Demostración constructiva del teorema de Barr

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Abstract

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INTRODUCTION

The theorem we prove in this work lies in between the realms of *Embedding Theorems* and *Sufficient Points Theorems*. Embedding theorems are those that fully represent an abstract mathematical object within a concrete one of the same nature. These theorems shed light on the nature of the abstraction itself. Some embeddings where the representation is in a way *canonical* are the following:

1. *Cayley*: $G \hookrightarrow S!$, $S = Underlying\ set\,(G)$.
(representation of a group as a subgroup of the symmetric group).

2. *Stone*: $B \hookrightarrow Sub(S)$, $S = Prime\ Ideals\,(B)$.
(representation of a boolean algebra as a subalgebra of the algebra of subsets)

3. *Gelfand*: $A \hookrightarrow \mathcal{S}et(S, \mathbb{C})$, $S = Closed\ Maximal\ Ideals\,(C)$
(representation of a C^*-algebra as a subalgebra of the algebra complex valued functions)

4. *Yoneda*: $\mathcal{C} \hookrightarrow \mathcal{S}et^{\mathcal{D}}$, $\mathcal{D} = \mathcal{C}^{op}$.
(representation of a small category $\mathcal{C}$ as a full subcategory of a category of $\mathcal{S}et$-valued functors.

The next example, *Barr's Representation Theorem*, is the theorem that concerns us in this work.

Theorem *For any small regular category $\mathcal{C}$ there exists a fully faithful regular functor $\mathcal{C} \overset{h}{\hookrightarrow} \mathcal{E}ns^{\mathcal{D}}$ into a set valued functor category, where the exponent has as objects the set $Sub(1)$ of subobjects of 1.*

Barr's original proof as well as all known proofs of this theorem are highly not constructive, using transfinite induction and the axiom of choice. The purpose of this work is to develop a constructive proof of a weaker form of Barr's theorem, namely that the functor h is *conservative* (reflects isomorphisms). This is in fact a *Sufficient Points Theorem.*

After the leading work of William Lawvere *completeness theorems* of logical theories were formulated in categorical terms as *Sufficient Points Theorems*. Informally, given a model $A \in \mathcal{C}$ of a theory $\mathcal{T}$ in an *appropiate* category $\mathcal{C}$, any formula $\varphi(x)$ has an extension in A which is a subobject $[\![x \in A \mid \varphi(x)\ holds]\!] \hookrightarrow A$. The idea is to associate to a theory $\mathcal{T}$ an *appropiate* category $\mathcal{C}_{\mathcal{T}}$ equipped with a model (*the generic model*) $G_{\mathcal{T}}$ of $\mathcal{T}$ that is generic in two senses:

1. It furnishes a (tautological by the very construction of $\mathcal{C}_{\mathcal{T}}$) completeness theorem for the theory $\mathcal{T}$. That is, a formula $\varphi(x)$ is provable in $\mathcal{T}$ if and only if $[\![x \in G_{\mathcal{T}} \mid \varphi(x)\ holds]\!] = G_{\mathcal{T}}$.

2. For any model $A \in \mathcal{C}$ of $\mathcal{T}$ in an appropiate category, there exist a unique *appropiate* functor $\mathcal{C}_{\mathcal{T}} \xrightarrow{F} \mathcal{C}$ such that $F(G_{\mathcal{T}}) = A$.

Clearly a *conservative* (thus a *monic-conservative*, i.e. if a monomorphism is sent by every functor in the family to an isomorphism it is itself an isomorphism) (see 1.3) family of appropriate set valued functors $\mathcal{C}_{\mathcal{T}} \xrightarrow{F} \mathcal{E}ns$ (in some contexts called *points*) yields a completeness theorem.

For first order intuicionistic geometric logic (that is admitting the intuicionistic propositional calculus and only the existencial quantifier), the appropriate categories are exactly the *regular categories* and the appropriate functors are *regular functors*. In this way, the weak version of Barr's theorem yields a completeness theorem for these logics using the following argument. Given any regular category $\mathcal{C}$ we wish to see that the family of regular functors $\mathcal{C} \longrightarrow \mathcal{E}ns$ is conservative. Barr's weak theorem guarantees us that there is a conservative regular functor $\mathcal{C} \xrightarrow{h} \mathcal{E}ns^{Sub(1)}$. Evaluations $\mathcal{E}ns^{Sub(1)} \longrightarrow \mathcal{E}ns$ are regular and the family of evaluations is conservative. The claim follows.

Independently of these considerations the purpose of this work is to develop a *constructive* proof of the weaker form of Barr's Theorem.

Outline of the Construction

For the construction we followed a guideline set by André Joyal in some unpublished talks given in Montreal in the early seventies. His proof was inspired in reinterpreting Leon Henkin's proof by adding constants of the *Gödel Completeness Theorem* of first order logic.

To carry out the whole proof constructively we need the additional hypothesis that the regular category $\mathcal{C}$ possesses a distinguished terminal object 1 and that distinguished subobject representatives for every subobject class exist in $\mathcal{C}$. These hypothesis will not affect our desired range of applications to logic. That is to say, for first order intuicionistic geometric logic theories the categories $\mathcal{C}_{\mathcal{T}}$ verify these additional hypothesis.

We start by constructing for any regular category $\mathcal{A}$ that possesses a distinguished terminal object 1 a regular functor $\mathcal{A} \longrightarrow \mathcal{E}ns$ that is conservative over monics with globally supported codomain (Section 4). This is achieved by constructing a regular functor $\mathcal{A} \xrightarrow{J_0} \mathcal{A}^{\infty}$ where $1 \in \mathcal{A}^{\infty}$ is weakly projective (thus the functor $\mathcal{A}^{\infty} \xrightarrow{[1,-]} \mathcal{E}ns$ is regular) and such that both J_0 and $[1,-]$ are conservative over monics with globally supported codomain. The construction of the functor $\mathcal{A} \xrightarrow{J_0} \mathcal{A}^\infty$ is carried out in two stages.

In the first stage we construct a functor $\mathcal{A} \xrightarrow{j} \mathcal{A}'$ in which we "add" a *generic global section* for each globally supported object in $\mathcal{A}$ (subsection 4.1). The basis of this construction lies on the following *idea* of how to construct a generic global section for a chosen globally supported object $B \overset{e}{\twoheadrightarrow} 1 \in \mathcal{A}$. If we had a *choice* of pullbacks along e we would have a faithful regular functor $\mathcal{A} \xrightarrow{e^*} \mathcal{A}/B$ and a global section of $e(B)$ described in the following diagram.

$$\begin{tikzcd}
e^*(1) = B \arrow[rr, dashed, "\Delta_B"] \arrow[dr, "1_B"'] & \equiv & B \times B = e^*(B) \arrow[dl, "\pi_2"] \\
& B &
\end{tikzcd}$$

The section Δ_B is generic in the following sense:

If $A \overset{m}{\rightarrowtail} B \in \mathcal{A}$ is such that Δ_B lifts along m,

$$\begin{tikzcd}
A \times B \arrow[r, tail, "e^*(m)"] & B \times B \\
\arrow[r, phantom, "\equiv"] & B \arrow[u, "\Delta_B"'] \arrow[ul, dotted, "\exists"]
\end{tikzcd}$$

it follows that m is an isomorphism.

That is to say *if Δ_B lifts to A, then **every** global section will lift to A* (As a note, in our proof we will not suppose that a choice of pullbacks can be made). In order to add these sections for every globally supported object of $\mathcal{A}$ leads to the calculation of the colimit of the following pseudo diagram in $\mathcal{C}at$ (that is the diagram commutes up to a unique isomorphism).

$$\begin{tikzcd}
& \mathcal{A}/B \arrow[dr] & \\
\mathcal{A} \arrow[ur] \arrow[rr] \arrow[dr] & & \mathcal{A}/B \times B' = (\mathcal{A}/B)/B' = (\mathcal{A}/B')/B \\
& \mathcal{A}/B' \arrow[ur] &
\end{tikzcd} \tag{0.1}$$

We construct a fibration whose fibres are isomorphic to the slice categories $\mathcal{A}/B$ (in fact, since we do not assume there is a choice of products the fibres cannot be single sliced categories but must be *multislice* categories 1.2) and whose cofiltered base category contains in *some* way the indexing category of the pseudo diagram above. The inclusion of the fibre $\mathcal{A}/1$ in the colimit of the fibration is what we take as $\mathcal{A} \xrightarrow{j} \mathcal{A}'$.

The second stage consists of iterating the previous construction, yielding a filtered diagram of regular functors and calculating the filtered colimit $\mathcal{A}^\infty$ of this diagram (In fact we do not take the filtered colimit $\mathcal{C}at$ but take the colimit of the fibration associated to this diagram which is in fact *equivalent* to it) (subsection 4.2.1).

$$\mathcal{A} \xrightarrow{j} \mathcal{A}' \xrightarrow{j'} (\mathcal{A}')' \longrightarrow \cdots \longrightarrow \mathcal{A}^\infty$$

For our initial category $\mathcal{C}$ we have thus obtained a regular functor, which we label $\mathcal{C} \xrightarrow{\Gamma_1} \mathcal{E}ns$, that is conservative over monics with globally supported codomain. In subsection 5.1 we construct a regular functor $\mathcal{C} \longrightarrow \mathcal{E}ns$ for each distinguished subobject $S \hookrightarrow 1$ the is conservative over monics whose codomain is supported in S(This is a monic $A \xrightarrow{m} B$ for which the strict factorization of $B \longrightarrow 1$ is through S). Thus that family of regular functors indexed by the the set $Sub(1)$ of subobjects of 1 yields a monic conservative family of functors(thus a conservative family (section 1.3)).

Lastly in subsection 5.2 we describe generic method of constructing a functor $\mathcal{C} \longrightarrow \mathcal{E}ns^{\mathcal{I}}$ from a given family of functors $\{h_i\}_{i \in I}$ which in our case will yield the desired result.

Results

In order to prove the weaker form of Barr's theorem constructively we developed the concept of *Regular Fibration* which does not apear in the literature and proved *constructively* the fundamental result that the colimit of a regular fibration over a cofiltered category yields a *regular* category. This in fact has as particular case that a filtered colimit of regular categories is a regular category.

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1 CATEGORICAL PRELIMINARIES

1.1 Regular Categories

We will denote with $\mathcal{C}at$ the category of small categories, and $\mathcal{E}ns$ the category of sets.

Let $\mathcal{C}$ be a small category. In what follows all the diagrams are in $\mathcal{C}$.

Recall the notion of *strict epimorphism* introduced in SGA4 [3][10.3, p. 180].

Definition 1.1. *Given a morphism $X \xrightarrow{f} Y$, a morphism $X \xrightarrow{g} Z$ is* compatible *if for every pair $Z \underset{y}{\overset{x}{\rightrightarrows}} X$ such that $fx = fy$ it follows that $gx = gy$.*

A morphism $X \xrightarrow{f} Y$ is a strict epimorphism *if for every compatible $X \xrightarrow{g} Z$, there exists a unique $Y \xrightarrow{h} Z$ such that $h \circ f = g$. The situation is described in a diagram:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend left, "\forall\, g\ compatible"] & Y \arrow[r, dashed, "\exists !\, h"] & Z \end{tikzcd}$$

A morphism $X \xrightarrow{f} Y$ is a monomorphism precisely when id_X is compatible. Strict epimorphisms are epimorphisms, if $X \xrightarrow{f} Y$ is a monomorphism and a strict epimorphism, then it is an isomorphism. Strict epimorphisms do not compose in general.

Remark 1.2. For any arrow $X \xrightarrow{f} Y$, let $R_f \rightrightarrows X$ be a kernel pair of f. Then f is a strict epimorphism if and only if the diagram $R_f \rightrightarrows X \xrightarrow{f} Y$ is a coequalizer. It follows that when pullbacks exists *a functor that preserves pull-backs and coequalizers preserves strict epimorphisms.*

Recall the definition of regular category:

Definition 1.3. *A category is* regular *if it has finite limits, any arrow can be factorized into a monomorphism composed with an strict epimorphism, and strict epimorphisms are universal.*

Remark 1.4. Given $X \xrightarrow{f} Y \xrightarrow{g} Z$ in a regular category, the following holds:
1) If f and g are strict epimorphisms, so it is the composite $g \circ f$.
2) If the composite $g \circ f$ is a strict epimorphisms, so it is g.

Definition 1.5. *A functor between regular categories is a* regular functor *if it preserves finite limits (hence it preserves monics) and either one of the two following equivalent conditions hold:*
1) F preserves strict factorizations.
2) F preserves strict epimorphisms.

Definition 1.6. *An object X is* of global support *if for all terminal objects the morphism $X \longrightarrow 1$ is a strict epimorphism, or, equivalently, if there exists a terminal object such that the morphism $X \longrightarrow 1$ is a strict epimorphism.*

Proposition 1.7. *A finite product is an object of global support if and only if each factor is of global support. Furthermore, the projections are strict epimorphisms.*

Proof. For a binary product the statement follows immediately from 1.3 and 1.4. Then proceed by induction. □

1.2 Multislice Categories of Regular Categories

In this section we will define what a multislice category is and prove that multislice categories of finitely complete categories and of regular categories are finitely complete and regular, respectively.

Definition 1.8. *For a category $\mathcal{C}$ and a family $\{B_i\}_{i\in[n]}$ of objects of $\mathcal{C}$ we define the* multislice category $\mathcal{C}_{/\{B_i\}_{i\in[n]}}$ *whose objects are families $\{X \xrightarrow{x_i} B_i\}_{i\in[n]}$ of arrows of $\mathcal{C}$ and morphisms $\{X \xrightarrow{x_i} B_i\}_{i\in[n]} \xrightarrow{f} \{Y \xrightarrow{y_i} B_i\}_{i\in[n]}$ are arrows $X \xrightarrow{f} Y$ in $\mathcal{C}$ such that for every $i \in [n]$*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "x_i"'] & \equiv & Y \arrow[dl, "y_i"] \\ & B_i & \end{tikzcd}$$

Remark 1.9. Clearly there is a forgetful functor, $\mathcal{C}_{/\{B_i\}_{i\in I}} \xrightarrow{\Sigma} \mathcal{C}$, $\Sigma(\{X \xrightarrow{x_i} B_i\}_{i\in[n]} = X$, $\Sigma(f) = f$, which is faithful and reflects isomorphisms. We will refer to this functor as *the functor* Σ. □

The following holds by definition of products:

Remark 1.10. Let $\{B \xrightarrow{\pi_i} B_i\}_{i\in[n]}$ be a product diagram in $\mathcal{C}$, then the functor:

$$\mathcal{C}_{/B} \xrightarrow{\Phi} \mathcal{C}_{/\{B_i\}_{i\in[n]}} : \qquad \Phi(X \xrightarrow{x} B) \;=\; \{X \xrightarrow{\pi_i x} B_i\}_{i\in[n]}$$

and $\Phi(f) = f$ on arrows, establishes an isomorphism of categories. Furthermore the family of its projections is a terminal object in the multislice category.

The reason we consider multislice categories is not that we want to work with categories lacking products, but lacking a distinguished choice of products.

For categories with finite products we can use this Remark to transport to multislice categories the properties of slice categories.

Though properties needed here may be proven directly for multislice categories, to simplify the notation we will do the proofs for slice categories and then use this remark.

