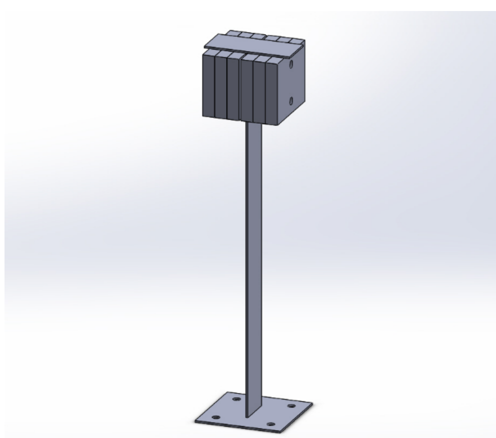
ASSIGNMENT : LINEAR SDOF SYSTEMS AND FREQUENCY-DOMAIN ANALYSIS

LGCIV2042
Dynamics of Structures

Professor : João Pacheco de Almeida
Teaching assistants : Benjamin Dardenne, Basile Payen



Group 5
Foret Félix
Stasse Carla

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Contents

The aim of this report is to analyze the dynamic behavior of a physical model. It is a column on top of which four masses are attached (see cover page).

All the calculation were done using Python scripts available, along with the results, on the following GitHub link: `https://github.com/feforet/LGCIV2042-Dynamics_of_Structures`

1 Dynamic properties and free vibration of the model

1.1 Model and dynamic properties

First, we know that the structure will only be loaded in the direction of its weak inertia. Therefore, we can model the problem with a planar situation.

Since the structure will only be loaded horizontally on the top and the displacements will stay sufficiently small and in the same direction as the load, we consider the whole structure as a linear system. This system is composed of a single vertical beam embedded at its base and hinged at the top.

Since the horizontal displacements will be significantly greater than the vertical ones, we consider its axial stiffness as infinite ($EA \approx \infty$) and block the vertical displacement at the top. We stay with one single degree of freedom, the horizontal one at the top.
We consider the element top extremity as the position of its center of mass.

The whole structure is made of steel with a density $\rho = 7900[kg/m^3]$. The inertia of the column along its weak axis is $I = \frac{0.03 \times 0.00142^3}{12} = 7.158 \times 10^{-12}[m^4]$.

In order to compute the required parameters, we first have to find the mass and the position of the center of mass of the system.

$$m = m_{mass} + m_{column} + m_{topPlate} = 1.2071[kg]$$

and

$$y_G = \frac{y_{G\_mass} \cdot m_{mass} + y_{G\_column} \cdot m_{column} + y_{G\_topPlate} \cdot m_{topPlate}}{m_{mass} + m_{column} + m_{topPlate}} = 0.2511[m]$$

with

$$m_{mass} = \rho \times 0.04 \times 0.05 \times 0.06 = 0.9480[kg]$$

$$m_{column} = \rho \times 0.3 \times 0.07 \times 0.00142 = 0.2355[kg]$$

$$m_{topPlate} = \rho \times 0.03 \times 0.07 \times 0.00142 = 0.0236[kg]$$

$$y_{G\_mass} = 0.275[m]$$

$$y_{G\_column} = 0.150[m]$$

$$y_{G\_topPlate} = 0.30071[m]$$

Then we can compute the parameters of the structure as follows:

$$k = 3\frac{E \cdot I}{L^3} = 284.818[N/m]$$

$$\omega_{n,th} = \sqrt{\frac{k}{m}} = 15.361[rad/s]$$

$$f_{n,th} = \frac{\omega_{n,th}}{2\pi} = 2.445[Hz]$$

$$T_{n,th} = \frac{1}{f_{n,th}} = 0.409[s]$$

1.2 Derive the displacement history from the experimental acceleration record (plot it in 1.4). Then, compute the initial displacement u0 of the column's head. Justify

The objective of this section is to determine the initial displacement imposed, u_0.

1.2.1 Methodology

1. **Pre-processing the experimental signal,** a_x
   The first step consists in processing the experimental acceleration signal, a_x. The signal is trimmed to remove unstable phases at the beginning and the end of the recording. Since the accelerometer may have a slight offset, the signal must then be re-centered around its mean value (assumed close to zero).

2. **Numerical integration on** a_x
   The acceleration signal is numerically integrated to obtain the velocity, v_x. A trapezoidal integration scheme is used, as expressed in equation 1.1:

$$y_n = y_{n-1} + \frac{x_n + x_{n-1}}{2}\Delta t \tag{1.1}$$

3. **Processing the velocity signal,** v_x
   Due to measurement errors and noise in the raw acceleration a_x, the computed velocity v_x is not centered around zero and exhibits an unrealistic drift, as shown in Figure 1.1. To correct this behavior, a high-pass filter is applied and the signal is re-centered around its mean value.

4. **Numerical integration on** v_x
   The same numerical integration method is then applied to the filtered velocity signal to obtain the displacement, u_x. After re-centering around the mean, the initial displacement u_0 can be extracted. Figure 1.2 illustrates the acceleration, velocity, and displacement signals obtained after filtering and processing the experimental data.

5. **Determine the initial displacement,** u_0
   The initial displacement u_0 corresponds to the first and highest peak of the displacement signal u_x. After discarding the first misaligned oscillations at the beginning of the record, the estimated initial displacement is $u_0 = 6.82$ cm in the direction of the weak axis.

1.2.2 Results

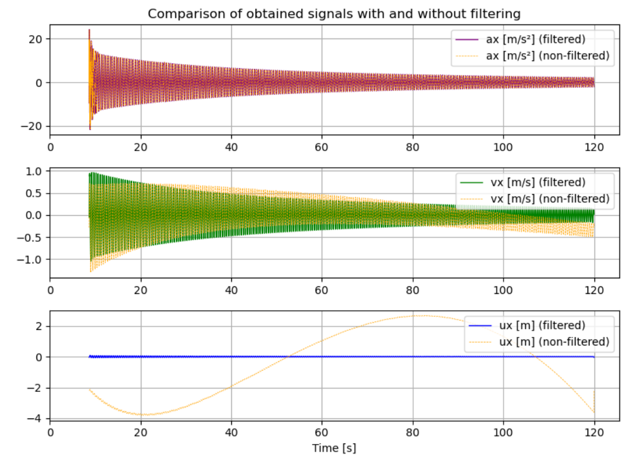


Figure 1.1: Comparison between numerical signals with and without filter applied on the velocity.

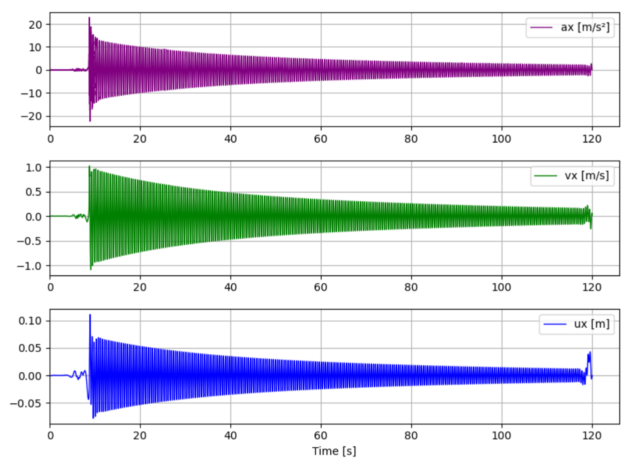


Figure 1.2: Calculated signals by numerical integration and filter applied.

1.3 Equation of motion of the undamped system

The equation of motion of the undamped single degree of freedom system, with its initial conditions, is written as follows:

$$m\ddot{u}(t) + ku(t) = 0, \qquad u(0) = u_0, \qquad \dot{u}(0) = \dot{u}_0 \tag{1.2}$$

Assuming that the initial velocity is zero ($\dot{u}_0 = 0$), the analytical solution of equation 1.2 is:

$$u(t) = u(0)\cos(\omega t) + \frac{\dot{u}(0)}{\omega}\sin(\omega t) = u_0\cos(\omega t), \tag{1.3}$$

where ω [rad/s] denotes the natural frequency of the system.

1.4 Plot the response in terms of acceleration, velocity and displacement from the questions (1.2) and (1.3) for 20s. Identify and discuss the sources of the energy loss. If needed, adjust the natural frequency of your model for the rest of the project.

Based on the theoretical stiffness, k and mass, m of the system, the theoretical natural frequency was calculated in section 1 1.1 ($\omega_{n,th} = 15.361[rad/s]$). An analytical response is then calculated with the theoretical frequency and the comparison with the experimental signals is shown in Figure 1.3.

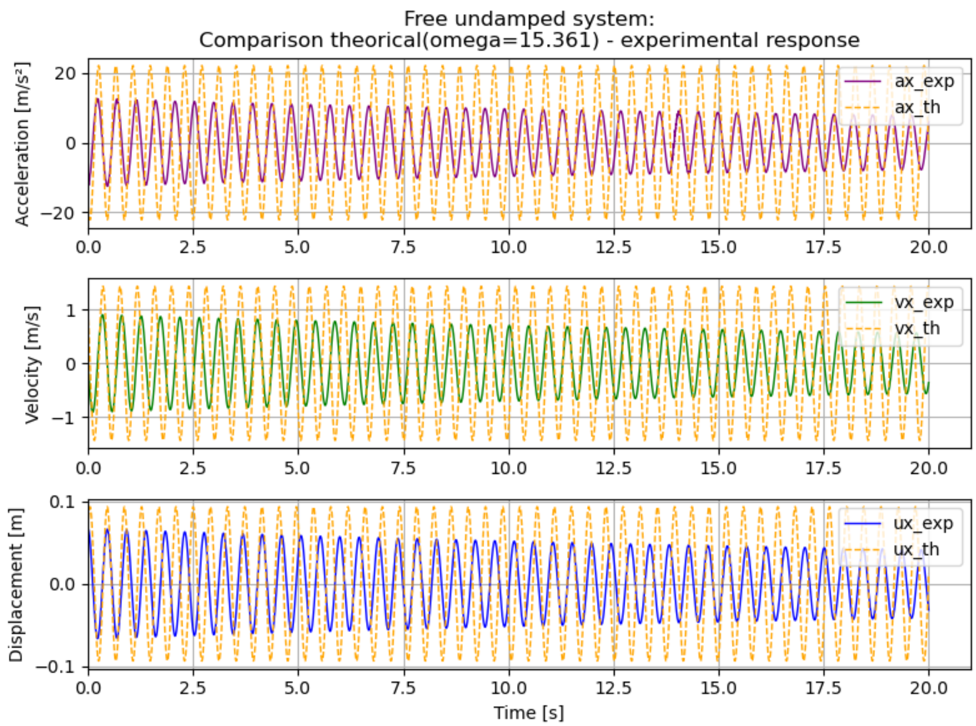


Figure 1.3: Comparison of the analytical and experimental signals.

The two major differences between the two signals are the period of the analytical signals who is slightly smaller than the experimental one and the theoretical amplitude which is greater than the experimental.

The energy loss linked with a smaller amplitude in case of the experimental signals can have multiple sources. The first cause is the natural damping of the system that drives the signal to rest whereas the analytical undamped system never decrease in amplitude. This damping is due to friction with the air and a non-elastic behaviour of the model. Indeed, after an imposed displcement, the model does not come back exactlly to its original position.

