1

Higher Categories and Homotopy Theory

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Introduction

I will summerize the main ideas in Cisinski's course on higher category theory.

Contents

Chapter 1

Homotopy Theory and Model Categories

Chapter 2

Construction of Model Categories

2.1 Argument of Small objects

Let C be a category admits inductive limits. Given a small set I of morphisms in C, we write $Sat(I)$ to be the smallest saturated class of morphisms in C containing I.

We say that an ordinal (set) α is I-admissible if, for each functor:

$$F : \alpha \to C$$

satisfies $\forall i \in \alpha, i > 0$, $colim_{j<i} F_j \to F_i \in Sat(I)$, then for all morphisms $A \xrightarrow{u} B \in I$, we have for $X = A$ or B:

$$colim_{i\in\alpha} Hom_C(X, F_i) \xrightarrow{\sim} Hom_C(X, colim_{i\in\alpha} F_i). \tag{2.1.1}$$

Definition 2.1.1. We say that I is admissible if there exists an ordinal which is I-admissible. We will sometimes say that I admits the argument of small objects. The object X satisfying (2.0.1) is called small with respect to I.

Definition 2.1.2. Let C be a cocomplete category and I a class of morphisms in C.

1. A morphism $p : X \to Y$ is called I-injective if it has the RLP with respect to all the morphisms in I. The set of I-injective morphisms will be noted as $r(I)$.

2. A morphism $i : A \to B$ is called an I-cofibration if it has the LLP with respect to all I-injective morphisms.

3. $f : A \to B$ is I-cellular if

$$A = B_0 \to B_1 \to \cdots \to colim B_i \simeq B$$

and

$$\begin{tikzcd} \amalg_\alpha U_{\alpha,i} \arrow[r] \arrow[d] & B_i \arrow[d] \\ \amalg_\alpha V_{\alpha,i} \arrow[r] & B_{i+1} \end{tikzcd}$$

is a pushout, with $U_{\alpha,i} \to V_{\alpha,i} \in I$.

Proposition 2.1.3 (Argument of Small Objects)**.** *Let C be a cocomplete category, I be a set of morphisms in C. Suppose that the source of the morphisms in I are small with respect to the class of I-cellular morphisms. Then all the morphisms $f : X \to Y$ in C factorizes functorially as:*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[rr, bend right, "f"'] & Z \arrow[r, "p"] & Y \end{tikzcd}$$

where i is cellular and p is I-injective.

proof:
Let $f : X \to Y \in C$. Let S_0 be the set (cause I is a set) of diagrams of the form:

$$S_0 = \left\{ \begin{tikzcd} A_i \arrow[r] \arrow[d, "f_i"] & X = Z_0 \arrow[d, "f"] \\ B_i \arrow[r] & Y \end{tikzcd} , f_i \in I \right\}$$

Construct the pushout Z_1 of the diagram:

$$\begin{tikzcd} \amalg_{s\in S_0} A_s \arrow[r] \arrow[d] & X \arrow[d, "i_0"'] \arrow[ddr] & \\ \amalg_{s\in S_0} B_s \arrow[r] \arrow[drr] & Z_1 \arrow[dr, dashed] & \\ & & Y \end{tikzcd}$$

We repeat this procedure by replacing X by Z_1m Z_1 by Z_2, etc... Then consider the diagram:

$$\begin{tikzcd} X \arrow[r, "i_0"] \arrow[d, "f"] \arrow[rrr, bend left, "i"] & Z_1 \arrow[r, "i_1"] \arrow[d] & \cdots \arrow[r] & colim Z_i =: Z \arrow[d, "p"] \\ Y \arrow[r, equal] & Y \arrow[r, equal] & \cdots \arrow[r, equal] & Y \end{tikzcd} .$$

Since by definition each morphism $Z_i \to Z_{i+1}$ is I-cellular, $i : X \to Z$ is I-cellular as well. It leaves us to prove that p is I-injective. Thus we consider the diagram:

$$\begin{tikzcd} A_i \arrow[r] \arrow[d, "f_i"] & Z \arrow[d, "p"] \\ B_i \arrow[r] & Y \end{tikzcd} , \textit{with } f_i \in I.$$

Since A_i is small with respect to the class of I-cellular morphisms, the morphism $A_i \to Z$ factorizes by certain Z_k as $A \to Z_k \to Z$. However, the diagram

$$\begin{tikzcd} A_i \arrow[r] \arrow[d] & Z_k \arrow[d] \\ B_i \arrow[r] & Y \end{tikzcd}$$

belongs to S_k and by definition belongs to S_{k+1}, so there exists a lifting diagram:

$$\begin{tikzcd} A_i \arrow[r] \arrow[d] & Z_k \arrow[r] \arrow[d] & Z_{k+1} \arrow[r] \arrow[d] & Z \arrow[d] \\ B_i \arrow[r] \arrow[urr, dashed] & Y \arrow[r, equal] & Y \arrow[r, equal] & Y \end{tikzcd} .$$

Lemma 2.1.4. *Let C be a cocomplete category and I a class of morphisms in C.*

1. *All I-cellular morphisms are I-cofibrations.*
2. *Under the hypothesis of the argument of small objects, each I-cofibration is a retract of an I-cellular morphism.*

proof:

For 1., note that by definition, we have $I \subset I$-cofibration. Since the class I-cofibration is defined by LLP, it is stable by compositions and pushouts, and then I-cell$\subset I$-cof.

For 2., let $f : X \to Y \in I$-cof. By the argument of small objects, there exists a factorization:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[rr, bend right, "f"'] & Z \arrow[r, "p"] & Y \end{tikzcd}$$

with i an I-cellular morphism and p an I-injective morphism. The diagram:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"] & Z \arrow[d, "p"] \\ Y \arrow[r, equal] \arrow[ur, dashed, "\phi"] & Y \end{tikzcd}$$

admits a lifting $\phi \in I$-cof and $p \in I$-inj. We thus have a diagram of retract:

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"] & X \arrow[r, equal] \arrow[d, "i"] & X \arrow[d, "f"] \\ Y \arrow[r, "\phi"] & Z \arrow[r, "p"] & Y \end{tikzcd} .$$

Actually, the class I-cofibration can be identified as $Sat(I)$, and if we write I-injectives as $r(I)$, we have a factorization system $(Sat(I), r(I))$.

Theorem 2.1.5. *If I is admissible (in the sense of definition 2.1.1), $(Sat(I), r(I))$ is a factorization system. Actually, $Sat(I) = l(r(I))$.*

2.2 Cofibrantly Generated Model Categories

Definition 2.2.1. A model category is said to be generated by cofibrations (cofibrantly generated), if there exists two small sets of morphisms I, J in C satisfying:

1. $Cof = Sat(I)$ and I is admissible.
2. $Cof \cap W = Sat(J)$ and J is admissible.

In this case, we have $Fib = r(J)$ and $Fib \cap W = r(I)$.

Example 2.2.2. In the case $C = Top$:

1. $I = \{S^{n-1} \hookrightarrow B^n, n \geqslant 0\}$.
2. $J = \{[0,1]^{n-1} \times \{0\} \hookrightarrow [0,1]^n\}$.

The weak equivalences in Top are the continues map $f : X \to Y$ such that:

1. $\pi_0(X) \xrightarrow{\sim} \pi_0(Y)$, where π_0 denotes the set of path connected components.
2. $\forall x \in X, \forall n \geq 1$, and $y = f(x)$,
$$\pi_n(X) \xrightarrow{\sim} \pi_n(Y).$$

   The elements of $r(J)$ are called Serre fibrations.

Example 2.2.3 ($C = Comp(Ab)$).

$$I = \{\mathbb{Z}[n-1] \hookrightarrow Cone(\mathbb{Z} \xrightarrow{Id} \mathbb{Z})[n-1], n \in \mathbb{Z}\}$$

.

$$J = \{0 \hookrightarrow Cone(\mathbb{Z} \xrightarrow{Id} \mathbb{Z})[n-1], n \in \mathbb{Z}\}$$

.

The weak equivalences are quasi-isomorphisms, which are morphisms of complexes $A^\bullet \to B^\bullet$ such that
$$\forall n \in \mathbb{Z} \qquad H^n(A^\bullet) \xrightarrow{\sim} H^n(B^\bullet).$$

The fibrations are the surjectives.

2.3 Construction of Model Categories in $\widehat{A}$

Theorem 2.3.1. *Let A be a small category, then each small set of morphisms I in $C = \widehat{A}$ is admissible, and thus defines a system of factorization $(Sat(I), r(I))$ in $\widehat{(A)}$.*

We fix a small category A.

Definition 2.3.2. A cellular model of $\widehat{A}$ is a small set of monomorphisms I (since all small sets of A are admissible and define factorization systems in $\widehat{A}$) in $\widehat{A}$ such that $Sat(I)$ is the class of monomorphisms of $\widehat{(A)}$.

Example 2.3.3. If $A = \Delta$, $I = \{\partial\Delta_n \hookrightarrow \Delta_n, n \leq 0\}$ is a cellular model of $\widehat{\Delta}$.

Proposition 2.3.4. *There always exists cellular models of $\widehat{A}$. More precisely, the monomorphisms $X \hookrightarrow Y$ with Y a quotient of representable presheaf forms a cellular model.*

Definition 2.3.5. An interval of $\widehat{A}$ is a presheaf I over A equipped with two separated global sections

$$\begin{tikzcd} * \arrow[r, bend left, "\partial^0"] \arrow[r, bend right, "\partial^1"'] & I \end{tikzcd} .$$

That is to say, we have a cartesian diagram:

$$\begin{tikzcd} \emptyset \arrow[r, hook] \arrow[d, hook] & * \arrow[d, "\partial^0"] \\ * \arrow[r, "\partial^1"] & I \end{tikzcd} .$$

Notations: $\{e\} = Im\{* \xrightarrow{\partial^e} I\}$ for $e = 0, 1$ and $\partial I = \{0\} \sqcup \{1\} \hookrightarrow I$.

Example 2.3.6. For $A = \Delta$, we could take

$$I = \Delta_1 \qquad \begin{tikzcd} * = \Delta_0 \arrow[r, bend left, "\partial^0"] \arrow[r, bend right, "\partial^1"'] & I \end{tikzcd} .$$

We have then $\partial I = \partial \Delta_1$.

Remark 2.3.7. Let

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[d, hook, "i"] & X' \arrow[d, "i'"] \\ Y \arrow[r, "j'"] & Y' \end{tikzcd}$$

be a cartesian diagram in $\widehat{A}$, with i', j' monomorphisms. Then $Y \sqcup_X Y' \to Y'$ is a monomorphism, and we note $Y \cup X'$ be its image in Y'.

If $X \hookrightarrow Y$ and $K \hookrightarrow L$ are two monomorphisms in $\widehat{A}$, the commutative diagram:

$$\begin{tikzcd} K \times X \arrow[r] \arrow[d] & L \times X \arrow[d] \\ K \times Y \arrow[r] & L \times Y \end{tikzcd}$$

is cartesian, and defines then a monomorphism

$$K \times Y \cup L \times X \hookrightarrow L \times Y.$$

Definition 2.3.8. A homotopy structure in $\widehat{A}$ is a couple (I, A_n), where I is an interval of $\widehat{A}$ and A_n is a class of Anodyne Extension with respect to I, that is to say, a class of morphisms in $\widehat{A}$ satisfying the following conditions:

$An\mathbf{0}$: There exists a set Λ of monomorphisms in $\widehat{A}$ such that $A_n = Sat(\Lambda)$ (In particular A_n is saturated, and all elements in A_n are monomorphisms.)

$An\mathbf{1}$: If $K \hookrightarrow L$ is a monomorphism, then

$$I \times K \cup \{e\} \times L \hookrightarrow I \times L \in A_n, \quad e = 0, 1.$$

$An\mathbf{2}$: If $K \hookrightarrow L \in A_n$, then

$$I \times K \cup \partial I \times L \hookrightarrow I \times L \in A_n.$$

Example 2.3.9. Let I be an interval of $\widehat{A}$, and S be a set of monomorphisms of $\widehat{A}$. We choose a cellular model $\mathcal{M}$ for $\widehat{A}$. We then write $\Lambda_I(S, \mathcal{M})$ be the set of morphisms in $\widehat{A}$ of the form:

$$I \times K \cup \{e\} \times L \hookrightarrow I \times L, \qquad K \hookrightarrow L \in \mathcal{M}, \qquad e = 0, 1,$$

and of the form:

$$I \times U \cup \partial I \times V \hookrightarrow I \times V, \qquad U \hookrightarrow V \in S.$$

Then let $An_I(S) = Sat(\Lambda_I(S, \mathcal{M}))$.

Proposition 2.3.10. *The couple $(ImAn_n(S))$ is a homotopy structure on $\widehat{A}$. Moreover, $An_I(S)$ is the smallest class of anodyne extensions with respect to I: for each homotopy structure of the form (I, An), we have actually:*

$$S \subset An \Leftrightarrow \Lambda_I(S, \mathcal{M}) \subset An \Leftrightarrow An_I(S) \subset An.$$

The proof of this proposition can be derived form the following observation:

Let $U \xrightarrow{i} V$, $K \xrightarrow{j} L$ and $X \xrightarrow{p} Y$ be morphisms in $\widehat{A}$. We have, by adjunction, the following correspondence:

$$\begin{tikzcd} K \times V \coprod_{K\times U} L \times U \arrow[r] \arrow[d, "k"] & X \arrow[d, "p"] \\ L \times K \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & \underline{Hom}(L,X) \arrow[d, "q"] \\ V \arrow[r] \arrow[ur, dashed] & \underline{Hom}(L,Y) \times_{\underline{Hom}(K,Y)} \underline{Hom}(K,X) \end{tikzcd}$$

where k and q are morphisms induced by the following commutative diagrams:

$$\begin{tikzcd} K \times U \arrow[r] \arrow[d] & L \times U \\ L \times V \arrow[r] & L \times V \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} \underline{Hom}(L,X) \arrow[r] \arrow[d] & \underline{Hom}(K,X) \arrow[d] \\ \underline{Hom}(L,Y) & \underline{Hom}(K,Y) \end{tikzcd} .$$

Thus we have $k \in l(p) \Leftrightarrow i \in l(q)$.

In particular, for fixed j and p, the class of morphisms $U \xrightarrow{i} V$ such that $k \in l(p)$ is saturated.

Moreover, if $X \to Y \in r(I)$ for certain class I, then $\underline{Hom}(L,X) \to \underline{Hom}(L,Y)$ belongs to $r(A)$, where $L \in C$.

Example 2.3.11. If I is an interval of $\widehat{A}$, by choosing $S = \emptyset$, we have thus $An = An_I(\emptyset)$.

Example 2.3.12. If we are given a set of monomorphisms S which we want to inverse without an privilege interval, we define:

$$An(S) = An_I(S)$$

where I is any injective interval in $\widehat{A}$. A such interval exists all the time: we factorize the codiagonal morphism $* \amalg * \hookrightarrow *$ into a monomorphism $* \amalg * \hookrightarrow I$ and a trivial fibration $I \to *$.

We could prove that the class $An(S)$ does not depend on the choice of I, and that the model structure associated to $(I, An(S))$ depends only on S.

Example 2.3.13. The homotopy structure of Gabriel-Zisman on $\widehat{\Delta}$ is:

$$(\Delta_1, An_{\Delta_1}).$$

Example 2.3.14. The homotopy structure of Joyal on Δ is:

$$(J, An_J(\Lambda^{int})),$$

where $J = N\pi_1([1])$ and $\Lambda^{int} = \{\wedge_n^k \hookrightarrow \Delta_n, n \geq 2, 0 < k < n\}$. Here, $\pi_1[1]$ is the groupoid whose objects are $\{0, 1\}$ such that

$$Hom(0,1) = Hom(1,0) = Hom(0,0) = Hom(1,1) = *.$$

Let us return to the general case: we call a morphism in $\widehat{A}$ to be a trivial fibration if it has the RLP with respect to monomorphisms in $\widehat{A}$ (whence we choose a cellular model $\mathcal{M}$ in $\widehat{A}$, the class of trivial fibrations is $r(\mathcal{M}) = r(Sat(\mathcal{M}))$). We call cofibrations to be the the monomorphisms in $\widehat{A}$.

Now we fix a homotopy structure (I, An) on $\widehat{A}$. We will associate to (I, An) a model (category) structure on $\widehat{A}$:

We say that two morphisms

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & Y \end{tikzcd}$$

are I-homotopic if there exists an I-homotopy of f to g, that is to say, a morphism $h : I \times X \to Y$ such that the following diagram:

$$\begin{tikzcd} X \times \{0\} \arrow[dr] \arrow[drr, "f"] & & \\ & I \times X \arrow[r, "h"] & Y \\ X \times \{1\} \arrow[ur] \arrow[urr, "g"'] & & \end{tikzcd} \qquad h|_{\{0\}\times X} = f, \quad h|_{\{1\}\times X} = g$$

commutes.

We note $[X, Y]$ the quotient of $Hom_{\widehat{A}}(X, Y)$ by the I-homotopic relation. This quotient is compatible with the composition of morphisms in $\widehat{A}$, thus we obtain a category $h_I(\widehat{A})$ whose objects are the objects of $\widehat{A}$, and

$$Hom_{h_I\widehat{A}}(X, Y) = [X, Y].$$

Therefore, we have a canonical functor $Q : \widehat{A} \to h_I(\widehat{A})$ who is identity on the objects is induced by the projection:

$$Hom_{\widehat{A}}(X, Y) \longrightarrow [X, Y].$$

Definition 2.3.15. A morphism of $\widehat{A}$ is a **naive fibration** if it satisfies the RLP with respect to An.

A presheaf X over A is fibrant if $X \to *$ is a naive fibration.

A morphism $X \xrightarrow{f} Y$ of $\widehat{A}$ is a weak equivalence if for each fibrant object Z, the map:

$$[Y, Z] \xrightarrow{f^*} [X, Z]$$

is bijective.

A morphism of $\widehat{A}$ is a trivial cofibration if it is a cofibration and a weak equivalence.

A morphism of $\widehat{A}$ is a fibration if it satisfies the RLP with respect to trivial cofibrations.

Remark 2.3.16. We have now two systems of factorization:

1. $$(\{cofibrations\}, \{trivial\ fibrations\}) \tag{2.3.1}$$
   .

2. $$(\{anodyne\ extensions\}, \{naive\ fibrations\}) \tag{2.3.2}$$
   .

It is important to distinguish the notions of anodyne extensions and trivial cofibrations, and of naive fibrations and fibrations.

Proposition 2.3.17. *Let X and Z be two presheaves on A with Z fibrant. The I-homotopic relation is an equivalence relation on $Hom_{\widehat{A}}(X,Z)$.*

proof:
The reflexibility is immediate: if $f : X \to Z$ is a morphism $I \times X \xrightarrow{\sigma_X} X \xrightarrow{f} Z$ is an I-homotopy of f to f.

Now we want to prove that if there is an I-homotopy $h : I \times X \to Z$ of v to um and an I-homotopy $k : I \times X \to Z$ of u to w, then there exists an I-homotopy $l : I \times X \to Z$ of v to w. Consider the diagram:

$$\begin{tikzcd}
(I \times \partial I \times X) \cup (\{0\} \times I \times X) \arrow[r, "(((h,k),u\sigma_X)"] \arrow[d, hook, "\in An"'] & Z \\
I \times I \times X \arrow[ur, dashed, "H"] &
\end{tikzcd}$$

with $(h,k) : I \times \partial I \times X \simeq (I \times X) \amalg (I \times X) \xrightarrow{(h,k)} Z$ and $u\sigma_X : \{0\} \times I \times X \simeq I \times X \xrightarrow{\sigma_X} X \xrightarrow{u} Z$.