Remark 1.11. Let $X \in \mathcal{C}$, let $B \xleftarrow{\pi_2} E \xrightarrow{\pi_1} X$ be a product diagram in $\mathcal{C}$, and let $Y \xrightarrow{y} B \in \mathcal{C}_{/B}$. Define $\Pi(X) = (E \xrightarrow{\pi_2} B)$, then we have:

$$\begin{tikzcd} Y \arrow[dr, "f"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "y"'] & & \\ & E \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & X \\ & B & \end{tikzcd} \qquad \frac{(Y \xrightarrow{y} B) \xrightarrow{f} \Pi(X)}{\Sigma(Y \xrightarrow{y} B) \xrightarrow{g} X}$$

By definition of product the diagram on the left establishes a bijective correspondence between the arrows f and g, which is the same that the correspondence indicated in the right diagram. This shows that the object $\Pi(X) \in \mathcal{C}_{/B}$ is the value at X of a right adjoint to the functor Σ, the defining universal property shows it is defined in arrows in a way that preserves composition, compare with Definition 2.4.

It follows that *when the product E exists the functor Σ will preserve any colimit that may exists in $\mathcal{C}_{/B}$.*

$\square$

The following is immediate and very easy to prove:

Proposition 1.12. *The category $\mathcal{C}_{/\{B_i\}_{i\in[n]}}$ inherits any pull-back that exists in $\mathcal{C}$, moreover the functor Σ preserves and reflects pull-backs. It follows it always preserves and reflects monomorphisms* $\square$

Since as observed in 1.10 for a category with finite products any multislice category has terminal objects, it follows:

Proposition 1.13. *If $\mathcal{C}$ is finitely complete, then $\mathcal{C}_{/\{B_i\}_{i\in[n]}}$ is finitely complete.* $\square$

From Remaks 1.2, 1.11, and Proposition 1.12 if follows:

Proposition 1.14. *If $\mathcal{C}$ has pullbacks and binary products, then the functor Σ preserves strict epimorphisms.*

Proposition 1.15. *The functor Σ always reflects strict epimorphisms.*

Proof. Let $X \xrightarrow{f} Y$ be a strict epimorphism in $\mathcal{C}$. We want to prove that any

$$(X \xrightarrow{x} B) \xrightarrow{f} (Y \xrightarrow{y} B) \qquad \begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, bend right, "x"'] & & Y \arrow[dl, bend left, "y"] \\ & B & \end{tikzcd}$$

is a strict epimorphism in $\mathcal{C}_B$. Let g be a f-compatible arrow in $\mathcal{C}_B$

$$(X \xrightarrow{x} B) \xrightarrow{g} (Z \xrightarrow{y} B) \qquad \begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"] \arrow[ddr, bend right, "x"'] & & Y \arrow[ddl, bend left, "y"] \\ & Z \arrow[d, "z"] & \\ & B & \end{tikzcd}$$

We want to see that g is f-compatible in $\mathcal{C}$. Let $C \underset{s}{\overset{r}{\rightrightarrows}} X$ in $\mathcal{C}$ be such that $fr = fs$. Since $xs = yfs = yfr = xr, \ say \ = c$, composing with $X \xrightarrow{x} B$ yields

$$\begin{tikzcd}
C \arrow[rr, shift left, "r"] \arrow[rr, shift right, "s"'] \arrow[dr, bend right, "c"'] & & X \arrow[dl, bend left, "x"] \\
& B &
\end{tikzcd}$$

Thus r and s determine arrows in $\mathcal{C}_{/B}$ such that $fr = fs$, and since g is f-compatible in $\mathcal{C}_{/B}$ it holds $gr = gs$. This shows that g is f-compatible in $\mathcal{C}$. It follows then that there exists a unique $Y \xrightarrow{h} Z$ such that $hf = g$ in $\mathcal{C}$. We have

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "g"] \arrow[ddr, bend right, "x"'] & & Y \arrow[dl, "h"'] \arrow[ddl, bend left, "y"] \\
& Z \arrow[d, "z"] & \\
& B &
\end{tikzcd}$$

It remains to see that h is an arrow in $\mathcal{C}_{/B}$, that is, $zh = y$. But $zhf = zg = x = yf$, and since f is in particular an epimorphism, this shows what we want. $\square$

Proposition 1.16. *For any arrow* $(X \xrightarrow{x} B) \xrightarrow{f} (Y \xrightarrow{y} B)$ *in* $\mathcal{C}_{/B}$*, a strict factorization of* f *in* $\mathcal{C}$ *determines a strict factorization of* f *in* $\mathcal{C}_{/B}$*. Thus if every arrow in* $\mathcal{C}$ *admits a strict factorization, so does every arrow in* $\mathcal{C}_{/\{B_i\}_{i\in[n]}}$.

Proof. Let $(X \xrightarrow{x} B) \xrightarrow{f} (Y \xrightarrow{y} B)$ be an arrow in $\mathcal{C}_{/B}$, and take a strict factorization of f in $\mathcal{C}$:

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "e"] \arrow[ddr, bend right, "x"'] & \equiv & Y \arrow[ddl, bend left, "y"] \\
& Z \arrow[ur, "m"'] \arrow[d, dashed, "z"] & \\
& B &
\end{tikzcd}$$

Since x is e-compatible, there is a unique arrow $Z \xrightarrow{z} B$ such that $ze = x$. Since e is epic it follows that $ym = z$. By 1.15 e is an strict epimorphism in $\mathcal{C}_{/B}$, and by 1.12 m is a monomorphism in $\mathcal{C}_{/B}$. $\square$

Proposition 1.17. *Strict epimorphisms are universal in* $\mathcal{C}_{/\{B_i\}_{i\in[n]}}$.

Proof. It follows from the fact that the functor Σ preserves and reflects pullbacks and strict epimorphism, 1.12, 1.14, 1.15. $\square$

Propositions 1.13, 1.16 and 1.17 put together show:

Proposition 1.18. *If $\mathcal{C}$ is regular, then $\mathcal{C}_{/\{B_i\}_{i\in[n]}}$ is regular.* $\square$

1.3 Families of Functors With Common Domain

In this section we will establish a result, Proposition 1.22 below, which is essential (although elementary) for the completeness theorem in this paper.

Everything in this section is based and contained in [3][Ex I, §6], for the convenience of the reader we recall the terminology and extract only the parts that we need.

Let $\mathcal{F}$ be a family of functors with common domain $\mathcal{C}$.

Definition 1.19.

1. $\mathcal{F}$ *preserves* pullbacks, equalizers, if for every $F \in \mathcal{F}$ the functor F preserves pullbacks, equalizers, respectively.
2. $\mathcal{F}$ *reflects* monomorphisms if for every arrow $X \xrightarrow{u} Y$ such that for every $F \in \mathcal{F}$ its image Fu is a monomorphism, it follows that u is a monomorphism.
3. $\mathcal{F}$ is *conservative* if $\mathcal{F}$ reflects isomorphisms.
4. $\mathcal{F}$ is *monic-conservative* if for every monomorphism $X \xrightarrow{u} Y$ such that for every $F \in \mathcal{F}$ its image Fu is an isomorphism, it follows that u is an isomorphism.
5. $\mathcal{F}$ is *faithful* if for every pair $X \underset{v}{\overset{u}{\rightrightarrows}} Y$ such that for every $F \in \mathcal{F}$ their images are equal, $Fu = Fv$, it follows that $u = v$.

Proposition 1.20. *If $\mathcal{F}$ reflects monomorphisms and is monic-conservative, then it is conservative.*

Proof. Let u be such that for every $F \in \mathcal{F}$, Fu is an isomorphism. Then in particular u is a monomorphism, and so by the first assumption it is a monomorphism, and in turn by the second assumption it is a isomorphism. $\square$

We now put a condition that assures us that F reflects monomorphisms.

Proposition 1.21. *If $\mathcal{C}$ has pullbacks, $\mathcal{F}$ preserves them and $\mathcal{F}$ is monic-conservative, then $\mathcal{F}$ reflects monomorphism.*

Proof. Let $X \xrightarrow{u} Y$ be such that for every $F \in \mathcal{F}$, Fu is monic. We will prove that the following diagram is a pullback

$$\begin{tikzcd} X \arrow[r, "id_X"] \arrow[d, "id_X"'] & X \arrow[d, "u"] \\ X \arrow[r, "u"'] & Y \end{tikzcd}$$

Take a pullback in $\mathcal{C}$ and δ as follows:

$$\begin{tikzcd} X \arrow[dr, dashed, "\exists!\delta"'] \arrow[drr, bend left, "id_X"] \arrow[ddr, bend right, "id_X"'] & & \\ & P \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "p.b."] & X \arrow[d, "u"] \\ & X \arrow[r, "u"'] & Y \end{tikzcd}$$

From this diagram we see δ is a monomorphism (since either projection is a section), and for every $F \in \mathcal{F}$, $F\delta$ is the isomorphism between the corresponding pullback diagrams. Therefore δ is an isomorphism. □

From 1.20 and 1.21 it follows:

Corollary 1.22. *If $\mathcal{C}$ has pullbacks, $\mathcal{F}$ preserves them and $\mathcal{F}$ is monic-conservative, then $\mathcal{F}$ is conservative.*

1.4 Weakly Projective Objects

Here we give a characterization of which hom-functors of a regular category $\mathcal{A}$ are regular functors.

Definition 1.23. *An object A in a category $\mathcal{A}$ is* weakly projective *if the functor $\mathcal{A} \xrightarrow{hom_{\mathcal{A}}(A,-)} \mathcal{E}ns$ preserves strict epimorphisms. That is, for any strict epimorphism $X \xrightarrow{u} Y$, any $A \xrightarrow{t} Y$ has a* lift *$A \xrightarrow{v} X$, as indicated in the diagram*

$$\begin{tikzcd} X \arrow[r, two heads, "u"] & Y \\ & A \arrow[u, "\forall t"'] \arrow[ul, dotted, bend left, "\exists v"] \arrow[ul, phantom, "\equiv"] \end{tikzcd}$$

Remark 1.24. In a regular category a object A is weakly projective if and only if every strict epimorphism $X \xrightarrow{u} A$ admits a section.

Proof. If A is weakly projective, a lift of id_A yields a section. For the converse let $X \overset{u}{\twoheadrightarrow} Y$ be a strict epimorphism. For a given $A \xrightarrow{t} Y$ take a pullback of u along t.

$$\begin{tikzcd} X \arrow[r, two heads, "u"] & Y \\ P \arrow[u, dotted, "p_1"] \arrow[r, dotted, two heads, "p_2"'] \arrow[ur, phantom, "p.b."] & A \arrow[u, "t"'] \end{tikzcd}$$

Since p_2 is a strict epimorphism it admits a section v. The composite $p_1 v$ is a lift of t. □

2 PREFIBRED CATEGORIES

2.1 Basic Notions

In this section we recall the context of fibered categories introduced in [2].

Consider a functor $\mathcal{E} \xrightarrow{F} \mathcal{G}$: We say that an object X in $\mathcal{E}$ *sits* over an object α in $\mathcal{G}$ if $F(X) = \alpha$. An arrow $X \xrightarrow{f} Y$ *sits* over an arrow $\alpha \xrightarrow{\varphi} \beta$ if $F(X \xrightarrow{f} Y) = \alpha \xrightarrow{\varphi} \beta$. Given α in $\mathcal{G}$, the *fiber* $\mathcal{E}_\alpha$ is the subcategory of $\mathcal{E}$ of objects X over α and arrows f over the identity id_α (we will say over α). We denote $hom_\varphi(X, Y) \subset hom(X, Y)$ the set of arrows sitting over φ. We will refer to arrows in a fibre as *vertical arrows*.

We display this situation in a double diagram:

$$\begin{tikzcd} Z \arrow[d, "h"'] & \\ X \arrow[r, "f"'] & Y \\ & \\ \alpha \arrow[r, "\varphi"'] & \beta \end{tikzcd}$$

2.1 (Basic Facts and Definitions).

1. Recall that an arrow $X \xrightarrow{f} Y$ is:

   cartesian, if for every arrow $Z \xrightarrow{h} Y$ over $\alpha \xrightarrow{\varphi} \beta$ there exists a unique $Z \xrightarrow{g} X$ over α such that $fg = h$. That is, postcomposing with f, $hom_\alpha(Z, X) \xrightarrow{f_*} hom_\varphi(Z, Y)$, is a bijection.

   strongly cartesian, if for every $\gamma \xrightarrow{\psi} \alpha$, Z over γ, and $Z \xrightarrow{h} Y$ over the composite $\varphi\psi$, there exists a unique g over ψ such that $fg = h$.

That is, postcomposing with f, $hom_{\psi}(Z,X) \xrightarrow{f_*} hom_{\varphi\psi}(Z,Y)$ is a bijection.

We see these definitions in the following diagrams:

$$
\begin{tikzcd}
Z \arrow[d, dashed, "\exists!"'] \arrow[dr, bend left, "\forall"] \arrow[d, phantom, shift left=4ex, "\equiv"] & \\
X \arrow[r, "f"'] & Y \\
\alpha \arrow[r, "\varphi"'] & \beta
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
Z \arrow[dr, dashed, "\exists!"'] \arrow[drr, bend left=10, "\forall"] \arrow[dr, phantom, shift left=3ex, "\equiv"] & & \\
& X \arrow[r, "f"'] & Y \\
\gamma \arrow[r, "\psi"'] & \alpha \arrow[r, "\varphi"'] & \beta
\end{tikzcd}
$$

2. Identity arrows are cartesian, and a vertical arrow is cartesian if and only if it is an isomorphism

3. Given a composite $X \xrightarrow{f} Y \xrightarrow{g} Z$, $h = g\,f$, then:

   If f and g are strongly cartesian, so it is h.

   If h and g are strongly cartesian, so it is f.

4. In general cartesian arrows do not compose, but they do if and only if they are strongly cartesian.

5. Recall that a functor $\mathcal{E} \xrightarrow{F} \mathcal{G}$ is:

   a prefibration if for every $\alpha \xrightarrow{\varphi} \beta \in \mathcal{G}$ and for every $Y \in \mathcal{E}_{\beta}$ there exists a cartesian morphism over φ with target Y.

   a fibration if it is a prefibration and the set of cartesian morphisms is closed under composition, equivalently, if cartesian morphisms are strongly cartesian.

6. In a fibration the concepts of cartesian and strongly cartesian coincide.

7. We write $Y^* \longrightarrow Y$ to indicate that we are labeling a cartesian morphism over φ. We omit the label φ in the usual notation φ^*Y to remind us we are making a choice of a single cartesian arrow and that we do not assume to have a clivage.

8. *We have the dual definitions of cocartesian morphism, precofibration and cofibration. We will freely use these notions and the dual theorems.*

Proposition 2.2. *For F prefibred and precofibred, F is a fibration if and only if it is a cofibration.*

Proof. We prove one side of the duality: *if F is a cofibration, then it is a fibration.* In fact, cartesian morphisms are strong cartesian. We see this as follows: Let $Y^* \xrightarrow{f} Y$

be a cartesian morphisms over $\alpha \xrightarrow{\varphi} \beta$. Given $\gamma \xrightarrow{\psi} \alpha \xrightarrow{\varphi} \beta \in \mathcal{G}$ and $Z \in \mathcal{E}_\gamma$, take $Z \xrightarrow{g} Z_*$ cocartesian over ψ, and consider the following diagram:

$$\begin{tikzcd} hom_\psi(Z, Y^*) \arrow[r, "f_*"] & hom_{\varphi\psi}(Z, Y) \\ hom_\alpha(Z_*, Y^*) \arrow[u, "g^*"] \arrow[r, "f_*"'] \arrow[ur, phantom, "\equiv"] & hom_\varphi(Z_*, Y) \arrow[u, "g^*"'] \end{tikzcd}$$

The bottom arrow is bijective because f is cartesian and the vertical arrows are bijections because g is strong cocartesian. It follows the top arrow is a bijection. $\square$

Proposition 2.3. *For $\mathcal{E} \xrightarrow{F} \mathcal{G}$ prefibered and precofibred, given $\beta \xrightarrow{\varphi} \alpha$ in $\mathcal{G}$, $X \in \mathcal{E}_\beta$ and $Y \in \mathcal{E}_\alpha$, there is a natural bijection $hom_\alpha(X, Y^*) \approx hom_\beta(X_*, Y)$.*

Proof. Consider the following diagram:

$$\begin{tikzcd} X \arrow[r, "b_X"] \arrow[d, "f"] & X_* \arrow[d, "g"] \\ Y^* \arrow[r, "a_Y"] & Y \end{tikzcd}$$

with b_X cocartesian and a_Y cartesian over φ. This establishes a natural bijection such that $a_Y \circ f = g \circ b_X$, see [2].VI.10. $\square$

The category of prefibrations over $\mathcal{G}$ is the subcategory of $\mathcal{C}at/\mathcal{G}$ whose objects are prefibrations and whose morphisms are $\mathcal{G}$-functors that transform cartesian morfisms into cartesian morphisms. We call these morphisms *cartesian* $\mathcal{G}$-functors and we will denote this category $Prefib(\mathcal{G})$. The category of fibrations over $\mathcal{G}$ is the full subcategory of $Prefib(\mathcal{G})$ whose objects are fibrations. We denote this category $Fib(\mathcal{G})$.