The difference in period can be caused by the weight of accelerometer on top of the masses which wasn't taken in account. The additionnal weight elevate the center of gravity of the upper node and thus increase the length parameter, L which then lower the overall stiffness, $k = \frac{12EI}{L^3}$. The decrease of stiffness and increase of mass result in a lower frequency, $\omega_n = \sqrt{k/m}$.

The difference in amplitude can be due to treatment errors which occur when the displacement is derived from the acceleration and when filters are applied causing a loss of information. The determination of the initial displacement, $u0$ might also be imprecise and too arbitrary. Indeed, the first peak found is not necessarily the actual initial displacement.

Since, the theoretical natural frequency ($\omega_{n,th}$) obtained is quite different from the experimental one ($\omega_{n,exp}$), the natural frequency is adjust to the real model. The following steps resume the calculation of the experimental frequency (ω_n) who will be used from now on.

1. **Find the peaks and troughs:**
   The natural frequency is derive from the recorded acceleration signal. Since the initial signal is not perfectly smooth, the maxima and minima are determine using a *Scipy.signal* function and the estimated period set at $\omega_{n,th}$.

2. **Calculate the period:**
   The distance between the peaks and the troughs give different values of period. The Figure 1.4 shows the distribution of those values. Finally, the natural period of the model is given by the mean value of the histogram and the corresponding frequency is $\omega_n = 13.63[rad/s]$.

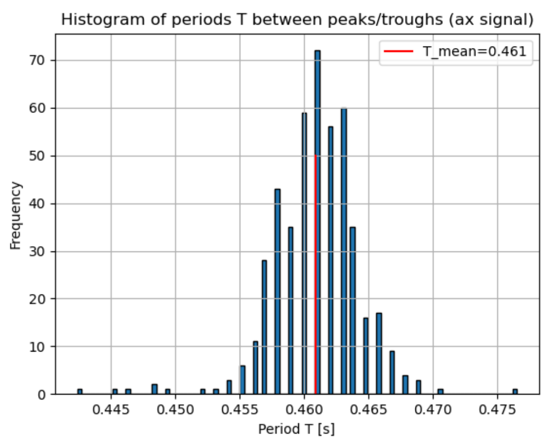


Figure 1.4: Histogram of the period obtained from the differences between peaks and troughs.

1.5 Compute the damping coefficient based on the recorded acceleration. Develop the method. Compare this value with the damping coefficient of steel structures.

The methodology to determine the damping coefficient from the acceleration is similar to the method used in section 1.4 for the period/frequency.

1. **Find the damping ratio, ξ:**
   After finding the peaks and troughs, the damping ratio is derived from the following formula:

$$\ln(\frac{\ddot{u}_i}{\ddot{u}_{i+1}}) = 2\pi\xi$$

   The obtained values of damping ratio, ξ are shown is Figure 1.5.

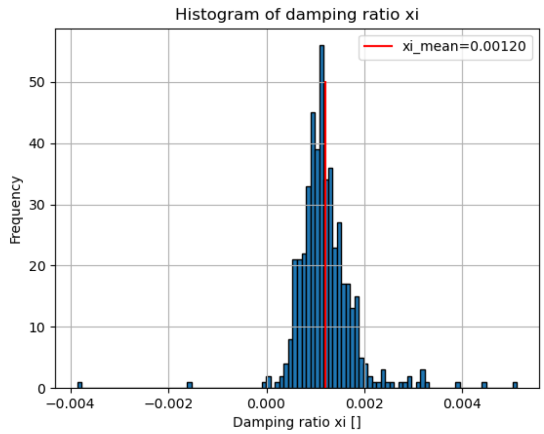


Figure 1.5: Histogram of the period obtained from the differences between peaks and troughs.

The retained value of the damping ratio is also the mean value, $\xi = 0.00120$.

2. **Calculate the damping coefficient, c:**
The damping coefficient is given by the formula:

$$\xi = \frac{c}{c_c} = \frac{c}{2m\omega} \iff c = \xi \cdot c_c = 0.0394$$

For the steel structure, the damping coefficient is generally about 1% to 2%. In case of our model, the system is underdamped (or undercritically-damped) since $c < c_c$.

1.6 Using the damping coefficient calculated in (1.5), write down the equation of motion of the damped system. Solve the differential equation analytically and plot the response of the structure in terms of acceleration, velocity and displacement during 20s. Compare with the experimental response in (1.2) and comment.

The equation of motion of a damped system is similar the undamped one but with a additionnal visious damping terme, $c\dot{u}$.

$$m\ddot{u}(t) + c\dot{u} + ku(t) = 0, \qquad u(0) = u_0, \qquad \dot{u}(0) = \dot{u}_0 \tag{1.4}$$

Like in section 1.3, the initial velocity is assumed to be null and since the damping ratio, $\xi < 0$, the under-critical case gives the following solution:

$$u(t) = e^{-\xi\omega t}[A\cos(\omega_d t) + B\sin(\omega_d t)] \tag{1.5}$$

with

- $\omega_d = \omega\sqrt{1 - \xi^2}$
- $A = u(0)$
- $B = \frac{\dot{u}(0) + \xi\omega u(0)}{\omega_d}$

Using the damping ratio (ξ) and the adjust frequency (ω_n) of sections 1.5 and 1.4, a comparison between the analytical and experimental solutions of a damped system is shown at Figure 1.6.

