Persistent Homotopy

Fei Sun
fei.mm.sun@gmail.com

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1 Persistent Homotopy

This note presents and proves the algorithm of persistent homotopy based on the well-known persistent homology method. The central idea of this algo based on the so called Hurewicz theorem.

Theorem 1 (Hurewicz)**.** *Let X be a path connected space and n a positive integer, there exists a group homomorphism*

$$h_* : \pi_n(X) \to H_n(X)$$

which is defined as follows. Let $f \in \pi_n(X)$ denotes an n-homotopy class of X and $[u] \in H_n(S^n)$ a canonical generator, the homomorphism h_* sends f to $f_*([u])$ where

$$f_* : H_n(S^n) \to H_n(X)$$

is the homology group homomorphism induced by the homotopy class $f : S^n \to X$.

Here's some more explanation and translation to our case. Let's first consider from CW-complex point of view, which may be thought of as the "smooth case". Let us consider path-connected spaces with a base point. After gluing all 1-simplices, we get a bouquet of S^1, which are the generators of the chain complex $C_1(X)$. Then, we attach all the 2-simplices, this step is the most important and makes all the difference between homology and homotopy.

Continue our CW-construction, we may consider our space is obtained by pasting standard 2-simplices (i.e, D^2) along 1-simplices (those S^1). Then, by Van-Kampen theorem, the fundamental group $\pi_1(X)$ is generated by those 2-simplices and the relation is given by how these 2-simplices are glued along 1-simplices. For example, consider a 2-torus, which can be thought of an octagon gluing boundaries according to the rule $aba^{-1}b^{-1}cdc^{-1}d^{-1}$. Imagine the octagon is covered by two subspaces: A an open disk contained in the octagon and B slightly thickened boundary so that $A \cap B$ is homotopically S^1. Since $\pi_1(S^1) = \mathbb{Z}$ and A is contractible, Van-Kampen theorem says the fundamental group of 2-torus is

$$\pi_1(T_2) = \mathbb{Z}[a,b,c,d]/ < aba^{-1}b^{-1}cdc^{-1}d^{-1} >= \mathbb{Z}[a,b,c,d]/R = G/R,$$

where G is the freely generated Abelian group by those S^1 and R is the relationship according to which 2-simplices are glued along 1-simpilices. And by Abelianization

$$H_1(T_2) = \mathbb{Z}[a,b,c,d] = \pi_1(T_2)/[\pi_1(T_2), \pi_1^{-1}(T_2)]$$

$$= G/ < R, [\pi_1(T_2), \pi_1^{-1}(T_2)] > .$$

The following results are the algebraic version of the above. It works for one-dimensional case since $H_1(X)$ is the Abelianization of $\pi_1(X)$.

Corollary 2. *There is an isomorphism induced by Hurewicz map:*

$$h_* : \pi_1(X)/[\pi_1(X), \pi_1(X)] \xrightarrow{\cong} H_1(X)$$

sending commutator subgroup of $\pi_1(X)$ *to* $0 \in H_1(X)$.

Corollary 3. *The Hurewicz isomorphism induces a short exact sequence*

$$0 \to [\pi_1(X), \pi_1^{-1}(X)] \to \pi_1(X) = \bigoplus_{generators} \mathbb{Z}/R \xrightarrow{h_*} H_1(X).$$

The algorithm of persistent homology gives generators with numbers equal to the first Betti number of X. Consider the projective space obtained by gluing a Mobius band to a disk along S^1. From persistent homology point of view, for 1-simplices, we have one generator which are the boundary of the disk. Then the face of disk kills this generator and the boundary of Mobius band ($2\mathbb{Z}$) is in R hence $\pi_1(\mathbb{P}^2) = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2 = H_1(\mathbb{P}^2)$. It's first Betti number is 0, that is, it's the "rank" of $H_1(X) \otimes \mathbb{Q}$ which ignores the torsion part. Together with our example on Torus, Corollary 3 covers all surface (dimension 2) cases.

