof. We know of the existence of a unique functor H that satisfies $HQ = G$. The fact that it preserves finite limits follows from the following diagram.

$$\begin{tikzcd}
\mathcal{E}[S^{-1}] \arrow[r, "H"] & \mathcal{I} \\
\mathcal{E} \arrow[u, "Q"] \arrow[ur, "G"'] \arrow[u, phantom, shift right=6ex, "\equiv"] \arrow[ur, phantom, "\equiv", near end] & \\
\mathcal{E}_\alpha \arrow[u, "j_\alpha"'] \arrow[uu, bend left=60, "J_\alpha"] &
\end{tikzcd}$$

∎

Corollary 0.4.5. *The construction determines a functor from the category of finitely complete fibrations into* $\mathcal{C}at_{fl}$.

Proof. Suppose we have a morphisms of finitely complete fibrations.

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "f"] \arrow[dr, "F"'] & \arrow[d, phantom, "\equiv"] & \mathcal{E}' \arrow[dl, "F'"] \\
& \mathcal{G} &
\end{tikzcd}$$

From the following commutative diagram it follows that for every $\alpha \in \mathcal{G}$ the functors $Q' f \cdot j_\alpha$ preserve finite limits.

$$\begin{tikzcd}
\mathcal{E}[S^{-1}] \arrow[r, dashed, "\exists!"] \arrow[dr, phantom, "\equiv"] & \mathcal{E}'[S'^{-1}] \\
\mathcal{E} \arrow[u, "Q"] \arrow[r, "f"] \arrow[dr, phantom, "\equiv"] & \mathcal{E}' \arrow[u, "Q'"'] \arrow[r, phantom, "\equiv"] & {} \\
\mathcal{E}_\alpha \arrow[u, "j_\alpha"] \arrow[r, "f_\alpha"'] & \mathcal{E}'_{f(\alpha)} \arrow[u, "j_{f(\alpha)}"] \arrow[uu, bend right=60, "J_{f(\alpha)}"']
\end{tikzcd}$$

The result follows. ∎

Proposition 0.4.6. *The functors* $\mathcal{E}_\alpha \xrightarrow{J_\alpha} \mathcal{E}[S^{-1}]$ *preserve terminal objects.*

Proof. Let 1_α be a terminal object in $\mathcal{E}_\alpha$ and $X \in \mathcal{E}[S^{-1}]$. Take a cone of the following diagram in $\mathcal{G}$.

$$\begin{tikzcd} & \beta \arrow[dl, dashed, "\varphi_0"'] \arrow[dr, dashed, "\varphi_1"] & \\ F(X) & & \alpha \end{tikzcd}$$

Take $X^* \xrightarrow{s} X$ and $(1_\alpha)^* \xrightarrow{t} 1_\alpha$ cartesian morphisms over φ_0 and φ_1 respectively. We obtain a morphism $X^* \xrightarrow{f} 1_\alpha$ in $\mathcal{E}$ going through the terminal object $(1_\alpha)^*$. This yields a morphism $X \xrightarrow{f/s} 1$ in $\mathcal{E}[S^{-1}]$. Now suppose we have $X \overset{f_0/s_0}{\underset{f_1/s_1}{\rightrightarrows}} 1_\alpha$ in $\mathcal{E}[S^{-1}]$. Take a cone of the following diagram in $\mathcal{G}$.

$$\begin{tikzcd} & F(X) & \\ {} \arrow[ur, "F(s_0)"] \arrow[dr, "F(f_0)"'] & \beta \arrow[u, dashed, "\varphi_0"'] \arrow[d, dashed, "\varphi_1"] \arrow[l, dashed, "\psi_0"'] \arrow[r, dashed, "\psi_1"] & {} \arrow[ul, "F(s_1)"'] \arrow[dl, "F(f_1)"] \\ & \alpha & \end{tikzcd}$$

Let $X^* \xrightarrow{s} X$ be a cartesian morphism over φ_0. Take $(i = 0, 1)$ a_i the unique morphism over ψ_i that factors s through s_i. It follows that $f_0 a_0 = f_1 a_1$ in $\mathcal{E}$ (0.3.30). Thus precomposing with $1/s$ we obtain $f_0/s_0 = f_1/s_1$.

□

Proposition 0.4.7. *The functors $\mathcal{E}_\alpha \xrightarrow{J_\alpha} \mathcal{E}[S^{-1}]$ preserve pullbacks.*

Proof. It is an immediate consequence of Proposition 0.3.28 and the fact that $\mathcal{E} \xrightarrow{Q} \mathcal{E}[S^{-1}]$ preserves finite limits [4]. □

Proposition 0.4.8. *Any diagram of type $\mathcal{P}$ in $\mathcal{E}[S^{-1}]$ is naturally isomorphic to one that can be factored through a fibre. More precisely, for every $\mathcal{P} \xrightarrow{G} \mathcal{E}[S^{-1}]$ there exists $\alpha \in \mathcal{G}$, G^* and η*

$$\begin{tikzcd} & \mathcal{P} \arrow[dl, dashed, "G^*"'] \arrow[dr, "G"] \arrow[d, phantom, "\overset{\eta}{\Longrightarrow}"] & \\ \mathcal{E}_\alpha \arrow[rr, "J_\alpha"'] & {} & \mathcal{E}[S^{-1}] \end{tikzcd}$$

where the natural transformation is composed of cartesian morphisms.

Proof. Let $\mathcal{P} \xrightarrow{G} \mathcal{E}[S^{-1}]$ be such a diagram. Suppose $G(a_i) = f_i/s_i$. Take a cone of the following diagam in $\mathcal{G}$.

$$\begin{tikzcd}
 & & \alpha \arrow[ddll, dashed, bend right, "\varphi_2"'] \arrow[dl, dashed, "\psi_1"'] \arrow[dd, dashed, "\varphi_1"] \arrow[dr, dashed, "\psi_0"] \arrow[ddrr, dashed, bend left, "\varphi_0"] & & \\
 & {} \arrow[dl, "F(s_1)"'] \arrow[dr, "F(f_1)"] & & {} \arrow[dl, "F(f_0)"'] \arrow[dr, "F(s_0)"] & \\
F(G_2) & & F(G_1) & & F(G_0)
\end{tikzcd}$$

Take $(i = 0, 1, 2)$ $G_i^* \xrightarrow{\eta_i} G_i$ a cartesian morphism over φ_i. Since $(i = 0, 1)$ s_i is cartesian there is a unique factorization of η_i through s_i over ψ_i, namely b_i. Set $G^*(a_i)$ to be the unique factorization of $f_i b_i$ through η_1 over α. □

Remark 0.4.9. Proposition 0.4.8 holds for any finite diagram $\mathcal{D}$ instead of $\mathcal{P}$, though the proof is harder when composition of nontrivial arrows exist in $\mathcal{D}$. We will not use the general case.