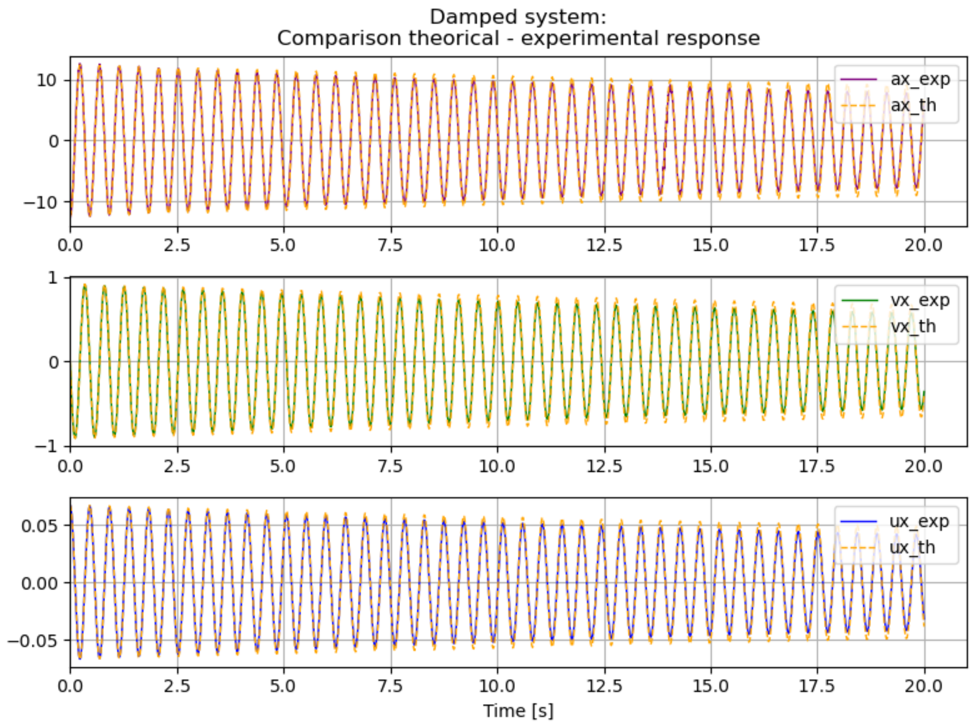


Figure 1.6: Comparison between theoretical and experimental responses.

The first element to compare is the period which is nearly the same for the three signals (u, $\dot{u}$, $\ddot{u}$).

In term of amplitude, both responses seem to begin at the same value which it expected since the initial displacement, $u(0)$ was determine based on the experimental displacement and then used in the analytical solution. However with time, the 3 experimental signals appear to have a higher damping as the amplitude decreases faster than the theoretical response. This small difference can be explain with the methodology to calculate the damping ratio, ξ. Indeed, this parameter is in reality the mean of all the rates of decrease calculated at different time in the recorded acceleration (values shown in Figure 1.5). In the end, the mean value might not be the most fit compare with the experimental signals.

2 Response to ground motion

For the following questions, we considered the natural periods and (angular-)frequencies derived from the experimental measurements and not the ones computed at the question 1.1.

2.1 Amplitude of the steady-state acceleration response as a function of the ratio $k = \frac{\omega_g}{\omega_n}$

We subjected the structure to five harmonic ground accelerations, changing the angular frequency of the loading. The equation of the ground acceleration is $\ddot{u}_g(t) = 0.1 \cdot g \cdot \sin(k \cdot \omega_n \cdot t)$ where $k \in \{0.5, 0.75, 1, 1.5, 2\}$ represents the ratio between the excitation frequency and the natural frequency. Using an accelerometer, we recorded the acceleration response at the top of the structure. Since we assumed a planar situation, we only focus on the component of the acceleration in the direction of the weak axis.

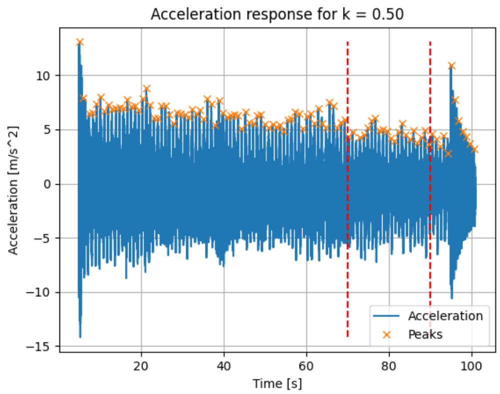
The five following graphs (Figure 2.1) represent the response of the structure to these loadings. As we can see, the acceleration of the structure exceeded the limit of the accelerometer. The record is thus limited to 410 $[m/s^2]$. We will keep this value knowing that it is a lower bound of the real response.

Since we focus on the amplitude of the steady-state acceleration response, we selected a time window where the system has reached steady-state conditions, characterized by stabilized oscillations. Within this interval, we determined the average value of the peaks. This average is then subtracted by the average value of the response. The results are written in Table 2.1. We plotted the amplitude of the steady-state acceleration as a function of the ratio k on the following graph (Figure 2.2).

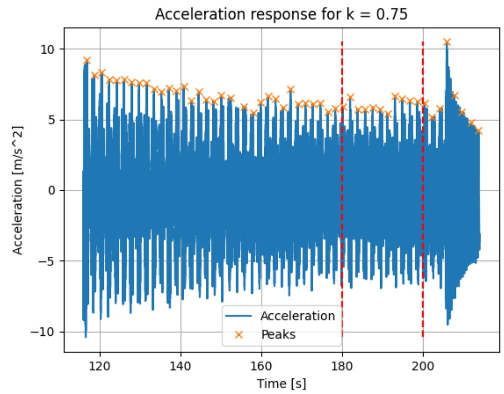
We observe that the shape of the curve is similar to the one of the acceleration response factor R_a as a function of the frequency ratio seen in the course. Unfortunately, as we do not know the static response of the structure p_0/m, we can not compute the acceleration response factor in our case.

k	Steady-state acceleration amplitude $[m/s^2]$
0.50	5.04
0.75	4.81
1.00	24.93
1.50	3.32
2.00	1.65

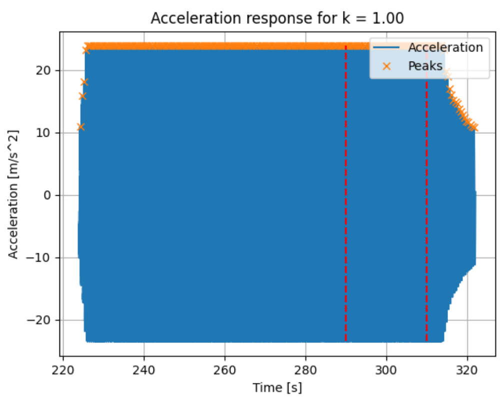
Table 2.1: Steady-state acceleration response for different values of k.



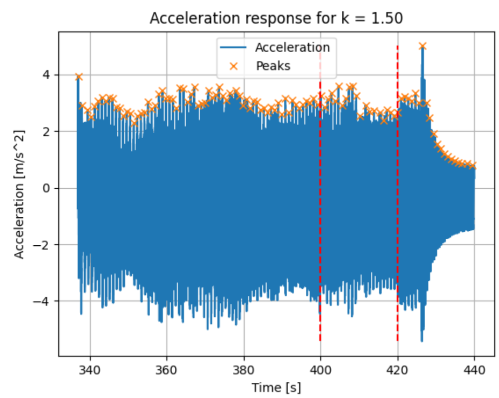
(a) Acceleration response for k = 0.50.



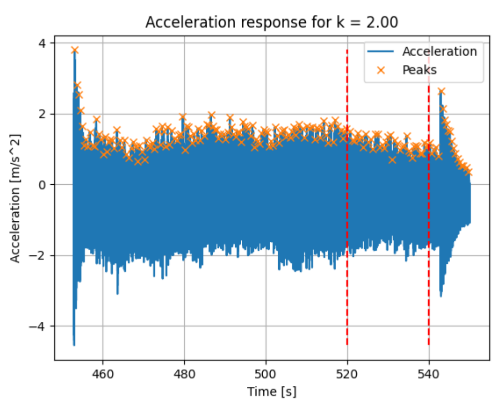
(b) Acceleration response for k = 0.75.



(c) Acceleration response for k = 1.00.



(d) Acceleration response for k = 1.50.



(e) Acceleration response for k = 2.00.

Figure 2.1: Acceleration responses for different values of k.
The orange crosses represent the peaks and the dashed lines the limits of the time period selected to compute the average peak value.

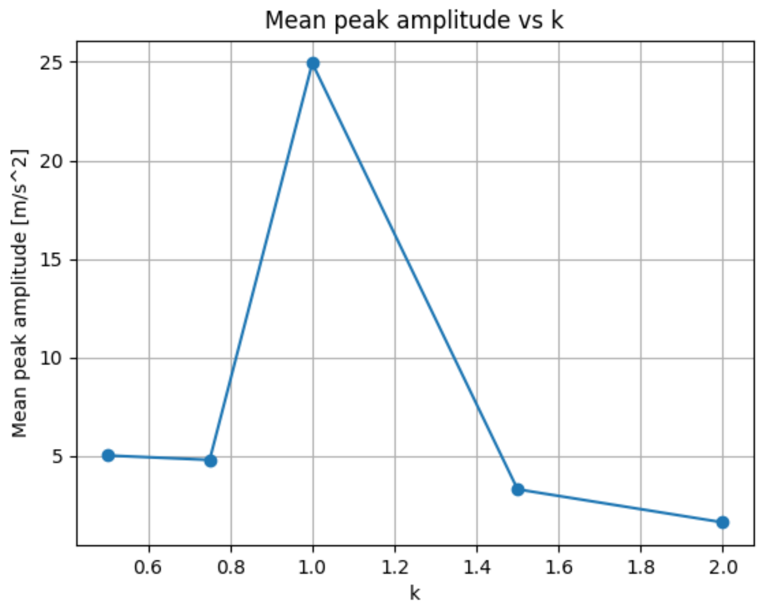


Figure 2.2: Steady-state acceleration response for different values of k.

2.2 For an arbitrary value of excitation frequency $\bar{\omega}$, compute the analytical expression of the steady-state displacement response of the structure. From the displacement response, compute the amplitude of the acceleration as a function of $\bar{\omega}$. Compare and comment with experimental values from (2.1). Comment the results for an extremely low and an extremely high frequency.

The absolute displacement, $u^t(t)$ of a system subjected to a ground motion $u_g(t)$ is the sum of this ground motion and the solution of the equation of motion subjected to the ground motion, $u(t)$. In the case of a viscously damped system subject to a ground motion of $\ddot{u}_g = 0.1g \cdot \sin(\bar{\omega}t)$, the equation of motion is as follow:

$$m\ddot{u}(t) + c\dot{u} + ku(t) = p(t), \qquad p(t) = -m \cdot \ddot{u}_g \qquad u(0) = u_0, \qquad \dot{u}(0) = \dot{u}_0 \tag{2.1}$$

The solution of this equation is the sum of two parts: the transient-state and the steady-state. After some manipulation, the steady-state of the response can be written as in equation 2.2:

$$\begin{cases} u(t) = u_0 \cdot \sin(\bar{\omega}t - \phi) \\ u_0 = \frac{p_0}{k} \cdot R_d \\ R_d = \left[\left[(1 - (\frac{\bar{\omega}}{\omega})^2 \right]^2 + \left[2\xi(\frac{\bar{\omega}}{\omega}) \right]^2 \right]^{-1/2} \\ \phi = \arctan(\frac{2\xi}{(1-(\frac{\bar{\omega}}{\omega})^2)}) \\ p_0 = -m \cdot 0.1g \end{cases}$$

Finally, the acceleration derived from the displacement has the form of $\ddot{u}(t) = -\frac{p_0}{m} \cdot R_a \cdot sin(\bar{\omega}t - \phi)$ with $R_a = (\frac{\bar{\omega}}{\omega})^2 \cdot R_d$.

The Figure **??** shows the steady-state part of the analytical acceleration.

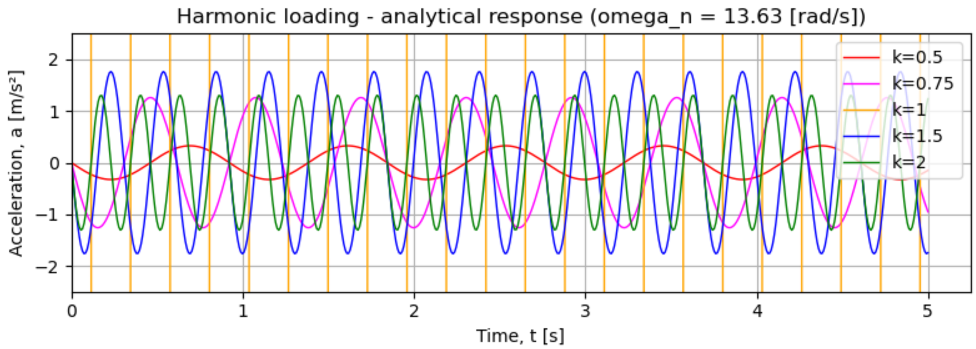
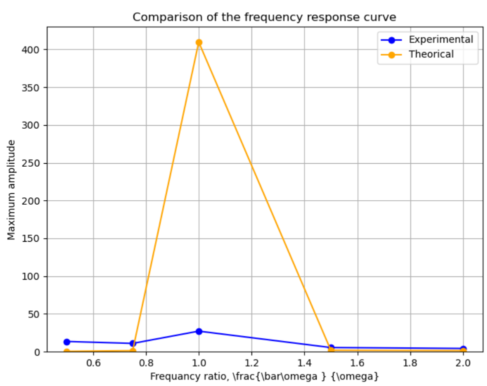
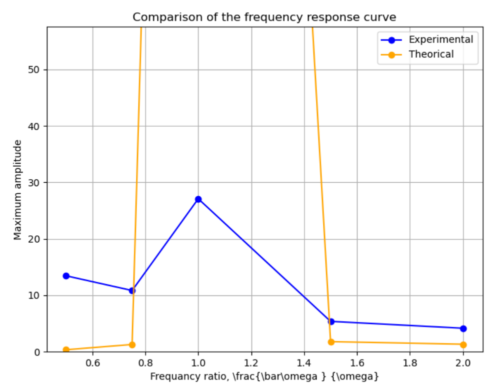


Figure 2.3: Comparison of the steady-state response for all different values of k for 5 seconds time.

The steady-state response for k=1 is not entirely shown on the plot since the amplitude is much greater than the other signals. Indeed, for k=1, the theoretical amplitudes of the acceleration is about $410m/s^2$ and $495m$ for the displacement.



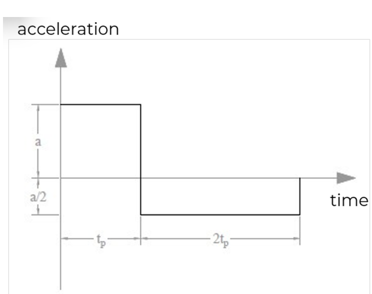
(a) Frequency response curves



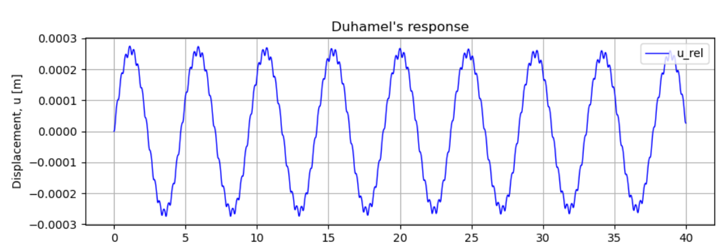
(b) Zoom on the Frequency response curves

2.3

For this sub-question, the Duhamel'integral is applied on a two-sided impulse (Figure 2.5b). The response calculated is shown in Figure **??**.



(a) Two-sided impulse with $a = 0.075g$ and $tp = 0.1s$.



(b) Duhamel's response in term of displacement.

3 Numerical time-domain integration

We programmed the central difference method to compute the response $U(t)$ and $\ddot{U}(t)$ of the structure subjected to the ground acceleration $\ddot{u}_g(t) = 0.1 \cdot g \cdot \sin(\omega_n \cdot t)$.

We used four different time steps: $\Delta t \in \{T_n/50, T_n/25, T_n/5, T_n/2\}$. The results are plotted, along with the analytical solution and the experimental data, on Figures 3.1 to 3.4.

We observe that the response for $\Delta t = T_n/2$ is increasing exponentially so we plotted it on separate graphs with logarithmic scale for readability (Figures 3.2 & 3.4). This is due to the fact that a time step of half the natural frequency captures very badly the shape of the excitation so the response can not be realistic. The same reason leads the response for $\Delta t = T_n/5$ to be oscillating with peaks significantly lower than the other two responses. These two response with smaller time steps are closer to each other and more realistic for an excitation at the natural frequency of the structure. These observations ar valuable for both the acceleration and the displacement response.

We can conclude that the choice of the time step is crucial in order to have a coherent response. Here, we can see that a time step of $T_n/5$ is too low but time steps of $T_n/25$ and $T_n/50$ give coherent shapes and the amplitude of the associated responses are sufficiently close to each other to consider that they are close to reality.

Comparing these results to the acceleration measured experimentally, we observe that the experimental one is limited to the maximum value permitted by the accelerometer. Although, we see that for the first oscillations during the first two seconds of recording, the experimental acceleration is greater than the one computed with the central difference method.

Comparing the results with the analytical solution derived in subsection 2.2, we see that the amplitude of the analytical acceleration response is greater than the one of the others. We also observe that a 40 second computation did not permit to reach a steady state using the central difference method. If we computed the response over a longer time period, we should be able to see if the steady state reached is equal or not to the analytical response. Considering the displacement response, the amplitude of the analytical solution is about $400m$. It is clear that such a value is not possible experimentally but, again, we can not determine if the central difference method give a similar value unless we compute the response for a longer time period.

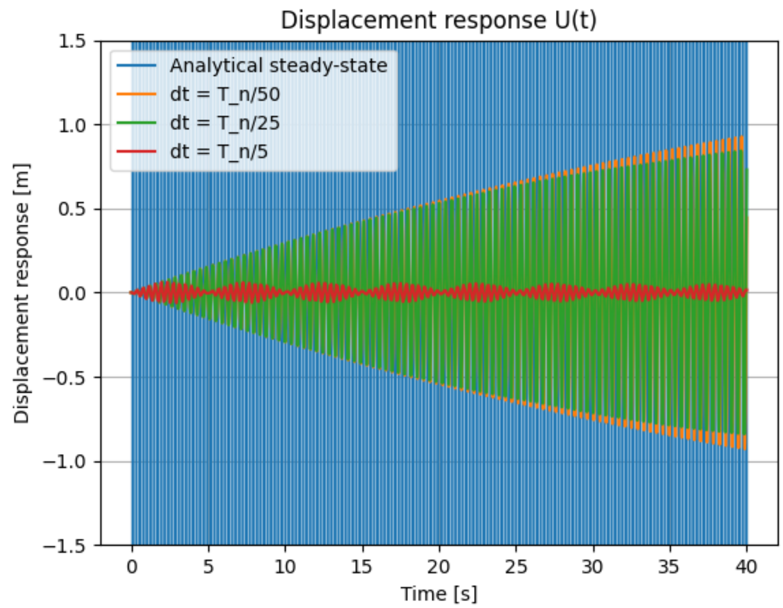


Figure 3.1: Displacement response.

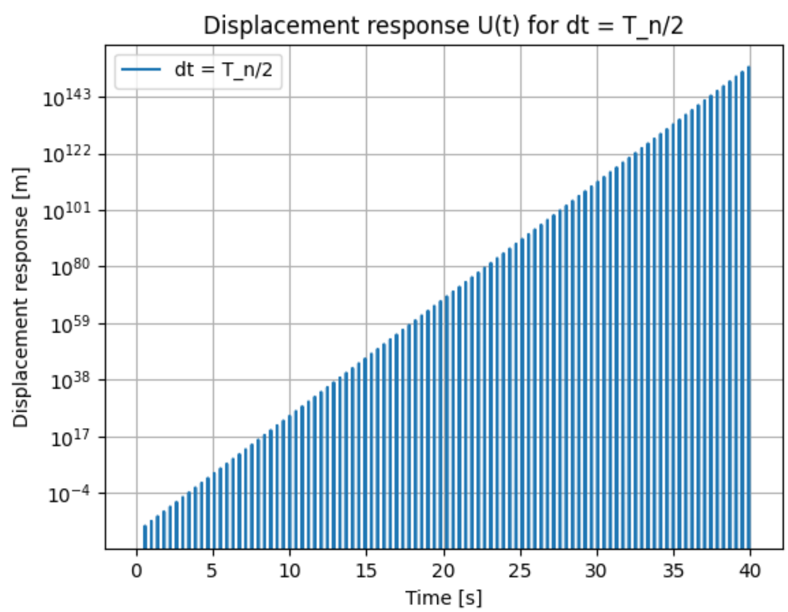


Figure 3.2: Displacement response for $\Delta t = T_n/2$.

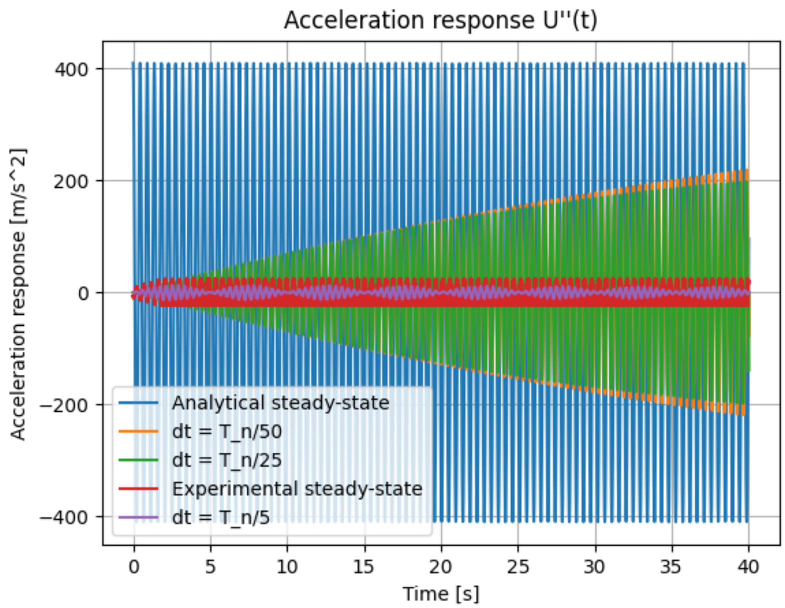


Figure 3.3: Acceleration response.

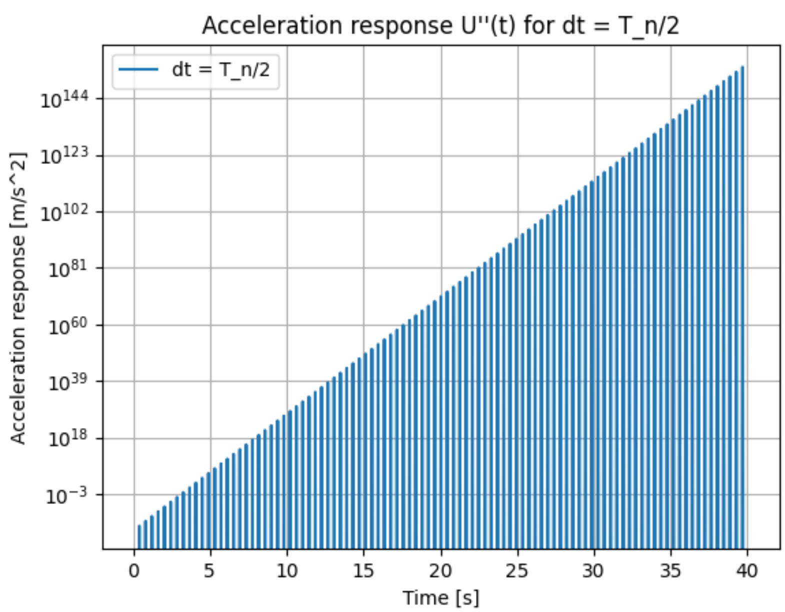


Figure 3.4: Acceleration response for $\Delta t = T_n/2$.

4 Frequency-domain analysis

The FFTs computed in this section are far from perfection. We should have computed them with the time as index and not the original indexes (1-2-3-...).

4.1 Undamped free vibration

The FFT of acceleration response from questions 1.2 and 1.3 are represented on Figure 4.1. They are the same. This is probably because we computed the FFT on the filtered signal and not on the raw output of the accelerometer. We expected a wider range of frequencies in the experimental data.

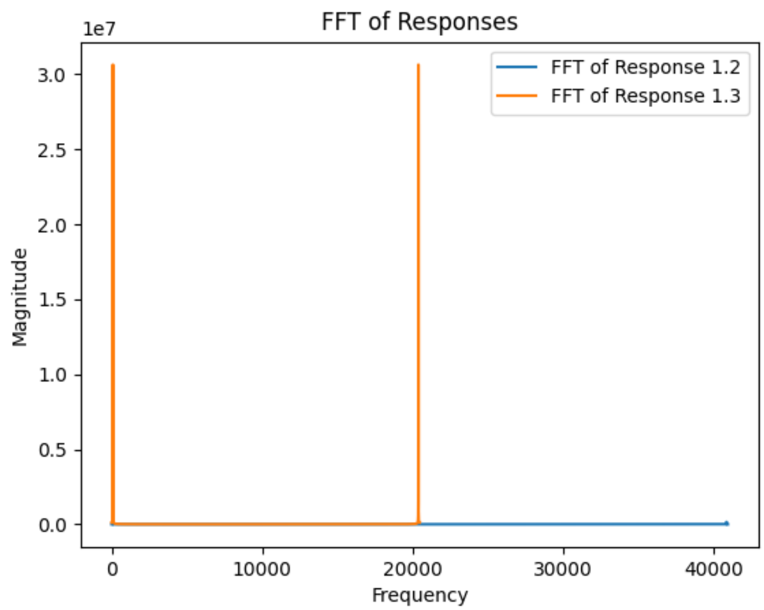


Figure 4.1: FFT of acceleration response from questions 1.2 and 1.3.

4.2 Damped free vibration

The FFT of acceleration response with damping are represented on Figure 4.2. We can observe that the peaks is a bit wider with higher rates of damping.

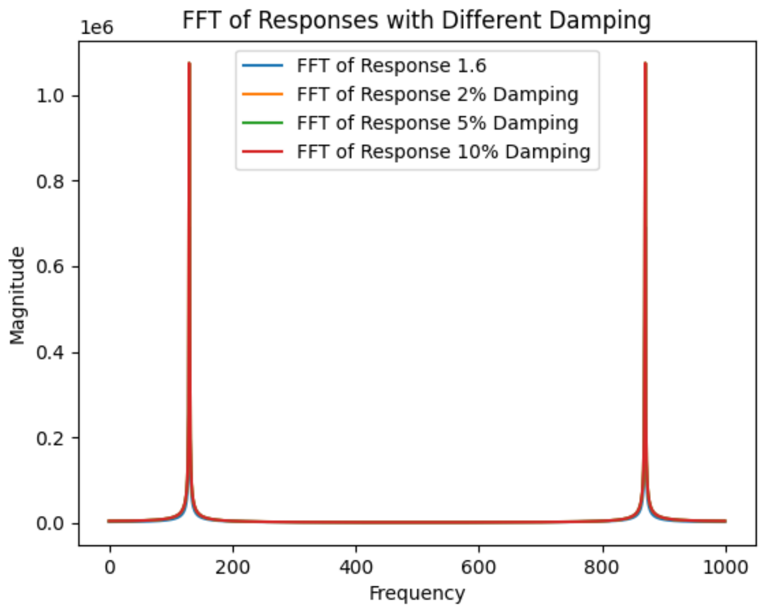


Figure 4.2: FFT of acceleration response with damping.

4.3 Response to ground acceleration

4.4 Response to ground acceleration - first 20s