Claim 4. *For CW-complexes, we can modify persistent homology algorithm to calculate persistent homotopy. We still pair 1-simplices as in persistent homology. For 2-simplices, follow persistent homology algorithm, if we find a 1-simplex generator that has not been paired, we pair it with the 2-simplex, otherwise we would get a bunch of 1-simplices who are the boundaries of certain 2-simplices, and these 1-simplices belong to R.*

However in reality, we would have to depend on subdivision/triangulation of our manifold. The subdivision may not guarantee the orientation we want. We may consider a square from which we get a torus by gluing edges, there exists a triangulation such that the boundary of it is not $aba^{-1}b^{-1}$. Hence in discrete case, we can't simply plug in Corollary 3 to get persistent homotopy. To address this problem, we pass everything to $\mathbb{Z}_2$ and don't need to bother orientation anymore. From Corollary 2, we have the following result. (Notice that we don't want to go into too much algebraic details, just to mention tensor product with $\mathbb{Z}_2$ is a not exact but commutative with colimits, so the following holds).

$$\begin{aligned} H_1(X) \otimes \mathbb{Z}_2 &\simeq \pi_1(X) \otimes \mathbb{Z}_2 / [\pi_1(X), \pi_1(X)] \otimes \mathbb{Z}_2 \\ &\simeq \left(\bigoplus_{generators} \mathbb{Z}_2 \right) / < R, [\pi_1(X), \pi_1(X)] > \otimes \mathbb{Z}_2. \end{aligned} \tag{1}$$

Claim 5 (Algorithm of persistent Homotopy)**.** *Suppose we have already paired vertices and edges. Then for 2-simplices, we do the following*

- *start with any 2-simplex σ, and calculate $c = \partial\sigma = \sigma_1 + \sigma_1 + \sigma_2$ and $c' = h_*(c)$.*
- *let τ be the youngest generator of c. While τ is paired and c' is not empty:*
  1. *get d such that (τ, d) is paired.*
  2. *replace c by $c + \partial d$.*
  3. *calculate $c' = h_*(c)$ and go back to top.*
- *if $c' = \emptyset$ then σ is a generator, otherwise pair (τ, σ).*

According to the above claim, if a 1-simplex is paired with some face, then homotopy case is exactly the same as homology case, as it'd be killed by some 2-simplex which means the loop it generates is contractible. However, if a 2-simplex is a generator, then $c' = \emptyset$ implies that the generator edges in c is in the relationship subgroup R up to an order.

2 Some Other Observations

Below is a simple observation, not used anywhere, yet.

Lemma 6. *In the above cycle chain c, a generator of any loop appears the latest in c, i.e, it will be added into c after all 1-simplices in that loop were added.*

Proof. First we notice that any loop has only one generator and it is the youngest edge in that loop. Suppose we start from some 2-simplex σ and reach a generator g of certain loop, and that generator hasn't been paired and we haven't gone through the whole loop yet, i.e, $c \neq$ loop. If the generator g is the youngest in c then it must generate c which is contradictory to our assumption, so there must exists another generator $g' \in c$ which is younger than g and this means σ doesn't pair with g. Therefore, if a 2-simplex kills a generator, it must travel through the 1-simplices of the loop that generator represent and reaches the generator the latest. □

Next, we show how to find canonical basis on surfaces. Suppose we have a 2-dimensional surface, after edge-face paring, there will be some generator faces left. For example, for orientable surfaces, there's only one such face. Note that we will work in $\mathbb{Z}$ not $\mathbb{Z}_2$ anymore.

We will find the tunnel/handle loops by iterating face $\rightarrow$ edge $\rightarrow$ face as follows:

1. Construct some containers: E for edges, F for faces and Visited to store visited faces. Visited and F contain the generator faces, and E contains the (oriented) boundary of F.

2. while !F.empty():

    - let temp_F be a temporary container for faces.
    - for $\forall e \in E$, use EdgeFaceIterator [1] to reflect a face in F along e and we revert orientation of the reflected face. If the new face is not visited, then push it to temp_F, if it has been visited, mark it.
    - for $\forall f \in F$, update $E \leftarrow E + \partial f$ in oriented way so that original edge in E got canceled. E should only contain the outer boundary of all visited faces.
    - update F = temp_F and add faces in temp_F to visited.

All the marked edges should consist a canonical basis.

[1]need to write it yourself!