Corollary 0.4.10. *Pullbacks exist in $\mathcal{E}[S^{-1}]$ and the rest of Theorem 0.4.3 follows.*

Proof. The way to calculate pullbacks in $\mathcal{P} \xrightarrow{G} \mathcal{E}[S^{-1}]$ is to take G^* as in Proposition 0.4.8 and take any pullback of G^* in $\mathcal{E}_\alpha$. This will yield the following diagram.

$$\begin{tikzcd}
P \arrow[r, "p_0"] \arrow[d, "p_2"'] \arrow[dr, phantom, "p.b."] & {G^*}_0 \arrow[d, "G^*(a_0)"] \arrow[dr, "\eta_0"] & \\
{G^*}_2 \arrow[r, "G^*(a_1)"'] \arrow[dr, "\eta_2"'] & {G^*}_1 \arrow[r, phantom, "\equiv"] \arrow[d, phantom, "\equiv"] \arrow[dr, "\eta_1"] & G_0 \arrow[d, "f_0/s_0"] \\
 & G_2 \arrow[r, "f_1/s_1"'] & G_1
\end{tikzcd} \tag{0.4.11}$$

The arrows η_i are isomorphisms and the square is a pullback in $\mathcal{E}_\alpha$ as well as in $\mathcal{E}[S^{-1}]$. □

0.4.2 Colimit of a Regular Fibration

Our objective in this section is to prove the following theorem.

Theorem 0.4.12. *If F is a regular fibration, then $\mathcal{E}[S^{-1}]$ is a regular category and the functors*

$$\begin{tikzcd}
\mathcal{E}_\alpha \arrow[rr, "J_\alpha"] \arrow[dr, "j_\alpha"'] & \arrow[d, phantom, "\equiv"] & \mathcal{E}[S^{-1}] \\
 & \mathcal{E} \arrow[ur, "Q"'] &
\end{tikzcd}$$

are regular. More so if $\mathcal{I} \in \mathcal{R}eg$ and $\mathcal{E}[S^{-1}] \xrightarrow{H} \mathcal{I}$ is a functor such that for every $\alpha \in \mathcal{G}$ the functors $H \cdot J_\alpha \in \mathcal{R}eg$, it follows that $H \in \mathcal{R}eg$.

The proofs of the following two corollaries ar identical to the proofs of Corollaries 0.4.4 and 0.4.5.

Corollary 0.4.13. *If $\mathcal{I} \in \mathcal{R}eg$ and $\mathcal{E} \xrightarrow{G} \mathcal{I}$ is such that G transforms cartesian morphisms into isomorphisms and for every $\alpha \in \mathcal{G}$ the functors $G \cdot j_\alpha \in \mathcal{R}eg$, then there exists a unique $\mathcal{E}[S^{-1}] \xrightarrow{H} \mathcal{I} \in \mathcal{R}eg$ such that $[HQ = G]$.*

Corollary 0.4.14. *The construction determines a functor from the category of regular fibrations into $\mathcal{R}eg$.*

Observation. If $A \overset{f}{\underset{g}{\rightrightarrows}} B$ are in $\mathcal{E}$, we have that $f = g$ in $\mathcal{E}[S^{-1}]$ if and only if there exists $s \in S$ such that $fs = gs$ in $\mathcal{E}$.

Proposition 0.4.15. *The functors $\mathcal{E}_\alpha \xrightarrow{J_\alpha} \mathcal{E}[S^{-1}]$ send strict epimorphisms to epimorphisms.*

Proof. Let $B \overset{f_0/s_0}{\underset{f_1/s_1}{\rightrightarrows}} C$ be such that $(f_0/s_0)f = (f_1/s_1)f$. Take a cone of the following diagram in $\mathcal{G}$.

$$\begin{tikzcd}
& \alpha & \\
{} \arrow[ur, "F(s_0)"] \arrow[dr, "F(f_0)"'] & \beta \arrow[u, dashed, "\varphi_0"'] \arrow[l, dashed, "\psi_0"'] \arrow[r, dashed, "\psi_1"] \arrow[d, dashed, "\varphi_1"] & {} \arrow[ul, "F(s_1)"'] \arrow[dl, "F(f_1)"] \\
& F(C) &
\end{tikzcd}$$

Let $B^* \xrightarrow{s} B$ be a cartesian morphism over φ_0. Take $(i = 0, 1)$ a_i the unique morphism over ψ_i that factors s through s_i. It suffices to prove that $f_0 a_0 = f_1 a_1$ in $\mathcal{E}[S^{-1}]$. Take $A^* \xrightarrow{t} A$ cartesian over φ_0 and f^* the corresponding pullback of f along φ_0.

$$\begin{tikzcd}
A^{**} \arrow[r, dashed, "u"] \arrow[d, "f^{**}"'] \arrow[dr, phantom, "\equiv"] & A^* \arrow[rr, "t"] \arrow[d, "f^*"] \arrow[drr, phantom, "\equiv"] & & A \arrow[d, "f"] & \\
B^{**} \arrow[r, dashed, "v"'] & B^* \arrow[rr, "s"'] \arrow[dr, "a_i"'] & & B & \\
& & {} \arrow[u, phantom, "\equiv"] \arrow[ur, "s_i"'] \arrow[drr, "f_i"] & & \\
& & & & C
\end{tikzcd}$$

$$\begin{tikzcd}
\gamma \arrow[r, dashed, "F(u)"] & \beta \arrow[r, "\varphi_0"] & \alpha
\end{tikzcd}$$

We have that $(f_0a_0)f^* = (f_1a_1)f^*$ in $\mathcal{E}[S^{-1}]$. Thus there is a cartesian morphism $A^{**} \xrightarrow{u} A^*$ such that $(f_0a_0f^*)u = (f_1a_1f^*)u$ in $\mathcal{E}$. Take $B^{**} \xrightarrow{v} B^*$ cartesian over $F(u)$ and call f^{**} the corresponding pullback of f^* along $F(u)$. Since $(f_0a_0v)f^{**} = (f_1a_1v)f^{**}$ in $\mathcal{E}$ and $F(f_1a_1v) = F(f_0a_0v) = \varphi_1F(u)$ from Lemma 0.3.35 we conclude $f_0a_0v = f_1a_1v$ in $\mathcal{E}$. The result follows. $\square$