We then take $l = H|_{\{1\}\times I \times X}$ and we have then:

$$l|_{\{0\}\times X} = v \qquad and \qquad l|_{\{1\}\times X} = w.$$

By letting $w = u$ and $k = u\sigma_X$, we obtain the symmetry.

Proposition 2.3.18. *Each anodyne extension is a trivial cofibration.*

proof:
Let $j : K \to L \in An$ and Z is fibrant, we prove that

$$j^* : [L,Z] \longrightarrow [K,Z]$$

is bijective.

Injectivity : Let $l_0, l_1 : L \to Z$ such that $Q(l_0 j) = Q(l_1 j)$. There exists an I-homotopy j of $l_0 j$ to $l_1 j$m and thus a diagram:

$$\begin{tikzcd}
I \times K \cup \partial I \times L \arrow[r, "(h,(l_0,l_1))"] \arrow[d, "\in An"'] & Z \\
I \times L \arrow[ur, dashed, "H"] &
\end{tikzcd}$$

such that $H|_{\{e\}\times L} = l_e$, $e = 0,1$, that is to say, $Q(l_0) = Q(l_1)$.

Surjectivity : If $k \in Hom_{\widehat{A}}(K,Z)$, we have a diagram:

$$\begin{tikzcd}
K \arrow[r, "k"] \arrow[d, "j"'] & Z \\
L \arrow[ur, dashed, "l"] &
\end{tikzcd}$$

and thus $k = j^*(l)$.

Corollary 2.3.19. *Each fibration is a naive fibration.*

Proposition 2.3.20. *A morphism between fibrant objects is a weak equivalence if and only if it is an I-homotopic equivalence. (i,e, an isomorphism in $h_I(\widehat{A})$.)*

proof: We use the Yoneda Lemma to the full subcategory of $h_I(\widehat{A})$ formed by fibrant objects.

Definition 2.3.21. A morphism $f : X \to Y$ in $\widehat{A}$ is a **deformation by retraction** (resp. **the dual of a deformation by retraction**) if there exists a morphism $g : Y \to X$ and a morphism $h : I \times Y \to Y$ (resp. $k : I \times X \to X$) such that:

1. $gf = Id_X$. (resp. $fg = Id_Y$.)

2.
$$h|_{\{0\}\times Y} = Id_Y \qquad \textit{and} \qquad h|_{\{1\}\times Y} = fg.$$
$$(\textit{resp.}\ \ k|_{\{0\}\times X} = Id_X \qquad \textit{and} \qquad k|_{\{1\}\times X} = gf.)$$

3. $h(Id_I \times f) = \sigma_Y(Id_Y \times f)$. (resp. $fk = f\sigma_X$.)

Proposition 2.3.22. *Each deformation by retraction is an anodyne extension and thus a trivial cofibration.*

proof:

Note first that a deformation by retraction is a monomorphism. Let $i : K \to L$ be a deformation by retraction. There exists then morphisms $r : L \to K$ and $h : I \times L \to L$ such that $ri = Id_K$, $h|_{\{0\}\times L} = Id_L$, $h|_{\{1\}\times L} = ir$ and $h(Id_I \times i) = \sigma_L(Id_I \times i)$. We obtain a commutative diagram:

$$\begin{tikzcd} I \times K \cup \{1\} \times L \arrow[r, "(u\sigma_K{,}ur)"] \arrow[d, hook] & X \arrow[d, "p"] \\ I \times L \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

associated to each commutative diagram:

$$\begin{tikzcd} K \arrow[r, "u"] \arrow[d, "i"] & X \arrow[d, "p"] \\ L \arrow[r, "v"] & Y \end{tikzcd}$$

with p a naive fibration.

By letting $l = k|_{\{0\}\times L}$ we have $li = u$ and $pl = v$.

Proposition 2.3.23. *Each anodyne extension has fibrant objects as source and target is a deformation by retraction.*

proof:

Let $i : K \to L$ be an anodyne extension with K and L fibrant. We thus have a diagram:

$$\begin{tikzcd} K \arrow[r, "="] \arrow[d, "i"] & K \\ L \arrow[ur, dashed, "r"] & \end{tikzcd}$$

and thus a diagram:

$$\begin{tikzcd} I \times K \cup \partial I \times L \arrow[r, "(i\sigma_K{,}(Id_L{,}ir))"] \arrow[d, hook] & L \\ I \times L \arrow[ur, dashed, "h"] & \end{tikzcd}$$

and the couple (r, h) makes i a deformation by retraction.

Proposition 2.3.24. *Each trivial fibration is the dual of a deformation by retraction.*

proof:
Let $p : X \to Y$ be a trivial fibration, we have then:

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ Y \arrow[r, "="'] \arrow[ur, dashed, "s"] & Y \end{tikzcd} \qquad ps = Id_Y,$$

thus we have the following diagram:

$$\begin{tikzcd} I \times Y \cup \partial I \times X \arrow[rr, "{(s\sigma_Y,(Id_X,sp))}"] \arrow[d, hook] & & X \arrow[d, "p"] \\ I \arrow[r, "\sigma_X"'] \arrow[urr, dashed, "h"] & X \arrow[r, "p"] & Y \end{tikzcd} .$$

The couple (s, h) makes p the dual of a deformation by retraction.

Corollary 2.3.25. *Each trivial fibration is a weak equivalence.*

proof:
Without lose of generality, we suppose that X, Y are fibrant.
Let $f : X \to Y$ be a trivial fibration. We have seen that a trivial fibration is the dual of a deformation by retraction, which proves that we have a morphism $k : I \times X \to X$ making $gf \simeq Id_X$.
However, $X \to Y$ is also a fibration, so that the diagram:

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X^f \\ Y \arrow[r, "Id"] \arrow[ur, dashed, "g"] & Y \end{tikzcd}$$

gives us $fg = Id_X$, which proves that $X \xrightarrow{f} Y$ is an I-homotopy equivalence and thus a weak equivalence.

Corollary 2.3.26. *A fibration in $\widehat{A}$ is a weak equivalence if and only if it is a trivial fibration.*

proof:
Each trivial fibration is a fibration and a weak equivalence, as we have already seen in the previous corollary.
On the other hand, let $p : X \to Y$ be a fibration and a weak equivalence, we have a factorization:

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d, "p"] & X' \arrow[dl, "q"] \\ Y & \end{tikzcd}$$

with i a cofibration and q a trivial fibration. Then i is a trivial cofibration. Thus, according to retract lemma, p is a retract of q. Thus p is a trivial fibration.

Lemma 2.3.27. *Let $p : X \to Y$ be a naive fibration. The following conditions are equivalent:*

1. *p is a trivial fibration.*
2. *p is the dual of a deformation by retraction.*

proof:
We have already proved (1.) $\Rightarrow$ (2.).
Now, suppose that (2.) is true. Consider the commutative diagram:

$$\begin{tikzcd} K \arrow[r, "a"] \arrow[d, hook, "i"] & X \arrow[d, "p"] \\ L \arrow[r, "b"] & Y \end{tikzcd}$$

with i a cofibration.

Let $s : Y \to X$ and $k : I \times X \to X$ makes p a dual of a deformation by retraction. Thus we have a diagram:

$$\begin{tikzcd}
I \times K \cup \{1\} \times L \arrow[rr, "(k(Id_I \times a), sb)"] \arrow[d, hook, "\in An"'] & & X \arrow[d, "p"] \\
I \times L \arrow[r, "\sigma_L"'] \arrow[urr, dashed, "h"] & L \arrow[r, "b"'] & Y
\end{tikzcd} .$$

We then let $h|_{\{0\}\times L} = l$. Then we have $li = a$ and $pl = b$.

Lemma 2.3.28. *A naive fibration with fibrant target is a weak equivalence if and only if it is a trivial fibration.*

proof:

Let $p : X \to Y$ be a naive fibration with fibrant target. We notice that since Y is fibrant, X is fibrant as well. Suppose that p is a weak equivalence, we want to prove that p is a dual of a deformation by retraction. Let $t : Y \to X$ and $k : I \times Y \to Y$ be two morphisms such that $k|_{\{0\}\times Y} = Id_Y$ and $k|_{\{1\}\times Y} = pt$ (t, k exist, xince p is an I-homotopy equivalence. i.e, an isomorphism in $h_I(\widehat{A})$.) Thus we have:

$$\begin{tikzcd}
Y \simeq \{1\} \times Y \arrow[r, "t"] \arrow[d, hook] & X \arrow[d, "p"] \\
I \times Y \arrow[r, "k"'] \arrow[ur, dashed, "k'"] & Y
\end{tikzcd}$$

and we let $s = k'|_{\{0\}\times Y}$. We have $Id_Y = k|_{\{0\}\times Y} = pk'|_{\{0\}\times Y} = ps$.

Since Qp is an isomorphism in $h_I(\widehat{A})$ and $Id_Y = Q(ps) = Q(p)Q(s)$, we have $Q(s)Q(p) = Q(sp) = Is_X$. Thus there exists $h : I \times X \to X$ such that:

$$h|_{\{0\}\times X} = Id_X \qquad \textit{and} \qquad h|_{\{1\}\times X} = sp.$$

We obtain a diagram of the form:

$$\begin{tikzcd}
(\{1\} \times I \times X) \cup (I \times \partial I \times X) \arrow[rrr, "(sp\sigma_X, (h, sph))"] \arrow[d, hook] & & & X \arrow[d, "p"] \\
I \times I \times X \arrow[r, "Id_I \times \sigma_X"'] \arrow[urrr, dashed, "H"] & I \times X \arrow[r, "h"'] & X \arrow[r, "p"'] & Y
\end{tikzcd}$$

with

$$sp\sigma_X : \{1\} \times I \times X \simeq I \times X \xrightarrow{\sigma_X} X \xrightarrow{p} Y \xrightarrow{a} X$$

and

$$(h, sph) : I \times \partial I \times X \simeq I \times X \amalg I \times X \longrightarrow X.$$

We set $K = H|_{\{0\}\times I \times X : I \times X \longrightarrow X}$. The couple (s, K) makes p a dual of a deformation by retraction.

Proposition 2.3.29. *A cofibration with fibrant target is a weak equivalence if and only if it is an anodyne extension.*

proof:

Let i be a cofibration with fibrant target. We have a factorization as $i = qj$, with j an anodyne extension and q a naive fibration. Then according to the previous lemma, i is a weak equivalence if and only if q is a trivial fibration. The proof can be finished by using the retract lemma.

From (2.3.28) and (2.3.29), we see that our two systems of factorization (2.3.1) and (2.3.2) can be related by considering the full subcategory of fibrant objects of $\widehat{A}$.

Theorem 2.3.30. *A cofibration is a weak equivalence if and only if it satisfies the LLP with respect to the class of naive fibrations with fibrant targets.*

In particular, each naive fibration with fibrant target is a fibration, and for each presheaf X on A, the morphism $X \to *$ is a fibration if and only if X is fibrant.

proof:
Let

$$\begin{tikzcd} K \arrow[r, "a"] \arrow[d, hook, "i"] & X \arrow[d, "p"] \\ L \arrow[r, "b"'] & Y \end{tikzcd}$$

where i is a trivial cofibration and p a naive fibration with fibrant target.

We choose an anodyne extension with fibrant target $j : L \to L'$. Then ji is an anodyne extension (because it is a cofibration with fibrant target and is a weak equivalence), and therefore we have

$$\begin{tikzcd} L \arrow[r, "b"] \arrow[d, hook, "j"] & Y \\ L' \arrow[ur, dashed, "b'"] & \end{tikzcd}$$

and

$$\begin{tikzcd} K \arrow[r, "a"] \arrow[d, hook, "i"] & X \arrow[d, "p"] \\ L \arrow[d, hook, "j"] & Y \\ L' \arrow[ur, "b'"] & \end{tikzcd}$$

We then let $j = l'j$ and then we have $li = a$ and $pl = b$.

Conversely, if i satisfies the LLP with respect to naive fibrations with fibrant targets, we choose $j : L \to L'$ be an anodyne extension with fibrant target, and a factorization $ji = pk$, where k is an anodyne extension and p a naive fibrant (with fibrant target). Then ji is a retract of k, and thus an anodyne extension. It follows that i is a weak equivalence.

Corollary 2.3.31. *The trivial cofibrations form a saturated class of morphisms in $\widehat{A}$.*

Proposition 2.3.32. *There exists a set J such that $Sat(J)$ is the class of trivial cofibrations. In fact, this set can be choose to be the set generated the class of anodyne extension with fibrant target.*

In other words, we have:

Theorem 2.3.33. *The homotopy structure (I, An) defines a model category structure on $\widehat{A}$, where the cofibrations are the monomorphisms, the fibrant objects are the presheaves X such that $X \to *$ is a naive fibration, and the fibrations between fibrant objects are the naive fibrations between fibrant objects. Moreover, this model structure is cofibrantly generated.*

Remark 2.3.34. The canonical functor $h_I(\widehat{A}) \to Ho(\widehat{A})$ has a right adjoint which is fully faithful and identifies $Ho(\widehat{A})$ with the localization of $h_i(\widehat{A})$ by the weak equivalences. In other words, if $h'_I(\widehat{\,})$ denotes the full sub-category of $h_I(\widehat{A})$ formed by the fibrant objects, then:

$$h'_I(\widehat{A}) \longrightarrow Ho(\widehat{A})$$

is an equivalence of categories.

The functor $An^{-1}\widehat{A} \to Ho(\widehat{A})$ is an equivalence of categories.

Actually, let $f : X \to Y$ be a morphism in $\widehat{A}$, we then have a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"] & X' \arrow[d, "p"] \\ Y \arrow[r, "j"'] & Y' \end{tikzcd}$$

where i, j are anodyne extensions, and p is a naive fibration with fibrant target. If f is a weak equivalence, then p is a trivial fibration, and thus p admits a section (cause it is the dual of a deformation by retraction) $s : Y' \to X'$ which is an anodyne extension (therefore is a trivial cofibration with fibrant target). Inversing i, j and s induces the inverse of f. This shows that $An^{=1}\widehat{A}$ and $Ho(\widehat{A})$ has the same universal property.

Remark 2.3.35. A morphism in $\widehat{A}$ is a weak equivalence if and only if its image in $Ho(\widehat{A})$ is an isomorphism.

Theorem 2.3.36. *Let C be a model category, and*

$$F : \widehat{A} \longrightarrow C$$

is a functor who commutes with inductive limits and sends monomorphisms to cofibrations. Then F is a left Quillen functor if and only if F sends anodyne extensions to trivial fofibrations.

Proposition 2.3.37. *Suppose that $An = Sat(\Lambda)$ and there is a cellular model $\mathcal{M}$ of $\widehat{A}$ such that for each $K \hookrightarrow L \in \mathcal{M}$ and $U \hookrightarrow V \in \Lambda$, we have:*

$$K \times V \cup L \times U \hookrightarrow L \times V \in An.$$

Then for each monomorphism $K \hookrightarrow L$ and each anodyne extension $U \hookrightarrow V$,

$$K \times V \cup L \times U \longrightarrow L \times V$$

is an anodyne extension. Moreover, the weak equivalences in $\widehat{A}$ are stable under finite products.

Example 2.3.38. For a given internal I, $An = An_I$ satisfies the conditions given in the previous proposition. In particular, the condition is verified by the homotopy structure of Gabriel-Zisman.

Remark 2.3.39. Under the condition of the previous proposition, we have then:
for each fibration $p : X \to Y$ and each cofibration $i : K \to L$, a fibration:

$$q : \underline{Hom}(L, X) \longrightarrow \underline{Hom}(K, X) \times_{\underline{Hom}(K,Y)} \underline{Hom}(L, Y).$$

If moreover, p or i is a weak equivalence, then q is also a weak equivalence.

In particular, if X is fibrant, then $\underline{Hom}(K, X)$ is fibrant for each K. We define $\mathbf{R}\underline{Hom}(K, X) = \underline{Hom}(K, X')$ where $X \to X'$ is a weak equivalence with fibrant target. Thus for all X, Y and Z in $\widehat{A}$:

$$Hom_{Ho(\widehat{A})}(X \times Y, Z) \simeq Hom_{Ho(\widehat{A})}(X, \mathbf{R}\underline{Hom}(Y, Z)).$$

2.4 Fundamental Anodyne Extension

2.4.1 Representation of the Nerve of a Finite Ordered Set

Let E be an ordered set. Let SdE the set of non-empty, totally ordered finite sub-sets of E. SdE is ordered by inclusions.

For $S, S' \in SdE$, we have $S \cap S' \in SdE$ or $S \cap S' = \emptyset$. We deduce thus $N(S) \cap N(S') = N(S \cap S')$ is either the nerve of a totally ordered finite set or empty. We could then write:

$$\bigcup_{\in SdE} N(S) \simeq colim_{S \in SdE} N(S).$$

On the other hand, all n-simplexes $\sigma : \Delta_n = N[n] \to NE$ is the nerve of a increasing map $[n] \xrightarrow{\sigma} E$ such that the image is an element of SdE. We could then deduce:

$$colim_{S \in SdE} N(S) \simeq N(E).$$

When E is finite, we thus obtain the following description of its nerve:

Let $S_1, S_2, \ldots, S_k$ be the maximal elements of SdE. For each index $i, 1 \leq i \leq k$, we have an isomorphism:

$$c_i : \Delta_{n_i} \xrightarrow{\sim} N(S_i),$$

and for $i < j$, and isomorphism:

$$c_{ij} : \Delta_{n_{ij}} \xrightarrow{\sim} N(S_i \cap S_j).$$

(where we let $\Delta_{n_{ij}} = \emptyset$ when $S_i \cap S_j = \emptyset$.) We thus obtain an exact sequence:

$$\coprod_{i<j} \Delta_{n_{ij}} \rightrightarrows \coprod_{i} \Delta_{n_i} \longrightarrow N(E) \tag{2.4.1}$$

where $u, v : \coprod_{i<j} \Delta_{n_{ij}} \rightrightarrows \coprod_i \Delta_{n_i}$ is induced by inclusions $S_i \cap S_j \subset S_i$ and $S_i \cap S_j \subset S_j$.

Let $p, q \geq 0$. We use the above discussion to the case $E = [p] \times [q]$. In order to describe a representation of $N([p] \times [q]) = \Delta_p \times Delta_q$, it suffices to describe the maximal ordered subsets of SdE. These are the images of the injective increasing maps $[p+q] \to [p] \times [q]$ (thus there are C^p_{p+q} many of them). This representation permits us to calculate certain sub-objects of $\Delta_p \times \Delta_q$.

Proposition 2.4.1 (Joyal)**.** *The following three families of monomorphisms generate the same saturated class of monomorphisms of simplicial sets:*

1. *The class* $A1$ *composed by the inclusions of inner horns:*

$$\wedge^k_n \hookrightarrow \Delta_n, \quad n \geq 2, \quad 0 < k < n$$

2. *The class* $A2$ *composed by the inclusions of the form:*

$$\Delta_m \times \wedge^1_2 \cup \partial\Delta_m \times \Delta_2 \hookrightarrow \Delta_m \times \Delta_2, \quad m \geq 0.$$

3. *The class* $A3$ *composed by the inclusions of the form:*

$$L \times \wedge^1_2 \cup K \times \Delta_2 \hookrightarrow L \times \Delta_2$$

for $K \hookrightarrow L$ *an arbitrary monomorphism.*

proof:

2.4.2 Inner Kan Fibrations and Inner Anodyne Extensions

Definition 2.4.2. An inner anodyne extension is an element of the saturated class of morphisms of simplicial sets generated by the inclusions $\wedge_n^k \hookrightarrow \Delta_n, \quad n \geq 2, \quad 0 < k < n$.

