

Abstract of thesis entitled

“Linear Programming Techniques for Algorithms with Applications in Economics”

Submitted by

Fei Chen

for the degree of Doctor of Philosophy
at The University of Hong Kong
in March, 2014

We study algorithms and models for several economics-related problems from the perspective of linear programming.

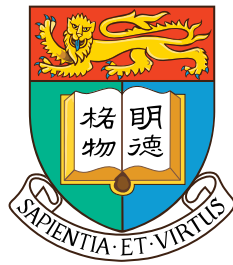
In network bargaining games, stable and balanced outcomes have been investigated in previous work. However, existence of such outcomes requires that the linear program relaxation of a certain maximum matching problem has integral optimal solution. We propose an alternative model for network bargaining games in which each edge acts as a player, who proposes how to split the weight of the edge among the two incident nodes. We show that the distributed protocol by Kanoria et. al can be modified to be run by the edge players such that the configuration of proposals will converge to a pure Nash Equilibrium, without the linear program integrality gap assumption. Moreover, ambiguous choices can be resolved in a way such that there exists a Nash Equilibrium that will not hurt the social welfare too much.

In the oblivious matching problem, an algorithm aims to find a maximum matching while it can only makes (random) decisions that are essentially oblivious to the input graph. Any greedy algorithm can achieve performance ratio 0.5, which is the expected number of matched nodes to the number of nodes in a maximum matching. We revisit the Ranking algorithm using the linear programming framework, where the constraints of the linear program are given by the structural properties of Ranking. We use continuous linear program relaxation to analyze the limiting behavior as the finite linear program grows. Of particular interest are new duality and complementary slackness characterizations that can handle monotone constraints and mixed evolving and boundary constraints in continuous linear program, which enable us to achieve a theoretical ratio of 0.523 on arbitrary graphs.

The J -choice K -best secretary problem, also known as the (J, K) -secretary problem, is a generalization of the classical secretary problem. An algorithm for the (J, K) -secretary problem is allowed to make J choices and the payoff to be maximized is the expected number of items chosen among the K best items. We use primal-dual continuous linear program techniques to analyze a class of infinite algorithms, which are general enough to capture the asymptotic behavior of the finite model with large number of items. Our techniques allow us to prove that the optimal solution can be achieved by a (J, K) -threshold algorithm, which has a nice “rational description” for the case $K = 1$.

An abstract of exactly 390 words

Linear Programming Techniques for Algorithms with Applications in Economics



by

Fei Chen

Department of Computer Science

The University of Hong Kong

Supervised by

Dr. Hubert Chan

A thesis submitted in partial fulfillment of the requirements for
the degree of *Doctor of Philosophy in Computer Science*
at *The University of Hong Kong*

March, 2014

Declaration

I declare that this thesis represents my own work, except where due acknowledgement is made, and that it has not been previously included in any thesis, dissertation or report submitted to this university or any institution for any diploma, degree or other qualifications.

Signed:

Fei Chen
March, 2014

Acknowledgements

I would like to thank my supervisor Dr. Hubert Chan for his advices and inspiration. He has given me the freedom to choose my research areas and continually provided invaluable guidance for my study.

I thank my friends and colleagues in the University of Hong Kong. Especially thanks to Li Ning for his help and suggestions at the beginning of my study.

I am grateful to my family for their understanding and unconditional support.

Finally, I would like to thank my lovely girlfriend who has been the major source of randomness in the past few years of my life.

Contents

Declaration	i
Acknowledgements	ii
Table of Contents	iii
List of Figures	v
List of Algorithms	vii
1 Introduction	1
1.1 Definitions and Notations	2
1.2 Outline of the Thesis	3
1.2.1 Optimizing Social Welfare for Network Bargaining Games	3
1.2.2 Analyzing Ranking via Continuous LP for the Oblivious Matching Problem	4
1.2.3 A Primal-dual Continuous Linear Programming Method for the Secretary Problem	5
2 Optimizing Social Welfare for Network Bargaining Games	7
2.1 Introduction	7
2.2 Preliminaries	11
2.3 A Distributed Protocol for Agents	14
2.3.1 Convergence of the Protocol to a Fixed Point	15
2.3.2 Properties of Fixed Points	16
2.4 Analyzing Social Welfare via the Linear Program LPB Relaxation	18
2.4.1 Finding Alternative Feasible Solution via Alternating Traversal	21
2.5 Achieving Social Welfare with Greedy and Spiteful Users	25

2.5.1	Handling Ambiguous Edges	26
2.6	The Rate of Convergence: ϵ -Greedy and ϵ -Spiteful Users . . .	30
2.6.1	The Rate of Convergence	32
2.6.2	δ -Fixed Point and LPB Optimum	33
3	Analyzing Ranking via Continuous LP for the Oblivious Matching Problem	41
3.1	Introduction	41
3.2	Preliminaries	44
3.3	The Linear Program LPM_n for Ranking	44
3.4	Analyzing LPM_n via Continuous LPM_∞ Relaxation	45
3.4.1	Continuous LP Relaxation	46
3.4.2	Primal-Dual for a General Class of Continuous LP . . .	48
3.4.3	Lower Bound for the Performance Ratio	50
4	A Primal-dual Continuous Linear Programming Method for the Secretary Problem	55
4.1	Introduction	55
4.2	Preliminaries	60
4.3	The Continuous Linear Program	63
4.4	A Primal-Dual Method for Finding Thresholds in the $(J, 1)$ -case	73
4.4.1	Fast Implementation for the $(J, 1)$ -case	78
4.5	A Primal-Dual Method for Finding Thresholds in the General (J, K) -case	78
4.5.1	An Illustration: the $(2, 2)$ -case	91
	Bibliography	95

List of Figures

2.1	Infinite Price of Anarchy on K_5	13
2.2	Graph with Equal Alternating Weight.	20
3.1	Optimal z and (w, y, γ)	52
4.1	Optimal Payoffs for $(J, 1)$ -secretary problem	59

List of Algorithms

2.1	A Distributed Protocol for Agents	14
4.1	(J, K) -Threshold Algorithm	57
4.2	J -Threshold Generator	78
4.3	θ_j 's Generator	79

Chapter 1

Introduction

With the popularization of online business and electronic commerce, problems and algorithms relating to economics have recently drawn much attention in computer science. Some economical problems admit a game nature, where a group of participants need to compete with each other for limited resources while the whole group is expected to achieve good social welfare, which is the collection of benefits gained by the participants. Moreover, distributed protocols are more favorable to coordinate individuals' behavior in order to reduce the cost of communication and computation. In some other economical problems, an authority is supposed to make a series of decisions with partial information only. The authority initially has little information in hand, and is required to make decisions in a step by step manner before more information is revealed. In general the partial information is insufficient for the authority to achieve the global optimal solution. With these restrictions, new techniques are often required to devise efficient algorithms with good performance guarantee for economical problems.

In this work, we theoretically study several economics-related problems: the bargaining network games, the oblivious matching problem and the secretary problem. The bargaining network games are, as the name suggests, a game theoretical problem. The oblivious matching problem and the secretary problem concern decision making with partial information.

Technically, we study the aforementioned problems from the perspective of linear programming. In particular, we propose a new model for the network bargaining games that can be described by a linear programming formulation, and develop new continuous linear programming framework for the oblivious matching problem and the secretary problem.

1.1 Definitions and Notations

We first introduce some basic definitions for linear programming.

Let m and n be positive integers. Define $[n] := \{1, 2, \dots, n\}$. For a vector $v \in \mathbb{R}^n$, let v_i be the i -th coordinate of v for each $i \in [n]$. In a *linear program* (LP) a linear function of unknown variables is to be optimized subject to certain constraints. A maximization linear program can be described as follows:

$$\begin{array}{ll} \text{LP} & \\ \max & c^T x \\ \text{s.t.} & Ax \leq b \\ & x \geq 0, \end{array}$$

where $x \in \mathbb{R}^n$ is the vector of variables, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ are vectors of coefficients, and $A \in \mathbb{R}^{m \times n}$ is a matrix of coefficients.

A vector $x \in \mathbb{R}^n$ is a *feasible* solution to LP if x satisfies the constraints of LP; that is, $Ax \leq b$ and $x \geq 0$. A vector x is an *optimal solution*, or *optimum*, to LP if x is feasible and achieves the maximum objective value; that is, x is feasible and $c^T x \geq c^T x'$ for every feasible solution x' . The *dual* program of the linear program LP is defined as follows:

$$\begin{array}{ll} \text{LD} & \\ \min & b^T y \\ \text{s.t.} & A^T y \geq c \\ & y \geq 0, \end{array}$$

where $y \in \mathbb{R}^m$ is the vector of dual variables.

Fact 1.1 (Weak Duality and Complementary Slackness). *Let x be a feasible solution to LP and let y be a feasible solution to LD, then $c^T x \leq b^T y$. Moreover, the vectors x and y are optimal to LP and LD, respectively, if they satisfy the following complementary slackness conditions:*

1. $(A_i x - b_i)y_i = 0$ for every $i \in [m]$,
2. $(A_i^T y - c_i)x_i = 0$ for every $i \in [n]$,

where A_i is the i -th row of the matrix A .

A linear program is a 0-1 *integer program* if each variable takes value in $\{0, 1\}$. The linear program *relaxation* of a 0-1 integer program is obtained by allowing each variable to take value in $[0, 1]$ instead of $\{0, 1\}$.

1.2 Outline of the Thesis

This work consists of three aspects: (1) optimizing social welfare for network bargaining games, (2) analyzing **Ranking** via continuous LP for the oblivious matching problem, and (3) a primal-dual continuous linear programming method for the secretary problem. The results on optimizing social welfare for network bargaining games appeared in the ESA 2012 paper [CCN12]. The work on analyzing **Ranking** via continuous LP for the oblivious matching problem is a collaboration with Hubert Chan, Xiaowei Wu and Zhichao Zhao, the results of which appeared in the SODA 2014 paper [CCWZ14]. The results on the primal-dual continuous linear programming method for the secretary problem appeared in the manuscript [CC13]. We summarize the background and results of each chapter as follows.

1.2.1 Optimizing Social Welfare for Network Bargaining Games

In Chapter 2, we study the network bargaining games. Bargaining games have been studied with a long history in economics [Nas50, CCF07] and sociology [CE78, CEGY83, CA01]. Recently in computer science, there has been a lot of attention on bargaining games in social exchange networks [KT08, ABC⁺09, BHIM10, KBB⁺11], in which users are modeled as nodes in an edge-weighted undirected graph, where the weight of an edge indicates a profit of a potential contract involving the two end points. The nodes bargain with one another to form an *outcome* which consists of the set of executed contracts and how the profit in each contract is distributed among the two participating nodes. The *outside option* of a node is the maximum profit the node can get from another node with whom there is no current contract. An outcome is *stable* if for every contract a node makes, the profit the node gets from that contract is at least its outside option. An outcome is *balanced* if, in addition to stability, for every contract made, after each participating node gets its outside option, the surplus is divided equally between the two nodes involved. Stable and balanced outcomes of network bargaining games have been investigated recently, but the existence of such outcomes requires that the linear program relaxation of a certain maximum matching problem has integral optimal solution.

We propose an alternative model for network bargaining games in which each edge acts as a player, who proposes how to split the weight of the edge among the two incident nodes. Based on the proposals made by all edges, a selection process will return a set of accepted proposals, subject to node capacities. An

edge receives a commission if its proposal is accepted. The social welfare can be measured by the weight of the matching returned.

The node users, as opposed to being rational players as in previous work, exhibit two characteristics of human nature: greed and spite. We define these notions formally and show that the distributed protocol by Kanoria et. al [KBB⁺11] can be modified to be run by the edge players such that the configuration of proposals will converge to a pure Nash Equilibrium, without the LP integrality gap assumption. Moreover, after the nodes have made their greedy and spiteful choices, the remaining ambiguous choices can be resolved in a way such that there exists a Nash Equilibrium that will not hurt the social welfare too much.

1.2.2 Analyzing Ranking via Continuous LP for the Oblivious Matching Problem

In Chapter 3, we study the oblivious matching problem. Maximum matching [MV80] in undirected graphs is a classical problem in computer science, where one is to obtain a set of disjoint edges with maximum cardinality or weight. However, as motivated by online advertising [GM08, AGKM11] and exchange settings [RSÜ05], information about the graphs can be incomplete or unknown. Different online or greedy versions of the problem [ADFS95, PS12, GT12] can be formulated by the oblivious matching problem, in which an algorithm is essentially oblivious to the input graph. In particular, given the set of nodes of a graph, an algorithm can only *probe* pairs of vertices: if there is an edge between a pair of nodes, then this edge must be included to the matching.

The **Ranking** algorithm (an early version appears in [KVV90]) first selects a permutation on the nodes uniformly at random, and then probes pairs of nodes in the lexicographical order induced by the permutation.

We revisit the **Ranking** algorithm using the linear programming framework. Given a finite LP with the constraints obtained from the structural properties of the **Ranking** algorithm, we use continuous linear program relaxation to analyze the limiting behavior as the finite LP grows. Of particular interest are new duality and complementary slackness characterizations that can handle the monotone and the boundary constraints in continuous LP. Our work achieves the currently best theoretical performance ratio of $\frac{2(5-\sqrt{7})}{9} \approx 0.523$ on arbitrary graphs.

1.2.3 A Primal-dual Continuous Linear Programming Method for the Secretary Problem

In Chapter 4, we study the secretary problem. In the classical secretary problem [Lin61, Dyn63, Fre83, Fer89], a known number of online items arrive in a random order. Upon arrival of each item, one must make an irrevocable decision whether to select or discard the item; the process terminates when an item is selected or the last item has arrived. One can observe the relative merits of items that have arrived so far for making decisions, with the goal of maximizing the probability of selecting the best item overall. The classical secretary problem has been popularized in the 1950s, and since then various versions and solutions for the problem have been studied.

The J -choice K -best secretary problem, also known as the (J, K) -secretary problem, is a generalization of the classical secretary problem. An algorithm for the (J, K) -secretary problem is allowed to make J choices and the payoff to be maximized is the expected number of items chosen among the K best items.

Previous work analyzed the case when the total number n of items is finite, and considered what happens when n grows. However, for general J and K , the optimal solution for finite n is difficult to analyze. Instead, we prove a formal connection between the finite model and the infinite model, where there are countably infinite number of items, each attached with a random arrival time drawn independently and uniformly from $[0, 1]$.

We use primal-dual continuous linear programming techniques to analyze a class of infinite algorithms, which are general enough to capture the asymptotic behavior of the finite model with large number of items. Our techniques allow us to prove that the optimal solution can be achieved by an algorithm involving JK thresholds, which has a nice “rational description” for the case $K = 1$.

Chapter 2

Optimizing Social Welfare for Network Bargaining Games

2.1 Introduction

In this chapter we study the network bargaining games. Users are modeled as nodes in an undirected graph $G = (V, E)$, whose edges are weighted. An edge $\{i, j\} \in E$ with weight $w_{ij} > 0$ means that users i and j can potentially form a contract with each other and split a profit of w_{ij} . A capacity vector $b \in \mathbb{Z}_+^V$ limits the maximum number b_i of contracts node i can form with its neighbors, and the set M of executed contracts form a b -matching in G .

In previous work, the nodes bargain with one another to form an *outcome* which consists of the set M of executed contracts and how the profit in each contract is distributed among the two participating nodes. The *outside option* of a node is the maximum profit the node can get from another node with whom there is no current contract. An outcome is *stable* if for every contract a node makes, the profit the node gets from that contract is at least its outside option. Hence, under a stable outcome, no node has motivation to break its current contract to form another one. Extending the notion of Nash bargaining solution [Nas50], Cook and Yamagishi [CY92] introduced the notion of *balanced* outcome. An outcome is balanced if, in addition to stability, for every contract made, after each participating node gets its outside option, the surplus is divided equally between the two nodes involved. For more notions of solutions, the reader can refer to [Bin98].

Although stability is considered to be an essential property, as remarked in [BHIM10, KBB⁺11], a stable outcome exists *iff* the linear program LPB relaxation (given in Section 2.4) for the b -matching problem on the given

graph G has integrality gap 1. Hence, even for very simple graphs like a triangle with unit node capacities and unit edge weights, there does not exist a stable outcome. Previous work simply assumed that LPB has integrality gap 1 [KBB⁺11, CDP10] or considered restriction to bipartite graphs [KT08, BHIM10], for which LPB always has integrality gap 1.

We consider the integrality gap condition as a limitation to the applicability of such framework in practice. We would like to consider an alternative model for network bargaining games, and investigate different notions of equilibrium, whose existence does not require the integrality gap condition.

Related Work. Kleinberg and Tardos [KT08] recently started the study of network bargaining games in the computer science community; they showed that a stable outcome exists *iff* a balanced outcome exists, and both can be computed in polynomial time, if they exist. Chakraborty et. al [CK08, CKK09] explored equilibrium concepts and experimental results for bipartite graphs. Celis et. al [CDP10] gave a tight polynomial bound on the rate of convergence for unweighted bipartite graphs with a unique balanced outcome. Kanoria [Kan10] considered *unequal division* (UD) solutions for bargaining games, in which stability is still guaranteed while the surplus is split with ratio $r : (1 - r)$, where $r \in (0, 1)$. They provided an FPTAS for the UD solutions assuming the existence of such solutions.

Azar et. al [ABC⁺09] considered a local dynamics that converges to a balanced outcome provided that it exists. Assuming that the LP relaxation for matching has a unique integral optimum, Kanoria et. al [KBB⁺11] designed a local dynamics that converges in polynomial time. Our distributed protocol is based on [KBB⁺11], but is generalized to general node capacities, run by edges and does not require the integrality condition on LPB.

Bateni et. al [BHIM10] also considered general node capacities; moreover, they showed that the network bargaining problem can be recast as an instance of the well-studied cooperative game [Dri88]. In particular, a stable outcome is equivalent to a point in the core of a cooperative game, while a balanced outcome is equivalent to a point in the core and the prekernel. Azar et. al [ADJR10] also studied bargaining games from the perspective of cooperative games, and proved some monotonicity property for several widely considered solutions.

Our Contribution. In this work, we let the edges take over the role of the “rational” players from the nodes. Each edge $e = \{i, j\} \in E$ corresponds to an agent, who proposes a way to divide up the potential profit w_{ij} among the two nodes. Formally, each edge $\{i, j\}$ has the action set $A_{ij} := \{(x, y) : x \geq 0, y \geq 0, x + y \leq w_e\}$, where a proposal (x, y) means that node i gets

amount x and j gets amount y .¹ Based on the configuration $m \in A_E := \times_{e \in E} A_e$ of proposals made by all the agents, a selection process (which can be randomized) will choose a b -matching M , which is the set of contracts formed. An agent e will receive a commission if his proposal is selected; his payoff $u_e(m)$ is the probability that edge e is in the matching M returned.² Observe that once the payoff function u is defined, the notion of (pure or mixed) Nash Equilibrium is also well-defined. We measure the social welfare $\mathcal{S}(m)$ by the (expected) weight $w(M)$ of the matching M returned, which reflects the volume of transaction.

We have yet to describe the selection process, which will determine the payoff function to each agent, and hence will affect the corresponding Nash Equilibrium. We mention earlier that the rational players in our framework will be the edges, as opposed to the nodes in previous work; in fact, in the selection process we assume the node users will exhibit two characteristics of human nature: greed and spite.

Greedy Users. For a node i with capacity b_i , user i will definitely want an offer that is strictly better than his $(b_i + 1)$ -st best offer. If this happens for both users forming an edge, then the edge will definitely be selected. We also say the resulting payoff function is greedy.

Spiteful Users. Spite is an emotion that describes the situation that once a person has seen a better offer, he would not settle for anything less, even if the original better offer is no longer available. If user i with capacity b_i sees that an offer is strictly worse than his b_i -th best offer, then the corresponding edge will definitely be rejected. We also say the resulting payoff function is spiteful.

One can argue that greed is a rational behavior (hence the regime of greedy algorithms), but spite is clearly not always rational. In fact, we shall see in Section 2.2 that there exist a spiteful payoff function and a configuration of agent proposals that is a pure Nash Equilibrium, in which all proposals are rejected by the users out of spite, even though no single agent can change the situation by unilaterally offering a different proposal. The important question is that: can the agents follow some protocol that can avoid such bad Nash Equilibrium? In other words, can they collaboratively find a Nash Equilibrium that achieves good social welfare?

We answer the above question in the affirmative. We modify the distributed protocol of Kanoria et. al [KBB⁺11] to be run by edge players and allow general node capacities b . As before, the protocol is iterative and the configuration

¹In case $x + y < w_{ij}$ the remaining amount is lost and not gained by anyone.

²The actual gain of an agent could be scaled according to the weight w_e , but this will not affect the Nash Equilibrium.

of proposals returned will converge to a fixed point m of some non-expansive function \mathcal{T} . In Section 2.3, we show that provided the payoff function u is greedy and spiteful, any fixed point m of \mathcal{T} is in the corresponding set \mathcal{N}_u of pure Nash Equilibria.

In Section 2.4, we analyze the social welfare through the linear program LPB relaxation of the maximum b -matching problem. As in [KBB⁺11], we investigate the close relationship between a fixed point of \mathcal{T} and LPB. However, we go beyond previous analysis and do not need the integrality gap assumption, i.e., LPB might not have an integral optimum. We show that when greedy users choose an edge, then all LPB optimal solutions must set the value of that edge to 1; on the other hand, when users reject an edge out of spite, then all LPB optimal solutions will set the value of that edge to 0. We do need some technical assumptions in order for our results to hold: either (1) LPB has unique optimum, or (2) the given graph G has no even cycle such that the sum of the weights of the odd edges equals that of the even edges; neither assumption implies the other, but both can be achieved by perturbing slightly the edge weights of the given graph. Unlike the case for simple 1-matching, we show that assumption (2) is necessary for general b -matching, showing that there is some fundamental difference between the two cases.

The greedy behavior states that some edges must be selected and the spiteful behavior requires that some edges must be rejected. However, there is still some freedom to deal with the remaining *ambiguous* edges.³ Observe that a fixed point will remain a Nash Equilibrium (for the edge players) no matter how the ambiguous edges are handled, so it might make sense at this point to maximize the total number of extra contracts made from the ambiguous edges. However, optimizing the cardinality of a matching can be arbitrarily bad in terms of weight, but a maximum weight matching is a 2-approximation in terms of cardinality. Therefore, in Section 2.5, we consider a greedy and spiteful payoff function u that corresponds to selecting a maximum matching (approximate or exact) among the ambiguous edges (subject to remaining node capacities b'); in reality, we can imagine this corresponds to a centralized clearing process or a collective effort performed by the users. We show that if a $(1+c)$ -approximation algorithm for maximum weight matching is used for the ambiguous edges, then the social welfare is at least $\frac{2}{3(1+c)}$ fraction of the social optimum, i.e., the price of stability is $1.5(1+c)$. Finally, observe that the iterative protocol we mention will converge to a fixed point, but might never get there exactly; hence, we relax the notions of greed and spite in Section 2.6 and show that the same guarantee on the price of stability can be achieved eventually (and quickly).

³As a side note, we remark that our results implies that under the unique integral LPB optimum assumption, there will be no ambiguous edges left.

We remark that if the topology of the given graph and the edge weights naturally indicate that certain edges should be selected while some should be rejected (both from the perspectives of social welfare and selfish behavior), then our framework of greed and spite can detect these edges. However, we do not claim that our framework is a silver bullet to all issues; in particular, for the triangle example given above, all edges will be ambiguous and our framework simply implies that one node will be left unmatched, but does not specify how this node is chosen. We leave as future research direction to develop notions of fairness in such situation.

In our selection process, we assume that the maximum weight b' -matching problem is solved on the ambiguous edges. This problem is well-studied and can be solved exactly in polynomial time [Sch04][Section 33.4]; moreover, the problem can be solved by a distributed algorithm [BBCZ11], and $(1 + c)$ -approximation for any $c > 0$ can be achieved in poly-logarithmic time [LPSP08, Nie08, KY09].

2.2 Preliminaries

Consider an undirected graph $G = (V, E)$, with vertex set V and edge set E . Each node $i \in V$ corresponds to a *user* i (vertex player), and each edge $e \in E$ corresponds to an *agent* e (edge player). Agents arrange contracts to be formed between users where each agent $e = \{i, j\}$ gains a commission when users i and j form a contract. Each edge $e = \{i, j\} \in E$ has weight $w_e = w_{ij} > 0$, which is the maximum profit that can be shared between users i and j if a contract is made between them. Given a node i , denoted by $N(i) := \{j \in V : \{i, j\} \in E\}$ the set of its neighbors in G , there exists a capacity vector $b \in \mathbb{Z}_+^V$ such that each node i can make at most b_i contracts with its neighbors in $N(i)$, where at most one contract can be made between a pair of users; hence, the set M of edges on which contracts are made is a b -matching in G .

Agent Proposal. For each $e = \{i, j\} \in E$, agent e makes a proposal of the form $(m_{j \rightarrow i}, m_{i \rightarrow j})$ from an action set A_e to users i and j , where $A_e := \{(x, y) : x \geq 0, y \geq 0, x + y \leq w_{ij}\}$, such that if users i and j accepts the proposal and form a contract with each other, user i will receive $m_{j \rightarrow i}$ and user j will receive $m_{i \rightarrow j}$ from this contract.

Selection Procedure and Payoff Function u . Given a configuration $m \in A_E := \times_{e \in E} A_e$ of all agents' proposals, some selection procedure is run on m to return a b -matching M , where an edge $e = \{i, j\} \in M$ means that a contract is made between i and j . The procedure can be (1) deterministic or randomized, (2) centralized or (more preferably) distributed.

If i and j are *matched* in M , i.e., $e = \{i, j\} \in M$, agent e will receive a commission, which can either be fixed or a certain percentage of w_e ; since an agent either gains the commission or not, we can assume that its payoff is 1 when a contract is made and 0 otherwise. Hence, the selection procedure defines a payoff function $u = \{u_e : A_E \rightarrow [0, 1] | e \in E\}$, such that for each $e \in E$, $u_e(m)$ is the probability that the edge e is in the b -matching M returned when the procedure is run on $m \in A_E$. We shall consider different selection procedures, which will lead to different payoff functions u . However, the selection procedure should satisfy several natural properties, which we relate to the human nature of the users as follows.

We use $\max^{(b)}$ to denote the b -th maximum value among a finite set of numbers (by convention it is 0 if there are less than b numbers). Given $m \in A_E$, we define $\hat{m}_i = \max_{j \in N(i)}^{(b_i)} m_{j \rightarrow i}$ and $\bar{m}_i = \max_{j \in N(i)}^{(b_i+1)} m_{j \rightarrow i}$.

Greedy Users. If both users i and j see that they cannot get anything better from someone else, then they will definitely make a contract with each other. Formally, we say that the payoff function u is *greedy* (or the users are greedy), if for each $e = \{i, j\} \in E$ and $m \in A_E$, if $m_{j \rightarrow i} > \bar{m}_i$ and $m_{i \rightarrow j} > \bar{m}_j$, then $u_e(m) = 1$.

Spiteful Users. It is human nature that once a person has seen the best, they will not settle for anything less. We try to capture this behavior formally. We say that the payoff function u is *spiteful* (or the users are spiteful) if for each $e = \{i, j\} \in E$ and $m \in A_E$, if $m_{j \rightarrow i} < \hat{m}_i$, then $u_e(m) = 0$, i.e., if user i cannot get the b_i -th best offer from j , then no contract will be formed between i and j .

Game Theory and Social Welfare. We have described a game between the agents, in which agent e has the action set A_e , and has payoff function u (determined by the selection procedure). In this work, we consider pure strategies and pure Nash Equilibria. A configuration $m \in A_E$ of actions is a Nash Equilibrium if no single player can increase its payoff by unilaterally changing its action.

Given a payoff function u , we denote by $\mathcal{N}_u \subset A_E$ the set of Nash Equilibria. Given a configuration $m \in A_E$ of proposals and a payoff function u , we measure social welfare by $\mathcal{S}_u(m) := \sum_{e \in E} w_e \cdot u_e(m)$, which is the expected weight of the b -matching returned. When there is no ambiguity, the subscript u is dropped. The optimal social welfare $\mathcal{S}^* := \max_{m \in A_E} \mathcal{S}(m)$ is the maximum weight b -matching; to achieve the social optimum, given a maximum weight b -matching M , every agent $e \in M$ proposes $(\frac{w_e}{2}, \frac{w_e}{2})$, while other agents propose $(0, 0)$. The weight of the b -matching can be an indicator of the volume of transactions or how active the market is. The Price of Anarchy

(PoA) is defined as $\frac{\mathcal{S}^*}{\min_{m \in \mathcal{N}} \mathcal{S}(m)}$ and the Price of Stability (PoS) is defined as $\frac{\mathcal{S}^*}{\max_{m \in \mathcal{N}} \mathcal{S}(m)}$.

Proposition 2.1 (Infinite Price of Anarchy). *There exists an instance of the game such that when the users are spiteful, there exists a Nash Equilibrium $m \in A_E$ under which no contracts are made.*

Proof. We take G to be the complete graph K_5 on five nodes, where each edge has unit weight, and each node has unit capacity. It is straight forward to construct a configuration $m \in A_E$ of proposals that has the following properties:

- (a) each agent splits the weight into $(0.4, 0.6)$;
- (b) each user gets two offers with profit 0.4 and two offers with profit 0.6.

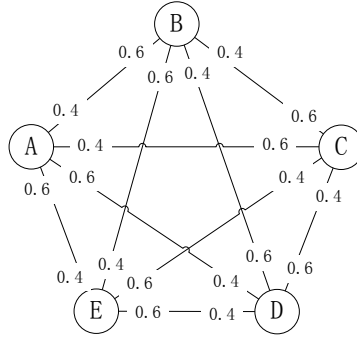


FIGURE 2.1: Infinite Price of Anarchy on K_5

Observe that for spiteful users, no contract will be accepted, because for each contract there will be a user getting 0.4, which is worse than his best choice 0.6. Hence, $\mathcal{S}(m) = 0$.

We next show that m is a Nash Equilibrium. Consider an edge $e = \{i, j\}$ where under m , user i gets 0.4 and user j gets 0.6. Since currently the best offer i receives is 0.6, in order for edge e to have any chance to be considered by user i , agent e must offer at least 0.6 to i , which means there is only at most 0.4 to be offered to user j , who will definitely reject e because user j still has another offer with 0.6. Hence, there is no way for any agent to change his strategy unilaterally to increase his payoff. \square

2.3 A Distributed Protocol for Agents

We describe a distributed protocol for the agents to update their actions in each iteration. The protocol is based on the one by Kanoria et. al [KBB⁺11], which is run by nodes and designed for (1-)matchings. The protocol can easily be generalized to be run by edges and for general b -matchings. In each iteration, two agents only need to communicate if their corresponding edges share a node. Given a real number $r \in \mathbb{R}$, we denote $(r)_+ := \max\{r, 0\}$. Moreover, as described in [KBB⁺11, BB96] a damping factor $\kappa \in (0, 1)$ is used in the update; we can think of $\kappa = \frac{1}{2}$.

Although later on we will also consider the LP relaxation of b -matching, unlike previous works [SMW07, BSS08, KBB⁺11], we do not require the assumption that the LP relaxation has a unique integral optimum.

Algorithm 2.1: A Distributed Protocol for Agents

Input: $G = (V, E, w)$

Initialization: For each $e = \{i, j\} \in E$, agent e picks arbitrary

$(m_{j \rightarrow i}^{(0)}, m_{i \rightarrow j}^{(0)}) \in A_e$.

for agent $e = \{i, j\} \in E$ **do**

$\alpha_{i \setminus j}^{(1)} := \max_{k \in N(i) \setminus j}^{(b_i)} m_{k \rightarrow i}^{(0)}$

$\alpha_{j \setminus i}^{(1)} := \max_{k \in N(j) \setminus i}^{(b_j)} m_{k \rightarrow j}^{(0)}$

end

for $t = 1, 2, 3, \dots$ **do**

for agent $e = \{i, j\} \in E$ **do**

$S_{ij} := w_{ij} - \alpha_{i \setminus j}^{(t)} - \alpha_{j \setminus i}^{(t)}$

$m_{j \rightarrow i}^{(t)} := (w_{ij} - \alpha_{j \setminus i}^{(t)})_+ - \frac{1}{2}(S_{ij})_+$, send $m_{j \rightarrow i}^{(t)}$ to edges incident to i

$m_{i \rightarrow j}^{(t)} := (w_{ij} - \alpha_{i \setminus j}^{(t)})_+ - \frac{1}{2}(S_{ij})_+$, send $m_{i \rightarrow j}^{(t)}$ to edges incident to j

end

for agent $e = \{i, j\} \in E$ **do**

$\alpha_{i \setminus j}^{(t+1)} := (1 - \kappa) \cdot \alpha_{i \setminus j}^{(t)} + \kappa \cdot \max_{k \in N(i) \setminus j}^{(b_i)} m_{k \rightarrow i}^{(t)}$

$\alpha_{j \setminus i}^{(t+1)} := (1 - \kappa) \cdot \alpha_{j \setminus i}^{(t)} + \kappa \cdot \max_{k \in N(j) \setminus i}^{(b_j)} m_{k \rightarrow j}^{(t)}$

end

end

In Algorithm 2.1, auxiliary variables $\alpha^{(t)} \in \mathbb{R}_+^{2|E|}$ are maintained. Intuitively, the parameter $\alpha_{i \setminus j}$ is meant to represent the b_i -th best offer user i can receive if user j is removed. Suppose $W := \max_{e \in E} w_e$ and we define a function $\mathcal{T} : [0, W]^{2|E|} \rightarrow [0, W]^{2|E|}$ as follows.

Given $\alpha \in [0, W]^{2|E|}$, for each $\{i, j\} \in E$, define the following quantities.

$$S_{ij}(\alpha) = w_{ij} - \alpha_{i \setminus j} - \alpha_{j \setminus i} \quad (2.1)$$

$$m_{j \rightarrow i}(\alpha) = (w_{ij} - \alpha_{j \setminus i})_+ - \frac{1}{2}(S_{ij}(\alpha))_+ \quad (2.2)$$

Then, we define $\mathcal{T}(\alpha) \in [0, W]^{2|E|}$ by $(\mathcal{T}(\alpha))_{i \setminus j} := \max_{k \in N(i) \setminus j}^{(b_i)} m_{k \rightarrow i}(\alpha)$. It follows that Algorithm 2.1 defines the sequence $\{\alpha^{(t)}\}_{t \geq 1}$ by $\alpha^{(t+1)} := (1 - \kappa)\alpha^{(t)} + \kappa\mathcal{T}(\alpha^{(t)})$.

2.3.1 Convergence of the Protocol to a Fixed Point

Given a vector space D , a function $T : D \rightarrow D$ is *non-expansive* under norm $\|\cdot\|$ if for all $x, y \in D$, $\|T(x) - T(y)\| \leq \|x - y\|$; a point $\alpha \in D$ is a fixed point of T if $T(\alpha) = \alpha$. As in [KBB⁺11], it can be proved that the function \mathcal{T} is non-expansive. The following result by Ishikawa [Ish76] shows that by applying a non-expansive function repeatedly, convergence to a fixed point can be obtained.

Fact 2.2 ([Ish76]). *Suppose $T : D \rightarrow D$ is a non-expansive function under norm $\|\cdot\|$ and for some $\kappa \in (0, 1)$ and some initial $\alpha^{(1)} \in D$, the sequence $\{\alpha^{(t)}\}$ is defined by $\alpha^{(t+1)} := (1 - \kappa) \cdot \alpha^{(t)} + \kappa \cdot T(\alpha^{(t)})$. Suppose further that the sequence $\{\alpha^{(t)}\}$ is bounded under norm $\|\cdot\|$. Then, the sequence $\{\alpha^{(t)}\}$ converges (under norm $\|\cdot\|$) to some fixed point of T .*

Claim 2.3. *Given a weighted graph $G = (V, E)$ with maximum weight W , the function $\mathcal{T} : [0, W]^{2|E|} \rightarrow [0, W]^{2|E|}$ is non-expansive under the ℓ_∞ norm.*

Proof. The claim can be proved in a similar way as in [KBB⁺11]. We observe the following facts.

- The ‘ $\max^{(b_i)}$ ’ in the mapping is non-expansive. To prove this, it suffices to show that given two vectors x and y with the same dimension d larger than b , the following holds

$$|\max_{i \in [d]}^{(b)} x_i - \max_{i \in [d]}^{(b)} y_i| \leq \max_{i \in [d]} |x_i - y_i|.$$

Let $b_1 := \arg \max_{i \in [d]}^{(b)} x_i$ and $b_2 := \arg \max_{i \in [d]}^{(b)} y_i$. Without loss of generality, assume $x_{b_1} \leq y_{b_2}$. If $y_{b_1} \geq y_{b_2}$, then $|x_{b_1} - y_{b_2}| \leq |x_{b_1} - y_{b_1}|$. If $y_{b_1} < y_{b_2}$, there exists $k \in [d]$ satisfying $y_k \geq y_{b_2}$ such that $x_k \leq x_{b_1}$. Then $|x_{b_1} - y_{b_2}| \leq |x_k - y_k|$. Therefore, $|x_{b_1} - y_{b_2}| \leq \max_{i \in [d]} |x_i - y_i|$.