Proposition 0.4.16. *If $A \xrightarrow{f} B$ is a strict epimorphism in $\mathcal{E}_\alpha$, then every compatible morphism with f in $\mathcal{E}[S^{-1}]$ factors through f.*

Proof. Let $A \xrightarrow{gr^{-1}} C$ be compatible with f. Take $K \underset{x_2}{\overset{x_1}{\rightrightarrows}} A$ a kernel pair of f in $\mathcal{E}_\alpha$ and $K^* \xrightarrow{s} K$ a cartesian morphisms over $F(r)$.

$$\begin{tikzcd}
K^{**} \arrow[r, dashed, "t"] \arrow[d, shift right, "x_2^{**}"'] \arrow[d, shift left, "x_1^{**}"] & K^* \arrow[r, "s"] \arrow[d, shift right, "x_2^*"'] \arrow[d, shift left, "x_1^*"] & K \arrow[d, shift right, "x_2"'] \arrow[d, shift left, "x_1"] \\
A^{**} \arrow[r, dashed, "u"] \arrow[d, "f^{**}"] & A^* \arrow[r, "r"] & A \arrow[d, "f"] \\
B^{**} \arrow[rr, dashed, "v"] & & B
\end{tikzcd}$$

$$F(K^{**}) \xrightarrow{F(t)} F(A^*) \xrightarrow{F(r)} \alpha$$

Since $(g/r)x_1 = (g/r)x_2$, we have that $gx_1^* = gx_2^*$ in $\mathcal{E}[S^{-1}]$. Thus there is a cartesian morphism $K^{**} \xrightarrow{t} K^*$ such that $(gx_1^*)t = (gx_2^*)t$ in $\mathcal{E}$. Take $A^{**} \xrightarrow{u} A^*$ cartesian over $F(t)$ and $B^{**} \xrightarrow{v} B$ cartesian over $F(r)F(t) = F(st)$. The morphisms $K^{**} \underset{x_2^{**}}{\overset{x_1^{**}}{\rightrightarrows}} A^{**}$ are a kernel pair of the strict epimorphism f^{**} in the fibre over $F(K^{**})$, so gu is compatible with f^{**} in $\mathcal{E}$. By Lemma 0.3.36 there is a morphism $h \in hom_{\mathcal{E}}(B^{**}, C)$ such that $gu = fh$ in $\mathcal{E}$. The morphism $B \xrightarrow{h/v} C$ yields the desired factorization. $\square$

Theorem 0.4.17. *The functors $\mathcal{E}_\alpha \xrightarrow{J_\alpha} \mathcal{E}[S^{-1}]$ preserve strict epimorphisms.*

Proof. It follows from Propositions 0.4.15 and 0.4.16. $\square$

Proposition 0.4.18. *Any morphisms $X \xrightarrow{f/s} Y \in \mathcal{E}[S^{-1}]$ admits a strict epic - monic factorization.*

Proof. For any morphism $X \xrightarrow{f/s} Y \in \mathcal{E}[S^{-1}]$ take a cartesian morphisms $Y^* \xrightarrow{t} Y$ over $F(f)$. We have the following situation.

$$\begin{tikzcd}
 & X^* \arrow[rr, "s"] \arrow[dl, dashed, two heads, "e"'] \arrow[dd, dashed, "\exists! f'"] \arrow[ddr, "f"] & & X \\
I \arrow[dr, dashed, tail, "m"'] \arrow[r, phantom, "\equiv"] & \arrow[r, phantom, "\equiv"] & \; & \\
 & Y^* \arrow[r, "t"'] & Y &
\end{tikzcd}$$

$$F(X^*) \xrightarrow{F(f)} F(Y)$$

The morphisms m and e form a strict epic - monic factorization of f' in the fibre over $F(X^*)$. From Theorem 0.4.17 and the fact that the J_α preserve monics we have that the morphisms e/s and tm yield a strict epic - monic factorization of f/s.

□

Proposition 0.4.19. *Strict epimorphisms are stable in $\mathcal{E}[S^{-1}]$.*

Proof. We will use as reference diagram 0.4.11. Suppose f_0/s_0 is a strict epimorphism. Then $G^*(a_0)$ is a strict epimorphism in $\mathcal{E}[S^{-1}]$. Take a strict epic - monic factorization of $G^*(a_0)$ in $\mathcal{E}_\alpha$. We will take a composite pullback of $G^*(a_0)$ along $G^*(a_1)$ in $\mathcal{E}_\alpha$.

$$\begin{tikzcd}
P \arrow[r, "p"] \arrow[d, two heads, "e'"'] \arrow[dr, phantom, "p.b."] & {G^*}_0 \arrow[d, two heads, "e"] \arrow[dd, bend left=60, "G^*(a_0)"] \\
I' \arrow[r] \arrow[d, tail, "m'"'] \arrow[dr, phantom, "p.b."] & I \arrow[d, tail, "m"] \arrow[r, phantom, "\equiv"] & \; \\
{G^*}_2 \arrow[r, "G^*(a_1)"'] & {G^*}_0
\end{tikzcd}$$

This diagram in fact is also true in $\mathcal{E}[S^{-1}]$. In fact in $\mathcal{E}[S^{-1}]$ we have that $G^*(a_0)$ is a strict epic and consequently m is an isomorphism in $\mathcal{E}[S^{-1}]$. Therefore m' is an isomorphism in in $\mathcal{E}[S^{-1}]$ and so $m'e'$ is a strict epimorphisms in $\mathcal{E}[S^{-1}]$. The result follows. □

0.4.3 Colimit of a Conservative Fibration Over $\mathcal{A}$

Take $\mathcal{A}$ a stable set of vertical arrows.

Theorem 0.4.20. *If F is conservative over $\mathcal{A}$, then for every $\alpha \in \mathcal{G}$ the functors $\mathcal{E}_\alpha \xrightarrow{J_\alpha} \mathcal{E}[S^{-1}]$ reflect isomorphisms that are already in $\mathcal{A}_\alpha$.*

That is to say that if $f \in \mathcal{A}_\alpha$ and $J_\alpha(f)$ is an isomorphism, then f is an isomorphisms.

Proof. Suppose $X \xrightarrow{f} Y \in \mathcal{A}_\alpha$ is such that $J_\alpha(f)$ is an isomorphism. Let $Y \xleftarrow{t} Y^* \xrightarrow{g} X$ represent its inverse. Take $\alpha \xrightarrow{\varphi} F(Y^*)$ such that $F(t) \cdot \varphi = F(g) \cdot \varphi = \psi$ and construct the following commutative diagram as indicated below.

$$\begin{tikzcd}
Y^{****} \arrow[r, "y"] \arrow[d, "g^{***}"'] & Y^{***} \arrow[r, "v"] \arrow[d, "g^{**}"'] & Y^{**} \arrow[r, "s"] \arrow[d, "g^{*}"'] & Y^{*} \arrow[r, "t"] \arrow[dr, "g"] & Y \\
X^{***} \arrow[r, "x"] \arrow[d, "f^{***}"'] & X^{**} \arrow[r, "w"] \arrow[d, "f^{**}"'] & X^{*} \arrow[rr, "u"] \arrow[d, "f^{*}"'] & & X \arrow[d, "f"] \\
Y^{****} \arrow[r, "y"] \arrow[d, "g^{***}"'] & Y^{***} \arrow[r, "v"] \arrow[d, "g^{**}"'] & Y^{**} \arrow[r, "s"] \arrow[d, "g^{*}"'] & Y^{*} \arrow[r, "t"] \arrow[dr, "g"] & Y \\
X^{***} \arrow[r, "x"] & X^{**} \arrow[r, "w"] & X^{*} \arrow[rr, "u"] & & X
\end{tikzcd}$$

$$\begin{tikzcd}
F(X^{***}) \arrow[r, "F(x)"] & F(Y^{***}) \arrow[r, "F(v)"] & \alpha \arrow[r, "\varphi"] & F(Y^{*}) \arrow[r, shift left, "F(t)"] \arrow[r, shift right, "F(g)"'] & F(X)
\end{tikzcd}$$

Take s cartesian over φ, u cartesian over ψ and the corresponding vertical arrows g^* and f^*. So it happens that $f^*g^* = 1_{Y^{**}}$ and $g^*f^* = 1_{X^*}$ in $\mathcal{E}[S^{-1}]$. Take v a cartesian morphism such that $(f^*g^*)v = 1_{Y^{**}}v$ in $\mathcal{E}$ followed by w cartesian over $F(v)$. For the corresponding vertical arrows we have that $f^{**}g^{**} = 1_{Y^{***}}$ in $\mathcal{E}$ and $g^{**}f^{**} = 1_{X^{**}}$ in $\mathcal{E}[S^{-1}]$. Take x a cartesian morphism such that $(f^{**}g^{**})x = 1_{Y^{***}}x$ in $\mathcal{E}$ and y cartesian over $F(x)$. It follows that f^{***} and g^{***} are inverse of eachother in the fibre over $F(X^{***})$. The result follows.

□

0.5 CONSTRUCTION OF A REGULAR SET VALUED FUNCTOR THAT IS CONSERVATIVE OVER MONICS WITH GLOBALLY SUPPORTED CODOMAIN FOR ANY REGULAR CATEGORY $\mathcal{A}$ THAT POSSESSES A DISTINGUISHED TERMINAL OBJECT

0.5.1 Construction of the Functor from $\mathcal{A}$ to $\mathcal{A}'$ That Sends Globally Supported Objects into Objects That Have a Generic Global Section

In this section $\mathcal{A}$ will denote a regular category that possesses a distinguished terminal object 1.

A fibration that has $\mathcal{A}$ as its fibres

For the following fibration we will have that $\mathcal{A}$ can be identified as de fibre over $\{1\}$.

The Cofilitered Base for the Fibration

Strict epimorphisms are closed under composition in $\mathcal{A}$ (0.2.11). Take $\mathcal{G}l_s(\mathcal{A})$ the category whose objects are the globally supported objects of $\mathcal{A}$ and whose morphisms are the strict epimorphisms in $\mathcal{A}$. We define $\mathcal{G}_{\mathcal{A}}$ to be the category whose objects are finite sequences of objects $\{B_i\}_{i\in[n]} \subset \mathcal{G}l_s(\mathcal{A})$

whose first term is $B_0 = 1$. A morphism $\{B_i\}_{i\in[n]} \xrightarrow{\varphi} \{C_j\}_{j\in[m]} \in \mathcal{G}_\mathcal{A}$ is a function $[m] \xrightarrow{\varphi} [n]$ that verifies $\varphi(0) = 0$ and that for every $j \in [m]$ it verifies $B_{\varphi(j)} = C_j$.

Remark 0.5.1. $\mathcal{G}_\mathcal{A}$ is a cofiltered category. More so it is finitely complete and has a *unique* terminal object. This can be verified interpreting $\mathcal{G}_\mathcal{A}^{op}$ embedded in $\mathcal{E}ns^*/\mathcal{G}l_s(\mathcal{A})$ where $\mathcal{E}ns^*$ denotes the category of pointed sets and where we distinguish $1 \in \mathcal{G}l_s(\mathcal{A})$.

A Finitely Complete Fibration

We will give an explicit description of Grothendiecks construction of a split cofibration associated to the covariant functor $D_\mathcal{A} : \mathcal{G}_\mathcal{A} \longrightarrow \mathcal{C}at$ that assigns to each object $\{B_i\}_{i\in[n]}$ the multislice category $\mathcal{A}_{/\{B_i\}_{i\in[n]}}$ and to each arrow $\{B_i\}_{i\in[n]} \xrightarrow{\varphi} \{C_j\}_{j\in[m]}$ the functor $\varphi_* : \mathcal{A}_{/\{B_i\}_{i\in[n]}} \longrightarrow \mathcal{A}_{/\{C_j\}_{j\in[m]}}$ that is defined as $\varphi_*(\{X \xrightarrow{x_i} B_i\}_{i\in[n]}) = \{X \xrightarrow{x_{\varphi(j)}} C_j\}_{j\in[m]}$ on objects and is the identity on arrows.