An inner Kan fibration is a morphism having the RLP with respect to inner anodyne extensions.

Remark 2.4.3. By definition, a simplicial set X is an ∞-category if and only if $X \to *$ is an inner Kan fibration.

Example 2.4.4. $u : A \to B \in Cat \Rightarrow Nu : NA \to NB$ is an inner Kan fibration.

Corollary 2.4.5. *A morphism of simplicial sets $X \to Y$ is an inner Kan fibration if and only if the morphism:*

$$\underline{Hom}(\Delta_2, X) \longrightarrow \underline{Hom}(\Delta_2, Y) \times_{\underline{Hom}(\wedge_2^1, Y)} \underline{Hom}(\wedge_2^1, X)$$

is a trivial fibration.

This assertion comes from the proposition (2.3.37) by noticing that $\{\partial\Delta_m \hookrightarrow \Delta_m\}$ is a cellular model of $\widehat{\Delta}$. Moreover, if we want to verify whether a morphism is an inner Kan fibration, we just verify whether it has the RLP with respect to $\wedge_2^1 \hookrightarrow \Delta_2$.

Corollary 2.4.6. *A simplicial set X is an ∞-category if and only if the morphism:*

$$\underline{Hom}(\Delta_2, X) \longrightarrow \underline{Hom}(\wedge_2^1, X) = \underline{Hom}(\Delta_1, X) \times_X \underline{Hom}(\Delta_1, X)$$

is a trivial fibration.

Corollary 2.4.7. *If A is an ∞-category, then $\underline{Hom}(X, A)$ is an ∞-category as well.*

Proposition 2.4.8. *The three following families of morphisms generate the same family of saturated class of monomorphisms:*

1. *The class $B1$ (resp. $B1'$) composed by the inclusions of inner horns:*

$$\wedge_n^k \hookrightarrow \Delta_n, \quad 0 \leq k < n. \ (resp. \ 0 < k \leq n).$$

2. *The class $B2$ (resp, $B2'$) composed by the inclusions of the form:*

$$\Delta_1 \times \partial\Delta_n \cup \{0\} \times \Delta_n \hookrightarrow \Delta_1 \times \Delta_n, \quad n \geq 0.$$

$$(rexp. \ \ \Delta_1 \times \partial\Delta_n \cup \{1\} \times \Delta_n \hookrightarrow \Delta_1 \times \Delta_n, \quad n \geq 0.)$$

3. *The class $B3$ (resp. $B3'$) composed by the inclusions of the form:*

$$\Delta_1 \times K \cup \{e\} \times L \hookrightarrow Delta_1 \times L$$

for $K \hookrightarrow L$ an arbitrary monomorphism and $e = 0$ (resp. $e = 1$).

proof:

Corollary 2.4.9. $An_{\Delta_1} = Sat\{\wedge_n^k \hookrightarrow \Delta_n \mid n \geq 1,\ 0 \leq k \leq n\}$.

proof:

$$An_{\Delta_1} = Sat\{\Delta_1 \times \partial\Delta_n \cup \{e\} \times \Delta_n \hookrightarrow \Delta_1 \times \Delta_n \mid n \geq 0, \quad e = 0, 1\}.$$

2.4.3 Quillen Model Structure and Joyal Model Structure

Definition 2.4.10. A morphism of simplicial sets is called an **anodyne extension** (resp. a **left anodyne extension**, resp. a **right anodyne extension**) if it belongs to the saturated class generated by the inclusions of the form $\wedge_n^k \hookrightarrow \Delta_n$ for $n \geq 1$ and $0 \leq k \leq n$ (resp. $0 \leq k < n$, resp. $0 < k \leq n$).

A morphism of simplicial sets is called a **Kan fibration** (resp. a **left Kan fibration**, reps. a **right Kan fibration**) if it has the RLP with respect to anodyne extensions (resp. left anodyne extensions, resp. right anodyne extensions).

$$\begin{tikzcd}
 & \{trivial\ fibrations\} \arrow[d] & \\
 & \{Kan\ fibrations\} \arrow[dl] \arrow[dr] & \\
\{left\ Kan\ fibrations\} \arrow[dr] & & \{right\ Kan\ fibrations\} \arrow[dl] \\
 & \{inner\ Kan\ fibrations\} &
\end{tikzcd}$$

Remark 2.4.11. By definition, a simplicial set X is an ∞-groupoid if and only if $X \to *$ is a Kan fibration.

The Kan fibrations are the naive fibrations with respect to the homotopy structure $(\Delta_1, An_{\Delta_1})$.

Definition 2.4.12. The Quillen model category structure on $\widehat{\Delta}$ is the model structure defined by the homotopy structure $(\Delta_1, An_{\Delta_1})$.

By definition, the fibrant objects of this structure are the ∞-groupoids, and the fibrations between fibrant objects are the Kan fibrations between ∞-groupoids.

We have the adjunction:

$$\begin{tikzcd} \widehat{\Delta} \arrow[r, shift left, "\tau"] & Cat \arrow[l, shift left, "N"] \end{tikzcd}$$

and the following facts:

1. if X is an ∞-groupoid, then $\tau(X) isagroupoid$.

2. if G is a groupoid, then $N(G)$ is an ∞-groupoid.

Thus we have an adjunction:

$$\begin{tikzcd} \infty - Gr \arrow[r, shift left, "\tau"] & Gr \arrow[l, shift left, "N"] \end{tikzcd} .$$

When X is an ∞-groupoid, we note $\pi_1(X) = \tau(X)$.

Let I be the groupoid defined as:

$$Ob(I) = \{0, 1\}$$

$$Hom(i, j) = *, \quad for\ all\ (i, j) \in Oh(I)^2.$$

We have a functor $[1] \to I$ who is identity on the objects. If we note $J = N(I)$, we have then a morphism

$$\Delta_1 \longrightarrow J$$

which is an anodyne extension: it is a monomorphism, $J \to *$ is a Δ_1-equivalence and J is an ∞-groupoid. in particular, $\Delta_1 \hookrightarrow J$ is a trivial cofibration with fibrant target, so it is in An_{Δ_1}.

The object J has a natural structure of an interval, thus we deduce:

Proposition 2.4.13. *A morphism of ∞-groupoids is a weak equivalence if and only if it is a J-homotopy equivalence.*

proof: The ∞-groupoids are the fibrant and cofibrant objects in the Quillen model structure of $\widehat{\Delta}$.

We call the equivalence of ∞-groupoids the morphisms of ∞-groupoids which are J-homotopy equivalences.

Proposition 2.4.14. *A functor of groupoids $G \to G'$ is an equivalence of groupoids if and only if $N(G) \to N(G')$ is an equivalence of ∞-groupoids.*

If $X \to Y$ is an equivalence of ∞-groupoids, then $\tau(X) \to \tau(Y)$ is an equivalence of groupoids.

proof: The functor τ commutes with products, and we have $\tau(J) = I$, so it sends the equivalences of ∞-groupoids to the I-homotopy equivalences, i.e., the equivalences of groupoids.

Definition 2.4.15. The Joyal model (category) structure on $\widehat{\Delta}$ is the model category structure associated to the homotopy structure $(J, An_J(S))$ where

$$S = \{\wedge_n^k \hookrightarrow \Delta_n \mid n \geq 2,\ 0 < k < n\}.$$

We call J-anodyne extensions to be the elements of $An_J(S)$, and J-fibrations to be the elements of $r(An_J(S))$. The weak equivalences in this model category structure are called categorical weak equivalences.

By definition, we have:

1. each inner anodyne extension is a J-anodyne extension.
2. each J-anodyne extension is an anodyne extension.
3. a morphism of simplicial sets $X \to Y$ is a J-fibration if and only if it is an inner Kan fibration and if $\underline{Hom}(J, X) \to \underline{Hom}(J, Y) \times_Y X$ is a trivial fibration.
4. categorical weak equivalences are stable under finite products.
5. all Kan fibrations are J-fibrations.
6. a morphism of ∞-groupoids is a categorical weak equivalence if and only if it is a equivalence of ∞-groupoids (i.e., J-homotopy equivalences).

We say that a simplicial set is J-fibrant if $X \to *$ is a J-fibration.

Proposition 2.4.16. *For a simplicial set X, the following conditions are equivalent:*

1. *X is J-fibrant.*
2. *X is an ∞-category, and the morphism:*

$$\underline{Hom}(J, X) \longrightarrow X = \underline{Hom}(\{0\}, X)$$

is a trivial fibration.

Corollary 2.4.17. *For each small category C, $N(C)$ is J-fibrant.*

This comes from:

Theorem 2.4.18 (Joyal)**.** *All ∞-categories are J-fibrants.*

The proof of this theorem requires more things than we have studied by now, which are we about to discuss in the next section.

2.5 Elementary Properties of Kan Fibrations

Definition 2.5.1. A functor $f : A \to B$ is an **isofibration** is, for each object $a_1 \in A$ and each isomorphism $\beta"b_0 \xrightarrow{\sim} f(a_1)$ in B, there exists an isomorphism $\alpha : a_0 \xrightarrow{\sim} a_1$ in A, such that $f(a_0) = b_0$ and $f(\alpha) = \beta$.

$$\begin{tikzcd} a_0 \arrow[r, dashed, "\alpha", "\sim"'] \arrow[d, dashed, "f"'] & a_1 \arrow[d, "f"] \\ b_0 \arrow[r, "\sim", "\beta"'] & f(a_1) \end{tikzcd} .$$

Let X be an ∞-category. We say that a morphism of X

$$u : \Delta_1 \longrightarrow X$$

is **quasi-invertible** or a **quasi-isomorphism** if the corresponding morphism in $\tau(X)$ is an isomorphism.

A morphism of ∞-categories $f : X \to Y$ is an **isofibration** if it is an inner Kan fibration, and if for each commutative diagram of the form:

$$\begin{tikzcd} \{1\} \arrow[r, "a_1"] \arrow[d, hook] & X \arrow[d, "f"] \\ \Delta_1 \arrow[r, "\beta"'] \arrow[ur, dashed, "k"] & Y \end{tikzcd}$$

such that β is quasi-invertible in Y, there exists $k : \Delta_1 \to X$ which is quasi-invertible such that $f\alpha = \beta$.

Proposition 2.5.2. *An inner Kan fibration $f : X \to Y$ is an isofibration if and only if $\tau f : \tau X \to \tau Y$ is an isofibration.*

proof: This comes from the description of the fundamental categories τX and τY.

Corollary 2.5.3. *A functor is an isofibration if and only if its nerve is an isofibration.*

Corollary 2.5.4. *A morphism of ∞-categories $f : X \to Y$ is an isofibration if and only if $f^{op} : X^{op} \to Y^{op}$ is an isofibration.*

Example 2.5.5. For each ∞-category X, the canonical morphism $\eta_X : X \to N\tau X$ is an isofibration.

In fact, η_X is an inner Kan fibration: if we have a commutative diagram

$$\begin{tikzcd} \wedge_n^k \arrow[r, "a"] \arrow[d, hook] & X \arrow[d, "\eta_X"] \\ \Delta_n \arrow[r, "b"'] & N\tau X \end{tikzcd} , \quad 0 < k < n.$$

we a lifting diagram:

$$\begin{tikzcd} \wedge_n^k \arrow[r, "a"] \arrow[d, hook] & X \\ \Delta_n \arrow[ur, dashed, "a'"] & \end{tikzcd}$$

and we could see that $\eta_X a' = b$ for $\eta_X a'|_{\wedge_n^k} = b|_{\wedge_n^k}$.

Moreover, we have: if X is an ∞-groupoid, then $\eta_X : X \to N\tau X$ is a Kan fibration.

Definition 2.5.6. A functor $f : A \to B$ is **conservative** is for all the morphisms in A whose images under f are isomorphisms in B are isomorphisms in A. That is to say, for any morphism $\alpha \in A$, $f(\alpha) \in B$ is an isomorphism$\Rightarrow$ $\alpha \in A$ is an isomorphism.

A morphism of ∞-categories is **conservative** if its image under τ is a conservative functor.

Example 2.5.7. For each ∞-category X, the morphism $\eta_X : X \to N\tau X$ is conservative.

Proposition 2.5.8. *Let X, Y be two ∞-categories. Let $p : X \to Y$ be a right Kan fibration (resp. left Kan fibration). Then p is a conservative isofibration.*

proof:
It suffices to consider the case when p is a right Kan fibration (p^{op}" $X^{op} \to Y^{op}$).
Since the inclusion $\{1\} \hookrightarrow \Delta_1$ is just the horn $\wedge_1^1 \hookrightarrow \Delta_1$, it suffices to prove that p is conservative. Let $f : \Delta_1 \to X$ such that $pf : \Delta_1 \to Y$ is quasi-invertible. We have thus a commutative triangle in Y of the form:

$$\begin{tikzcd} & \cdot \arrow[dr, "v"] & \\ \cdot \arrow[ur, "pf"] \arrow[rr, "Id"'] \arrow[rr, phantom, shift left=3, "c"] & & \cdot \end{tikzcd} \qquad c : \Delta_2 \to Y.$$

We have a morphism $\wedge_2^2 \xrightarrow{b} X$ defined by:

$$\begin{tikzcd} & \cdot & \\ \cdot \arrow[ur, "f"] \arrow[rr, "Id"'] & & \cdot \end{tikzcd}$$

such that we obtain a commutative diagram:

$$\begin{tikzcd} \wedge_2^2 \arrow[r, "b"] \arrow[d, hook] & X \arrow[d, "p"] \\ \Delta_2 \arrow[r, "c"'] \arrow[ur, dashed, "l"] & Y \end{tikzcd} .$$

In other words, we have a commutative triangle in X of the form:

$$\begin{tikzcd} & \cdot \arrow[dr, "u"] & \\ \cdot \arrow[ur, "f"] \arrow[rr, "Id"'] & & \cdot \end{tikzcd}$$

with $pu = v$. Since v is quasi-invertible in Y, we apply the above discussion to u, which gives a commutative triangle in X of the form:

$$\begin{tikzcd} & \cdot \arrow[dr, "u'"] & \\ \cdot \arrow[ur, "u"] \arrow[rr, "Id"'] & & \cdot \end{tikzcd} ,$$

which proves that f is quasi-invertible in X.

Chapter 3

Properties of Model Categories

3.1 Gluing Operations and Fibre Products

3.1.1 Augmented Simplicial Sets

Let Δ_+ be the category whose objects are totally ordered sets $[n] = \{0, 1, \dots, n\}$ for $n \geq -1$ (we have then $[-1] = \emptyset$), and whose morphisms are the increasing maps.

An **augmented simplicial set** is a presheaf on Δ_+.

We have a natural inclusion $i : \Delta \hookrightarrow \Delta_+$, form which we deduce three adjoint functors $(i_!, i^*, i_*)$:

$$\begin{tikzcd} \widehat{\Delta} \arrow[r, bend left, "i_!"] \arrow[r, bend right, "i_*"'] & \widehat{\Delta_+} \arrow[l, "i^*"'] \end{tikzcd}$$

Given an augmented simplicial set $\Delta_+^{op} \to Sets$ is equivalent to be given a triple (X, ε, E) where X is a simplicial set, E is a set, and $\varepsilon : X_0 \to E$ is a map called the **augmentation** satisfying $\varepsilon d^0 = \varepsilon d^1$. In this point of view, the morphisms

$$(X, \varepsilon, E) \longrightarrow (X', \varepsilon', E')$$

are simply the couples (f, g) where $f : X \to X'$ is a morphism of simplicial sets, and $g : E \to E'$ is a map, such that the diagram

$$\begin{tikzcd} X_0 \arrow[r, "f_0"] \arrow[d, "\varepsilon"'] & X'_0 \arrow[d, "\varepsilon'"] \\ E \arrow[r, "g"'] & E' \end{tikzcd}$$

commutes.

Proposition 3.1.1 (Explicit description of $i_!$ and i_*)**.** *We could explicitly describe the functor $i_!$ and i_* for a simplicial set X. Actually, $i_!(X)$ (resp. $i_*(X)$) is the simplicial set X equipped with the augmentation*

$$X_0 \xrightarrow{Id} X_0 \qquad (resp,\ X_0 \to *).$$

proof: Let C be a monoidal category, we have the following commutative diagram:

$$\begin{tikzcd} & \widehat{C} \times \widehat{C} \arrow[dr, "\theta"] & \\ C \times C \arrow[ur, "h_C \times h_C"] \arrow[rr, "h_{C \times C}"] \arrow[d, "*"] & & \widehat{C \times C} \arrow[d, "*_!"] \\ C \arrow[rr, "h_C"'] & & \widehat{C} \end{tikzcd} \tag{3.1.1}$$

We have a monoidal structure (not symmetric) on $\varDelta_+$ induced by:

$$\begin{array}{ccc} \varDelta_+ \times \varDelta_+ & \to & \varDelta_+ \\ ([m],[n]) & \longmapsto & [m+1+n] \end{array}.$$

This structure can be extended uniquely by inductive limits to a monoidal structure on $\widehat{\varDelta}$:

$$\begin{array}{ccc} \widehat{\varDelta_+} \times \widehat{\varDelta_+} & \to & \widehat{\varDelta_+} \\ (X,Y) & \longmapsto & X * Y \end{array}.$$

in which the unit object is $[-1]$. We thus can describe $X * Y$ as:

Proposition 3.1.2. *If X and Y are two augmented simplicial sets, the for each $n \geq -1$, we have:*

$$(X * Y)_n = \coprod_{i+1+j=n} X_i \times Y_j.$$

proof:

For simplicity, we will denote the ordinal $[n-1]$ as n and the set X_{n-1} as $X(n)$. According to the diagram (2.6.1), we imbed $[n] \in \varDelta_+$ into $\widehat{\varDelta_+}$ by Yoneda functor. Thus we have actually the following diagram:

$$\begin{tikzcd} C \times C \arrow[r] \arrow[d, "*"] & \widehat{\varDelta_+} \times \widehat{\varDelta_+} \arrow[r, "\theta(X,Y)"] \arrow[d, dashed] & Sets \\ C \arrow[r] & \widehat{\varDelta_+} \arrow[ur, dashed, "*_! \ \theta(X,Y)"'] & \end{tikzcd}$$

for $X, Y \in \widehat{\varDelta_+}$, which makes $X*Y$ to be the left Kan extension of the functor $(p,q) \mapsto X(p) \times Y(q)$ along the functor $* : \varDelta_+ \times \varDelta_+ \to \varDelta_+$.