- The variable $m = m(\alpha)$ is non-expansive according to its definition. Let $m_{i \rightarrow j} = f(\alpha_{i \setminus j}, \alpha_{j \setminus i})$, where $f(x, y)$ is given by

$$f(x, y) = \begin{cases} \frac{w_{ij} - x + y}{2} & x + y \leq w_{ij}, \\ (w_{ij} - x)_+ & \text{otherwise.} \end{cases}$$

It can be checked that f is continuous in \mathbb{R}_+^2 . Also, it is differentiable except in $\{(x, y) \in \mathbb{R}_+^2 : x + y = w_{ij} \text{ or } x = w_{ij}\}$, and satisfies $|\frac{\partial f}{\partial x}| + |\frac{\partial f}{\partial y}| \leq 1$. Therefore, f is Lipschitz continuous in the ℓ_∞ norm with Lipschitz constant 1 and is non-expansive.

□

Fact 2.2 and Claim 2.3 give Theorem 2.4.

Theorem 2.4 (Convergence to a Fixed Point). *The distributed protocol shown in Algorithm 2.1 maintains the sequence $\{\alpha^{(t)}\}$ which converges to a fixed point of the function \mathcal{T} under the ℓ_∞ norm.*

2.3.2 Properties of Fixed Points

Given a fixed point α of the function \mathcal{T} , the quantities $S \in \mathbb{R}^{|E|}$ and $m \in A_E$ are defined according to equations (2.1) and (2.2). We also say that (m, α, S) or (m, α) or m is a fixed point (of \mathcal{T}). Similar to [KBB⁺11], we give several important properties of a fixed point. Recall that $\widehat{m}_i := \max_{k \in N(i)}^{(b_i)} m_{k \rightarrow i}$ and $\overline{m}_i := \max_{k \in N(i)}^{(b_i+1)} m_{k \rightarrow i}$, and in addition to equation (2.2), a fixed point (m, α) also satisfies $\alpha_{i \setminus j} = \max_{k \in N(i) \setminus j}^{(b_i)} m_{k \rightarrow i}$.

Proposition 2.5 (Outside option α). *Suppose for $\{i, j\} \in E$, we have $\alpha_{i \setminus j} = \max_{k \in N(i) \setminus j}^{(b_i)} m_{k \rightarrow i}$. Then, the following properties hold.*

- (a) *If $m_{j \rightarrow i} < \widehat{m}_i$, then $\alpha_{i \setminus j} = \widehat{m}_i$; if $m_{j \rightarrow i} \geq \widehat{m}_i$, then $\alpha_{i \setminus j} = \overline{m}_i$.*
- (b) *$m_{j \rightarrow i} \geq \widehat{m}_i$ iff $m_{j \rightarrow i} \geq \alpha_{i \setminus j}$; $m_{j \rightarrow i} \leq \overline{m}_i$ iff $m_{j \rightarrow i} \leq \alpha_{i \setminus j}$;*

Proof. (a) If $m_{j \rightarrow i} < \widehat{m}_i$, that is, the offer that user i gets from j is not one of its best b_i offers, then $\alpha_{i \setminus j} = \max_{k \in N(i) \setminus j}^{(b_i)} m_{k \rightarrow i} = \max_{k \in N(i)}^{(b_i)} m_{k \rightarrow i} = \widehat{m}_i$. If $m_{j \rightarrow i} \geq \widehat{m}_i$, that is, the offer that user i gets from j is at least as good as its b_i -th best offer, then $\alpha_{i \setminus j} = \max_{k \in N(i) \setminus j}^{(b_i)} m_{k \rightarrow i} = \max_{k \in N(i)}^{(b_i+1)} m_{k \rightarrow i} = \overline{m}_i$. Note that the equalities hold even if $\widehat{m}_i = \overline{m}_i$.

(b) Proposition 2.5(a) and its proof imply the following:

- If $m_{j \rightarrow i} \geq \widehat{m}_i$, then $\alpha_{i \setminus j} = \overline{m}_i \leq \widehat{m}_i \leq m_{j \rightarrow i}$. If $m_{j \rightarrow i} < \widehat{m}_i$, then $\alpha_{i \setminus j} = \widehat{m}_i > m_{j \rightarrow i}$.
- If $m_{j \rightarrow i} \leq \overline{m}_i$, then $\alpha_{i \setminus j} = \widehat{m}_i \geq \overline{m}_i \geq m_{j \rightarrow i}$. If $m_{j \rightarrow i} > \overline{m}_i$, then $m_{j \rightarrow i} \geq \widehat{m}_i$ and thus $\alpha_{i \setminus j} = \overline{m}_i < m_{j \rightarrow i}$.

□

Proposition 2.6 (α defining (S, m)). *Suppose given $\alpha \in [0, W]^{2|E|}$, the quantities $S \in \mathbb{R}^{|E|}$ and $m \in A_E$ are defined as in Equations (2.1) and (2.2). Then, the following properties hold for each $\{i, j\} \in E$.*

(a) $S_{ij} > 0$ iff $m_{j \rightarrow i} > \alpha_{i \setminus j}$;

(b) $S_{ij} = 0$ iff $m_{j \rightarrow i} = \alpha_{i \setminus j}$ and $m_{i \rightarrow j} = \alpha_{j \setminus i}$.

Proof. (a) Recall that $S_{ij} = w_{ij} - \alpha_{i \setminus j} - \alpha_{j \setminus i}$ and $m_{j \rightarrow i} = (w_{ij} - \alpha_{j \setminus i})_+ - \frac{1}{2}(S_{ij})_+$. If $S_{ij} > 0$, then $w_{ij} - \alpha_{j \setminus i} = S_{ij} + \alpha_{i \setminus j} > 0$. Therefore $m_{j \rightarrow i} = (S_{ij} + \alpha_{i \setminus j})_+ - \frac{1}{2}(S_{ij})_+ = \alpha_{i \setminus j} + \frac{1}{2}S_{ij} > \alpha_{i \setminus j}$.

On the other hand, if $S_{ij} \leq 0$, then $m_{j \rightarrow i} = (S_{ij} + \alpha_{i \setminus j})_+ = \max(S_{ij} + \alpha_{i \setminus j}, 0) \leq \alpha_{i \setminus j}$.

(b) If $S_{ij} = 0$, then $m_{j \rightarrow i} = (S_{ij} + \alpha_{i \setminus j})_+ - \frac{1}{2}(S_{ij})_+ = \alpha_{i \setminus j}$ and similarly $m_{i \rightarrow j} = \alpha_{j \setminus i}$.

If $m_{j \rightarrow i} = \alpha_{i \setminus j}$ and $m_{i \rightarrow j} = \alpha_{j \setminus i}$, then from Proposition 2.6(a) we have $S_{ij} \leq 0$. Therefore $m_{j \rightarrow i} + m_{i \rightarrow j} = \alpha_{i \setminus j} + \alpha_{j \setminus i} = w_{ij} - S_{ij} \geq w_{ij}$. Since $m_{j \rightarrow i} + m_{i \rightarrow j} \leq w_{ij}$, we have $m_{j \rightarrow i} + m_{i \rightarrow j} = w_{ij}$ and therefore $S_{ij} = 0$.

□

Proposition 2.7 (Fixed Point (m, α)). *Suppose (m, α) is a fixed point of \mathcal{T} . Then, for each $\{i, j\} \in E$, the following properties hold.*

(a) If $m_{j \rightarrow i} > 0$ and $m_{j \rightarrow i} \geq \widehat{m}_i$, then $S_{ij} \geq 0$; if $S_{ij} \geq 0$, then $m_{j \rightarrow i} \geq \widehat{m}_i$.

(b) $m_{j \rightarrow i} \leq \overline{m}_i$ iff $S_{ij} \leq 0$.

Proof. (a) Suppose $m_{j \rightarrow i} > 0$ and $m_{j \rightarrow i} \geq \widehat{m}_i$. Then from Proposition 2.5(b), we have $m_{j \rightarrow i} \geq \alpha_{i \setminus j}$. Consider the following two cases:

- $m_{j \rightarrow i} > \alpha_{i \setminus j}$. Then, from Proposition 2.6(a) we have $S_{ij} > 0$.
- $m_{j \rightarrow i} = \alpha_{i \setminus j}$. Then, from Proposition 2.6(a) we have $S_{ij} \leq 0$. Since $m_{j \rightarrow i} > 0$, it follows that $\alpha_{i \setminus j} = m_{j \rightarrow i} = (w_{ij} - \alpha_{j \setminus i})_+ - \frac{1}{2}(S_{ij})_+ = w_{ij} - \alpha_{j \setminus i}$ and so $S_{ij} = 0$.

Therefore, $S_{ij} \geq 0$.

For the converse, $S_{ij} \geq 0$ implies from Proposition 2.6 that $m_{j \rightarrow i} \geq \alpha_{i \setminus j}$, which implies from Proposition 2.5(b) that $m_{j \rightarrow i} \geq \hat{m}_i$.

- (b) We have by Proposition 2.5(b) that $m_{j \rightarrow i} \leq \bar{m}_i$ iff $m_{j \rightarrow i} \leq \alpha_{i \setminus j}$, which is equivalent to $S_{ij} \leq 0$ by Proposition 2.6(a).

□

Theorem 2.8 (Fixed Point is NE). *Suppose the payoff function u is greedy and spiteful. Then, any fixed point $m \in A_E$ of \mathcal{T} is a Nash Equilibrium in \mathcal{N}_u .*

Proof. Let (m, α, S) be a fixed point of \mathcal{T} . We show that for each $e = \{i, j\} \in E$, agent e cannot increase $u_e(m)$ by changing its action m_e unilaterally.

If $S_{ij}(m) < 0$, i.e. $\alpha_{i \setminus j} + \alpha_{j \setminus i} > w_{ij}$, then any proposal $(m'_{j \rightarrow i}, m'_{i \rightarrow j}) \in A_e$ must satisfy $m'_{j \rightarrow i} + m'_{i \rightarrow j} \leq w_{ij} < \alpha_{i \setminus j} + \alpha_{j \setminus i}$, which implies that $m'_{i \rightarrow j} < \alpha_{j \setminus i}$ or $m'_{j \rightarrow i} < \alpha_{i \setminus j}$. Since the payoff function is spiteful, it follows that i and j cannot be matched. Hence, $u_e = 0$ if other agents maintain their actions.

If $S_{ij}(m) > 0$, then by Proposition 2.7(b), $m_{j \rightarrow i} > \bar{m}_i$ and $m_{i \rightarrow j} > \bar{m}_j$. Since the payoff function u is greedy, we already have $u_e(m) = 1$ and there is no more room for improvement.

If $S_{ij}(m) = 0$, then by Proposition 2.6(b), $m_{j \rightarrow i} = \alpha_{i \setminus j}$ and $m_{i \rightarrow j} = \alpha_{j \setminus i}$. This also implies that $m_{i \rightarrow j} + m_{j \rightarrow i} = w_{i,j}$. Then, any change of $(m_{i \rightarrow j}, m_{j \rightarrow i})$ will lead to either $m'_{j \rightarrow i} < \alpha_{i \setminus j}$ or $m'_{i \rightarrow j} < \alpha_{j \setminus i}$, which means there is no chance for $\{i, j\}$ to be matched for spiteful payoff function u . □

Theorems 2.4 and 2.8 imply that as long as the payoff function is greedy and spiteful, the game defined between the agents (edge players) always has a pure Nash Equilibrium.

2.4 Analyzing Social Welfare via the Linear Program LPB Relaxation

Theorem 2.8 states that a fixed point (m, α) of the function \mathcal{T} is a Nash Equilibrium in \mathcal{N}_u , as long as the underlying payoff function is greedy and spiteful. Our goal is to show that there exists some greedy and spiteful u such that the fixed point m also achieves good social welfare $\mathcal{S}_u(m) = \sum_{e \in E} w_e \cdot u_e(m)$.

As observed by Kanoria et. al [KBB⁺11], the network bargain games are closely related to the following linear program relaxation of the b -matching problem:

$$\begin{aligned}
 & \text{LPB} \\
 \max \quad & w(x) := \sum_{\{i,j\} \in E} x_{ij} w_{ij} \\
 \text{s.t.} \quad & \sum_{j: \{i,j\} \in E} x_{ij} \leq b_i, \quad \forall i \in V \\
 & 0 \leq x_{ij} \leq 1, \quad \forall \{i,j\} \in E.
 \end{aligned}$$

Let \mathcal{S}_{LPB} be the optimal objective value of LPB. Let \mathcal{L} be the set of feasible solutions to LPB. Given a feasible solution x of LPB, we say a node i is *saturated* under x if $\sum_{j: \{i,j\} \in E} x_{ij} = b_i$, and otherwise *unsaturated*.

Kanoria et. al showed that when the LPB relaxation has a unique integral optimum, a fixed point (m, α, S) corresponds naturally to the unique maximum (1-)matching. However, their analysis cannot cover the case when the optimal solution is fractional or when the maximum matching is not unique.

In this section, we fully exploit the relationship between a fixed point and the LPB relaxation, from which we show that good social welfare can be achieved. Note that we do not require the unique integral optimum assumption. On the other hand, we assume that either (1) LPB has a unique optimum or (2) the following technical assumption.

No Cycle with Equal Alternating Weight. We say that a cycle has *equal alternating weight* if it is even, and the sum of the weights of the odd edges equals that of the even edges. We assume that the given weighted graph G has no such cycle. The weights of any given graph can be perturbed slightly such that this condition holds. Observe that the optimum of LPB might not be unique even with this assumption.

The main technical properties are as follows.

Theorem 2.9 (Fixed Point and LPB). *Suppose LPB has a unique optimum or the graph G has no cycle with equal alternating weight, and (m, α, S) is a fixed point of \mathcal{T} . Then, for any edge $\{i, j\} \in E$, the following holds.*

- (a) *Suppose LPB has a unique integral optimum corresponding to the maximum b -matching M^* . Then, $S_{ij} \geq 0$ implies that $\{i, j\} \in M^*$.*
- (b) *Suppose $S_{ij} > 0$. Then, any optimal solution x to LPB must satisfy $x_{ij} = 1$.*

- (c) Suppose $S_{ij} < 0$. Then, any optimal solution x to LPB must satisfy $x_{ij} = 0$.

For the 1-matching case, the conclusions listed in Theorem 2.9 have been derived by Kanoria et. al [KBB⁺11] without the no-alternating-cycle assumption. However, for general b -matching, this assumption in Theorem 2.9 is necessary for cases (b) and (c). We show that without this technical assumption, there is a counter example for Theorem 2.9(b).

Consider the graph given in Fig 2.2, which contains a cycle with equal alternating weight, e.g. $\{C, D, E, F, C\}$. Each node has capacity 2. The offers specified on the edges form a fixed point of \mathcal{T} . It can be checked that $S_{AC} > 0$, while every optimal solution of the corresponding LPB has $x_{AC} = 0$. Thus, case (b) of Theorem 2.9 is incorrect for this given graph. By slight modification of the given graph, a counter example for Theorem 2.9(c) can be constructed.

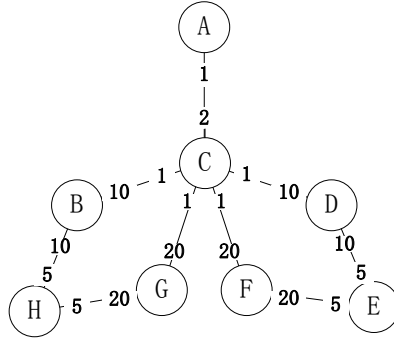


FIGURE 2.2: Graph with Equal Alternating Weight.

Although the three statements in Theorem 2.9 look quite different, they can be implied by the three similar-looking corresponding statements in the following lemma.

Lemma 2.10 (Fixed Point and LPB). *Suppose (m, α, S) is a fixed point of \mathcal{T} , and x is a feasible solution to LPB. Then, for each $\{i, j\} \in E$, the following properties hold.*

- (a) *If $S_{ij} \geq 0$ and $x_{ij} = 0$, then there exists $\hat{x} \in \mathcal{L}$ such that $\hat{x} \neq x$ and $w(\hat{x}) \geq w(x)$.*
- (b) *If $S_{ij} > 0$ and $x_{ij} < 1$, then there exists $\hat{x} \in \mathcal{L}$ such that $\hat{x} \neq x$ and $w(\hat{x}) \geq w(x)$.*
- (c) *If $S_{ij} < 0$ and $x_{ij} > 0$, then there exists $\hat{x} \in \mathcal{L}$ such that $\hat{x} \neq x$ and $w(\hat{x}) \geq w(x)$.*

Moreover, strict inequality holds for (b) and (c), if in addition the graph G has no cycle with equal alternating weight.

2.4.1 Finding Alternative Feasible Solution via Alternating Traversal

Lemma 2.10 shows the existence of alternative feasible solutions under various conditions. We use the unifying framework of *alternating traversal* to show its existence.

Alternating Traversal. Given a fixed point (m, α, S) of \mathcal{T} and a feasible solution $x \in \mathcal{L}$, we define a structure called *alternating traversal* as follows.

(1) An alternating traversal \mathcal{Q} (with respect to (m, α, S) and x) is a path or circuit (not necessarily simple and might contain repeated edges), which alternates between two disjoint edge sets \mathcal{Q}^+ and \mathcal{Q}^- (hence \mathcal{Q} can be viewed as a multiset which is the disjoint union of \mathcal{Q}^+ and \mathcal{Q}^-) such that $\mathcal{Q}^+ \subset S^+$ and $\mathcal{Q}^- \subset S^-$, where $S^+ := \{e \in E : S_e \geq 0\}$ and $S^- := \{e \in E : S_e \leq 0\}$.

The alternating traversal is called *feasible* if in addition $\mathcal{Q}^+ \subset E^+$ and $\mathcal{Q}^- \subset E^-$, where $E^+ := \{e \in S^+ : x_e < 1\}$ and $E^- := \{e \in S^- : x_e > 0\}$.

An edge e is called **critical** if e is in exactly one of E^+ and E^- , and is called **strict** if $S_e \neq 0$. Given an edge $e \in E$, we denote by $r_{\mathcal{Q}}(e)$ the number of times e appears in \mathcal{Q} , and by $\text{sgn}_{\mathcal{Q}}(e)$ to be $+1$ if $e \in \mathcal{Q}^+$, -1 if $e \in \mathcal{Q}^-$ and 0 otherwise. Given a multiset \mathcal{U} of edges, we denote by $w(\mathcal{U}) := \sum_{e \in \mathcal{U}} r_{\mathcal{U}}(e)w_e$ the sum of the weights of the edges in \mathcal{U} in accordance with each edge's multiplicity.

(2) The following additional properties must be satisfied if the traversal \mathcal{Q} is a path. If one end of the path has edge $\{i, j\} \in \mathcal{Q}^+$ and end node i , then i is unsaturated under x , i.e., $\sum_{e: i \in e} x_e < b_i$; if the end has edge $\{i, j\} \in \mathcal{Q}^-$ and end node i , then $\alpha_{i \setminus j} = 0$. Observe that there is a special case where the path starts and ends at the same node i ; we still consider this as the path case as long as the end node conditions are satisfied for both end edges (which could be the same).

Lemma 2.11 (Alternative Feasible Solution.). *Suppose \mathcal{Q} is a feasible alternating traversal with respect to some feasible $x \in \mathcal{L}$. Then, there exists feasible $\hat{x} \neq x$ such that $w(\hat{x}) - w(x)$ has the same sign ($\{-1, 0, +1\}$) as $w(\mathcal{Q}^+) - w(\mathcal{Q}^-)$.*

Proof. Suppose \mathcal{Q} is a feasible alternating traversal. Then, for some $\lambda > 0$, we can define an alternative feasible solution $\hat{x} \neq x$ by $\hat{x}_e := x_e + \lambda \cdot \text{sgn}_{\mathcal{Q}}(e) \cdot r_{\mathcal{Q}}(e)$. Moreover, $w(\hat{x}) - w(x) = \lambda(w(\mathcal{Q}^+) - w(\mathcal{Q}^-))$. \square

Lemma 2.12 (Alternating Traversal Weight). *Suppose \mathcal{Q} is an alternating traversal. Then, the following holds.*

- (a) *We have $w(\mathcal{Q}^+) \geq w(\mathcal{Q}^-)$, where strict inequality holds if \mathcal{Q} contains a strict edge.*
- (b) *If \mathcal{Q} is a simple cycle with no strict edges, then $w(\mathcal{Q}^+) = w(\mathcal{Q}^-)$, i.e., \mathcal{Q} is a cycle with equal alternating weight; in particular, with the “no cycle with alternating weight” assumption, any alternating traversal that is an even cycle must contain a strict edge.*

Proof. Consider consecutive edges $\{i, j\} \in \mathcal{Q}^+$ and $\{j, k\} \in \mathcal{Q}^-$ in the alternating traversal.

Since $S_{ij} \geq 0$, by Proposition 2.7(a), we have $m_{i \rightarrow j} \geq \hat{m}_j$. Since $\alpha_{j \setminus k}$ is either \hat{m}_j or \bar{m}_j , we have $\alpha_{j \setminus k} \leq \hat{m}_j$. Therefore, $m_{i \rightarrow j} \geq \alpha_{j \setminus k}$.

Suppose $\{j, k\}$ is strict, i.e., $S_{jk} < 0$. Then, by definition, we have $\alpha_{j \setminus k} + \alpha_{k \setminus j} > w_{jk}$.

Suppose $\{i, j\}$ is strict, i.e., $S_{ij} > 0$. Then we show that either $m_{i \rightarrow j} > \alpha_{j \setminus k}$ or $\alpha_{j \setminus k} + \alpha_{k \setminus j} > w_{jk}$ holds. From Proposition 2.6(a), we have $m_{i \rightarrow j} > \alpha_{i \setminus j}$, which implies by Proposition 2.5(b) that $m_{i \rightarrow j} > \bar{m}_j$. If $\hat{m}_j = \bar{m}_j$, then $m_{i \rightarrow j} > \hat{m}_j \geq \alpha_{j \setminus k}$. If $\hat{m}_j > \bar{m}_j$, then $S_{jk} \leq 0$ together with Proposition 2.7(b) implies $m_{k \rightarrow j} \leq \bar{m}_j < \hat{m}_j$. Then from Proposition 2.5(a) we have $\alpha_{j \setminus k} = \hat{m}_j$ and thus $m_{k \rightarrow j} < \alpha_{j \setminus k}$. It follows from Proposition 2.6 that $S_{jk} < 0$, that is, $\alpha_{j \setminus k} + \alpha_{k \setminus j} > w_{jk}$.

We next show that $w(\mathcal{Q}^+) \geq w(\mathcal{Q}^-)$ by a “paying” argument. Observe that for $S_{ij} \geq 0$, we have $w_{ij} = m_{j \rightarrow i} + m_{i \rightarrow j}$, and for $S_{jk} \leq 0$, we have $\alpha_{j \setminus k} + \alpha_{k \setminus j} \geq w_{jk}$. Since we have $m_{i \rightarrow j} \geq \alpha_{j \setminus k}$, it follows that we can split the weight of every edge into two parts, such that each part of the weight from an edge in \mathcal{Q}^+ can be used to pay for a part of the weight in a neighboring edge in \mathcal{Q}^- . Observe that if $\{j, k\} \in \mathcal{Q}^-$ and node k is an end point of the traversal, then $\alpha_{k \setminus j} = 0$ and so there is no need for k to have a neighboring edge in \mathcal{Q}^+ to pay for this part. Hence, we have $w(\mathcal{Q}^+) \geq w(\mathcal{Q}^-)$. Observe that if \mathcal{Q} contains a strict edge, then at least one of the inequalities $m_{i \rightarrow j} \geq \alpha_{j \setminus k}$ and $\alpha_{j \setminus k} + \alpha_{k \setminus j} \geq w_{jk}$ becomes strict and hence we have $w(\mathcal{Q}^+) > w(\mathcal{Q}^-)$.

Finally, if \mathcal{Q} is a simple cycle with no strict edges, then all edges $\{i, j\} \in \mathcal{Q}$ satisfy $S_{ij} = 0$. Hence, the roles of \mathcal{Q}^+ and \mathcal{Q}^- can be exchanged and so $w(\mathcal{Q}^+) = w(\mathcal{Q}^-)$ follows. \square

Growing Procedure. Using Lemma 2.13, we can grow an alternating traversal with edges alternating between E^+ and E^- . We start the procedure from a critical edge $e = \{i, j\}$, i.e., in exactly one of E^+ and E^- .

1. If the growth process stops at both ends eventually without revisiting nodes, then we have a simple alternating traversal path containing the edge e .
2. Suppose the growth process revisits a node starting from one end, say for the one from node j . Then, at some point a node is revisited, and suppose k is the first node to be revisited and we have a simple cycle C . If C is even, then C forms an alternating traversal \hat{Q} that is an even cycle.
3. If C is odd and $k \neq i$, then we continue to grow from k ; we next consider the case $k = i$, and suppose l is the node before i is revisited again. Observe that since C is odd, the edges $\{i, j\}$ and $\{i, l\}$ are either both in \hat{E}^+ or both in \hat{E}^- . Suppose we continue to grow from i with respect to the edge $\{l, i\}$ and the next node is h ; since $\{i, j\}$ is in exactly one of \hat{E}^+ and \hat{E}^- , it follows that $h \neq j$, and so we can continue the growth process. At this point, the partial traversal forms a “lollipop” graph with the odd cycle C , and we are growing the stem.
4. If the growth process at the stem stops at some node g without revisiting nodes, then we have an alternating traversal \hat{Q} that is considered to be a path starting at g , traveling along the stem to k , then along the cycle C back to k , and finally returning to g along the stem again. Note that the traversal \hat{Q} contains edge e .
5. Suppose the growth process at the stem revisits some node g . If g is a node in the cycle C other than k , then since C is odd, we must have formed an alternating traversal that is an even cycle.
6. Finally, suppose the revisited node g is on the stem (including k) forming another cycle C' . If C' is even, then C' forms an alternating traversal; otherwise, we have two odd cycles C and C' connected by the stem between g and k . In this case, we have an alternating traversal \hat{Q} that is a circuit going through each cycle once and the stem back and forth, which contains edge e .

Note that an alternating traversal obtained from the above growing procedure has one of the following forms. Apart from the first case of simple even cycle, the edge e from which the procedure starts is contained in the alternating traversal.

- (a) Simple Even Cycle.
- (b) Simple Path.
- (c) Lollipop with Odd Cycle. The traversal is a path starting from the end of the stem, traveling along the stem to some node, then along an odd cycle back to this node, and then along the stem, finally returning to the end of the stem.
- (d) Dumbbell - Two Odd Cycles connected with a path.

Lemma 2.13 (Growing Feasible Alternating Traversal). *Suppose a fixed point (m, α, S) and a feasible $x \in \mathcal{L}$ are given as above.*

1. *Suppose $\{i, j\} \in E^+$ and node j is saturated (we stop if j is unsaturated). Then, there exists some node $k \in N(j) \setminus i$ such that $\{j, k\} \in E^-$.*
2. *Suppose $\{j, k\} \in E^-$ and $\alpha_{k \setminus j} > 0$ (we stop if $\alpha_{k \setminus j} = 0$). Then, there exists some node $l \in N(k) \setminus j$ such that $\{k, l\} \in E^+$.*

Proof. 1. Suppose $\{i, j\} \in E^+$ and node j is saturated. Since $\sum_{k \in N(j)} x_{jk} = b_j$ and $x_{ij} < 1$, there are at least b_j nodes k in $N(j) \setminus i$ such that $x_{jk} > 0$. We pick the k such that $m_{k \rightarrow j}$ is the smallest. Since $S_{ij} \geq 0$, we conclude from Proposition 2.7(a) that $m_{i \rightarrow j} \geq \widehat{m}_j$. It follows that $m_{k \rightarrow j}$ is at most as large as the minimum offer to j among the $b_j + 1$ best offers. Hence, $m_{k \rightarrow j} \leq \overline{m}_j$, which implies from Proposition 2.7(b) that $S_{jk} \leq 0$. Hence, $\{j, k\} \in E^-$.

2. Suppose $\{j, k\} \in E^-$ and $\alpha_{k \setminus j} > 0$. By Proposition 2.7(b), $S_{jk} \leq 0$ implies that $m_{j \rightarrow k} \leq \overline{m}_k$, i.e., node j 's offer to k is as worse as the $(b_k + 1)$ -st offer and so $\widehat{m}_k = \alpha_{k \setminus j} > 0$. Moreover $x \in \mathcal{L}$ and $x_{jk} > 0$ implies that there are at most $b_k - 1$ neighbors $i \in N(k) \setminus j$ such that $x_{ik} = 1$. Suppose $l \in N(k) \setminus j$ such that $x_{kl} < 1$ and $m_{l \rightarrow k}$ is the largest. It follows that node l 's offer to k is at least as good as the b_k -th offer and hence $m_{l \rightarrow k} \geq \widehat{m}_k > 0$, which implies that $S_{kl} \geq 0$, by Proposition 2.7(a). Hence, we have $\{k, l\} \in E^+$. \square

Lemma 2.14 (Unifying Structural Lemma). *Suppose edge $e \in E$ is critical (with respect to some fixed point (m, α) and feasible $x \in \mathcal{L}$). Then, there exists a feasible alternating traversal \mathcal{Q} ; if in addition e is strict and there is no cycle with equal alternating weight, then \mathcal{Q} contains a strict edge.*

Proof. To find a feasible alternating traversal \mathcal{Q} , we apply the growing procedure that starts from the critical edge $e = \{i, j\}$. Moreover, if \mathcal{Q} is a simple even cycle, then by Lemma 2.12(b), \mathcal{Q} contains a strict edge under the “no cycle with equal alternating weight” assumption; otherwise, \mathcal{Q} contains the edge e , in which case e being strict implies that \mathcal{Q} contains a strict edge. \square

Proof of Lemma 2.10: It suffices to check the given edge $\{i, j\}$ is critical in each of the three cases. Then, Lemma 2.13 promises the existence of a feasible alternating traversal, which contains a strict edge where appropriate. Then, Lemmas 2.12 and 2.11 guarantee the existence of feasible $\widehat{x} \neq x$ such that $w(\widehat{x}) \geq w(x)$, where strict inequality holds where appropriate. \square

2.5 Achieving Social Welfare with Greedy and Spiteful Users

We saw in Proposition 2.1 that a Nash Equilibrium m can result in zero social welfare if users are spiteful. In this section, we investigate under what conditions can a fixed point (m, α, S) of \mathcal{T} achieve good social welfare, even if the underlying payoff function u is greedy and spiteful. Given $m \in A_E$, recall that for each node i , \widehat{m}_i is the b_i -th best offer to i and \overline{m}_i is the $(b_i + 1)$ -st best offer to i . Observe that each edge $e = \{i, j\} \in E$ falls into exactly one of the following three categories.

1. **Greedy Edges:** $m_{j \rightarrow i} > \overline{m}_i$ and $m_{i \rightarrow j} > \overline{m}_j$. Edge e will be selected and $u_e(m) = 1$.
2. **Spiteful Edges:** $m_{j \rightarrow i} < \widehat{m}_i$ or $m_{i \rightarrow j} < \widehat{m}_j$. Edge e will be rejected and $u_e(m) = 0$.
3. **Ambiguous Edges:** these are the remaining edges that are neither greedy nor spiteful.

Given a fixed point (m, α, S) , by Propositions 2.5 and 2.6, the category of an edge $e \in E$ can be determined by the sign of S_e : greedy (+1), spiteful (-1), ambiguous (0). Observe that after the greedy edges are selected and the spiteful edges are rejected, even if ambiguous edges are chosen arbitrarily (deterministic or randomized) to form a b -matching, the resulting payoff function is still greedy and spiteful. We first recover the result in [KBB⁺11] for the special case where LPB has a unique integral optimum, for which we shall see that the ambiguous edges are handled trivially.

Given a fixed point (m, α, S) , the following result implies that there are no ambiguous edges in this case. Hence, after all the greedy edges are selected, the optimal social welfare is achieved automatically, and so the price of stability is 1.

Theorem 2.15 (Fixed Point and Integral LPB). *Suppose (m, α, S) is a fixed point of \mathcal{T} and LPB has a unique integral optimum x corresponding to the maximum b -matching M^* . Then, for each edge $\{i, j\} \in E$, $S_{ij} > 0$ if and only if $x_{ij} = 1$.*

Proof. For each edge $\{i, j\} \in E$, if $S_{ij} > 0$, then it follows from Theorem 2.9 (a) that $x_{ij} = 1$.

If $x_{ij} = 1$, then from Theorem 2.9 (c) we have $S_{ij} \geq 0$. Suppose $S_{ij} = 0$. Then, from Proposition 2.6 (b) we have $m_{j \rightarrow i} = \alpha_{i \setminus j}$ and $m_{i \rightarrow j} = \alpha_{j \setminus i}$. Since

$m_{j \rightarrow i} + m_{i \rightarrow j} = \alpha_{i \setminus j} + \alpha_{j \setminus i} = w_{ij} > 0$, at least one of $m_{j \rightarrow i}$ and $m_{i \rightarrow j}$ is positive. Assume $m_{j \rightarrow i} > 0$ without loss of generality. Then, $\hat{m}_i = \bar{m}_i > 0$. Let node k be a neighbor of i such that $m_{k \rightarrow i} = \alpha_{i \setminus j}$. Then, we have $m_{k \rightarrow i} = m_{j \rightarrow i} = \alpha_{i \setminus k} > 0$ and thus $S_{ik} = 0$. Then, from Theorem 2.9 (a) we have $x_{ik} = 1$. Since from Proposition 2.7(a) any $l \in N(i)$ such that $m_{l \rightarrow i} \geq \hat{m}_i > 0$ satisfies $S_{il} \geq 0$, we know that there are more than b_i edges incident to node i appear in M^* , which is a contradiction. Therefore, S_{ij} must be positive. \square

2.5.1 Handling Ambiguous Edges

In general, given $m \in A_E$, there will be ambiguous edges, and how the ambiguous edges are handled will affect the payoff function u and the social welfare. However, observe that a fixed point m of \mathcal{T} will remain a Nash Equilibrium no matter how the ambiguous edges are handled. For the rest of the section, we analyze the social welfare of a fixed point m .

Since no agent (edge player) has motivation to unilaterally change his action for fixed point m , and any contract made for an ambiguous edge will be within the best b_i offers for a node i (i.e., if $\{i, j\} \in E$ is ambiguous, then $m_{j \rightarrow i} = \hat{m}_i$ and $m_{i \rightarrow j} = \hat{m}_j$), we can optimize the following, subject to remaining node capacity constraints b' (after greedy edges are selected).

- Find a maximum cardinality b' -matching among the ambiguous edges, hence optimizing the number of contracts made.
- Find a maximum weight b' -matching among the ambiguous edges, hence optimizing the social welfare.

Choosing Maximum Weight Matching among Ambiguous Edges.

Observe that a maximum cardinality matching can be arbitrarily bad in terms of weight, but a maximum weight matching must be maximal and so is a 2-approximation for maximum cardinality. Hence, we argue that it makes sense to find a maximum weight b' -matching among the ambiguous edges. We can imagine this step to be performed centrally or as a collective decision by the users. From now on, we consider a payoff function u that results from this selection procedure and analyze the social welfare $\mathcal{S}_u(m)$ for a fixed point m of \mathcal{T} . Recall that we assume at least one of the following holds: (1) LPB has unique optimum, (2) the graph has no cycle with equal alternating weight.

Analyzing $\mathcal{S}(m)$ for Fixed Point m . Suppose (m, α, S) is a fixed point and let $\hat{E} := \{e \in E : S_e > 0\}$ and $\bar{E} := \{e \in E : S_e = 0\}$. As observed before, \hat{E} is the set of greedy edges and \bar{E} is the set of ambiguous edges.

Suppose $H \subseteq \bar{E}$ is the maximum weight b' -matching that is chosen in \bar{E} by the selection process. Then, $\mathcal{S}(m) = \sum_{e \in \hat{E}} w_e + \sum_{f \in H} w_f$, which we compare with the value of an optimal LPB solution, which we can assume is half-integral from a standard fact. We include its proof for completeness.