Take $\mathcal{E}_\mathcal{A}$ the category whose objects are ordered pairs (X, α) where $\alpha \in \mathcal{G}_\mathcal{A}$ and $X \in D_\mathcal{A}(\alpha)$. Its arrows are ordered pairs $(X, \alpha) \xrightarrow{(f,\varphi)} (Y, \beta)$ where $\alpha \xrightarrow{\varphi} \beta \in \mathcal{G}_\mathcal{A}$ and $f : \varphi_* X \longrightarrow Y$. Composition is defined for $(Y, \beta) \xrightarrow{(g,\psi)} (Z, \gamma)$ as $(g, \psi)(f, \varphi) = (g \cdot \psi_*(f), \psi\varphi)$. Take $F_\mathcal{A}$ be the projection in the second coordinate. The arrow $(X, \alpha) \xrightarrow{(1_{\varphi_* X},\varphi)} (\varphi_* X, \beta)$ is cocartesian over φ with source X and these arrows are closed under composition. The projection in the first coordinate restricted to a fiber $(\mathcal{E}_\mathcal{A})_\alpha \xrightarrow{\pi_1} D_\mathcal{A}(\alpha)$ is an isomorphism. If φ_* denoted the (co) pullback functor along φ we have in fact this isomorphism that is natural in the following sense:

$$\begin{tikzcd} (\mathcal{E}_\mathcal{A})_\alpha \arrow[r, "\varphi_*"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\equiv"] & (\mathcal{E}_\mathcal{A})_\beta \arrow[d, "\pi_1"] \\ D_\mathcal{A}(\alpha) \arrow[r, "D_\mathcal{A}(\varphi)"'] & D_\mathcal{A}(\beta) \end{tikzcd} \tag{0.5.2}$$

Thus we can make the abuse of language of identifying the fiber of the split cofibration $\mathcal{E}_\mathcal{A} \xrightarrow{F_\mathcal{A}} \mathcal{G}_\mathcal{A}$ over $\{B_i\}_{i\in[n]}$ with $\mathcal{A}_{/\{B_i\}_{i\in[n]}}$ and similarly identify the cotransport functor along φ with $D_\mathcal{A}(\varphi)$.

Proposition 0.5.3. *$F_\mathcal{A}$ is a fibration.*

Proof. FIX-001.02 START Dar referencia directa a SGA1 en lugar de proposition 2.2

It suffices to prove that $F_\mathcal{A}$ is prefibered (see [5, page 143]).

FIX-001.02 END

Take $\{B_i\}_{i\in[n]} \xrightarrow{\varphi} \{C_j\}_{j\in[m]} \in \mathcal{G}_\mathcal{A}$ and $\{Y \xrightarrow{y_j} C_j\}_{j\in[m]}$ over $\{C_j\}_{j\in[m]}$. Let $\mathcal{D}_\varphi$ be the finite graph whose objects are $[n+1]$ and whose arrows are identified with $[m]$. The arrow $j \in [m]$ has source n and target $\varphi(j)$. The object $\{Y \xrightarrow{y_j} C_j\}_{j\in[m]}$ induces a functor $\mathcal{D}_\varphi \xrightarrow{\tilde{Y}} \mathcal{A}$ defined as $\tilde{Y}n = Y$, as $\tilde{Y}i = B_i$ for any other $i \in [n]$ and $\tilde{Y}j = y_j$ on arrows.

A cone for this functor is a family of arrows $\{X \xrightarrow{x_i} \tilde{Y}i\}_{i\in[n+1]}$ such that for every $j \in [m]$ the following diagram is commutative.

$$\begin{tikzcd} & X \arrow[dl, "x_n"'] \arrow[dr, "x_{\varphi(j)}"] \arrow[d, phantom, "\equiv"] & \\ Y \arrow[rr, "y_j"'] & & B_{\varphi(j)} \end{tikzcd}$$

Thus it is naturally identified with a morphism $\{X \xrightarrow{x_i} \tilde{Y}i\}_{i\in[n]} \xrightarrow{(x_n,\varphi)} \{Y \xrightarrow{y_j} C_j\}_{j\in[m]}$ over φ with target $\{Y \xrightarrow{y_j} C_j\}_{j\in[m]}$. A limit cone corresponds to a cartesian morphism. Since $\mathcal{A}$ is finitely complete the result follows. $\square$

Proposition 0.5.4. *$F_{\mathcal{A}}$ is finitely complete.*

Proof. It follows from Theorem 0.3.29 and that the fibers are multislice categories of a regular category, in particular finitely complete. $\square$

A Regular Fibration

We will in fact prove that $\mathcal{E}_{\mathcal{A}} \xrightarrow{F_{\mathcal{A}}} \mathcal{G}_{\mathcal{A}}$ is a regular fibration.

Lemma 0.5.5. *In $\mathcal{E}_{\mathcal{A}}$ if*

$$\begin{tikzcd}
\{W \xrightarrow{w_i} B_i\}_{i\in[n]} \arrow[r, "(f,\varphi)"] \arrow[d, "(a,1)"'] \arrow[dr, phantom, "\equiv"] & \{X \xrightarrow{x_j} C_j\}_{j\in[m]} \arrow[d, "(b,1)"] \\
\{Z \xrightarrow{z_i} B_i\}_{i\in[n]} \arrow[r, "(g,\varphi)"] & \{Y \xrightarrow{y_j} C_j\}_{j\in[m]}
\end{tikzcd}$$

$$\{B_i\}_{i\in[n]} \xrightarrow{\quad\varphi\quad} \{C_j\}_{j\in[m]}$$

is such that (f,φ) and (g,φ) are cartesian, then

$$\begin{tikzcd}
W \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\equiv"] & X \arrow[d, "b"] \\
Z \arrow[r, "g"'] & Y
\end{tikzcd}$$

is a pullback in $\mathcal{A}$.

Proof. For any cone $\{V \xrightarrow{h} X, V \xrightarrow{c} Z\}$ in $\mathcal{A}$ we have the object $\{V \xrightarrow{z_i c} B_i\}_{i\in[n]}$ together with the cone $\{(c,1),(h,\varphi)\}$. A factorization of the former cone in $\mathcal{A}$ is identified with a factorization of the latter in $\mathcal{E}_{\mathcal{A}}$ over $\{B_i\}_{i\in[n]}$. $\square$

Proposition 0.5.6. *Strict epimorphisms are stable in the fibration $F_{\mathcal{A}}$.*

Proof. It follows from Lemma 0.5.5 and the fact that the domain functors Σ in multislice categories preserve and reflect strict epimorphisms. $\square$

Corollary 0.5.7. *$F_{\mathcal{A}}$ is a regular fibration.*

Construction of the colimit $\mathcal{A}'$ of the fibration and proof that including the first fibre is conservative over monics with globally supported codomain

Theorem 0.4.12 guarantees that the colimit of this fibration is a regular category and in particular the functors $J_{\{1\}}$ in the diagram below is regular.

$$\begin{tikzcd}
(\mathcal{E}_{\mathcal{A}})_{\{1\}} \arrow[rr, "J_{\{1\}}"] \arrow[dr, "j_{\{1\}}"'] & \equiv & \mathcal{E}_{\mathcal{A}}[S^{-1}] \\
& \mathcal{E}_{\mathcal{A}} \arrow[ur, "Q"'] &
\end{tikzcd}$$

Identifying $(\mathcal{E}_{\mathcal{A}})_{\{1\}}$ with $\mathcal{A}$ we will label to top arrow in the diagram with $\mathcal{A} \xrightarrow{j} \mathcal{A}'$. For a morphism $X \xrightarrow{f} Y \in \mathcal{A}$ will use the abuse of language of saying $X \xrightarrow{f} Y$ in $\mathcal{A}'$ referring to the morphism $j(X) \xrightarrow{j(f)} j(Y) \in \mathcal{A}'$. Taking into consideration that j transforms 1 into a terminal object, preserves monics and preserves strict epimorphisms makes the abuse coherent with these objects.

A generic section for every $B \twoheadrightarrow 1 \in \mathcal{A}$

Take a globally supported object $B \overset{\pi}{\twoheadrightarrow} 1 \in \mathcal{A}$. The fiber over $\{1, B\}$ is naturally identified with $\mathcal{A}_{/B}$. Choose a product $\{B \times B \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} B\}$ of B with itself in $\mathcal{A}$ and take $B \xrightarrow{\Delta} B \times B$ the diagonal morphism. We obtain the following diagram in $\mathcal{E}_{\mathcal{A}}$.

$$\begin{tikzcd}
\{P \xrightarrow{\pi_2} B\} \arrow[r, "(\pi_1,\varphi)"] & B \\
\{B \xrightarrow{id_B} B\} \arrow[u, "(\Delta,1)"] \arrow[r, "(\pi,\varphi)"'] & 1 \\
\{1, B\} \arrow[r, "\varphi"] & \{1\}
\end{tikzcd}$$

Lemma 0.5.8. *$\{B \xrightarrow{id_B} B\} \xrightarrow{(\pi,\varphi)} 1$ and $\{P \xrightarrow{\pi_2} B\} \xrightarrow{(\pi_1,\varphi)} B$ are cartesian morphisms.*

Proof. This follows immediately using the characterization of cartesian morphisms given in Proposition 0.5.3. $\square$

Remark 0.5.9. We have a section $\frac{(\pi_1\Delta,\varphi)}{(\pi,\varphi)} = \frac{(1_B,\varphi)}{(\pi,\varphi)}$ of $B \overset{\pi}{\twoheadrightarrow} 1$ in $\mathcal{A}'$. This section is in fact *canonical* in the sense that any choice of product $\{B \times B \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} B\}$ of B with itself in $\mathcal{A}$ will induce the *same* arrow in $\mathcal{A}'$ built this way. This follows from the fact that (π_1, φ) is cartesian. We will label this uniquely determined arrow $1 \xrightarrow{\Delta_B} B$.

Separating B from its subobjects in $\mathcal{A}$

We will prove is that $1 \xrightarrow{\Delta_B} B$ *separates* B from its subobjects in $\mathcal{A}$, in the sense of Theorem 0.5.11.

Lemma 0.5.10. *For any $\{B_i\}_{i\in[n]} \xrightarrow{\psi} \{1, B\}$ in $\mathcal{G}$, if $\{X \xrightarrow{x_i} B_i\}_{i\in[n]} \xrightarrow{(f,\psi)} \{B \xrightarrow{id_B} B\}$ is cartesian, then f is a strict epimorphism in $\mathcal{A}$.*

Proof. Note that $\{X \xrightarrow{x_i} B_i\}_{i\in[n]}$ is a product of the family $\{B_i\}_{i\in[n]}$ and that f is in fact one of the projections. The result follows (recall every B_i is globally supported and 0.2.14). $\square$

Theorem 0.5.11. *If $A \rightarrowtail^{m} B \in \mathcal{A}$ is such that Δ_B lifts along m,*

$$\begin{tikzcd} A \arrow[r, tail, "m"] & B \\ & 1 \arrow[u, "\Delta_B"'] \arrow[ul, dotted, "\exists"] \arrow[ul, phantom, "\equiv"] \end{tikzcd}$$

it follows that m is an isomorphism.

Proof. In our context the existence of such a lifting of Δ_B reduces to there being a morphism $\{B_i\}_{i\in[n]} \xrightarrow{\psi} \{1, B\}$ in $\mathcal{G}_{\mathcal{A}}$, a cartesian morphism $\{X \xrightarrow{x_i} B_i\}_{i\in[n]} \xrightarrow{(f,\psi)} \{B \xrightarrow{id_B} B\}$ over ψ and a morphism $\{X \xrightarrow{x_i} B_i\}_{i\in[n]} \xrightarrow{(g,\varphi\circ\psi)} A$ such that the following diagram is commutative.

$$\begin{tikzcd} & & A \arrow[d, tail, "m"] \\ & \{P \xrightarrow{\pi_2} B\} \arrow[r, "(\pi_1,\varphi)"] & B \\ \{X \xrightarrow{x_i} B_i\}_{i\in[n]} \arrow[r, "(f,\psi)"] \arrow[urr, bend left, "(g,\varphi\circ\psi)"] & \{B \xrightarrow{id_B} B\} \arrow[u, "(\Delta,1)"] \arrow[r, "(\pi,\varphi)"'] & 1 \end{tikzcd}$$

$$\{B_i\}_{i\in[n]} \xrightarrow{\psi} \{1, B\} \xrightarrow{\varphi} \{1\}$$

It follows that $mg = f$ and together with Lemma 0.5.10 m must be an isomorphism. $\square$

Corollary 0.5.12. *The functor $j_{\{1\}}$ is conservative over monics with globally supported codomain.*

0.5.2 Construction of $\mathcal{A}^\infty$ Where 1 is Weakly Projective

A new fibration that has $\mathcal{A}$ as its first fibre, proof that the inclusion of any fibre into the colimit $\mathcal{A}^\infty$ is conservative over monics with globally supported codomain and $1 \in \mathcal{A}^\infty$ is weakly projective

Iterating the construction in Section 0.5.1 we obtain the following sequence of regular functors.

$$\mathcal{A} \xrightarrow{j} \mathcal{A}' \xrightarrow{j'} (\mathcal{A}')' \longrightarrow \cdots$$

Using the dual construction in 0.5.1 we obtain from this diagram a split regular fibration $\mathcal{E} \xrightarrow{\mathcal{F}} {\mathcal{N}_0}^{op}$ whose base is cofiltered. $\mathcal{A}^{(n)}$ will denote the fiber over n and j_n the $(n+1)^{th}$ functor of the diagram. We will make the same identification between j_n and the transfer functor along $n+1 \longrightarrow n$. The hypothesis of Theorem 0.4.12 are satisfied and it follows that the colimit of this fibration $\mathcal{A}^\infty$ is a regular category and the morphisms $\mathcal{A}^{(n)} \xrightarrow{J_n} \mathcal{A}^\infty$ are regular.

Remark 0.5.13. The functors j_n preserve monics with globally supported codomain and are conservative over them (see 0.5.11). It follows that the fibration is conservative over vertical monics with globally supported codomain.

Corollary 0.5.14. *The functors $\mathcal{A}^{(n)} \xrightarrow{J_n} \mathcal{A}^{\infty}$ are conservative over monics with globally supported codomain.*

Proof. See 0.4.20. □

Proposition 0.5.15. *The functors $\mathcal{A}^{(n)} \xrightarrow{J_n} \mathcal{A}^{\infty}$ reflect globally suported objects.*

Proof. Let $X \in cc A^{(n)}$ be such that $J_n(X)$ has global support in $\mathcal{A}^{\infty}$. Take a strict epic-monic factorization of $X \xrightarrow{f} 1$ in $\mathcal{A}^{(n)}$.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "e"'] & \equiv & 1 \\ & S \arrow[ur, tail, "m"'] & \end{tikzcd}$$

Since $J_n(f)$ and $J_n(e)$ are strict epimorphisms, it follows that $J_n(m)$ is an isomorphism. Corollary 0.5.14 guarantees that m is an isomorphism. The result follows. □

Remark 0.5.16. For $A \xrightarrow{f} B$ in $\mathcal{A}^{(n)}$, $j_n(A) \xrightarrow{j_n(f)} j_n(B)$ is a transfer of f along $n+1 \longrightarrow n$.

Theorem 0.5.17. *1 is weakly projective in $\mathcal{A}^{\infty}$.*

Proof. Let $B \in \mathcal{A}^{\infty}$ be a globally supported object. Because of Proposition 0.5.15 it is globally supported in the fiber over $n = \mathcal{F}(B)$. Since $j_n(B)$ is isomorphic to B in $\mathcal{A}^{\infty}$ (0.5.16) and we have the generic section $1 \xrightarrow{\Delta_B} j_{\mathcal{A}^{(n)}}(B)$, the result follows. □

The representable functor of $1 \in \mathcal{A}^{\infty}$ is conservative over monics with globally supported codomain

Theorem 0.5.18. *The functor $\mathcal{A}^{\infty} \xrightarrow{[1,\_]} \mathcal{E}ns$ is conservative over monics with globally supported codomain. In particular we have constructed a functor $\mathcal{A} \xrightarrow{J_0} \mathcal{A}^{\infty} \xrightarrow{[1,\_]} \mathcal{E}ns$ that is conservative over monics with globally supported codomain.*

Proof. It suffices to prove that the regular functors $\mathcal{A}^{(n)} \xrightarrow{J_n} \mathcal{A}^{\infty} \xrightarrow{[1,\_]} \mathcal{E}ns$ are conservative over monics with globally supported codomain. Let Γ represent the functor $[1,\_]$. We will use the abuse of notation of suppressing the symbol J_k when it is clear that we are viewing an element of a fiber inside of $\mathcal{A}^{\infty}$. Let $A \overset{m}{\rightarrowtail} B \in \mathcal{A}^{(n)}$ be monic with globally supported codomain in $\mathcal{A}^{(n)}$ such that $\Gamma(m)$ is an isomorphism (that is to say $\Gamma(J_n(m))$). Since $A \overset{m}{\rightarrowtail} B$ is isomorphic to $j_n(A) \overset{j_n(m)}{\rightarrowtail} j_n(B)$ in $\mathcal{A}^{\infty}$ (Remark 0.5.16) we have that $\Gamma(j_n(m))$ is an isomorphism, which in particular means that any section of $j_n(B) \longrightarrow 1$ lifts along $j_n(m)$ in $\mathcal{A}^{\infty}$. Take a pullback of $j_n(m)$ along Δ_B in $\mathcal{A}^{(n+1)}$.

$$\begin{tikzcd} j_n(A) \arrow[r, tail, "j_n(m)"] & j_n(B) \\ P \arrow[u, dotted, "p_1"] \arrow[r, tail, dotted, "p_2"'] \arrow[ur, phantom, "p.b."] & 1 \arrow[u, "\Delta_B"'] \end{tikzcd}$$

It suffices to prove that p_2 admits a section in $\mathcal{A}^{(n+1)}$. This diagram viewed in $\mathcal{A}^\infty$ is still a pullback and p_2 is still monic. Since Δ_B lifts along $j_n(m)$ in $\mathcal{A}^\infty$ we obtain a section of p_2 in $\mathcal{A}^\infty$. Thus p_2 is an isomorphism in $\mathcal{A}^\infty$. Thus we conclude p_2 is an isomorphism in $\mathcal{A}^{(n+1)}$. The result follows. □

0.6 REDUCTIONS

0.6.1 A Regular Functor That is Conservative Over Monics With Globally Supported Codomain Suffices

In this section we will prove that by making the following assumption on our category $\mathcal{C}$ we will obtain the *Sufficient Points* theorem if we are able to construct a function that associates to each regular category $\mathcal{A}$ that possesses a distinguished terminal object, a regular functor $\mathcal{A} \longrightarrow \mathcal{E}ns$ that is conservative over monics with globally supported codomain.

Assumption 0.6.1. *$\mathcal{C}$ possesses a distinguished terminal object which we denote with 1, and a distinguished representative for each subobject class in $\mathcal{C}$.*

Remark 0.6.2. This assumption does not affect our desired range of applicability when proving completeness theorems in logic.

We will denote the distinguished representatives of a subobject class with a curly arrow $\hookrightarrow$ and for every object $X \in \mathcal{C}$ we will choose $X \overset{1_X}{\hookrightarrow} X$ as the distinguished representative of its subobject class.

Observation. For every $X \in \mathcal{C}$ the slice category $\mathcal{C}_{/X}$ has the distinguished terminal object $X \xrightarrow{1_X} X$. Additionally since the domain functor $\mathcal{C}_{/X} \xrightarrow{\Sigma} \mathcal{C}$ preserves and reflects monics there are distinguished representatives for each subobject class in $\mathcal{C}_{/X}$.

Definition 0.6.3. *For $S \hookrightarrow 1$ we say $X \in \mathcal{C}$ **has support in** S if*

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, dashed, "\exists"'] & \arrow[d, phantom, "\equiv"] & 1 \\ & S \arrow[ur, hook] & \end{tikzcd}$$

*and that S **is the support of** X if the dashed arrow is a strict epimorphism.*

Observation. X is globally supported if and only if 1 is the support of X (see 0.2.12).

Pullback functor along a monomorphism

Proposition 0.6.4. *For $A \overset{m}{\rightarrowtail} B$ there are distinguished pullbacks along m.*

Proof. Pullbacks are well defined on subobject classes. For $W \xrightarrow{w} B$ take a pullback that uses the representative of the corresponding subobject class of m along w and label it $m^*(W) \hookrightarrow W$.

$$\begin{tikzcd} m^*(W) \arrow[r, hook] \arrow[d, dashed] \arrow[dr, phantom, "p.b."] & W \arrow[d, "w"] \\ A \arrow[r, tail, "m"'] & B \end{tikzcd}$$

The dashed arrow is uniquely determined because m is monic. That arrow will be denoted $m^*(w)$.
□

Remark 0.6.5. Having this choice of pullbacks along m determines a functor $\mathcal{C}_{/B} \xrightarrow{m^*} \mathcal{C}_{/A}$ which is right adjoint to the functor defined as postcomposing by m. Thus it preserves all limits. Since the following diagram is a pullback and the domain functor Σ preserves and reflects strict epimorphisms we have that m^* is a regular functor.

$$\begin{tikzcd} m^*(X) \arrow[r, hook] \arrow[d, "m^*(f)"'] \arrow[dr, phantom, "p.b."] & X \arrow[d, "f"] \\ m^*(Y) \arrow[r, hook] & Y \end{tikzcd}$$

Observation. If $A \overset{m}{\hookrightarrow} B$ is a distinguished subobject, we have that $m^*(1_B) = 1_A$. That is m^* transforms the distinguished terminal object of $\mathcal{C}_{/B}$ into the distinguished terminal object in $\mathcal{C}_{/A}$. If the distinguished subobjects in $\mathcal{C}$ are closed under composition and from

$$\begin{tikzcd} A \arrow[rr, hook] \arrow[dr, "u"'] & \equiv & B \\ & A' \arrow[ur, hook] & \end{tikzcd}$$

it follows that u is a distinguished subobject, then m^* transforms distinguished subobjects into distinguished subobjects.

In the particular case where we take a distinguished subobject of 1 we will use the following notation. For $S \hookrightarrow 1$ we have the pullback functor $\mathcal{C} \xrightarrow{S^\wedge} \mathcal{C}_{/S}$ and denote its action as follows.

$$S^\wedge(X \xrightarrow{f} Y) = \begin{tikzcd} X \wedge S \arrow[rr, "f \wedge S"] \arrow[dr] & \equiv & Y \wedge S \arrow[dl] \\ & S & \end{tikzcd}$$

Lemma 0.6.6. *If $Y \in \mathcal{C}$ has support in S, then for $X \xrightarrow{f} Y$ we have that $Y \wedge S = Y$, $X \wedge S = X$ and $f \wedge S = f$.*

Proof. It follows from the fact that for such a Y the following diagram is a pullback.

$$\begin{tikzcd} Y \arrow[r, hook, "1_Y"] \arrow[d] \arrow[dr, phantom, "p.b."] & Y \arrow[d] \\ S \arrow[r, hook, "i"'] & 1 \end{tikzcd}$$

and that X has support in S as well. □

Corollary 0.6.7. *$S^\wedge(S)$ is a terminal object in $\mathcal{C}_{/S}$ and $S^\wedge$ is conservative over morphisms whose target has support in S.*

Proof. In fact $S^\wedge(S) = 1_S$. □

The result

Theorem 0.6.8. *If for every $S \hookrightarrow 1$ we are given a regular functor $\mathcal{C}_{/S} \xrightarrow{\Gamma_S} \mathcal{E}ns$ that is conservative over monics with globally supported codomain, then the family of regular functors of $\mathcal{C} \longrightarrow \mathcal{E}ns$ is monic-conservative.*

Proof. Consider the following family of regular functors of $\mathcal{C} \longrightarrow \mathcal{E}ns$.

$$\mathcal{C} \xrightarrow{S^\wedge} \mathcal{C}_{/S} \xrightarrow{\Gamma_S} \mathcal{E}ns$$

It is a family of functors $\{h_S\}$ indexed by the set $Sub(1)$ of subobject classes of 1. It suffices to prove that this family is monic-conservative. Let $X \overset{f}{\rightarrowtail} Y$ in $\mathcal{C}$ be such that its image through all these functors is an isomorphism. Take S the support of Y. It is enough to prove that $f \wedge S$ is an isomorphism. But this follows from the fact that $f \wedge S$ is monic, $Y \wedge S \longrightarrow S$ has global support in $\mathcal{C}_{/S}$ and $\Gamma_S(f \wedge S)$ is an isomorphism. $\square$

Corollary 0.6.9. *The family of regular functors of $\mathcal{C} \longrightarrow \mathcal{E}ns$ is conservative.*

Proof. It follows from Theorem 0.5.18 and the Remark 0.2.36. $\square$

0.6.2 From a Family $\{h_S\}$ to h

Consider the following general construction for a family of set-valued functors $\{h_i\}_{i \in I}$ with common domain $\mathcal{C}$. Let $\mathcal{I}$ denote the category whose object set is I and for $i, j \in I$ we define $hom_{\mathcal{I}}(i, j) = Nat(h_i, h_j)$ with composition defined naturally. Let $\mathcal{C} \xrightarrow{h} \mathcal{E}ns^{\mathcal{I}}$ denote de functor defined as $h(C)(i) = h_i(C)$.

Remark 0.6.10. This asignment is functorial in both variables. We have the following commutative diagram for every $i \in I$.

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "h"] \arrow[dr, "h_i"'] & \equiv & \mathcal{E}ns^{\mathcal{I}} \arrow[dl, "ev_i"] \\ & \mathcal{E}ns & \end{tikzcd}$$

Given the pointwise structure of the regular category $\mathcal{E}ns^{\mathcal{I}}$ we have that h preserves finite limits if and only if for every $i \in I$ the functors h_i preserve finite limits, h preserves strict epimorphisms if and only if for every $i \in I$ the functors h_i preserve strict epimorphisms and h is conservative if and only if $\{h_i\}_{i \in I}$ is a conservative family as in 0.2.3.

In our particular case we have constructed a conservative family of regular functors $\{\mathcal{C} \xrightarrow{h_S} \mathcal{E}ns\}_{S \in Sub(1)}$. Using the previous construction we give $Sub(1)$ a structure and obtain a regular conservative functor $\mathcal{C} \xrightarrow{h} \mathcal{E}ns^{Sub(1)}$. The End.

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