Thus we have:

$$(X * Y)_n = colim_{p+q \to n} X(p) \times Y(q),$$

where the colimit is taken over the category E_n of the functor $(p,q) \mapsto \varDelta_+(p+q,n)$. But every arrow $f : p+q \to n$ can be decomposed to be the form $f = u + v : p + q \to i + j$ with $(u,v) \in \varDelta_+ \times \varDelta_+$ and $i = u(p)$ and $j = v(q)$. This means that the set of decompositions $n = i+j$ is terminal in the category E_n. Thus, we have:

$$colim_{p+q \to n} X(p) \times Y(q) = \bigsqcup_{i+j=n} X(i) \times Y(j).$$

In the sequel, we will identify $\widehat{\varDelta}$ with the full subcategory of $\widehat{\varDelta_+}$ formed by the augmented simplicial sets (X, ε, E) such that $E = *$. By the previous proposition, we have:

$$X, Y \in \widehat{\varDelta} \Rightarrow X * Y \in \widehat{\varDelta}.$$

We have then a gluing functor:

$$\begin{array}{ccc} \widehat{\varDelta} \times \widehat{\varDelta} & \to & \widehat{\varDelta} \\ (X,Y) & \longmapsto & X * Y \end{array}$$

We have the formula $X * \emptyset = \emptyset * X = X$ (since $i_*(\emptyset) = [-1]$), and the natural inclusion:

$$X \sqcup Y \hookrightarrow X * Y.$$

For a fixed simplicial set T, we obtain the functors:

$$\begin{array}{rcl} (-) * T : \widehat{\varDelta} & \to & T \backslash \widehat{\varDelta} \\ X & \longmapsto & (T \to X * T) \end{array}$$

$$\begin{array}{rcl} T * (-) : \widehat{\varDelta} & \to & T \backslash \widehat{\varDelta} \\ X & \longmapsto & (T \to T * X) \end{array}$$

Proposition 3.1.3. *The functors $(-) * T$ and $T * (-)$ commute with inductive limits.*

proof: The functors respect the initial object. Thus it suffices to verify if they commute to connected inductive limits, which comes from the description of $X * Y$.

The existence of adjoint functors????

If $t : T \to X$ is a morphism of simplicial sets, we note:

$$X/t,\ or X/T \qquad (resp.\ t\backslash X\ or\ T\backslash X)$$

the image of $(X, t) \in T\backslash\widehat{\Delta}$ by the right adjoint of the functor $(-) * T$ (resp. $T * (-)$).

Remark 3.1.4. We have $(X/T)^{op} = T^{op}\backslash X^{op}$.

The gluing operation is associative:

$$(A * B) * C \simeq A * (B * C),$$

and we have the following formula:

$$X/S * Y \simeq (X/T)/S.$$

Given a morphism $S * T \to X$, we also have:

$$(S\backslash X)/T \simeq S\backslash(X/T).$$

Lemma 3.1.5. *If $K \hookrightarrow L$ and $U \hookrightarrow V$ are monomorphisms of simplicial sets, the the commutative diagram:*

$$\begin{tikzcd} K * U \arrow[r, hook] \arrow[d] & K * V \arrow[d] \\ L * U \arrow[r, hook] & L * V \end{tikzcd}$$

is a Cartesian diagram in which all the morphisms are monomorphisms. We shall write:

$$L * U \cup K * V \simeq L * U \coprod_{K*U} K * V \subset L * V.$$

We have

$$\partial\Delta_m * \Delta_n \cup \Delta_m * \partial\Delta_n = \partial\Delta_{m+1+n}$$

$$\wedge^k_m * \Delta_n \cup \Delta_m * \partial\Delta_n = \wedge^k_{m+1+n}$$

$$\partial\Delta_m * \Delta_n \cup \Delta_m * \wedge^k_n = \wedge^{m+1+k}_{m+1+n}$$

(as sub-objects of $\Delta_{m+1+n} = \Delta_m * \Delta_n$).

Theorem 3.1.6 (Joyal)**.** *Let X, Y be two ∞-categories, and $p : X \to Y$ be an inner Kan fibration. We consider the commutative diagram of the form*

$$\begin{tikzcd} \Delta_1 \arrow[r, hook, "{(0,1)}"] \arrow[rr, bend left] & \wedge^0_n \arrow[r, "a"] \arrow[d, hook] & X \arrow[d, "p"] \\ & \Delta_n \arrow[r, "b"'] \arrow[ru, dashed, "\exists"] & Y \end{tikzcd}, \qquad n \geq 2.$$

We suppose that $a(0, 1) : \Delta_1 \to X$ is quasi-invertible. Then there exists $l : \Delta_n \to X$ such that $pl = b$ and $l|_{\wedge^0_n} = a$.

($a(0, 1)$ is the composition of a with the morphism $\Delta_1 \to \wedge^0_n$ induced by the inclusion $[1] \hookrightarrow [n]$).

This theorem tells us that if $a(0,1)$ is quasi-invertible (for example, we will see later, it is the case when X is an ∞-groupoid), the there exists a lifting morphism for $\wedge_n^k \to \Delta_n \to X$ for $k = 0, n$ and $n \geq 2$.

This theorem comes from the more general assertion, by regarding the identification:

$$\wedge_1^0 * \Delta_m \cup \Delta_1 * \partial\Delta_m = \wedge_{m+2}^0$$

with $n = m+2$ and $m \geq 0$:

Theorem 3.1.7 (Coherence Theorem)**.** *Let X, Y be two ∞-categories and $p : X \to Y$ be an inner Kan fibration. Given a monomorphism $S \hookrightarrow T$ and a commutative diagram of the form*

$$\begin{tikzcd}
\{0\} * T \cup \Delta_1 * S \arrow[r, "a"] \arrow[d, hook, "i"] & X \arrow[d, "p"] \\
\Delta_1 * T \arrow[r, "b"'] & Y
\end{tikzcd} .$$

We suppose moreover that $a(0,1) : \Delta_1 \to X$ is quasi-invertible (where $a(0,1)$ is the composition of a and the inclusion $\Delta_1 \hookrightarrow \Delta_1 * S \hookrightarrow \{0\} * T \cup \Delta_1 * S$). Then there exists $l : \Delta_1 * T \to X$ such that $pl = b$ and $li = a$.

The proof of this theorem needs some preparations.

Lemma 3.1.8. *Let $A \xrightarrow{i} B$ and $S \xrightarrow{j} T$ be two monomorphisms of simplicial sets. We write*

$$A * T \cup B * S \xrightarrow{u} B * T$$

to be the corresponding inclusion. The for each morphism $X \to Y$, we have a bijection between the following two diagrams:

$$\begin{tikzcd}
A * T \cup B * S \arrow[r] \arrow[d, hook, "u"] & X \arrow[d, "f"] \\
B * T \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd}
A \arrow[r] \arrow[d, hook, "i"] & X/T \arrow[d, "f/j"] \\
B \arrow[r] \arrow[ur, dashed] & Y/T \times_{Y/S} X/S
\end{tikzcd}$$

Lemma 3.1.9. *Let $p : X \to Y$ be an inner Kan fibration. Then for each $n \geq 1$ and $0 \leq k < n$, and each morphism $t : \Delta_n \to X$, the projection:*

$$X/\Delta_n \longrightarrow X/\wedge_n^k \times_{Y/\wedge_n^k} Y/\Delta_n$$

is a trivial fibration.

proof: For each $m \geq 0$, we have:

$$\Delta_m * \wedge_n^k \cup \partial\Delta_m * \Delta_n = \wedge_{m+n+1}^{k+m+1}$$

with $0 < k+m+1 < m+n+1$.

Theorem 3.1.10. *Let $p : X \to Y$ be an inner Kan fibration. We consider a monomorphism $j : S \hookrightarrow T$, and a morphism $t : T \to X$. Then the projection:*

$$p/j : X/T \longrightarrow X/S \times_{Y/S} Y/T$$

is a right Kan fibration.

If moreover Y is an ∞-category, then X/S and $X/S \times_{Y/S} Y/T$ are also ∞-categories.

proof: Consider $\wedge_n^k \hookrightarrow \Delta_n$ with $0 < k \leq n$. We have the following correspondence:

$$\begin{tikzcd}
\wedge_n^k \arrow[r] \arrow[d, hook] & X/T \arrow[d] \\
\Delta_n \arrow[r] \arrow[ur, dashed] & X/S \times_{Y/S} Y/T
\end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd}
\wedge_n^k * T \cup \Delta_n * S \arrow[r] \arrow[d, hook] & X \arrow[d] \\
\Delta_n * T \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd}
S \arrow[r] \arrow[d, hook] & \Delta_n \backslash X \arrow[d, "q"] \\
T \arrow[r] \arrow[ur, dashed] & \wedge_n^k \backslash X \times_{\wedge_n^k \backslash Y} \Delta_n \backslash Y
\end{tikzcd} .$$

While q is a trivial fibration, we obtain the lifting arrow.

When $Y = *$ we have $X/T \to X/S$ is a right Kan fibration, and when $S = \emptyset$ we have $X/T \to Y/T$ is a right Kan fibration. If X is an ∞-category, then X/T is as well an ∞-category. Moreover, the projection $Y/T \times_{Y/S} X/S$ on X/S is a right Kan fibration (obtained from $Y/T \to Y/S$ by base change), and we deduce that $Y/T \times_{Y/S} X/S$ is an ∞-category as well.

proof of the coherence theorem:Let $p : X \to Y$ be an inner Kan fibration, with X and Y ∞-categories. We consider the commutative diagram

$$\begin{tikzcd} \{0\} * T \cup \Delta_1 * S \arrow[r, "a"] \arrow[d, hook] & X \arrow[d, "p"] \\ \Delta_1 * T \arrow[r] & Y \end{tikzcd}$$

where $S \hookrightarrow T$ is a monomorphism, and $a(0,1)$ is quasi-invertible. In order to prove the existence of the lifting morphism $l : \Delta_1 * T \to X$ in this diagram, it suffices to find a lifting diagram in the commutative diagram:

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d] & X/T \arrow[d] \\ \Delta_1 \arrow[r, "\alpha"'] & X/S \times_{Y/S} Y/T \end{tikzcd}$$

However, we have a commutative diagram of the form

$$\begin{tikzcd} & X/S \times_{Y/S} Y/T \arrow[r] \arrow[d, "q"] & Y/T \arrow[d] \\ \Delta_1 \arrow[ur, "\alpha"] \arrow[dr, "a(0,1)"'] & X/S \arrow[r] \arrow[d, "\pi"] & Y/S \\ & X & \end{tikzcd}$$

Since q and π are right Kan fibration, they are conservative such that α is quasi-invertible.

On the other hand, g is also a right Kan fibration and thus an isofibration, which proves the existence of the lifting morphism we want.

Theorem 3.1.11. *An ∞-category X is an ∞-groupoid if and only if all its morphisms are quasi-invertible. (This is equivalent to say that $\tau(X)$ is a groupoid.)*

This comes from the theorem of Joyal (2.6.6).

The inclusion $i : Gr \to Cat$ of the category of groupoids into the category of small categories admits a left adjoint

$$\pi_1 : Cat \longrightarrow Gr$$

and a right adjoint:

$$k : Cat \longrightarrow Gr.$$

We have $\pi_1(C) = C^{-1}C$ (the localization of C with respect to all its morphisms):

$$\begin{tikzcd} \coprod_{f \in Mor(C)} \Delta_1 \arrow[r] \arrow[d, hook] & C \arrow[d] \\ \coprod_{f \in Mor(C)} J \arrow[r] & \pi_1(C) \end{tikzcd}$$

is cocartesian.

We have $k(C)$ is defined by $Ob(k(C)) = Ob(C)$ and $Mor(k(C)) = \{isomorphisms \in C\}$.

If X is an ∞-category, we define $k(X)$ by the following cartesian diagram:

$$\begin{tikzcd} k(X) \arrow[r] \arrow[d] & X \arrow[d, "\eta_X"] \\ Nk\tau X \arrow[r] & N\tau X \end{tikzcd} .$$

Corollary 3.1.12. *$k(X)$ is an ∞-groupoid, moreover, $k(X)$ is the biggest ∞-groupoid contained in X.*

An n-simplex $\Delta_n \xrightarrow{x} X$ factorizes through $k(X)$ if and only if, for all $0 \leq i < j \leq n$, the induced morphism $x_i \xrightarrow{x(i,j)} x_j$ is quasi-invertible in X.

Moreover, we have a canonical isomorphism:

$$\tau k(X) \simeq k\tau(X).$$

Corollary 3.1.13. *The functor*

$$\infty - Cat \xrightarrow{k} \infty - Gr$$

is a right adjoint of the inclusion functor $\infty - Gr \hookrightarrow \infty - Cat$.

Corollary 3.1.14. *Let X be an ∞-category. A morphism in X*

$$\Delta_1 \xrightarrow{x} X$$

is quasi-invertible if and only if there exists a commutative diagram of the form:

$$\begin{tikzcd} \Delta_1 \arrow[r, "x"] \arrow[d, hook] & C \\ J \arrow[ur, dashed] & \end{tikzcd} .$$

proof: If x is quasi-invertible, we have a factorization:

$$\begin{tikzcd} \Delta_1 \arrow[r] \arrow[d, hook] \arrow[rr, bend left, "x"] & k(X) \arrow[r, hook] \arrow[d] & X \\ J \arrow[r] & * & \end{tikzcd}$$

and since $\Delta_1 \to J$ is an anodyne extension, we obtain the lifting morphism.

Proposition 3.1.15. *For each ∞-category X, the inclusion $k(X) \hookrightarrow X$ is an isofibration (conservative).*

proof: Since $k(X)$ is an ∞-groupoid, all its morphisms are quasi-invertible. The inclusion $k(X) \hookrightarrow X$ has the RLP with respect to the inclusions of the form

$$\wedge_n^k \hookrightarrow \Delta_n, \quad n \geq 2, \quad 0 \leq k \leq n.$$

Proposition 3.1.16. *Let $p : X \to Y$ be a left Kan fibration (resp. right Kan fibration). If X and Y are ∞-groupoids, then p is a Kan fibration.*

Corollary 3.1.17. *The fibres of a left (resp. right) Kan fibration are ∞-groupoids.*

3.2 ∞-Category of Functors

Let $p : X \to Y$ be an inner Kan fibration. For each morphism $A \to B$, the morphism:

$$\underline{Hom}(B, X) \longrightarrow \underline{Hom}(A, X) \times_{\underline{Hom}(A,Y)} \underline{Hom}(B, Y)$$

is still an inner Kan fibration.

In particular, for each ∞-category X and each simplicial set A, $\underline{Hom}(A, X)$ is an ∞-category, and for each monomorphism $A \hookrightarrow B$,

$$\underline{Hom}(B, X) \longrightarrow \underline{Hom}(A, X)$$

is an inner Kan fibration.

If A is a simplicial set, we shall write

$$Ob(A) = \coprod_{a \in A_0} \Delta_0.$$

We have a canonical inclusion $ObA \hookrightarrow A$ which is identity on the 0-simplexes.

If X is an ∞-category, we have then an inner Kan fibration:

$$\begin{array}{rcl} ev : \underline{Hom}(A, X) & \to & \underline{Hom}(ObA, X) \simeq \prod_{a \in A_0} X \\ F & \longmapsto & (F_a)_{a \in A_0} \end{array}$$

where

$$\begin{tikzcd} A \arrow[r, "F"] & X . \\ \Delta_0 \arrow[u, "a"] \arrow[ur, "F_a"'] & \end{tikzcd}$$

We define the ∞-category $k(A, X)$ by the cartesian diagram:

$$\begin{tikzcd} k(A, X) \arrow[r] \arrow[d] & \underline{Hom}(A, X) \arrow[d, "ev"] \\ \prod_{a \in A_0} kX \simeq k\underline{Hom}(ObA, X) \arrow[r] & \underline{Hom}(ObA, X) \simeq \prod_{a \in A_0} X \end{tikzcd}$$

Since $k\underline{Hom}(ObA, X)$ is a Kan complex, we deduce that $k(A, X)$ is an ∞-category.

We could think $k(A, X)$ to be generated by the morphisms of $\underline{Hom}(A, X)$ which are quasi-invertible term by term. Actually, an n-simplex $\Delta_n \xrightarrow{x} \underline{Hom}(A, X)$ lies in $k(A, X)$ if and only if the induced morphism

$$\Delta_n \times A \xrightarrow{\bar{x}} X$$

satisfies: for each $a \in A_0$, the n-simplex

$$ev_a(x) : \Delta_n \times \Delta_0 \xrightarrow{Id \times a} \Delta_n \times A \xrightarrow{\bar{x}} X$$

factorizes through $k(X)$.

If B is a simplicial set, we shall write $h(B, X)$ to be the sub-simplicial set of $\underline{Hom}(B, X)$ formed by the n-simplexes

$$\Delta_n \longrightarrow \underline{Hom}(B, X)$$

such that the corresponding morphism $B \to \underline{Hom}(\Delta_n, X)$ factorizes through $k(\Delta_n, X)$.

$$\begin{tikzcd} B \arrow[rr] \arrow[dr, dashed] & & \underline{Hom}(\Delta_n, X) . \\ & k(\Delta_n, X) \arrow[ur, dashed] & \end{tikzcd}$$

We have then, for each ∞-category X and all simplicial sets A and B, a bijection:

$$Hom_{\widehat{\Delta}}(A, h(B,X)) \simeq Hom_{\widehat{\Delta}}(B, k(A,X)).$$

We could also verify that $h(B,X) \hookrightarrow \underline{Hom}((B,X)$ is an inner Kan fibration, and thus $h(B,X)$ is an ∞-category as well.

Let $p : X \to Y$ be an inner Kan fibration between ∞-categories. We have then a morphism:

$$ev_1 : h(\Delta_1, X) \longrightarrow X \times_Y h(\Delta_1, Y)$$

induced by the inclusion $\{1\} \hookrightarrow \Delta_1$. Saying that p is an isofibration is equivalent to saying that ev_1 has the RLP with respect to the inclusion $\emptyset = \partial\Delta_0 \hookrightarrow \Delta_0$. In fact, we have the following correspondence:

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & h(\Delta_1, X) \arrow[d, "ev_1"] \\ \Delta_0 \arrow[r] \arrow[ur, dashed] & X \times_Y h(\Delta_1, Y) \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \{1\} \arrow[r] \arrow[d, hook] & X \arrow[d, "p"] \\ \Delta_1 \arrow[r, "y"'] \arrow[ur, dashed, "x"] & Y \end{tikzcd}$$

where x and y are quasi-invertible.

Theorem 3.2.1. *The morphism $ev_1 : h(\Delta_1, X) \longrightarrow X \times_Y h(\Delta_1, Y)$ has the RLP with respect to the inclusions $\partial\Delta_n \hookrightarrow \Delta_n$ for $n \geq 1$. In particular, p is an isofibration if and only if ev_1 is a trivial fibration.*

proof:
First, we notice that if $A \hookrightarrow B$ is an inclusion of simplicial sets induced by a bijection on 0-simplexes $A_0 \simeq B_0$, then the commutative diagram:

$$\begin{tikzcd} h(B,X) \arrow[r] \arrow[d] & \underline{Hom}(B,X) \arrow[d] \\ h(A,X) \arrow[r] & \underline{Hom}(A,X) \end{tikzcd}$$

is cartesian.
If $n \geq 1$, the inclusion $\partial\Delta_n \to \Delta_n$ is identity on 0-simplexes, so that we have the correspondence:

$$\begin{tikzcd} \partial\Delta_n \arrow[r] \arrow[d, hook] & h(\Delta_1, X) \arrow[d, "ev_1"] \\ \Delta_0 \arrow[r] \arrow[ur, dashed] & X \times_Y h(\Delta_1, Y) \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \Delta_1 \times \partial\Delta_n \cup \{1\} \times \Delta_n \arrow[r, "a"] \arrow[d, hook] & X \arrow[d, "p"] \\ \Delta_1 \times \Delta_n \arrow[r, "b"'] \arrow[ur, dashed] & Y \end{tikzcd}$$

where we have $b \in k(\Delta_n, Y)_1$ and $a|_{\Delta_1 \times \partial\Delta_n} \in k(\Delta_n, X)_1$.

