Persistent Homotopy

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This note presents and proves the algorithm of persistent homotopy based on the well-known persistent homology method. The central idea of this algo based on the so called Hurewicz theorem.

Theorem 1 (Hurewicz)**.** *Let X be a path connected space and n a positive integer, there exists a group homomorphism*

$$h_* : \pi_n(X) \to H_n(X)$$

which is defined as follows. Let $f \in \pi_n(X)$ denotes an n-homotopy class of X and $[u] \in H_n(S^n)$ a canonical generator, the homomorphism h_* sends f to $f_*([u])$ where

$$f_* : H_n(S^n) \to H_n(X)$$

is the homology group homomorphism induced by the homotopy class $f : S^n \to X$.

Here's some more explanation and translation to our case. Let's first consider from CW-complex point of view, which may be thought of as the "smooth case". Since we are dealing with path connected spaces, up to homotopy equivalence, we can always think our space has only one 0-simplex, and all the 1-simplices are glued to the only 0-simplex by pasting two of their end points. After gluing all 1-simplices, we get a bouquet of S^1, which are the generators of the chain complex $C_1(X)$. Then, we attach all the 2-simplices, this step is the most important and makes all the difference between homology and homotopy.

Continue our CW-construction, we may consider our space is obtained by pasting standard 2-simplices (i.e, D^2) along 1-simplices (those S^1). Then, by Van-Kampen theorem, the fundamental group $\pi_1(X)$ is generated by those 2-simplices and the relation is given by how these 2-simplices are glued along 1-simplices. For example, consider a 2-torus, which can be thought of an octagon gluing boundaries according to the rule $aba^{-1}b^{-1}cdc^{-1}d^{-1}$. Imagine the octagon is covered by two subspaces: A an open disk contained in the octagon and B slightly thickened boundary so that $A \cap B$ is homotopically S^1. Since $\pi_1(S^1) = \mathbb{Z}$ and A is contractible, Van-Kampen theorem says the fundamental group of 2-torus is

$$\pi_1(T_2) = \mathbb{Z}[a,b,c,d]/ < aba^{-1}b^{-1}cdc^{-1}d^{-1} >= \mathbb{Z}[a,b,c,d]/R = G/R,$$

where G is the freely generated Abelian group by those S^1 and R is the relationship according to which 2-simplices are glued along 1-simpilices. And by Abelianization

$$\begin{aligned} H_1(T_2) &= \mathbb{Z}[a,b,c,d] = \pi_1(T_2)/[\pi_1(T_2), \pi_1^{-1}(T_2)] \\ &= G/ < R, [\pi_1(T_2), \pi_1^{-1}(T_2)] > . \end{aligned}$$

From the above observation, we get the following corollaries which will later give us the persistent homotopy algorithm:

Corollary 2. *There is an isomorphism induced by Hurewicz map:*

$$h_* : \pi_1(X)/[\pi_1(X), \pi_1(X)] \xrightarrow{\simeq} H_1(X).$$

Corollary 3. *The Hurewicz isomorphism induces a map*

$$h_* : G = \bigoplus_{generators} \mathbb{Z} \to H_1(X)$$

sending the generator relationship R to 0*.*

Proof. By Corollary 2, we identify

$$H_1(X) \simeq \pi_1(X)/[\pi_1(X), \pi_1(X)] = \bigoplus_{generators} \mathbb{Z}/ < R, [\pi_1(X), \pi_1^{-1}(X)] > .$$

We still denote ∂ the map from 2-CW-simplices to 1-CW-simplices. If the image of ∂ lands in R then it will sent to 0 in $H_1(X)$, i.e,

$$\begin{tikzcd} G = \bigoplus_g \mathbb{Z} \arrow[rr] \arrow[dr, dashed, "\partial"'] & & H_1(X) \\ & R \arrow[ur, "0"'] & \end{tikzcd}$$

□

The algorithm of persistent homology gives generators with numbers equal to the first Betti number of X. Consider the projective space, whose fundamental group is $\mathbb{Z}_2$ (again via Van-Kampen theorem by gluing a Mobius band to a disk along S^1). It's first Betti number is 0, that is, it's the "rank" of $H_1(X) \otimes \mathbb{Q}$ which ignores the torsion part. From persistent homology point of view, in dim=1 case, we have two generators which are the boundary of the disk and the boundary of the Mobius band. Then the face of disk and Mobius band surface kill these two generators so there will be 0 generators left after paring the faces. Corollary 3 however says $2\mathbb{Z}$ is the fundamental group relationship and hence $\pi_1(\mathbb{P}^2) = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$. Together with our example on Torus, Corollary 3 covers all surface (dimension 2) cases.

Claim 4. *For CW-complexes, we can modify persistent homology algorithm to calculate persistent homotopy. We still pair 1-simplices as in persistent homology. For 2-simplices, follow persistent homology algorithm, if we find a 1-simplex generator that has not been paired, we pair it with the 2-simplex, otherwise we would get a bunch of 1-simplices who are the boundaries of certain 2-simplices, and these 1-simplices belong to R.*

However in reality, we would have to depend on subdivision/triangulation of our manifold. The subdivision may not guarantee the orientation we want. We may consider a square from which we get a torus by gluing edges, there exists a triangulation such that the boundary of it is not $aba^{-1}b^{-1}$. Hence in discrete case, we can't simply plug in Corollary 3 to get persistent homotopy. To address this problem, we pass everything to $\mathbb{Z}_2$ and don't need to bother orientation anymore. From Corollary 2, we have the following result. (Notice that we don't want to go into too much algebraic details, just to mention tensor product with $\mathbb{Z}_2$ is a not exact but commutative with colimits, so the following holds).

$$\begin{aligned} H_1(X) \otimes \mathbb{Z}_2 &\simeq \pi_1(X) \otimes \mathbb{Z}_2 / [\pi_1(X), \pi_1(X)] \otimes \mathbb{Z}_2 \\ &\simeq \left(\bigoplus_{generators} \mathbb{Z}_2 \right) / < R, [\pi_1(X), \pi_1(X)] > \otimes \mathbb{Z}_2. \end{aligned} \tag{1}$$

The following algorithm is based on Corollary 3 (CW-complex, smooth case) and Equation (1) (triangle mesh, discrete case). We still denote $h_* : \bigoplus \mathbb{Z}_2 \to H_1(X, \mathbb{Z}_2)$.

Claim 5 (Algorithm of persistent Homotopy)**.** *Suppose we have already paired edges and get a set of generators (2g of them in the case of 2-dim closed orientable surfaces).*

- *start with any 2-simplex σ, and calculate $c = \partial\sigma = \sigma_1 + \sigma_1 + \sigma_2$ and $c' = h_*(c)$.*
- *let τ be the youngest generator of c. While τ is paired and c' is not empty:*
    1. *get d such that (τ, d) is paired.*
    2. *replace c by $c + \partial d$.*
    3. *calculate $c' = h_*(c)$ and go back to top.*
- *if $c' = \emptyset$ then σ is a generator, otherwise pair (τ, σ).*

According to the above claim, if a 1-simplex is paired with some face, then homotopy case is exactly the same as homology case, as it'd be killed by some 2-simplex which means the loop it generates is contractible. However, if a 2-simplex is a generator, then $c' = \emptyset$ implies that the generator edges in c is in the relationship subgroup R up to an order.

Below is a simple observation, not used anywhere, yet.

Lemma 6. *In the above cycle chain c, a generator of any loop appears the latest in c, i.e, it will be added into c after all 1-simplices in that loop were added.*

Proof. First we notice that any loop has only one generator and it is the youngest edge in that loop. Suppose we start from some 2-simplex σ and reach a generator g of certain loop, and that generator hasn't been paired and we haven't gone through the whole loop yet, i.e, $c \neq$ loop. If the generator g is the youngest in c then it must generate c which is contradictory to our assumption, so there must exists another generator $g' \in c$ which is younger than g and this means σ doesn't pair with g. Therefore, if a 2-simplex kills a generator, it must travel through the 1-simplices of the loop that generator represent and reaches the generator the latest. $\square$