2.2 Stability in a Prefibration

The following concepts are inspired in defining properties of the pullback functors of a cleaved prefibration without using clivages.

2.4 (Pulling back along an arrow of $\mathcal{G}$).

Let $\mathcal{E} \xrightarrow{F} \mathcal{G}$ be a prefibration and $\alpha \xrightarrow{\varphi} \beta$ an arrow in $\mathcal{G}$.

1. Given X in $\mathcal{E}_\beta$ and $X^* \multimap X$ cartesian over $\alpha \xrightarrow{\varphi} \beta$, we say that X^* in $\mathcal{E}_\alpha$ is a *pull-back* of X along φ.

2. *Vertical arrows are pulled back as follows:* For $X \xrightarrow{f} Y \in \mathcal{E}_\beta$, choose cartesian morphisms $X^* \;{-\!\!\circ\!\!\!\rightarrow}\; X$ and $Y^* \;{-\!\!\circ\!\!\!\rightarrow}\; Y$ over $\alpha \xrightarrow{\varphi} \beta$. These determine a unique arrow $X^* \xrightarrow{f^*} Y^* \in \mathcal{E}_\alpha$ as in the following diagram.

$$\begin{tikzcd}
X^* \arrow[r, "\circ" marking] \arrow[d, dashed, "f^*"'] \arrow[dr, phantom, "\equiv"] & X \arrow[d, "f"] \\
Y^* \arrow[r, "\circ" marking] & Y
\end{tikzcd}$$

$$\alpha \xrightarrow{\varphi} \beta$$

We say that f^* is a *pull-back* of f along φ.

3. *Finite vertical diagrams are pulled back as follows:*

Let $\mathcal{D}$ be a finite category and $\mathcal{D} \xrightarrow{X} \mathcal{E}_\beta$ a diagram:

Choose for each $i \in \mathcal{D}$ a cartesian arrow $X_i^* \;{-\!\!\circ\!\!\!\rightarrow}\; X_i$. For each $i \xrightarrow{f} j$ we have an arrow $X_i \xrightarrow{X_f} X_j$ in $\mathcal{E}$ which determines an arrow $X_i^* \xrightarrow{X_f^*} X_j^*$ as in 2. For $i \xrightarrow{f} j \xrightarrow{g} k$ in $\mathcal{D}$ the equation $X_{gf}^* = X_g^* X_f^*$ follows by uniqueness. In a diagram:

$$\begin{tikzcd}
X^* \arrow[r, "\circ" marking] \arrow[d, dashed, "X_f^*"] \arrow[dd, dashed, bend right=60, "X_{gf}^*"'] & X \arrow[d, "X_f"] \arrow[dd, bend left=60, "X_{gf}"] \\
Y^* \arrow[r, "\circ" marking] \arrow[d, dashed, "X_g^*"] & Y \arrow[d, "X_g"] \\
Z^* \arrow[r, "\circ" marking] & Z
\end{tikzcd}$$

$$\alpha \xrightarrow{\varphi} \beta$$

Thus we have a diagram $\mathcal{D} \xrightarrow{X^*} \mathcal{E}_\alpha$ in $\mathcal{E}_\alpha$, and a cartesian natural transformation $j_\alpha X^* \;{-\!\!\circ\!\!\!\rightarrow}\; j_\beta X$ over φ, that is, for all i, $X_i^* \;{-\!\!\circ\!\!\!\rightarrow}\; X_i$ is a cartesian arrow over φ.

4. Recall that a *cone* is the same thing that a natural transformation $\Delta C \xrightarrow{p} X$, where ΔC is the *constant diagram*:

$\forall i \in \mathcal{D}$, $(\Delta C)_i = C$, $(\Delta C)(i \xrightarrow{m} j) = C \xrightarrow{id_C} C$.

Given $A \xrightarrow{f} C$, $\Delta A \xrightarrow{\Delta_f} \Delta C$ is the constant natural transformation: $(\Delta_f)_i = f$.

5. *Vertical cones of a finite vertical diagram are pulled back as follows:*

Let $C \xrightarrow{p_i} X_i$ be a cone in $\mathcal{E}_\beta$.

Choose a cartesian arrow $C^* \multimap C$, pulling back p_i we get arrows $C^* \xrightarrow{p_i^*} X_i^*$ in $\mathcal{E}_\alpha$. The cone equations follow by uniqueness as in 3.

2.5 (Pulling back along a cone of $\mathcal{G}$).

Here it is necessary to assume that the cartesian arrows compose, which implies that in 2.4, 3 we have in addition the equations $X^{**} = X^*$, $f^{**} = f^*$.

Let $\mathcal{E} \xrightarrow{F} \mathcal{G}$ be a fibration.

1. *Non vertical arrows are pulling back as follows:* Let $X \xrightarrow{f} Y$ an arrow in $\mathcal{E}$ over $\alpha \xrightarrow{\varphi} \beta$. Let $\delta \xrightarrow{\phi} \alpha$, $\delta \xrightarrow{\mu} \beta$ arrows in $\mathcal{G}$ such that $\varphi\phi = \mu$. Then choosing cartesian arrows, f is pulled back along μ to $\mathcal{E}_\delta$, as shown in the following diagram:

$$\begin{tikzcd}
X^* \arrow[r, "\circ" marking] \arrow[d, dashed, "f^*"] & X \arrow[d, dashed, "f^*"] \arrow[dr, bend left, "f"] & \\
Y^* \arrow[r, "\circ" marking] & Y^* \arrow[r, "\circ" marking] & Y
\end{tikzcd}$$

$$\delta \xrightarrow{\phi} \alpha \xrightarrow{\varphi} \beta$$

Remark: Notice that from 2.1, 3 and 2, it follows that if f is cartesian, then the arrows f^* are isomorphisms.

2. *Finite diagrams are pulled back as follows:* Let $\mathcal{D}$ be a finite category and $\mathcal{D} \xrightarrow{X} \mathcal{E}$ a diagram in $\mathcal{E}$. Denote the diagram FX in $\mathcal{G}$ by $FX_i = \alpha_i$ and $FX_f = \varphi_f$. Let $\delta \xrightarrow{\phi_i} \alpha_i$ be a cone of FX in $\mathcal{G}$.

   Choose for each $i \in \mathcal{D}$ a cartesian arrow $X_i^* \multimap X_i$. For each $i \xrightarrow{f} j$ we have an arrow $X_i \xrightarrow{X_f} X_j$ in $\mathcal{E}$. Pulling back X_f along ϕ_j we get arrows $X_i^* \xrightarrow{X_f^*} X_j^*$ in $\mathcal{E}_\delta$ as in **1.** For $i \xrightarrow{f} j \xrightarrow{g} k$ in $\mathcal{D}$ the equation $X_{gf}^* = X_g^* X_f^*$ follows by uniqueness. In a diagram:

$$\begin{tikzcd}
X_i^* \arrow[r, "\circ" marking] \arrow[d, dashed, "X_f^*"] \arrow[dd, dashed, bend right=60, "X_{gf}^*"'] & X_i \arrow[d, dashed, "X_f^*"] \arrow[dr, "X_f"] \arrow[ddrr, dashed, bend left, "X_{gf}"] & & \\
X_j^* \arrow[r, "\circ" marking] \arrow[d, dashed, "X_g^*"] & X_j^* \arrow[r, "\circ" marking] \arrow[d, dashed, "X_g^*"] & X_j \arrow[d, dashed, "X_g^*"] \arrow[dr, "X_g"] & \\
X_k^* \arrow[r, "\circ" marking] & X_k^* \arrow[r, "\circ" marking] & X_k^* \arrow[r, "\circ" marking] & X_k
\end{tikzcd}$$

$$\delta \xrightarrow{\phi_i} \alpha_i \xrightarrow{\varphi_f} \alpha_j \xrightarrow{\varphi_g} \alpha_k$$

Thus we have a diagram $\mathcal{D} \xrightarrow{X^*} \mathcal{E}_\delta$ in $\mathcal{E}_\delta$, and a cartesian natural transformation $j_\delta X^* \longrightarrow\!\!\circ\!\!\to X$ over the cone $\delta \xrightarrow{\phi_i} \alpha_i$, that is, $X_i^* \longrightarrow\!\!\circ\!\!\to X_i$ is a cartesian arrow over ϕ_i for all i.

3. *Cones of a finite diagram are pulled back as follows:*

   Considerar eliminar este item ya que no es necesario mas adelante

   Let $C \xrightarrow{p_i} X_i$ be a cone of X in $\mathcal{E}$, which sits over a cone $\alpha \xrightarrow{\pi_i} \alpha_i$ of FX in $\mathcal{G}$. Let $\delta \xrightarrow{\phi} \alpha$ in $\mathcal{G}$, and choose a cartesian arrow $C^* \longrightarrow\!\!\circ\!\!\to C$ over ϕ. Pulling back p_i along $\pi_i\phi$ we get arrows $C^* \xrightarrow{p_i^*} X_i^*$ in $\mathcal{E}_\alpha$. The cone equations follow by uniqueness as in 2.

To study the stability properties of the pulling back operations, for clarity of exposition and proofs we find convenient to introduce the following construction.

Let $\mathcal{D}$ be a finite category, $\mathcal{E} \xrightarrow{F} \mathcal{G}$ be a prefibration and $\alpha \xrightarrow{\varphi} \beta$ an arrow in $\mathcal{G}$. We consider the following prefibered subcategory of the functor category $\mathcal{E}^{\mathcal{D}}$, that we denote $\mathcal{E}^{(\mathcal{D})}$.

Definition 2.6 (The Δ-exponential). We omit the label j_α.

Objects: Pairs $(X,\, \alpha)$, $X \in \mathcal{E}_\alpha^{\mathcal{D}} \subset \mathcal{E}^{\mathcal{D}},\ \alpha \in \mathcal{G}$.

Morphisms: Pairs $(\eta,\, \varphi)$

$$\frac{(X,\, \alpha) \xrightarrow{(\eta,\, \varphi)} (Y,\, \beta)}{(X \xrightarrow{\eta} Y,\, \alpha \xrightarrow{\varphi} \beta),\ \ F(\eta_i) = \varphi\ \forall i \in \mathcal{D}}$$

The composition is clear, $\mathcal{E}^{(\mathcal{D})} \subset \mathcal{E}^{\mathcal{D}} \times \mathcal{G}$. Since the fibers are disjoint the first projection determines $\mathcal{E}^{(\mathcal{D})}$ as a subcategory of the functor category $\mathcal{E}^{\mathcal{D}}$, $\mathcal{E}^{(\mathcal{D})} \xrightarrow{i} \mathcal{E}^{\mathcal{D}}$.

The prefibered structure. The second projection defines a functor $\mathcal{E}^{(\mathcal{D})} \xrightarrow{F^{(\mathcal{D})}} \mathcal{G}$ whose fibers are $(\mathcal{E}^{(\mathcal{D})})_\alpha = \mathcal{E}_\alpha^{\mathcal{D}}$. The functor $F^{(\mathcal{D})}$ is prefibred, we define a morphism to be cartesian if and only if for every $i \in \mathcal{D}$ the morphisms η_i are cartesian. Then, given $(X,\, \beta)$ and $\alpha \xrightarrow{\varphi} \beta$, pulling back X as in Definition 2.4, 2., determines an object $(X^*,\, \alpha)$ and a cartesian arrow $(X^*,\, \alpha) \longrightarrow (X,\, \beta)$, which is the pull-back in $\mathcal{E}^{(\mathcal{D})}$. It is clear that if F is a fibration, so it is $\mathcal{E}^{(\mathcal{D})}$. $\square$

Definition 2.7 (The diagonal functor). There is a cartesian $\mathcal{G}$-functor $\mathcal{E} \xrightarrow{(\Delta, F)} \mathcal{E}^{(\mathcal{D})}$ which sends an object C into the pair $(\Delta C, FC)$, where ΔC is the constant diagram, see definition 2.4, 3..

Although in this paper we have no use of this remark, it is pertinent to mention the following:

Remark 2.8. Let $\mathcal{E}^{\mathcal{D}} \xrightarrow{F^{\mathcal{D}}} \mathcal{G}^{\mathcal{D}}$ be the functor defined as postcomposing with F. The functor $F^{(\mathcal{D})}$ is a pullback of $F^{\mathcal{D}}$ along $\mathcal{G} \xrightarrow{\Delta} \mathcal{G}^{\mathcal{D}}$ in $\mathcal{C}at$ as indicated in the following diagram, where $\delta = (\Delta, F)$:

$$\begin{tikzcd}
\mathcal{E} \arrow[dr, "\delta"] \arrow[drr, bend left, "\Delta"] \arrow[ddr, bend right, "F"'] & & \\
& \mathcal{E}^{(\mathcal{D})} \arrow[r, "i"] \arrow[d, "F^{(\mathcal{D})}"'] \arrow[dr, phantom, "p.b."] & \mathcal{E}^{\mathcal{D}} \arrow[d, "F^{\mathcal{D}}"] \\
& \mathcal{G} \arrow[r, "\Delta"'] & \mathcal{G}^{\mathcal{D}}
\end{tikzcd}$$

□

Observation 2.9. For a diagram $\mathcal{D} \xrightarrow{X} \mathcal{E}_\beta$, note that a cone $C \xrightarrow{p_i} X_i$ in the fiber $\mathcal{E}_\beta$ is the same thing that a natural transformation $\Delta C \xrightarrow{p} X$ such that the pair $(\Delta C,\, \beta) \xrightarrow{(p,\, id_\beta)} (X, \beta)$ is an arrow in $\mathcal{E}^{(\mathcal{D})}$, $\;c\,\mathcal{E}_\beta(C, X) = (\mathcal{E}^{(\mathcal{D})})_\beta[(\Delta C, \beta), (X, \beta)]$.

More generally, given $\alpha \xrightarrow{\varphi} \beta$ in $\mathcal{G}$, and a object A over α, a cone $A \xrightarrow{\pi_i} X_i$ over φ is the same thing that a natural transformation $\Delta A \xrightarrow{p} X$ such that the pair $(\Delta A,\, \alpha) \xrightarrow{(p,\, \varphi)} (X, \beta)$ is an arrow in $\mathcal{E}^{(\mathcal{D})}$, $\;c\,\mathcal{E}_\varphi(A, X) = (\mathcal{E}^{(\mathcal{D})})_\varphi[(\Delta A, \alpha), (X, \beta)]$.

Definition 2.10. *For a category $\mathcal{D}$, we say that limits of type $\mathcal{D}$ are* stable *if for every $\alpha \xrightarrow{\varphi} \beta$ in $\mathcal{G}$, and limit cone $C \xrightarrow{\pi_i} X$ in $\mathcal{E}_\beta$, the pull-back cone $C^* \xrightarrow{\pi_i^*} X_i^*$ is a limit cone in $\mathcal{E}_\alpha$.*

Proposition 2.11. *If $\mathcal{E} \xrightarrow{F} \mathcal{G}$ is prefibered and precofibred, then limits of type $\mathcal{D}$ are stable for any category $\mathcal{D}$ with a finite set of objects, including $\mathcal{D} = \emptyset$.*

Proof. Using Proposition 2.3 the proof follows by the usual argument that functors with a left adjoint preserves limits. □

Observation 2.12. Recall that a cone $C \xrightarrow{\pi_i} X_i$ in a fiber $\mathcal{E}_\beta$ is a *limit* cone in $\mathcal{E}_\beta$ if for any object A over β, the map $\mathcal{E}_\beta(A, C) \xrightarrow{\pi_*} c\,\mathcal{E}_\beta(A, X)$ is a bijection.

From Observation 2.9 we see that this is the same as to say that the map $(\mathcal{E}^{(D)})_\beta[(\Delta A,\, \beta), (\Delta C,\, \beta)]_+ \xrightarrow{(\pi,\, id_\beta)_*} (\mathcal{E}^{(D)})_\beta[(\Delta A,\, \beta), (X,\, \beta)]$ is a bijection, where by $[\;\;]_+$ we denote the subset of constant natural transformations. □

Definition 2.13. *We say that a cone $C \xrightarrow{\pi_i} X_i$ in a fiber $\mathcal{E}_\beta$ is a F-limit if for every $\alpha \xrightarrow{\varphi} \beta$ in $\mathcal{G}$, and a object A over α, the map $\mathcal{E}_\varphi(A, C) \xrightarrow{\pi_*} c\,\mathcal{E}_\varphi(A, X)$ is a bijection.*

From Observation 2.9 we see that this is the same as to say that the map $(\mathcal{E}^{(D)})_\varphi[(\Delta A,\, \alpha), (\Delta C,\, \beta)]_+ \xrightarrow{(\pi,\, id_\beta)_*} (\mathcal{E}^{(D)})_\varphi[(\Delta A,\, \alpha), (X,\, \beta)]$ is a bijection.

Clearly by the respective definition it follows

Observation. *limit in $\mathcal{E}$ $\quad\Rightarrow\quad$ F-limit $\quad\Rightarrow\quad$ limit in $\mathcal{E}_\beta$*

The following is a characterization of stable limits.

Proposition 2.14. *A cone $C \xrightarrow{\pi_i} X_i$ is a F-limit cone if and only if, all, or any, of its pullback cones $C^* \xrightarrow{\pi_i^*} X_i^*$ are limit cones.*

Proof. Let $\Delta C \xrightarrow{p} X$ be a cone in $\mathcal{E}_\beta$, and consider the following commutative diagram:

$$\begin{tikzcd}
(\mathcal{E}^{(D)})_\alpha[(\Delta A,\, \alpha), (\Delta C^*,\, \alpha)]_+ \arrow[r, "\approx"] \arrow[d, "(\pi^*{,}id_\alpha)_*"] & (\mathcal{E}^{(D)})_\varphi[(\Delta A,\, \alpha), (\Delta C,\, \beta)]_+ \arrow[d, "(\pi{,}\, id_\beta)_*"] \\
(\mathcal{E}^{(D)})_\alpha[(\Delta A,\, \alpha), (X^*,\, \alpha)] \arrow[r, "\approx"] & (\mathcal{E}^{(D)})_\varphi[(\Delta A,\, \alpha), (X,\, \beta)]
\end{tikzcd}$$

The horizontal arrows are bijections by definition of cartesian arrows, which shows the equivalence between the bijectivity condition for the vertical arrows. $\square$

Since limits in $\mathcal{E}$ are in particular F-limits, we have:

Corollary 2.15. *If the functors $\mathcal{E}_\alpha \xrightarrow{j_\alpha} \mathcal{E}$ preserve limits of type $\mathcal{D}$, then limits of type $\mathcal{D}$ are stable.* $\square$

The reverse implication does not hold in general. For $\mathcal{D} = \emptyset$, trivially the identity functor of any category with two non isomorphic objects is a fibration where the fibers have stable terminal objects which can not all be terminal in the whole category. Next we show an example with a non empty $\mathcal{D}$.

Example 2.16. The following describes a fibration in which products are stable but the inclusion functors of the fibres don't preserve products. Take $\mathcal{G} = \{\alpha \overset{\varphi_1}{\underset{\varphi_2}{\rightrightarrows}} \beta\}$, and consider the prefibration $\mathcal{E} \longrightarrow \mathcal{G}$ described in the following diagram:

$$\begin{tikzcd}
W \arrow[d, "c_i"'] & \\
P_i \arrow[r, "p_i"] \arrow[d, "a_i"'] \arrow[dd, bend right=60, "b_i"'] & P \arrow[d, "a"] \arrow[dd, bend left=60, "b"] \\
X_i \arrow[r, "x_i"] & X \\
Y_i \arrow[r, "y_i"] & Y
\end{tikzcd}$$

$$\alpha \xrightarrow{\varphi_i} \beta$$

Where $\mathcal{E}$ is the finite category generated by the arrows p_i, a_i, b_i, c_i, x_i and y_i, $i = 0, 1$, satisfying the relations $a.p_i = x_i.a_i$, $b.p_i = x_i.b_i$. We have a functor F defined by the equations $F(a_i) = F(b_i) = F(c_i) = id_\alpha$, $F(a) = F(b) = id_\beta$, and $F(p_i) = F(x_i) = F(y_i) = \varphi_i$.

There are no arrows directly connecting the two diagrams $\{P_0 \xrightarrow{a_0} X_0, P_0 \xrightarrow{b_0} Y_0\}$ and $\{P_1 \xrightarrow{a_1} X_1, P_1 \xrightarrow{b_1} Y_1\}$. It can be verified that these diagrams are products in $\mathcal{E}_\alpha$ and that the diagram $\{P \xrightarrow{a} X, P \xrightarrow{b} Y\}$ is a product in $\mathcal{E}_\beta$. F is a fibration where the arrows p_i, x_i and y_i are cartesian. $\{P_0 \xrightarrow{a_0} X_0,\, P_0 \xrightarrow{b_0} Y_0\}$ and $\{P_1 \xrightarrow{a_1} X_1,\, P_1 \xrightarrow{b_1} Y_1\}$ are pullbacks of the diagram $\{P \xrightarrow{a} X, P \xrightarrow{b} Y\}$ along the arrows φ_0 and φ_1 respectively. Products are stable in this fibration but the diagram $\{P \xrightarrow{a} X, P \xrightarrow{b} Y\}$ is not a product en $\mathcal{E}$. This can be verified by the fact that the diagram $\{W \xrightarrow{x_0 a_0 c_0} X,\, W \xrightarrow{y_1 b_1 c_1} Y\}$ has no factorization through P. □

Next we consider a sufficient condition under which the functors j_α preserve stable limits.

Proposition 2.17. *Let $\mathcal{D}$ be a non empty connected category and $\mathcal{D} \xrightarrow{X} \mathcal{E}_\beta$ a vertical diagram. Then:*

1. For any cone $A \xrightarrow{p_i} X_i$, $(\Delta C,\, \beta) \xrightarrow{(p,\, id_\beta)} (X, \beta)$, with A sitting over α, there exists $\alpha \xrightarrow{\varphi} \beta$ such that $F(p_i) = \varphi \,\forall\, i$.

2. The diagonal functors $\mathcal{E} \xrightarrow{\Delta} \mathcal{E}^{\mathcal{D}}$ and $\mathcal{E} \xrightarrow{(\Delta, F)} \mathcal{E}^{(\mathcal{D})}$ are full and faithful.

3 The functor $\mathcal{E}^{(\mathcal{D})} \xrightarrow{i} \mathcal{E}^{\mathcal{D}}$ is full and faithful.

Proof. By assumption for any two $i, j \in \mathcal{D}$, there is a zig-zag of arrows $i \overset{m}{\rightsquigarrow} j$.

1. Let $i, j \in \mathcal{D}$, we have a zig-zag $X_i \overset{X_m}{\rightsquigarrow} X_j$ in $\mathcal{E}_\beta$, such that $p_j = X_m \circ p_i$. Thus $F(p_j) = id_\beta \circ F(p_i)$, $F(p_j) = F(p_i)$. .

2. Let $X, Y \in \mathcal{E}$ over α, β respectively, and $\Delta X \xrightarrow{\eta} \Delta Y$ in $\mathcal{E}^{\mathcal{D}}$. Let $i, j \in \mathcal{D}$, the naturality condition on η means $\Delta Y(m) \circ \eta_i = \eta_j \circ \Delta X(m)$. Since $\Delta X(m) = id_X$, $\Delta Y(m) = id_Y$, we have $\eta_i = \eta_j$. For the functor (Δ, F) we do in the same way.

3. We have a factorization $\Delta = i \circ (\Delta, F)$, the statement follows. □

Proposition 2.18. *Let $\mathcal{D}$ be any non empty connected category. Then limits of type $\mathcal{D}$ are stable if and only if the the functors $\mathcal{E}_\alpha \xrightarrow{j_\alpha} \mathcal{E}$ preserve limits of type $\mathcal{D}$, that is, they remain universal in the category $\mathcal{E}$.*

Proof. Let $C \xrightarrow{\pi_i} X_i$ be a limit cone in a fiber $\mathcal{E}_\beta$, then by proposition 2.14 and Proposition 2.17, 1., 3. (recall Observation 2.12), it follows that this limit is stable if and only if the map $\mathcal{E}^{\mathcal{D}}[\Delta A,\, \Delta C]_+ \xrightarrow{\pi_*} \mathcal{E}^{\mathcal{D}}[\Delta A,\, X]$ is a bijection. In turn by

Proposition 2.17, 2., it follows that this map is a bijection if and only if the map $\mathcal{E}(A, C) \xrightarrow{\pi_*} \mathcal{E}^{\mathcal{D}}(\Delta A, X)$ is a bijection, that is, if and only if the cone $C \xrightarrow{\pi_i} X_i$ is a limit cone in the category $\mathcal{E}$. $\square$

Definition 2.19. *A prefibration $\mathcal{E} \xrightarrow{F} \mathcal{G}$ is* ***finitely complete*** *if the categories $\mathcal{E}_\alpha$ are finitely complete and finite limits are stable.*

Accordingly we have the category of finitely complete prefibrations whose morphisms are cartesian $\mathcal{G}$-functors $f \in hom_{\mathcal{G}}(\mathcal{E}, \mathcal{E}')$ such that for every $\alpha \in \mathcal{G}$ the restrictions

$$\mathcal{E}_\alpha \xrightarrow{f_\alpha} \mathcal{E}'_\alpha$$

preserve finite limits.

Definition 2.20. *A prefibration $\mathcal{E} \xrightarrow{F} \mathcal{G}$ is* ***regular*** *if the categories $\mathcal{E}_\alpha$ are regular, finite limits are stable and strict epimorphisms are stable.*

The category of regular prefibrations over $\mathcal{G}$ is the category that has regular prefibrations (fibrations) as objects and whose morphisms are cartesian $\mathcal{G}$-functors $f \in hom_{\mathcal{G}}(\mathcal{E}, \mathcal{E}')$ such that for every $\alpha \in \mathcal{G}$ the restrictions

$$\mathcal{E}_\alpha \xrightarrow{f_\alpha} \mathcal{E}'_\alpha$$

are regular functors. This category is included in the category if finitely complete prefibrations. The restrictions to fibrations are natural and coherent. que significa esta ultima frase

Este rojo es para eliminar y/o modificar

2.2.1 Reflection properties in a prefibration

Properties of functors such as reflecting limits and other types of categorical objects can be defined in a prefibration. Here we define without using clivages the property corresponding to the fact that the pullback functors are conservative over the specific set of morphisms as defined in section 1.3.

We say that a subset $\mathcal{A} \subset \mathcal{E}$ of vertical arrows is *stable* if for any $X \xrightarrow{m} Y \in \mathcal{A}$ and pullback $X^* \xrightarrow{m^*} Y^*$, it follows that $m^* \in \mathcal{A}$

Consider $\mathcal{A}$ a *stable* set of vertical arrows. We will use the notation $\mathcal{A}_\alpha$ to represent the subset $\mathcal{A} \cap \mathcal{E}_\alpha$.

Definition 2.21. *A prefibration $\mathcal{E} \xrightarrow{F} \mathcal{G}$ is* ***conservative over*** *$\mathcal{A}$ if for every $X \xrightarrow{f} Y \in \mathcal{A}$ such that $X^* \xrightarrow{f^*} Y^*$ is an isomorphism, it follows that f is an isomorphism.*

We will say the prefibration $\mathcal{E} \xrightarrow{F} \mathcal{G}$ is *conservative* if it is conservative over the complete set of arrows of $\mathcal{E}$.
Para que definir estable y considerar que $\mathcal{A}$ es estable en la definicion de conservative ?

2.3 Two facts about epimorphisms in a prefibration

Lemma 2.22. *The functors j_α preserve epimorphisms and strict epimorphisms. That is, epimorphisms and strict epimorphisms in $\mathcal{E}_\alpha$ remain so in $\mathcal{E}$.*

Proof. Let $A \xrightarrow{f} B$ over α, consider morphisms $B \longrightarrow C$, C over β, they sit over $\alpha \xrightarrow{\varphi} \beta$. Take $C^* \longrightarrow C$ cartesian over φ. We have the following diagram, where the vertical arrows are bijections:

$$\begin{tikzcd} hom_\varphi(B,C) \arrow[r, "f^*"] & hom_\varphi(A,C) \\ hom_\alpha(B,C^*) \arrow[u] \arrow[r, "f^*"'] \arrow[ur, phantom, "\equiv"] & hom_\alpha(A,C^*) \arrow[u] \end{tikzcd}$$

Assuming that the bottom is injective, it follows that so it is the top.

For the second statement, let $A \xrightarrow{g} C$ be compatible with f. Take $C^* \longrightarrow C$ cartesian over $F(g) = \alpha \xrightarrow{\varphi} \beta$. It is straightforward that the unique factorization of g through $C^* \longrightarrow C$ over α is compatible with f. The result follows: In a diagram:

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[d, dashed, "\exists! a"] \arrow[dr, "g"] & \\ B \arrow[r, phantom, "\equiv"] \arrow[r, dashed, "\exists! b"'] & C^* \arrow[r, "s"'] & C \end{tikzcd}$$

$$\alpha \xrightarrow{id_\alpha} \alpha \xrightarrow{\varphi} \beta$$

□

3 COLIMIT OF A FIBRATION WITH COFILTERED BASE

In this section we will study the structure of the colimit of a fibration developed in [3] for the particular case where the base category is cofiltered.

Let $\mathcal{E} \xrightarrow{F} \mathcal{G}$ be a fibration where $\mathcal{G}$ is cofiltered. If S denotes the set of cartesian morphisms in $\mathcal{E}$, the *colimit* of the fibration is defined as the Gabriel Zisman category of fractions $\mathcal{E}[S^{-1}]$, [4], characterized by a functor $\mathcal{E} \xrightarrow{Q} \mathcal{E}[S^{-1}]$ that satisfies the following universal property in $\mathcal{C}at$.

Definition 3.1. *For every functor $\mathcal{E} \xrightarrow{G} \mathcal{X}$ such that G sends cartesian morphisms into isomorphisms, there exists a unique functor $\mathcal{E}[S^{-1}] \xrightarrow{H} \mathcal{X}$ such that $HQ = G$.*

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "Q"] \arrow[dr, "\forall G"'] & \equiv & \mathcal{E}[S^{-1}] \arrow[dl, "\exists ! H"] \\
& \mathcal{X} &
\end{tikzcd}$$

Definition 3.2. Since $\mathcal{G}$ is cofiltered S admits a calculus of right fractions [3]. For internal references we recall the Gabriel-Zisman construction of $\mathcal{E}[S^{-1}]$ in the presence of a calculus of right fractions.

1. *Objects*: A *object* is simply a object of $\mathcal{E}$.

2. *Premorphisms*: A *premorphism* $X \longrightarrow Y$ is a spam $X \xleftarrow{s} A \xrightarrow{f} Y$, with $s \in S$. In case that there is no risk of confusion we will simply write (f, s).

3. *Composition*: For $X \xrightarrow{(f_1,s_1)} Y \xrightarrow{(f_2,s_2)} Z$, having a calculus of right fractions guarantees that there is a pair $A_1 \xleftarrow{s_{21}} A_{21} \xrightarrow{f_{21}} A_2$, $s \in S$, such that $f_1 s_{21} = s_2 f_{21}$, $s_1 s_{21} \in S$. This is visualized in the commutative diagram:

$$\begin{tikzcd}
& & A_{21} \arrow[dl, "s_{21}"'] \arrow[dr, "f_{21}"] & & \\
& A_1 \arrow[dl, "s_1"'] \arrow[dr, "f_1"] & & A_2 \arrow[dl, "s_2"'] \arrow[dr, "f_2"] & \\
X & & Y & & Z
\end{tikzcd}$$

We define a *composition* by $(f_2, s_2) \circ (f_1, s_1) = (f_2 f_{21}, s_1 s_{21})$.

4. *Morphisms*: A morphism is an equivalence class of premorphisms under the following equivalence: (f_1, s_1), (f_2, s_2) are equivalent if there exists $A \xrightarrow{a_1} A_1$, $A \xrightarrow{a_2} A_2$ in $\mathcal{E}$ such that $s_1 a_1 = s_2 a_2\,, say = s \in S$, and $f_1 a_1 = f_2 a_2$. Note that by basic facts 2.1, 3. it follows that a_1 and a_2 are cartesian.

We denote $(f_1, s_1) \sim (f_2, s_2)$. This is visualized in the commutative diagram:

$$\begin{tikzcd}
& & A_1 \arrow[dl, "s_1"'] \arrow[dr, bend left, "f_1"] & \\
A \arrow[urr, bend left, "a_1"] \arrow[r, "s"] \arrow[drr, bend right, "a_2"'] & X & & Y \\
& & A_2 \arrow[ul, "s_2"] \arrow[ur, bend right, "f_2"'] &
\end{tikzcd}$$

Equivalence is indeed an equivalence relation and the composition of premorphisms becomes unique and well defined on equivalent classes.

We will denote the class of a premorhism (f, s) by f/s. For $A = X$ we adopt the abuse of notations $f/id_X = f$,

The functor Q is defined as $Q(X) = X$, and for $X \xrightarrow{f} Y$, $Q(f) = f/id_X = f$.

For verifications and details see [4][Ch I, 2.].

Remark 3.3. Let $X \xrightarrow{f/s} Y$ be a morphism in $\mathcal{E}[S^{-1}]$ with s already invertible, then $f/s = fs^{-1}/id_X = Q(fs^{-1})$.

Proof. The reader can easily check the equivalence $(f, s) \sim (fs^{-1}, id_X)$. $\square$

From 2.1, 2, and Remark 3.3 it follows:

Remark 3.4. The functors J_α defined as the composite
$$\begin{tikzcd} \mathcal{E}_\alpha \arrow[rr, "J_\alpha"] \arrow[dr, bend right, "j_\alpha"'] & & \mathcal{E}[S^{-1}] \\ & \mathcal{E} \arrow[ur, bend right, "Q"'] & \end{tikzcd},$$
$J_\alpha = Qj_\alpha$, are fully-faithful. $\square$

The next remark follows directly by the definition of the composition and of the equivalence relation, Definition 3.2, items 3 and 4.

Remark 3.5 (Lifting of triangles)**.**

For $X \xrightarrow{(f_1,s_1)} Y \xrightarrow{(f_2,s_2)} Z$, $X \xrightarrow{(f_3,s_3)} Z$, and the equation $f_3/s_3 = f_2/s_2 \circ f_1/s_1$ in $\mathcal{E}[S^{-1}]$, there is a diagram in $\mathcal{E}$ witnessing this equation:

$$\begin{tikzcd}
 & & & A_{21} \arrow[dl, "s_{21}"'] \arrow[dr, "f_{21}"] & & \\
 & & A_1 \arrow[dl, "s_1"'] \arrow[dr, "f_1"] & & A_2 \arrow[dl, "s_2"'] \arrow[dr, "f_2"] & \\
A \arrow[r, "s"] \arrow[urrr, bend left, "a_1"] \arrow[drrr, bend right, "a_2"'] & X & & Y & & Z \\
 & & & A_3 \arrow[ull, "s_3"'] \arrow[urr, "f_3"'] & &
\end{tikzcd}$$

Vice-versa, any such diagram in $\mathcal{E}$ corresponds to a commutative triangle in $\mathcal{E}[S^{-1}]$. $\square$

3.1 Colimit of a finitely complete fibration

Our objective in this section is to prove the following theorem, which we do in 3.9 below.

Theorem 3.6. *If $\mathcal{E} \xrightarrow{F} \mathcal{G}$ is finitely complete, then $\mathcal{E}[S^{-1}]$ is a finitely complete category and the functors $J_\alpha = Q\, j_\alpha$*

$$\begin{tikzcd}
\mathcal{E}_\alpha \arrow[rr, "J_\alpha"] \arrow[dr, bend right, "j_\alpha"'] & & \mathcal{E}[S^{-1}] \\
& \mathcal{E} \arrow[ur, bend right, "Q"'] &
\end{tikzcd}$$

preserve finite limits. Moreover for $\mathcal{X} \in \mathcal{C}at_{fl}$ and $\mathcal{E}[S^{-1}] \xrightarrow{H} \mathcal{X}$, H preserves finite limits if and only if for every $\alpha \in \mathcal{G}$ the functors $H J_\alpha$ preserves finite limits. □

Corollary 3.7. *If $\mathcal{X} \in \mathcal{C}at_{fl}$ and $\mathcal{E} \xrightarrow{G} \mathcal{X}$ is such that G sends cartesian morphisms into isomorphisms, and for every $\alpha \in \mathcal{G}$ the functors $G\, j_\alpha \in \mathcal{C}at_{fl}$, then there exists a unique functor $\mathcal{E}[S^{-1}] \xrightarrow{H} \mathcal{X} \in \mathcal{C}at_{fl}$ such that $HQ = G$.*

Proof. Consider the diagram

$$\begin{tikzcd}
\mathcal{E}_\alpha \arrow[r, "j_\alpha"] \arrow[rrr, bend left, "J_\alpha"] & \mathcal{E} \arrow[rr, "Q"] \arrow[dr, "G"'] & & \mathcal{E}[S^{-1}] \arrow[dl, "H"] \\
& & \mathcal{X} &
\end{tikzcd}$$

By Definition 3.1 it only remains to see that H preserves finite limits, which follows by Theorem 3.6. □

Corollary 3.8. *The assignment $\mathcal{E} \xrightarrow{F} \mathcal{G} \;\mapsto\; \mathcal{E}[S^{-1}]$ determines a functor from the category of finitely complete fibrations into $\mathcal{C}at_{fl}$.*

Proof. Let $\begin{tikzcd}[column sep=small] \mathcal{E} \arrow[rr, "f"] \arrow[dr, "F"'] & & \mathcal{E}' \arrow[dl, "F'"] \\ & \mathcal{G} & \end{tikzcd}$ be a morphism of finitely complete fibrations. Consider the diagram

$$\begin{tikzcd}
\mathcal{E}[S^{-1}] \arrow[r, dashed, "\exists!"] & \mathcal{E}'[S'^{-1}] \\
\mathcal{E} \arrow[u, "Q"] \arrow[r, "f"] & \mathcal{E}' \arrow[u, "Q'"'] \\
\mathcal{E}_\alpha \arrow[u, "j_\alpha"] \arrow[r, "f_\alpha"'] & \mathcal{E}'_\alpha \arrow[u, "j'_\alpha"'] \arrow[uu, bend right=60, "J'_\alpha"']
\end{tikzcd}$$

The functor $Q'f$ sends cartesian morphisms into isomorphisma, and for every $\alpha \in \mathcal{G}$ the functors $Q' f j_\alpha = J'_\alpha f_\alpha$ preserve finite limits. The result follows. □

3.9 (Proof of theorem 3.6).

Proposition 3.10. *The functors $\mathcal{E}_\alpha \xrightarrow{J_\alpha} \mathcal{E}[S^{-1}]$ preserve terminal objects, in particular terminal objects exist in $\mathcal{E}[S^{-1}]$.*

Proof. Let $\alpha \in \mathcal{G}$, 1_α a terminal object in $\mathcal{E}_\alpha$, and $X \in \mathcal{E}$ over some $\beta \in \mathcal{G}$. With the following diagram we show the existence of an arrow $X \longrightarrow 1_\alpha$ in $\mathcal{E}[S^{-1}]$ (recall that terminal objects are stable $1_\alpha^* = 1_\gamma$).

$$\begin{tikzcd}
 & X^* \arrow[dl, bend right, "s"'] \arrow[d, dashed] & \\
X & 1_\alpha^* \arrow[r, "t"] & 1_\alpha
\end{tikzcd}$$

$$\begin{tikzcd}
\beta & \gamma \arrow[l, dashed, "\mu"'] \arrow[r, dashed, "\nu"] & \alpha
\end{tikzcd}$$

Take μ, ν as indicated, and s cartesian over μ, t cartesian over ν.

Now suppose we have $X \underset{f_2/s_2}{\overset{f_1/s_1}{\rightrightarrows}} 1_\alpha$ in $\mathcal{E}[S^{-1}]$. With the following diagrams we show that they are equivalent, $(f_1, s_1) \sim (f_2, s_2)$. We refer to 3.2.

$$\begin{tikzcd}
 & A_1 \arrow[dl, "s_1"'] \arrow[dr, bend left, "f_1"] & \\
X & & 1_\alpha \\
 & A_2 \arrow[ul, "s_2"] \arrow[ur, bend right, "f_2"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & A_1 \arrow[dl, "s_1"'] \arrow[dr, bend left, "f_1"] & \\
X^* \arrow[urr, bend left, "a_1"] \arrow[r, "s"] \arrow[drr, bend right, "a_2"'] & X & & 1_\alpha \\
 & & A_2 \arrow[ul, "s_2"] \arrow[ur, bend right, "f_2"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \gamma_1 \arrow[dl, "\mu_1"'] \arrow[dr, bend left, "\nu_1"] & \\
\beta & & \alpha \\
 & \gamma_2 \arrow[ul, "\mu_2"] \arrow[ur, bend right, "\nu_2"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & \gamma_1 \arrow[dl, "\mu_1"'] \arrow[dr, bend left, "\nu_1"] & \\
\gamma \arrow[urr, dashed, bend left, "\rho_1"] \arrow[r, dashed, "\mu"] \arrow[drr, dashed, bend right, "\rho_2"'] & \beta & & \alpha \\
 & & \gamma_2 \arrow[ul, "\mu_2"] \arrow[ur, bend right, "\nu_2"'] &
\end{tikzcd}$$

We have the two diagrams on the left. Take a cone γ as indicated in the bottom right diagram. Let s be cartesian over μ. Since s_1, s_2 are cartesian, there are a_1, a_2 uniques over ρ_1, ρ_2 respectively that factor s through s_1, s_2 respectively. It only remains to check $f_1 a_1 = f_2 a_2$, which we see as follows: Let $\nu := \nu_1 \rho_1 = \nu_2 \rho_2$, and consider $1_\alpha^* \longrightarrow 1_\alpha$ over ν. Then $f_1 a_1$, $f_2 a_2$ would be equal if the corresponding unique factorizations through 1_α^* are equal in $\mathcal{E}_\gamma$, which is clear since $1_\alpha^* = 1_\gamma$. $\square$

Proposition 3.11. *Any diagram $\mathcal{D} \xrightarrow{X} \mathcal{E}[S^{-1}]$ in $\mathcal{E}[S^{-1}]$ is naturally isomorphic to one that can be factored through a fibre. More precisely, there exists $\alpha \in \mathcal{G}$, X^* and η*

$$\begin{tikzcd}
 & \mathcal{D} \arrow[dl, dashed, "X^*"'] \arrow[dr, "X"] \arrow[d, phantom, "\overset{\eta}{\Longrightarrow}"] & \\
\mathcal{E}_\alpha \arrow[rr, "J_\alpha"'] & {} & \mathcal{E}[S^{-1}]
\end{tikzcd}$$

where the natural transformation is composed of cartesian arrows. In particular, since cartesian arrows are invertible, there is a natural isomorphism between X and X^* in $\mathcal{E}[S^{-1}]$ (we omit J_α and write $J_\alpha X^* = X^*$).

Proof. For every $i \xrightarrow{f} j$ in $\mathcal{D}$, take a premorphism $L_f = (X_i \xleftarrow{s_f} A_f \xrightarrow{p_f} X_j)$ for each $X_i \xrightarrow{X_f} X_j$, $X_f = p_f/s_f$. Having done these choices, we obtain a function $\mathcal{D} \xrightarrow{L} \mathcal{E}$, $L_i = X_i$, and L_f in place of X_f. Then for every composable pair $i \xrightarrow{f} j \xrightarrow{g} k$, $h = gf$, take a witnessing diagram $L_{g,f}$ as in 3.5 for the equation $X_h = X_g X_f$, $p_h/s_h = p_g/s_g \circ p_f/s_f$.

$$\begin{tikzcd}
 & & & B_h \arrow[dl, "t_h"'] \arrow[dr, "q_h"] & & \\
 & & A_f \arrow[dl, "s_f"'] \arrow[dr, "p_f"] & & A_g \arrow[dl, "s_g"'] \arrow[dr, "p_g"] & \\
C_h \arrow[r, "c_h"] \arrow[urrr, bend left, "b_h"] \arrow[drrr, bend right, "a_h"'] & X_i & & X_j & & X_k \\
 & & & A_h \arrow[ull, "s_h"'] \arrow[urr, "p_h"] & &
\end{tikzcd}$$

All these data can be seen as a diagram $\mathcal{L} = \{L_i,\, L_f,\, L_{g,f}\}$ in $\mathcal{E}$, which sits over the diagram $F\mathcal{L}$ in $\mathcal{G}$. Since $\mathcal{G}$ is cofiltered we can take a cone with vertex α for $F\mathcal{L}$, and pull back $\mathcal{L}$ along this cone as in 2.5, 2. In this way we obtain a diagram $\mathcal{L}^* = \{L_i^*,\, L_f^*,\, L_{g,f}^*\}$ in the fiber $\mathcal{E}_\alpha$ and a natural transformation $j_\alpha \mathcal{L}^* \overset{\eta}{\Longrightarrow} \mathcal{L}$ whose components are cartesian arrows in $\mathcal{E}$. The data in $\mathcal{L}^*$ determines a diagram $\mathcal{D} \xrightarrow{X^*} \mathcal{E}[S^{-1}]$ with all the vertices $Q(X_i) = X_i$ in $\mathcal{E}_\alpha$, then Remark 3.4 finishes the proof. $\square$

Proposition 3.12. *For any finite non empty category $\mathcal{D}$, the functors $\mathcal{E}_\alpha \xrightarrow{J_\alpha} \mathcal{E}[S^{-1}]$ preserve limits of type $\mathcal{D}$, and limits of type $\mathcal{D}$ (in particular pull-backs) exist in $\mathcal{E}[S^{-1}]$.*

Proof. The first assertion is an immediate consequence of Proposition 2.18 and the fact that $\mathcal{E} \xrightarrow{Q} \mathcal{E}[S^{-1}]$ preserves finite limits [4]. ver GZ seccion 3.6 p 18.

Given $\mathcal{D} \xrightarrow{X} \mathcal{E}[S^{-1}]$, we consider α and $\mathcal{D} \xrightarrow{X^*} \mathcal{E}_\alpha$ as in Proposition 3.11, and take a limit of X^* in $\mathcal{E}_\alpha$. This remains a limit in $\mathcal{E}[S^{-1}]$, and since X^* is isomorphic to X, it will be a limit of X. $\square$

3.13 (end of Proof of theorem 3.6).

Clearly, since all finite limits are constructed with terminal objects and pull-backs, propositions 3.10 and 3.12 finish the proof. $\square$

3.2 Colimit of a Regular Fibration

Our objective in this section is to prove the following theorem.

Theorem 3.14. *If F is a regular fibration, then $\mathcal{E}[S^{-1}]$ is a regular category and the functors*

$$\begin{tikzcd} \mathcal{E}_\alpha \arrow[rr, "J_\alpha"] \arrow[dr, "j_\alpha"'] & \arrow[d, phantom, "\equiv"] & \mathcal{E}[S^{-1}] \\ & \mathcal{E} \arrow[ur, "Q"'] & \end{tikzcd}$$

are regular. More so if $\mathcal{I} \in \mathcal{R}eg$ and $\mathcal{E}[S^{-1}] \xrightarrow{H} \mathcal{I}$ is a functor such that for every $\alpha \in \mathcal{G}$ the functors $H \cdot J_\alpha \in \mathcal{R}eg$, it follows that $H \in \mathcal{R}eg$.

The proofs of the following two corollaries ar identical to the proofs of Corollaries 3.7 and 3.8.

Corollary 3.15. *If $\mathcal{I} \in \mathcal{R}eg$ and $\mathcal{E} \xrightarrow{G} \mathcal{I}$ is such that G transforms cartesian morphisms into isomorphisms and for every $\alpha \in \mathcal{G}$ the functors $G \cdot j_\alpha \in \mathcal{R}eg$, then there exists a unique $\mathcal{E}[S^{-1}] \xrightarrow{H} \mathcal{I} \in \mathcal{R}eg$ such that $[HQ = G]$.*

Corollary 3.16. *The construction determines a functor from the category of regular fibrations into $\mathcal{R}eg$.*

Observation. If $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ are in $\mathcal{E}$, we have that $f = g$ in $\mathcal{E}[S^{-1}]$ if and only if there exists $s \in S$ such that $fs = gs$ in $\mathcal{E}$.

Proposition 3.17. *The functors $\mathcal{E}_\alpha \xrightarrow{J_\alpha} \mathcal{E}[S^{-1}]$ send strict epimorphisms to epimorphisms.*

Proof. Let $Z \xrightarrow{f} X$ be a strict epimorphism in $\mathcal{E}_\alpha$, and $X \overset{f_1/s_1}{\underset{f_2/s_2}{\rightrightarrows}} Y$ be such that $(f_1/s_1)f = (f_2/s_2)f$. We have to show that $(f_1/s_1) = (f_2/s_2)$ in $\mathcal{E}[S^{-1}]$, that is $(f_1, s_1) \sim (f_2, s_2)$. We consider the diagram obtained in the proof of Proposition 3.10, and, as in that proposition, we have to see that $f_1 a_1 = f_2 a_2$. It suffices to prove that $f_1 a_1 = f_2 a_2$ in $\mathcal{E}[S^{-1}]$.

no entiendo porque vale que es suficiente

We expand the diagram as follows:

$$
\begin{tikzcd}
Z^{**} \arrow[r, "u"] \arrow[dd, "f^{**}"] & Z^{*} \arrow[r, "t"] \arrow[dd, "f^{*}"] & Z \arrow[dd, "f"] & & \\
& & & A_1 \arrow[dl, "s_1"'] \arrow[dr, bend left, "f_1"] & \\
X^{**} \arrow[r, "v"] & X^{*} \arrow[r, "s"] \arrow[urr, bend left, "a_1"] \arrow[drr, bend right, "a_2"'] & X & & Y \\
& & & A_2 \arrow[ul, "s_2"] \arrow[ur, bend right, "f_2"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& & \gamma_1 \arrow[dl, "\mu_1"'] \arrow[dr, bend left, "\nu_1"] & \\
\gamma \arrow[r, "F(u)"] & \beta \arrow[ur, dashed, bend left, "\phi_1"] \arrow[r, dashed, "\varphi_1"] \arrow[dr, dashed, bend right, "\phi_2"'] & \alpha & \delta \\
& & \gamma_2 \arrow[ul, "\mu_2"] \arrow[ur, bend right, "\nu_2"'] &
\end{tikzcd}
$$

Take $Z^* \xrightarrow{t} Z$ cartesian over φ_1 and f^* the corresponding pullback of f along φ_1. We have that $(f_1a_1)f^* = (f_2a_2)f^*$ in $\mathcal{E}[S^{-1}]$ explicar en detalle esto. Thus there is a cartesian morphism $Z^{**} \xrightarrow{u} Z^*$ such that $(f_1a_1f^*)u = (f_2a_2f^*)u$ in $\mathcal{E}$. Take $X^{**} \xrightarrow{v} X^*$ cartesian over $F(u)$ and call f^{**} the corresponding pullback of f^* along $F(u)$. Since $(f_1a_1v)f^{**} = (f_2a_2v)f^{**}$ in $\mathcal{E}$ and $F(f_2a_2v) = F(f_1a_1v) = \varphi_2F(u)$ para que esta ecuacion ? from Lemma 2.22 we conclude $f_1a_1v = f_2a_2v$ in $\mathcal{E}$ porque strict epis son estables, luego f^{**} es srtrict epi. The result follows. □

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It suffices to prove that $f_1a_1 = f_2a_2$ in $\mathcal{E}[S^{-1}]$. Take a cone of the following diagram in $\mathcal{G}$. I

$$
\begin{tikzcd}
& \alpha & \\
{} \arrow[ur, "F(s_0)"] \arrow[dr, "F(f_0)"'] & \beta \arrow[u, dashed, "\varphi_0"'] \arrow[l, dashed, "\psi_0"'] \arrow[r, dashed, "\psi_1"'] \arrow[d, dashed, "\varphi_1"] & {} \arrow[ul, "F(s_1)"'] \arrow[dl, "F(f_1)"] \\
& F(C) &
\end{tikzcd}
$$

Let $B^* \xrightarrow{s} B$ be a cartesian morphism over φ_0. Take $(i = 0, 1)$ a_i the unique morphism over ψ_i that factors s through s_i. It suffices to prove that $f_0a_0 = f_1a_1$ in $\mathcal{E}[S^{-1}]$. Take $A^* \xrightarrow{t} A$ cartesian over φ_0 and f^* the corresponding pullback of f along φ_0.

$$
\begin{tikzcd}
A^{**} \arrow[r, dashed, "u"] \arrow[d, "f^{**}"'] \arrow[dr, phantom, "\equiv"] & A^{*} \arrow[rr, "t"] \arrow[d, "f^{*}"] \arrow[drr, phantom, "\equiv"] & & A \arrow[d, "f"] & \\
B^{**} \arrow[r, dashed, "v"'] & B^{*} \arrow[rr, "s"'] \arrow[dr, "a_i"'] & \arrow[d, phantom, "\equiv"] & B & \\
& & {} \arrow[ur, "s_i"'] \arrow[drr, "f_i"] & & \\
& & & & C
\end{tikzcd}
$$

$$
\gamma \overset{F(u)}{\dashrightarrow} \beta \xrightarrow{\varphi_0} \alpha
$$

We have that $(f_0a_0)f^* = (f_1a_1)f^*$ in $\mathcal{E}[S^{-1}]$. Thus there is a cartesian morphism $A^{**} \xrightarrow{u} A^*$ such that $(f_0a_0f^*)u = (f_1a_1f^*)u$ in $\mathcal{E}$. Take $B^{**} \xrightarrow{v} B^*$ cartesian over $F(u)$ and call f^{**} the corresponding pullback of f^* along $F(u)$. Since $(f_0a_0v)f^{**} = (f_1a_1v)f^{**}$ in $\mathcal{E}$ and $F(f_1a_1v) = F(f_0a_0v) = \varphi_1 F(u)$ from Lemma 2.22 we conclude $f_0a_0v = f_1a_1v$ in $\mathcal{E}$. The result follows.

==
==

Proposition 3.18. *If $A \xrightarrow{f} B$ is a strict epimorphism in $\mathcal{E}_\alpha$, then every compatible morphism with f in $\mathcal{E}[S^{-1}]$ factors through f.*

Proof. Let $A \xrightarrow{gr^{-1}} C$ be compatible with f. Take $K \underset{x_2}{\overset{x_1}{\rightrightarrows}} A$ a kernel pair of f in $\mathcal{E}_\alpha$ and $K^* \xrightarrow{s} K$ a cartesian morphisms over $F(r)$.

$$
\begin{tikzcd}
K^{**} \arrow[r, dashed, "t"] \arrow[d, shift right, "x_2^{**}"'] \arrow[d, shift left, "x_1^{**}"] & K^{*} \arrow[r, "s"] \arrow[d, shift right, "x_2^{*}"'] \arrow[d, shift left, "x_1^{*}"] & K \arrow[d, shift right, "x_2"'] \arrow[d, shift left, "x_1"] \\
A^{**} \arrow[r, dashed, "u"] \arrow[d, "f^{**}"] & A^{*} \arrow[r, "r"] & A \arrow[d, "f"] \\
B^{**} \arrow[rr, dashed, "v"] & & B
\end{tikzcd}
$$

$$
F(K^{**}) \xrightarrow{F(t)} F(A^*) \xrightarrow{F(r)} \alpha
$$

Since $(g/r)x_1 = (g/r)x_2$, we have that $gx_1^* = gx_2^*$ in $\mathcal{E}[S^{-1}]$. Thus there is a cartesian morphism $K^{**} \xrightarrow{t} K^*$ such that $(gx_1^*)t = (gx_2^*)t$ in $\mathcal{E}$. Take $A^{**} \xrightarrow{u} A^*$ cartesian over $F(t)$ and $B^{**} \xrightarrow{v} B$ cartesian over $F(r)F(t) = F(st)$. The morphisms $K^{**} \overset{x_1^{**}}{\underset{x_2^{**}}{\rightrightarrows}} A^{**}$ are a kernel pair of the strict epimorphism f^{**} in the fibre over $F(K^{**})$, so gu is compatible with f^{**} in $\mathcal{E}$. By Lemma 2.22 there is a morphism $h \in hom_{\mathcal{E}}(B^{**}, C)$ such that $gu = fh$ in $\mathcal{E}$. The morphism $B \xrightarrow{h/v} C$ yields the desired factorization. $\square$

Theorem 3.19. *The functors $\mathcal{E}_\alpha \xrightarrow{J_\alpha} \mathcal{E}[S^{-1}]$ preserve strict epimorphisms.*

Proof. It follows from Propositions 3.17 and 3.18. porque 3.17 $\square$

Proposition 3.20. *Any morphisms $X \xrightarrow{f/s} Y \in \mathcal{E}[S^{-1}]$ admits a strict epic - monic factorization.*

Proof. For any morphism $X \xrightarrow{f/s} Y \in \mathcal{E}[S^{-1}]$ take a cartesian morphisms $Y^* \xrightarrow{t} Y$ over $F(f)$. We have the following situation.

$$\begin{tikzcd}
& X^* \arrow[rr, "s"] \arrow[dl, dashed, "e"'] \arrow[dd, dashed, "\exists! f'"] \arrow[ddr, "f"] & & X \\
I \arrow[dr, dashed, "m"'] \arrow[r, phantom, "\equiv"] & {} & & \\
& Y^* \arrow[r, "t"'] & Y &
\end{tikzcd}$$

$$F(X^*) \xrightarrow{F(f)} F(Y)$$

The morphisms m and e form a strict epic - monic factorization of f' in the fibre over $F(X^*)$. From Theorem 3.19 and the fact that the J_α preserve monics we have that the morphisms e/s and tm yield a strict epic - monic factorization of f/s.

$\square$

Proposition 3.21. *Strict epimorphisms are stable in $\mathcal{E}[S^{-1}]$.*

Proof. We will use as reference diagram **??**. Suppose f_0/s_0 is a strict epimorphism. Then $G^*(a_0)$ is a strict epimorphism in $\mathcal{E}[S^{-1}]$. Take a strict epic - monic factorization of $G^*(a_0)$ in $\mathcal{E}_\alpha$. We will take a composite pullback of $G^*(a_0)$ along $G^*(a_1)$ in $\mathcal{E}_\alpha$.

$$
\begin{tikzcd}
P \arrow[r, "p"] \arrow[d, two heads, "e'"'] \arrow[dr, phantom, "p.b."] & {G^*}_0 \arrow[d, two heads, "e"] \arrow[dd, bend left=60, "G^*(a_0)"] \\
I' \arrow[r] \arrow[d, tail, "m'"'] \arrow[dr, phantom, "p.b."] & I \arrow[d, tail, "m"] \\
{G^*}_2 \arrow[r, "G^*(a_1)"'] & {G^*}_0
\end{tikzcd}
$$

This diagram in fact is also true in $\mathcal{E}[S^{-1}]$. In fact in $\mathcal{E}[S^{-1}]$ we have that $G^*(a_0)$ is a strict epic and consequently m is an isomorphism in $\mathcal{E}[S^{-1}]$. Therefore m' is an isomorphism in in $\mathcal{E}[S^{-1}]$ and so $m'e'$ is a strict epimorphisms in $\mathcal{E}[S^{-1}]$. The result follows. $\square$

3.3 Colimit of a Conservative Fibration Over $\mathcal{A}$

Take $\mathcal{A}$ a stable set of vertical arrows.

Theorem 3.22. *If F is conservative over $\mathcal{A}$, then for every $\alpha \in \mathcal{G}$ the functors $\mathcal{E}_\alpha \xrightarrow{J_\alpha} \mathcal{E}[S^{-1}]$ reflect isomorphisms that are already in $\mathcal{A}_\alpha$.*

That is to say that if $f \in \mathcal{A}_\alpha$ and $J_\alpha(f)$ is an isomorphism, then f is an isomorphisms.

Proof. Suppose $X \xrightarrow{f} Y \in \mathcal{A}_\alpha$ is such that $J_\alpha(f)$ is an isomorphism. Let $Y \xleftarrow{t} Y^* \xrightarrow{g} X$ represent its inverse. Take $\alpha \xrightarrow{\varphi} F(Y^*)$ such that $F(t) \cdot \varphi = F(g) \cdot \varphi = \psi$ and construct the following commutative diagram as indicated below.

$$\begin{tikzcd}
Y^{****} \arrow[r, "y"] \arrow[d, "g^{***}"'] & Y^{***} \arrow[r, "v"] \arrow[d, "g^{**}"'] & Y^{**} \arrow[r, "s"] \arrow[d, "g^{*}"'] & Y^{*} \arrow[r, "t"] \arrow[dr, "g"'] & Y \\
X^{***} \arrow[r, "x"] \arrow[d, "f^{***}"'] & X^{**} \arrow[r, "w"] \arrow[d, "f^{**}"'] & X^{*} \arrow[rr, "u"] \arrow[d, "f^{*}"'] & & X \arrow[d, "f"] \\
Y^{****} \arrow[r, "y"] \arrow[d, "g^{***}"'] & Y^{***} \arrow[r, "v"] \arrow[d, "g^{**}"'] & Y^{**} \arrow[r, "s"] \arrow[d, "g^{*}"'] & Y^{*} \arrow[r, "t"] \arrow[dr, "g"'] & Y \\
X^{***} \arrow[r, "x"] & X^{**} \arrow[r, "w"] & X^{*} \arrow[rr, "u"] & & X
\end{tikzcd}$$

$$\begin{tikzcd}
F(X^{***}) \arrow[r, "F(x)"] & F(Y^{***}) \arrow[r, "F(v)"] & \alpha \arrow[r, "\varphi"] & F(Y^{*}) \arrow[r, shift left, "F(t)"] \arrow[r, shift right, "F(g)"'] & F(X)
\end{tikzcd}$$

Take s cartesian over φ, u cartesian over ψ and the corresponding vertical arrows g^* and f^*. So it happens that $f^*g^* = 1_{Y^{**}}$ and $g^*f^* = 1_{X^*}$ in $\mathcal{E}[S^{-1}]$. Take v a cartesian morphism such that $(f^*g^*)v = 1_{Y^{**}}v$ in $\mathcal{E}$ followed by w cartesian over $F(v)$. For the corresponding vertical arrows we have that $f^{**}g^{**} = 1_{Y^{***}}$ in $\mathcal{E}$ and $g^{**}f^{**} = 1_{X^{**}}$ in $\mathcal{E}[S^{-1}]$. Take x a cartesian morphism such that $(f^{**}g^{**})x = 1_{Y^{***}}x$ in $\mathcal{E}$ and y cartesian over $F(x)$. It follows that f^{***} and g^{***} are inverse of eachother in the fibre over $F(X^{***})$. The result follows.

$\square$

4 CONSTRUCTION OF A REGULAR SET VALUED FUNCTOR THAT IS CONSERVATIVE OVER MONICS WITH GLOBALLY SUPPORTED CODOMAIN FOR ANY REGULAR CATEGORY $\mathcal{A}$ THAT POSSESSES A DISTINGUISHED TERMINAL OBJECT

4.1 Construction of the Functor from $\mathcal{A}$ to $\mathcal{A}'$ That Sends Globally Supported Objects into Objects That Have a Generic Global Section

In this section $\mathcal{A}$ will denote a regular category that possesses a distinguished terminal object 1.

4.1.1 A fibration that has $\mathcal{A}$ as its fibres

For the following fibration we will have that $\mathcal{A}$ can be identified as de fibre over $\{1\}$.

The Cofilitered Base for the Fibration

Strict epimorphisms are closed under composition in $\mathcal{A}$ (**??**). Take $\mathcal{G}l_s(\mathcal{A})$ the category whose objects are the globally supported objects of $\mathcal{A}$ and whose morphisms are the strict epimorphisms in $\mathcal{A}$. We define $\mathcal{G}_{\mathcal{A}}$ to be the category whose objects are finite sequences of objects $\{B_i\}_{i\in[n]} \subset \mathcal{G}l_s(\mathcal{A})$ whose first term is $B_0 = 1$. A morphism $\{B_i\}_{i\in[n]} \xrightarrow{\varphi} \{C_j\}_{j\in[m]} \in \mathcal{G}_{\mathcal{A}}$ is a function $[m] \xrightarrow{\varphi} [n]$ that verifies $\varphi(0) = 0$ and that for every $j \in [m]$ it verifies $B_{\varphi(j)} = C_j$.

Remark 4.1. $\mathcal{G}_{\mathcal{A}}$ is a cofilitered category. More so it is finitely complete and has a *unique* terminal object. This can be verified interpreting $\mathcal{G}_{\mathcal{A}}^{op}$ embedded in $\mathcal{E}ns^*/\mathcal{G}l_s(\mathcal{A})$ where $\mathcal{E}ns^*$ denotes the category of pointed sets and where we distinguish $1 \in \mathcal{G}l_s(\mathcal{A})$.

A Finitely Complete Fibration

We will give an explicit description of Grothendiecks construction of a split cofibration associated to the covariant functor $D_{\mathcal{A}} : \mathcal{G}_{\mathcal{A}} \longrightarrow \mathcal{C}at$ that assigns to each object $\{B_i\}_{i\in[n]}$ the multislice category $\mathcal{A}_{/\{B_i\}_{i\in[n]}}$ and to each arrow $\{B_i\}_{i\in[n]} \xrightarrow{\varphi} \{C_j\}_{j\in[m]}$ the functor $\varphi_* : \mathcal{A}_{/\{B_i\}_{i\in[n]}} \longrightarrow \mathcal{A}_{/\{C_j\}_{j\in[m]}}$ that is defined as $\varphi_*(\{X \xrightarrow{x_i} B_i\}_{i\in[n]}) = \{X \xrightarrow{x_{\varphi(j)}} C_j\}_{j\in[m]}$ on objects and is the identity on arrows.

Take $\mathcal{E}_\mathcal{A}$ the category whose objects are ordered pairs (X, α) where $\alpha \in \mathcal{G}_\mathcal{A}$ and $X \in D_\mathcal{A}(\alpha)$. Its arrows are ordered pairs $(X, \alpha) \xrightarrow{(f,\varphi)} (Y, \beta)$ where $\alpha \xrightarrow{\varphi} \beta \in \mathcal{G}_\mathcal{A}$ and $f : \varphi_* X \longrightarrow Y$. Composition is defined for $(Y, \beta) \xrightarrow{(g,\psi)} (Z, \gamma)$ as $(g, \psi)(f, \varphi) = (g \cdot \psi_*(f), \psi\varphi)$. Take $F_\mathcal{A}$ be the projection in the second coordinate. The arrow $(X, \alpha) \xrightarrow{(1_{\varphi_* X},\varphi)} (\varphi_* X, \beta)$ is cocartesian over φ with source X and these arrows are closed under composition. The projection in the first coordinate restricted to a fiber $(\mathcal{E}_\mathcal{A})_\alpha \xrightarrow{\pi_1} D_\mathcal{A}(\alpha)$ is an isomorphism. If φ_* denoted the (co) pullback functor along φ we have in fact this isomorphism that is natural in the following sense:

$$\begin{tikzcd}
(\mathcal{E}_\mathcal{A})_\alpha \arrow[r, "\varphi_*"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\equiv"] & (\mathcal{E}_\mathcal{A})_\beta \arrow[d, "\pi_1"] \\
D_\mathcal{A}(\alpha) \arrow[r, "D_\mathcal{A}(\varphi)"'] & D_\mathcal{A}(\beta)
\end{tikzcd} \tag{4.2}$$

Thus we can make the abuse of language of identifying the fiber of the split cofibration $\mathcal{E}_\mathcal{A} \xrightarrow{F_\mathcal{A}} \mathcal{G}_\mathcal{A}$ over $\{B_i\}_{i\in[n]}$ with $\mathcal{A}_{/\{B_i\}_{i\in[n]}}$ and similarly identify the cotransport functor along φ with $D_\mathcal{A}(\varphi)$.

Proposition 4.3. *$F_\mathcal{A}$ is a fibration.*

Proof. FIX-001.02 START Dar referencia directa a SGA1 en lugar de proposition 2.2

It suffices to prove that $F_\mathcal{A}$ is prefibered (see [2, page 143]).

FIX-001.02 END

Take $\{B_i\}_{i\in[n]} \xrightarrow{\varphi} \{C_j\}_{j\in[m]} \in \mathcal{G}_\mathcal{A}$ and $\{Y \xrightarrow{y_j} C_j\}_{j\in[m]}$ over $\{C_j\}_{j\in[m]}$. Let $\mathcal{D}_\varphi$ be the finite graph whose objects are $[n+1]$ and whose arrows are identified with $[m]$. The arrow $j \in [m]$ has source n and target $\varphi(j)$. The object $\{Y \xrightarrow{y_j} C_j\}_{j\in[m]}$ induces a functor $\mathcal{D}_\varphi \xrightarrow{\tilde{Y}} \mathcal{A}$ defined as $\tilde{Y}n = Y$, as $\tilde{Y}i = B_i$ for any other $i \in [n]$ and $\tilde{Y}j = y_j$ on arrows.

A cone for this functor is a family of arrows $\{X \xrightarrow{x_i} \tilde{Y}i\}_{i\in[n+1]}$ such that for every $j \in [m]$ the following diagram is commutative.

$$\begin{tikzcd}
& X \arrow[dl, "x_n"'] \arrow[dr, "x_{\varphi(j)}"] \arrow[d, phantom, "\equiv"] & \\
Y \arrow[rr, "y_j"'] & {} & B_{\varphi(j)}
\end{tikzcd}$$

Thus it is naturally identified with a morphism $\{X \xrightarrow{x_i} \tilde{Y}i\}_{i\in[n]} \xrightarrow{(x_n,\varphi)} \{Y \xrightarrow{y_j} C_j\}_{j\in[m]}$ over φ with target $\{Y \xrightarrow{y_j} C_j\}_{j\in[m]}$. A limit cone corresponds to a cartesian morphism. Since $\mathcal{A}$ is finitely complete the result follows. $\square$

Proposition 4.4. *$F_{\mathcal{A}}$ is finitely complete.*

Proof. It follows from Theorem 2.11 and that the fibers are multislice categories of a regular category, in particular finitely complete. $\square$

A Regular Fibration

We will in fact prove that $\mathcal{E}_{\mathcal{A}} \xrightarrow{F_{\mathcal{A}}} \mathcal{G}_{\mathcal{A}}$ is a regular fibration.

Lemma 4.5. *In $\mathcal{E}_{\mathcal{A}}$ if*

$$\begin{tikzcd}
\{W \xrightarrow{w_i} B_i\}_{i\in[n]} \arrow[r, "(f,\varphi)"] \arrow[d, "(a,1)"'] \arrow[dr, phantom, "\equiv"] & \{X \xrightarrow{x_j} C_j\}_{j\in[m]} \arrow[d, "(b,1)"] \\
\{Z \xrightarrow{z_i} B_i\}_{i\in[n]} \arrow[r, "(g,\varphi)"] & \{Y \xrightarrow{y_j} C_j\}_{j\in[m]}
\end{tikzcd}$$

$$\{B_i\}_{i\in[n]} \xrightarrow{\quad\varphi\quad} \{C_j\}_{j\in[m]}$$

is such that (f,φ) and (g,φ) are cartesian, then

$$\begin{tikzcd}
W \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\equiv"] & X \arrow[d, "b"] \\
Z \arrow[r, "g"'] & Y
\end{tikzcd}$$

is a pullback in $\mathcal{A}$.

Proof. For any cone $\{V \xrightarrow{h} X, V \xrightarrow{c} Z\}$ in $\mathcal{A}$ we have the object $\{V \xrightarrow{z_i c} B_i\}_{i\in[n]}$ together with the cone $\{(c,1),(h,\varphi)\}$. A factorization of the former cone in $\mathcal{A}$ is identified with a factorization of the latter in $\mathcal{E}_{\mathcal{A}}$ over $\{B_i\}_{i\in[n]}$. $\square$

Proposition 4.6. *Strict epimorphisms are stable in the fibration $F_{\mathcal{A}}$.*

Proof. It follows from Lemma 4.5 and the fact that the domain functors Σ in multislice categories preserve and reflect strict epimorphisms. $\square$

Corollary 4.7. *$F_{\mathcal{A}}$ is a regular fibration.*

4.1.2 Construction of the colimit $\mathcal{A}'$ of the fibration and proof that including the first fibre is conservative over monics with globally supported codomain

Theorem 3.14 guarantees that the colimit of this fibration is a regular category and in particular the functors $J_{\{1\}}$ in the diagram below is regular.

$$\begin{tikzcd}
(\mathcal{E}_{\mathcal{A}})_{\{1\}} \arrow[rr, "J_{\{1\}}"] \arrow[dr, "j_{\{1\}}"'] & \equiv & \mathcal{E}_{\mathcal{A}}[S^{-1}] \\
& \mathcal{E}_{\mathcal{A}} \arrow[ur, "Q"'] &
\end{tikzcd}$$

Identifying $(\mathcal{E}_{\mathcal{A}})_{\{1\}}$ with $\mathcal{A}$ we will label to top arrow in the diagram with $\mathcal{A} \xrightarrow{j} \mathcal{A}'$. For a morphism $X \xrightarrow{f} Y \in \mathcal{A}$ will use the abuse of language of saying $X \xrightarrow{f} Y$ in $\mathcal{A}'$ referring to the morphism $j(X) \xrightarrow{j(f)} j(Y) \in \mathcal{A}'$. Taking into consideration that j transforms 1 into a terminal object, preserves monics and preserves strict epimorphisms makes the abuse coherent with these objects.

A generic section for every $B \twoheadrightarrow 1 \in \mathcal{A}$

Take a globally supported object $B \overset{\pi}{\twoheadrightarrow} 1 \in \mathcal{A}$. The fiber over $\{1, B\}$ is naturally identified with $\mathcal{A}_{/B}$. Choose a product $\{B \times B \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} B\}$ of B with itself in $\mathcal{A}$ and take $B \xrightarrow{\Delta} B \times B$ the diagonal morphism. We obtain the following diagram in $\mathcal{E}_{\mathcal{A}}$.

$$\begin{tikzcd}
\{P \xrightarrow{\pi_2} B\} \arrow[r, "(\pi_1,\varphi)"] & B \\
\{B \xrightarrow{id_B} B\} \arrow[u, "(\Delta,1)"] \arrow[r, "(\pi,\varphi)"'] & 1
\end{tikzcd}$$

$$\{1, B\} \xrightarrow{\varphi} \{1\}$$

Lemma 4.8. *$\{B \xrightarrow{id_B} B\} \xrightarrow{(\pi,\varphi)} 1$ and $\{P \xrightarrow{\pi_2} B\} \xrightarrow{(\pi_1,\varphi)} B$ are cartesian morphisms.*

Proof. This follows immediately using the characterization of cartesian morphisms given in Proposition 4.3. □

Remark 4.9. We have a section $\frac{(\pi_1\Delta,\varphi)}{(\pi,\varphi)} = \frac{(1_B,\varphi)}{(\pi,\varphi)}$ of $B \overset{\pi}{\twoheadrightarrow} 1$ in $\mathcal{A}'$. This section is in fact *canonical* in the sense that any choice of product $\{B \times B \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} B\}$ of B with itself in $\mathcal{A}$ will induce the *same* arrow in $\mathcal{A}'$ built this way. This follows from the fact that (π_1, φ) is cartesian. We will label this uniquely determined arrow $1 \xrightarrow{\Delta_B} B$.

Separating B from its subobjects in $\mathcal{A}$

We will prove is that $1 \xrightarrow{\Delta_B} B$ *separates* B from its subobjects in $\mathcal{A}$, in the sense of Theorem 4.11.

Lemma 4.10. *For any $\{B_i\}_{i\in[n]} \xrightarrow{\psi} \{1, B\}$ in $\mathcal{G}$, if $\{X \xrightarrow{x_i} B_i\}_{i\in[n]} \xrightarrow{(f,\psi)} \{B \xrightarrow{id_B} B\}$ is cartesian, then f is a strict epimorphism in $\mathcal{A}$.*

Proof. Note that $\{X \xrightarrow{x_i} B_i\}_{i\in[n]}$ is a product of the family $\{B_i\}_{i\in[n]}$ and that f is in fact one of the projections. The result follows (recall every B_i is globally supported and **??**). $\square$

Theorem 4.11. *If $A \xrightarrow{m} B \in \mathcal{A}$ is such that Δ_B lifts along m,*

$$\begin{tikzcd}
A \arrow[r, tail, "m"] & B \\
& 1 \arrow[u, "\Delta_B"'] \arrow[ul, dotted, "\exists"] \arrow[ul, phantom, "\equiv", shift left=3]
\end{tikzcd}$$

it follows that m is an isomorphism.

Proof. In our context the existence of such a lifting of Δ_B reduces to there being a morphism $\{B_i\}_{i\in[n]} \xrightarrow{\psi} \{1, B\}$ in $\mathcal{G}_{\mathcal{A}}$, a cartesian morphism $\{X \xrightarrow{x_i} B_i\}_{i\in[n]} \xrightarrow{(f,\psi)} \{B \xrightarrow{id_B} B\}$ over ψ and a morphism $\{X \xrightarrow{x_i} B_i\}_{i\in[n]} \xrightarrow{(g,\varphi\circ\psi)} A$ such that the following diagram is commutative.

$$\begin{tikzcd}
& & A \arrow[d, tail, "m"] \\
& \{P \xrightarrow{\pi_2} B\} \arrow[r, "(\pi_1{,}\varphi)"] & B \\
\{X \xrightarrow{x_i} B_i\}_{i\in[n]} \arrow[r, "(f{,}\psi)"] \arrow[uurr, bend left, "(g{,}\varphi\circ\psi)"] & \{B \xrightarrow{id_B} B\} \arrow[r, "(\pi{,}\varphi)"'] \arrow[u, "(\Delta{,}1)"'] & 1
\end{tikzcd}$$

$$\{B_i\}_{i\in[n]} \xrightarrow{\quad\psi\quad} \{1, B\} \xrightarrow{\quad\varphi\quad} \{1\}$$

It follows that $mg = f$ and together with Lemma 4.10 m must be an isomorphism. $\square$

Corollary 4.12. *The functor $j_{\{1\}}$ is conservative over monics with globally supported codomain.*

4.2 Construction of $\mathcal{A}^\infty$ Where 1 is Weakly Projective

4.2.1 A new fibration that has $\mathcal{A}$ as its first fibre, proof that the inclusion of any fibre into the colimit $\mathcal{A}^\infty$ is conservative over monics with globally supported codomain and $1 \in \mathcal{A}^\infty$ is weakly projective

Iterating the construction in Section 4.1 we obtain the following sequence of regular functors.

$$\mathcal{A} \xrightarrow{j} \mathcal{A}' \xrightarrow{j'} (\mathcal{A}')' \longrightarrow \cdots$$

Using the dual construction in 4.1.1 we obtain from this diagram a split regular fibration $\mathcal{E} \xrightarrow{\mathcal{F}} {\mathcal{N}_0}^{op}$ whose base is cofiltered. $\mathcal{A}^{(n)}$ will denote the fiber over n and j_n the $(n+1)^{th}$ functor of the diagram. We will make the same identification between j_n and the transfer functor along $n+1 \longrightarrow n$. The hypothesis of Theorem 3.14 are satisfied and it follows that the colimit of this fibration $\mathcal{A}^\infty$ is a regular category and the morphisms $\mathcal{A}^{(n)} \xrightarrow{J_n} \mathcal{A}^\infty$ are regular.

Remark 4.13. The functors j_n preserve monics with globally supported codomain and are conservative over them (see 4.11). It follows that the fibration is conservative over vertical monics with globally supported codomain.

Corollary 4.14. *The functors $\mathcal{A}^{(n)} \xrightarrow{J_n} \mathcal{A}^\infty$ are conservative over monics with globally supported codomain.*

Proof. See 3.22. $\square$

Proposition 4.15. *The functors $\mathcal{A}^{(n)} \xrightarrow{J_n} \mathcal{A}^\infty$ reflect globally suported objects.*

Proof. Let $X \in cc A^{(n)}$ be such that $J_n(X)$ has global support in $\mathcal{A}^\infty$. Take a strict epic-monic factorization of $X \xrightarrow{f} 1$ in $\mathcal{A}^{(n)}$.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "e"'] & \equiv & 1 \\ & S \arrow[ur, tail, "m"'] & \end{tikzcd}$$

Since $J_n(f)$ and $J_n(e)$ are strict epimorphisms, it follows that $J_n(m)$ is an isomorphism. Corollary 4.14 guarantees that m is an isomorphism. The result follows. $\square$

Remark 4.16. For $A \xrightarrow{f} B$ in $\mathcal{A}^{(n)}$, $j_n(A) \xrightarrow{j_n(f)} j_n(B)$ is a transfer of f along $n+1 \longrightarrow n$.

Theorem 4.17. *1 is weakly projective in $\mathcal{A}^\infty$.*

Proof. Let $B \in \mathcal{A}^\infty$ be a globally supported object. Because of Proposition 4.15 it is globally supported in the fiber over $n = \mathcal{F}(B)$. Since $j_n(B)$ is isomorphic to B in $\mathcal{A}^\infty$ (4.16) and we have the generic section $1 \xrightarrow{\Delta_B} j_{\mathcal{A}^{(n)}}(B)$, the result follows. □

4.2.2 The representable functor of $1 \in \mathcal{A}^\infty$ is conservative over monics with globally supported codomain

Theorem 4.18. *The functor $\mathcal{A}^\infty \xrightarrow{[1,\_]} \mathcal{E}ns$ is conservative over monics with globally supported codomain. In particular we have constructed a functor $\mathcal{A} \xrightarrow{J_0} \mathcal{A}^\infty \xrightarrow{[1,\_]} \mathcal{E}ns$ that is conservative over monics with globally supported codomain.*

Proof. It suffices to prove that the regular functors $\mathcal{A}^{(n)} \xrightarrow{J_n} \mathcal{A}^\infty \xrightarrow{[1,\_]} \mathcal{E}ns$ are conservative over monics with globally supported codomain. Let Γ represent the functor $[1, \_]$. We will use the abuse of notation of suppressing the symbol J_k when it is clear that we are viewing an element of a fiber inside of $\mathcal{A}^\infty$. Let $A \overset{m}{\rightarrowtail} B \in \mathcal{A}^{(n)}$ be monic with globally supported codomain in $\mathcal{A}^{(n)}$ such that $\Gamma(m)$ is an isomorphism (that is to say $\Gamma(J_n(m))$). Since $A \overset{m}{\rightarrowtail} B$ is isomorphic to $j_n(A) \overset{j_n(m)}{\rightarrowtail} j_n(B)$ in $\mathcal{A}^\infty$ (Remark 4.16) we have that $\Gamma(j_n(m))$ is an isomorphism, which in particular means that any section of $j_n(B) \longrightarrow 1$ lifts along $j_n(m)$ in $\mathcal{A}^\infty$. Take a pullback of $j_n(m)$ along Δ_B in $\mathcal{A}^{(n+1)}$.

$$\begin{tikzcd}
j_n(A) \arrow[r, tail, "j_n(m)"] & j_n(B) \\
P \arrow[u, dotted, "p_1"] \arrow[r, tail, dotted, "p_2"'] \arrow[ur, phantom, "p.b."] & 1 \arrow[u, "\Delta_B"']
\end{tikzcd}$$

It suffices to prove that p_2 admits a section in $\mathcal{A}^{(n+1)}$. This diagram viewed in $\mathcal{A}^\infty$ is still a pullback and p_2 is still monic. Since Δ_B lifts along $j_n(m)$ in $\mathcal{A}^\infty$ we obtain a section of p_2 in $\mathcal{A}^\infty$. Thus p_2 is an isomorphism in $\mathcal{A}^\infty$. Thus we conclude p_2 is an isomorphism in $\mathcal{A}^{(n+1)}$. The result follows. □

5 REDUCTIONS

5.1 A Regular Functor That is Conservative Over Monics With Globally Supported Codomain Suffices

In this section we will prove that by making the following assumption on our category $\mathcal{C}$ we will obtain the *Sufficient Points* theorem if we are able to construct a function that associates to each regular category $\mathcal{A}$ that possesses a distinguished terminal object, a regular functor $\mathcal{A} \longrightarrow \mathcal{E}ns$ that is conservative over monics with globally supported codomain.

Assumption 5.1. *$\mathcal{C}$ possesses a distinguished terminal object which we denote with 1, and a distinguished representative for each subobject class in $\mathcal{C}$.*

Remark 5.2. This assumption does not affect our desired range of applicability when proving completeness theorems in logic.

We will denote the distinguished representatives of a subobject class with a curly arrow $\hookrightarrow$ and for every object $X \in \mathcal{C}$ we will choose $X \overset{1_X}{\hookrightarrow} X$ as the distinguished representative of its subobject class.

Observation. For every $X \in \mathcal{C}$ the slice category $\mathcal{C}_{/X}$ has the distinguished terminal object $X \xrightarrow{1_X} X$. Additionally since the domain functor $\mathcal{C}_{/X} \xrightarrow{\Sigma} \mathcal{C}$ preserves and reflects monics there are distinguished representatives for each subobject class in $\mathcal{C}_{/X}$.

Definition 5.3. *For $S \hookrightarrow 1$ we say $X \in \mathcal{C}$* ***has support in*** *S if*

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, dashed, "\exists"'] & \arrow[d, phantom, "\equiv"] & 1 \\ & S \arrow[ur, hook] & \end{tikzcd}$$

and that S ***is the support of*** *X if the dashed arrow is a strict epimorphism.*

Observation. X is globally supported if and only if 1 is the support of X (see 1.6).

5.1.1 Pullback functor along a monomorphism

Proposition 5.4. *For $A \overset{m}{\rightarrowtail} B$ there are distinguished pullbacks along m.*

Proof. Pullbacks are well defined on subobject classes. For $W \xrightarrow{w} B$ take a pullback that uses the representative of the corresponding subobject class of m along w and label it $m^*(W) \hookrightarrow W$.

$$\begin{tikzcd} m^*(W) \arrow[r, hook] \arrow[d, dashed] \arrow[dr, phantom, "p.b."] & W \arrow[d, "w"] \\ A \arrow[r, tail, "m"'] & B \end{tikzcd}$$

The dashed arrow is uniquely determined because m is monic. That arrow will be denoted $m^*(w)$. □

Remark 5.5. Having this choice of pullbacks along m determines a functor $\mathcal{C}_{/B} \xrightarrow{m^*} \mathcal{C}_{/A}$ which is right adjoint to the functor defined as postcomposing by m. Thus it preserves all limits. Since the following diagram is a pullback and the domain functor Σ preserves and reflects strict epimorphisms we have that m^* is a regular functor.

$$\begin{tikzcd}
m^*(X) \arrow[r, hook] \arrow[d, "m^*(f)"'] \arrow[dr, phantom, "p.b."] & X \arrow[d, "f"] \\
m^*(Y) \arrow[r, hook] & Y
\end{tikzcd}$$

Observation. If $A \overset{m}{\hookrightarrow} B$ is a distinguished subobject, we have that $m^*(1_B) = 1_A$. That is m^* transforms the distinguished terminal object of $\mathcal{C}_{/B}$ into the distinguished terminal object in $\mathcal{C}_{/A}$. If the distinguished subobjects in $\mathcal{C}$ are closed under composition and from

$$\begin{tikzcd}
A \arrow[rr, hook] \arrow[dr, "u"'] & \equiv & B \\
& A' \arrow[ur, hook] &
\end{tikzcd}$$

it follows that u is a distinguished subobject, then m^* transforms distinguished subobjects into distinguished subobjects.

In the particular case where we take a distinguished subobject of 1 we will use the following notation. For $S \hookrightarrow 1$ we have the pullback functor $\mathcal{C} \xrightarrow{S^\wedge} \mathcal{C}_{/S}$ and denote its action as follows.

$$S^\wedge(X \xrightarrow{f} Y) = \begin{tikzcd}
X \wedge S \arrow[rr, "f \wedge S"] \arrow[dr] & \equiv & Y \wedge S \arrow[dl] \\
& S &
\end{tikzcd}$$

Lemma 5.6. *If $Y \in \mathcal{C}$ has support in S, then for $X \xrightarrow{f} Y$ we have that $Y \wedge S = Y$, $X \wedge S = X$ and $f \wedge S = f$.*

Proof. It follows from the fact that for such a Y the following diagram is a pullback.

$$\begin{tikzcd}
Y \arrow[r, hook, "1_Y"] \arrow[d] \arrow[dr, phantom, "p.b."] & Y \arrow[d] \\
S \arrow[r, hook, "i"'] & 1
\end{tikzcd}$$

and that X has support in S as well. $\square$

Corollary 5.7. *$S^\wedge(S)$ is a terminal object in $\mathcal{C}_{/S}$ and $S^\wedge$ is conservative over morphisms whose target has support in S.*

Proof. In fact $S^\wedge(S) = 1_S$. □

5.1.2 The result

Theorem 5.8. *If for every $S \hookrightarrow 1$ we are given a regular functor $\mathcal{C}_{/S} \xrightarrow{\Gamma_S} \mathcal{E}ns$ that is conservative over monics with globally supported codomain, then the family of regular functors of $\mathcal{C} \longrightarrow \mathcal{E}ns$ is monic-conservative.*

Proof. Consider the following family of regular functors of $\mathcal{C} \longrightarrow \mathcal{E}ns$.

$$\mathcal{C} \xrightarrow{S^\wedge} \mathcal{C}_{/S} \xrightarrow{\Gamma_S} \mathcal{E}ns$$

It is a family of functors $\{h_S\}$ indexed by the set $Sub(1)$ of subobject classes of 1. It suffices to prove that this family is monic-conservative. Let $X \overset{f}{\rightarrowtail} Y$ in $\mathcal{C}$ be such that its image through all these functors is an isomorphism. Take S the support of Y. It is enough to prove that $f \wedge S$ is an isomorphism. But this follows from the fact that $f \wedge S$ is monic, $Y \wedge S \longrightarrow S$ has global support in $\mathcal{C}_{/S}$ and $\Gamma_S(f \wedge S)$ is an isomorphism. □

Corollary 5.9. *The family of regular functors of $\mathcal{C} \longrightarrow \mathcal{E}ns$ is conservative.*

Proof. It follows from Theorem 4.18 and the Remark **??**. □

5.2 From a Family $\{h_S\}$ to h

Consider the following general construction for a family of set-valued functors $\{h_i\}_{i \in I}$ with common domain $\mathcal{C}$. Let $\mathcal{I}$ denote the category whose object set is I and for $i, j \in I$ we define $hom_{\mathcal{I}}(i, j) = Nat(h_i, h_j)$ with composition defined naturally. Let $\mathcal{C} \xrightarrow{h} \mathcal{E}ns^{\mathcal{I}}$ denote de functor defined as $h(C)(i) = h_i(C)$.

Remark 5.10. This asignment is functorial in both variables. We have the following commutative diagram for every $i \in I$.

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "h"] \arrow[dr, "h_i"'] & {} \arrow[d, phantom, "\equiv"] & \mathcal{E}ns^{\mathcal{I}} \arrow[dl, "ev_i"] \\
& \mathcal{E}ns &
\end{tikzcd}$$

Given the pointwise structure of the regular category $\mathcal{E}ns^{\mathcal{I}}$ we have that h preserves finite limits if and only if for every $i \in I$ the functors h_i preserve finite limits, h

preserves strict epimorphisms if and only if for every $i \in I$ the functors h_i preserve strict epimorphisms and h is conservative if and only if $\{h_i\}_{i \in I}$ is a conservative family as in 1.3.

In our particular case we have constructed a conservative family of regular functors $\{\mathcal{C} \xrightarrow{h_S} \mathcal{E}ns\}_{S \in Sub(1)}$. Using the previous construction we give $Sub(1)$ a structure and obtain a regular conservative functor $\mathcal{C} \xrightarrow{h} \mathcal{E}ns^{Sub(1)}$. The End.

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