Recall that we have a filtration:

$$\Delta_1 \times \partial\Delta_n \cup \{1\} \times \Delta_n = \mathcal{A}_{-1} \subset A_0 \subset \cdots \subset A_n = \Delta_1 \times \Delta_n$$

with the cocartesian diagram of the following form:

$$\begin{tikzcd} \Lambda^{i+1}_{n+1} \arrow[r, hook] \arrow[d, hook] & A_{i-1} \arrow[d] \\ \Delta_{n+1} \arrow[r, "c_i"'] & A_i \end{tikzcd}$$

with $0 \leq i \leq n$, and c_i is induced by $[n+1] \hookrightarrow [1] \times [n]$, the unique injection with value $(0, i)$ and (i, i).

By letting $a_{-1} = a$, we obtain the following lifting diagram:

$$\begin{tikzcd}
 & & \Delta_1 \times \partial\Delta_n \cup \{1\} \times \Delta_n \arrow[r, "a"] \arrow[d, hook] & X \arrow[ddd] \\
\wedge^{i+1}_{n+1} \arrow[r, hook] \arrow[d, hook] & A_{i-1} \arrow[d, hook] \arrow[urr, dashed, "a_{i-1}"] & & \\
\Delta_{n+1} \arrow[r] \arrow[ur, dashed] & A_i \arrow[uurr, dashed, "a_i"'] & & \\
 & & \Delta_1 \times \Delta_n \arrow[r, "b"'] & Y
\end{tikzcd}$$

The cases $0 \leq i \leq n-1$ induces the case $0 < i+1 < n+1$ and so that p is an inner Kan fibration, and the case $i = n$ comes from the coherence theorem because $a_{n=1}(n, n+1)$ is invertible by hypothesis.

Theorem 3.2.2. *Let $p : X \to Y$ be an isofibration between ∞-categories. Then for each monomorphism of simplicial sets $A \hookrightarrow B$, the morphism:*

$$q : k(B, X) \longrightarrow k(A, X) \times_{k(A,Y)} k(B, Y)$$

is a Kan fibration between ∞-groupoids.

proof:
We start by the following cartesian diagram:

$$\begin{tikzcd}
k(B,X) \arrow[r, hook] \arrow[d, "q"] & \underline{Hom}(B,X) \arrow[d] \\
k(A,X) \times_{k(A,Y)} k(B,Y) \arrow[r, hook] & \underline{Hom}(A,X) \times_{\underline{Hom}(A,Y)} \underline{Hom}(B,Y)
\end{tikzcd} ,$$

from which we deduce that q is an inner Kan fibration between ∞-categories.

The correspondence

$$C \to k(A, X) \leftrightsquigarrow A \to h(C, X)$$

induces a correspondence:

$$\begin{tikzcd}
\{1\} \arrow[r] \arrow[d, hook] & k(B,X) \arrow[d, "q"] \\
\Delta_1 \arrow[r] \arrow[ur, dashed] & k(A,X) \times_{k(A,Y)} k(B,Y)
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
A \arrow[r] \arrow[d, hook] & h(\Delta_1,X) \arrow[d] \\
B \arrow[r] \arrow[ur, dashed] & X \times_Y h(\Delta_1,Y)
\end{tikzcd}$$

Thus it left we to prove that q has the RLP with respect ot the inclusions

$$\Delta_1 \times \partial\Delta_n \cup \{1\} \times \Delta_n \hookrightarrow \Delta_1 \times \Delta_n$$

for $n \geq 1$ case we have just verified the case $n = 0$.

Corollary 3.2.3. *If X is an ∞-category, then for each simplicial set A, we have:*

$$k(A, X) = k\underline{Hom}(A, X)$$

. Specifically, if $p : X \to Y$ is an isofibration between ∞-categories, then for each monomorphism $A \hookrightarrow B$, we have

$$k(A, X) \times_{k(A,Y)} k(B, Y) = k(\underline{Hom} \times_{\underline{Hom}(A,Y)} \underline{Hom}(B, Y)).$$

proof: We have $k(\underline{Hom}(A, X)) \subset k(A, X)$ by definition, and since $k(A, X)$ is a Kan complex (obtained from the previous theorem by letting $Y = *$ and $A = \emptyset$), we obtain the equality.

Corollary 3.2.4. *Let $p : X \to Y$ be an isofibration between ∞-categories. Then for each anodyne extension $A \hookrightarrow B$, the morphism $h(B, X) \to h(A, X) \times_{h(A,Y)} h(B, Y)$ is a trivial fibration.*

proof: We have the following correspondence:

$$\begin{tikzcd} \partial\Delta_n \arrow[r] \arrow[d, hook] & h(B,X) \arrow[d] \\ \Delta_n \arrow[r] \arrow[ur, dashed] & h(A,X) \times_{h(A,Y)} h(B,Y) \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} A \arrow[r] \arrow[d, hook] & k(\Delta_1, X) \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & k(\partial\Delta_n, X) \times_{k(\partial\Delta_n,Y)} h(\Delta_n, Y) \end{tikzcd} .$$

Theorem 3.2.5 (Joyal)**.** *A simplicial set is an ∞-category if and only if it is J-fibrant.*
A morphism between ∞-categories is an isofibration if and only if it is a J-fibration.

proof: Let $p : X \to Y$ is an isofibration between ∞-categories. We want to prove that the inclusion $\{1\} \hookrightarrow J$ induces a trivial fibration

$$\underline{Hom}(J, X) \longrightarrow X \times_Y \underline{Hom}(J, Y).$$

However, $h(J, X) = \underline{Hom}(J, X)$ and $h(\{1\}, X) = \underline{Hom}(\{1\}, X) = X$ (since J and $\{1\}$ are Kan complexes). We conclude by noticing that $\{1\} \hookrightarrow J$ is an anodyne extension and the previous corollary.

Corollary 3.2.6. *The class of categorical weak equivalences is the smallest class W of morphisms satisfy the following properties:*

1. *W satisfies the "2 out of 3" property.*
2. *all inner anodyne extensions belong to W.*
3. *all trivial fibrations between ∞-categories belong to W.*

proof: Let $f : X \to Y$ be a morphism of simplicial sets. We apply the argument of small objects to $\{\wedge_n^k \hookrightarrow \Delta_n \mid 0 < k < n\}$ allows us to construct the following commutative diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & X' \arrow[d, "f'"] \\ Y \arrow[r] & Y' \end{tikzcd}$$

where the horizontal arrows are inner anodyne extensions. We have thus:

$$f \in W \quad \Leftrightarrow \quad f' \in W.$$

Moreover, from the previous theorem and the Brown lemma, we know that all categorical weak equivalences between ∞-categories are in W, which conclude the proof.

Corollary 3.2.7. *A morphism between ∞-categories is a categorical weak equivalence if and only if it is an equivalence of ∞-categories. (i.e. a J-homotopy equivalence.)*
In particular, if $f : A \to B$ is a functor, then f is an equivalence of categories if and only if $Nf : NA \to NB$ is a categorical weak equivalence.

Corollary 3.2.8. *The isofibrations are stable under base changes in the category of ∞-categories.*

Corollary 3.2.9. *The functor $\tau : \widehat{\Delta} \to Cat$ sends the categorical weak equivalences to the equivalences of categories.*

proof: τ sends anodyne extensions to isomorphisms. The restriction of τ to ∞-categories commutes to finite products and $\tau(J) = I$, which implies that τ sends the equivalences of ∞-categories to the equivalences of categories.

Theorem 3.2.10. *Let $A \to B$ be a morphism between simplicial sets. The following conditions are equivalent:*

1. *the morphism $A \to B$ is a categorical weak equivalence.*
2. *for each ∞-category X, the morphism*
$$\underline{Hom}(B, X) \longrightarrow \underline{Hom}(A, X)$$
is an equivalence of ∞-categories.
3. *for each ∞-category X, the functor*
$$\tau\underline{Hom}(B, X) \longrightarrow \tau\underline{Hom}(A, X)$$
is an equivalence of categories.
4. *for each ∞-category X, the morphism*
$$k(B, X) \longrightarrow k(A, X)$$
is an equivalence of ∞-groupoids.

proof: We notice that form any ∞-category X, we have:

$$Hom_{Ho_J(\widehat{\Delta})}(A, X)$$
$$\simeq \{J - homotopy\ classes\ of\ morphisms\ A \to X\}$$
$$\simeq \pi_0 k(A, X)$$
$$\simeq \{classes\ of\ isomorphisms\ of\ objects\ in\ \tau\underline{Hom}(A, X)\}.$$

Each of the condition 3. and 4. implies 1.. The condition 2. implies the condition 3. (since $\tau : \widehat{\Delta} \to Cat$ sends the categorical weak equivalences to the equivalences of categories), and the condition 1. implies the condition 2 (since $\underline{Hom}(-.X)$ is a Quillen functor).

It leaves us to prove that the condition 3. implies the condition 4. According to Brown's lemma, we could suppose that $A \to B$ is a trivial cofibration. In this case, $\underline{Hom}(B, X) \to \underline{Hom}(A, X)$ is a trivial cofibration. In particular, this is a conservative isofibration, which implies the following commutative diagram is cartesian:

$$\begin{tikzcd} k(B, X) \arrow[r, "="] & k\underline{Hom}(B, X) \arrow[r] \arrow[d] & \underline{Hom}(B, X) \arrow[d] \\ k(A, X) \arrow[r, "="] & k\underline{Hom}(A, X) \arrow[r] & \underline{Hom}(A, X) \end{tikzcd}.$$

We could then conclude the proof.

Similarly, we have:

Theorem 3.2.11. *Let $X \to Y$ be a morphism between ∞-categories. The following conditions are equivalent:*

1. *the morphism $X \to Y$ is an equivalence of ∞-categories.*

2. *for each simplicial set A, the morphism*

$$\underline{Hom}(A, X) \longrightarrow \underline{Hom}(A, Y)$$

is an equivalence of ∞-categories.

3. *for each simplicial set A, the functor:*

$$\tau\underline{Hom}(A, X) \longrightarrow \tau\underline{Hom}(A, Y)$$

is an equivalence of categories.

4. *for each simplicial set A, the morphism*

$$k(A, X) \longrightarrow k(A, Y)$$

is an equivalence of ∞-groupoids.

3.3 Homotopy Limits in Model Categories

3.3.1 Homotopy Limits and Homotopy Colimits

Let C be a category admitting projective limits.

If I is a small category, the functor lim:

$$lim_I : \underline{Hom}(I, C) \longrightarrow C$$

is a right adjoint to the "constant diagram" functor:

$$\begin{array}{rcl} (-)_I : C & \to & \underline{Hom}(I, C) \\ X & \longrightarrow & X_I \end{array}$$

Suppose now that C is moreover a model category.

We say that a morphism $F \to G$ in $\underline{Hom}(I, C)$ is a **termwise weak equivalence** if, for each object $i \in I$, the morphism $F_i \to G_i$ is a weak equivalence in C.

We will denote $Ho\underline{Hom}(I, C)$ the localization of $\underline{Hom}(I, C)$ by the termwise weak equivalence. The constant diagram functor preserve the weak equivalences (sends weak equivalences to termwise weak equivalences) and thus induces a *homotopic constant diagram* functor:

$$\begin{array}{rcl} (-)_I : Ho(C) & \to & Ho\underline{Hom}(I, C) \\ X & \longmapsto & X_I \end{array}$$

Theorem 3.3.1. *Suppose that C admits projective limits (resp. inductive limits). The functor*

$$(-)_I : Ho(C) \longrightarrow Ho\underline{Hom}(I, C)$$

admits a right adjoint (resp, a left adjoint):

$$holim_I \text{ (resp. } hocolim_I) : \; Ho\underline{Hom}(I, C) \longrightarrow Ho(C).$$

*If $F : I \to C$ is a functor considered as an object in the category $Ho\underline{Hom}(I, C)$, $holim_I F$ (resp. $hocolim_I F$) is called the **homotopy limit** (resp. **homotopy colimit**) of F.*

The proof of this theorem is very technical and difficult: we do not know how to construct a Quillen model structure on $\underline{Hom}(I, C)$, so that we can not use Quillen functors, adjoint functors, derived functors.......

I will compile the treatment of Klingler or Hirschhorn here later.

Example 3.3.2. If the category I is discrete (all the morphisms are identities such that we could view I as a set), we have $\underline{Hom}(I,C) \simeq \prod_{i\in I} C$ has a model category structure, where the weak equivalences, cofibrations and fibrations are defined termwise. The constant diagram functor is thus a left and right Quillen functor, so that:

$$Ho(C) \longrightarrow Ho\underline{Hom}(I,C) = \prod_{i\in I} Ho(C)$$

admits a right adjoint and a left adjoint.

We have thus a functor *homotopy sum indexed by* I:

$$\begin{array}{rcl} \prod_{i\in I} Ho(C) & \to & Ho(C) \\ (X_i)_{i\in I} & \longmapsto & \coprod^{\mathbf{L}}_{i\in I} X_i = \coprod_{i\in I} X'_i \end{array}$$

where X'_i is a cofibrant replacement of X_i, and a functor *homotopy product indexed by* I:

$$\begin{array}{rcl} \prod_{i\in I} Ho(C) & \to & Ho(C) \\ (X_i)_{i\in I} & \longmapsto & \prod^{\mathbf{R}}_{i\in I} X_i = \prod_{i\in I} X'_i \end{array}$$

where X'_i is a fibrant replacement of X_i.

3.3.2 Homotopy Fibre Products

Let $\lrcorner$ be the category generated by the oriented graphs:

$$\begin{tikzcd} & b \arrow[d] \\ a \arrow[r] & c \end{tikzcd}$$

Given a functor $\lrcorner \xrightarrow{X} C$ is equivalent to be given a diagram in C of the following form:

$$X = \begin{tikzcd} & X_b \arrow[d] \\ X_a \arrow[r] & X_c \end{tikzcd}$$

The object $lim_{\lrcorner} X$ is therefore the fibre product

$$X_a \times_{X_c} X_b.$$

A morphism in $\underline{Hom}(\lrcorner, C)$ is a commutative diagram of the form:

$$\begin{tikzcd} X_a \arrow[r, "f_a"] \arrow[d] & Y_a \arrow[d] \\ X_c \arrow[r, "f_c"] & Y_c \\ X_b \arrow[r, "f_b"] \arrow[u] & Y_b \arrow[u] \end{tikzcd}$$

Theorem 3.3.3. *The category $\underline{Hom}(\lrcorner, C)$ has a model category structure, in which the weak equivalences are termwise weak equivalences and the fibrations and cofibrations can be describe as:*

A morphism $X \to Y$ is a fibration (resp. a cofibration) if and only if it satisfies the following conditions:

1. *f_b and f_c are fibrations (resp. f_a and f_c are cofibrations).*
2. *the morphism $X_a \to X_c \times_{Y_c} Y_a$ is a fibration (resp. $X_c \coprod_{X_b} Y_b \to Y_c$ is a cofibration).*

In particular, the fibrant objects are the diagram in C of the form:

$$\begin{tikzcd} & X_b \arrow[d] \\ X_a \arrow[r] & X_c \end{tikzcd}$$

such that X_a, X_b and X_c are fibrant and $X_a \to X_c$ is a fibration.

Moreover, the functor $lim_{\lrcorner} : \underline{Hom}(\lrcorner, C) \to C$ is a right Quillen functor.

proof:

Definition 3.3.4. A commutative diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

is **homotopy cartesian** if the induced morphism

$$X' \to holim_\lrcorner \left(\begin{tikzcd} & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd} \right)$$

is an isomorphism in $Ho(C)$.

The commutative diagram is called **homotopy cocartesian** in C if it is homotopy cartesian in C^{op}.

We will write

$$Y' \times^h_Y X = holim_\lrcorner \left(\begin{tikzcd} & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd} \right)$$

to be the **homotopy fibre product** of the diagram

$$\begin{tikzcd} & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

. We have the similar notation of the **homotopy amalgamated sum** which is written as $A \amalg^h_B B'$.

If in a diagram

$$\begin{tikzcd} & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd},$$

all the objects are fibrant and if $Y' \to Y$ or $X \to Y$ is a fibration (which is equivalence to say that the diagram is a fibrant object in $\underline{Hom}(\lrcorner, C)$), then we have:

$$Y' \times_Y X \simeq Y' \times^h_Y X.$$

Up to weak equivalence, all homotopy fibre products are of this form.

We have the following properties:

1. if we have a commutative diagram of the form"

$$\begin{tikzcd} X'' \arrow[r] \arrow[d] \arrow[dr, phantom, "(C')"] & X' \arrow[r] \arrow[d] \arrow[dr, phantom, "(C')"] & X \arrow[d] \\ Y'' \arrow[r] & Y' \arrow[r] & Y \end{tikzcd}$$

   the ((C) homotopy cartesian, then (C') is homotopy cartesian if and only if the big diagram $(C) + (C')$ is homotopy cartesian.

2. if we have a commutative cube in C:

$$\begin{tikzcd}
A' \arrow[rr] \arrow[dr, "a'"] \arrow[dd] & & A \arrow[dr, "a"] \arrow[dd, dashed] & \\
& X' \arrow[rr] \arrow[dd] & & X \arrow[dd] \\
B' \arrow[rr, dashed] \arrow[dr, "b'"] & & B \arrow[dr, "b"] & \\
& Y' \arrow[rr] & & Y
\end{tikzcd}$$

where the front face and the back face are homotopy cartesian, and if a, b, b' are weak equivalences, then a' is a weak equivalence.

3. if

$$\begin{tikzcd}
X' \arrow[r, "g"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Y' \arrow[r, "h"'] & Y
\end{tikzcd}$$

is homotopy cartesian, and if f is a weak equivalence, then f' is a weak equivalence.

Theorem 3.3.5. *Let $\mathbb{N}$ to be the normal ordinal set. The category $\underline{Hom}(\mathbb{N}^{op}, C)$ admits a closed model category structure where the weak equivalences (resp. cofibrations) are termwise. The fibrations are the morphisms:*

$$\begin{tikzcd}
\cdots \arrow[r] & X_{n+1} \arrow[r] \arrow[d, "p_{n+1}"] & X_n \arrow[r] \arrow[d, "p_n"] & \cdots \arrow[r] & X_1 \arrow[r] \arrow[d, "p_1"] & X_0 \arrow[d, "p_0"] & X \arrow[d, "p"] \\
\cdots \arrow[r] & Y_{n+1} \arrow[r] & Y_n \arrow[r] & \cdots \arrow[r] & Y_1 \arrow[r] & Y_0 & Y
\end{tikzcd}$$

such that p_0 is a fibration and $X_{n+1} \to Y_{n+1} \times_{Y_n} X_n$ is a fibration for all $b \geq 0$.

The functor $lim_{\mathbb{N}^{op}} : \underline{Hom}(\mathbb{N}^{op}, C) \to C$ is a right Quillen functor.

Corollary 3.3.6. *For any diagram in C of the form*

$$\begin{tikzcd}
\cdots \arrow[r] & X_{n+1} \arrow[r] \arrow[d, "p_{n+1}"] & X_n \arrow[r] \arrow[d, "p_n"] & \cdots \arrow[r] & X_1 \arrow[r] \arrow[d, "p_1"] & X_0 \arrow[d, "p_0"] & X \arrow[d, "p"] \\
\cdots \arrow[r] & Y_{n+1} \arrow[r] & Y_n \arrow[r] & \cdots \arrow[r] & Y_1 \arrow[r] & Y_0 & Y
\end{tikzcd},$$

if all the objects are fibrant, and if all the morphisms $X_{n+1} \to X_n$ and $Y_{n+1} \to Y_n$ are fibrations, and if $\forall n$ $X_n \to Y_n$ is a weak equivalence, then $lim_n X_n \to lim_n Y_n$ is a weak equivalence.