Fact 2.16 (Half-Integral LP Optimum). *There exists a half-integral optimal solution x to LPB, i.e., for all $e \in E$, $x_e \in \{0, \frac{1}{2}, 1\}$.*

Proof. Let A be the incident matrix of G . It suffices to show that the polytope $P = \{x : Ax \leq b, x \geq 0 \text{ and } x \leq 1\}$ is half-integral, that is, the vertices of P are half-integral. Note that P is the set of points x satisfying

$$\begin{bmatrix} A \\ I \\ -I \end{bmatrix} x \leq \begin{bmatrix} b \\ 1 \\ 0 \end{bmatrix}.$$

Let x be a vertex of P , and without loss of generality assume $x = (x_f, x_g)$, where x_f of size l and x_g of size $(m-l)$ are vectors consisting of fractional and integral entries of x , respectively. Let A' be the submatrix of A consisting of the first l columns of A and $b' = b - A \cdot (0, x_g)$. Then the inequality $A'x_f \leq b'$ holds and there is a subsystem (A'', b'') of (A', b') where A'' is nonsingular satisfying $A''x_f = b''$. Note that A' is formed from A by deleting columns in A with integral x values. Therefore, A' is the incident matrix of a subgraph of G which consists of all edges with fractional x values. Suppose the subgraph has k connected components C_1, C_2, \dots, C_k , and A_1, A_2, \dots, A_k are their incident matrices, respectively.

In the following proof, when we intend to argue that x is not a vertex of P , we show that for arbitrarily small $\delta > 0$, we can construct x'_f of size l satisfying $0 < \|x'_f - x_f\|_\infty \leq \delta$ with $A''x'_f = b''$ and $A'x'_f \leq b'$, such that there exists x''_f satisfying $x_f = \frac{x'_f + x''_f}{2}$ with $A''x''_f = b''$ and $A'x''_f \leq b'$. We say that x'_f satisfies the *closeness condition*. For each edge e with fractional x_e , define $g_e := \min(x_e, 1 - x_e)$.

Consider the connected component C_1 consisting of n_1 nodes and m_1 edges, together with the corresponding fractional vector x_1 of size m_1 and subsystem (A_1, b_1) . Recall that (A'', b'') is a subsystem of (A', b') such that A'' is nonsingular and $A''x_f = b''$. Then there is a subsystem (A''_1, b''_1) of (A_1, b_1) such that A''_1 is nonsingular and $A''_1x_1 = b''_1$. Note that A_1 is of dimension $n_1 \times m_1$. If $n_1 < m_1$, then there exists a non-zero y_1 such that $A''_1(x_1 + y_1) = b''_1$ and $A''_1(x_1 - y_1) = b''_1$ while both $x_1 + y_1$ and $x_1 - y_1$ are in the interval $(0, 1)$. Extending y_1 to y of size m with other entries being 0, we see that both $x + y$ and $x - y$ are in P , which contradicts the fact that x is a vertex of P . Therefore we have $n_1 \geq m_1$. On the other hand, since C_1 is connected, we have

$m_1 \geq n_1 - 1$. Then $m_1 = n_1 - 1$ or $m_1 = n_1$. That is, C_1 is a single node, a tree or a graph containing one cycle. We define $\delta_m := \min_{e \in E(C_1)} g_e$.

We show that C_1 cannot contain an even cycle. Assume C_1 contains an even cycle with consecutive edges e_1, e_2, \dots, e_{2j} . Then for any positive $\delta < \delta_m$ we can get a vector x'_1 defined as

$$(x'_1)_e = \begin{cases} (x_1)_e + \delta & \text{if } e = e_i, \text{ where } i \in [2j] \text{ and } i \text{ is odd,} \\ (x_1)_e - \delta & \text{if } e = e_i, \text{ where } i \in [2j] \text{ and } i \text{ is even,} \\ (x_1)_e & \text{otherwise.} \end{cases}$$

Then we know that $A_1 x'_1 \leq b_1$ and $A''_1 x'_1 = b''_1$. Let x'_f be the vector obtained from x_f by replacing x_1 with x'_1 , then x'_f satisfies the closeness condition and thus x is not a vertex of P , which is a contradiction.

Next we show that if C_1 is not a single node, each node in C_1 is of degree at least 2. Suppose, on the contrary, that C_1 contains at least one node with degree 1. Below we consider two cases.

If there are at least two nodes in C_1 with degree 1, let $D = \{q, e_1, v_2, e_2, \dots, w\}$ be the simple path between any of such two nodes q and w . Since each of q and w is incident to only one edge with fractional value and b_1 is integral, their corresponding inequalities strictly hold. Given any positive $\delta < \delta_m$ define vector x'_1 of size m_1 as follows:

$$(x'_1)_e = \begin{cases} (x_1)_e + \delta & \text{if } e = e_i \in E(D) \text{ and } i \text{ is odd,} \\ (x_1)_e - \delta & \text{if } e = e_i \in E(D) \text{ and } i \text{ is even.} \end{cases}$$

Then we can get a vector x'_f satisfying the closeness condition and leading to a contradiction.

If C_1 contains exactly one node that has degree 1, then it must be an odd cycle $L = \{v_1, e_1^L, v_2, e_2^L, \dots, v_{2j+1}, e_{2j+1}^L, v_1\}$ connected to a simple path $D = \{w_1, e_1^D, w_2, e_2^D, \dots\}$. Assume $w_1 = v_1$ without loss of generality. Given any positive $\delta < \frac{\delta_m}{2}$, define vector x'_1 of size m_1 as follows:

$$(x'_1)_e = \begin{cases} (x_1)_e + \delta & \text{if } e = e_i^L \text{ and } i \text{ is odd,} \\ (x_1)_e - \delta & \text{if } e = e_i^L \text{ and } i \text{ is even,} \\ (x_1)_e - 2\delta & \text{if } e = e_i^D \text{ and } i \text{ is odd,} \\ (x_1)_e + 2\delta & \text{if } e = e_i^D \text{ and } i \text{ is even.} \end{cases}$$

Again we can get a vector x'_f satisfying the closeness condition where a contradiction occurs.

Recall that C_1 can not contain any even cycle. Now we know that if C_1 is not a single node, then it must be an odd cycle. In the latter case, the incident square matrix A_1 is nonsingular. Therefore we have $A_1 x_1 = b_1$. It can be shown by induction that the determinant $|A_1| = 2$. Since b_1 is integral, x_1 is half-integral by Cramer's rule.

Generally, for each connected component C_i , where $i \in [k]$, the corresponding edge vector is either empty or half-integral. In conclusion, x_f must be half-integral and thus x is half-integral. \square

Propositions 2.9(b) and (c) state that any optimal LPB solution x must set $x_e = 1$ for a greedy edge $e \in \widehat{E}$, and set $x_e = 0$ for a spiteful edge e . Hence, we analyze the contribution of the ambiguous edges to the optimal value.

Lemma 2.17 (Integrality Gap). *Suppose x is a half-integral solution to LPB (with node capacity vector b') that takes non-zero values on the edge set E' . Then, there exists a b' -matching H in E' such that $w(H) \geq \frac{2}{3} \sum_{e \in E'} w_e \cdot x_e$.*

Proof. Since x is half-integral, we can first include all edges $e \in E'$ such that $x_e = 1$ in H ; this ensures that for edges e such that $x_e = 1$, they contribute the same to $w(H)$ and $w(x)$. We next transform the solution x , if necessary, such that the set J of $\frac{1}{2}$ -edges form vertex-disjoint odd cycles.

Observe that if there is a node i such that its degree in J is odd, then there must exist a path in J from i to another odd degree node j ; moreover, both i and j are unsaturated in x . Hence, using standard alternating path argument, we can transform the solution x without decreasing $w(x)$ such that all edges on the path has value either 0 or 1.

We can now assume that all degrees of nodes in J are even. Hence, each connected component in J has an Euler circuit. If the circuit is even, then again we can use alternating circuit argument to choose a solution (without decreasing its value) such that all edges on the circuit is either 0 or 1. If the circuit is odd but not simple, then there must be an even circuit, which can be eliminated again; hence, any remaining edges in J form vertex-disjoint odd cycles.

Consider each odd cycle C . Observe that the edges in C can be partitioned into three sets C_1 , C_2 and C_3 such that each set forms a (1-)matching; moreover, the contribution of C to the value of $w(x)$ solution is $\frac{1}{2}w(C)$. Hence, if we pick the C_r with the largest weight and include it in H , then we have $w(C_r) \geq \frac{2}{3} \cdot \frac{1}{2}w(C)$.

Since H gets the same contribution as $w(x)$ from integral edges and at least $\frac{2}{3}$ fraction from the fractional edges, it follows that $w(H)$ is at least $\frac{2}{3}w(x)$, as required. \square

We summarize the main result of this section in the following theorem.

Theorem 2.18 (Price of Stability). *Suppose the given graph has no cycle with equal alternating weight or LPB has unique optimum. Then, there exists a greedy and spiteful payoff function u such that any fixed point m of \mathcal{T} is a Nash Equilibrium in \mathcal{N}_u ; moreover, the social welfare $\mathcal{S}_u(m) \geq \frac{2}{3}\mathcal{S}_{LP} \geq \frac{2}{3}\max_{m' \in A_E} \mathcal{S}_u(m')$, showing that the Price of Stability is at most 1.5*

Proof. The statements about fixed point and Nash Equilibrium follow from Theorem 2.8. We focus on analyzing the social welfare of a fixed point m under the payoff function u that results from choosing a maximum weight matching H from the set of ambiguous edges \bar{E} . Suppose x is an optimal solution to LPB. Since greedy edges \hat{E} will be chosen under m , we have $\mathcal{S}(m) = \sum_{e \in \hat{E}} w_e + \sum_{f \in H} w_f$.

Suppose $x \in \mathcal{L}$ is an optimal solution to LPB, i.e., $w(x) = \mathcal{S}_{LP}$. Observe that Theorem 2.9(b) implies that for $e \in \hat{E}$, $x_e = 1$, and Theorem 2.9(c) implies that for $e \in E$ such that $x_e > 0$, $e \in \hat{E} \cup \bar{E}$. Hence, it follows that $\mathcal{S}_{LP} = \sum_{e \in \hat{E}} w_e + \sum_{f \in \bar{E}} x_f \cdot w_f$.

Observe that x restricted to \bar{E} is still an optimal solution to the LPB restricted to \bar{E} with remaining node capacity vector b' (after accounting for the greedy edges). Then, from Lemma 2.17, we have $\sum_{f \in H} w_f \geq \frac{2}{3} \sum_{f \in \bar{E}} x_f \cdot w_f$, and hence $\mathcal{S}(m) \geq \frac{2}{3}\mathcal{S}_{LP}$, as required. \square

Remark 2.19. We remark that if we use a distributed algorithm such as [LPSP08, Nie08, KY09] to find a $(1+c)$ -approximate maximum matching among the ambiguous edges, then we can show that the resulting price of stability is at most $1.5(1+c)$.

2.6 The Rate of Convergence: ϵ -Greedy and ϵ -Spiteful Users

Although the iterative protocol described in Figure 2.1 will converge to some fixed point (m, α, S) , it is possible that a fixed point will never be exactly reached. However, results in Section 2.4 and 2.5 can be extended if we relax the notions of greedy and spiteful users.

Suppose $\epsilon \geq 0$. We say the payoff function u is ϵ -greedy (or the users are ϵ -greedy), if for each $e = \{i, j\} \in E$ and $m \in A_E$, if $m_{j \rightarrow i} > \bar{m}_i + \epsilon$ and $m_{i \rightarrow j} > \bar{m}_j + \epsilon$, then $u_e(m) = 1$. We say the payoff function u is ϵ -spiteful, if for each $e = \{i, j\} \in E$ and $m \in A_E$, if $m_{j \rightarrow i} < \hat{m}_i - \epsilon$, then $u_e(m) = 0$. Given

$m \in A_E$, we can place each edge $e = \{i, j\} \in E$ in the following ϵ -categories (with respect to m).

1. **ϵ -Greedy Edges:** $m_{j \rightarrow i} > \bar{m}_i + \epsilon$ and $m_{i \rightarrow j} > \bar{m}_j + \epsilon$. If u is ϵ -greedy, then $u_e(m) = 1$.
2. **ϵ -Spiteful Edges:** $m_{j \rightarrow i} < \hat{m}_i - \epsilon$ or $m_{i \rightarrow j} < \hat{m}_j - \epsilon$. If u is ϵ -spiteful, then $u_e(m) = 0$.
3. **ϵ -Ambiguous Edges:** these are the remaining edges that are neither ϵ -greedy nor ϵ -spiteful.

As in Section 2.5, we consider an ϵ -greedy and ϵ -spiteful payoff function u that corresponds to the selection process of finding a maximum weight matching among the ϵ -ambiguous edges, after accepting the ϵ -greedy edges and rejecting the ϵ -spiteful edges.

Recall that given $\alpha \in [0, W]^{2|E|}$, $m = m(\alpha) \in A_E$ is defined by Equation (2.2). We shall use the following fact about convergence.

Fact 2.20 (Convergence). *Suppose the sequence $\{\alpha^{(t)}\}$ converges to $\alpha \in [0, W]^{2|E|}$ under the ℓ_∞ norm, where each $\alpha^{(t)}$ and α define $m^{(t)}$ and m respectively. Then, for all $\epsilon > 0$, there exists $T > 0$ such that for all $t \geq T$, the following holds.*

1. For all $\{i, j\} \in E$, $|m_{j \rightarrow i} - m_{j \rightarrow i}^{(t)}| \leq \epsilon$ and $|\alpha_{i \setminus j} - \alpha_{i \setminus j}^{(t)}| \leq \epsilon$.
2. For all $i \in V$, $|\hat{m}_i - \hat{m}_i^{(t)}| \leq \epsilon$ and $|\bar{m}_i - \bar{m}_i^{(t)}| \leq \epsilon$.

Theorem 2.21 (Convergence of Social Welfare). *Suppose the given graph has no cycle with equal alternating weight or LPB has a unique optimum. For any $\epsilon > 0$, there exists an ϵ -greedy and ϵ -spiteful payoff function u such that the following holds. For any sequence $\{(m^{(t)}, \alpha^{(t)})\}$ produced by the iterative protocol in Figure 2.1, there exists $T > 0$ such that for all $t \geq T$, the social welfare $\mathcal{S}_u(m^{(t)}) \geq \frac{2}{3}\mathcal{S}_{\text{LPB}}$.*

Proof. Fix $\epsilon > 0$. By Theorem 2.4, the sequence $\{(m^{(t)}, \alpha^{(t)})\}$ converges to (m, α) .

Consider the ϵ -greedy and ϵ -spiteful payoff function u that, when acting on $m^{(t)}$, corresponds to selecting a maximum matching among the ϵ -ambiguous edges $\bar{E}^{(t)}$, after selecting the ϵ -greedy edges $\hat{E}^{(t)}$ (both with respect to $m^{(t)}$).

Suppose \hat{E} is the set of greedy edges and \bar{E} is the set of ambiguous edges, both with respect to m . Observe that as long as the payoff function u is

conservative in selecting greedy edges (i.e., $\widehat{E}^{(t)} \subseteq \widehat{E}$) and rejecting spiteful edges (i.e., $\widehat{E} \cup \overline{E} \subseteq \widehat{E}^{(t)} \cup \overline{E}^{(t)}$), then $\mathcal{S}_u(m^{(t)}) \geq \mathcal{S}_{\widehat{u}}(m) \geq \frac{2}{3}\mathcal{S}_{\text{LPB}}$, where \widehat{u} is the greedy and spiteful payoff function as in Theorem 2.18.

Hence, it suffices to show that for large enough t , both $\widehat{E}^{(t)} \subseteq \widehat{E}$ and $\widehat{E} \cup \overline{E} \subseteq \widehat{E}^{(t)} \cup \overline{E}^{(t)}$ hold.

Let $T > 0$ be large enough such that the upper bounds in Fact 2.20 hold with $\frac{\epsilon}{2}$. Consider $t \geq T$.

To prove $\widehat{E}^{(t)} \subseteq \widehat{E}$, it suffices to show that any ϵ -greedy edge $e = \{i, j\}$ with respect to $m^{(t)}$ is greedy with respect to m . Observe that $m_{j \rightarrow i}^{(t)} > \overline{m}_i^{(t)} + \epsilon$, $|m_{j \rightarrow i}^{(t)} - m_{j \rightarrow i}| \leq \frac{\epsilon}{2}$ and $|\overline{m}_i^{(t)} - \overline{m}_i| \leq \frac{\epsilon}{2}$ imply that $m_{j \rightarrow i} > \overline{m}_i$.

Similarly, to prove that $\widehat{E} \cup \overline{E} \subseteq \widehat{E}^{(t)} \cup \overline{E}^{(t)}$, it suffices to show that any ϵ -spiteful edge $e = \{i, j\}$ with respect to $m^{(t)}$ is spiteful with respect to m . One also observes that $m_{j \rightarrow i}^{(t)} < \widehat{m}_i^{(t)} - \epsilon$, $|m_{j \rightarrow i}^{(t)} - m_{j \rightarrow i}| \leq \frac{\epsilon}{2}$ and $|\widehat{m}_i^{(t)} - \widehat{m}_i| \leq \frac{\epsilon}{2}$ imply that $m_{j \rightarrow i} < \widehat{m}_i$. \square

2.6.1 The Rate of Convergence

Although the result in [Ish76] shows that the configurations given by the iterative protocol will converge to a fixed point, it does not give the rate of convergence. However, a result by Baillon and Bruck [BB96] tells us how fast the protocol can arrive at an approximate fixed point.

Given $\delta \geq 0$, a δ -fixed point α for a function \mathcal{T} satisfies $\|\alpha - \mathcal{T}(\alpha)\|_\infty \leq \delta$.

Proposition 2.22 ([BB96]). *Suppose the maximum edge weight is W . Then, after iteration t of the distributed protocol in Figure 2.1, the $\alpha^{(t)} \in [0, W]^{2|E|}$ returned is a $O(\frac{W}{\sqrt{t}})$ -fixed point of \mathcal{T} .*

We also need a modified technical assumption on the network topology. Given $\lambda \geq 0$, a cycle is said to have λ -equal alternating weight if it is even, and the sum of the even edge weights differs from the sum of the odd edge weights by at most λ . We remark that all our arguments in Sections 2.4 and 2.5 can be extended to δ -fixed points, and ϵ -greedy and ϵ -spiteful payoff functions in a straightforward manner. We state the following result and defer the proof to the next section.

Theorem 2.23 (δ -Fixed Point, ϵ -Greedy and ϵ -Spiteful, and LPB Optimum). *Suppose $\epsilon > 0$ and the given graph has no cycle with ϵ -equal alternating weight. Suppose further that (m, α, S) is a δ -fixed point of \mathcal{T} , where $\delta = O(\frac{\epsilon}{|V|^2})$, and $x \in \mathcal{L}$ is an optimal solution to LPB. Then, the following holds.*

1. If an edge e is ϵ -greedy with respect to m , then $x_e = 1$.
2. If an edge e is ϵ -spiteful with respect to m , then $x_e = 0$.

Applying Proposition 2.22 and Theorem 2.23 to previous analysis can give the following theorem.

Theorem 2.24 (Rate of Convergence). *Suppose $\epsilon > 0$, and the given graph has maximum edge weight W and has no cycle with ϵ -equal alternating weight. Then, there exists an ϵ -greedy and ϵ -spiteful payoff function u such that the following holds. For any sequence $\{(m^{(t)}, \alpha^{(t)})\}$ produced by the iterative protocol in Figure 2.1, and for all $t \geq \Theta(\frac{W^2|V|^4}{\epsilon^2})$, the social welfare $\mathcal{S}_u(m^{(t)}) \geq \frac{2}{3}\mathcal{S}_{\text{LPB}}$.*

2.6.2 δ -Fixed Point and LPB Optimum

In this section we prove Theorem 2.23. Note that Proposition 2.6 holds for a δ -fixed point of \mathcal{T} . We further state the following properties for a δ -fixed point.

Proposition 2.25 (δ -Fixed Point). *Suppose (m, α, S) is a δ -fixed point of \mathcal{T} . Then for each $\{i, j\} \in E$ and $c \geq 0$, the following properties hold.*

- (a) $\alpha_{i \setminus j} \leq \widehat{m}_i + \delta$ and $\alpha_{i \setminus j} \geq \overline{m}_i - \delta$.
- (b) If $m_{j \rightarrow i} \geq \alpha_{i \setminus j} - c$, then $m_{j \rightarrow i} \geq \widehat{m}_i - c - \delta$.
- (c) If $m_{j \rightarrow i} \leq \alpha_{i \setminus j} + c$, then $m_{j \rightarrow i} \leq \overline{m}_i + c + \delta$.
- (d) If $m_{j \rightarrow i} > 0$ and $m_{j \rightarrow i} \geq \widehat{m}_i - c$, then $m_{i \rightarrow j} \geq \widehat{m}_j - c - 2\delta$.
- (e) If $m_{j \rightarrow i} > 0$ and $m_{j \rightarrow i} \geq \widehat{m}_i - c$, then $\widehat{m}_i + \widehat{m}_j \leq w_{ij} + 2c + 2\delta$.
- (f) If $m_{j \rightarrow i} \leq \overline{m}_i + c$, then $m_{i \rightarrow j} \leq \overline{m}_j + c + 2\delta$.
- (g) If $m_{j \rightarrow i} \leq \overline{m}_i + c$, then $\widehat{m}_i + \widehat{m}_j \geq w_{ij} - 2c - 2\delta$.
- (h) Suppose further that $c \geq \delta$. If $m_{j \rightarrow i} < \widehat{m}_i - c$, then $\widehat{m}_i + \widehat{m}_j > w_{ij} + c - \delta$.

Proof. (a) This follows directly from the definition of α and δ -fixed point, observing that $\overline{m}_i \leq \mathcal{T}(\alpha)_{i \setminus j} \leq \widehat{m}_i$.

- (b) The result is true, if $m_{j \rightarrow i} \geq \widehat{m}_i$; hence we assume $m_{j \rightarrow i} < \widehat{m}_i$, which implies $\mathcal{T}(\alpha)_{i \setminus j} = \widehat{m}_i$. Therefore, by the definition of δ -fixed point, we have $|\alpha_{i \setminus j} - \widehat{m}_i| \leq \delta$ and thus $m_{j \rightarrow i} \geq \alpha_{i \setminus j} - c \geq \widehat{m}_i - \delta - c$. Therefore, we always have $m_{j \rightarrow i} \geq \widehat{m}_i - c - \delta$.

- (c) The result is true, if $m_{j \rightarrow i} \leq \bar{m}_i$; hence we assume $m_{j \rightarrow i} > \bar{m}_i$, which implies $\mathcal{T}(\alpha)_{i \setminus j} = \bar{m}_i$. Therefore, by the definition of δ -fixed point, we have $|\alpha_{i \setminus j} - \bar{m}_i| \leq \delta$ and thus $m_{j \rightarrow i} \leq \alpha_{i \setminus j} + c \leq \bar{m}_i + \delta + c$. Therefore, we always have $m_{j \rightarrow i} \leq \bar{m}_i + c + \delta$.
- (d) Consider the following cases.
- If $m_{j \rightarrow i} > \alpha_{i \setminus j}$, then from Proposition 2.6 we have $S_{ij} > 0$ and thus $m_{i \rightarrow j} > \alpha_{j \setminus i}$. From (b) we have $m_{i \rightarrow j} \geq \hat{m}_j - \delta$.
 - If $m_{j \rightarrow i} \leq \alpha_{i \setminus j}$, then from Proposition 2.6 we have $S_{ij} \leq 0$ and thus $m_{j \rightarrow i} = w_{ij} - \alpha_{j \setminus i} = \alpha_{i \setminus j} + S_{ij}$. We also have $m_{j \rightarrow i} \geq \hat{m}_i - c \geq \alpha_{i \setminus j} - \delta - c$ from (a). So $S_{ij} \geq -c - \delta$. Then $m_{i \rightarrow j} = (w_{ij} - \alpha_{i \setminus j})_+ \geq \alpha_{j \setminus i} + S_{ij} \geq \alpha_{j \setminus i} - c - \delta$. From (b), we have $m_{i \rightarrow j} \geq \hat{m}_j - c - 2\delta$.
- (e) From (d) we have $\hat{m}_i + \hat{m}_j \leq m_{j \rightarrow i} + m_{i \rightarrow j} + 2\delta + 2c \leq w_{ij} + 2\delta + 2c$.
- (f) Consider the following cases.
- If $m_{j \rightarrow i} > \alpha_{i \setminus j}$, then from Proposition 2.6 we have $S_{ij} > 0$ and thus $m_{j \rightarrow i} = \alpha_{i \setminus j} + \frac{1}{2}S_{ij}$. We also have $m_{j \rightarrow i} \leq \bar{m}_i + c \leq \alpha_{i \setminus j} + \delta + c$, from (a). Hence, $\frac{1}{2}S_{ij} \leq c + \delta$. Then $m_{i \rightarrow j} = \alpha_{j \setminus i} + \frac{1}{2}S_{ij} \leq \alpha_{j \setminus i} + c + \delta$. From (c), we have $m_{i \rightarrow j} \leq \bar{m}_j + c + 2\delta$.
 - If $m_{j \rightarrow i} \leq \alpha_{i \setminus j}$, then from Proposition 2.6 we have $S_{ij} \leq 0$ and thus $m_{i \rightarrow j} \leq \alpha_{j \setminus i}$. From (c) we have $m_{i \rightarrow j} \leq \bar{m}_j + \delta$.
- (g) Consider the following cases.
- If $S_{ij} > 0$, then from (f) we have $\hat{m}_i + \hat{m}_j \geq \bar{m}_i + \bar{m}_j \geq m_{j \rightarrow i} + m_{i \rightarrow j} - 2c - 2\delta = w_{ij} - 2c - 2\delta$.
 - If $S_{ij} \leq 0$, then from (a) we have $\hat{m}_i + \hat{m}_j \geq \alpha_{i \setminus j} + \alpha_{j \setminus i} + 2\delta \geq w_{ij} + 2\delta$.
- (h) Since $m_{j \rightarrow i} < \hat{m}_i - c$ implies that $\mathcal{T}(\alpha)_{i \setminus j} = \hat{m}_i$, then by the definition of δ -fixed point we have $|\alpha_{i \setminus j} - \hat{m}_i| \leq \delta$. Since $c \geq \delta$, we have $m_{j \rightarrow i} < \hat{m}_i - c \leq \alpha_{i \setminus j} + \delta - c \leq \alpha_{i \setminus j}$, that is $m_{j \rightarrow i} < \alpha_{i \setminus j}$. From Proposition 2.6, we have $S_{ij} < 0$ and thus $m_{j \rightarrow i} = (w_{ij} - \alpha_{j \setminus i})_+$. Observe that $\hat{m}_j \geq \alpha_{j \setminus i} - \delta$, from (a). Therefore, we have $\hat{m}_i + \hat{m}_j > (m_{j \rightarrow i} + c) + (\alpha_{j \setminus i} - \delta) = (m_{j \rightarrow i} + \alpha_{j \setminus i}) + c - \delta = ((w_{ij} - \alpha_{j \setminus i})_+ + \alpha_{j \setminus i}) + c - \delta \geq w_{ij} + c - \delta$.

□

Observe that Theorem 2.23 can be implied by the following lemma.

Lemma 2.26 (δ -Fixed Point and LPB Optimum). *Suppose $\epsilon > 0$ and the given graph has no cycle with ϵ -equal alternating weight and further that x is a feasible solution of LPB. Then there exists a δ -fixed point (m, α, S) of \mathcal{T} , where $\delta = O(\frac{\epsilon}{|V|^2})$, such that for each $\{i, j\} \in E$, the following properties hold.*

- (a) If there exists an edge e such that e is ϵ -greedy and $x_e < 1$, then there exists $\hat{x} \in \mathcal{L}$ such that $\hat{x} \neq x$ and $w(\hat{x}) > w(x)$.
- (b) If there exists an edge e such that e is ϵ -spiteful and $x_e > 0$, then there exists $\hat{x} \in \mathcal{L}$ such that $\hat{x} \neq x$ and $w(\hat{x}) > w(x)$.

In the rest of this chapter we assume ϵ is sufficiently larger than δ , say $\epsilon = \Omega(n^2\delta)$. Similarly to the argument used in Section 2.4, we use a unifying framework of (ϵ, δ) -alternating traversal.

(ϵ, δ) -Alternating Traversal. Given a δ -fixed point (m, α, S) of \mathcal{T} and a feasible solution $x \in \mathcal{L}$, we define a structure called (ϵ, δ) -alternating traversal as follows.

1. A (ϵ, δ) -alternating traversal \mathcal{Q} (with respect to (m, α, S) and x) is a path or circuit (not necessarily simple and might contain repeated edges). \mathcal{Q} alternates between two disjoint edge sets \mathcal{Q}^+ and \mathcal{Q}^- (hence \mathcal{Q} can be viewed as a multiset which is the disjoint union of \mathcal{Q}^+ and \mathcal{Q}^-) such that $\mathcal{Q}^+ \subset S^+$ and $\mathcal{Q}^- \subset S^-$, where $S^+ := \{\{i, j\} \in E : m_{j \rightarrow i} \geq \hat{m}_i - \epsilon \text{ and } m_{i \rightarrow j} \geq \hat{m}_j - \epsilon\}$ is the set of edges that are **not** ϵ -spiteful, and $S^- := \{\{i, j\} \in E : m_{j \rightarrow i} \leq \bar{m}_i + \epsilon \text{ or } m_{i \rightarrow j} \leq \bar{m}_j + \epsilon\}$ is the set of edges that are **not** ϵ -greedy. We require that an edge appearing for multiple times in \mathcal{Q} cannot appear both in \mathcal{Q}^+ and in \mathcal{Q}^- .
The (ϵ, δ) -alternating traversal is called *feasible* if in addition $\mathcal{Q}^+ \subset E^+$ and $\mathcal{Q}^- \subset E^-$, where $E^+ := \{e \in S^+ : x_e < 1\}$ and $E^- := \{e \in S^- : x_e > 0\}$. An edge e is called **critical** if e is in exactly one of E^+ and E^- , and is called **ϵ -strict** if e is either ϵ -greedy or ϵ -spiteful. Hence, the edges in statements (a) and (b) of Lemma 2.26 are both critical and ϵ -strict.
2. The following additional properties must be satisfied if the traversal \mathcal{Q} is a path. If one end of the path has edge $\{i, j\} \in \mathcal{Q}^+$ and end node i , then i is unsaturated under x , i.e., $\sum_{e: i \in e} x_e < b_i$; if the end has edge $\{i, j\} \in \mathcal{Q}^-$ and end node i , then $(\mathcal{T}(\alpha))_{i \setminus j} = 0$.
Observe that there is a special case where the path starts and ends at the same node i ; we still consider this as the path case as long as the end node conditions are satisfied for both end edges (which could be the same).
3. As described in Section 2.4, the alternating traversal is obtained from the growing procedure starting from some **seed** edge, which in this case is both critical and ϵ -strict. Observe that the alternating traversal might not contain the seed edge.

Lemma 2.27 ((ϵ, δ) -Alternative Feasible Solution.). *Suppose \mathcal{Q} is a feasible (ϵ, δ) -alternating traversal with respect to some feasible $x \in \mathcal{L}$. Then, there exists feasible $\hat{x} \neq x$ such that $w(\hat{x}) - w(x)$ has the same sign ($\{-1, 0, +1\}$) as $w(\mathcal{Q}^+) - w(\mathcal{Q}^-)$.*

Proof. The proof is exactly the same as that of Lemma 2.11. \square

In view of Lemma 2.27, it suffices to show that we can find a feasible (ϵ, δ) -traversal \mathcal{Q} such that $w(\mathcal{Q}^+) - w(\mathcal{Q}^-) > 0$. We make use of the following slack variables for each edge to analyze the weights of the edges.

Slack Variables. For each edge $e \in \mathcal{Q}$, there exists some $c_e \geq 0$ such that

- (a) for $e = \{i, j\} \in \mathcal{Q}^+$ that is grown from i , $m_{j \rightarrow i} \geq \hat{m}_i - c_e$; define $d_e := \hat{m}_i + \hat{m}_j - w_{ij}$.
- (b) for $e = \{j, k\} \in \mathcal{Q}^-$ that is grown from j , $m_{k \rightarrow j} \leq \bar{m}_j + c_e$; define $d_e := w_{jk} - (\hat{m}_j + \hat{m}_k)$.

Observe that by Proposition 2.25(e) and (g), we have $d_e \leq 2c_e + 2\delta$.

We show that there is some way to grow the alternating traversal such that the slack variables can be kept small. Given an edge e , its *hop number* is the number of steps away from the seed edge in the growing procedure. For instance, the seed edge has hop number 0 and the next edge grown adjacent to the seed edge has hop number 1, and so on.

Lemma 2.28 (Growing Feasible (ϵ, δ) -Alternating Traversal). *Suppose a δ -fixed point (m, α, S) and a feasible $x \in \mathcal{L}$ are given as above. Assume that $c \geq 0$ is a constant.*

1. *Suppose $\{i, j\} \in E^+$ such that $m_{j \rightarrow i} > 0$ and $m_{j \rightarrow i} \geq \hat{m}_i - c$. Suppose further $c \leq \epsilon - 2\delta$ and that node j is saturated (we stop if j is unsaturated). Then, there exists some node $k \in N(j) \setminus i$ such that $\{j, k\} \in E^-$ and $m_{k \rightarrow j} \leq \bar{m}_j + c + 2\delta$.*
2. *Suppose $\{j, k\} \in E^-$ such that $m_{k \rightarrow j} \leq \bar{m}_j + c$. Suppose further $c \leq \epsilon - 4\delta$ and $(\mathcal{T}(\alpha))_{k \setminus j} > 0$ (we stop if $(\mathcal{T}(\alpha))_{k \setminus j} = 0$). Then, there exists some node $l \in N(k) \setminus j$ such that $\{k, l\} \in E^+$, $m_{l \rightarrow k} > 0$ and $m_{l \rightarrow k} \geq \hat{m}_k - c - 2\delta$.*

In particular, it follows that an edge e with hop number t has $c_e \leq (2t + 2)\delta$.

Proof. 1. Suppose $\{i, j\} \in E^+$ such that $m_{j \rightarrow i} > 0$ and $m_{j \rightarrow i} \geq \hat{m}_i - c$ and node j is saturated. Since $x_{ij} < 1$ and $\sum_{k \in N(j)} x_{jk} = b_j$, there are at least b_j nodes k in $N(j) \setminus i$ such that $x_{jk} > 0$. We pick the k such that $m_{k \rightarrow j}$ is the smallest. Then it follows that $m_{k \rightarrow j} \leq (\mathcal{T}(\alpha))_{j \setminus i}$. Since $m_{j \rightarrow i} > 0$ and $m_{j \rightarrow i} \geq \hat{m}_i - c$, from Proposition 2.25(d) we have $m_{i \rightarrow j} \geq \hat{m}_j - c - 2\delta$. If $m_{i \rightarrow j} \geq \hat{m}_j$, then $m_{k \rightarrow j} \leq (\mathcal{T}(\alpha))_{j \setminus i} = \bar{m}_j$. Otherwise, $m_{i \rightarrow j} \leq \bar{m}_j$ and thus $\hat{m}_j - \bar{m}_j \leq c + 2\delta$. Then $m_{k \rightarrow j} \leq (\mathcal{T}(\alpha))_{j \setminus i} = \hat{m}_j \leq \bar{m}_j + c + 2\delta$.

To show $\{j, k\} \in E^-$, it suffices to prove that $\{j, k\} \in S^-$. With the condition $c \leq \epsilon - 2\delta$, we have $m_{k \rightarrow j} \leq \bar{m}_j + \epsilon$. Therefore $\{j, k\} \in S^-$.

2. Suppose $\{j, k\} \in E^-$ such that $m_{k \rightarrow j} \leq \bar{m}_j + c$ and $(\mathcal{T}(\alpha))_{k \setminus j} > 0$. Then $x \in \mathcal{L}$ and $x_{jk} > 0$ implies that there are at most $b_k - 1$ neighbors $i \in N(k) \setminus j$ such that $x_{ik} = 1$. Suppose $l \in N(k) \setminus j$ such that $x_{kl} < 1$ and $m_{l \rightarrow k}$ is the largest. Then it follows that $m_{l \rightarrow k} \geq (\mathcal{T}(\alpha))_{k \setminus j} > 0$. Since $m_{k \rightarrow j} \leq \bar{m}_j + c$, from Proposition 2.25(f) we have $m_{j \rightarrow k} \leq \bar{m}_k + c + 2\delta$. If $m_{j \rightarrow k} \leq \bar{m}_k$, then $m_{l \rightarrow k} \geq (\mathcal{T}(\alpha))_{k \setminus j} = \hat{m}_k$. Otherwise, $m_{j \rightarrow k} \geq \hat{m}_k$ and thus $\hat{m}_k - \bar{m}_k \leq c + 2\delta$. Then $m_{l \rightarrow k} \geq (\mathcal{T}(\alpha))_{k \setminus j} = \bar{m}_k \geq \hat{m}_k - c - 2\delta$.

To show $\{k, l\} \in E^+$, it suffices to prove that $\{k, l\} \in S^+$. With the condition $c \leq \epsilon - 2\delta$ and Proposition 2.25(d), we have $m_{l \rightarrow k} \geq \hat{m}_k - c - 2\delta \geq \hat{m}_k - \epsilon$ and $m_{k \rightarrow l} \geq \hat{m}_l - c - 4\delta \geq \hat{m}_l - \epsilon$. Therefore $\{k, l\} \in S^+$.

Moreover, if $e_0 := \{x, y\}$ is the seed edge corresponding to some (ϵ, δ) -alternating traversal, then either i) e_0 is ϵ -greedy, i.e., $m_{y \rightarrow x} > \bar{m}_x + \epsilon$ and $m_{x \rightarrow y} > \bar{m}_y + \epsilon$, which implies $m_{y \rightarrow x} \geq \hat{m}_x$ and $m_{x \rightarrow y} \geq \hat{m}_y$; or ii) e_0 is ϵ -spiteful, i.e., $m_{y \rightarrow x} < \hat{m}_x - \epsilon$ or $m_{x \rightarrow y} < \hat{m}_y - \epsilon$, which implies $m_{y \rightarrow x} \leq \bar{m}_x$ or $m_{x \rightarrow y} \leq \bar{m}_y$, indicating that $m_{y \rightarrow x} \leq \bar{m}_x + 2\delta$ and $m_{x \rightarrow y} \leq \bar{m}_y + 2\delta$ by Proposition 2.25(f). Hence, we have $c_{e_0} \leq 2\delta$. By induction, it is obvious that an edge e with hop number t satisfies $c_e \leq (2t + 2)\delta$. \square

Observe that each edge appears at most twice in the traversal and has hop number at most $2n - 1$ from the seed edge. Hence, from Lemma 2.28, for all edges $e \in \mathcal{Q}$, we have $c_e \leq 4n\delta$ and $d_e \leq (8n + 2)\delta$.

Lemma 2.29 (Property of End Node). *Suppose an (ϵ, δ) -alternating traversal \mathcal{Q} is a path and i is an end node with $\{i, j\} \in \mathcal{Q}^-$. If $m_{j \rightarrow i} \leq \bar{m}_i + c$, then $\hat{m}_i \leq c$, where $c \geq 0$ is a constant.*

Proof. From the definition of (ϵ, δ) -alternating traversal, we have $(\mathcal{T}(\alpha))_{i \setminus j} = 0$. If $m_{j \rightarrow i} < \hat{m}_i$, then $\hat{m}_i = (\mathcal{T}(\alpha))_{i \setminus j} = 0 \leq c$. Suppose $m_{j \rightarrow i} \geq \hat{m}_i$. Then $\bar{m}_i = (\mathcal{T}(\alpha))_{i \setminus j} = 0$. Therefore $\hat{m}_i \leq m_{j \rightarrow i} \leq \bar{m}_i + c = c$. \square

Lemma 2.30 (Analyzing Weight with Slack Variables). *Given an (ϵ, δ) -alternating traversal \mathcal{Q} , we have $w(\mathcal{Q}^+) - w(\mathcal{Q}^-) \geq -\sum_{e \in \mathcal{Q}} r_{\mathcal{Q}}(e) \cdot d_e - 8n\delta$, where $r_{\mathcal{Q}}(e)$ is the number of times e appears in \mathcal{Q} .*

Proof. Let $V_{\mathcal{Q}}$ be the set of nodes that appear in the traversal \mathcal{Q} . For all $v \in V_{\mathcal{Q}}$, define $E_v := \{e \in \mathcal{Q} : e \text{ is incident to } v\}$. Consider the following two cases.

- (1) \mathcal{Q} is a circuit. Then for all $v \in V_{\mathcal{Q}}$, we have $|\mathcal{Q}^+ \cap E_v| = |\mathcal{Q}^- \cap E_v|$. By the definition of the slack variable d , we have $\sum_{e_1 \in \mathcal{Q}^+} r_{\mathcal{Q}^+}(e_1)(w_{e_1} + d_{e_1}) = \sum_{e_2 \in \mathcal{Q}^-} r_{\mathcal{Q}^-}(e_2)(w_{e_2} - d_{e_2})$. Rearranging the equality implies $w(\mathcal{Q}^+) - w(\mathcal{Q}^-) = -\sum_{e \in \mathcal{Q}} r_{\mathcal{Q}}(e)d_e \geq -\sum_{e \in \mathcal{Q}} r_{\mathcal{Q}}(e)d_e - 8n\delta$.
- (2) \mathcal{Q} is a path. Define $V_{\mathcal{Q}^+} := \{x \in V_{\mathcal{Q}} : x \text{ is an end point of } \mathcal{Q} \text{ with } \{x, y\} \in \mathcal{Q}^+\}$ and $V_{\mathcal{Q}^-} := \{x \in V_{\mathcal{Q}} : x \text{ is an end point of } \mathcal{Q} \text{ with } \{x, y\} \in \mathcal{Q}^-\}$. Then, for any $v \in V_{\mathcal{Q}}$, we have the following claims.
- (a) If $v \in V_{\mathcal{Q}^+}$, then $|\mathcal{Q}^+ \cap E_v| - |\mathcal{Q}^- \cap E_v| = 1$.
 - (b) If $v \in V_{\mathcal{Q}^-}$, then $|\mathcal{Q}^- \cap E_v| - |\mathcal{Q}^+ \cap E_v| = 1$.
 - (c) if $v \in V_{\mathcal{Q}} \setminus (V_{\mathcal{Q}^+} \cup V_{\mathcal{Q}^-})$, then $|\mathcal{Q}^+ \cap E_v| = |\mathcal{Q}^- \cap E_v|$.

Note that for all $e \in \mathcal{Q}$, we have $c_e \leq 4n\delta$. Then from Lemma 2.29, for any $v \in V_{\mathcal{Q}^-}$ we have $\hat{m}_v \leq 4n\delta$. Also observe that $|V_{\mathcal{Q}^-}| \leq 2$. Then, by the definition of the slack variable d , we have $w(\mathcal{Q}^+) - w(\mathcal{Q}^-) = -\sum_{e \in \mathcal{Q}} r_{\mathcal{Q}}(e)d_e + \sum_{v_1 \in V_{\mathcal{Q}^+}} \hat{m}_{v_1} - \sum_{v_2 \in V_{\mathcal{Q}^-}} \hat{m}_{v_2} \geq -\sum_{e \in \mathcal{Q}} r_{\mathcal{Q}}(e)d_e + 0 - 2 \cdot 4n\delta = -\sum_{e \in \mathcal{Q}} r_{\mathcal{Q}}(e)d_e - 8n\delta$.

□

We can obtain an (ϵ, δ) -alternating traversal \mathcal{Q} by applying the growing procedure as described in Section 2.4, which in this case starts from a seed edge e_0 that is critical and ϵ -strict, and goes with the rules indicated in Lemma 2.28. Observe that either \mathcal{Q} is a simple even cycle or it contains the seed edge e_0 .

Lemma 2.31 ((ϵ, δ) -Alternating Traversal Weight). *Suppose that we have a δ -fixed point (m, α, S) and a feasible solution x to LPB. Suppose further $\delta = \frac{\epsilon}{22n^2}$ and there is no cycle with ϵ -equal alternating weight. Then, the growing procedure gives an (ϵ, δ) -alternating traversal \mathcal{Q} such that $w(\mathcal{Q}^+) > w(\mathcal{Q}^-)$.*

Proof. We consider the following two cases.

- (1) The traversal \mathcal{Q} is a simple even cycle. By the no cycle with ϵ -equal alternating weight assumption, we have $|w(\mathcal{Q}^+) - w(\mathcal{Q}^-)| > \epsilon$. Observe that any $e \in \mathcal{Q}$ satisfies $d_e \leq (8n + 2)\delta$. Also note that $|\mathcal{Q}| \leq 2n$. Then from Lemma 2.30, we have $w(\mathcal{Q}^+) - w(\mathcal{Q}^-) \geq -\sum_{e \in \mathcal{Q}} r_{\mathcal{Q}}(e) \cdot d_e \geq -2n(8n + 2)\delta \geq -\epsilon$. Therefore, $w(\mathcal{Q}^+) - w(\mathcal{Q}^-) \geq \epsilon > 0$.
- (2) The traversal \mathcal{Q} is a simple path, lollipop with odd cycle or dumbbell. In this case $e_0 := \{x, y\} \in \mathcal{Q}$. Then one of the following two cases happens.

- (a) The seed edge e_0 is ϵ -spiteful. Then either $m_{y \rightarrow x} < \hat{m}_x - \epsilon$ or $m_{x \rightarrow y} < \hat{m}_y - \epsilon$. From Proposition 2.25(h) we have $\hat{m}_x + \hat{m}_y > w_{xy} + \epsilon - \delta$. Since $e_0 \in \mathcal{Q}^-$, we have $d_{e_0} < -(\epsilon - \delta) < 0$. Therefore, from Lemma 2.30 we have $w(\mathcal{Q}^+) - w(\mathcal{Q}^-) \geq -\sum_{e \in \mathcal{Q}} r_{\mathcal{Q}}(e) \cdot d_e - 8n\delta \geq -\sum_{e \in \mathcal{Q} \setminus \{e_0\}} r_{\mathcal{Q}}(e) \cdot d_e - d_{e_0} - 8n\delta > -(2n-1)(8n+2)\delta + (\epsilon - \delta) - 8n\delta \geq 0$.
- (b) The seed edge e_0 is ϵ -greedy. Then $m_{y \rightarrow x} > \bar{m}_x + \epsilon$ and $m_{x \rightarrow y} > \bar{m}_y + \epsilon$, which implies $m_{y \rightarrow x} \geq \hat{m}_x$ and $m_{x \rightarrow y} \geq \hat{m}_y$. Note that $e_0 \in \mathcal{Q}^+$. Let $e_1 := \{y, z\} \in \mathcal{Q}^-$ be the edge next to e_0 in the traversal \mathcal{Q} . If we know that either $d_{e_0} < -(\epsilon - 4\delta)$ or $d_{e_0} + d_{e_1} < -(\epsilon - 4\delta)$, then from Lemma 2.30, we have $w(\mathcal{Q}^+) - w(\mathcal{Q}^-) \geq -\sum_{e \in \mathcal{Q}} r_{\mathcal{Q}}(e) \cdot d_e - 8n\delta > -(2n-1)(8n+2)\delta + (\epsilon - 4\delta) - 8n\delta \geq 0$.

To finish the proof, we only need to show that $d_{e_0} < -(\epsilon - 4\delta)$ or $d_{e_0} + d_{e_1} < -(\epsilon - 4\delta)$.

Note that $(\mathcal{T}(\alpha))_{y \setminus x} = \bar{m}_y$ and thus $|\alpha_{y \setminus x} - \bar{m}_y| \leq \delta$. Then $m_{x \rightarrow y} > \bar{m}_y + \epsilon > \bar{m}_y + \delta \geq \alpha_{y \setminus x}$, which implies $S_{xy} > 0$, from Proposition 2.6. Hence, $m_{x \rightarrow y} + m_{y \rightarrow x} = w_{xy}$. Define $g_y := \hat{m}_y - \bar{m}_y \geq 0$. Then $d_{e_0} = \hat{m}_x + \hat{m}_y - w_{xy} \leq m_{y \rightarrow x} + (g + \bar{m}_y) - w_{xy} < m_{y \rightarrow x} + (g + m_{x \rightarrow y} - \epsilon) - w_{xy} = -(\epsilon - g)$.

- If $g \leq 4\delta$, then $d_{e_0} < -(\epsilon - g) \leq -(\epsilon - 4\delta)$.
- If $g > 4\delta$, then from the growing procedure we know that $m_{z \rightarrow y} \leq \bar{m}_y + 2\delta = \hat{m}_y - g + 2\delta < \hat{m}_y - (g - 3\delta)$. Since $g - 3\delta \geq \delta$, then from Proposition 2.25(h) we have $d_{e_1} < -(g - 3\delta) + \delta = -(g - 4\delta)$. Therefore, $d_{e_0} + d_{e_1} < -(\epsilon - g) - (g - 4\delta) = -(\epsilon - 4\delta)$.

In both cases (a) and (b), we have $w(\mathcal{Q}^+) - w(\mathcal{Q}^-) > 0$.

□

Chapter 3

Analyzing Ranking via Continuous LP for the Oblivious Matching Problem

3.1 Introduction

In this chapter we study the oblivious matching problem. Maximum matching [MV80] in undirected graphs is a classical problem in computer science. However, as motivated by online advertising [GM08, AGKM11] and exchange settings [RSÜ05], information about the graphs can be incomplete or unknown. Different online or greedy versions of the problem [ADFS95, PS12, GT12] can be formulated by the following problem, in which the algorithm is essentially oblivious to the input graph.

Oblivious Matching Problem. An *adversary* commits to a graph $G(V, E)$ and reveals the nodes V (where $n = |V|$) to the (possibly randomized) *algorithm*, while keeping the edges E secret. The algorithm returns a list L that gives a permutation of the set $\binom{V}{2}$ of unordered pairs of nodes. Each pair of nodes in G is probed according to the order specified by L to form a matching greedily. In the round when a pair $e = \{u, v\}$ is probed, if both nodes are currently *unmatched* and the edge e is in E , then the two nodes will be *matched* to each other; otherwise, we skip to the next pair in L until all pairs in L are probed. The goal is to maximize the *performance ratio* of the (expected) number of nodes matched by the algorithm to the number of nodes in a maximum matching in G .

Observe that any ordering of the pairs $\binom{V}{2}$ will result in a maximal matching in $G(V, E)$, giving a trivial performance ratio at least 0.5. However, for any deterministic algorithm, the adversary can choose a graph such that ratio 0.5

is attained. The interesting question is: how much better can randomized algorithms perform on arbitrary graphs? (For bipartite graphs, there are theoretical analysis of randomized algorithms [KMT11, MY11] achieving ratios better than 0.5.)

The **Ranking** algorithm (an early version appears in [KVV90]) is simple to describe: a permutation σ on V is selected uniformly at random, and naturally induces a lexicographical order on the unordered pairs in $\binom{V}{2}$ used for probing. Although by experiments, the **Ranking** algorithm and other randomized algorithms seem to achieve performance ratios much larger than 0.5, until very recently, the best theoretical performance ratio $0.5 + \epsilon$ (where $\epsilon = \frac{1}{400000}$) on arbitrary graphs was proved in the mid-nineties by Aronson et al. [ADFS95], who analyzed the **Modified Randomized Greedy** algorithm (MRG), which can be viewed as a modified version of the **Ranking** algorithm.

After more than a decade of research, two papers were published in FOCS 2012 that attempted to give theoretical ratios significantly better than the $0.5 + \epsilon$ bound. Poloczek and Szegedy [PS12] also analyzed the MRG algorithm to give ratio $0.5 + \frac{1}{256} \approx 0.5039$. Goel and Tripathi [GT12] analyzed the **Ranking** algorithm and claimed that ratio 0.56 can be achieved, but they later announced the withdrawal of the paper on arXiv [GT13] because of a crucial bug in their proof. Both papers used a common framework which has been successful for analyzing bipartite graphs: (i) utilize the structural properties of the matching problem to form a minimization linear program that gives a lower bound on the performance ratio; (ii) analyze the LP theoretically and/or experimentally to give a lower bound.

Related Work. We describe and compare the most relevant related work. Please refer to [PS12, GT12] and the references therein for a more comprehensive background of the problem. We describe the **Oblivious Matching Problem** general enough so that we can compare different works that are studied under different names and settings. Dyer and Frieze [DF91] showed that picking a permutation of unordered pairs uniformly at random cannot produce a constant ratio that is strictly greater than 0.5. On the other hand, this framework also includes the MRG algorithm, which was analyzed by Aronson et al. [ADFS95] to prove the first non-trivial constant performance ratio crossing the 0.5 barrier. One can also consider *adaptive* algorithms in which the algorithm is allowed to change the order in the remaining list after seeing the probing results; although hardness results have been proved for adaptive algorithms [GT12], no algorithm in the literature seems to utilize this feature yet.

Running **Ranking** on bipartite graphs for the **Oblivious Matching Problem** is

equivalent to running ranking [KVV90] for the Online Bipartite Matching problem with *random arrival order* [KMT11]. From Karande, Mehta and Tripathi [KMT11], one can conclude that **Ranking** achieves ratio 0.653 on bipartite graphs. Moreover, they constructed a hard instance in which **Ranking** performs no better than 0.727.

On a high level, most work on analyzing **Ranking** or similar randomized algorithms on matching are based on variations of the framework by Karp et al. [KVV90]. The basic idea is to relate different bad and good events to form constraints in an LP, whose asymptotic behavior is analyzed when n is large. For **Online Bipartite Matching**, Karp et al. [KVV90] showed that ranking achieves performance ratio $1 - \frac{1}{e}$; similarly, Aggarwal et al. [AGKM11] also showed that a modified version of **Ranking** achieves the same ratio for the node-weighted version of the problem.

Sometimes very sophisticated mappings are used to relate different events, and produce LPs whose asymptotic behavior is difficult to analyze. Mahdian and Yan [MY11] developed the technique of strongly factor-revealing LP. The idea is to consider another family of LPs whose optimal values are all below the asymptotic value of the original LP. Hence, the optimal value of any LP (usually a large enough instance) in the new family can be a lower bound on the performance ratio. The results of [MY11] imply that for the **Oblivious Matching Problem** on bipartite graphs, **Ranking** achieves performance ratio 0.696.

No attempts have been made in the literature to theoretically improve the $0.5 + \epsilon$ ratio for arbitrary graphs until two recent papers appeared in FOCS 2012. Poloczek and Szegedy [PS12] used a technique known as *contrast analysis* to analyze the MRG algorithm and gave ratio $\frac{1}{2} + \frac{1}{256} \approx 0.5039$. Goel and Tripathi [GT12] showed a hardness result of 0.7916 for any algorithm and 0.75 for adaptive vertex-iterative algorithms. They also analyzed the **Ranking** algorithm for a better performance ratio, but later withdrew the paper [GT13] due to a crucial bug in the proof.

Our Contribution. In this work, we revisit the **Ranking** algorithm using a continuous LP framework: given a finite linear program LPM_n the constraints of which are produced from new structural properties, we develop new primal-dual techniques for continuous LP to analyze the limiting behavior as the finite LPM_n grows. Of particular interest are new duality and complementary slackness results that can handle monotone constraints and boundary conditions in continuous LP. Our work achieves the currently best theoretical performance ratio of $\frac{2(5-\sqrt{7})}{9} \approx 0.523$ on arbitrary graphs.

Theorem 3.1. *For the Oblivious Matching Problem on arbitrary graphs, the Ranking algorithm achieves performance ratio at least $\frac{2(5-\sqrt{7})}{9} \approx 0.523$.*

As in previous works, the optimal value of LPM_n decreases as n increases. Hence, to obtain a theoretical proof, one needs to analyze the asymptotic behavior of LPM_n . It could be tedious to find the optimal solution of LPM_n and investigate its limiting behavior. One could also use experiments (for example using strongly factor-revealing LP [MY11]) to give a proof. We instead observe that the LPM_n has a continuous LPM_∞ relaxation (in which normal variables becomes a function variable). However, the monotone constraints in LPM_n require that the function in LPM_∞ be monotonically decreasing. Moreover, the boundary constraint of LPM_n has its counterpart in LPM_∞ . To the best of our knowledge, such continuous LPs have not been analyzed in the literature.

3.2 Preliminaries

Let Ω be the set of all permutations of the nodes in V , where each permutation is a bijection $\sigma : V \rightarrow [n]$. The *rank* of node u in σ is $\sigma(u)$, where smaller rank means higher priority.

The Ranking algorithm. For the Oblivious Matching Problem, the algorithm selects a permutation $\sigma \in \Omega$ uniformly at random, and returns a list L of unordered pairs according to the lexicographical order induced by σ . Specifically, given two pairs e_1 and e_2 (where for each i , $e_i = \{u_i, v_i\}$ and $\sigma(u_i) < \sigma(v_i)$), the pair e_1 has higher priority than e_2 if (i) $\sigma(u_1) < \sigma(u_2)$, or (ii) $u_1 = u_2$ and $\sigma(v_1) < \sigma(v_2)$. Each pair of nodes in $G(V, E)$ is probed according to the order given by L ; initially, all nodes are *unmatched*. In the round when the pair $e = \{u, v\}$ is probed, if both nodes are currently *unmatched* and the edge e is in E , then each of u and v is *matched*, and they are each other's *partner* in σ ; moreover, if $\sigma(u) < \sigma(v)$ in this case, we say that u *chooses* v . Otherwise, if at least one of u and v is already matched or there is no edge between them in G , we skip to the next pair in L until all pairs in L are probed.

The *performance ratio* of **Ranking** on G is the expected number of nodes matched by the algorithm to the number of nodes in a maximum matching in G , where the randomness comes from the random permutation in Ω . By Corollary 2 of [PS12] we can assume that G has a perfect matching.

3.3 The Linear Program LPM_n for Ranking

The structural properties of **Ranking** gives the following LP constraints.

Proposition 3.2 (LP Constraints [CCWZ14]). *For each $1 \leq t \leq n$, let x_t be the probability that a node at rank t is matched, over the random choice of permutation σ . Then we have the following:*

1. *Monotone constraints:* $x_{t-1} \geq x_t$ for $2 \leq t \leq n$;
2. *Evolving constraints:* $\left(1 - \frac{t-1}{n}\right) x_t + \frac{2}{n} \sum_{i=1}^{t-1} x_i \geq 1$ for $2 \leq t \leq n$;
3. *Boundary constraint:* $x_n + \frac{3}{2n} \sum_{i=1}^n x_i \geq 1$.

Note that the performance ratio is $\frac{1}{n} \sum_{t=1}^n x_t$, which will be the objective function of our minimization LP. Observe that all x_t 's are between 0 and 1; in particular, $x_1 = 1$. Combining all the constraints in Proposition 3.2, the following LPM_n gives a lower bound on the performance ratio when **Ranking** is run on a graph with n nodes. It is not surprising that the optimal value of LPM_n decreases as n increases (although our proof does not rely on this). In Section 3.4, we analyze the continuous LPM_∞ relaxation in order to give a lower bound for all finite LPM_n , thereby proving a lower bound on the performance ratio of **Ranking**.

$$\begin{aligned}
& \text{LPM}_n \\
\min \quad & \frac{1}{n} \sum_{t=1}^n x_t \\
\text{s.t.} \quad & x_1 = 1, \\
& x_{t-1} - x_t \geq 0, \quad 2 \leq t \leq n \\
& \left(1 - \frac{t-1}{n}\right) x_t + \frac{2}{n} \sum_{i=1}^{t-1} x_i \geq 1, \quad 2 \leq t \leq n \\
& x_n + \frac{3}{2n} \sum_{t=1}^n x_t \geq 1, \\
& x_t \geq 0, \quad 1 \leq t \leq n.
\end{aligned}$$

3.4 Analyzing LPM_n via Continuous LPM_∞ Relaxation

In this section, we analyze the limiting behavior of LPM_n by solving its continuous LPM_∞ relaxation, which contains both monotone and boundary condition constraints. We develop new duality and complementary slackness characterizations to solve for the optimal value of LPM_∞ , thereby giving a lower bound on the performance ratio of **Ranking**.

3.4.1 Continuous LP Relaxation

To form a continuous linear program LPM_∞ from LPM_n , we replace the variables x_t 's with a function variable z that is continuous in $[0, 1]$ and differentiable almost everywhere in $[0, 1]$. The dual LDS_∞ contains a real variable γ , and function variables w and y , where y is continuous in $[0, 1]$ and differentiable almost everywhere in $[0, 1]$. In the rest of this chapter, we use “ $\forall\theta$ ” to denote “for almost all θ ”, which means for all but a measure zero set.

$$\begin{aligned}
& \text{LPM}_\infty \\
\min \quad & \int_0^1 z(\theta) d\theta \\
\text{s.t.} \quad & z(0) = 1 \\
& z'(\theta) \leq 0, \quad \forall\theta \in [0, 1] \\
& (1 - \theta)z(\theta) + 2 \int_0^\theta z(\lambda) d\lambda \geq 1, \quad \forall\theta \in [0, 1] \\
& z(1) + \frac{3}{2} \int_0^1 z(\theta) d\theta \geq 1 \\
& z(\theta) \geq 0, \quad \forall\theta \in [0, 1].
\end{aligned}$$

$$\begin{aligned}
& \text{LDM}_\infty \\
\max \quad & \int_0^1 w(\theta) d\theta + \gamma - y(0) \\
\text{s.t.} \quad & (1 - \theta)w(\theta) + 2 \int_\theta^1 w(\lambda) d\lambda + \frac{3\gamma}{2} + y'(\theta) \leq 1, \quad \forall\theta \in [0, 1] \\
& \gamma - y(1) \leq 0 \\
& \gamma, y(\theta), w(\theta) \geq 0, \quad \forall\theta \in [0, 1].
\end{aligned}$$

Continuity Requirement. In other literature [Tyn65, Lev66] concerning continuous LP, it is often only required that the functions concerned are measurable. However, we require z and y to be continuous *everywhere* in $[0, 1]$, which is essential in deriving weak duality for LPM_∞ and LDM_∞ .

It is not hard to see that x_i corresponds to $z(\frac{i}{n})$, but perhaps it is less obvious how LDM_∞ is formed. We remark that one could consider the limiting behavior of the dual of LPM_n to conclude that LDM_∞ is the resulting program. We show in Section 3.4.2 that the pair $(\text{LPM}_\infty, \text{LDM}_\infty)$ is actually a special case of a more general class of primal-dual continuous LPs. First, we show in Lemma 3.3 that LPM_∞ is a relaxation of LPM_n .

Lemma 3.3 (Continuous LP Relaxation). *The optimal value of LPM_n is at least the optimal value of LPM_∞ .*

Proof. We fix n , and let p_n and p_∞ be the optimal values for LPM_n and LPM_∞ , respectively. For the sake of contradiction, suppose $p_\infty = p_n + \delta$ for some $\delta > 0$, which may be dependent on n . Let x be an optimal solution for LPM_n . In order to obtain a contradiction, our goal is to construct a feasible solution z (from x) for LPM_∞ that has an objective value smaller than $p_n + \delta$.

The rest of the proof proceeds in the following manner. We first construct a natural step function \hat{z} in $[0, 1]$ corresponding to x . Although \hat{z} is not continuous, it satisfies the constraints of LPM_∞ and the objective function evaluates to $\int_0^1 \hat{z}(\theta) d\theta = p_n$. Then we modify \hat{z} into a feasible solution z for LPM_∞ , increasing the objective value by less than δ .

Recall that x is an optimal solution for LPM_n . Define a step function \hat{z} in interval $[0, 1]$ as follows: $\hat{z}(0) := 1$ and $\hat{z}(\theta) := x_t$ for $\theta \in (\frac{t-1}{n}, \frac{t}{n}]$ and $t \in [n]$. It follows that

$$\int_0^1 \hat{z}(\theta) d\theta = \sum_{t=1}^n \int_{\frac{t-1}{n}}^{\frac{t}{n}} \hat{z}(\theta) d\theta = \frac{1}{n} \sum_{t=1}^n x_t = p_n.$$

We now prove that \hat{z} satisfies the constraints of LPM_∞ . Clearly $\hat{z}(0) = 1$ and $\hat{z}'(\theta) = 0$ for $\theta \in [0, 1] \setminus \{\frac{t}{n} : 0 \leq t \leq n, t \in \mathbb{Z}\}$.

Evolving constraint: For every $\theta \in (0, 1]$, suppose $\theta \in (\frac{t-1}{n}, \frac{t}{n}]$, and we have

$$\begin{aligned} & (1 - \theta)\hat{z}(\theta) + 2 \int_0^\theta \hat{z}(\lambda) d\lambda \\ &= (1 - \theta)x_t + 2 \sum_{i=1}^{t-1} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \hat{z}(\lambda) d\lambda + 2 \int_{\frac{t-1}{n}}^\theta \hat{z}(\lambda) d\lambda \\ &= (1 - \theta)x_t + \frac{2}{n} \sum_{i=1}^{t-1} x_i + 2 \left(\theta - \frac{t-1}{n} \right) x_t \\ &= \left(1 - \frac{t-1}{n} + \left(\theta - \frac{t-1}{n} \right) \right) x_t + \frac{2}{n} \sum_{i=1}^{t-1} x_i \\ &\geq \left(1 - \frac{t-1}{n} \right) x_t + \frac{2}{n} \sum_{i=1}^{t-1} x_i \\ &\geq 1, \end{aligned}$$

where the last inequality follows from the feasibility of x to LPM_n . The above inequality holds trivially at $\theta = 0$.

Boundary constraint: Using the fact that $\int_0^1 \hat{z}(\theta) d\theta = \frac{1}{n} \sum_{t=1}^n x_t$ we have

$$\hat{z}(1) + \frac{3}{2} \int_0^1 \hat{z}(\theta) d\theta = x_n + \frac{3}{2n} \sum_{t=1}^n x_t \geq 1,$$

where the last inequality follows from the feasibility of x to LPM_n .

Achieving Continuity. Next we define a continuous function z as follows. Let $\epsilon := \min\{\delta, \frac{1}{2n}\}$. The idea is that for $2 \leq t \leq n$, at the transition point $\frac{t-1}{n}$, we let the function drop gradually from x_{t-1} to x_t , as θ increases from $\frac{t-1}{n}$ to $\frac{t-1}{n} + \epsilon$.

Formally, let $z(\theta) := x_1 = 1$ for $\theta \in [0, \frac{1}{n}]$. For each $t \in \{2, \dots, n\}$, let

$$z(\theta) := \begin{cases} x_t + \frac{x_{t-1}-x_t}{\epsilon} \left(\frac{t-1}{n} + \epsilon - \theta \right), & \theta \in \left(\frac{t-1}{n}, \frac{t-1}{n} + \epsilon \right] \\ x_t, & \theta \in \left(\frac{t-1}{n} + \epsilon, \frac{t}{n} \right]. \end{cases}$$

Observe that z is continuous on $[0, 1]$. Moreover, it is differentiable almost everywhere, and has non-positive derivative whenever it is differentiable. To check that z is feasible, observe that $z \geq \hat{z}$ on $[0, 1]$, and so z also satisfies the evolving and the boundary constraints.

Finally, observe that for each $2 \leq t \leq n$, when we let the function z drop gradually at the transition point $\frac{t-1}{n}$, the difference in area under the curves z and \hat{z} on the interval $[\frac{t-1}{n}, \frac{t-1}{n} + \epsilon]$ is $\frac{(x_{t-1}-x_t)\epsilon}{2}$. Hence, the total difference in area under the curves z and \hat{z} is $\sum_{t=2}^n \frac{(x_{t-1}-x_t)\epsilon}{2} = \frac{(x_1-x_n)\epsilon}{2} \leq \frac{\epsilon}{2}$.

It follows that $\int_0^1 z(\theta) d\theta \leq \int_0^1 \hat{z}(\theta) d\theta + \frac{\epsilon}{2} = p_n + \frac{\epsilon}{2} < p_n + \delta$, obtaining the desired contradiction. \square

3.4.2 Primal-Dual for a General Class of Continuous LP

We study a class of continuous linear program CP that includes LPM_∞ as a special case. In particular, CP contains monotone and boundary conditions as constraints. Let $K, L > 0$ be two real constants. Let A, B, C, F be measurable functions on $[0, 1]$. Let D be a non-negative measurable function on $[0, 1]^2$. We first describe CP and its dual CD , and then present weak duality and complementary slackness conditions in Lemma 3.4. In CP , the variable is a function z that is continuous on $[0, 1]$ and differentiable almost everywhere in $[0, 1]$; in CD , the variables are a real number γ , and measurable functions

w and y , where y is continuous on $[0, 1]$ and differentiable almost everywhere in $[0, 1]$.

CP

$$\min \quad p(z) = \int_0^1 A(\theta)z(\theta)d\theta \quad (3.1)$$

$$\text{s.t.} \quad z(0) = K \quad (3.2)$$

$$z'(\theta) \leq 0, \forall \theta \in [0, 1] \quad (3.2)$$

$$B(\theta)z(\theta) + \int_0^\theta D(\theta, \lambda)z(\lambda)d\lambda \geq C(\theta), \forall \theta \in [0, 1] \quad (3.3)$$

$$z(1) + \int_0^1 F(\theta)z(\theta)d\theta \geq L \quad (3.4)$$

$$z(\theta) \geq 0, \forall \theta \in [0, 1].$$

CD

$$\max \quad d(w, y, \gamma) = \int_0^1 C(\theta)w(\theta)d\theta + L\gamma - Ky(0) \quad (3.5)$$

$$\text{s.t.} \quad B(\theta)w(\theta) + \int_\theta^1 D(\lambda, \theta)w(\lambda)d\lambda + F(\theta)\gamma + y'(\theta) \leq A(\theta), \forall \theta \in [0, 1] \quad (3.5)$$

$$\gamma - y(1) \leq 0 \quad (3.6)$$

$$\gamma, y(\theta), w(\theta) \geq 0, \forall \theta \in [0, 1].$$

Lemma 3.4 (Weak Duality and Complementary Slackness). *Suppose z and (w, y, γ) are feasible solutions to CP and CD respectively. Then, $d(w, y, \gamma) \leq p(z)$. Moreover, suppose z and (w, y, γ) satisfy the following complementary slackness conditions $\forall \theta \in [0, 1]$:*

$$z'(\theta)y(\theta) = 0 \quad (3.7)$$

$$\left[B(\theta)z(\theta) + \int_0^\theta D(\theta, \lambda)z(\lambda)d\lambda - C(\theta) \right] w(\theta) = 0 \quad (3.8)$$

$$\left[z(1) + \int_0^1 F(\theta)z(\theta)d\theta - L \right] \gamma = 0 \quad (3.9)$$

$$\left[B(\theta)w(\theta) + \int_\theta^1 D(\lambda, \theta)w(\lambda)d\lambda + F(\theta)\gamma + y'(\theta) - A(\theta) \right] z(\theta) = 0 \quad (3.10)$$

$$(\gamma - y(1))z(1) = 0. \quad (3.11)$$

Then, z and (w, y, γ) are optimal for CP and CD, respectively, and achieve the same optimal value.

Proof. Using the primal and dual constraints, we obtain

$$\begin{aligned}
& d(w, y, \gamma) \\
&= \int_0^1 C(\theta)w(\theta)d\theta + L\gamma - Ky(0) \\
&\leq \int_0^1 \left[B(\theta)z(\theta) + \int_0^\theta D(\theta, \lambda)z(\lambda)d\lambda \right] w(\theta)d\theta + L\gamma - Ky(0) && \text{by (3.3)} \\
&= \int_0^1 \left[B(\theta)w(\theta) + \int_\theta^1 D(\lambda, \theta)w(\lambda)d\lambda \right] z(\theta)d\theta + L\gamma - Ky(0) && (*) \\
&\leq \int_0^1 [A(\theta) - F(\theta)\gamma - y'(\theta)] z(\theta)d\theta + L\gamma - Ky(0) && \text{by (3.5)} \\
&= \int_0^1 A(\theta)z(\theta)d\theta - \int_0^1 y'(\theta)z(\theta)d\theta \\
&\quad + \left[L - \int_0^1 F(\theta)z(\theta)d\theta \right] \gamma - Ky(0) \\
&\leq \int_0^1 A(\theta)z(\theta)d\theta - \int_0^1 y'(\theta)z(\theta)d\theta + z(1)\gamma - Ky(0) && \text{by (3.4)} \\
&= \int_0^1 A(\theta)z(\theta)d\theta - y(1)z(1) + y(0)z(0) + \int_0^1 z'(\theta)y(\theta)d\theta \\
&\quad + z(1)\gamma - Ky(0) && (**) \\
&\leq \int_0^1 A(\theta)z(\theta)d\theta + (\gamma - y(1))z(1) && \text{by (3.1), (3.2)} \\
&\leq \int_0^1 A(\theta)z(\theta)d\theta && \text{by (3.6)} \\
&= p(z),
\end{aligned}$$

where in (*) we change the order of integration by using Tonelli's Theorem on non-negative measurable function g : $\int_0^1 \int_0^\theta g(\theta, \lambda)d\lambda d\theta = \int_0^1 \int_\theta^1 g(\lambda, \theta)d\lambda d\theta$; and in (**) we use integration by parts and the Fundamental Theorem of Calculus, as both y and z are continuous everywhere in $[0, 1]$. Moreover, if z and (w, y, γ) satisfy conditions (3.7) – (3.11), then all the inequalities above hold with equality. Hence, $d(w, y, \gamma) = p(z)$; so z and (w, y, γ) are optimal for CP and CD, respectively. \square

3.4.3 Lower Bound for the Performance Ratio

The performance ratio of **Ranking** is lower bounded by the optimal value of LPM_∞ . We analyze this optimal value by applying the primal-dual method to LPM_∞ . In particular, we construct a primal feasible solution z and a dual

feasible solution (w, y, γ) that satisfy the complementary slackness conditions presented in Lemma 3.4. Note that LPM_∞ and LDM_∞ are achieved from **CP** and **CD** by setting $K := 1$, $L := 1$, $A(\theta) := 1$, $B(\theta) := 1 - \theta$, $C(\theta) := 1$, $D(\lambda, \theta) := 2$, $F(\theta) := \frac{3}{2}$.

We give some intuition on how z is constructed. An optimal solution to LPM_∞ should satisfy the primal constraints with equality for some θ . Setting the constraint $(1 - \theta)z(\theta) + 2 \int_0^\theta z(\lambda) d\lambda \geq 1$ to equality, we get $z(\theta) = 1 - \theta$. However this function violates the last constraint $z(1) + \frac{3}{2} \int_0^1 z(\theta) d\theta \geq 1$. Since z is decreasing, we need to balance between $z(1)$ and $\int_0^1 z(\theta) d\theta$.

The intuition is that we set $z(\theta) := 1 - \theta$ for $\theta \in [0, \mu]$ and allow z to decrease until θ reaches some value $\mu \in (0, 1)$, and then $z(\theta) := 1 - \mu$ stays constant for $\theta \in [\mu, 1]$. To determine the value of μ , note that the equation $z(1) + \frac{3}{2} \int_0^1 z(\theta) d\theta = 1$ should be satisfied, since otherwise we could construct a feasible solution with smaller objective value by decreasing the value of $z(\theta)$ for $\theta \in (\mu, 1]$. It follows that $(1 - \mu) + \frac{3}{2} \left(1 - \mu + \frac{\mu^2}{2}\right) = 1$, that is, the value of $\mu \in (0, 1)$ is determined by the equation $3\mu^2 - 10\mu + 6 = 0$.

After setting z , we construct (w, y, γ) carefully to fit the complementary slackness conditions. Formally, we set z and (w, y, γ) as follows with their plots in Figure 3.1.

$$z(\theta) = \begin{cases} 1 - \theta, \\ 1 - \mu, \end{cases} \quad w(\theta) = \begin{cases} \frac{2(1-\mu)^2}{(5-3\mu)(1-\theta)^3}, \\ 0, \end{cases} \quad y(\theta) = \begin{cases} 0, \\ \frac{2(\theta-\mu)}{5-3\mu}, \end{cases} \quad \begin{matrix} 0 \leq \theta \leq \mu \\ \mu < \theta \leq 1 \end{matrix}$$

$$\gamma = \frac{2(1-\mu)}{5-3\mu},$$

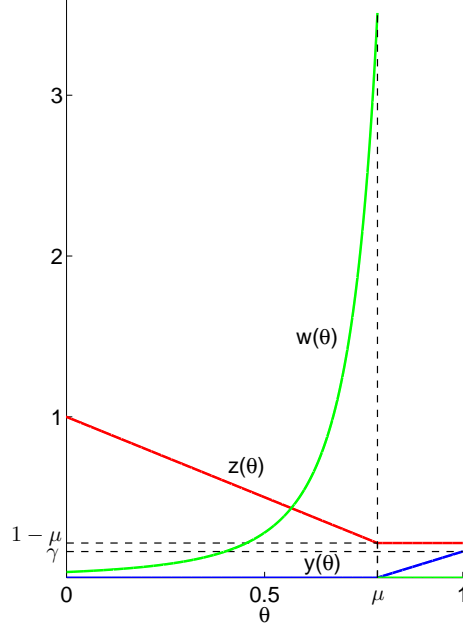
where $\mu = \frac{5-\sqrt{7}}{3}$ is a root of the equation

$$3\mu^2 - 10\mu + 6 = 0.$$

Lemma 3.5 (Optimality of z and (w, y, γ)). *The solutions z and (w, y, γ) constructed above are optimal for LPM_∞ and LDM_∞ , respectively. In particular, the optimal value of LPM_∞ is $\frac{2(5-\sqrt{7})}{9} \approx 0.523$.*

Proof. We list the complementary slackness conditions and check that they are satisfied by z and (w, y, γ) . Then Lemma 3.4 gives the optimality of z and (w, y, γ) .

$$(3.7) \quad z'(\theta)y(\theta) = 0: \text{ we have } y(\theta) = 0 \text{ for } \theta \in [0, \mu) \text{ and } z'(\theta) = 0 \text{ for } \theta \in (\mu, 1].$$


 FIGURE 3.1: Optimal z and (w, y, γ)

$$(3.8) \quad \left[(1 - \theta)z(\theta) + 2 \int_0^\theta z(\lambda) d\lambda - 1 \right] w(\theta) = 0: \text{ we have}$$

$$(1 - \theta)z(\theta) + 2 \int_0^\theta z(\lambda) d\lambda - 1 = (1 - \theta)^2 + 2 \left(\theta - \frac{\theta^2}{2} \right) - 1 = 0$$

for $\theta \in [0, \mu)$ and $w(\theta) = 0$ for $\theta \in (\mu, 1]$.

$$(3.9) \quad \left[z(1) + \frac{3}{2} \int_0^1 z(\theta) d\theta - 1 \right] \gamma = 0: \text{ we have}$$

$$z(1) + \frac{3}{2} \int_0^1 z(\theta) d\theta - 1 = (1 - \mu) + \frac{3}{2} \left(1 - \mu + \frac{\mu^2}{2} \right) - 1 = 0$$

by the definition of μ .

$$(3.10) \quad \left[(1 - \theta)w(\theta) + 2 \int_\theta^1 w(\lambda) d\lambda + \frac{3\gamma}{2} + y'(\theta) - 1 \right] z(\theta) = 0: \text{ for } \theta \in [0, \mu),$$

we have

$$\begin{aligned} & (1 - \theta)w(\theta) + 2 \int_\theta^1 w(\lambda) d\lambda + \frac{3\gamma}{2} + y'(\theta) - 1 \\ &= \frac{2(1 - \mu)^2}{(5 - 3\mu)(1 - \theta)^2} + 2 \int_\theta^\mu w(\lambda) d\lambda + \frac{3(1 - \mu)}{5 - 3\mu} + 0 - 1 \\ &= 0, \end{aligned}$$

and for $\theta \in (\mu, 1]$, we have

$$\begin{aligned} & (1 - \theta)w(\theta) + 2 \int_{\theta}^1 w(\lambda) d\lambda + \frac{3\gamma}{2} + y'(\theta) - 1 \\ &= \frac{3\gamma}{2} + y'(\theta) - 1 = \frac{3(1 - \mu)}{5 - 3\mu} + \frac{2}{5 - 3\mu} - 1 = 0. \end{aligned}$$

(3.11) $(\gamma - y(1))z(1) = 0$: we have

$$\gamma - y(1) = \frac{2(1 - \mu)}{5 - 3\mu} - \frac{2(1 - \mu)}{5 - 3\mu} = 0.$$

Moreover, the optimal value of LPM_{∞} is

$$\int_0^1 z(\theta) d\theta = 1 - \mu + \frac{\mu^2}{2} = \frac{2(5 - \sqrt{7})}{9} \approx 0.523.$$

□

Proof of Theorem 3.1: By Lemma 3.3, the performance ratio of Ranking is lower bounded by the optimal value of LPM_n . By Lemma 3.5, the optimal value of LPM_n is $\frac{2(5 - \sqrt{7})}{9} \approx 0.523$. □

Chapter 4

A Primal-dual Continuous Linear Programming Method for the Secretary Problem

4.1 Introduction

In this chapter, we study the secretary problem. In computer science, guarantees are often expressed in terms of big-O notation, but occasionally the optimal performance ratio for some problems can be narrowed down to the precise number. One such example is the classical secretary problem [Lin61, Dyn63, Fre83, Fer89], which has been popularized in the 1950s, and since then various versions and solutions for the problem have been studied. Buchbinder, Jain and Singh [BJS10] proposed the following generalized secretary problem.

(J, K) -Secretary Problem. There are n items (whose merits are given by a total ordering) that arrive in a random order, i.e., the arrival order is picked uniformly at random among all permutations of the n items. An *algorithm* (which knows the parameter n) has J *quotas* for selecting (or choosing) items. It can observe the relative merits of items that have arrived so far, and must decide irrevocably if an item is selected when it arrives. The objective is to maximize the expected *payoff*, which is the number of items selected among the best K items, where expectation is over the random arrival permutation. The *performance ratio* is the expected payoff divided by $\min\{J, K\}$. Observe that no adversary is involved in the problem, and hence randomization is unnecessary to achieve optimality, although randomized algorithms can often help the analysis, in which case the performance ratio is an expectation over randomness from both the algorithm and the arrival order.

Related Work. The simple $(1, 1)$ -case is the classical secretary problem, where both the expected payoff and the performance ratio is the probability that the best item is selected. Buchbinder et al. [BJS10] gave a linear programming formulation $\text{LPS}_n(J, K)$ that completely characterizes the generalized secretary problem in the sense that the optimal performance ratio can be inferred from the optimal value of the LP, which gives the optimal expected payoff, and every feasible solution gives a randomized algorithm with the same objective value. Indeed, for the $(1, 1)$ -case, they showed that the optimal solution corresponds to the well-known algorithm that discards the first $\frac{n}{e}$ items¹, and after that selects the first arriving item that is the best among all already observed items. It is well-known that the asymptotic ratio approaches to $\frac{1}{e}$ from above as n tends to infinity (indeed for finite n , the optimal ratio must be rational). For the general (J, K) -case, it is tedious to analyze the optimal LP solution and its asymptotic behavior. Moreover, the authors in [BJS10] did not analyze the structure of the optimal LP for the general case to derive a “simple” algorithm. They could show that the $(2, 1)$ -case has optimal asymptotic ratio $\frac{1}{e} + \frac{1}{e^{1.5}}$ that is achievable by a similar algorithm involving the $\frac{n}{e}$ -th and $\frac{n}{e^{1.5}}$ -th items. However, for the $(1, 2)$ -case, the optimal LP solution is already very complicated, and they only claimed a ratio of 0.572284, which was shown by Gusein-Zade [GZ66] (and also confirmed by our results) to be actually 0.573567.

Continuous Model. Bruss introduced the *continuous model* [Bru84], in which there is still a totally ordered set of n items, but each item picks an arrival time independently and uniformly at random from $[0, 1]$. Any algorithm in the previous *step* model can still work in the continuous model by simply ignoring the arrival times, whereas any algorithm in the continuous model can be implemented as a randomized algorithm in the finite step model by first artificially generating n independent time-stamps uniformly at random from $[0, 1]$, and giving the i -th arriving item the i -th smallest time-stamp. Although the two models are equivalent in terms of performance ratio, the continuous model is a step towards developing a convenient tool for analyzing the asymptotic ratio, because algorithms in the continuous model might not need to know n in advance, and can rely on the current time to infer what fraction of items have already been sampled. For instance, for the $(1, 1)$ -case, an asymptotically optimal algorithm selects the first arriving item after time $\frac{1}{e}$ that is the best so far.

Infinite Model. Immorlica, Kleinberg and Mahdian [IKM06] extended the continuous model to the *infinite model*, in which the totally ordered set of items is the set of positive integers, with a smaller integer having better merit. Each item still picks an arrival time uniformly at random from $[0, 1]$, and an algorithm again can only observe the relative merits of arriving items. The

¹From experiments, depending on n , the optimal threshold $\frac{n}{e}$ can be either $\lfloor \frac{n}{e} \rfloor$ or $\lceil \frac{n}{e} \rceil$.

authors considered multiple employers competing for the best item under the infinite model, but there was no formal treatment for the connection with the finite case.

Our Contribution. We give a formal treatment of the infinite model and define a special class \mathcal{A} of *piecewise continuous* infinite algorithms. Since the infinite model is ultimately just a tool to analyze the finite (continuous) model, the special class \mathcal{A} does not need to include all conceivable notions of infinite algorithms. In fact, we just need it to include the class of (J, K) -*threshold algorithms*, which we describe as follows.

Quotas. It will be helpful to imagine that there are J quotas Q_J, Q_{J-1}, \dots, Q_1 available for selecting items, where a quota with larger index is used first. For instance, Q_J is used first, and Q_1 last.

Potentials. For $k \geq 1$, during the execution of the secretary problem, an arriving item is a k -*potential* or has *potential* k , if it is the k -th best item among all those (including itself) that have already arrived. We say an item is a k_{\geq} -potential (pronounced as “at least k -potential”) if it is a k' -potential for some $k' \leq k$. Given a positive integer m , we use the notation $[m] := \{1, 2, \dots, m\}$.

Algorithm 4.1: (J, K) -Threshold Algorithm

The algorithm is characterized by JK thresholds $(\tau_{j,k})_{j \in [J], k \in [K]}$ such that

- (i) for all $j \in [J]$, $0 < \tau_{j,1} \leq \tau_{j,2} \leq \dots \leq \tau_{j,K} \leq 1$; and
- (ii) for all $k \in [K]$, $0 < \tau_{J,k} \leq \tau_{J-1,k} \leq \dots \leq \tau_{1,k} \leq 1$.

Quotas are used to select items according to the following rules:

- (a) For each $j \in [J]$ and $k \in [K]$, after time $\tau_{j,k}$, the algorithm will select a k_{\geq} -potential, if for some $j' \geq j$, quota $Q_{j'}$ is still available when the item arrives, in which case we require the available quota $Q_{j'}$ with the largest j' be used.
 - (b) Selection is done greedily, i.e., the algorithm will select an arriving item whenever it is possible according to rule (a).
-

Interpretation. We can imagine that each quota Q_j has different maturity times. For instance, at time $\tau_{j,1}$, quota Q_j can only be used for selecting 1-potential. Hence, condition (i) means that there are K progressive maturity times, where after time $\tau_{j,k}$, quota Q_j can be used for selecting k_{\geq} -potentials. Condition (ii) means that quotas with larger indices mature to the next stage earlier.

Finite Step Model. Observe that similar thresholds $(T_{j,k})_{j \in [J], k \in [K]}$ can be defined for the finite step model, where a rule is applied after step $T_{j,k}$ instead of after time $\tau_{j,k}$. In fact, any result for threshold algorithms in the continuous model can be viewed as a result for the step model in which the thresholds are randomized such that each $T_{j,k}$ is concentrated around $\tau_{j,k}n$ by Chernoff Bound.

Optimality in a Nutshell. We first show (in Proposition 4.13) that any infinite algorithm in \mathcal{A} , in particular a threshold algorithm, is feasible for some maximization primal continuous linear program LPS_∞ . To achieve this step, we use an important property of the infinite model that for any $t \in [0, 1]$, the sample space of arrivals in $[0, t]$ has exactly the same structure as that in $[0, 1]$. Next, we describe a procedure (in Section 4.5) to construct a dual feasible solution q and pick thresholds at the same time such that the corresponding threshold algorithm gives a primal solution p , which together with q satisfies the complementary slackness conditions under the continuous primal-dual LP framework by Tyndall [Tyn65] and Levinson [Lev66]. This establishes the optimality of some threshold algorithm in LPS_∞ .

Connection with the Finite Model. Observe that a threshold algorithm can be applied readily to the finite (continuous) model for any n . We show in Theorem 4.10 that if it has ratio x under the infinite model, then it has ratio at least x in the finite model for any n , implying that $\text{LPS}_n \geq \text{LPS}_\infty$ for all n . Coming to a full circle, we conclude from Proposition 4.14 that $\lim_{n \rightarrow \infty} \text{LPS}_n = \text{LPS}_\infty$, thereby proving that the optimal threshold algorithm in the infinite model is also asymptotically optimal in the finite model.

Theorem 4.1 (Optimal Thresholds for (J, K) -Secretary Problem). *There is a procedure to find appropriate thresholds $(\tau_{j,k})_{j \in [J], k \in [K]}$ such that the corresponding (J, K) -threshold algorithm is optimal under the infinite model and has a performance ratio of $\rho_{J,K} := \frac{1}{\min\{J, K\}} \cdot (J - \sum_{j=1}^J (1 - \tau_{j,1})^K)$. Moreover, under the finite model for any n , the same threshold algorithm achieves a performance ratio of at least $\rho_{J,K}$, which is asymptotically optimal. Furthermore, a result from Kleinberg [Kle05] implies that $\rho_{K,K} \geq 1 - O(\frac{1}{\sqrt{K}})$.*

As a by-product, we discover that the optimal threshold algorithm for the $(J, 1)$ -case has a nice representation using rational numbers (see Table 4.1). We believe it would be too tedious to directly analyze the limiting behavior of the finite model to reach the same conclusion.

Theorem 4.2 ($(J, 1)$ -Secretary Problem). *There is a procedure to construct an increasing sequence $\{\theta_j\}_{j \geq 1}$ of rational numbers such that for any $J \geq 1$, the optimal $(J, 1)$ -threshold algorithm uses thresholds $\{\tau_{j,1} := \frac{1}{e^{\theta_j}} \mid 1 \leq j \leq J\}$ (that can be computed in $O(J^3)$ time) and has expected payoff $\sum_{j=1}^J \tau_{j,1}$.*

J	Payoff	θ_J
1	0.367879	1
2	0.591010	$\frac{3}{2} = 1.5$
3	0.732103	$\frac{47}{24} \approx 1.958333$
4	0.823121	$\frac{2761}{1152} \approx 2.396701$
5	0.882550	$\frac{4162637}{1474560} \approx 2.822969$
6	0.921675	$\frac{380537052235603}{117413668454400} \approx 3.240994$
7	0.947588	$\frac{705040594914523588948186792543}{193003573558876719588311040000} \approx 3.652992$
8	0.964831	$\frac{302500210177484374840641189918370275991590974715547528765249}{74500758812993473612938854416966977838930799571763200000000} \approx 4.060364$

FIGURE 4.1: Optimal Payoffs for $(J, 1)$ -secretary problem

For general $K \geq 2$, the optimal solution does not have such a regular structure, but the continuous LP still allows us to compute the exact solution. To describe the optimal solution, we use part of the principal branch of the *Lambert W function* [Lam58] $W : [-\frac{1}{e}, 0] \rightarrow [-1, 0]$, where $z = W(z)e^{W(z)}$ for all $z \in [-\frac{1}{e}, 0]$. Gusein-Zade [GZ66] used a recursion to obtain the thresholds $\tau_{1,2}$ and $\tau_{1,1}$; to the best of our knowledge, no previous work has computed the exact values for thresholds $\tau_{2,1}$ and $\tau_{2,2}$.

Theorem 4.3 ($(J, 2)$ -Secretary Problem for $J = 1, 2$). *Define the thresholds: $\tau_{1,2} = \frac{2}{3}$; $\tau_{1,1} = -W(-\frac{2}{3e}) \approx 0.346982$;*

$\tau_{2,2} \approx 0.517297$ is the solution of: $x \ln x + \ln x - (2 + 3 \ln \frac{2}{3})x + 1 - \ln \frac{2}{3} = 0$; $\tau_{2,1} = -W(-e^{-c/2}) \approx 0.227788$, where $c := -(\ln \tau_{1,1})^2 + 2 \ln \frac{2}{3} \ln \tau_{1,1} + (\ln \tau_{2,2})^2 - 2 \ln \frac{2}{3} \ln \tau_{2,2} - 2\tau_{2,2} + 4 - 2 \ln \frac{2}{3}$.

Then, these thresholds can be used to achieve the following optimal performance ratios:

$$(a) \rho_{1,2} := 2\tau_{1,1} - \tau_{1,1}^2 \approx 0.573567.$$

$$(b) \rho_{2,2} := \frac{1}{2} \cdot ((2\tau_{1,1} - \tau_{1,1}^2) + (2\tau_{2,1} - \tau_{2,1}^2)) \approx 0.488628.$$

We remark that from experiments, the finite LPS_n can be solved to reveal the exact optimal thresholds for that particular n by observing precisely at which steps certain variables switch from zero to positive. However, to prove directly that the optimal solution has this threshold structure, one would need to consider messier recurrence relations instead of differential equations. Furthermore, it would be even more difficult to analyze its asymptotic behavior directly.

4.2 Preliminaries

We use the infinite model as a tool to analyze the secretary problem when the number n of items is large. We shall describe the properties of our “infinite” algorithms, which can still be applied to finite instances to obtain conventional algorithms. We consider countably infinite number of items, whose ranks are indexed by the set \mathbb{N} of positive integers, where lower rank means better merit. Hence, the item with rank 1 is the best item. The arrival time of each item is a real number drawn independently and uniformly at random from $[0, 1]$ (where the probability that two items arrive at the same time is 0); the (random) function $\rho : \mathbb{N} \rightarrow [0, 1]$ gives the arrival time of each item, where $\rho(i)$ is the arrival time of the item with rank i . For a positive integer m , we denote $[m] := \{1, 2, \dots, m\}$.

Input Sample Space. An algorithm can observe the arrival times Σ of items and their relative merit, which can be given by a total ordering \prec on Σ . Given $\rho : \mathbb{N} \rightarrow [0, 1]$, we have the set $\Sigma_\rho := \{\rho(i) \mid i \in \mathbb{N}\}$, and a total ordering \prec_ρ on Σ_ρ defined by $\rho(i) \prec_\rho \rho(j)$ if and only if $i < j$. The sample space is $\Omega := \{(\Sigma_\rho, \prec_\rho) \mid \rho : \mathbb{N} \rightarrow [0, 1]\}$, with a probability distribution induced by the randomness of ρ ; we say each $\omega = (\Sigma, \prec) \in \Omega$ is an *arrival sequence*. We sometimes use time $t \in \Sigma$ to mean the item arriving at time t , for instance we might say “the algorithm selects $t \in \Sigma$.”

Fact 4.4 (Every Non-Zero Interval Contains Infinite Number of Items). *For every interval $\mathcal{I} \subseteq [0, 1]$ with non-zero length, the probability that there exist infinitely many items arriving in \mathcal{I} is 1.*

Infinite Algorithm. When an item arrives, an algorithm must decide immediately whether to select that item. Moreover, an algorithm does not know the absolute ranks of the items, but can observe only the relative merit of the items seen so far. These properties are captured for our infinite algorithms as follows. Given $\omega = (\Sigma, \prec) \in \Omega$, and $t \in [0, 1]$, let $\Sigma^{[t]} := \{x \in \Sigma \mid x \leq t\}$ be the arrival times up to time t , with the total ordering inherited from \prec , which strictly speaking can be denoted by $\prec_{\Sigma^{[t]}}$. However, for notational convenience, we write $\omega^{[t]} = (\Sigma^{[t]}, \prec)$, dropping the subscript for \prec . We denote $\Omega^{[t]} := \{\omega^{[t]} \mid \omega \in \Omega\}$. Also we denote $\omega^{(t)} := (\Sigma^{[t]} \setminus \{t\}, \prec)$ and $\Omega^{(t)} := \{\omega^{(t)} \mid \omega \in \Omega\}$.

An infinite algorithm \mathcal{A} is an ensemble of functions $\{A^{[t]} : \Omega^{[t]} \rightarrow \{0, 1\} \mid t \in [0, 1]\}$, where for time t , the value 1 means an item is chosen at time t and 0 otherwise. Observe that if no item arrives at time t , an algorithm cannot select an item at that time; this means for all $\omega = (\Sigma, \prec) \in \Omega$, if $t \notin \Sigma$, then $\mathcal{A}^{[t]}(\omega^{[t]}) = 0$. Any J -choice algorithm \mathcal{A} must also satisfy that for any $\omega \in \Omega$, there can be at most J values of t such that $\mathcal{A}^{[t]}(\omega^{[t]}) = 1$.

An algorithm $\mathcal{A} : \Omega \rightarrow \{0, 1, \dots, K\}$ can also be interpreted as a function, which returns the number of items selected among the K best items. Since we wish to maximize the expected payoff of an algorithm where randomness comes from Ω , we can consider only deterministic algorithms without loss of generality.

Definition 4.5 (Outcome and Payoff). Let \mathcal{A} be an (infinite) algorithm. For $\omega \in \Omega$, the *outcome* $\mathcal{A}(\omega)$ is the number of items selected among the K best items. The expected *payoff* of \mathcal{A} is defined as $P(\mathcal{A}) := \mathbf{E}_\omega [\mathcal{A}(\omega)]$.

The reason we consider the infinite model is that for any $0 < t \leq 1$, the sample space $\Omega^{(t)}$ observed before t has the same structure as Ω in the sense described in the following Proposition 4.6. This allows us to analyze the recursive behavior of any infinite algorithm.

Proposition 4.6 (Isomorphism between $\Omega^{(t)}$ and Ω). *For any $0 < t \leq 1$, the sample space $\Omega^{(t)}$ (with distribution inherited from Ω) rescaled to $[0, 1]$ (by dividing each arrival time by t) has the same distribution as Ω .*

Proof. Recall that the probability distribution over Ω is induced by the randomness of all the infinite arrival times, each of which is a random number drawn independently and uniformly from $[0, 1]$. Similarly, the probability distribution over $\Omega^{(t)}$ is induced by the randomness of arrival times before t , the number of which is infinite by Fact 4.4. Moreover, each arrival time in $[0, t)$ is drawn independently and uniformly from $[0, t)$, which after rescaling is independently and uniformly distributed in $[0, 1]$. \square

Recall that for $k \geq 1$, an arriving item is a k -potential if it is the k -th best item among all those that have already arrived, and a k_\geq -potential if it is a k' -potential for some $k' \leq k$.

Proposition 4.7 (Distribution of Potentials). *For every $k \geq 1$ and $t > 0$, with probability 1, the following conditions hold.*

1. *There exists a potential in $[0, t)$.*
2. *There are finitely many k -potentials in $[t, 1]$.*

Proof. 1. From Fact 4.4, with probability 1, there exists an item arriving in $[0, t)$. This implies that there exists $i \in \mathbb{N}$ such that the item with rank i arrives at $\rho(i) \in [0, t)$. If $\rho(i)$ is not a potential, then a non-empty subset S of items with ranks in $\{1, \dots, i-1\}$ must have arrived before $\rho(i)$. Since S is finite, the item among them with smallest arrival time is a potential. Thus, there is a potential in $[0, t)$ with probability 1.

2. Similarly, from Fact 4.4, there exist k items in $[0, t)$ and let r be the maximum rank among those k ranks. Every k -potential in $[t, 1]$ must have a rank in $\{1, \dots, r-1\}$; that is, there are finite number of k -potentials in $[t, 1]$ with probability 1. \square

Remark 4.8. (Validity of Threshold Algorithms) Consider a strange scenario when no items arrive at $\tau_{j,1}$, but, for every n , the item with rank n arrives at time $\tau_{1,1} + \frac{1}{n}$. It follows that every item is a 1-potential, but there is no “first” item arriving after $\tau_{j,1}$. One way to resolve this situation is that quota Q_j is simply “lost”. On the other hand, by Proposition 4.7, with probability 1, there are only a finite number of K_{\geq} -potentials arriving after $\tau_{j,1}$, and hence this strange scenario we describe happens with probability 0.

Piecewise Continuous. Given an algorithm \mathcal{A} , $j \in [J]$ and $k \in [K]$, define the function $p_{j|k}^{\mathcal{A}} : [0, 1] \rightarrow [0, 1]$ such that $p_{j|k}^{\mathcal{A}}(x)$ is the probability that \mathcal{A} selects time x using quota Q_j given that x is a k -potential. Let $p^{\mathcal{A}} := (p_{j|k}^{\mathcal{A}})_{j \in [J], k \in [K]}$ be the collection of functions for \mathcal{A} . We say \mathcal{A} is *piecewise continuous* if every function $p_{j|k}^{\mathcal{A}}$ is piecewise continuous. We denote by \mathcal{A} the class of piecewise continuous algorithms; as we shall see, this class of algorithms is general enough to capture the asymptotic behavior for finite models with large n number of items.

Proposition 4.9. *Every threshold algorithm is piecewise continuous.*

Proof. Consider the first threshold $\tau_{J,1}$ and the function $p_{J|1}(x)$ giving the probability that a potential arriving at time x is selected by using quota Q_J , which is 0 for $x < \tau_{J,1}$, and there is a discontinuity at $\tau_{J,1}$. At time $x > \tau_{J,1}$, the probability $p_{J|1}(x)$ is the same as that for the event of the best item before time x arriving before $\tau_{J,1}$, and so $p_{J|1}(x) = \frac{\tau_{J,1}}{x}$ for $x > \tau_{J,1}$. Other $p_{j|k}$ ’s can be analyzed similarly. In conclusion, for each $p_{j|k}$ discontinuities can only appear at the JK thresholds. \square

As discussed in the introduction, there is a linear program LPS_n for the finite step model such that the optimal value of the LP gives the optimal performance ratio, and every feasible solution gives a randomized algorithm with the same objective value. As we will see in Section 4.3, for the infinite model, any infinite algorithm in \mathcal{A} , in particular a threshold algorithm, is feasible for some maximization primal continuous LPS_{∞} . With slight abuse of notation, we let LPS_n and LPS_{∞} denote the optimal values of the corresponding LPs.

Theorem 4.10 (Finite Optimal is at Least Infinite Optimal). *Let $\text{LPS}_n(J, K)$ and $\text{LPS}_{\infty}(J, K)$ be the LPs as defined in Section 4.3. Then the optimal value of $\text{LPS}_n(J, K)$ is at least the optimal value of $\text{LPS}_{\infty}(J, K)$ for each J and K .*

Proof. We fix J and K . Note that an infinite algorithm can be applied when the number n of items is finite, which can be further implemented as a randomized algorithm in the finite step model. As we will show in Section 4.5, there is a threshold algorithm that achieves the optimal payoff, that is, the optimal value of $\text{LPS}_\infty(J, K)$. It remains to prove that if a threshold algorithm has payoff ρ in the infinite model, then for each n , the payoff of applying the algorithm to the finite continuous model with n items is at least ρ .

We suppose a threshold algorithm \mathcal{A} has payoff ρ in the infinite model and fix n . For an item $t \in [0, 1]$, let $\omega = (\Sigma, \prec) \in \Omega^{[t]}$ be the arrival sequence up to time t . Note that if item t has a rank larger than n , then $\mathcal{A}^{[t]}(\omega) = 0$, i.e., item t cannot be selected by \mathcal{A} . Suppose item t has a rank at most n . Let $\omega_n = (\Sigma_n, \prec)$, where Σ_n is obtained from Σ by removing all items with ranks larger than n . Then item t is a k -potential in ω_n if and only if it is a k -potential in ω . Then we have $\mathcal{A}^{[t]}(\omega_n) = \mathcal{A}^{[t]}(\omega)$. Therefore, $\mathcal{A}^{[t]}(\omega) = 1$ implies $\mathcal{A}^{[t]}(\omega_n) = 1$ for all $t \in [0, 1]$. It follows that the payoff of applying \mathcal{A} to the finite continuous model with n items is at least ρ . \square

Main Approach. Note that LPS_n is the optimal payoff for the finite model with n items. In Section 4.3, we consider the continuous LPS_∞ , which gives a connection to the (finite) LPS_n . From Proposition 4.14 in Section 4.3, we can conclude that $\limsup_{n \rightarrow \infty} \text{LPS}_n \leq \text{LPS}_\infty$. In Sections 4.4 and 4.5, we show that the optimal payoff for the infinite algorithm can be achieved by a threshold algorithm, establishing Theorem 4.10. In particular, we show that the optimal payoff for the infinite model is LPS_∞ . Hence, it follows that $\lim_{n \rightarrow \infty} \text{LPS}_n = \text{LPS}_\infty$, and there exist thresholds such that the corresponding threshold algorithm achieves the asymptotic optimal payoff for the finite model for large n , as stated in Theorem 4.1.

4.3 The Continuous Linear Program

For the finite model with random permutation, Buchbinder et al. [BJS10] showed that there exists a linear program $\text{LPS}_n(J, K)$ such that there is a one-to-one correspondence between an algorithm for the (J, K) -secretary problem with n items and a feasible solution of $\text{LPS}_n(J, K)$; the payoff of the algorithm is exactly the objective of $\text{LPS}_n(J, K)$. Therefore, the optimal value of the $\text{LPS}_n(J, K)$ gives the maximum payoff of the (J, K) -secretary problem with n items. We rewrite their LP in a convenient form; recall that the quotas are used in the order Q_J, Q_{J-1}, \dots, Q_1 . The variable $z_{j|k}(i)$ represents that the probability that the i -th item is selected using quota Q_j given that it is a

k -potential.

$$\begin{aligned}
& \text{LPS}_n(J, K) \\
\max \quad & v_n(z) = \sum_{j=1}^J \sum_{k=1}^K \sum_{i=1}^n \frac{1}{n} \sum_{\ell=k}^K \frac{\binom{n-i}{\ell-k} \binom{i-1}{k-1}}{\binom{n-1}{\ell-1}} z_{j|k}(i) \\
\text{s.t.} \quad & z_{j|k}(i) \leq \sum_{m=1}^{i-1} \frac{1}{m} \sum_{\ell=1}^K [z_{(j+1)|\ell}(m) - z_{j|\ell}(m)], \\
& \forall i \in [n], k \in [K], 1 \leq j < J \\
& z_{J|k}(i) \leq 1 - \sum_{m=1}^{i-1} \frac{1}{m} \sum_{\ell=1}^K z_{J|\ell}(m), \quad \forall i \in [n], k \in [K] \\
& z_{j|k}(i) \geq 0, \quad \forall i \in [n], k \in [K], j \in [J].
\end{aligned}$$

For the (J, K) -secretary problem in the infinite model, we construct a continuous linear program $\text{LPS}_\infty(J, K)$ such that every piecewise continuous algorithm corresponds to a feasible solution, whose objective value is the payoff of the algorithm. Hence, the optimal value of $\text{LPS}_\infty(J, K)$ gives an upper bound for the maximum payoff; we later show that there exist thresholds such that the corresponding threshold algorithm can achieve this optimal value.

For each $j \in [J]$ and $k \in [K]$, let $p_{j|k}(x)$ be a function of x that is piecewise continuous in $[0, 1]$. In the rest of this chapter, we use “ $\forall x$ ” to denote “for almost all x ”, which means for all but a measure zero set. Define $\text{LPS}_\infty(J, K)$ as follows.

$$\begin{aligned}
& \text{LPS}_\infty(J, K) \\
\max \quad & w(p) = \sum_{j=1}^J \sum_{k=1}^K \int_0^1 \left(\sum_{\ell=k}^K \binom{\ell-1}{k-1} (1-x)^{\ell-k} \right) x^{k-1} p_{j|k}(x) dx \\
\text{s.t.} \quad & p_{j|k}(x) \leq \int_0^x \frac{1}{y} \sum_{\ell=1}^K [p_{(j+1)|\ell}(y) - p_{j|\ell}(y)] dy, \\
& \forall x \in [0, 1], k \in [K], 1 \leq j < J \\
& p_{J|k}(x) \leq 1 - \int_0^x \frac{1}{y} \sum_{\ell=1}^K p_{J|\ell}(y) dy, \quad \forall x \in [0, 1], k \in [K] \\
& p_{j|k}(x) \geq 0, \quad \forall x \in [0, 1], k \in [K], j \in [J].
\end{aligned}$$

For brevity we denote by LPS_n and LPS_∞ the optimal values of $\text{LPS}_n(J, K)$ and $\text{LPS}_\infty(J, K)$, respectively, when there is no ambiguity. Fix an algorithm $\mathcal{A} \in \mathcal{A}$. For each x and j and k , the events E_x^j , Z_x^j , V_x^k and W_x^k are defined

as follows. Let E_x^j be the event that time x is selected using quota Q_j . Let Z_x^j be the event that quota Q_j has already been used before time x , i.e., all quotas $Q_{j'}$ for $j' \geq j$ have been used. Let V_x^k be the event that time x is a k -potential. Let W_x^k be the event that time x is the k -th best item overall. Note that Z_x^j implies Z_x^{j+1} , and $Z_x^{j+1} \wedge \overline{Z_x^j}$ is the event that quota Q_j is the next quota available to be used at time x , for $1 \leq j < J$. Also observe that E_x^j implies $Z_x^{j+1} \wedge \overline{Z_x^j}$.

Lemma 4.11 (Independence between a Potential and Past History). *For $0 < x \leq 1$, and positive integer k , the event V_x^k that x is a k -potential is independent of the arrival sequence observed before time x . In particular, this implies that for any $K > 1$, the event that x is a K_{\geq} -potential is also independent of the arrival sequence observed before time x .*

Proof. By Proposition 4.6 the arrival sequence observed before time x can be generated by sampling a random arrival time for each integer in \mathbb{N} independently and uniformly in $[0, x)$. We distinguish two cases: (1) without knowledge of x , this sequence is generated for all integers in \mathbb{N} ; (2) given that x is a k -potential for some $k \in [K]$, this sequence is generated for all integers in $\mathbb{N} \setminus \{k\}$. Since the total ordering on a sequence observed before x is inherited from \mathbb{N} and there is a bijection between \mathbb{N} and $\mathbb{N} \setminus \{k\}$, the sequences generated in the two cases have the same distribution. Hence, the event V_x^k is independent of $\Omega^{(x)}$. Since the V_x^k 's for $k \in [K]$ are disjoint, the event that x is a K_{\geq} -potential and $\Omega^{(x)}$ are independent. \square

Lemma 4.12. *For all $j \in [J]$ and $x \in [0, 1]$, we have*

$$\Pr(Z_x^j) = \int_0^x \frac{1}{y} \sum_{\ell=1}^K p_{j|\ell}(y) dy.$$

Proof. For $\ell \in [K]$, let y_ℓ be the arrival time of the ℓ -th best item in $[0, x]$. Define $Y := \max_{\ell \in [K]} \{y_\ell\}$. Then for each $y \in [0, x]$ we have $\Pr(Y \leq y) = \frac{y^K}{x^K}$. It follows that the probability density function of Y is $f(y) = \frac{Ky^{K-1}}{x^K}$. Also note that given $Y = y$, we have $\Pr(y_\ell = y | Y = y) = \frac{1}{K}$ for all $\ell \in [K]$. It follows that $\Pr(E_y^j | Y = y) = \sum_{\ell=1}^K \Pr(E_y^j | V_y^\ell) \Pr(y_\ell = y | Y = y) = \frac{1}{K} \sum_{\ell=1}^K p_{j|\ell}(y)$.

Given $Y = y$, there is no K_{\geq} -potential in $(y, x]$ and hence no item is selected. Thus Z_x^j happens if and only if either Z_y^j or E_y^j (i.e. $Z_y^{j+1} \wedge \overline{Z_y^j} \wedge E_y^j$) happens. By Lemma 4.11, whether the event $Y = y$ happens or not, the distribution of sample space $\Omega^{(y)}$ of arrival time observed before time y remains the same; in particular, the events Z_y^j and $Y = y$ are independent. Moreover the events Z_y^j

and E_y^j are disjoint. By the law of total probability we have

$$\begin{aligned} \Pr(Z_x^j) &= \int_0^x \Pr(Z_x^j | Y = y) f(y) dy \\ &= \int_0^x [\Pr(Z_y^j | Y = y) + \Pr(E_y^j | Y = y)] \frac{Ky^{K-1}}{x^K} dy \\ &= \frac{K}{x^K} \int_0^x \left[\Pr(Z_y^j) + \frac{1}{K} \sum_{\ell=1}^K p_{j|\ell}(y) \right] y^{K-1} dy. \end{aligned}$$

Fix j and let $g(x) := \Pr(Z_x^j)$ be a function with respect to x . Taking derivatives on both sides of $x^K g(x) = K \int_0^x [g(y) + \frac{1}{K} \sum_{\ell=1}^K p_{j|\ell}(y)] y^{K-1} dy$ and using piecewise continuity, we have $g'(x) = \frac{1}{x} \sum_{\ell=1}^K p_{j|\ell}(x)$ for almost all x . Then $g(x) = \int_0^x \frac{1}{y} \sum_{\ell=1}^K p_{j|\ell}(y) dy + c$ for some constant c . By definition $g(0) = 0$ and thus $c = 0$. Therefore we have $\Pr(Z_x^j) = \int_0^x \frac{1}{y} \sum_{\ell=1}^K p_{j|\ell}(y) dy$. \square

Proposition 4.13 (The Optimal Payoff is at Most $\text{LPS}_\infty(J, K)$). *Let $\mathcal{A} \in \mathcal{A}$ be an algorithm for the (J, K) -secretary problem. Let $p = (p_{j|k})_{j \in [J], k \in [K]}$ be the functions such that for $j \in [J]$ and $k \in [K]$ and $x \in [0, 1]$, the probability that time x is selected by \mathcal{A} using quota Q_j given that time x is a k -potential is $p_{j|k}(x)$. Then p is a feasible solution of $\text{LPS}_\infty(J, K)$. Moreover, the payoff of \mathcal{A} is exactly the objective function*

$$w(p) = \sum_{j=1}^J \sum_{k=1}^K \int_0^1 \left(\sum_{\ell=k}^K \binom{\ell-1}{k-1} (1-x)^{\ell-k} \right) x^{k-1} p_{j|k}(x) dx.$$

Proof. We first show that the payoff $P(\mathcal{A}) = w(p)$. Consider the relation between $\Pr(E_x^j | V_x^k)$ and $\Pr(E_x^j | W_x^\ell)$. If time x is the ℓ -th best item overall, then it must be a k -potential for some $k \leq \ell$. Moreover, we have $\Pr(V_x^k | W_x^\ell) = \binom{\ell-1}{k-1} x^{k-1} (1-x)^{\ell-k}$ (by convention $0^0 = 1$). Then

$$\begin{aligned} \Pr(E_x^j | W_x^\ell) &= \sum_{k=1}^{\ell} \Pr(E_x^j | V_x^k \wedge W_x^\ell) \Pr(V_x^k | W_x^\ell) \\ &= \sum_{k=1}^{\ell} \Pr(E_x^j | V_x^k) \Pr(V_x^k | W_x^\ell) \\ &= \sum_{k=1}^{\ell} \binom{\ell-1}{k-1} x^{k-1} (1-x)^{\ell-k} p_{j|k}(x). \end{aligned}$$

Let \mathbb{I}_ℓ be the indicator that the ℓ -th best item is selected. Since the probability density function of each arrival time is uniform in $[0, 1]$, the payoff of the

algorithm is

$$\begin{aligned}
P(\mathcal{A}) &= \mathbf{E} \left[\sum_{\ell=1}^K \mathbb{I}_\ell \right] = \sum_{\ell=1}^K \mathbf{E}[\mathbb{I}_\ell] = \sum_{\ell=1}^K \Pr(\ell\text{-th best item selected}) \\
&= \sum_{\ell=1}^K \sum_{j=1}^J \int_0^1 1 \cdot \Pr(E_x^j | W_x^\ell) dx \\
&= \sum_{j=1}^J \int_0^1 \sum_{\ell=1}^K \sum_{k=1}^\ell \binom{\ell-1}{k-1} x^{k-1} (1-x)^{\ell-k} p_{j|k}(x) dx \\
&= \sum_{j=1}^J \sum_{k=1}^K \int_0^1 \left(\sum_{\ell=k}^K \binom{\ell-1}{k-1} (1-x)^{\ell-k} \right) x^{k-1} p_{j|k}(x) dx.
\end{aligned}$$

For the constraints, by Lemma 4.12 we have $p_{J|k}(x) = \Pr(E_x^J | V_x^k) \leq \Pr(\overline{Z_x^J}) = 1 - \int_0^x \frac{1}{y} \sum_{\ell=1}^K p_{J|\ell}(y) dy$, and $p_{j|k}(x) = \Pr(E_x^j | V_x^k) \leq \Pr(Z_x^{j+1} \wedge \overline{Z_x^j}) = \Pr(Z_x^{j+1}) - \Pr(Z_x^j) = \int_0^x \frac{1}{y} \sum_{\ell=1}^K [p_{(j+1)|\ell}(y) - p_{j|\ell}(y)] dy$ for $1 \leq j < J$, where the second last equality follows since Z_x^j implies Z_x^{j+1} . \square

Proposition 4.14 (Relation between $\text{LPS}_n(J, K)$ and $\text{LPS}_\infty(J, K)$). *For every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\text{LPS}_\infty \geq \text{LPS}_n - \epsilon$ for all $n \geq N$.*

Proof. Our proof strategy is as follows. We start from an optimal solution y of $\text{LPS}_n(J, K)$, with objective value $v_n(y) = \text{LPS}_n$. Our goal is to construct a feasible solution p of $\text{LPS}_\infty(J, K)$ such that the difference $v_n(y) - w(p)$ of objective values is small for sufficiently large n . The idea is to transform y into p by interpolation. However, a piecewise continuous solution directly constructed from y might not be feasible for $\text{LPS}_\infty(J, K)$; intuitively, the constructed functions at points x for small (constant) $i \in [n]$ may violate the constraints by large (constant) values. We introduce two intermediate solutions: z with respect to $\text{LPS}_n(J, K)$ and r with respect to $\text{LPS}_\infty(J, K)$. To avoid constraint violation due to small i , the solution z is obtained by shifting y by a distance of $s \leq n$ such that $z_{j|k}(i) = 0$ for all $i < s$. Then, we construct r from z by interpolation, which is not necessarily feasible to $\text{LPS}_\infty(J, K)$ but can only violate the constraints for $i \geq s$. Finally, we reduce r by some multiplicative factors and obtain a feasible solution p of $\text{LPS}_\infty(J, K)$. The parameter s is carefully selected such that the difference $v_n(y) - w(p)$ remains small.

Let y be an optimal solution of $\text{LPS}_n(J, K)$. Let s with $3K \leq s \leq n$ be an integer to be determined later. Define a solution z to $\text{LPS}_n(J, K)$ as follows: for each $1 \leq j \leq J$ and $1 \leq k \leq K$, set $z_{j|k}(i) := 0$ for $1 \leq i < s$ and $z_{j|k}(i) := y_{j|k}(i - s + 1)$ for $s \leq i \leq n$. For $1 \leq i < s$, obviously the constraints

hold for $z_{j|k}(i)$. Suppose $s \leq i \leq n$, then we have

$$\begin{aligned} z_{j|k}(i) &= y_{j|k}(i - s + 1) \leq 1 - \sum_{m=1}^{i-s} \frac{1}{m} \sum_{\ell=1}^K y_{j|\ell}(m) \\ &= 1 - \sum_{m=s}^{i-1} \frac{1}{m} \sum_{\ell=1}^K z_{j|\ell}(m) = 1 - \sum_{m=1}^{i-1} \frac{1}{m} \sum_{\ell=1}^K z_{j|\ell}(m) \end{aligned}$$

and

$$\begin{aligned} z_{j|k}(i) &= y_{j|k}(i - s + 1) \leq \sum_{m=1}^{i-s} \frac{1}{m} \sum_{\ell=1}^K [y_{(j+1)|\ell}(m) - y_{j|\ell}(m)] \\ &= \sum_{m=1}^{i-1} \frac{1}{m} \sum_{\ell=1}^K [z_{(j+1)|\ell}(m) - z_{j|\ell}(m)] \end{aligned}$$

for $1 \leq j < J$. Therefore z is a feasible solution of $\text{LPS}_n(J, K)$. Next we analyze $v_n(z)$. First observe that

$$\begin{aligned} v_n(z) &= \sum_{j=1}^J \sum_{k=1}^K \sum_{i=s}^n \frac{1}{n} \sum_{\ell=k}^K \frac{\binom{n-i}{\ell-k} \binom{i-1}{k-1}}{\binom{n-1}{\ell-1}} z_{j|k}(i) \\ &= \sum_{j=1}^J \sum_{k=1}^K \sum_{i=1}^{n-s+1} \frac{1}{n} \sum_{\ell=k}^K \frac{\binom{n-i-s+1}{\ell-k} \binom{i+s-2}{k-1}}{\binom{n-1}{\ell-1}} y_{j|k}(i). \end{aligned}$$

For $1 \leq k \leq \ell \leq K$ and $1 \leq i < n - K$ and positive integer m with $K \leq m \leq n - i$, we have

$$\begin{aligned} \frac{\binom{n-i-m}{\ell-k} \binom{i+m-1}{k-1}}{\binom{n-i}{\ell-k} \binom{i-1}{k-1}} &\geq \frac{\binom{n-i-m}{\ell-k}}{\binom{n-i}{\ell-k}} \geq \frac{(n-i-m-\ell+k)^{\ell-k}}{(n-i)^{\ell-k}} = \left(1 - \frac{m+\ell-k}{n-i}\right)^{\ell-k} \\ &\geq 1 - \frac{(\ell-k)(m+\ell-k)}{n-i} \geq 1 - \frac{K(m+K)}{n-i} \geq 1 - \frac{2Km}{n-i}. \end{aligned}$$

Then it follows that

$$\frac{\binom{n-i}{\ell-k} \binom{i-1}{k-1}}{\binom{n-1}{\ell-1}} - \frac{\binom{n-i-m}{\ell-k} \binom{i+m-1}{k-1}}{\binom{n-1}{\ell-1}} \leq \frac{2Km}{n-i} \cdot \frac{\binom{n-i}{\ell-k} \binom{i-1}{k-1}}{\binom{n-1}{\ell-1}}. \quad (4.1)$$

Since $0 \leq y_{j|k}(i) \leq 1$ for all i, j, k and $\sum_{k=1}^K \sum_{\ell=k}^K \frac{\binom{n-i}{\ell-k} \binom{i-1}{k-1}}{\binom{n-1}{\ell-1}} = K$, we have

$$\begin{aligned}
& v_n(y) - v_n(z) \\
& \leq \sum_{j=1}^J \sum_{k=1}^K \sum_{i=1}^{n-s^2} \frac{1}{n} \sum_{\ell=k}^K \left(\frac{\binom{n-i}{\ell-k} \binom{i-1}{k-1}}{\binom{n-1}{\ell-1}} - \frac{\binom{n-i-s+1}{\ell-k} \binom{i+s-2}{k-1}}{\binom{n-1}{\ell-1}} \right) y_{j|k}(i) \\
& \quad + \sum_{j=1}^J \sum_{k=1}^K \sum_{i=n-s^2+1}^n \frac{1}{n} \sum_{\ell=k}^K \frac{\binom{n-i}{\ell-k} \binom{i-1}{k-1}}{\binom{n-1}{\ell-1}} y_{j|k}(i) \\
& \leq \sum_{j=1}^J \sum_{k=1}^K \sum_{i=1}^{n-s^2} \frac{1}{n} \sum_{\ell=k}^K \frac{2Ks}{n-i} \cdot \frac{\binom{n-i}{\ell-k} \binom{i-1}{k-1}}{\binom{n-1}{\ell-1}} y_{j|k}(i) + \frac{JKs^2}{n} \\
& \leq \frac{2K}{s} v_n(y) + \frac{JKs^2}{n} \\
& \leq \frac{2JK}{s} + \frac{JKs^2}{n},
\end{aligned}$$

where the second inequality follows from (4.1), the third from $i \leq n - s^2$ and the last from $v_n(y) = \text{LPS}_n \leq J$. Then we have

$$v_n(z) \geq \text{LPS}_n - \frac{JKs^2}{n} - \frac{2JK}{s}.$$

For each j and k , define function $r_{j|k}(x)$ as follows: set $r_{j|k}(x) := 0$ when $0 \leq x \leq \frac{s}{n}$ and $r_{j|k}(x) := z_{j|k}(i)$ when $\frac{i}{n} < x \leq \frac{i+1}{n}$ for $s \leq i \leq n-1$. Then we have

$$\begin{aligned}
w(r) &= \sum_{j=1}^J \sum_{k=1}^K \int_0^1 \left(\sum_{\ell=k}^K \binom{\ell-1}{k-1} (1-x)^{\ell-k} \right) x^{k-1} r_{j|k}(x) dx \\
&= \sum_{j=1}^J \sum_{k=1}^K \sum_{m=s}^{n-1} \int_{\frac{m}{n}}^{\frac{m+1}{n}} \left(\sum_{\ell=k}^K \binom{\ell-1}{k-1} (1-x)^{\ell-k} \right) x^{k-1} z_{j|k}(m) dx \\
&\geq \sum_{j=1}^J \sum_{k=1}^K \sum_{m=s}^{n-1} \frac{1}{n} \left(\sum_{\ell=k}^K \binom{\ell-1}{k-1} \left(1 - \frac{m+1}{n}\right)^{\ell-k} \right) \left(\frac{m}{n}\right)^{k-1} z_{j|k}(m).
\end{aligned}$$

We wish to show $\frac{\binom{n-m}{\ell-k}\binom{m-1}{k-1}}{\binom{n-1}{\ell-1}} = \binom{\ell-1}{k-1}(1 - \frac{m+1}{n})^{\ell-k}(\frac{m}{n})^{k-1} + O(\frac{1}{n})$ for each k, ℓ, m .

Observe that $\frac{\ell-1}{n-\ell+1} \leq \frac{K}{n-K} \leq \frac{2K}{n}$ whenever $n \geq 2K$. We have

$$\begin{aligned}
& \frac{\binom{n-m}{\ell-k}\binom{m-1}{k-1}}{\binom{n-1}{\ell-1}} \\
&= \binom{\ell-1}{k-1} \cdot \frac{(n-m)!}{(n-m-\ell+k)!} \cdot \frac{(m-1)!}{(m-k)!} \cdot \frac{(n-\ell)!}{(n-1)!} \\
&\leq \binom{\ell-1}{k-1} (n-m)^{\ell-k} m^{k-1} \left(\frac{1}{n-\ell+1}\right)^{\ell-1} \\
&= \binom{\ell-1}{k-1} m^{k-1} \left\{ \sum_{u=0}^{\ell-k} \binom{\ell-k}{u} (n-m-1)^u \sum_{u=0}^{\ell-1} \binom{\ell-1}{u} \left(\frac{1}{n}\right)^u \left(\frac{\ell-1}{n(n-\ell+1)}\right)^{\ell-u-1} \right\} \\
&\leq \binom{\ell-1}{k-1} m^{k-1} \left\{ [(n-m+1)^{\ell-k} + K \cdot 2^K (n-m+1)^{\ell-k-1}] \right. \\
&\quad \left. \cdot \left[\left(\frac{1}{n}\right)^{\ell-1} + K \cdot 2^K \left(\frac{2K}{n}\right)^\ell\right] \right\} \\
&\leq \binom{\ell-1}{k-1} m^{k-1} \left\{ (n-m+1)^{\ell-k} \left(\frac{1}{n}\right)^{\ell-1} + \frac{c_0}{n^k \binom{\ell-1}{k-1}} \right\} \\
&\leq \binom{\ell-1}{k-1} \left(1 - \frac{m+1}{n}\right)^{\ell-k} \left(\frac{m}{n}\right)^{k-1} + \frac{c_0}{n},
\end{aligned}$$

where $c_0 = c_0(K)$ is some constant. Then we have

$$\begin{aligned}
w(r) &\geq \sum_{j=1}^J \sum_{k=1}^K \sum_{m=s}^{n-1} \frac{1}{n} \sum_{\ell=k}^K \left(\frac{\binom{n-m}{\ell-k}\binom{m-1}{k-1}}{\binom{n-1}{\ell-1}} - \frac{c_0}{n} \right) z_{j|k}(m) \\
&= v_n(z) - \sum_{j=1}^J \sum_{k=1}^K \frac{1}{n} \sum_{\ell=k}^K \frac{\binom{0}{\ell-k}\binom{n-1}{k-1}}{\binom{n-1}{\ell-1}} z_{j|k}(n) - \sum_{j=1}^J \sum_{k=1}^K \sum_{m=s}^{n-1} \frac{1}{n} \sum_{\ell=k}^K \frac{c_0}{n} z_{j|k}(m) \\
&\geq v_n(z) - \frac{JK}{n} - \frac{JK^2 c_0}{n} \\
&\geq \text{LPS}_n - \frac{JK(s^2 + c)}{n} - \frac{2JK}{s},
\end{aligned}$$

where $c := Kc_0 + 1$ is a constant.

Suppose $\frac{s}{n} < x \leq 1$. Let i be the integer such that $\frac{i}{n} < x \leq \frac{i+1}{n}$. Observe that for each j and k , we have

$$\begin{aligned}
\int_0^x \frac{r_{j|k}(y)}{y} dy &= \sum_{m=s}^{i-1} \int_{\frac{m}{n}}^{\frac{m+1}{n}} \frac{z_{j|k}(m)}{y} dy + \int_{\frac{i}{n}}^x \frac{z_{j|k}(i)}{y} dy \\
&\leq \sum_{m=s}^{i-1} \frac{z_{j|k}(m)}{m/n} \cdot \frac{1}{n} + \frac{z_{j|k}(i)}{i/n} \cdot \frac{1}{n} = \sum_{m=s}^i \frac{z_{j|k}(m)}{m},
\end{aligned}$$

and

$$\begin{aligned} \int_0^x \frac{r_{j|k}(y)}{y} dy &= \sum_{m=s}^{i-1} \int_{\frac{m}{n}}^{\frac{m+1}{n}} \frac{z_{j|k}(m)}{y} dy + \int_{\frac{i}{n}}^x \frac{z_{j|k}(i)}{y} dy \\ &\geq \sum_{m=s}^{i-1} \frac{z_{j|k}(m)}{(m+1)/n} \cdot \frac{1}{n} \geq \frac{s}{s+1} \sum_{m=s}^{i-1} \frac{z_{j|k}(m)}{m}, \end{aligned}$$

where the last inequality follows from $\frac{m}{m+1} \geq \frac{s}{s+1}$ for $m \geq s$. Then we have

$$\begin{aligned} r_{J|k}(x) = z_{J|k}(i) &\leq 1 - \sum_{m=s}^{i-1} \frac{1}{m} \sum_{\ell=1}^K z_{J|\ell}(m) \\ &\leq 1 - \int_0^x \frac{1}{y} \sum_{\ell=1}^K r_{J|\ell}(y) dy + \frac{1}{i} \sum_{\ell=1}^K z_{J|\ell}(i) \\ &= 1 - \int_0^x \frac{1}{y} \sum_{\ell=1}^K r_{J|\ell}(y) dy + \frac{1}{i} \sum_{\ell=1}^K r_{J|k}(x) \\ &\leq \frac{s+K}{s} - \int_0^x \frac{1}{y} \sum_{\ell=1}^K r_{J|\ell}(y) dy. \end{aligned} \tag{4.2}$$

For $1 \leq j < J$, we have

$$\begin{aligned} r_{j|k}(x) = z_{j|k}(i) &\leq \sum_{m=s}^{i-1} \frac{1}{m} \sum_{\ell=1}^K [z_{(j+1)|\ell}(m) - z_{j|\ell}(m)] \\ &\leq \frac{s+1}{s} \int_0^x \frac{1}{y} \sum_{\ell=1}^K r_{(j+1)|\ell}(y) dy - \int_0^x \frac{1}{y} \sum_{\ell=1}^K r_{j|\ell}(y) dy + \frac{1}{s} \sum_{\ell=1}^K r_{j|\ell}(x). \end{aligned}$$

Summing up the above inequalities over k yields

$$\sum_{\ell=1}^K r_{j|\ell}(x) \leq \frac{K(s+1)}{s-K} \int_0^x \frac{1}{y} \sum_{\ell=1}^K r_{(j+1)|\ell}(y) dy,$$

then it follows that

$$r_{j|k}(x) \leq \int_0^x \frac{1}{y} \sum_{\ell=1}^K [r_{(j+1)|\ell}(y) - r_{j|\ell}(y)] dy + \frac{K+1}{s-K} \int_0^x \frac{\sum_{\ell=1}^K r_{(j+1)|\ell}(y)}{y} dy. \tag{4.3}$$

Now we define a solution p for $\text{LPS}_\infty(J, K)$ as follows. For each j and k , set $p_{j|k}(x) := (1 - (J-j+1)\delta) \cdot r_{j|k}(x)$ for $x \in [0, 1]$, where $\delta \in (0, \frac{1}{J})$ is determined later. Note that $p_{j|k}(x) = 0$ for $0 \leq x \leq \frac{s}{n}$. Since $r_{j|k}(x) \leq 1$ for all j and k ,

and $\sum_{k=1}^K \sum_{\ell=k}^K \binom{\ell-1}{k-1} (1-x)^{\ell-k} x^{k-1} = K$, we have

$$\begin{aligned} w(p) &= w(r) - \sum_{j=1}^J \sum_{k=1}^K \int_{\frac{s}{n}}^1 \left(\sum_{\ell=k}^K \binom{\ell-1}{k-1} (1-x)^{\ell-k} \right) x^{k-1} (J-j+1) \delta r_{j|k}(x) dx \\ &\geq w(r) - J^2 K \delta \geq \text{LPS}_n - \frac{JK(s^2+c)}{n} - \frac{2JK}{s} - J^2 K \delta. \end{aligned}$$

For $0 \leq x \leq \frac{s}{n}$, obviously the constraints of $\text{LPS}_\infty(J, K)$ hold for $p_{j|k}(x)$. Suppose $\frac{s}{n} < x \leq 1$. Then from (4.2) we have

$$\begin{aligned} p_{J|k}(x) &= (1-\delta)r_{J|k}(x) \leq \frac{(1-\delta)(s+K)}{s} - \int_0^x \frac{1}{y} \sum_{\ell=1}^K p_{J|\ell}(y) dy \\ &\leq 1 - \int_0^x \frac{1}{y} \sum_{\ell=1}^K p_{J|\ell}(y) dy, \end{aligned}$$

assuming $\delta \geq \frac{K}{s}$ (and hence $\frac{(1-\delta)(s+K)}{s} \leq 1$). For $1 \leq j < J$, from (4.3) we have

$$\begin{aligned} p_{j|k}(x) &= (1 - (J-j+1)\delta) r_{j|k}(x) \\ &\leq \int_0^x \frac{1}{y} \left[\sum_{\ell=1}^K (1 - (J-j)\delta - \delta) r_{(j+1)|\ell}(y) - (1 - (J-j+1)\delta) r_{j|\ell}(y) \right] dy \\ &\quad + \frac{K+1}{s-K} \int_0^x \frac{1}{y} \sum_{\ell=1}^K r_{(j+1)|\ell}(y) dy \\ &\leq \int_0^x \frac{1}{y} \left[\sum_{\ell=1}^K p_{(j+1)|\ell}(y) - p_{j|\ell}(y) \right] dy + \left(\frac{K+1}{s-K} - \delta \right) \int_0^x \frac{1}{y} \sum_{\ell=1}^K r_{(j+1)|\ell}(y) dy \\ &\leq \int_0^x \frac{1}{y} \left[\sum_{\ell=1}^K p_{(j+1)|\ell}(y) - p_{j|\ell}(y) \right] dy, \end{aligned}$$

assuming $\delta \geq \frac{3K}{s}$ (and hence $\frac{K+1}{s-K} - \delta \leq 0$). Therefore p is a feasible solution for $\text{LPS}_\infty(J, K)$, whenever $\frac{3K}{s} \leq \delta < \frac{1}{J}$, with objective value $w(p) \geq \text{LPS}_n - \frac{JK(s^2+c)}{n} - \frac{2JK}{s} - J^2 K \delta$. Set $s := \lceil \sqrt[3]{n} \rceil$ and $\delta := \frac{3K}{s} \leq \frac{3K}{\sqrt[3]{n}}$. For every $\epsilon \in (0, 1)$, let $N := \lceil \max\{\frac{512J^6K^6}{\epsilon^3}, c^{3/2}\} \rceil$. Then for all $n \geq N$, we have $\delta \leq \frac{3\epsilon}{8J^2K} < \frac{1}{J}$

and

$$\begin{aligned} \text{LPS}_n - \text{LPS}_\infty &\leq \text{LPS}_n - w(p) \leq \frac{JK(s^2 + c)}{n} + \frac{2JK}{s} + J^2K\delta \\ &\leq \frac{3JKn^{2/3}}{n} + \frac{2JK}{\sqrt[3]{n}} + \frac{3J^2K^2}{\sqrt[3]{n}} \leq \frac{8J^2K^2}{\sqrt[3]{N}} \leq \epsilon, \end{aligned}$$

as required. \square

4.4 A Primal-Dual Method for Finding Thresholds in the $(J, 1)$ -case

We give a primal-dual procedure that finds appropriate thresholds for which the corresponding (J, K) -threshold algorithm corresponds to an optimal solution in the continuous linear program $\text{LPS}_\infty(J, K)$. To illustrate our primal-dual method, we first consider the special case $K = 1$ as described in Theorem 4.2; the general case is given in Section 4.5.

A J -threshold algorithm is a special case with $t_j := \tau_{j,1}$, and recall that any algorithm in the class \mathcal{A} corresponds to a feasible solution in the following primal continuous LP:

$$\begin{aligned} &\text{LPS}_\infty(J) \\ \max \quad &w(p) = \sum_{j=1}^J \int_0^1 p_j(x) dx \\ \text{s.t.} \quad &p_j(x) \leq \int_0^x \frac{1}{y} [p_{j+1}(y) - p_j(y)] dy, \quad \forall x \in [0, 1], 1 \leq j < J \\ &p_J(x) \leq 1 - \int_0^x \frac{p_J(y)}{y} dy, \quad \forall x \in [0, 1] \\ &p_j(x) \geq 0, \quad \forall x \in [0, 1], j \in [J]. \end{aligned}$$

The dual LP for $\text{LPS}_\infty(J)$ is as follows (see [Lev66] for details on primal-dual continuous LP):

$$\begin{aligned}
& \text{LDS}_\infty(J) \\
\min \quad & \int_0^1 q_J(x) dx \\
\text{s.t.} \quad & q_1(x) + \frac{1}{x} \int_x^1 q_1(y) dy \geq 1, \quad \forall x \in [0, 1] \\
& q_j(x) + \frac{1}{x} \int_x^1 [q_j(y) - q_{j-1}(y)] dy \geq 1, \quad \forall x \in [0, 1], 1 < j \leq J \\
& q_j(x) \geq 0, \quad \forall x \in [0, 1], j \in [J].
\end{aligned}$$

Similar to normal linear program, $\text{LPS}_\infty(J)$ and $\text{LDS}_\infty(J)$ satisfies weak duality and the complementary slackness conditions. The following result is captured by Lemma 4.17 as a special case.

Lemma 4.15 (Weak Duality and Complementary Slackness Conditions). *Let $p = (p_1, \dots, p_J)$ and $q = (q_1, \dots, q_J)$ be feasible solutions of $\text{LPS}_\infty(J)$ and $\text{LDS}_\infty(J)$, respectively. Then we have*

$$\sum_{j=1}^J \int_0^1 p_j(x) dx \leq \int_0^1 q_J(x) dx.$$

Moreover, p and q are primal and dual optimal, respectively, if they satisfy the following complementary slackness conditions $\forall x \in [0, 1]$:

$$\begin{aligned}
& \left(p_J(x) + \int_0^x \frac{1}{y} p_J(y) dy - 1 \right) q_J(x) = 0 \\
& \left(p_j(x) + \int_0^x \frac{1}{y} [p_j(y) - p_{j+1}(y)] dy \right) q_j(x) = 0, \quad 1 \leq j < J \\
& \left(q_1(x) + \frac{1}{x} \int_x^1 q_1(y) dy - 1 \right) p_1(x) = 0 \\
& \left(q_j(x) + \frac{1}{x} \int_x^1 [q_j(y) - q_{j-1}(y)] dy - 1 \right) p_j(x) = 0, \quad 1 < j \leq J.
\end{aligned}$$

Proof. Follows from Lemma 4.17. □

Primal-Dual Method. We start from a primal feasible solution p corresponding to a J -threshold algorithm, whose thresholds are to be determined. We can determine the values of the thresholds one by one in order to construct a dual q such that complementary slackness conditions hold. Then

Lemma 4.15 implies that with those found thresholds the corresponding J -threshold algorithm is optimal.

(1) Forming Feasible Primal Solution p . Suppose p is a (feasible) primal corresponding to a J -threshold algorithm with thresholds $0 < t_J \leq t_{J-1} \leq \dots \leq t_1 \leq 1$. We denote E_x^j to be the event that item at x is selected by using quota Q_j (where quotas with larger j 's are used first), V_x to be the event that x is a potential, and Z_x^j to be the event that at time x , quota Q_j has already been used (and so have the quotas with indices larger than j). For notational convenience, Z_x^{J+1} is the whole sample space, i.e., an always true event.

For each $j \in [J]$, consider the conditional probability $\Pr(E_x^j | Z_x^{j+1} \wedge \overline{Z_x^j} \wedge V_x)$ of the event that item at x is selected by using quota Q_j , given that x is a potential and quota Q_j is the next available quota at time x . By definition of threshold algorithms, this conditional probability is 0 if $x < t_j$ and is 1 if $x \geq t_j$. Hence, we have the following.

$$\Pr(E_x^j | Z_x^{j+1} \wedge \overline{Z_x^j} \wedge V_x) = \frac{\Pr(E_x^j | V_x)}{\Pr(Z_x^{j+1} \wedge \overline{Z_x^j} | V_x)} = \frac{p_j(x)}{\Pr(Z_x^{j+1} \wedge \overline{Z_x^j} | V_x)} = \begin{cases} 0, & 0 \leq x < t_j \\ 1, & t_j \leq x \leq 1, \end{cases}$$

where by independence of V_x and Z_x^j from Lemma 4.11, and by Lemma 4.12, we have:

$$\Pr(Z_x^{j+1} \wedge \overline{Z_x^j} | V_x) = \Pr(Z_x^{j+1} \wedge \overline{Z_x^j}) = \begin{cases} \int_0^x \frac{1}{y} [p_{j+1}(y) - p_j(y)] dy, & 1 \leq j < J \\ 1 - \int_0^x \frac{1}{y} p_J(y) dy, & j = J. \end{cases}$$

This implies that in the primal $\text{LPS}_\infty(J)$, the j -th constraint is equality in the range $[t_j, 1]$, but might be strict inequality in the range $[0, t_j)$ (hence forcing q_j to 0); the function p_j is zero in the range $[0, t_j)$, but might be strictly positive in the range $[t_j, 1]$ (hence forcing equality for the j -th constraint in dual).

(2) Finding Feasible Dual q to Satisfy Complementary Slackness.

To ensure that a dual solution q satisfies complementary slackness together with the above primal p , we require the following for each $j \in [J]$, where for notational convenience we write $q_0 \equiv 0$.

$$\begin{cases} q_j(x) = 0, & x \in [0, t_j]; \\ q_j(x) + \frac{1}{x} \int_x^1 [q_j(y) - q_{j-1}(y)] dy = 1, & x \in [t_j, 1]. \end{cases} \quad (4.4)$$

The astute reader might notice that we have imposed an extra condition $q_j(t_j) = 0$. This will ensure that as long as (4.4) is satisfied by some non-negative q_j , the j -th constraint in $\text{LDS}_\infty(J)$ is also automatically satisfied. For $x \in [t_j, 1]$, the constraint is clearly satisfied with equality; for $x \in [0, t_j)$,

observing that both q_j and q_{j-1} vanishes below t_j the left hand side reduces to $\frac{1}{x} \int_{t_j}^1 [q_j(y) - q_{j-1}(y)] dy$, which is larger than $q_j(t_j) + \frac{1}{t_j} \int_{t_j}^1 [q_j(y) - q_{j-1}(y)] = 1$.

As we shall see soon, in the recursive equations (4.4), the function q_1 and the threshold t_1 does not depend on J . In particular, the thresholds t_j 's and functions q_j 's found for $\text{LDS}_\infty(J)$ can be used to extend to the solution for $\text{LDS}_\infty(J+1)$. This explains the nice structure of the solution that appears in Theorem 4.2.

Objective Value. The objective value of $\text{LDS}_\infty(J)$ is $\int_0^1 q_J(y) dy = \int_{t_J}^1 q_J(y) dy$. From the second equation of (4.4) evaluating at $x = t_j$, we have the recursive definition $\int_{t_j}^1 q_j(y) dy = \int_{t_{j-1}}^1 q_{j-1}(y) dy + t_j$, which immediately implies that the objective value for $\text{LDS}_\infty(J)$ is $\sum_{j=1}^J t_j$, as stated in Theorem 4.2. Hence, it suffices to show the existence of dual functions as required in (4.4).

Lemma 4.16 (Existence of Feasible Dual Satisfying Complementary Slackness). *There is a procedure to generate an increasing sequence $\{\theta_j\}_{j \geq 1}$ of rational numbers producing $t_j := \frac{1}{e^{\theta_j}}$, and a sequence $\{q_j : [0, 1] \rightarrow \mathbb{R}^+\}_{j \geq 1}$ of non-negative functions that satisfy (4.4).*

Proof. We show the existence result by induction; our induction proof actually gives a method to generate such t_j 's and q_j 's. We explicitly describe the method in Section 4.4.1, and it can be seen that the time to generate the first J thresholds is $O(J^3)$.

For convenience, we denote $q_0(x) \equiv 0$ and set $\theta_0 := 0$ and $t_0 := 1$. Suppose for some $j \geq 1$ we have constructed the function q_{j-1} which is continuous and can be positive only in $[t_{j-1}, 1]$. We next wish to find continuous function q_j and threshold $t_j < t_{j-1}$ satisfying (4.4). If such q_j and t_j exist, then we must have the following for $x \in [t_j, 1]$:

$$\begin{aligned} q_j(x) + \frac{1}{x} \int_x^1 [q_j(y) - q_{j-1}(y)] dy &= 1 \\ xq_j(x) + \int_x^1 [q_j(y) - q_{j-1}(y)] dy &= x \\ (xq_j'(x) + q_j(x)) - q_j(x) + q_{j-1}(x) &= 1 \\ q_j'(x) &= \frac{1}{x}(1 - q_{j-1}(x)). \end{aligned}$$

Since $q_j(1) = 1$, we must have

$$q_j(x) = 1 + \ln x + \int_x^1 \frac{q_{j-1}(y)}{y} dy, \quad \forall x \in [t_j, 1] \quad (4.5)$$

To show that both q_j and t_j exist, we need a stronger induction hypothesis. Hence, we first explicitly solve for q_1 and t_1 , and state what properties we can assume. Since $q_0(x) \equiv 0$, we have $q_1(x) = 1 + \ln x$ on $[t_1, 1]$. In order to have $q_1(t_1) = 0$, we must have $t_1 := \frac{1}{e}$ and $\theta_1 := 1$. We give our induction hypothesis, which is true for $j = 1$.

Induction Hypothesis. Suppose for some $j \geq 1$, there exist functions $\{q_i\}_{i=0}^j$ and thresholds $\{t_i\}_{i=0}^j$ satisfying (4.4) such that the following holds.

1. The function q_j is non-negative and continuous.
2. There exists an increasing sequence $\{\theta_i\}_{i=0}^j$ of rational numbers that defines the thresholds $t_i := \exp(-\theta_i)$ such that q_j is 0 on $[0, t_j]$ and between successive thresholds, $q_j(x)$ is given by a polynomial in $\ln x$ with rational coefficients.
3. For $x \in (t_j, 1)$, $q_j(x) > q_{j-1}(x)$.

We next show the existence of q_{j+1} and t_{j+1} .

Finding q_{j+1} . From (4.5), $q_{j+1}(x)$ must agree on $[t_{j+1}, 1]$ with the function $q(x)$ given by $q(x) = 1 + \ln x + \int_x^1 \frac{q_j(y)}{y} dy$, which is continuous.

We first check that we can set $q_{j+1}(x) := q(x)$ for $x \in [t_j, 1]$. Since from the induction hypothesis we have $q_j > q_{j-1}$ on $(t_j, 1)$, we immediately have $\forall x \in [t_j, 1)$, $q(x) = 1 + \ln x + \int_x^1 \frac{q_j(y)}{y} dy > 1 + \ln x + \int_x^1 \frac{q_{j-1}(y)}{y} dy = q_j(x)$. In particular, we have $q(t_j) > q_j(t_j) = 0$, and also $q(x) \geq q_j(x) \geq 0$ for $x \in [t_j, 1]$.

From the induction hypothesis on q_j , we can conclude that between successive thresholds in $[t_j, 1]$, $q_{j+1}(x)$ can also be represented by a polynomial in $\ln x$ with rational coefficients. Hence, it follows that $d_j := \int_{t_j}^1 \frac{q_j(y)}{y} dy$ is rational.

Finding t_{j+1} . We next consider the behavior of q for $x \leq t_j$. Observe that in this range, $q(x) = 1 + \ln x + d_j$, which is a polynomial in $\ln x$ with rational coefficients, and strictly increasing in x . Moreover, we have $q(t_j) > 0$ and as x tends to 0, $q(x)$ tends to negative infinity. Hence, there is a unique $t_{j+1} \in (0, t_j)$ such that $q(t_{j+1}) = 0$; we set $t_{j+1} := \exp(-\theta_{j+1})$, where $\theta_{j+1} := 1 + d_j$, which is rational.

Hence, we can set $q_{j+1}(x) := q(x)$ for $x \in [t_{j+1}, 1]$ and 0 for $x \in [0, t_{j+1}]$. We can check that the conditions in the induction hypothesis hold for q_{j+1} and t_{j+1} as well. This completes the induction proof. \square

4.4.1 Fast Implementation for the $(J, 1)$ -case

For $K = 1$ with thresholds denoted by $t_j = \tau_{j,1}$ for $j \in [J]$, the proof of Lemma 4.16 gives a method to generate the dual variables q_j 's and thresholds t_j 's, which we describe in Algorithm 4.2.

Algorithm 4.2: J -Threshold Generator

Input: J

Initialization: Set $\theta_1 := 1$ and $t_1 := e^{-\theta_1}$. Let q_1 be a function defined on $[0, 1]$ such that

$$q_1(x) = \begin{cases} 0, & 0 \leq x < t_1 \\ 1 + \ln x, & t_1 \leq x \leq 1 \end{cases}$$

for $j = 1, 2, \dots, J - 1$ **do**

Set $\theta_{j+1} := 1 + \int_{t_j}^1 \frac{q_j(y)}{y} dy$ and $t_{j+1} := e^{-\theta_{j+1}} \in (0, t_j)$.

Let q_{j+1} be a function defined on $[0, 1]$ such that

$$q_{j+1}(x) = \begin{cases} 0, & 0 \leq x < t_{j+1} \\ 1 + \ln x + \int_{t_j}^1 \frac{q_j(y)}{y} dy, & t_{j+1} \leq x < t_j \\ 1 + \ln x + \int_x^1 \frac{q_j(y)}{y} dy, & t_j \leq x \leq 1. \end{cases}$$

end

Output: $\{t_j : j \in [J]\}$

Lemma 4.16 implies that Algorithm 4.2 is well defined. That is, for each $1 \leq j \leq J - 1$, the threshold t_{j+1} is guaranteed to be in the interval $(0, t_j)$. Observe that the $q_j(x)$'s are polynomials in $\ln x$ with rational coefficients and consequently the θ_j 's are all rational numbers. Hence we can describe the algorithm by maintaining the rational coefficients of the polynomials. We present in Algorithm 4.3 an efficient procedure to generate the θ_j 's which only uses additions and multiplications instead of integrations. It can be verified that generating the first J thresholds takes $O(J^3)$ time; see Figure 4.1 for several initial θ values.

4.5 A Primal-Dual Method for Finding Thresholds in the General (J, K) -case

In Section 4.4 we have shown that there exist thresholds such that the corresponding J -threshold algorithm is optimal. We apply our primal-dual method to the general (J, K) -case following a similar framework. After proving Theorem 4.1, we apply the construction procedure to the $(2, 2)$ -case as an illustration, and hence prove Theorem 4.3, in Section 4.5.1.

Algorithm 4.3: θ_j 's Generator**Input:** J **Initialization:**For integer $n \geq 1$, let 0^n be the zero vector with n coordinates.For positive integers j and k , let $c_{j,k} \in \mathbb{R}^{J+1}$ be a vector (corresponding to q_j in interval $[t_k, t_{k-1}]$ where $t_0 := 1$). Denote by $c_{j,k}(i)$ the i -th coordinate of $c_{j,k}$ for $i \in [J+1]$.Set $c_{1,1} := (1, 1, 0^{J-1})$. Set $\theta_0 := 0$ and $\theta_1 := 1$.**for** $j = 1, 2, \dots, J-1$ **do** Let $\alpha \in \mathbb{R}$ be an auxiliary variable with initial value $\alpha := 0$. **for** $k = 1, 2, \dots, j$ **do** Let $d \in \mathbb{R}^J$ be such that $d(i) = -\frac{c_{j,k}(i)}{i}$ for each $i \in [J]$. If $k > 1$, then set $\alpha := \alpha + \sum_{i=1}^{J+1} \frac{c_{j,k-1}(i)}{i} [(-\theta_{k-2})^i - (-\theta_{k-1})^i]$. Set $c_{j+1,k} := (1, 1, 0^{J-1}) + (\sum_{i=1}^{J+1} \frac{c_{j,k}(i)}{i} (-\theta_{k-1})^i, d) + (\alpha, 0^J)$. **end** Set $\alpha := \alpha + \sum_{i=1}^{J+1} \frac{c_{j,j}(i)}{i} [(-\theta_{j-1})^i - (-\theta_j)^i]$. Set $c_{j+1,j+1} := (1, 1, 0^{J-1}) + (\alpha, 0^J)$. Set $\theta_{j+1} := c_{j+1,j+1}(1)$.**end****Output:** $\{\theta_j : j \in [J]\}$ For $k \in [K]$ and $x \in [0, 1]$, define

$$\alpha_k(x) := \sum_{\ell=k}^K \binom{\ell-1}{k-1} (1-x)^{\ell-k} x^{k-1}.$$

Recall from Section 4.3 that the primal continuous linear program $\text{LPS}_\infty(J, K)$ for the (J, K) -case is:

$$\begin{aligned} & \text{LPS}_\infty(J, K) \\ \max \quad & w(p) = \sum_{j=1}^J \sum_{k=1}^K \int_0^1 \alpha_k(x) p_{j|k}(x) dx \\ \text{s.t.} \quad & p_{j|k}(x) \leq \int_0^x \frac{1}{y} \sum_{\ell=1}^K [p_{(j+1)|\ell}(y) - p_{j|\ell}(y)] dy, \\ & \forall x \in [0, 1], k \in [K], 1 \leq j < J \\ & p_{J|k}(x) \leq 1 - \int_0^x \frac{1}{y} \sum_{\ell=1}^K p_{J|\ell}(y) dy, \quad \forall x \in [0, 1], k \in [K] \\ & p_{j|k}(x) \geq 0, \quad \forall x \in [0, 1], k \in [K], j \in [J]. \end{aligned}$$

We derive the dual $\text{LDS}_\infty(J, K)$ as follows:

$$\begin{aligned}
& \text{LDS}_\infty(J, K) \\
\min \quad & \sum_{k=1}^K \int_0^1 q_{J|k}(x) dx \\
\text{s.t.} \quad & q_{1|k}(x) + \frac{1}{x} \int_x^1 \sum_{\ell=1}^K q_{1|\ell}(y) dy \geq \alpha_k(x), \quad \forall x \in [0, 1], k \in [K] \\
& q_{j|k}(x) + \frac{1}{x} \int_x^1 \sum_{\ell=1}^K [q_{j|\ell}(y) - q_{(j-1)|\ell}(y)] dy \geq \alpha_k(x), \\
& \quad \forall x \in [0, 1], k \in [K], 1 < j \leq J \\
& q_{j|k}(x) \geq 0, \quad \forall x \in [0, 1], k \in [K], j \in [J].
\end{aligned}$$

For $\text{LPS}_\infty(J, K)$ and $\text{LDS}_\infty(J, K)$, we say a constraint is the (j, k) -th constraint if $p_{j|k}$ or $q_{j|k}$ is the concerned function in the constraint. For instance, the (j, k) -th constraint in the dual with $1 < j \leq J$ is $q_{j|k}(x) + \frac{1}{x} \int_x^1 \sum_{\ell=1}^K [q_{(j)|\ell}(y) - q_{(j-1)|\ell}(y)] dy \geq \alpha_k(x)$, $\forall x \in [0, 1]$. We still have weak duality and the following complementary slackness conditions.

Lemma 4.17 (Weak Duality and Complementary Slackness Conditions). *Let $p = (p_{j|k})_{j \in [J], k \in [K]}$ and $q = (q_{j|k})_{j \in [J], k \in [K]}$ be feasible solutions to $\text{LPS}_\infty(J, K)$ and $\text{LDS}_\infty(J, K)$, respectively. Then we have*

$$\sum_{j=1}^J \sum_{k=1}^K \int_0^1 \alpha_k(x) p_{j|k}(x) dx \leq \sum_{k=1}^K \int_0^1 q_{J|k}(x) dx.$$

Moreover, p and q are primal and dual optimal, respectively, if they satisfy the following complementary slackness conditions $\forall x \in [0, 1], k \in [K]$:

$$\begin{aligned}
& \left(p_{j|k}(x) + \int_0^x \frac{1}{y} \sum_{\ell=1}^K [p_{j|\ell}(y) - p_{(j+1)|\ell}(y)] dy \right) q_{j|k}(x) = 0, \quad 1 \leq j < J \\
& \left(p_{J|k}(x) + \int_0^x \frac{1}{y} \sum_{\ell=1}^K p_{J|\ell}(y) dy - 1 \right) q_{J|k}(x) = 0 \\
& \left(q_{j|k}(x) + \frac{1}{x} \int_x^1 \sum_{\ell=1}^K [q_{j|\ell}(y) - q_{(j-1)|\ell}(y)] dy - \alpha_k(x) \right) p_{j|k}(x) = 0, \quad 1 < j \leq J \\
& \left(q_{1|k}(x) + \frac{1}{x} \int_x^1 \sum_{\ell=1}^K q_{1|\ell}(y) dy - \alpha_k(x) \right) p_{1|k}(x) = 0.
\end{aligned}$$

Proof. In the primal objective $\sum_{j=1}^J \sum_{k=1}^K \int_0^1 \alpha_k(x) p_{j|k}(x) dx$, for each j and k , we substitute the coefficient $\alpha_k(x)$ of the function $p_{j|k}(x)$ according to the (j, k) -th constraint in the dual, and obtain

$$\begin{aligned}
& \sum_{j=1}^J \sum_{k=1}^K \int_0^1 \alpha_k(x) p_{j|k}(x) dx \\
&= \sum_{k=1}^K \left(\int_0^1 \alpha_k(x) p_{1|k}(x) dx + \sum_{j=2}^J \int_0^1 \alpha_k(x) p_{j|k}(x) dx \right) \\
&\leq \sum_{k=1}^K \left(\int_0^1 \left(q_{1|k}(x) + \frac{1}{x} \int_x^1 \sum_{\ell=1}^K q_{1|\ell}(y) dy \right) p_{1|k}(x) dx \right. \\
&\quad \left. + \sum_{k=1}^K \sum_{j=2}^J \int_0^1 \left(q_{j|k}(x) + \frac{1}{x} \int_x^1 \sum_{\ell=1}^K (q_{j|\ell}(y) - q_{(j-1)|\ell}(y)) dy \right) p_{j|k}(x) dx \right) \\
&= \sum_{k=1}^K \sum_{j=1}^J \int_0^1 q_{j|k}(x) p_{j|k}(x) dx + \sum_{k=1}^K \sum_{\ell=1}^K \left(\int_0^1 \frac{1}{x} \int_x^1 q_{1|\ell}(y) p_{1|k}(x) dy dx \right. \\
&\quad \left. + \sum_{j=2}^J \int_0^1 \frac{1}{x} \int_x^1 (q_{j|\ell}(y) - q_{(j-1)|\ell}(y)) p_{j|k}(x) dy dx \right) \\
&= \sum_{k=1}^K \sum_{j=1}^J \int_0^1 q_{j|k}(x) p_{j|k}(x) dx + \sum_{k=1}^K \sum_{\ell=1}^K \left(\int_0^1 \frac{1}{x} \int_x^1 q_{1|k}(y) p_{1|\ell}(x) dy dx \right. \\
&\quad \left. + \sum_{j=2}^J \int_0^1 \frac{1}{x} \int_x^1 (q_{j|k}(y) - q_{(j-1)|k}(y)) p_{j|\ell}(x) dy dx \right), \tag{4.6}
\end{aligned}$$

where the inequality follows from the (j, k) -th constraint in the dual for each j and k , and the last equality from simply swapping the subscripts k and ℓ in the second group of summations. Using Tonelli's Theorem on non-negative measurable function g : $\int_0^1 \int_0^\theta g(\theta, \lambda) d\lambda d\theta = \int_0^1 \int_\theta^1 g(\lambda, \theta) d\lambda d\theta$, we change the order of integration for each j, k and ℓ :

$$\begin{aligned}
\int_0^1 \frac{1}{x} \int_x^1 q_{j|k}(y) p_{j|\ell}(x) dy dx &= \int_0^1 \int_0^y \frac{1}{x} q_{j|k}(y) p_{j|\ell}(x) dx dy \\
&= \int_0^1 q_{j|k}(x) \int_0^x \frac{1}{y} p_{j|\ell}(y) dy dx.
\end{aligned}$$

Then from equation (4.6) we have

$$\begin{aligned}
& \sum_{k=1}^K \sum_{j=1}^J \int_0^1 q_{j|k}(x) p_{j|k}(x) dx + \sum_{k=1}^K \sum_{\ell=1}^K \left(\int_0^1 \frac{1}{x} \int_x^1 q_{1|k}(y) p_{1|\ell}(x) dy dx \right. \\
& \quad \left. + \sum_{j=2}^J \int_0^1 \frac{1}{x} \int_x^1 (q_{j|k}(y) - q_{(j-1)|k}(y)) p_{j|\ell}(x) dy dx \right) \\
&= \sum_{k=1}^K \sum_{j=1}^J \int_0^1 q_{j|k}(x) p_{j|k}(x) dx + \sum_{k=1}^K \sum_{\ell=1}^K \left(\int_0^1 q_{1|k}(x) \int_0^x \frac{1}{y} p_{1|\ell}(y) dy dx \right. \\
& \quad \left. + \sum_{j=2}^J \int_0^1 (q_{j|k}(x) - q_{(j-1)|k}(x)) \int_0^x \frac{1}{y} p_{j|\ell}(y) dy dx \right) \\
&= \sum_{k=1}^K \sum_{j=1}^J \int_0^1 q_{j|k}(x) p_{j|k}(x) dx + \sum_{k=1}^K \sum_{\ell=1}^K \left(\int_0^1 q_{J|k}(x) \int_0^x \frac{1}{y} p_{J|\ell}(y) dy dx \right. \\
& \quad \left. - \sum_{j=1}^{J-1} \int_0^1 q_{j|k}(x) \int_0^x \frac{1}{y} (p_{(j+1)|\ell}(y) - p_{j|\ell}(y)) dy dx \right) \\
&= \sum_{k=1}^K \sum_{j=1}^J \int_0^1 q_{j|k}(x) p_{j|k}(x) dx + \sum_{k=1}^K \left(\int_0^1 q_{J|k}(x) \int_0^x \frac{1}{y} \sum_{\ell=1}^K p_{J|\ell}(y) dy dx \right. \\
& \quad \left. - \sum_{j=1}^{J-1} \int_0^1 q_{j|k}(x) \int_0^x \frac{1}{y} \sum_{\ell=1}^K (p_{(j+1)|\ell}(y) - p_{j|\ell}(y)) dy dx \right). \tag{4.7}
\end{aligned}$$

We now substitute $\int_0^x \frac{1}{y} \sum_{\ell=1}^K p_{J|\ell}(y) dy$ and $\int_0^x \frac{1}{y} \sum_{\ell=1}^K (p_{(j+1)|\ell}(y) - p_{j|\ell}(y)) dy$ according to the (j, k) -th constraint in the primal for each j and k . Then from equation (4.7) we get

$$\begin{aligned}
& \sum_{k=1}^K \sum_{j=1}^J \int_0^1 q_{j|k}(x) p_{j|k}(x) dx + \sum_{k=1}^K \left(\int_0^1 q_{J|k}(x) \int_0^x \frac{1}{y} \sum_{\ell=1}^K p_{J|\ell}(y) dy dx \right. \\
& \quad \left. - \sum_{j=1}^{J-1} \int_0^1 q_{j|k}(x) \int_0^x \frac{1}{y} \sum_{\ell=1}^K (p_{(j+1)|\ell}(y) - p_{j|\ell}(y)) dy dx \right) \\
&\leq \sum_{k=1}^K \sum_{j=1}^J \int_0^1 q_{j|k}(x) p_{j|k}(x) dx + \sum_{k=1}^K \left(\int_0^1 q_{J|k}(x) (1 - p_{J|k}(x)) dx \right. \\
& \quad \left. - \sum_{j=1}^{J-1} \int_0^1 q_{j|k}(x) p_{j|k}(x) dx \right) \\
&= \sum_{k=1}^K \int_0^1 q_{J|k}(x). \tag{4.8}
\end{aligned}$$

Combining equations (4.6), (4.7) and (4.8) we obtain the weak duality

$$\sum_{j=1}^J \sum_{k=1}^K \int_0^1 \alpha_k(x) p_{j|k}(x) dx \leq \sum_{k=1}^K \int_0^1 q_{J|k}(x) dx. \quad (4.9)$$

Moreover, if p and q satisfy the complementary slackness conditions, then both the inequalities in equations (4.6) and (4.8) are tight. This implies that equation (4.9) holds with equality; hence p and q are primal and dual optimal respectively. \square

Primal-Dual Method. We start from a primal feasible solution p corresponding to a (J, K) -threshold algorithm, whose thresholds are to be determined. We can determine the values of the thresholds one by one in order to construct a dual q such that complementary slackness conditions hold. Then Lemma 4.17 implies that with those found thresholds the corresponding (J, K) -threshold algorithm is optimal.

(1) Forming Feasible Primal Solution p . Suppose p is a (feasible) primal correspond to a (J, K) -threshold algorithm with JK thresholds $\tau_{j,k}$ such that $0 < \tau_{J,k} \leq \tau_{J-1,k} \leq \dots \leq \tau_{1,k} \leq 1$ for $k \in [K]$ and $0 < \tau_{j,1} \leq \tau_{j,2} \leq \dots \leq \tau_{j,K} \leq 1$ for $j \in [J]$. Suppose E_x^j is the event that the item at x is selected by using quota Q_j (where quotas with larger j 's are used first), V_x^k is the event that x is a k -potential, and Z_x^j is the event that at time x , quota Q_j has already been used (and so have the quotas with indices larger than j). For notational convenience, Z_x^{J+1} is the whole sample space, i.e., an always true event.

For each $j \in [J]$, consider the conditional probability $\Pr(E_x^j | Z_x^{j+1} \wedge \overline{Z_x^j} \wedge V_x^k)$ of the event that item at x is selected by using quota Q_j , given that x is a k -potential and quota Q_j is the next available quota at time x . By definition of threshold algorithms, this conditional probability is 0 if $x < \tau_{j,k}$ and is 1 if $x \geq \tau_{j,k}$. Hence, we have the following.

$$\Pr(E_x^j | Z_x^{j+1} \wedge \overline{Z_x^j} \wedge V_x^k) = \frac{\Pr(E_x^j | V_x^k)}{\Pr(Z_x^{j+1} \wedge \overline{Z_x^j} | V_x^k)} = \frac{p_{j|k}(x)}{\Pr(Z_x^{j+1} \wedge \overline{Z_x^j} | V_x^k)} = \begin{cases} 0, & 0 \leq x < \tau_{j,k} \\ 1, & \tau_{j,k} \leq x \leq 1, \end{cases}$$

where from independence of V_x^k and Z_x^j (Lemma 4.11), and Lemma 4.12, we have:

$$\Pr(Z_x^{j+1} \wedge \overline{Z_x^j} | V_x^k) = \begin{cases} \int_0^x \frac{1}{y} \sum_{\ell=1}^K [p_{(j+1)|\ell}(y) - p_{j|\ell}(y)] dy, & 1 \leq j < J \\ 1 - \int_0^x \frac{1}{y} \sum_{\ell=1}^K p_{J|\ell}(y) dy, & j = J. \end{cases}$$

This implies that in the primal $\text{LPS}_\infty(J, K)$, the (j, k) -th constraint is equality in the range $[\tau_{j,k}, 1]$, but might be strict inequality in the range $[0, \tau_{j,k})$ (hence

forcing $q_{j|k}$ to be 0); the function $p_{j|k}$ is zero in the range $[0, \tau_{j,k})$, but might be strictly positive in the range $[\tau_{j,k}, 1]$ (hence forcing equality for the (j, k) -th constraint in dual).

(2) Finding Feasible Dual q to Satisfy Complementary Slackness. To ensure that a dual solution q satisfies complementary slackness together with the above primal p . We require the following for each $j \in [J]$ and $k \in [K]$, where for notational convenience we write $q_{0|k} \equiv 0$ for all $k \in [K]$.

$$\begin{cases} q_{j|k}(x) = 0, & x \in [0, \tau_{j,k}); \\ q_{j|k}(x) + \frac{1}{x} \int_x^1 \sum_{\ell=1}^K [q_{j|\ell}(y) - q_{(j-1)|\ell}(y)] dy = \alpha_k(x), & x \in [\tau_{j,k}, 1]. \end{cases} \quad (4.10)$$

Here the extra condition $q_{j|k}(\tau_{j,k}) = 0$ ensures that as long as (4.10) is satisfied by some non-negative $q_{j|k}$, the (j, k) -th constraint in $\text{LDS}_\infty(J)$ is also automatically satisfied. Proof for this indication is not straightforward and requires stronger conditions for the dual functions, which we provide along the way we prove Theorem 4.1. From the recursive equations (4.10), the thresholds $\tau_{j,k}$'s and functions $q_{j|k}$'s found for $\text{LDS}_\infty(J, K)$ can be used to extend to the solution for $\text{LDS}_\infty(J+1, K)$.

Objective Value. The objective value of $\text{LDS}_\infty(J, K)$ is $\int_0^1 \sum_{k=1}^K q_{J|k}(x) dx = \int_{\tau_{J,1}}^1 \sum_{k=1}^K q_{J|k}(x) dx$. From equations (4.10) we have for $j \in [J]$,

$$\int_{\tau_{j,1}}^1 \sum_{k=1}^K q_{j|k}(x) dx - \int_{\tau_{j-1,1}}^1 \sum_{k=1}^K q_{(j-1)|k}(x) dx = \tau_{j,1} \alpha_1(\tau_{j,1}),$$

where $\tau_{0,1} = 1$. This together with $\alpha_1(x) = \frac{1-(1-x)^K}{x}$ implies that the objective value of $\text{LDS}_\infty(J, K)$ is

$$\int_{\tau_{J,1}}^1 \sum_{k=1}^K q_{J|k}(x) dx = \sum_{j=1}^J \tau_{j,1} \alpha_1(\tau_{j,1}) = J - \sum_{j=1}^J (1 - \tau_{j,1})^K.$$

To prove Theorem 4.1, it suffices to show the existence of dual functions as required in (4.10). Before showing this result we first give two useful observations.

Lemma 4.18. *Let $b > 0$ and c be real numbers, N be a positive integer, and $g(x)$ and $\gamma(x)$ be two functions of x continuous in $(0, b]$. Then, the equation $f(x) + \frac{N}{x} \int_x^b [f(y) - g(y)] dy + \frac{c}{x} = \gamma(x)$ with respect to f has a continuous*

solution in $(0, b]$, which can be expressed as

$$f(x) = x^{N-1} \left[\frac{b\gamma(b) - c}{b^N} - \int_x^b \frac{(y\gamma(y))'}{y^N} dy + N \int_x^b \frac{g(y)}{y^N} dy \right].$$

In particular, if we replace c with \hat{c} and g with \hat{g} such that $\hat{c} < c$ and $\hat{g} > g$, then the resulting solution \hat{f} satisfies $\hat{f} > f$.

Proof. Substituting x with b into the equation we get $f(b) = \gamma(b) - \frac{c}{b}$. Taking derivatives on both sides of $xf(x) + N \int_x^b [f(y) - g(y)] dy + c = x\gamma(x)$ we get

$$xf'(x) - (N-1)f(x) + Ng(x) = (x\gamma(x))'.$$

It suffices to show there exists a continuous $f(x)$ satisfying the above equation with condition $f(b) = \gamma(b) - \frac{c}{b}$. The above equation is equivalent to $\left(\frac{f(x)}{x^{N-1}}\right)' = \frac{(x\gamma(x))' - Ng(x)}{x^N}$, which gives $f(x) = x^{N-1}[c_0 - \int_x^b \frac{(y\gamma(y))' - Ng(y)}{y^N} dy]$ for some constant c_0 . Using the initial condition we get $c_0 = \frac{b\gamma(b) - c}{b^N}$ and hence a continuous function $f(x) = x^{N-1}[\frac{b\gamma(b) - c}{b^N} - \int_x^b \frac{(y\gamma(y))'}{y^N} dy + N \int_x^b \frac{g(y)}{y^N} dy]$. \square

Lemma 4.19. *Let $K > 1$ be an integer. For $k \in [K]$ and $x \in [0, 1]$, define $\alpha_k(x) := x^{k-1} \sum_{\ell=k}^K \binom{\ell-1}{k-1} (1-x)^{\ell-k}$. Then for all $x \in (0, 1)$ we have the following.*

- (a) $(x\alpha_k(x))' > 0$ for $1 \leq k \leq K$;
- (b) $\alpha_k(x) > \alpha_{k+1}(x)$ for $1 \leq k < K$.

Proof. Observe that $\alpha_K(x) = x^{K-1}$. Then $(x\alpha_K(x))' = Kx^{K-1} > 0$. In what follows, we first show that $(x\alpha_k(x))' > 0$ if and only if $\alpha_k(x) > \alpha_{k+1}(x)$ for $1 \leq k < K$. Then we prove $\alpha_k(x) > \alpha_{k+1}(x)$ by induction starting with $k = K-1$.

Let $1 \leq k < K$. Define $\beta_k(x) := \frac{\alpha_k(x)}{x^{k-1}}$; note $\beta_K(x) = 1$. We show $(x^k \beta_k(x))' > 0$ if and only if $\beta_k(x) > x\beta_{k+1}(x)$. By definition we have

$$\begin{aligned} \beta'_k(x) &= \left(\sum_{\ell=k+1}^K \binom{\ell-1}{k-1} (1-x)^{\ell-k} \right)' = - \sum_{\ell=k+1}^K \binom{\ell-1}{k-1} (\ell-k) (1-x)^{\ell-k-1} \\ &= -k \sum_{\ell=k+1}^K \binom{\ell-1}{k} (1-x)^{\ell-k-1} = -k\beta_{k+1}(x). \end{aligned}$$

It follows that

$$(x^k \beta_k(x))' > 0 \iff k\beta_k(x) + x\beta'_k(x) > 0 \iff \beta_k(x) > x\beta_{k+1}(x).$$

Next we prove $\beta_k(x) > x\beta_{k+1}(x)$ by backward induction starting at $k = K - 1$. Note that we have $\beta_{K-1}(x) = 1 + (K - 1)(1 - x) > x\beta_K(x)$. Suppose $1 \leq k < K - 1$ and the inequality holds for $k + 1$, i.e., $\beta_{k+1}(x) - x\beta_{k+2}(x) > 0$.

Define $\lambda_k(x) := \beta_k(x) - x\beta_{k+1}(x)$. Note that $\beta_k(1) = \alpha_k(1) = 1$ and hence $\lambda_k(1) = 0$ for all $1 \leq k < K$. Moreover, we have

$$\begin{aligned}\lambda'_k(x) &= \beta'_k(x) - \beta_{k+1}(x) - x\beta'_{k+1}(x) \\ &= -k\beta_{k+1}(x) - \beta_{k+1}(x) + (k+1)x\beta_{k+2}(x) \\ &= -(k+1)(\beta_{k+1}(x) - x\beta_{k+2}(x)) < 0,\end{aligned}$$

where the last inequality follows from the induction hypothesis. Therefore, we have $\lambda_k(x) > \lambda_k(1) = 0$, which implies $\beta_k(x) > x\beta_{k+1}(x)$. \square

Lemma 4.20 (Existence of Feasible Dual Satisfying Complementary Slackness). *There is a procedure to find appropriate thresholds $(\tau_{j,k})_{j \in [J], k \in [K]}$ and a collection $(q_{j|k})_{j \in [J], k \in [K]}$ of non-negative functions that satisfy (4.10).*

Proof. We show the result by induction; our induction proof gives a method to generate the thresholds $\tau_{j,k}$'s and the functions $q_{j|k}$'s. For convenience we denote $q_{0|k}(x) \equiv 0$, and set $\tau_{0,k} := 1$ and $\tau_{j,K+1} := 1$ for all $k \in [K+1]$ and $j \in [J]$. Also, define $r_{j|k}(x) := \sum_{\ell=1}^k q_{j|\ell}(x)$ and $\gamma_k(x) = \sum_{\ell=1}^k \alpha_\ell(x)$ for all $j \in [J]$ and $k \in [K]$. Observe that $\gamma_K(x) \equiv K$ and $\gamma_k(0) = K$ for $k \in [K]$. The induction process is over $j \in [J]$. For each j , since each constraint involves the functions $q_{j|k}$ for all $k \in [K]$, we do not find $q_{j|k+1}$ on the whole interval $[0, 1]$ before going to $q_{j|k}$; instead, we consider the intervals $[\tau_{j,k}, \tau_{j,k+1}]$ one by one and study the behavior of all functions $(q_{j|\ell})_{\ell \in [K]}$ within each interval.

Base Case ($j = 1$). First consider the base case with $j = 1$. To find thresholds $\tau_{1,k}$'s and non-negative functions $q_{1,k}$'s for $k \in [K]$ satisfying (4.10), we use another induction procedure on k . Suppose $k = K$ and $x \in [\tau_{1,K}, 1]$. Summing up the equalities $q_{1|k}(x) + \frac{1}{x} \int_x^1 \sum_{\ell=1}^K q_{(j)|\ell}(y) dy = \alpha_k(x)$ over k we get $r_{1|K}(x) + \frac{K}{x} \int_x^1 r_{1|K}(y) dy = K$. By Lemma 4.18 we have $r_{1|K}(x) = \frac{K^2}{K-1} x^{K-1} - \frac{K}{K-1}$. Then it follows that $q_{1|K}(x) = \alpha_K(x) + \frac{K}{K-1} x^{K-1} - \frac{K}{K-1}$. From the equation $q_{1|K}(x) + \frac{1}{x} \int_x^1 r_{1|K}(y) dy = \alpha_K(x)$ and Lemma 4.19 we have $r_{1|K}(x) \geq Kq_{1|K}(x)$ and hence $q'_{1|K}(x) = \frac{(x\alpha_K(x))' + r_{1|K}(x) - q_{1|K}(x)}{x} \geq 0$. Recall $\alpha_K(x) = x^{K-1}$. Setting $q_{1|K}(\tau_{1,K}) = 0$ yields $\tau_{1,K} = \sqrt[K-1]{\frac{K}{2K-1}}$. Then $q_{1|K}(x) \geq 0$ for $x \in [\tau_{1,K}, 1]$. By Lemma 4.19(b) we have $q_{1|k}(x) > q_{1|K}(x) \geq 0$ for $1 \leq k \leq K - 1$ and $x \in [\tau_{1,K}, 1]$.

Suppose for some $k \leq K - 1$ we have found $\tau_{1,k+1} < \tau_{1,k+2}$ and constructed continuous $q_{1|\ell}$ for all $\ell \in [K]$ in $[\tau_{1,k+1}, 1]$, where $q_{1|\ell}$ can be positive only in $[\tau_{1,\ell}, 1]$ for $\ell > k$ and $q_{1|\ell}(\tau_{1,k+1}) > q_{1|\ell+1}(\tau_{1,k+1})$ for $\ell < k + 1$. Consider the

case $x < \tau_{1,k+1}$. Set $d := \int_{\tau_{1,k+1}}^1 r_{1|K}(x)dx$. Then by $q_{1|k+1}(\tau_{1,k+1}) + \frac{d}{\tau_{1,k+1}} = \alpha_{k+1}(\tau_{1,k+1})$ and $q_{1|k+1}(\tau_{1,k+1}) = 0$ we get $d = \tau_{1,k+1}\alpha_{k+1}(\tau_{1,k+1})$. If there exist $\tau_{1,k} < \tau_{1,k+1}$ and $q_{1|\ell}$ for $\ell \in [k]$ that satisfy (4.10), then we must have the following:

$$\begin{aligned} q_{1|\ell}(x) + \frac{1}{x} \int_x^{\tau_{1,k+1}} r_{1|\ell}(y)dy + \frac{d}{x} &= \alpha_\ell(x), \quad \ell \in [k] \\ r_{1|k}(x) + \frac{k}{x} \int_x^{\tau_{1,k+1}} r_{1|k}(y)dy + \frac{kd}{x} &= \gamma_k(x). \end{aligned}$$

By Lemma 4.18 and the above equations we must have

$$r_{1|k}(x) = x^{k-1} \left[\frac{\gamma_k(\tau_{1,k+1}) - k\alpha_{k+1}(\tau_{1,k+1})}{\tau_{1,k+1}^{k-1}} - \int_x^{\tau_{1,k+1}} \frac{(y\gamma_k(y))'}{y^k} dy \right] \quad (4.11)$$

$$q_{1|\ell}(x) = \frac{r_{1|k}(x) - \gamma_k(x)}{k} + \alpha_\ell(x), \quad \ell \in [k]. \quad (4.12)$$

From (4.11) and (4.12) the function $q_{1,k}$ must agree on $[\tau_{1,k}, \tau_{1,k+1}]$ with $q(x)$, where $q(x)$ is given by $q(x) = \frac{r(x) - \gamma_k(x)}{k} + \alpha_k(x)$ and $r(x)$ is given by $r(x) = x^{k-1} \left[\frac{\gamma_k(\tau_{1,k+1}) - k\alpha_{k+1}(\tau_{1,k+1})}{\tau_{1,k+1}^{k-1}} - \int_x^{\tau_{1,k+1}} \frac{(y\gamma_k(y))'}{y^k} dy \right]$. Observe that $q(\tau_{1,k+1}) + \frac{d}{\tau_{1,k+1}} = \alpha_k(\tau_{1,k+1})$ and hence $q(\tau_{1,k+1}) = q_{1|k}(\tau_{1,k+1}) > q_{1|k+1}(\tau_{1,k+1}) = 0$. On the other hand, we have $r(x) > kq(x)$ and thus $q'(x) > 0$. In (4.11), the term $(y\gamma_k(y))'$ is a positive polynomial in y and takes value K when $y = 0$; thus $r(x)$ is negative when x is close to 0. It follows that $q(x)$ is also negative when x is close to 0 and hence the equation $q(x) = 0$ has a unique solution in $(0, \tau_{1,k+1})$. Let $\tau_{1,k} \in (0, \tau_{1,k+1})$ be such that $q(\tau_{1,k}) = 0$. Then, we set $q_{1|\ell} := \frac{r(x) - \gamma_k(x)}{k} + \alpha_\ell(x)$ for $\ell \in [k]$ and $x \in [\tau_{1,k}, \tau_{1,k+1}]$. By Lemma 4.19(b) for $\ell < k$ we have $q_{1|\ell}(x) > q_{1|k}(x)$ and in particular $q_{1|\ell}(\tau_{1,k}) > 0$. It can be easily checked that the functions $q_{1|\ell}$'s are continuous in $[\tau_{1,k}, 1]$. The base case with $j = 1$ is completed.

Inductive Step (j, k) . Let $j \geq 2$ and $k \leq K$. We state the induction hypothesis; the base case we proved above corresponds to $j = 2$ and $k = K$. For each of the conditions, we also state (in parentheses) what we need to prove for the inductive step.

Induction Hypothesis. Let $j \geq 2$ and $k \leq K$. Suppose we have constructed the following thresholds and functions satisfying (4.10): (1) Thresholds $\tau_{i,\ell}$ for all $i < j$, $\ell \in [K]$ and $\tau_{j,\ell}$ for $\ell > k$, where $\tau_{i,\ell} < \tau_{i,\ell+1}$ and $\tau_{i,\ell} < \tau_{i-1,\ell}$ for appropriate i, ℓ . (2) Functions $q_{i,\ell}$ for all $i < j$, $\ell \in [K]$ and $q_{j,\ell}$ for $\ell > k$. Moreover, the following holds.

1. The functions $q_{i|\ell}$'s for all $i < j$, $\ell \in [K]$ are non-negative and continuous in $[0, 1]$. The functions $q_{j|\ell}$'s for $\ell > k$ are non-negative and continuous in

- [0, 1]. (We shall find $\tau_{j,k} < \min\{\tau_{j,k+1}, \tau_{j-1,k}\}$ and complete the definition of the function $q_{j|k}$.)
2. In interval $[\tau_{j,k+1}, 1]$, the functions $q_{j|\ell}$'s for $\ell \in [k]$ are determined and they are non-negative and continuous. Moreover, they satisfy equation (4.13). (We shall extend these functions to the range $[\tau_{j,k}, 1]$.)
 3. It holds that $r_{j|K}(x) > r_{j-1|K}(x)$ for $x \in (\tau_{j,k+1}, 1)$, and $r_{i|K}(x) > r_{i-1|K}(x)$ for $1 \leq i < j$ and $x \in (\tau_{i,1}, 1)$. (We shall show that $r_{j|K}(x) > r_{j-1|K}(x)$ for $x \in (\tau_{j,k}, 1)$.)
 4. In interval $[\tau_{j,\ell}, 1]$, where $k < \ell \leq K$, it holds that $q_{j|m}(x) > q_{j|m+1}(x)$ for $1 \leq m < \ell$. (We shall show that in the interval $[\tau_{j,k}, 1]$, for $1 \leq m < k$, $q_{j|m}(x) > q_{j|m+1}(x)$.)

Dual Feasibility. Before we prove the inductive step, we first show that the above hypothesis implies dual feasibility.

Lemma 4.21 (Dual Feasibility). *Consider dual functions and thresholds that satisfy (4.10), and in particular suppose for some $i > 0$, we have $r_{i|K}(x) > r_{i-1|K}(x)$ on $(\tau_{i,1}, 1)$ (Condition 3). Then, for all $k \in [K]$, the (i, k) -th constraint in $\text{LDS}_\infty(J, K)$ is satisfied; moreover, strict inequality holds for $x \in [0, \tau_{i,k})$.*

Proof. From (4.10), the (i, k) -th constraint is equality for $x \in [\tau_{i,k}, 1]$. Since in particular (4.10) holds at $x = \tau_{i,k}$, we have $\frac{1}{\tau_{i,k}} \int_{\tau_{i,k}}^1 [r_{i|K}(y) - r_{i-1|K}(y)] dy = \alpha_k(\tau_{i,k})$. Suppose $x \in [0, \tau_{i,k})$, then by Lemma 4.19(a) we have $\tau_{i,k} \alpha_k(\tau_{i,k}) > x \alpha_k(x)$. Observe that condition 3 implies that $r_{i|K}(x) \geq r_{i-1|K}(x)$, as the two functions are equal outside $(\tau_{i,1}, 1)$. It follows that $q_{i|k}(x) + \frac{1}{x} \int_x^1 [r_{i|K}(y) - r_{i-1|K}(y)] dy \geq \frac{1}{x} \int_{\tau_{i,k}}^1 [r_{i|K}(y) - r_{i-1|K}(y)] dy = \frac{\tau_{i,k} \alpha_k(\tau_{i,k})}{x} > \alpha_k(x)$. \square

Conditions 1 and 2. Now we wish to determine the threshold $\tau_{j,k}$ and define functions $q_{j,\ell}$'s for $\ell \in [k]$ in $[\tau_{j,k}, \tau_{j,k+1}]$. Define $d_j := \int_{\tau_{j,k+1}}^1 [r_{j|K}(y) - r_{j-1|K}(y)] dy$. If such $\tau_{j,k}$ and $q_{j,\ell}$ for $\ell \in [k]$ exist, then by (4.10) we must have the following for $\ell \in [k]$ and $x \in [\tau_{j,k}, \tau_{j,k+1}]$:

$$q_{j|\ell}(x) + \frac{1}{x} \int_x^{\tau_{j,k+1}} [r_{j|K}(y) - r_{j-1|K}(y)] dy + \frac{d_j}{x} = \alpha_\ell(x). \quad (4.13)$$

Since $q_{j|k+1}$ satisfies (4.10) at point $\tau_{j,k+1}$, we have $d_j = \tau_{j,k+1} \alpha_{k+1}(\tau_{j,k+1})$. Summing up the above equation over $\ell \in [k]$, and observing that $r_{j|K}(x) =$

$r_{j|k}(x)$ for $x \leq \tau_{j,k+1}$, we get

$$r_{j|k}(x) + \frac{k}{x} \int_x^{\tau_{j,k+1}} [r_{j|k}(y) - r_{j-1|K}(y)] dy + \frac{k\tau_{j,k+1}\alpha_{k+1}(\tau_{j,k+1})}{x} = \gamma_k(x). \quad (4.14)$$

By Lemma 4.18, we can conclude that $r_{j|k}(x)$ must agree on $[\tau_{j,k}, \tau_{j,k+1}]$ with the following function $r(x)$:

$$r(x) = x^{k-1} \left[\frac{\gamma_k(\tau_{j,k+1}) - k\alpha_{k+1}(\tau_{j,k+1})}{\tau_{j,k+1}^{k-1}} - \int_x^{\tau_{j,k+1}} \frac{(y\gamma_k(y))' - kr_{j-1|K}(y)}{y^k} dy \right]. \quad (4.15)$$

Comparing equations (4.13) and (4.14), we conclude that for $\ell \in [k]$, for $x \in [\tau_{j,k}, \tau_{j,k+1}]$,

$$q_{j|\ell}(x) = \frac{r_{j|k}(x) - \gamma_k(x)}{k} + \alpha_\ell(x). \quad (4.16)$$

In particular, $q_{j|k}$ must agree on $[\tau_{j,k}, \tau_{j,k+1}]$ with the function $q(x) = \frac{r(x) - \gamma_k(x)}{k} + \alpha_k(x)$.

Let $\hat{\tau} := \min\{\tau_{j-1,k}, \tau_{j,k+1}\}$. We wish to extend the $q_{j|\ell}$'s to $[\hat{\tau}, \tau_{j,k+1}]$ first. For $\hat{\tau} = \tau_{j,k+1} < \tau_{j-1,k}$, this is done and by induction hypothesis, we have $q(\hat{\tau}) = q_{j|k}(\tau_{j,k+1}) > q_{j|k+1}(\tau_{j,k+1}) = 0$. Next, we assume $\hat{\tau} = \tau_{j-1,k} \leq \tau_{j,k+1}$. Suppose $x \in [\tau_{j-1,k}, \tau_{j,k+1}]$. Note that $\tau_{j,k+1} < \tau_{j-1,k+1}$. Thus we have

$$r_{j-1|k}(x) + \frac{k}{x} \int_x^{\tau_{j,k+1}} [r_{j-1|k}(y) - r_{j-2|K}(y)] dy + \frac{k\tau_{j-1,k+1}\alpha_{k+1}(\tau_{j-1,k+1}) + kc}{x} = \gamma_k(x),$$

where $c = \int_{\tau_{j,k+1}}^{\tau_{j-1,k+1}} [r_{j-1|K}(y) - r_{j-2|K}(y)] dy > 0$, because by the hypothesis $r_{j-1|K}(y) > r_{j-2|K}(y)$ for $y \geq \tau_{j-1,k} > \tau_{j-1,1}$.

Comparing the above equation with (4.14) and using Lemmas 4.18 and 4.19(a) we have $r(x) > r_{j-1|k}(x)$ for all $x \in [\tau_{j-1,k}, \tau_{j,k+1}]$. Also, we have $q(x) = \frac{r(x) - \gamma_k(x)}{k} + \alpha_k(x) > \frac{r_{j-1|k}(x) - \gamma_k(x)}{k} + \alpha_k(x) = q_{j-1|k}(x) \geq 0$. Now, for $x \in [\tau_{j-1,k}, \tau_{j,k+1}]$, we set $q_{j|\ell}(x) := \frac{r(x) - \gamma_k(x)}{k} + \alpha_\ell(x)$ for $\ell \in [k]$. Then

$$r_{j|K}(x) = r_{j|k}(x) = r(x) > r_{j-1|K}(x). \quad (4.17)$$

Also $q_{j|\ell}(x) > 0$ for $\ell \in [k]$ by Lemma 4.19(b). It can be easily checked that the functions $q_{j|\ell}$'s are continuous. In particular, we also have $q(\hat{\tau}) > 0$.

Hence, we have show that for $x \in [\widehat{\tau}, \tau_{j,k+1}]$, $q(x) > 0$. We next analyze the behavior of $r(x)$ and $q(x)$ as x tends to 0. In the definition of $r(x)$, the function $r_{j-1|K}(y)$ has value 0 when $y < \tau_{j-1,1}$; the term $(y\gamma_k(y))' > 0$ is by definition a polynomial of y , which takes value K when $y = 0$. Then $\frac{(y\gamma_k(y))' - kr_{j-1|K}(y)}{y^k} > 0$ is unbounded and hence $r(x)$ is negative when x is close to 0. Since $q(x) < kr(x)$, it holds that $q(x)$ is also negative when x is close to 0. Since q is continuous in $(0, \widehat{\tau}]$, there exists $x \in (0, \widehat{\tau})$ such that $q(x) = 0$. Let $\tau_{j,k}$ be the largest value in $(0, \min\{\tau_{j,k+1}, \tau_{j-1,k}\})$ such that $q(\tau_{j,k}) = 0$. Then $q(x) > 0$ for $x \in (\tau_{j,k}, \widehat{\tau}]$.

Now, for $x \in [\tau_{j,k}, \widehat{\tau}]$, we set $q_{j|\ell}(x) := \frac{r(x) - \gamma_k(x)}{k} + \alpha_\ell(x)$ for $\ell \in [k]$. It can be easily checked that the functions $q_{j|\ell}$'s are continuous.

Condition 4. By Lemma 4.19(b), we have $\alpha_m(x) > \alpha_{m+1}(x)$ for $x \in (0, 1)$, and so we have $q_{j|m}(x) > q_{j|m+1}(x)$ for $1 \leq m < k$.

Condition 3. If $\tau_{j-1,k} \leq \tau_{j,k+1}$, then from (4.17) we already have $r_{j|K}(x) > r_{j-1|K}(x)$ for $x \in [\tau_{j-1,k}, \tau_{j,k+1}]$; we are left to show $r_{j|K}(x) > r_{j-1|K}(x)$ for $x \in (\tau_{j,k}, \widehat{\tau})$. If $\tau_{j-1,k} > \tau_{j,k+1}$, then we are left to show $r_{j|K}(x) > r_{j-1|K}(x)$ for $x \in (\tau_{j,k}, \widehat{\tau}]$. We next show that $r_{j|K}(x) = r_{j|k}(x) > r_{j-1|K}(x)$ for all $x \in \mathcal{I}$, where \mathcal{I} is defined as

$$\mathcal{I} = \begin{cases} x \in (\tau_{j,k}, \widehat{\tau}), & \tau_{j-1,k} \leq \tau_{j,k+1} \\ x \in (\tau_{j,k}, \widehat{\tau}], & \tau_{j-1,k} > \tau_{j,k+1}. \end{cases}$$

Suppose on the contrary $r_{j|K}(x) \leq r_{j-1|K}(x)$ for some $x \in \mathcal{I}$. Since $r_{j|K}$ and $r_{j-1|K}$ are continuous, there exists $z \in \mathcal{I}$ with $z \geq x$ such that $0 < r_{j|K}(z) = r_{j-1|K}(z)$. Let $m < k$ be the integer such that $\tau_{j-1,m} \leq z < \tau_{j-1,m+1}$. Then $r_{j|k}(z) = r_{j-1|m}(z)$. By Lemma 4.21 we have

$$\begin{aligned} q_{j-1|\ell}(z) + \frac{1}{z} \int_z^1 [r_{j-1|K}(y) - r_{j-2|K}(y)] dy &= \alpha_\ell(z), & 1 \leq \ell \leq m \\ q_{j-1|\ell}(z) + \frac{1}{z} \int_z^1 [r_{j-1|K}(y) - r_{j-2|K}(y)] dy &> \alpha_\ell(z), & m < \ell \leq k. \end{aligned}$$

Since at least one strict inequality holds for $\ell \in [k]$, we have

$$r_{j-1|k}(z) + \frac{k}{z} \int_z^1 [r_{j-1|K}(y) - r_{j-2|K}(y)] dy > \gamma_k(z).$$

Then, for all $l \in [m]$, we get $q_{j-1|l}(z) < \frac{r_{j-1|k}(z) - \gamma_k(z)}{k} + \alpha_\ell(z) = \frac{r_{j|k}(z) - \gamma_k(z)}{k} + \alpha_\ell(z)$, because we have $r_{j-1|k}(z) = r_{j-1|m}(z) = r_{j|k}(z)$. From (4.16), we have $q_{j-1|\ell}(z) < q_{j|\ell}(z)$ for $\ell \in [m]$. But this implies $r_{j-1|m}(z) < r_{j|m}(z) \leq r_{j|k}(z)$, a contradiction.

This completes the induction proof. \square

4.5.1 An Illustration: the $(2, 2)$ -case

As an example, we solve the $(2, 2)$ -secretary problem using the primal-dual method. In particular, we settle thresholds for an optimal $(2, 2)$ -threshold algorithm, and then calculate the corresponding primal variables and expected payoff.

Finding Optimal Thresholds. We follow the primal-dual method described above to find optimal thresholds $\tau_{j,k}$ for $j, k \in \{1, 2\}$ for the $(2, 2)$ -case. In particular, we use the equations (4.10) to set the dual $q_{j|k}$'s. Let $r_1(x) := q_{1|1}(x) + q_{1|2}(x)$ and $r_2(x) := q_{2|1}(x) + q_{2|2}(x)$. Observe that $\alpha_1(x) = 2 - x$ and $\alpha_2(x) = x$.

- Finding $\tau_{1,2}$. Note that in $[\tau_{1,2}, 1]$, $r_1(x)$ must agree with the function $r(x)$ such that

$$r(x) + \frac{2}{x} \int_x^1 r(y) dy = 2.$$

Solving the equation we get $r(x) = 4x - 2$. Also note that $q_{1,2}(x)$ in $[\tau_{1,2}, 1]$ must agree with $q(x)$ such that

$$q(x) + \frac{1}{x} \int_x^1 r(y) dy = x.$$

Solving the equation we get $q(x) = 3x - 2$. Setting $q(x) = 0$ we get $x = \frac{2}{3} \in (0, 1)$. Hence $\tau_{1,2} = \frac{2}{3}$. Also, for $x \in [\tau_{1,2}, 1]$, we have

$$\int_x^1 r_1(y) dy = 2x - 2x^2.$$

In the following calculation, we will use the substitution $\tau_{1,2} = \frac{2}{3}$ when necessary.

- Finding $\tau_{1,1}$. Note that $q_{1|2}(x) = 0$ in $[0, \tau_{1,2}]$, and $q_{1|1}(x) = r_1(x)$ in $[\tau_{1,1}, \tau_{1,2}]$ must agree with $q(x)$ such that

$$q(x) + \frac{1}{x} \int_x^{\tau_{1,2}} q(y) dy + \frac{1}{x} \int_{\tau_{1,2}}^1 r_1(y) dy = 2 - x.$$

Solving the equation we get $q(x) = 2 \ln \frac{x}{\tau_{1,2}} - 2x + 3\tau_{1,2}$. Setting $q(x) = 0$ we get $x = -W(-\frac{2}{3e}) \approx 0.346982 \in (0, \tau_{1,2})$. Hence $\tau_{1,1} = -W(-\frac{2}{3e})$.

Also, for $x \in [\tau_{1,1}, \tau_{1,2}]$, we have

$$\int_x^{\tau_{1,2}} r_1(y) dy = x^2 - 2x \ln \frac{x}{\tau_{1,2}} - \frac{4}{9}$$

- Finding $\tau_{2,2}$. For $x \in [\tau_{1,2}, 1]$, let $q(x)$ and $r(x)$ be functions such that

$$\begin{aligned} r(x) + \frac{2}{x} \int_x^1 r(y) dy - \frac{2}{x} \int_x^1 r_1(y) dy &= 2 \\ q(x) + \frac{1}{x} \int_x^1 r(y) dy - \frac{1}{x} \int_x^1 r_1(y) dy &= x. \end{aligned}$$

Solving the equations we get $r(x) = 8x - 8x \ln x - 6$ and $q(x) = 5x - 4x \ln x - 4$. Since $q(\tau_{1,2}) = -\frac{8}{3} \ln \frac{2}{3} - \frac{2}{3} \approx 0.414573 > 0$, the function $r_2(x)$ agrees with $r(x)$ in $[\tau_{1,2}, 1]$. Also, we have

$$\int_{\tau_{1,2}}^1 r_2(y) dy = \frac{4}{3} + \frac{16}{9} \ln \tau_{1,2}.$$

Now consider the case $x \in [\tau_{1,1}, \tau_{1,2}]$. Let $q(x)$ and $r(x)$ be functions such that

$$\begin{aligned} r(x) + \frac{2}{x} \int_x^{\tau_{1,2}} r(y) dy + \frac{2}{x} \int_{\tau_{1,2}}^1 r_2(y) dy - \frac{2}{x} \int_x^1 r_1(y) dy &= 2 \\ q(x) + \frac{1}{x} \int_x^{\tau_{1,2}} r(y) dy + \frac{1}{x} \int_{\tau_{1,2}}^1 r_2(y) dy - \frac{1}{x} \int_x^1 r_1(y) dy &= x. \end{aligned}$$

Solving the equations we get $r(x) = 4x \ln x + 4 \ln \frac{x}{\tau_{1,2}} - (10 + 12 \ln \tau_{1,2})x + 6$ and $q(x) = 2x \ln x + 2 \ln \frac{x}{\tau_{1,2}} - (4 + 6 \ln \tau_{1,2})x + 2$. Setting $q(x) = 0$ we get $x \approx 0.517297 \in [\tau_{1,1}, \tau_{1,2}]$. Hence $\tau_{2,2} \approx 0.517297$ is the solution of $x \ln x + \ln x - (2 + 3 \ln \frac{2}{3})x + 1 - \ln \frac{2}{3} = 0$. Then, in $[\tau_{2,2}, \tau_{1,2}]$, the functions $r_2(x)$ agree with $r(x)$. Also, we have

$$\begin{aligned} \int_{\tau_{2,2}}^{\tau_{1,2}} r_2(y) dy &= -\frac{4}{3} - \frac{16}{9} \ln \tau_{1,2} + 6\tau_{2,2}^2 \ln \tau_{1,2} - 2\tau_{2,2}^2 \ln \tau_{2,2} \\ &\quad + 6\tau_{2,2}^2 - 4\tau_{2,2} \ln \frac{\tau_{2,2}}{\tau_{1,2}} - 2\tau_{2,2}. \end{aligned}$$

- Finding $\tau_{2,1}$. Note that $q_{2|2}(x) = 0$ in $[0, \tau_{2,2}]$. For $x \in [\tau_{1,1}, \tau_{2,2}]$, let $q(x)$ be a function such that

$$q(x) + \frac{1}{x} \int_x^{\tau_{2,2}} q(y) dy + \frac{1}{x} \int_{\tau_{2,2}}^1 r_2(y) dy - \frac{1}{x} \int_x^1 r_1(y) dy = 2 - x.$$

Solving the equation we get $q(x) = -(\ln \frac{x}{\tau_{1,2}})^2 + (\ln \frac{\tau_{2,2}}{\tau_{1,2}})^2 + 2 \ln \frac{\tau_{2,2}}{\tau_{1,2}} - 6\tau_{2,2} \ln \tau_{1,2} + 2\tau_{2,2} \ln \tau_{2,2} - 6\tau_{2,2} + 4$. Since $q(\tau_{1,1}) > 0$, the function $q_{2|1}(x) = r_2(x)$ agrees with $q(x)$ in $[\tau_{1,1}, \tau_{2,2}]$. Also, we have

$$\begin{aligned} \int_{\tau_{1,1}}^{\tau_{2,2}} r_2(y) dy &= 4\tau_{2,2} \ln \frac{\tau_{2,2}}{\tau_{1,2}} - 6\tau_{2,2}^2 \ln \tau_{1,2} + 2\tau_{2,2}^2 \ln \tau_{2,2} - 6\tau_{2,2}^2 + 2\tau_{2,2} \\ &\quad - \tau_{1,1} (\ln \frac{\tau_{2,2}}{\tau_{1,2}})^2 - 2\tau_{1,1} \ln \frac{\tau_{2,2}}{\tau_{1,2}} + 6\tau_{1,1}\tau_{2,2} \ln \tau_{1,2} - 2\tau_{1,1}\tau_{2,2} \ln \tau_{2,2} \\ &\quad + 6\tau_{1,1}\tau_{2,2} - 2\tau_{1,1} + \tau_{1,1} (\ln \frac{\tau_{1,1}}{\tau_{1,2}})^2 - 2\tau_{1,1} \ln \frac{\tau_{1,1}}{\tau_{1,2}}. \end{aligned}$$

Now consider the case $x \in [0, \tau_{1,1}]$, where $r_1(x) = 0$. Let $q(x)$ be a function such that

$$q(x) + \frac{1}{x} \int_x^{\tau_{1,1}} q(y) dy + \frac{1}{x} \int_{\tau_{1,1}}^1 r_2(y) dy - \frac{1}{x} \int_{\tau_{1,1}}^1 r_1(y) dy = 2 - x.$$

Solving the equation we get $q(x) = 2 \ln x - 2x + c$, where c is a constant defined as

$$\begin{aligned} c &= - \left(\ln \frac{\tau_{1,1}}{\tau_{1,2}} \right) + (\ln \frac{\tau_{2,2}}{\tau_{1,2}})^2 + 2 \ln \frac{\tau_{2,2}}{\tau_{1,2}} - 6\tau_{2,2} \ln \tau_{1,2} \\ &\quad + 2\tau_{2,2} \ln \tau_{2,2} - 6\tau_{2,2} + 2\tau_{1,1} - 2 \ln \tau_{1,1} + 4 \\ &= -(\ln \tau_{1,1})^2 + 2 \ln \frac{2}{3} \ln \tau_{1,1} + (\ln \tau_{2,2})^2 - 2 \ln \frac{2}{3} \ln \tau_{2,2} - 2\tau_{2,2} + 4 - 2 \ln \frac{2}{3}, \end{aligned}$$

where the last equation follows from the definitions of $\tau_{1,2}$, $\tau_{1,1}$ and $\tau_{2,2}$. Setting $q(x) = 0$ we get $x = -W(-e^{-c/2}) \approx 0.227788 \in (0, \tau_{1,1})$. Hence $\tau_{2,1} = -W(-e^{-c/2})$. The function $q_{2,1}(x)$ agrees with $q(x)$ in $[\tau_{2,1}, \tau_{1,1}]$.

Calculating Primal Variables and Expected Payoff. Given thresholds $\tau_{j,k}$'s, the primal variables can be calculated using the following recursion. For $0 \leq x < \tau_{j,k}$, $p_{j|k}(x) = 0$; for $\tau_{j,k} \leq x \leq 1$,

$$p_{j|k}(x) = \begin{cases} \int_0^x \frac{1}{y} \sum_{\ell=1}^K [p_{(j+1)|\ell}(y) - p_{j|\ell}(y)] dy, & 1 \leq j < J \\ 1 - \int_0^x \frac{1}{y} \sum_{\ell=1}^K p_{J|\ell}(y) dy, & j = J. \end{cases}$$

For the (2, 2)-case, setting $J = K = 2$, we obtain the following:

$$\begin{aligned}
 p_{2|1}(x) &= \begin{cases} \frac{\tau_{2,1}}{x}, & x \in [\tau_{2,1}, \tau_{1,1}] \\ \frac{\tau_{2,1}}{x}, & x \in [\tau_{1,1}, \tau_{2,2}] \\ \frac{\tau_{2,1}\tau_{2,2}}{x^2}, & x \in [\tau_{2,2}, \tau_{1,2}] \\ \frac{\tau_{2,1}\tau_{2,2}}{x^2}, & x \in [\tau_{1,2}, 1] \end{cases} \\
 p_{1|1}(x) &= \begin{cases} \frac{\tau_{1,1}-\tau_{2,1}}{x} + \frac{\tau_{2,1}}{x} \ln \frac{x}{\tau_{1,1}}, & x \in [\tau_{1,1}, \tau_{2,2}] \\ \frac{\tau_{1,1}+\tau_{2,1}}{x} + \frac{\tau_{2,1}}{x} \ln \frac{\tau_{2,2}}{\tau_{1,1}} - \frac{2\tau_{2,1}\tau_{2,2}}{x^2}, & x \in [\tau_{2,2}, \tau_{1,2}] \\ \frac{2\tau_{2,1}\tau_{2,2}}{x^2} \ln \frac{x}{\tau_{1,2}} + \frac{\tau_{1,2}(\tau_{2,1}+\tau_{1,1})-2\tau_{2,1}\tau_{2,2}}{x^2} + \frac{\tau_{2,1}\tau_{1,2}}{x^2} \ln \frac{\tau_{2,2}}{\tau_{1,1}}, & x \in [\tau_{1,2}, 1] \end{cases} \\
 p_{2|2}(x) &= \begin{cases} \frac{\tau_{2,1}\tau_{2,2}}{x^2}, & x \in [\tau_{2,2}, \tau_{1,2}] \\ \frac{\tau_{2,1}\tau_{2,2}}{x^2}, & x \in [\tau_{1,2}, 1] \end{cases} \\
 p_{1|2}(x) &= \frac{2\tau_{2,1}\tau_{2,2}}{x^2} \ln \frac{x}{\tau_{1,2}} + \frac{\tau_{1,2}(\tau_{2,1}+\tau_{1,1})-2\tau_{2,1}\tau_{2,2}}{x^2} + \frac{\tau_{2,1}\tau_{1,2}}{x^2} \ln \frac{\tau_{2,2}}{\tau_{1,1}}, \quad x \in [\tau_{1,2}, 1]
 \end{aligned}$$

The objective function of LPS_∞ , which is the expected payoff of the threshold algorithm, can be expressed as

$$J - \sum_{j=1}^J (1 - \tau_{j,1})^K = 2 - (1 - \tau_{1,1})^2 - (1 - \tau_{2,1})^2 \approx 0.977256.$$

Bibliography

- [ABC⁺09] Yossi Azar, Benjamin E. Birnbaum, L. Elisa Celis, Nikhil R. Devanur, and Yuval Peres. Convergence of local dynamics to balanced outcomes in exchange networks. In *FOCS*, pages 293–302, 2009.
- [ADFS95] Jonathan Aronson, Martin E. Dyer, Alan M. Frieze, and Stephen Suen. Randomized greedy matching ii. *Random Struct. Algorithms*, pages 55–74, 1995.
- [ADJR10] Yossi Azar, Nikhil R. Devanur, Kamal Jain, and Yuval Rabani. Monotonicity in bargaining networks. In *SODA*, pages 817–826, 2010.
- [AGKM11] Gagan Aggarwal, Gagan Goel, Chinmay Karande, and Aranyak Mehta. Online vertex-weighted bipartite matching and single-bid budgeted allocations. In *SODA*, pages 1253–1264, 2011.
- [BB96] J. Baillon and R. E. Bruck. The rate of asymptotic regularity is $O(1/\sqrt{n})$. *Theory and applications of nonlinear operators of accretive and monotone type, Lecture Notes in Pure and Appl. Math.*, pages 51–81, 1996.
- [BBCZ11] Mohsen Bayati, Christian Borgs, Jennifer T. Chayes, and Riccardo Zecchina. Belief propagation for weighted b-matchings on arbitrary graphs and its relation to linear programs with integer solutions. *SIAM J. Discrete Math.*, pages 989–1011, 2011.
- [BHIM10] MohammadHossein Bateni, MohammadTaghi Hajiaghayi, Nicole Immorlica, and Hamid Mahini. The cooperative game theory foundations of network bargaining games. In *ICALP (1)*, pages 67–78, 2010.
- [Bin98] K. Binmore. *Game Theory and the Social Contract*, volume 2. MIT Press, 1998.
- [BJS10] Niv Buchbinder, Kamal Jain, and Mohit Singh. Secretary problems via linear programming. In *IPCO*, pages 163–176, 2010.

- [Bru84] F. Thomas Bruss. A unified approach to a class of best choice problems with an unknown number of options. *The Annals of Probability*, 12(3):pp. 882–889, 1984.
- [BSS08] Mohsen Bayati, Devavrat Shah, and Mayank Sharma. Max-product for maximum weight matching: Convergence, correctness, and LP duality. *IEEE Transactions on Information Theory*, pages 1241–1251, 2008.
- [CA01] Antoni Calv-Armengol. Bargaining power in communication networks. *Mathematical Social Sciences*, pages 69–87, 2001.
- [CC13] T.-H. Hubert Chan and Fei Chen. A primal-dual continuous lp method on the multi-choice multi-best secretary problem. *CoRR*, abs/1307.0624, 2013.
- [CCF07] Gary Charness, Margarida Corominas-Bosch, and Guillaume Fréchette. Bargaining and network structure: An experiment. *J. Economic Theory*, pages 28–65, 2007.
- [CCN12] T.-H. Hubert Chan, Fei Chen, and Li Ning. Optimizing social welfare for network bargaining games in the face of unstability, greed and spite. In *ESA*, pages 265–276, 2012.
- [CCWZ14] T.-H. Hubert Chan, Fei Chen, Xiaowei Wu, and Zhichao Zhao. Ranking on arbitrary graphs: Rematch via continuous LP with monotone and boundary condition constraints. In *SODA*, pages 1112–1122, 2014.
- [CDP10] L. Elisa Celis, Nikhil R. Devanur, and Yuval Peres. Local dynamics in bargaining networks via random-turn games. In *WINE*, pages 133–144, 2010.
- [CE78] Karen S. Cook and Richard M. Emerson. Power, equity and commitment in exchange networks. *American Sociological Review*, 43(5):pp. 721–739, 1978.
- [CEGY83] Karen S. Cook, Richard M. Emerson, Mary R. Gillmore, and Toshio Yamagishi. The distribution of power in exchange networks: Theory and experimental results. *American Journal of Sociology*, 89(2):pp. 275–305, 1983.
- [CK08] Tanmoy Chakraborty and Michael Kearns. Bargaining solutions in a social network. In *WINE*, pages 548–555, 2008.
- [CKK09] Tanmoy Chakraborty, Michael Kearns, and Sanjeev Khanna. Network bargaining: algorithms and structural results. In *EC*, pages 159–168, 2009.

- [CY92] K. S. Cook and T. Yamagishi. Power in exchange networks: A power-dependence formulation. *Social Networks*, 14:245–265, 1992.
- [DF91] Martin E. Dyer and Alan M. Frieze. Randomized greedy matching. *Random Struct. Algorithms*, pages 29–46, 1991.
- [Dri88] T.S.H. Driessen. *Cooperative Games: Solutions and Applications*. Kluwer Academic Publishers, 1988.
- [Dyn63] E. B. Dynkin. The optimum choice of the instant for stopping a Markov process. *Soviet Math. Dokl*, 4, 1963.
- [Fer89] Thomas S. Ferguson. Who solved the secretary problem? *Statistical Science*, 4(3):pp. 282–289, 1989.
- [Fre83] P. R. Freeman. The secretary problem and its extensions: A review. *International Statistical Review*, 51(2):pp. 189–206, 1983.
- [GM08] Gagan Goel and Aranyak Mehta. Online budgeted matching in random input models with applications to adwords. In *SODA*, pages 982–991, 2008.
- [GT12] Gagan Goel and Pushkar Tripathi. Matching with our eyes closed. In *FOCS*, pages 718–727, 2012.
- [GT13] Gagan Goel and Pushkar Tripathi. Matching with our eyes closed. *CoRR*, abs/1306.2988, 2013.
- [GZ66] S. Gusein-Zade. The problem of choice and the optimal stopping rule for a sequence of independent trials. *Theory of Probability & Its Applications*, 11(3):472–476, 1966.
- [IKM06] Nicole Immorlica, Robert D. Kleinberg, and Mohammad Mahdian. Secretary problems with competing employers. In *WINE*, pages 389–400, 2006.
- [Ish76] Shiro Ishikawa. Fixed points and iteration of a nonexpansive mapping in a banach space. *Proceedings of the American Mathematical Society*, 59(1):pp. 65–71, 1976.
- [Kan10] Yashodhan Kanoria. An fptas for bargaining networks with unequal bargaining powers. In *WINE*, pages 282–293, 2010.
- [KBB⁺11] Yashodhan Kanoria, Mohsen Bayati, Christian Borgs, Jennifer T. Chayes, and Andrea Montanari. Fast convergence of natural bargaining dynamics in exchange networks. In *SODA*, pages 1518–1537, 2011.

- [Kle05] Robert D. Kleinberg. A multiple-choice secretary algorithm with applications to online auctions. In *SODA*, pages 630–631, 2005.
- [KMT11] Chinmay Karande, Aranyak Mehta, and Pushkar Tripathi. Online bipartite matching with unknown distributions. In *STOC*, pages 587–596, 2011.
- [KT08] Jon M. Kleinberg and Éva Tardos. Balanced outcomes in social exchange networks. In *STOC*, pages 295–304, 2008.
- [KVV90] Richard M. Karp, Umesh V. Vazirani, and Vijay V. Vazirani. An optimal algorithm for on-line bipartite matching. In *STOC*, pages 352–358, 1990.
- [KY09] Christos Koufogiannakis and Neal E. Young. Distributed fractional packing and maximum weighted b-matching via tail-recursive duality. In *DISC*, pages 221–238, 2009.
- [Lam58] Johann Heinrich Lambert. *Acta Helvetica Physico-Mathematico-Anatomico-Botanico-Medica.*, volume v. 3 (1758). Basileae., 1758.
- [Lev66] N. Levinson. A class of continuous linear programming problems. *Journal of Mathematical Analysis and Applications*, 16:73–83, 1966.
- [Lin61] D. V. Lindley. Dynamic programming and decision theory. *Journal of the Royal Statistical Society. Series C (Applied Statistics)*, 10(1):pp. 39–51, 1961.
- [LPSP08] Zvi Lotker, Boaz Patt-Shamir, and Seth Pettie. Improved distributed approximate matching. In *SPAA*, pages 129–136, 2008.
- [MV80] Silvio Micali and Vijay V. Vazirani. An $O(\sqrt{V}E)$ algorithm for finding maximum matching in general graphs. In *FOCS*, pages 17–27, 1980.
- [MY11] Mohammad Mahdian and Qiqi Yan. Online bipartite matching with random arrivals: an approach based on strongly factor-revealing LPs. In *STOC*, pages 597–606, 2011.
- [Nas50] J. Nash. The bargaining problem. *Econometrica*, 18:155–162, 1950.
- [Nie08] Tim Nieberg. Local, distributed weighted matching on general and wireless topologies. In *DIALM-POMC*, pages 87–92, 2008.
- [PS12] Matthias Poloczek and Mario Szegedy. Randomized greedy algorithms for the maximum matching problem with new analysis. In *FOCS*, pages 708–717, 2012.

-
- [RSÜ05] Alvin E. Roth, Tayfun Sönmez, and M. Utku Ünver. Pairwise kidney exchange. *J. Economic Theory*, pages 151–188, 2005.
- [Sch04] Alexander Schrijver. *Combinatorial optimization: polyhedra and efficiency*. Springer, 2004.
- [SMW07] Sujay Sanghavi, Dmitry M. Malioutov, and Alan S. Willsky. Linear programming analysis of loopy belief propagation for weighted matching. In *NIPS*, 2007.
- [Tyn65] William F. Tyndall. A duality theorem for a class of continuous linear programming problems. *Journal of the Society for Industrial and Applied Mathematics*, 13(3):pp. 644–666, 1965.