Chapter 1

Induction

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1.1 Introduction

Induction is an important proof technique which is used, in different forms, in almost all areas of logic, theoretical computer science, and mathematics. It is needed to prove many of the results in logic.

Induction is often contrasted with deduction, and characterized as the inference from the particular to the general. For instance, if we observe many green emeralds, and nothing that we would call an emerald that's not green, we might conclude that all emeralds are green. This is an inductive inference, in that it proceeds from many particlar cases (this emerald is green, that emerald is green, etc.) to a general claim (all emeralds are green). *Mathematical* induction is also an inference that concludes a general claim, but it is of a very different kind than this "simple induction."

Very roughly, an inductive proof in mathematics concludes that all mathematical objects of a certain sort have a certain property. In the simplest case, the mathematical objects an inductive proof is concerned with are natural numbers. In that case an inductive proof is used to establish that all natural numbers have some property, and it does this by showing that

1. 0 has the property, and
2. whenever a number k has the property, so does $k + 1$.

Induction on natural numbers can then also often be used to prove general claims about mathematical objects that can be assigned numbers. For instance, finite sets each have a finite number n of elements, and if we can use induction to show that every number n has the property "all finite sets of size n are …" then we will have shown something about all finite sets.

Induction can also be generalized to mathematical objects that are *inductively defined*. For instance, expressions of a formal language such as those of first-order logic are defined inductively. *Structural induction* is a way to prove results about all such expressions. Structural induction, in particular, is very useful—and widely used—in logic.

1.2 Induction on $\mathbb{N}$

In its simplest form, induction is a technique used to prove results for all natural numbers. It uses the fact that by starting from 0 and repeatedly adding 1 we eventually reach every natural number. So to prove that something is true for every number, we can (1) establish that it is true for 0 and (2) show that whenever it is true for a number n, it is also true for the next number $n + 1$. If we abbreviate "number n has property P" by $P(n)$ (and "number k has property P" by $P(k)$, etc.), then a proof by induction that $P(n)$ for all $n \in \mathbb{N}$ consists of:

1. a proof of $P(0)$, and
2. a proof that, for any k, if $P(k)$ then $P(k + 1)$.

To make this crystal clear, suppose we have both (1) and (2). Then (1) tells us that $P(0)$ is true. If we also have (2), we know in particular that if $P(0)$ then $P(0 + 1)$, i.e., $P(1)$. This follows from the general statement "for any k, if $P(k)$ then $P(k + 1)$" by putting 0 for k. So by modus ponens, we have that $P(1)$. From (2) again, now taking 1 for n, we have: if $P(1)$ then $P(2)$. Since we've just established $P(1)$, by modus ponens, we have $P(2)$. And so on. For any number n, after doing this n times, we eventually arrive at $P(n)$. So (1) and (2) together establish $P(n)$ for any $n \in \mathbb{N}$.

Let's look at an example. Suppose we want to find out how many different sums we can throw with n dice. Although it might seem silly, let's start with 0 dice. If you have no dice there's only one possible sum you can "throw": no dots at all, which sums to 0. So the number of different possible throws is 1. If you have only one die, i.e., $n = 1$, there are six possible values, 1 through 6. With two dice, we can throw any sum from 2 through 12, that's 11 possibilities. With three dice, we can throw any number from 3 to 18, i.e., 16 different possibilities. 1, 6, 11, 16: looks like a pattern: maybe the answer is $5n + 1$? Of course, $5n + 1$ is the maximum possible, because there are only $5n + 1$ numbers between n, the lowest value you can throw with n dice (all 1's) and $6n$, the highest you can throw (all 6's).

Theorem 1.1. *With n dice one can throw all $5n + 1$ possible values between n and $6n$.*

Proof. Let $P(n)$ be the claim: "It is possible to throw any number between n and $6n$ using n dice." To use induction, we prove:

1. The *induction basis* $P(1)$, i.e., with just one die, you can throw any number between 1 and 6.
2. The *induction step*, for all k, if $P(k)$ then $P(k + 1)$.

(1) Is proved by inspecting a 6-sided die. It has all 6 sides, and every number between 1 and 6 shows up one on of the sides. So it is possible to throw any number between 1 and 6 using a single die.

To prove (2), we assume the antecedent of the conditional, i.e., $P(k)$. This assumption is called the *inductive hypothesis*. We use it to prove $P(k + 1)$. The hard part is to find a way of thinking about the possible values of a throw of $k + 1$ dice in terms of the possible values of throws of k dice plus of throws of the extra $k + 1$-st die—this is what we have to do, though, if we want to use the inductive hypothesis.

The inductive hypothesis says we can get any number between k and $6k$ using k dice. If we throw a 1 with our $(k + 1)$-st die, this adds 1 to the total. So we can throw any value between $k + 1$ and $6k + 1$ by throwing k dice and then rolling a 1 with the

$(k+1)$-st die. What's left? The values $6k+2$ through $6k+6$. We can get these by rolling k 6s and then a number between 2 and 6 with our $(k+1)$-st die. Together, this means that with $k+1$ dice we can throw any of the numbers between $k+1$ and $6(k+1)$, i.e., we've proved $P(k+1)$ using the assumption $P(k)$, the inductive hypothesis. □

Very often we use induction when we want to prove something about a series of objects (numbers, sets, etc.) that is itself defined "inductively," i.e., by defining the $(n+1)$-st object in terms of the n-th. For instance, we can define the sum s_n of the natural numbers up to n by

$$\begin{aligned} s_0 &= 0 \\ s_{n+1} &= s_n + (n+1) \end{aligned}$$

This definition gives:

$$\begin{aligned} s_0 &= 0, & & \\ s_1 &= s_0 + 1 & &= 1, \\ s_2 &= s_1 + 2 & &= 1 + 2 = 3 \\ s_3 &= s_2 + 3 & &= 1 + 2 + 3 = 6, \text{ etc.} \end{aligned}$$

Now we can prove, by induction, that $s_n = n(n+1)/2$.

Proposition 1.2. $s_n = n(n+1)/2$.

Proof. We have to prove (1) that $s_0 = 0 \cdot (0+1)/2$ and (2) if $s_k = k(k+1)/2$ then $s_{k+1} = (k+1)(k+2)/2$. (1) is obvious. To prove (2), we assume the inductive hypothesis: $s_k = k(k+1)/2$. Using it, we have to show that $s_{k+1} = (k+1)(k+2)/2$.

What is s_{k+1}? By the definition, $s_{k+1} = s_k + (k+1)$. By inductive hypothesis, $s_k = k(k+1)/2$. We can substitute this into the previous equation, and then just need a bit of arithmetic of fractions:

$$\begin{aligned} s_{k+1} &= \frac{k(k+1)}{2} + (k+1) = \\ &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \\ &= \frac{k(k+1) + 2(k+1)}{2} = \\ &= \frac{(k+2)(k+1)}{2}. \end{aligned}$$

□

The important lesson here is that if you're proving something about some inductively defined sequence a_n, induction is the obvious way to go. And even if it isn't (as in the case of the possibilities of dice throws), you can use induction if you can somehow relate the case for $k+1$ to the case for k.