Lemma 3.3.7 (Transfer Lemma, Kan-Thomason-Crans)**.** *We consider a pair of adjoint functors*

$$\begin{tikzcd}
C \arrow[r, shift left, "L"] & C' \arrow[l, shift left, "R"]
\end{tikzcd}.$$

Suppose that C is equipped with a cofibrantly generated model category structure. We choose an admissible set I (resp. J) which generates the cofibrations (resp. trivial cofibrations), and we suppose that the following conditions are satisfied:

1. *the set of morphisms $L(I)$ and $L(J)$ are admissible in C'.*

2. *the right adjoint D sends the elements of $Sat(L(J))$ to weak equivalences.*

If moreover C' admits inductive and projective limits, then C' is equipped with a cofibrantly generated model category structure, where $L(I)$ generates the cofibrations and $L(J)$ generates trivial cofibrations, and the weak equivalences (resp. fibrations) in C' are the morphisms that are sent to the weak equivalences (resp. fibrations) by the functor R.

Furthermore, the couple (L, R) is a pair of Quillen functors.

Theorem 3.3.8. *Let C be a cofibrantly generated model category, and I a small category.*

The category of functors $\underline{Hom}(I, C)$ has a cofibrantly generated model category structure, in which the weak equivalences (resp. the fibrations) are the termwise weak equivalences (resp. fibrations). The functor $colim_I : \underline{Hom}(I, C) \to C$ is a left Quillen functor.

proof:
The evaluation functor:

$$\begin{array}{rcl} ev : \underline{Hom}(I, C) & \to & \prod_{i \in Ob(I)} C \\ F & \longmapsto & (F_i)_{i \in I} \end{array}$$

has a left adjoint

$$L : \prod_{i \in Ob(I)} C \longrightarrow \underline{Hom}(I, C)$$

who associate a family $(X_i)_{i \in ObI}$ to the functor defined by

$$j \longmapsto \coprod_{i \in ObI} \coprod_{Hom_I(i,j)} X_i.$$

We could verify that this pair of adjoint functors verify the hypothesis of the transfer lemma and thus conclude the proof.

3.4 Properness of Model Categories

Definition 3.4.1. A model category C is said to be **right proper** if for each cartesian diagram of the form

$$\begin{tikzcd} X' \arrow[r, "u"] \arrow[d, "p'"] & X \arrow[d, "p"] \\ Y' \arrow[r, "v"'] & Y \end{tikzcd}$$

with p a fibration and v a weak equivalence, we then obtain that u is a weak equivalence.

A model category C is said to be **left proper** if C^{op} is right proper.

A model category is said to be **proper** if it is left and right proper.

For a model category C, the following conditions are equivalent:

1. C is right proper.
2. each cartesian diagram of the form

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

   with p a fibration is homotopy cartesian. This can be proved my using the *cube* in the last section.

3. for each weak equivalence $X \xrightarrow{u} Y$, the functor

$$u_! : Ho(C/X) \longrightarrow Ho(C/Y)$$

   is an equivalence of categories.

Example 3.4.2. If, in a model category C, all the objects are fibrant (resp. cofibrant), then C is right proper (resp. left proper).

Theorem 3.4.3. *Let A be a small category, and (I, An) be a homotopy structure on $\widehat{A}$.*

We choose a set S of morphisms in $\widehat{A}$ such that $An_I(S) = An$.

The model category structure on $\widehat{A}$ associated to (I, An) is (right) proper if and only if, for all cartesian diagrams of the form:

$$\begin{tikzcd} X'' \arrow[r, "u"] \arrow[d] & X' \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ Y'' \arrow[r, "v"'] & Y' \arrow[r] & Y \end{tikzcd}$$

with p a fibration between fibrant objects and $v \in S$, u is a weak equivalence.

Please refer to Ast. 308, theorem 1.5.1 for the proof.

Example 3.4.4. $(I, An_I(\emptyset))$ gives us always a proper model category structure (e.g. the Quillen structure on $\widehat{\Delta}$).

Example 3.4.5. The Joyal model category structure is left proper but **not** right proper.

3.5 Fibrewise Kan Fibrations

Lemma 3.5.1. *Let A be a small category, and (I, An) be a homotopy structure on $\widehat{A}$. Consider the cartesian diagram in $\widehat{A}$ of the form*

$$\begin{tikzcd} Z \times_Y X \arrow[r, hook, "j"] \arrow[d, "q"] & X \arrow[d, "p"] \\ Z \arrow[r, "i"'] & Y \end{tikzcd}$$

with p a naive fibration. Then if i is a retract by deformation, j is also a retract by deformation.

proof: Consider $h : I \times Y \to Y$ and $r : Y \to Z$ such that $ri = Id_Y$ and $h|_{\{0\}\times Y} = Id_Y$, $h|_{\{1\}\times Y} = ir$, and $h(Id_I \times i) = \sigma_Y(Id_I \times i)$. We have a commutative diagram:

$$\begin{tikzcd} \{0\} \times (X \times_Y X) \arrow[r] \arrow[d] & \{0\} \times X \simeq X \\ I \times (Z \times_Y X) \arrow[r, "\sigma_{Z\times_Y X}"'] & Z \times_Y X \arrow[u, "j"] \end{tikzcd}$$

and a commutative diagram admitting a lifting k:

$$\begin{tikzcd} I \times (Z \times_Y X) \cup \{0\} \times X \arrow[r, "(j\sigma_{Z\times_Y X}, Id_X)"] \arrow[d, hook] & X \arrow[d, "p"] \\ I \times X \arrow[r, "h(Id_I \times p)"'] \arrow[ur, dashed] & Y \end{tikzcd}$$

Thus we have $k(Id_I \times j) = j\sigma_{Z\times_Y X} = \sigma_X(Id_I \times j)$. Let

$$s = (rp, k\partial^1_X) : X \longrightarrow Z \times_Y X$$

we have:

$$\begin{tikzcd} X \arrow[r, "k\partial^1_X"] \arrow[d, "rp"'] & X \arrow[d, "p"] \\ Z \arrow[r, "i"'] & Y \end{tikzcd} .$$

Thus $sj = Id_{Z\times_Y X}$, $kk\partial^0_X = Id_X$, $kk\partial^1_X = js$.

Definition 3.5.2. Let A be a small category. A class C of objects in $\widehat{A}$ is **saturated by monomorphisms** if it satisfies the following properties:

1. the class C is stable by arbitrary (small) sums.
2. for each cocartesian diagram in $\widehat{A}$ of the form:

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook, "i"] & X' \arrow[d, hook] \\ Y \arrow[r] & Y' \end{tikzcd}$$

with i a monomorphism and $X, X', Y \in C$, we have $Y' \in C$. (stable by pushouts.)

3. for each sequence of monomorphisms in $\widehat{A}$ of the form:

$$X_0 \hookrightarrow X_1 \hookrightarrow \ldots \hookrightarrow X_n \hookrightarrow X_{n+1} \hookrightarrow \ldots \ , \quad n \geq 0$$

if $\forall n \geq 0$, $X_n \in C$, then $X_\infty = colim_n X_n \in C$.

Proposition 3.5.3. *Let B be a simplicial set. The smallest class of objects saturated by monomorphism in $\widehat{\Delta}/B \simeq \widehat{\Delta/B}$ which contains all the representables $\Delta_n \xrightarrow{b} B, n \geq 0, b \in B_n$ is the the class of all the simplicial sets above B.*

proof: Let C be a class of objects saturated by monomorphism in $\widehat{\Delta}/B \simeq \widehat{\Delta/B}$ contains the representables $\Delta_n \xrightarrow{b} B, n \geq 0, b \in B_n$.

We start by proving that for each integer $m \geq 0$ and each subset $I \subset \{0, 1, \ldots, m\}$, each m-simplex $b \in B_m$, the object

$$B \xleftarrow{\quad b \quad} K = \bigcup_{i \in I} Im(\Delta_{m-1} \xrightarrow{\delta_m^i} \Delta_m) \subset \Delta_m$$

is in C. We deduce on m:

If $m \leq 1$, the K is a finite sum of representables,

If $m > 1$, we deduce on the number of objects of I. If $\sharp I \leq 1$, K is empty and representable. If $\sharp > 1$, let $i = max\{I\}$, and $J = I - \{i\}$, $L = cup_{j \in J} Im(\delta_m^j) \subset \Delta_m$. We thus have a cartesian diagram of the form:

$$\begin{tikzcd} L \cap \Delta_{m-1} \arrow[r, hook] \arrow[d, hook] & \Delta_{m-1} \arrow[d, hook, "\delta_m^i"] \\ L \arrow[r, hook] & K \end{tikzcd}$$

and

$$L \cap \Delta_{m-1} = \bigcup_{j \in J} Im(\delta_m^i) \cap Im(\delta_m^i) = \bigcup_{j \in J} Im(\delta_{m-1}^{j-1}).$$

We have then $L \cap \Delta_{m-1} \to B \in C$, $L \to B \in C$ and $\Delta_{m-1} \to B \in C$, which implies that $K \to B \in C$. In particular, for all $m \geq 0$ and each $b \in B_n$, $\partial \Delta_m \xrightarrow{b} B \in C$.

If X/B is a simplicial set above B, we have the cocartesian diagram of the form:

$$\begin{tikzcd} \coprod_\Sigma \partial \Delta_m \arrow[r] \arrow[d, hook] & Sk_{m-1}(X) \arrow[d, hook] \\ \coprod_\Sigma \Delta_m \arrow[r] & Sk_m(X) \end{tikzcd}$$

and $X = \bigcup_m Sk_m(X)$ and $X \to B \in C$.

Proposition 3.5.4. *Let $p : X \to Y$ be a Kan fibration and Y is an ∞-groupoid. Then each cartesian diagram of the form*

$$\begin{tikzcd} \Delta_n \times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta_n \arrow[r] & Y \end{tikzcd}$$

is homotopy cartesian. (in the sense of Quillen model category structure.)

proof: The assertion is evident for $n = 0$, since we have Δ_0, and X, Y are fibrant.

If $n > 0$, we have a retract by deformation $\Delta_0 \to \Delta_n, 0 \mapsto 0$, and thus a retract by deformation $\Delta_0 \times_Y X \hookrightarrow \Delta_n \times_Y X$. We conclude the proof by considering the following diagram:

$$\begin{tikzcd}
\Delta_0 \times_Y X \arrow[r, "\sim"] \arrow[d] & \Delta_n \times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\
\Delta_0 \arrow[r, "\sim"'] & \Delta_n \arrow[r] & Y
\end{tikzcd}$$

Remark 3.5.5. We could prove that the Kan fibrations are exactly the fibrations in the Quillen model structure, so the hypothesis of Y to be an ∞-groupoid is not necessary.

Proposition 3.5.6. *Consider the commutative triangle of simplicial sets*

$$\begin{tikzcd}
X \arrow[rr, "u"] \arrow[dr, "p"'] & & X' \arrow[dl, "p'"] \\
& Y &
\end{tikzcd}$$

Suppose that p and p' are Kan fibrations and Y is an ∞-groupoid.

Then u is a weak equivalence if and only if for each n-simplex $\Delta_n \to Y$, the morphism

$$\Delta_n \times_Y X \longrightarrow \Delta_n \times_Y X'$$

is a weak equivalence.

proof: For each morphism $Z \xrightarrow{q} Y$, the functor

$$\begin{array}{cccc} q^* : & \widehat{\Delta}/Y & \to & \widehat{\Delta}/Z \\ & (E \to Y) & \longmapsto & (Z \times_Y E \to Z) \end{array}$$

commutes with inductive limits and respect the monomorphisms. We deduce that the class of objects in $\widehat{\Delta}/Y$ such that $E \times_Y X \to E \times_Y X'$ is a weak equivalence is saturated by monomorphisms, which leads to the result.

Remark 3.5.7. Here, the hypothesis on Y is not necessary neither. (Consult to the previous remark.)

3.6 Local Characterization of equivalences of ∞-categories

Theorem 3.6.1. *Let $X \to Y$ be a morphism of ∞-categories. The following conditions are equivalent:*

1. *the morphisms $k(\Delta_n, X) \to k(\Delta_n, Y)$ are equivalences of ∞-groupoids for $n = 0, 1$.*
2. *the morphisms $k(\Delta_n, X) \to k(\Delta_n, Y)$ are equivalences of ∞-groupoids for $n \geq 0$.*
3. *the morphisms $k(A, X) \to k(A, Y)$ are the equivalences of ∞-categories for all simplicial sets A.*
4. *the morphism $X \to Y$ is an equivalence of ∞-categories.*

We need some preparations for this proof.

Lemma 3.6.2. *For each trivial cofibration $K \hookrightarrow L$ (in the Joyal model category structure) and each monomorphism $A \hookrightarrow B$, the inclusions:*

$$B * K \cup A * L \hookrightarrow B * L \quad \text{and} \quad L * A \cup K * B \hookrightarrow L * B$$

are trivial cofibrations.

proof: We know that, for each inner Kan fibration $X \xrightarrow{p} Y$, and each morphism $B \to X$, the morphism

$$X/B \longrightarrow X/A \times_{Y/A} Y/B$$

is a right Kan fibration, and so a fibration (due to the condition what says Y is an ∞-category).

However, we have the correspondence:

$$\begin{tikzcd} L * A \cup K * B \arrow[r] \arrow[d, hook] & X \arrow[d] \\ L * B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \qquad \rightsquigarrow \qquad \begin{tikzcd} K \arrow[r] \arrow[d, hook] & X/B \arrow[d] \\ L \arrow[r] \arrow[ur, dashed] & X/A \times_{Y/A} Y/B \end{tikzcd}$$

However, we know that the trivial cofibrations in the Joyal model category structure are the morphisms having the RLP with respect to fibrations between fibrant objects (i.e. isofibrations between ∞-categories). This completes the proof.

Recall that, for $n \geq 2$, $I_n \subset \Delta_n$ is defined as:

$$I_n = \bigcup_{1 \leq i \leq n} Im(\Delta_1 \xrightarrow{u_i} \Delta_n)$$

where u_i is defined by $u(0) = i - 1$ and $u(1) = i$.

Proposition 3.6.3. *The inclusions $I_n \hookrightarrow \Delta_n,\ n \geq 2$ are categorical weak equivalent.*

proof: We proceed by deducing on $n \geq 2$:

$n = 2$: $I_n = \wedge_2^1 \hookrightarrow \Delta_2$ is an inner anodyne extension.

$n > 2$: we write $\Delta_n = \Delta_{n-1} * \Delta_0$. Since the inclusion $I_{n-1} \hookrightarrow \Delta_{n-1}$ is a categorical weak equivalence, the previous lemma assures the inclusion

$$I_{n-1} * \Delta_0 \hookrightarrow \Delta_{n-1} * \Delta_0$$

is a categorical weak equivalence. However, we could write $I_{n-1} * \Delta_0 \subset \Delta_n$ as:

$$I_{n-1} * \Delta_0 = (I_{n-1} * \emptyset) \cup Im(\Delta_2 \xrightarrow{v} \Delta_n)$$

where v is defined by

$$v(0) = n - 2, \qquad v(1) = n - 1, \qquad v(2) = n.$$

Thus we have a cartesian and cocartesian diagram

$$\begin{tikzcd} \wedge_2^1 \arrow[r] \arrow[d, hook] & I_n \arrow[d, hook] \\ \Delta_2 \arrow[r] & I_{n-1} * \Delta_0 \end{tikzcd}$$

The composed morphism $I_n \to I_{n-1} * \Delta_0 \to \Delta_n$ is thus a categorical weak equivalence.

Lemma 3.6.4. *Let X be an ∞-category.*

1. *for each cocartesian diagram of simplicial sets*

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook, "i"] & A' \arrow[d, hook, "i'"] \\ B \arrow[r] & B' \end{tikzcd} .$$

with i a monomorphism, the diagram:

$$\begin{tikzcd} k(B',X) \arrow[r] \arrow[d, "i'^*"] & k(B,X) \arrow[d, "i^*"] \\ k(A',X) \arrow[r] & k(A,X) \end{tikzcd}$$

is cartesian, and i^* is a Kan fibration between ∞-groupoids. We have a homotopy equivalence

$$k(B',X) \simeq k(A',X) \times^h_{k(A,X)} k(B,X).$$

2. *for each sequence of monomorphisms of simplicial sets:*

$$A_0 \hookrightarrow A_1 \hookrightarrow A_2 \hookrightarrow\hookrightarrow \ldots \hookrightarrow A_n \hookrightarrow A_{n+1} \hookrightarrow \ldots$$

If we let $A_\infty = colim_i A_i$, then we have an isomorphism

$$k(A_\infty, X) \simeq lim_n k(A_n, X)$$

and a homotopy equivalence

$$k(A_\infty, X) \simeq holim_n k(A_n, X).$$

3. *for each family of simplicial sets $(A_i)_{i\in I}$, we have:*

$$k(\coprod_i A_i, X) \simeq \prod_i k(A_i, X).$$

proof:

1. We have a cartesian diagram of ∞-categoires:

$$\begin{tikzcd} \underline{Hom}(B',X) \arrow[r] \arrow[d, "i'^*"] & \underline{Hom}(B,X) \arrow[d, "i^*"] \\ \underline{Hom}(A',X) \arrow[r] & \underline{Hom}(A,X) \end{tikzcd}$$

with i^* an isofibration. We conclude 1. by noticing that the functor k preserves cartesian diagrams.

2. We notice that $lim_n k(A_n, X)$ is an ∞-groupoid contained in $\underline{Hom}(A_\infty, X) = lim_n \underline{Hom}(A_n, X)$ which contains $k(\underline{Hom}(A_\infty, X)) = k(A_\infty, X)$, form which we get the isomorphism of simplicial sets

$$k(A_\infty, X) \simeq lim_n k(A_n, X).$$

Since the morphisms $k(A_{n+1}, X) \to k(A_n, X)$ are Kan fibrations between ∞-groupoids, we have an equivalence

$$lim_n k(A_n, X) \simeq holim_n k(A_n, X).$$

3. use the similar methods as 2.

Lemma 3.6.5. *If a class of objects in $\widehat{\Delta}$ is saturated by monomorphisms and contains Δ_0 and Δ_1, then it contains I_n, $n \geq 2$ as well.*

proof: We have the cocartesian diagram of the form:

$$\begin{tikzcd} \Delta_0 \arrow[r] \arrow[d] & I_{n-1} \arrow[d] \\ \Delta_1 \arrow[r] & I_n \end{tikzcd} , \qquad n \geq 2 \ \ (\textit{with } I_1 = \Delta_1)$$

proor of the theorem:
Let $X \to Y$ be a morphisms of ∞-categories.
Note C to be the class of simplicial sets A such that

$$k(A, X) \to k(A, Y)$$

is an equivalence of ∞-groupoids. We have proven that the class C is saturated by monomorphisms.

$1 \Rightarrow 2$: if $\Delta_0, \Delta_1 \in C$, according to the previous lemma, we have

$$\forall n \geq 2, \quad I_n \in C.$$

However,, for each ∞-category C, the morphisms

$$k(\Delta_n, C) \longrightarrow k(I_n, C)$$

are trivial fibrations (since $I_n \to \Delta_n$ is a categorical weak equivalence). The commutative diagram:

$$\begin{tikzcd} k(\Delta_n, X) \arrow[r] \arrow[d, "\sim"] & k(\Delta_n, Y) \arrow[d, "\sim"] \\ k(I_n, X) \arrow[r] & k(I_n, Y) \end{tikzcd}$$

proves that $\Delta_n \in C$ for $n \geq 2$.

$2 \Rightarrow 3$: this comes from the fact that C is saturated by monomorphisms.

$3 \Leftrightarrow 4$: we have already proven this.

Since $3 \Rightarrow 1$ is clear, we conclude the proof of the theorem.

Consider an ∞-category X.
We have a morphism

$$\underline{Hom}(\Delta_1, X) \xrightarrow{(s,t)} X \times X$$

induced by the inclusion $\{0\} \bigsqcup \{1\} = \partial\Delta_1 \hookrightarrow \Delta_1$ and the canonical isomorphism

$$\underline{Hom}(\partial\Delta_1.X) = \underline{Hom}(\{0\}, X) \times \underline{Hom}(\{1\}, X) \simeq X \times X.$$

If $(a, b) \in X_0^2$ are two 0-simplexes of X (i.e. two objects of the ∞-category X), we define

$$X(a, b)$$

by the following cartesian diagram:

$$\begin{tikzcd} X(a,b) \arrow[r] \arrow[d] & \underline{Hom}(\Delta_1, X) \arrow[d, "(s,t)"] \\ * = \Delta_0 \arrow[r, "(a,b)"'] & X \times X \end{tikzcd}$$

The 0-simplexes of $X(a, b)$ are thus the morphisms of a to b in X.

Proposition 3.6.6. *$X(a, b)$ is an ∞-groupoid.*

proof:
The equality $k\underline{Hom}(\Delta_1, X) = k(\Delta_1, X)$ and the fact that the inclusion $\partial\Delta_1 \hookrightarrow \Delta_1$ is bijective between objects imply that we have a cartesian diagram:

$$\begin{tikzcd}
k\underline{Hom}(\Delta_1, X) \arrow[r] \arrow[d, "k(s,t)"] & \underline{Hom}(\Delta_1, X) \arrow[d, "(s,t)"] \\
k(X) \times k(X) \arrow[r] & X \times X
\end{tikzcd}$$

and we know that $k(s,t)$ is a Kan fibration between ∞-groupoids. We then obtain a cartesian diagram

$$\begin{tikzcd}
X(a,b) \arrow[r] \arrow[d] & k\underline{Hom}(\Delta_1, X) \arrow[r] \arrow[d] & \underline{Hom}(\Delta_1, X) \arrow[d, "(s,t)"] \\
\Delta_0 \arrow[r, "(a,b)"'] & k(X) \times k(X) \arrow[r, hook] & X \times X
\end{tikzcd}$$

Definition 3.6.7. A morphism of ∞-categories $X \xrightarrow{u} Y$ is fully faithful if, for each couple of objects (a,b) of X, the induced morphism

$$X(a,b) \longrightarrow Y(u(a), u(b))$$

is an equivalence of ∞-groupoids.

A morphism of ∞-groupoids $X \xrightarrow{u} Y$ is essentially surjective if, for each object b of Y, there exists an object $a \in X$, and a morphism $\Delta_1 \to Y$ quasi-invertible with source $u(a)$ and target b.

Remark 3.6.8. A morphism $X \xrightarrow{u} Y$ is essentially surjective if and only if the functor $\tau(u) : \tau(X) \to \tau(Y)$ is essentially surjective of categories.

Proposition 3.6.9. *Let X be an ∞-category, and a, b be two objects in X. The canonical map*

$$\pi_0(X(a,b)) \longrightarrow Hom_{hoX}(a,b) \simeq Hom_{\tau X}(a,b)$$

is bijective.

proof: The subjectivity of this map is evident.

Consider two 0-simplexes $f, g \in X(a,b)_0$, the images of whom in $Hom_{hoX}(a,b)$ coincide. We have thus a commutative triangle in X of the form

$$\begin{tikzcd}
a \arrow[dr, "f"] \arrow[dd, "Id_a"'] & \\
\arrow[r, phantom, "c"] & b \\
a \arrow[ur, "g"'] &
\end{tikzcd} \qquad c : \Delta_2 \to X.$$

We have also the commutative triangle of the form:

$$\begin{tikzcd}
 & b \arrow[dd, "Id_b"] \\
a \arrow[ur, "f"] \arrow[dr, "f"'] \arrow[r, phantom, "c'"] & \\
 & b
\end{tikzcd} \qquad c' : \Delta_2 \to X.$$

The triangulation of $\Delta_1 \times \Delta_1$ corresponds to the diagram

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] \arrow[dr] \arrow[dr, phantom, "c", shift left=2] \arrow[dr, phantom, "c'"', shift right=2] & (0,1) \arrow[d] \\
(1,0) \arrow[r] & (1,1)
\end{tikzcd}$$

gives us a morphism

$$(c,c') : \Delta_1 \times \Delta_1 \longrightarrow X$$

and thus a morphism $\Delta_1 \to \underline{Hom}(\Delta_1, X)$ who factorizes through $X(a,b)$ and connects f and g, which proves the injectivity.

We could describe the composition of the morphisms in X as the following:
We have a Kan fibration between ∞-groupoids

$$k\underline{Hom}(\Delta_2, X) = k(\Delta_2, X) \longrightarrow k(X) \times k(X) \times k(X)$$

induced by the inclusion $\{0\} \coprod \{1\} \coprod \{2\} \hookrightarrow \Delta_2$.

If a, b, c are three objects of X, we define $X(a,b,c)$ by the following cartesian diagram:

$$\begin{tikzcd}
X(a,b,c) \arrow[r] \arrow[d] & k(\Delta_2, X) \arrow[d] \\
* = \Delta_0 \arrow[r, "(a,b,c)"'] & k(X)^3
\end{tikzcd} .$$

We have thus a commutative cube whose front and back faces are homotopy cartesian:

$$\begin{tikzcd}
k(\wedge^1_2, X) \arrow[rr] \arrow[dr] \arrow[dd] & & k(\Delta_1, X) \arrow[dr, "(s,t)"] \arrow[dd, dashed, "s" near start] & \\
& k(X)^3 \arrow[rr] \arrow[dd, "t" near start] & & k(X)^2 \arrow[dd] \\
k(\Delta_1, X) \arrow[rr, dashed] \arrow[dr, "(s,t)"] & & k(X) \arrow[dr, "="] & \\
& k(X)^2 \arrow[rr, "pr_2"'] & & k(X)
\end{tikzcd}$$

We deduce thus a cartesian diagram

$$\begin{tikzcd}
X(a,b) \times X(b,c) \arrow[r] \arrow[d] & k(\wedge^1_2, X) \arrow[d] \\
* \arrow[r, "(a,b,c)"'] & k(X)^3
\end{tikzcd} ,$$

and a cartesian diagram of the form

$$\begin{tikzcd}
X(a,b,c) \arrow[r] \arrow[d, two heads, "\sim"] & k(\Delta_2, X) \arrow[d, two heads, "\sim"] \\
X(a,b) \times X(b,c) \arrow[r] & k(\wedge^1_2, X)
\end{tikzcd}$$

in which the two vertical morphisms are trivial fibrations.

The morphism $\Delta_1 \xrightarrow{\delta^1_2} \Delta_2$ induces a composition

$$k(\Delta_2, X) \longrightarrow k(\Delta_1, X)$$

which induces a morphism

$$X(a,b,c) \longrightarrow X(a,c).$$

We obtain thus a diagram

$$\begin{tikzcd}
X(a,b) \times X(b,c) & X(a,b,c) \arrow[l, two heads, "\sim"'] \arrow[r] & X(a,c)
\end{tikzcd}$$

The choice of a section of $X(a,b,c) \to X(a,b) \times X(b,c)$ defines thus the composition

$$X(a,b) \times X(b,c) \longrightarrow X(a,c).$$

By applying the functor π_0 (which commutes to finite products), according to the previous proposition we have a map

$$Hom_{\tau X}(a,b) \times Hom_{\tau X}(b,c) \longrightarrow Hom_{\tau X}(a,c)$$

which is exactly the composition in the category τX.

Remark 3.6.10. The ∞-groupoid of sections of a trivial fibration is contractible. In particular, all the choices of the composition law are canonically equivalent.

3.6.1 Higher Homotopy Groups

Now, suppose that X is an ∞-groupoid.

if $x \in X_0$ is an object of X (if we view X as a space, we say that x is a point of X), we write

$$\Omega(X,x) = X(x,x)$$

the ∞-groupoid of endomorphisms of x, which is also called the loop space with base point x in X.

The composition law

$$X(x,x) \times X(x,x) \longrightarrow X(x,x)$$

induces a map

$$\pi_0(X(x,x)) \times \pi_0(X(x,x)) \longrightarrow \pi_0(X(x,x)).$$

We call $\pi_0(X(x,x))$ the fundamental group of X with respect to the point x, and we will note

$$\pi_1(X,x) = \pi_0(X(x,x)).$$

If we note by abusing the notation that x is the identity of x viewed as an object in $X(x,x)$, we could repeat this construction and define the n-th loop space of X with point x:

$$\Omega^n(X,x) = \Omega(\Omega^{n-1}(X,x),x), \quad n \geq 1.$$

We write $\pi_n(X,x) = \pi_0(\Omega^n(X,x))$ to be the n-th homotopy group of X with base point x.

Proposition 3.6.11. *The group $\pi_n(X,x)$ is abelian for $n \geq 2$.*

Consider $\widehat{\Delta}$ as a model category with Quillen model structure, and note $\widehat{\Delta}_\bullet = \Delta_0/\widehat{\Delta}$ to be the model category of the pointed simplicial sets. We have then a Quillen adjunction:

$$\begin{tikzcd} \widehat{\Delta} \arrow[r, shift left, "(-)_+"] & \widehat{\Delta}_\bullet \arrow[l, shift left, "u"] \end{tikzcd}$$

where u is the functor forgetting the base point, and $(-)_+$ is the functor adjoining a base point to a simplicial set:

$$X_+ = X \coprod \Delta_0 \ \ (\textit{pointed by } \Delta_0).$$

The cartesian product induces a symmetric monoidal structure on $\widehat{\Delta}_\bullet$ defined by the smash-product:

$$X \wedge Y.$$

The simplicial set $X \wedge Y$ is obtained via the cocartesian diagrams

$$\begin{tikzcd} \Delta_0 \arrow[r, "y"] \arrow[d, "x"] & Y \arrow[d] \\ X \arrow[r] & X \vee Y \end{tikzcd} \qquad \begin{tikzcd} X \vee Y \arrow[r, hook] \arrow[d] & X \times Y \arrow[d] \\ \Delta_0 \arrow[r] & X \wedge Y \end{tikzcd},$$

and the unit object is $S^0 = \Delta_{0+}$.

We define the simplicial circle S^1 by

$$S^1 = \Delta_1 / \partial \Delta_1$$

which is viewed as a pointed object. We thus check that

$$Hom_{\widehat{\Delta}_\bullet}(A \wedge S^1, (X, a)) = Hom_{\widehat{\Delta}_\bullet}(A, \Omega(X, x))$$

and since $(-) \wedge S^1$ is a left Quillen functor, we deduce that:

Theorem 3.6.12. *For each ∞-groupoid X with a base point x, we have a canonical bijection*

$$Hom_{Ho(\widehat{\Delta}_\bullet)}(S^n, (X, x)) \simeq \pi_n(X, x), \quad n \geq 0$$

where $S^n = S^1 \times \cdots \times S^1$ n times.

Proposition 3.6.13. $\partial \Delta_{n+1} \simeq S^n$ *in* $Ho(\widehat{\Delta}_\bullet)$ *for* $n \geq 0$.

proof: We have the following homotopy cocartesian diagrams:

$$\begin{tikzcd} S^n = S^n \wedge S^0 \arrow[r, hook] \arrow[d] & S^n \wedge \Delta_1 \simeq * \arrow[d] \\ * = S^n \wedge * \arrow[r, hook] & S^{n+1} \end{tikzcd} \qquad \begin{tikzcd} \partial \Delta_{n+1} \arrow[r] \arrow[d, hook] & \Lambda^{n+1}_{n+1} \arrow[d, hook] \\ * \sim \Delta_{n+1} \arrow[r, "s^{n+1}_{n+2}"'] & \partial \Delta_{n+2} \end{tikzcd}$$

Thus it suffices to check the trivial case $n = 0$.

Lemma 3.6.14. *Let X to be an ∞-groupoid. The morphism $X \to *$ is a weak equivalence if and only if $\pi_0(X) \simeq *$ and there exists a base point $x \in X_0$ such that $\forall n \geq 1$, $\pi_n(X, x) \simeq *$.*

proof:

Theorem 3.6.15 (Serre)**.** *Let $X \xrightarrow{p} Y$ be a morphism of ∞-groupoids. Given $x \in X_0$ and $p(x) = y \in Y_0$, and we note F to be the fibre of p at y. Then we have a long exact sequence:*

$$\cdots \to \pi_{n+1}(Y,y) \to \pi_n(F,x) \to \pi_n(X,x) \to \pi_n(Y,y) \to \cdots \to \pi_0(F) \to \pi_0(X) \to \pi_0(Y)$$

proof:
Prove that the sequence

$$\pi_0(F) \to \pi_0(X) \to \pi_0(Y)$$

is an exact sequence of pointed simplicial sets:
The composed map is constant on y.
If $a \in X_0$ is sent to the connected component of y, we have a morphism $\Delta_1 \to Y$ with source $p(a)$ and target y, and the commutative diagram:

$$\begin{tikzcd} \{0\} \arrow[r, "a"] \arrow[d, hook] & X \arrow[d] \\ \Delta_1 \arrow[r, "u"'] & Y \end{tikzcd} .$$

The weak equivalence $\{1\} \times_{\Delta_1} \Delta_1 \times_Y X \simeq \Delta_1 \times_Y X \simeq F$ induces the bijection:

$$\pi_0(\{1\} \times_{\Delta_1} \Delta_1 \times_Y X) \simeq \pi_0(\Delta_1 \times_Y X) \simeq \pi_0(F),$$

which proves that a is in the image of $\pi_0(F) \to \pi_0(X)$.
To prove the general case, we consider the homotopy cartesian diagram

$$\begin{tikzcd} \Omega(X,x) \arrow[r] \arrow[d] & P(X) \arrow[r] \arrow[d, "q"] & \underline{Hom}(\Delta_1, X) \arrow[d, "(s,t)"] \\ * \arrow[r, "x"'] & X \arrow[r] \arrow[d] & X \times X \arrow[d] \\ & * \arrow[r, "x"'] & X \end{tikzcd} ,$$

where $P(X)$ is the path space of X. However, $p(X) \to *$ is a trivial fibration, for $\underline{Hom}(\Delta_1, X) \to X$ is a trivial fibration (since X is an ∞-groupoid).The loop space is thus defined by a homotopy cartesian diagram of the form:

$$\begin{tikzcd} \Omega(X,x) \arrow[r] \arrow[d] & * \arrow[d, "x"] \\ * \arrow[r, "x"'] & X \end{tikzcd}$$

The infinite diagram of homotopy cartesian diagrams:

$$\begin{tikzcd} \Omega^{n+2}(Y,y) \arrow[r] \arrow[d] & \Omega^{n+1}(F,x) \arrow[r] \arrow[d] & * \arrow[d] & \\ * \arrow[r] & \Omega^{n+1}(X,x) \arrow[r] \arrow[d] & \Omega^{n+1}(Y,y) \arrow[r] \arrow[d] & * \arrow[d] \\ & * \arrow[r] & \Omega^n(F,x) \arrow[r] \arrow[d] & \Omega^n(X,x) \arrow[d] \\ & & * \arrow[r] & \Omega^n(Y,y) & \cdots \end{tikzcd}$$

finishes the proof by induction.

Corollary 3.6.16. *Let $X \xrightarrow{p} Y$ be a morphism between ∞-groupoids. Then p is an equivalence of ∞-groupoids if and only if $\pi_0(X) \to \pi_0(Y)$ is bijective and $\forall x \in X_0$, $\pi_n(X,x) \to \pi_n(Y,p(x))$ is a isomorphism of groups.*

proof: We could suppose that p is a Kan fibration. The long exact sequence of the previous theorem proves that this condition is equivalent to saying that all the (homotopy) fibres of p are contractible.

Proposition 3.6.17. *A morphism of ∞-categories $X \xrightarrow{p} Y$ is fully faithful if and only if the commutative diagram of ∞-groupoids*

$$\begin{tikzcd} k(\Delta_1, X) \arrow[r] \arrow[d, "(s,t)"] & k(\Delta_1, Y) \arrow[d, "(s,t)"] \\ k(X)^2 \arrow[r] & k(Y)^2 \end{tikzcd}$$

is homotopy cartesian.

proof:
Consider the fibre product:

$$\begin{tikzcd} k(X)^2 \times_{k(Y)^2} k(\Delta_1, Y) \arrow[r] \arrow[d, "q"] & k(\Delta_1, Y) \arrow[d] \\ k(X)^2 \arrow[r] & k(Y)^2 \end{tikzcd}$$

The fibre of q over $(a,b) \in k(X)^2_0 = X_0 \times X_0$ is $Y(p(a),p(b))$.

Saying that the diagram is homotopy cartesian is equivalent to saying that the morphism

$$\begin{tikzcd} k(\Delta_1, X) \arrow[rr, "\phi"] \arrow[dr, "(s,t)"'] & & k(X)^2 \times_{k(Y)^2} k(\Delta_1, Y) \arrow[dl, "q"] \\ & k(X)^2 & \end{tikzcd}$$

is a weak equivalence above $k(X)^2$.

Since (s,t) and q are Kan fibrations, this is equivalent to saying that ϕ is a fibrewise weak equivalence, which concludes the proof.

Theorem 3.6.18. *A morphism of ∞-categories is an equivalence of ∞-categories if and only if it is fully faithful and essentially surjective.*

3.7 Joins

Let A and B be two simplicial sets. We define the **join** $A \diamond B$ by the following cocartesian diagram:

$$\begin{tikzcd} A \times \partial\Delta_1 \times B \arrow[r, "\pi"] \arrow[d, hook] & A \coprod B \arrow[d, hook] \\ A \times \Delta_1 \times B \arrow[r] & A \diamond B \end{tikzcd} , \tag{3.7.1}$$

where π is induced by the projections:

$$A \times \{0\} \times B \to A \qquad A \times \{1\} \times B \to B.$$

By the construction, we have:

Proposition 3.7.1. *The functor $\widehat{\Delta} \to A/\widehat{\Delta}$ (resp. $\widehat{\Delta} \to B/\widehat{\Delta}$) defined by $B \mapsto A \diamond B$ (resp. $A \mapsto A \diamond B$) commutes with inductive limits and respect categorical weak equivalences.*

proof: The exactness property gives us the property of cartesian product, i.e., commute with inductive limits for each variable and the sum. The compatibility with categorical weak equivalences comes from the fact that the diagram defining joins is homotopy cocartesian.

For $m, n \geq 0$, we have a natural morphism:

$$\phi_{m,n} : \Delta_m \diamond \Delta_n \longrightarrow \Delta_m * \Delta_n = \Delta_{m+1+n}.$$

This morphism is induced by the canonical inclusions:

$$\Delta_m \hookrightarrow \Delta_m * \Delta_n \hookleftarrow \Delta_n$$

and by the morphism:

$$\Delta_m \times \Delta_1 \times \Delta_n \longrightarrow \Delta_{m+1+n}$$

obtained as the nerve of the increasing map

$$[m] \times [1] \times [n] \xrightarrow{\widetilde{\phi_{m,n}}} [m+1+n]$$

defined by:

$$\widetilde{\phi_{m,n}}(i, e, j) = \begin{cases} i & if \quad e = 0 \\ j & if \quad e = 1 \end{cases}$$

Proposition 3.7.2. *There exists a canonical functorial morphism*

$$\phi_{A,B} : A \diamond B \longrightarrow A * B.$$

such that $\phi_{\Delta_m, \Delta_n} = \phi_{m,n}$ for $m, n \geq 0$.

proof: This comes from the fact that all simplicial sets are the inductive limits of representables.

Lemma 3.7.3. *The morphism $\Delta_0 \diamond \Delta_n \longrightarrow \Delta_0 * \Delta_n = \Delta_{n+1}$ is a categorical weak equivalence for $n \geq 0$.*

proof: The morphism $[1] \times [n] = [0] \times [1] \times [n] \xrightarrow{\widetilde{\phi_{0,n}}} [n+1]$ has a right adjunction

$$\widetilde{s} : [n+1] \longrightarrow [1] \times [n]$$

defined by

$$\widetilde{s}(k) = (inf(k, 1), sup(k-1, 0))$$

. The induced morphism:

$$s : \Delta_{n+1} \to \Delta_0 \times \Delta_1 \times \Delta_n \to \Delta_0 \diamond \Delta_n$$

is a section of $p = \phi_{0,n} : \Delta_0 \diamond \Delta_n \to \Delta_{n+1}$.

We verify that the natural transformation $\widetilde{s\widetilde{\phi}_{0,n}} \Rightarrow Id_{[1]\times[n]}$ defines a morphism

$$h : \Delta_1 \times (\Delta_0 \diamond \Delta_n) \longrightarrow \Delta_0 \diamond \Delta_n$$

such that $h_0 = sp$ and $h_1 = Id_{\Delta_0 \diamond \Delta_n}$.

Moreover, $\tau(p) : \tau(\Delta_0 \diamond \Delta_n) \to \tau(\Delta_{n+1}) = [n+1]$ is an isomorphism. We then deduce that, for any 0-simplex x of $\Delta_0 \diamond \Delta_n$, the morphism $h_x : \Delta_1 \to \Delta_0 \diamond \Delta_n$ is quasi-invertible (in the sense that it induces an identity in $\tau(\Delta_0 \diamond \Delta_n)$).

Choose a factorization of p by an inner anodyne extension i and an inner Kan fibration q:

$$\begin{tikzcd}
\Delta_0 \diamond \Delta_n \arrow[rr, "p"] \arrow[dr, hook, "i"'] & & \Delta_{n+1} . \\
& X \arrow[ur, "q"'] &
\end{tikzcd}$$

Since X is an ∞-category, we have a lifting diagram:

$$\begin{tikzcd}
\Delta_1 \times (\Delta_0 \diamond \Delta_n) \arrow[r, "h"] \arrow[d, hook, "Id_{\Delta_1} \times i"] & \Delta_0 \diamond \Delta_n \arrow[r, hook, "i"] & X \\
\Delta_1 \times X \arrow[urr, dashed, "k"'] & &
\end{tikzcd}$$

However, for each $x \in X_0$, the morphism $\Delta_1 \xrightarrow{k_x} X$ is quasi-invertible in X (since we have the identification $\tau X = \tau(\Delta_0 \diamond \Delta_n)$ so that $\tau(k) = \tau(h)$). In other words, k corresponds to a quasi-invertible morphisms in the ∞-category $\underline{Hom}(X, X)$. This implies the existence of a lifting k':

$$\begin{tikzcd}
\Delta_1 \times X \arrow[r, "k"] \arrow[d, hook] & X . \\
J \times X \arrow[ur, dashed, "k'"] &
\end{tikzcd}$$

To conclude, we have the commutative diagram of the form:

$$\begin{tikzcd}
\{0\} \times (\Delta_0 \diamond \Delta_n) \arrow[d, hook, "i"] \arrow[dr, "sp"] & . \\
\{0\} \times X \arrow[d, hook] & \Delta_0 \diamond \Delta_n \arrow[d, hook, "i"'] \\
J \times X \arrow[r, "k'"] & X \\
\{1\} \times X \arrow[u, hook] & \Delta_0 \diamond \Delta_n \arrow[u, hook, "i"] \\
\{1\} \times (\Delta_0 \diamond \Delta_n) \arrow[u, hook] \arrow[ur, "="'] &
\end{tikzcd}$$

The commutativity of the lower diagram makes k' is a categorical weak equivalence, and so the upper triangle proves that sp is a categorical weak equivalence too. Since $ps = Id_{\Delta_{n+1}}$, this proves that p and s are categorical weak equivalences (since they are isomorphic in $Ho(\widehat{\Delta})$).

Proposition 3.7.4. *For each simplicial set A, the morphism*

$$\Delta_0 \diamond A \longrightarrow \Delta_0 * A$$

is a categorical weak equivalence.

proof: The functors $\Delta_0 \diamond (-)$ and $\Delta_0 * (-)$ commute to inductive limits and respect monomorphisms. We then deduce that the class C of simplicial sets A such that $\Delta_0 \diamond A \to \Delta_0 * A$ is a categorical weak equivalence is saturated by monomorphisms. Since C contains the representables, it contains all the simplicial sets.

Theorem 3.7.5 (Joyal). *For all simplicial sets A and B, the morphism*

$$A \diamond B \longrightarrow A * B$$

is a categorical weak equivalence.

idea of the proof: By the argument of saturation by monomorphisms, we turn to the case $A = \Delta_m$ and $B = \Delta_n$. Since the inclusions $I_m \hookrightarrow \Delta_m$ and $I_n \hookrightarrow \Delta_n$ are categorical weak equivalences, we thus consider the case $m = 0, 1$ and $n = 0, 1$. If $m = 0$ or $n = 0$, we have already done. If $m = n = 1$, we have the similar argument.

If we fix B, the functor

$$\begin{array}{ccc} \widehat{\Delta} & \to & B\backslash\widehat{\Delta} \\ A & \longmapsto & A \diamond B \end{array}$$

has a right adjunction:

$$\begin{array}{ccc} B\backslash\widehat{\Delta} & \to & \widehat{\Delta} \\ B \xrightarrow{u} X & \longmapsto & X//B = X//u \end{array}$$

We have then a pair of Quillen adjunctions, and then

Corollary 3.7.6. *If X is an ∞-category, for each morphism $B \xrightarrow{u} X$, the morphism $X/B \to X//B$ is an equivalence of ∞-categories.*

A particular case: if X is an ∞-category, and $x \in X_0$ is an object of X. Then we have a cartesian diagram:

$$\begin{tikzcd} X//x \arrow[r] \arrow[d] & \underline{Hom}(\Delta_1, X) \arrow[d, "t"] \\ \Delta_0 \arrow[r, "x"'] & X \end{tikzcd} .$$

More specific, we have the diagram:

$$\begin{tikzcd} X//x \arrow[r] \arrow[d] & \underline{Hom}(\Delta_1, X) \arrow[d, "{(s,t)}"] \\ X = X \times \Delta_0 \arrow[r, "{(Id_X, x)}"'] \arrow[d] & X \times X \arrow[d, "pr_2"] \\ \Delta_0 \arrow[r, "x"'] & X \end{tikzcd}$$

We have thus a canonical morphism

$$s : X//x \longrightarrow X$$

and by construction, for each object $y \in X_0$, we have a cartesian (actually homotopy cartesian) diagram:

$$\begin{tikzcd} X(y,x) \arrow[r] \arrow[d] & X//x \arrow[d, "s"] \\ \Delta_0 \arrow[r, "y"'] & X \end{tikzcd}$$

Corollary 3.7.7. *The homotopy fibre of $X/x \to X$ above $y \in X_0$ is canonical equivalent to the ∞-groupoid $X(y,x)$.*

proof: We consider the commutative diagram

$$\begin{tikzcd} X/x \arrow[rr] \arrow[dr, "\sigma"'] & & X//x \arrow[dl, "s"] \\ & X & \end{tikzcd}$$

where s is a fibration (restriction of the fibration $\underline{Hom}(\Delta_1, X) \to X \times X = \underline{Hom}(\partial\Delta_1, X)$) and σ is a right Kan fibration (since X is an ∞-category). Since $X/x \to X//x$ is an equivalence of ∞-categories, it is a fibrewise equivalence: for any $y \in X_0$, we have thus an equivalence of ∞-groupoids:

$$\sigma^{-1}(y) \xrightarrow{\sim} s^{-1}(y) = X(y, x).$$

3.8 Initial Objects

Definition 3.8.1. Let X be an ∞-category. We say that an object $a \in X_0$ is initial, if for each morphism $\partial\Delta_n \xrightarrow{x} X$, $n > 0$, if $x(0) = a$, then there exists a morphism $\Delta_n \xrightarrow{\bar{x}} X$ such that $\bar{x}|_{\partial\Delta_n} = x$.

We say that an object is final if it is initial in X^{op}.

Proposition 3.8.2. *Let X be an ∞-category, and $a \in X_0$ be an object of X. The following assertions are equivalent:*

1. *the object a is initial in X.*
2. *the projection $a\backslash X \to X$ is a trivial fibration.*
3. *the projection $a\backslash\backslash X \to X$ is a trivial fibration.*
4. *the projection $a\backslash X \to X$ admits a section $X \xrightarrow{s} a\backslash X$ such that $s(a) = Id_a$.*
5. *the projection $a\backslash\backslash X \to X$ admits a section $X \xrightarrow{s} a\backslash\backslash X$ such that $s(a) = Id_a$.*
6. *the inclusion $X \hookrightarrow \Delta_0 * X$ admits a retraction $\Delta_0 * X \xrightarrow{r} X$ such that $r(0 * a) = Id_a$.*

proof:

We start by clarifying some notations.The object $Id_a \in (a\backslash X)_0$ is the morphism $\Delta_0 \xrightarrow{Id_a} a\backslash X$ corresponding by adjunction to $\Delta_0 * \Delta_0 = \Delta_1 \to \Delta_0 \xrightarrow{a} X$.

The object $Id_a \in (a\backslash\backslash X)_0 \subset \underline{Hom}(\Delta_1, X)_0$ is $\Delta_1 \to \Delta_0 \xrightarrow{a} X$.

In particular, the morphism

$$a\backslash X \xrightarrow{\phi} a\backslash\backslash X$$

sends Id_a to Id_a.

Since ϕ is a weak equivalence between fibrations over X, we see that 2.,3.,4.,5. are equivalent, since $a\backslash X \to X$ is a trivial fibration, the commutative diagram

$$\begin{tikzcd} \Delta_0 \arrow[r, "Id_a"] \arrow[d, hook, "a"] & a\backslash X \arrow[d] \\ X \arrow[r, "Id_X"'] & X \end{tikzcd}$$

admits a lifting.

Prove 1. $\Leftrightarrow$ 2.: by observing the decomposition

$$\partial\Delta_{n+1} = \Delta_0 * \partial\Delta_n \cup \emptyset * \Delta_n,$$

we see that given a morphism $\partial' Delta_n \xrightarrow{x} X$ such that $x(0) = a$ is equivalent to be given a commutative diagram

$$\begin{tikzcd} \partial\Delta_n \arrow[r] \arrow[d, hook] & a\backslash X \arrow[d] \\ \Delta_n \arrow[r] \arrow[ur, dashed] & X \end{tikzcd} ,$$

since X is an ∞-category, and we have the lifting diagram

$$\begin{tikzcd} \Delta_0 * \partial\Delta_n \cup \emptyset * \Delta_n \arrow[r] \arrow[d] & X \\ \Delta_n * \Delta_0 \arrow[ur, dashed] & \end{tikzcd} .$$

This proves 1. $\Leftrightarrow$ 2..

For 2. $\Rightarrow$ 3.: if $a\backslash X \to X$ is a trivial fibration, the following commutative diagram admits a lifting

$$\begin{tikzcd} \Delta_0 \arrow[r, "Id_a"] \arrow[d, hook, "a"] & a\backslash X \arrow[d] \\ X \arrow[r, "="'] \arrow[ur, dashed] & X \end{tikzcd} .$$

For 4. $\Rightarrow$ 6.: By adjunction, the section $s : X \to a\backslash X$ corresponds to a morphism $r : \Delta_0 * X \to X$ such that $r(0) = a$, where $0 \in (\Delta_0 * X)_0$ is the 0-simplex $\Delta_0 \hookrightarrow \Delta_0 * X$. The fact that s is a section implies that r is a retraction by functoriality.

For 6. $\Rightarrow$ 1.: The morphism

$$y : \Delta_0 * \partial\Delta_n \xrightarrow{Id_{\Delta_0} * x} \Delta_0 * X \xrightarrow{r} X$$

extends $x : \partial\Delta_n \to X$, and we have $y(0 * 0) = r(0 * a) = Id_a$. However, we also have $\Delta_0 * \partial\Delta_n = \wedge^0_{n+1}$, and since Id_a is invertible, there exists $z : \Delta_{n+1} \to X$ such that $z|_{\wedge^0_{n+1}} = y$. The restriction of z to $\partial^0 : \Delta_n \hookrightarrow \Delta_{n+1}$ is a lifting of x.

Proposition 3.8.3. *Let X be an ∞-category. We write $initial(X)$ the full sub-∞-category of X formed by initial objects.*

Then $initial(X)$ is either empty or a contractible ∞-groupoid (thus an injective object in $\widehat{\Delta}$).

Each object of X who is quasi-isomorphic (i.e. isomorphic in τX) to an initial object is initial. Moreover, each initial object in X is an initial object in the category $hoX = \tau X$. Inversely, if X admits an initial object, then each object in X which is initial in τX is initial in X.

proof: Recall that the morphism $X \to N\tau X$ is an isofibration. If A is a set of objects in X, we note $\tau(X)_A$ to be the full sub-category of τX generated by A, and we define X_A to be the full sub-∞-category of X generated by A. By the following cartesian diagram (which is also homotopy cartesian)

$$\begin{tikzcd} X_A \arrow[r] \arrow[d] & X \arrow[d] \\ N\tau(X)_A \arrow[r] & N\tau(X) \end{tikzcd} ,$$

we know that $\tau(X_A) = \tau(X)_A$. The ∞-category $initial(X)$ is obtained by taking the set of initial objects in X as A.

It is clear that, if $initial(X)$ is not empty, then the morphism $initial(X) \to \Delta_0$ has the RLP with respect to the inclusions $\partial\Delta_n \to \Delta_n,\ n \geq 0$.

Prove that all initial objects in X are initial in $\tau(X)$. Let $a \in X_0$ is an initial object. We know that, for each $x \in X_0$, $X(a, x)$ is the homotopy fibre of $a\backslash X \to X$ above x. In particular, $X(a, x)$ is thus a contractible ∞-groupoid. Therefore

$$Hom_{\tau X}(a, x) = \pi_0 X(a, x) \simeq pt.$$

Suppose that X admits an initial object a, and consider an object $b \in X_0$ which is initial in $hoX = \tau X$. Thus there exists an isomorphism $a \xrightarrow{\sim} b$ in hoX such that we could lift it to a quasi-invertible $\Delta_1 \xrightarrow{f} X$ with source a and target b.

Consider a morphism $\partial\Delta_n \xrightarrow{x} X$ such that $x(0) = b$. Let $r : \Delta_0 * X \to X$ be a retraction of $X \hookrightarrow \Delta_0 * X$ such that $r(0 * a) = Id_a$. We define x' by

$$x' : \Delta_0 * \partial\Delta_n \xrightarrow{Id_{\Delta_0} * x} \Delta_0 * X \xrightarrow{r} X,$$

and $x'|_{\partial\Delta_n} = x$, $x'(0) = a$. Since $\Delta_0 * \partial\Delta_n = \wedge^0_{n+1}$ and $x'(0,1) = f$, there exists then $z : \Delta_{n+1} \to X$ whichi extends x'. The restriction of z to $\Delta_0 \xrightarrow{\partial^0} \Delta_{n+1}$ is a lifting for x.

Each object of $\tau(X)$ which is isomorphic to an initial object in $\tau(X)$ is initial in $\tau(X)$, which implies that each object of X which is quasi-isomorphic to an initial object is initial in X.

Lemma 3.8.4. *Let X be an ∞-category and $a \in X_0$. If I is a simplicial set, we note $a_I \in \underline{Hom}(I, X)_0$ the constant diagram of value a:*

$$a_I : I \to \Delta_0 \xrightarrow{a} X.$$

Then we have an isomorphism of simplicial sets:

$$a_I \backslash\backslash \underline{Hom}(I, X) \simeq \underline{Hom}(I, a\backslash\backslash X).$$

proof: It suffices to check that for each simplicial set K, we have:

$$(\Delta_0 \diamond K) \times I \simeq \Delta_0 \diamond (K \times I).$$

Theorem 3.8.5. *Let X be an ∞-category admitting an initial object a. Then, for each simplicial set I, the ∞-category $\underline{Hom}(I, X)$ has a_I as an initial object.*

proof: If $a\backslash\backslash X \to X$ is a trivial fibration, then it is the same for $\underline{Hom}(I, a\backslash\backslash X) \to \underline{Hom}(I, X)$. We conclude the proof by using the previous lemma.

Let $\Delta_1 \xrightarrow{u} X$ be an arrow in an ∞-category. We note a to be the source of u and b to be the target. We want to describe

$$X(x, a) \longrightarrow X(x, b).$$

Lemma 3.8.6. *The projection $X/u \to X/a$ is a trivial fibration.*

proof:

A commutative diagram of the form

$$\begin{tikzcd} K \arrow[r] \arrow[d, hook] & X/u \arrow[d] \\ L \arrow[r] \arrow[ur, dashed] & X/a \end{tikzcd}$$

is equivalent to a diagram of the following form

$$\begin{tikzcd} L * \{0\} \cup K * \Delta_1 \arrow[r] \arrow[d, hook, "i"] & X . \\ L * \Delta_1 \arrow[ur] & \end{tikzcd}$$

Thus it suffices to verify whether i is an inner anodyne extension when $K \hookrightarrow L$ is $\partial\Delta_n \hookrightarrow \Delta_n$, which is well-known.

For $x \in X_0$, we note $\underline{X}(x, a)$ to be the fibre of $X/u \to X$ above x. We have then a trivial fibration

$$\underline{X}(x, a) \xrightarrow{\sim} X(x, a)$$

and the right Kan fibration $X/u \to X/b$ induces a Kan fibration

$$\underline{X}(x,a) \longrightarrow X(x,b).$$

The Zig-Zag

$$X(x,a) \overset{\sim}{\twoheadleftarrow} \underline{X}(x,a) \longrightarrow X(x,b)$$

corresponds to $a \xrightarrow{u} b$.

We could view $a \xrightarrow{u} b$ as a morphism

$$\Delta_0 * \Delta_0 \longrightarrow X,$$

and thus as an object of X/a. We could then check that

$$(X/a)/u = X/u.$$

Let $a \xrightarrow{u} b$ and $a' \xrightarrow{u'} b$ be two morphisms of X, we could view u and u' as objects in X/b.

Lemma 3.8.7. *We have an isomorphism of simplicial sets between $X/b(u,u')$ and the fibre of the morphism $\underline{X}(a,a') \to X(a,b)$ above u.*

proof: We have a commutative cube in which the two lateral faces are cartesian:

$$\begin{tikzcd}
\underline{X}(a,a') \arrow[rr] \arrow[dr, "a'"] \arrow[dd] & & X(a,b) \arrow[dr] \arrow[dd, dashed] & \\
& X/u' \arrow[rr] \arrow[dd] & & X/b \arrow[dd, "q"] \\
* \arrow[rr, dashed] \arrow[dr] & & * \arrow[dr] & \\
& X/b \arrow[rr, "q"'] & & X
\end{tikzcd} .$$

Theorem 3.8.8. *Let X be an ∞-category, and $a \in X_0$. Then a is an final object if and only if for each object $x \in X_0$, the ∞-groupoid $X(x,a)$ is contractible.*

proof: