Homotopy Theory of Schemes

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Introduction

I summarize the basic definitions and constructions of $\mathbf{A^1}$-homotopy theory of Schemes, including unstable and stable homotopy theory.

The first chapter of this article reviews the constructions of unstable $\mathbf{A^1}$-homotopy categories. The ideas given by [13] and [21] will be introduced, and we will see that these two constructions are equivalent at the end of the first section. Most of the techniques used in the first chapter are given in [18].

In the second chapter, the construction of stable $\mathbf{A^1}$-homotopy category is presented. I will first introduce the classical stable homotopy theory and spectra, and then use the language of Jardine's [5] to construct *motivic stable homotopy theory*. Some of the theorems and properties mentioned in the classical spectra theory will be proved in motivic background in detail. After constructing the general theory of motivic stable homotopy theory, I will focus on S^1-spectra and the Morel-Voevodsky spectra.

The last section treats with the relations to Voevodsky's triangulated category of motives and motivic cohomologies, using motivic Eilenberg-MacLane spaces. Most of the contents in this chapter are given in [23]

Classical simplicial techniques, model categories, and grothendieck topology, Nisnevich topology, and the Dold-Kan theorem are given in the appendix, and can be found in texts good enough or in the references.

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Contents

Chapter 1

The Construction of Unstable $\mathbf{A^1}$-Homotopy Theory

1.1 The Construction in [21]

1.1.1 Introduction and Motivation

In [21], Voevodsky made an outline of how to do abstract homotopy theory on the category of S-schemes, where S is the smooth base scheme. Here I will make a brief introduction.

$\mathbf{A^1}$-homotopy theory is based on the idea of doing homotopy theory in algebro-geometrical background using the affine line $\mathbf{A^1}$ instead of the ordinary unit interval. In order to do abstract homotopy theory, we consider a category of *spaces* denoted by *spec*. The problem we encounter in the category Sm/S, smooth schemes over a base S, is that this category is not closed under arbitrary colimits, and so Sm/S is not what we want. Therefore we have to add colimits and extend the category Sm/S. There is a standard way to do this. By Yoneda fully faithful embedding functor $h_X : Y \longrightarrow Hom(Y, X)$, we can identify a space X with a representable presheaf $Hom(*, X)$. The category of presheaves $Psh(\mathcal{C})$ admits all colimits (and limits) because:

Lemma 1.1.1 *Every presheaf is the colimit of a canonical diagram of representable presheaves.*

proof : Let F be a functor $\mathcal{D} \longrightarrow Set$. Define $\mathcal{D}/F$ the category with objects $\alpha_X : h_X \longrightarrow F$ and morphisms of commutative diagrams $\alpha_X = t \circ \alpha_Y$ where $t : h_X \longrightarrow h_Y$. Then consider the natural functor

$$\phi : \mathcal{D}/F \longrightarrow \mathcal{D}^{op}\textit{Set},\ (h_X \longrightarrow F) \longmapsto h_X.$$

We have

$$colim_{\mathcal{D}/F}(\phi) = colim_{\{h_X \to F\}} h_X = F.$$

According to the lemma, $Psh(\mathcal{C})$ is in a sense the category obtained from $\mathcal{C}$ by adding formally all small colimits. One may think now to develop homotopy theory by taking $Psh(\mathcal{C})$ the category we want. But there is one problem. Consider a pushout square in Sm/S:

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & \mathrm{U} \arrow[d] \\ \mathrm{V} \arrow[r] & \mathrm{X} \end{tikzcd}$$

where U and V are Zariski open subsets of X and $X = U \cup V$. However, the corresponding square of representable sheaves:

$$\begin{tikzcd} h_U \cap h_V \arrow[r] \arrow[d] & h_U \arrow[d] \\ h_V \arrow[r] & h_X \end{tikzcd}$$

is not a pushout square anymore unless U=X or V=X. Thus if we define spaces as presheaves, the union $U \cup_{Psh} V$ of U and V as spaces is not the same of X.

Definition 1.1.1 *An elementary distinguished square in Sm/S is a square of the form:*

$$\begin{tikzcd} p^{-1}(U) \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

where $p : V \longrightarrow X$ is an etale morphism, $j : U \longrightarrow X$ is an open immersion and $p^{-1}(X \backslash U) \longrightarrow X \backslash U$ is an isomorphism. (Here we consider $X \backslash U$ the maximal reduced subscheme with support in the closed subset $X \backslash U$)

We notice that for a distinguished square, the family $\{U \xrightarrow{i} X, V \xrightarrow{p} X\}$ is a covering family in the Nisnevich topology. Observe also if p is an open immersion, the last condition exactly means that U and V cover X. An important class of elementary distinguished square is provided by coverings $X = U \cup V$ by two Zariski open subsets. In this case $p = i_v$ is an open immersion and the condition that $p^{-1}(X \backslash U) \longrightarrow X \backslash U$ is an isomorphism is equivalent to the condition that $U \cup V = X$. We then notice that X is the colimit of the diagram:

$$\begin{array}{ccc} p^{-1}(U) & \longrightarrow & V \\ \downarrow & & \\ U & & \end{array}$$

We want to define our category of spaces in such a way that elementary distinguished squares remain pushout squares when considered in these category of spaces.

Definition 1.1.2 *A contravariant functor $F : Sm/S \longrightarrow$ Set (a presheaf on Sm/S) is called a sheaf on Nisnevich topology if the following two conditions hold:*

1. $F(\emptyset) =$ *pt*
2. *for any elementary distinguished square as in the* **Definition 1.1.1**, *the square of sets:*

$$\begin{array}{ccc} F(X) & \longrightarrow & F(V) \\ \downarrow & & \downarrow \\ F(U) & \longrightarrow & F(p^{-1}(U)) \end{array}$$

is Cartesian.

We define the category of spaces, spc, as: the full subcategory of Psh(Sm/S) consists of sheaves in Nisnevich topology. Because the elementary distinguished squares are pushout squares in Sm/S, any representable presheaf belongs to $Sh_{Nis}(Sm/S)$. Thus by Yoneda lemma and our definition the square of representable sheaves corresponding to an elementary distinguished square is a pushout square in $Sh_{Nis}(Sm/S)$. The following theorem is a corollary of the general theory of sheaves on a site:

Theorem 1.1.1 *The category $Sh_{Nis}(Sm/S)$ has all small limits and colimits. The inclusion functor $Sh_{Nis}(Sm/S) \longrightarrow Psh_{Nis}(Sm/S)$ has a left adjoint $a_{Nis} : PSh_{Nis}(Sm/S) \longrightarrow Sh_{Nis}(Sm/S)$ which commutes with both limits and colimits.*

The functor a_{Nis} is called the functor of sheafification.

Let $S(X, Y) \in sSet$ induces the adjoint

$$Mor_{sSh}(K \otimes X, Y) = Mor_{sSet}(K, S(X, Y))$$

in the simplicial model structure, and $\underline{Hom}(X, Y) \in sSh_{Nis}$ induces the adjoint

$$Mor_{sSh}(X \otimes Y, Z) = Mor_{sSh}(X, \underline{Hom}(Y, Z))$$

in the symmetric monoidal structure.

We will need a definition of a subcategory *spc*^{ft} of spaces of finite type in *spc* whose objects play the role of *compact object* in the sense of categorical definition:

Definition 1.1.3 *A space X is called of finite type if for any filtered system of spaces* X_α *the canonical map* $colim_\alpha \underline{Hom}(X, X_\alpha) \longrightarrow \underline{Hom}(X, colim_\alpha X_\alpha)$ *is a bijection.*

Remark 1.1.1 *We will see in the second chapter that we need more conditions to define compact objects (motivic flasque). The reason is we want to pass it to stable homotopy theory. However, the above definition of finite type is enough for us now. Also notice that all schemes in the underlying site* $(Sm/S)_{Nis}$ *are of finite type.*

1.1.2 Unstable $\mathbf{A}^1$-Homotopy Category

In order to make the context coherent, in the following, we will use $Mor_{\mathcal{C}}(*, *)$ to emphasize the morphisms in a category $\mathcal{C}$.

Definition 1.1.4 *The algebraic standard cosimplicial simplicial simplex* Δ_s^n *is defined as a subscheme of* $\mathbf{A}^{n+1}$ *by* $\sum_{1=0}^{n} x_i = 0$.

Definition 1.1.5 *Suppose that X is a smooth scheme belonging to spc and U is another smooth scheme, in order to identify* X *with the associated simplicial presheaf, we define*

$$Sing_*(X)(U) = Mor_{spc}(U \times \Delta_s^*, X).$$

Since *spc* admits arbitrary colimits, this functor gives a *generalized geometrization*

$$| * |_s\colon \Delta^{op}\mathit{Set} \longrightarrow \mathit{spc}$$

which is characterized by $| \Delta^n |_s = \Delta_s^n$.

Example 1.1.1 *Let* $\partial\Delta^2$ *the boundary of the standard simplicial 2-simplex, then* $| \partial\Delta^2 |_s$ *is the space replace the ordinary intervals with affine lines.*

Definition 1.1.6 *X is called almost fibrant if for any open immersion* $V \longrightarrow U$*, the simplicial set* $Sing_*(X)(U) \longrightarrow Sing_*(X)(V)$ *is a Kan fibration.*

For any open immersion $V \longrightarrow U$, consider the morphism

$$i_{n,k,j} : U\times \mid \wedge^n_k \mid_s \cup_{V\times|\wedge^n_k|_s} V\times \mid \Delta^n \mid_s \longrightarrow U\times \mid \Delta^* \mid_s .$$

This morphism plays a role of *anodyne morphism*. For a space X, denote A_x the set of all triples

$$(\wedge^n_k \longrightarrow \Delta^n, j : V \hookrightarrow U, f : U\times \mid \wedge^n_k \mid_s \cup_{V\times|\wedge^n_k|_s} V\times \mid \Delta^n \mid_s \longrightarrow X).$$

We have the diagram:

$$\begin{tikzcd} \amalg_{A_X} U\times \mid \wedge^n_k \mid_s \cup_{V\times|\wedge^n_k|_s} V\times \mid \Delta^n \mid_s \arrow[r] \arrow[d] & X \\ \amalg_{A_x} U\times \mid \Delta^n \mid_s & \end{tikzcd}$$

and make $Ex^1(X)$ the pushout of this diagram. Set $Ex^n(X) = Ex^1(Ex^{n-1}(X))$, and $Ex^\infty(X) = colim_n(Ex^n(X))$.

Definition 1.1.3 immediately implies:

Lemma 1.1.2 *The space $Ex^\infty(X)$ is almost fibrant for any space X.*

The Model Structure

Definition 1.1.7 *Let $Sing_0(X)(U) = Mor_{\mathcal{S}}(U, X)$. $\pi_{i,U}^{\mathbf{A}^1}(X, x)$ the homotopy group of the Kan simplicial set $Sing_*(Ex^\infty(X)(U))$ with respect to the base point x. The morphism $f : X \longrightarrow Y \in spc$ is a weak equivalence if and only if $\pi_{i,U}^{\mathbf{A}^1}(X, x) \longrightarrow \pi_{i,U}^{\mathbf{A}^1}(Y, f(x))$ is an isomorphism for any smooth scheme U.*

Remark 1.1.2

- *Given the definition of weak equivalences, we notice that the functor $Sing_*$ sends $X \in spc$ to a simplicial presheaf $Sing_*(X)$ and the weak equivalences are defined as global equivalences in the category of simplicial presheaves.*
- *Another approach to homotopy theory of spaces over S will be introduced in the next subsection, which is based on the use of simplicial (pre)sheaves. In the* **Section 1.3**, *we will prove that these two constructions are equivalent.*

Definition 1.1.8 *$H^{\mathbf{A}^1}(S)$ of smooth scheme over S is defined as the localization of the category of spaces over S with respect to the class of weak equivalence.*

Remark 1.1.3 *The category spc is actually a category of sheaves in Nisnevich topology, and we denote by $sSh(Sm/S)_{Nis}$ the category of simplicial Nisnevich sheaves.*

1.2 The Construction in [13]

1.2.1 The Simplicial Homotopy Category of (Pre)sheaves

We equip the category of simplicial presheaves with the local injective structure. The B.G-property we will see discussing the relation between local equivalences and global equivalences. For more details about local and global structures, please refer to the **Appendix A**.

Definition 1.2.1 *A B.G-class of objects in Sm/S is a class $\mathcal{B}$ of objects in Sm/S such that:*

1. *for any X in $\mathcal{B}$ and any open immersion $U \longrightarrow X$ we have $U \in \mathcal{B}$*
2. *any smooth S-scheme X has a Nisnevich covering which consists of objects in $\mathcal{B}$*

Definition 1.2.2 *A simplicial presheaf F on $(Sm/S)_{Nis}$ is said to have the B.G-property with respect to $\mathcal{B}$ if for any elementary distinguish square such that X and V belong to $\mathcal{B}$, the square of simplicial sets*

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & F(V) \arrow[d] \\ F(U) \arrow[r] & F(U \times_X V) \end{tikzcd}$$

is homotopy cartesian.

Note that having the B.G-property is invariant with respect to weak equivalence. For any simplicial sheaf F and an elementary distinguished square the corresponding square of simplicial sets is cartesian. Thus if F is a simplicial sheaf such that for any open immersion $U \longrightarrow V$ with $V \in \mathcal{B}$ the map of simplicial sets $F(V) \longrightarrow F(U)$ is a fibration, then F has the B.G-property with respect to $\mathcal{B}$. Especially, a simplicial fibrant F has this property.

Lemma 1.2.1 *Suppose $\mathcal{F} \in sPsh((Sm/S)_{Nis})_{loc}$, then F is a stack if and only if it has the B.G-property with respect to Nisnevich covering families.*

The discuss of stacks and model structures can be found in **Appendix A**.

Lemma 1.2.2 *Let $F \longrightarrow G$ be a morphism of simplicial presheaves such that the associated morphism of simplicial sheaves is a weak equivalence (local weak equivalence in Nisnevich topology) and suppose that both F and G have the B.G-property with respect to $\mathcal{B}$. Then for any U in $\mathcal{B}$ the morphism of simplicial sets $F(U) \longrightarrow G(U)$ is a weak equivalence.*

Note that these two lemmas conclude that a local weak equivalence between stacks is a global weak equivalence.

proof of **lemma 2.6**:

1. **Definition 1.2.3** *A* **B.G***-functor on X_{Nis} is a family of contravariant functors T_q from X_{Nis} to the category of pointed sets, together with pointed maps ∂_q : $T_{q+1}(U \times_X V) \longrightarrow T_q(X)$ for all elementary distinguish square, such that the following conditions hold:*

   (a) the morphisms ∂_q are natural with respect to morphisms of elementary distinguished squares

   (b) for any $q \geq 0$ the sequence of pointed sets

   $$T_{q+1}(U \times_X V) \longrightarrow T_q(X) \longrightarrow T_q(U) \times T_q(V)$$

   is exact

2. **Lemma 1.2.3** *Let (T_q,∂_q) be a B.G-functor on X_{Nis} such that the Nisnevich sheaves associated with T_q are trivial (i.e. isomorphic to the pointed sheaf pt) for all q, then $T_q =$ pt for all q.*

   admit this lemma and give the proof of $\mathbf{Lemma 2.1}$: For any smooth scheme U over S, according to **CM5**, there is a factorization of $F(U) \longrightarrow G(U)$ into an acyclic cofibration and a fibration. So, we can assume that $F(U) \longrightarrow G(U)$ is a fibration between Kan fibrants. Because $F \longrightarrow G$ is a weak equivalence (in Nisnevich topology), so each of its fibers is a simplicial weak equivalence. Therefore let's assume moreover that G is trivial. Assume $F(S) \neq \emptyset$ and $a \in F(S)$ be an element. Consider the family of functors Ti on S*Nis* of the form

   $$U \longmapsto \pi_i(F(U), a_U).$$

   It is a B.G-functor and the associated Nisnevich sheaves are trivial since $F \longrightarrow$ *pt* is a weak equivalence. We conclude that $F(S)$ is contractible.

1.2.2 Unstable $\mathbf{A}^1$-Homotopy Theory

Denote $H_s(S)$ the simplicial homotopy category of presheaves on $(Sm/S)_{Nis}$.

Theorem 1.2.1 *If F is a simplicial presheaf with B.G-property, then for any $U \in (Sm/S)_{Nis}$, the map $\pi_0(F(U)) \longrightarrow Hom_{H_s(S)}(U, a_{Nis}(F))$ is bijective. If moreover F is pointed with B.G-property, then for any $n \geq 0$, $U \in (Sm/S)_{Nis}$, the morphism $\pi_n(F(U)) \longrightarrow Hom_{H_s(S)}(U_+ \wedge S^n, a_{Nis}(F))$ is bijective. $a_{Nis}(F)$ denotes the sheaf associated to the presheaf F.*

proof: Notice that $\pi_0(F(U)) = \pi_0(S(U,F)) = \pi(U,F)$, Therefore:

$$\pi(U,F) \cong Hom_{H_s(S)}(U, a_{nis}(F)_f) \cong Hom_{H_s(S)}(U, a_{nis}(F))$$

comes from well-known results of simplicial homotopy theory and the previous lemma. Note that in simplicial homotopy

$$\pi_n(X) = \pi_0(\underline{Hom}(S_s^n, X)) = Hom(S_s^n, X)/\sim .$$

The Model Structure

Definition 1.2.4

1. *An object $K \in \Delta^{op}PSh((Sm/S)_{Nis})$ is called $\mathbf{A^1}$-local if and only if for any $F \in \Delta^{op}PSh((Sm/S)_{Nis})$, the projection $F \times \mathbf{A^1} \longrightarrow F$ induces a bijection:*

$$Hom_{H_s(S)}(F, K) \xrightarrow{\sim} Hom_{H_s(S)}(F \times \mathbf{A^1}, K).$$

2. *A morphism $f : F \to G$ is called an $\mathbf{A^1}$-weak equivalence if and only if for any $\mathbf{A^1}$-local K, the map:*

$$Hom_{H_s(S)}(F, K) \xrightarrow{\sim} Hom_{H_s(S)}(G, K)$$

is a bijective.

Given the definition of $\mathbf{A^1}$-weak equivalence, we therefore have a model structure on the category of simplicial presheaves on $(Sm/S)_{Nis}$. The class of weak equivalences $W_{\mathbf{A^1}}$ is the class of $\mathbf{A^1}$-equivalences, the class of cofibrations is the class of monomorphisms, and the class of fibrations $F_{\mathbf{A^1}}$ is the class of morphisms have the RLP with respect to morphisms that are both $\mathbf{A^1}$-equivalence and cofibration.

We will see later in the next chapter (also in Jardine's paper [5]) that we can replace every *presheaf* by *sheaf* in the above discussions. We still denote $H_s(S)$ the simplicial homotopy category of sheaves on $(Sm/S)_{Nis}$.

Remark 1.2.1

1. *Any simplicial weak equivalence is obviously an $\mathbf{A^1}$-weak equivalence.*

2. *For any F, $F \times \mathbf{A^1} \longrightarrow F$ is an $\mathbf{A^1}$-weak equivalence (but not a simplicial weak equivalence).*

3. *A simplicial fibrant simplicial sheaf F is $\mathbf{A^1}$-fibrant if and only if it is $\mathbf{A^1}$-local.*

Denote $H_s(S)$ the simplicial homotopy category of sheaves on $(Sm/S)_{Nis}$, and $H^{\mathbf{A^1}}(S)$ the $\mathbf{A^1}$-homotopy category of simplicial sheaves on $(Sm/S)_{Nis}$.

Let $H_{s,\mathbf{A^1}}((Sm/S)_{Nis})$ be the full subcategory of $H_s(S)$ consists of $\mathbf{A^1}$-local objects. The inclusion $H_{s,\mathbf{A^1}}((Sm/S)_{Nis}) \longrightarrow H_s((Sm/S)_{Nis})$ admits a left adjoint $L_{\mathbf{A^1}}(-) : H_s((Sm/S)_{Nis}) \longrightarrow H_{s,\mathbf{A^1}}((Sm/S)_{Nis})$, which is called the $\mathbf{A^1}$-localization functor, identifying $H_{s,\mathbf{A^1}}(S)$ with the homotopy category $H^{\mathbf{A^1}}(S)$.

1.3 The Equivalence of Previous Constructions

1.3.1 Homotopy Theory of a Site with Interval

Definition 1.3.1 *An interval in a site $\mathcal{C}$ is a sheaf of set I together with morphisms:*

- $\mu : I \times I \longrightarrow I$
- $i_0, i_1 : pt \longrightarrow I$

satisfying the following conditions:

- *Let p be the canonical morphism $S \xrightarrow{p} pt$ then*
  - $\mu(i_0 \times Id) = \mu(Id \times i_0) = i_0 p$
  - $\mu(i_1 \times Id) = \mu(Id \times i_1) = Id$
- *the morphism $i_0 \coprod i_1 : pt \coprod pt \longrightarrow I$ is a monomorphism.*

Definition 1.3.2

1. *Let ($\mathcal{C}$,I) be a site with interval. A simplicial sheaf F is called I-local if for any simplicial sheaf G, the map: $Hom_{H_s(S)}(G \times I, F) \longrightarrow Hom_{H_s(S)}(G, F)$ induced by $i_0 : pt \longrightarrow I$ is a bijection.*
2. *A morphism $f : F \longrightarrow G$ is called an I-weak equivalence if for any I-local K the corresponding map: $Hom_{H_s(S)}(G, K) \longrightarrow Hom_{H_s(S)}(F, K)$ is a bijection*

The homotopy category $H(\mathcal{C}, I)$ of a site with interval ($\mathcal{C}$,I) is the localization of $\Delta^{op} Sh(\mathcal{C})$ with respect to the class of I-weak equivalence.

Denote the class of I-weak equivalence by W_I and define the class F_I of I-fibration as the class of morphisms having the Right Lifting Property (RLP) with respect to I-acyclic cofibration, where I-cofibration is the class of monomorphisms. According to the structure of model category, we have the following theorem:

Theorem 1.3.1 *Let $(\mathcal{C}, I)$ be the site with interval. Then the category of simplicial sheaves on $\mathcal{C}$ together with three classes of morphisms (W_I, C, F_I) is a proper model category. The inclusion $H_{s,I}(\mathcal{C}) \longrightarrow H_s(\mathcal{C})$,where $H_{s,I}(\mathcal{C})$ denotes the full subcategory with $I - local$ objects, admits a left adjoint L_I which identifies $H_{s,I}(\mathcal{C})$ with the homotopy category $H(\mathcal{C}, I)$.*

We define a functor and denote it by $Sing_*^I(F)(U)$ for any $U \in T$, which is defined by the following equation:

$$Hom(U \otimes \Delta^n, F) = Hom(\Delta^n, \underline{Hom}(U, F)) = Hom(\Delta^n, Sing_*^I(F)(U)) \qquad (1.1)$$

Let $f, g : F \longrightarrow G$ be two morphisms of simplicial sheaves. An elementary I-homotopy from f to g is a morphism $H : F \times I \longrightarrow G$ such that $Hi_0 = f$ and $Hi_1 = g$. Two morphisms of simplicial sheaves are called I-homotopic if they can be connected by a sequence of I-homotopies. A morphism $f : F \longrightarrow G$ is called a strict homotopy equivalence (to different from weak equivalence) if there is a morphism $g : G \longrightarrow F$ such that $fg \simeq Id_G$ and $gf \simeq Id_F$. Respectively, we can replace I by Δ^1 to get elementary simplicial homotopy, simplicial homotopy morphisms and strict simplicial equivalence.

Proposition 1.3.1 *Let $f, g : F \longrightarrow G$ be two morphisms and H be an elementary I-homotopy between f and g. Then there exists an elementary simplicial homotopy from $Sing_*^I(F)$ to $Sing_*^I(G)$.*

proof: We can see from the definition of $Sing_*^I$ that it commutes with products. It is then sufficient to prove that the morphisms $Sing_*^I(i_0) and Sing_*^I(i_1) : pt = Sing_*^I(pt) \longrightarrow Sing_*^I(I)$ are elementary simplicial homotopies. The required homotopy is given by the morphism $pt \longrightarrow Sing_{*,1}^I(I) = \underline{Hom}(I, I)$ which corresponds to the identity of I.

Corollary 1.3.1 *For any simplicial sheaf F the morphism*

$$Sing_*^I(F) \stackrel{Id \times i_0}{\longrightarrow} Sing_*^I(F \times I).$$

is a simplicial homotopy equivalence.

proof: By **Proposition 2.4** it is sufficient to show that the composition $F \times \mathbf{A^1} \stackrel{pr}{\longrightarrow} F \stackrel{Id \times i_0}{\longrightarrow} F \times I$ is an elementary homotopy to the identity. This homotopy is given by the morphism $Id \times \mu : F \times I \times I \longrightarrow F \times I$.

Lemma 1.3.1 *Any strict I-homotopy equivalence is an I-weak equivalence.*

proof: Let $f : F \longrightarrow G$ be a strict I-homotopy equivalence and g be an inverse of f. Because it is easy to see from the definition that two elementary I-homotopic morphisms coincide in the I-homotopy category, the result is concluded.

Lemma 1.3.2 *For any F the canonical morphism $F \longrightarrow \underline{Hom}(I, F)$ is a strict I-homotopy equivalence, and thus an I-weak equivalence.*

proof: The morphism $\underline{Hom}(I, F) \times I \longrightarrow \underline{Hom}(I, F)$ whose adjoint corresponds to μ defines a strict I-homotopy from $\underline{Hom}(p, F) \circ \underline{Hom}(i_0, F)$ to $Id_{\underline{Hom}(I,F)}$. Since $\underline{Hom}(i_0, F) \circ \underline{Hom}(p, F) = Id_{pt}$, the lemma is proven.

Corollary 1.3.2 *For any F the canonical morphism $F \longrightarrow Sing^I_*(F)$ is an I-weak equivalence.*

proof: It is easy to see that the i-th term of the simplicial sheaf $\underline{Hom}(I^i, F)$ and the canonical morphism $F \longrightarrow Sing_*(F)$ coincides with the canonical morphism $F_i \longrightarrow \underline{Hom}(I^i, F)$ from **Lemma 2.6**. Our result comes from the simplicial algebraic topology. The last few lemmas and corrolaries prove the following results:

1. $Sing^I_*$ preserves limits
2. $Sing^I_*$ takes the morphism i_0 to a weak equivalence
3. for any F the morphism $s_F : F \longrightarrow Sing^I_*(F)$ is a monomorphism and an I-weak equivalence

1.3.2 $\mathbf{A^1}$-localization Functor

We have already seen the affine line $\mathbf{A^1}$ plays a basic role in $\mathbf{A^1}$-homotopy theory. In the sequal, we will view $\mathbf{A^1}$ as a pointed object with base point 0. For any $X \in \Delta^{op}Sh_\bullet(Sm/S)_{Nis}$ let's denote by $ev_1 : \underline{Hom}_\bullet(\mathbf{A^1}, X) \longrightarrow X$ the evaluation at 1.

Lemma 1.3.3 *Let X be a simplicial fibrant sheaf. Then the following conditions are equivalent:*

(i) *X is $\mathbf{A^1}$-local*

(ii) *The morphism of simplicial sheaves: $X \longrightarrow \underline{Hom}(\mathbf{A^1}, X)$ induced by the projection map $X \times \mathbf{A^1} \longrightarrow X$ is a simplicial weak equivalence*

(iii) *The morphism of simplicial sheaves: $ev_0 : \underline{Hom}(\mathbf{A^1}, X) \longrightarrow X$ is a simplicial weak equivalence.*

Assume moreover that X is 0-connected and pointed. Then the above conditions are equivalent to the followings:

(iv) *the functional object $\underline{Hom}_\bullet(\mathbf{A^1}, X)$ is weakly contractible*

(v) *For any smooth k-scheme U, and integer $n \in N$, the map*

$$Hom_{H_{\bullet,s}(S)}((U_+) \wedge S^n, \underline{Hom}_\bullet(\mathbf{A^1}), X) \longrightarrow Hom_{H_{\bullet,s}(S)}((U_+) \wedge S^n, X)$$

is trivial (as a map of pointed sets).

sketch of proof:

The first three conditions are easily seen to be equivalent. X being pointed simplicial fibrant and 0-connected, then the *evaluation at 0*:

$$\underline{Hom}(\mathbf{A^1}, X) \longrightarrow X$$

be a simplicial weak equivalence if and only if each of its fibers is weakly contractible. Then the equivalence of the first four conditions is the consequence of the fiber of $\underline{Hom}(\mathbf{A^1}, X) \longrightarrow X$ at the base point of X is actually the object $\underline{Hom}_\bullet(\mathbf{A^1}, X)$. The equivalent of all the five conditions comes from the lemma:

Lemma 1.3.4 *A pointed simplicial sheaf X is weakly contractible if and only if for any $n \geq 0$, any $U \in Sm(k)$, the $H_{s,\bullet}(S)$-morphism $f : S^n \wedge (U_+) \longrightarrow X$ is trivial.*

Let $X \longrightarrow X_f$ be the functorial fibrant replacement. Suppose that X is a pointed simplicial sheaf, and $L^{(1)}(X)$ be the cone of the morphism $ev_1 : \underline{Hom}_\bullet(\mathbf{A^1}, X_f) \longrightarrow X_f$. Let $L_f^{(1)}(X)$ be $L^{(1)}(X)_f$. We let $X \longrightarrow L_f^{(1)}(X)$ be the obvious morphism of pointed simplicial sheaf. The define by induction for $n \geq 0, L^{(n)} = L^{(1)} \circ L^{(n-1)}$. So we have a natural morphism for any X, $X \longrightarrow L^{(n)}(X)$ and we set $L^\infty(X) = colim_n L^{(n)}(X)$.

Proposition 1.3.2 *Let X be a pointed connected simplicial sheaf. Then the simplicial sheaf $L^\infty(X)$ is $\mathbf{A^1}$-local and the morphism:*

$$X \longrightarrow L^\infty(X)$$

is an $\mathbf{A^1}$-weak equivalence. We call the functor $L^\infty = L_{\mathbf{A^1}}$ the $\mathbf{A^1}$-localization functor.

proof: Consider $\mathbf{A^1}$ being the *interval*, and denote i_0, i_1, μ be morphisms as we described in last section. Then the product μ gives a morphism:

$$\underline{Hom}_\bullet(\mathbf{A^1}, X_f) \wedge \mathbf{A^1} \longrightarrow \underline{Hom}_\bullet(\mathbf{A^1}, X_f)$$

which is left inverse to the section at 1

$$\underline{Hom}_\bullet(\mathbf{A^1}, X_f) \longrightarrow \underline{Hom}_\bullet(\mathbf{A^1}, X_f) \wedge \mathbf{A^1}.$$

So $\underline{Hom}_\bullet(\mathbf{A^1}, X_f)$ being a retract of $\underline{Hom}_\bullet(\mathbf{A^1}, X_f) \wedge \mathbf{A^1}$, and it is $\mathbf{A^1}$-weakly contractible. Therefore, the morphisms $X \longrightarrow L^{(1)}(X)$ is an $\mathbf{A^1}$-weak equivalence, and thus the composition $X \longrightarrow L^\infty(X)$ is an $\mathbf{A^1}$-weak equivalence as well.

The fact that $L^\infty(X)$ is $\mathbf{A^1}$-local comes from the **Lemma 2.10** and the following lemma.

Lemma 1.3.5 *Let $X^0 \longrightarrow X^1 \longrightarrow X^2 \longrightarrow \ldots$ be a direct system of pointed simplicial sheaves. The for any integer n and any $U \in Sm/k$ the map:*

$$colim_m Hom_{H_s(C_{Nis})}(S^n \wedge (U_+), X^m) \longrightarrow Hom_{H_s(C_{Nis})}(S^n \wedge (U_+), colim_m X^m)$$

is a bijection.

This lemma is equivalent to say, the objects $S^n \wedge (U_+)$ are compact objects.

1.3.3 The Main Result

We begin to describe $\mathbf{A^1}$-homotopy groups and consider the pointed simplicial sheaves.

When we know the description of model category structure on the category of simplicial sheaves, we can then define the induced structure of the pointed version. A morphism of pointed spaces is called a weak equivalence, fibration or cofibration if it is a weak equivalence, fibration or cofibration as a morphism of spaces with forgotten base points. Therefore, we can define further a model category structure. Let us denote the category of pointed spaces by $Spc_\bullet$, and the corresponding homotopy categories by $H_\bullet^{\mathbf{A^1}}(S)$ and $H_{s,\bullet}(S)$. Furthermore, we can define a smash product on the category of $Spc_\bullet$ by first defining the wedge of $X, Y \in Spc_\bullet$ by:

$$\begin{tikzcd} * \arrow[r] \arrow[d] & X \\ Y & \end{tikzcd}$$

and is denoted by $X \vee Y$. Then de pointed quotient space $X \times Y / X \vee Y$ is denoted by $X \wedge Y$ and is called the *smash-product* of X and Y.

Moreover, the smash-product is we defined gives the category $Spc_\bullet$ a symmetric monoidal structure where the unit object is the space $(S \amalg S, i_S)$ which is noted as S_0 and called the 0-sphere.

For any pointed simplicial sheaf (F, x) and any $i \geq 0$ we get three types of presheaves of homotopy groups (sets):

1. the naive homotopy groups (sets) $\underline{\pi}_i^{naive}(F, x)(U) = \pi_i(F(U), x)$
2. the simplicial homotopy groups $\underline{\pi}_i(F, x)(U) = \pi_i(Ex(F)(U), x)$

3. the $\mathbf{A^1}$-homotopy group $\underline{\pi}_i^{\mathbf{A^1}}(F,x)(U) = \pi_i(L_{\mathbf{A^1}}(F(U)),x)$,

where $Ex(F)$ denote the functorial fibrant replacement of F.

We will also denote $a\pi_i(F,x)$ the sheaf associated to the presheaf $\pi_i(F,x)$, and $a\underline{\pi}_i^{\mathbf{A^1}}(F,x)$ the sheaf associated to the presheaf $\underline{\pi}_i^{\mathbf{A^1}}(F,x)$. Note that $a\underline{\pi}_i(F,x)$ is isomorphic to the sheaf associated to the presheaf of naive homotopy groups $\underline{\pi}_i(F,x)$ according to **Theorem 2.2**. We say that F is $\mathbf{A^1}$-connected if $a\underline{\pi}_0^{\mathbf{A^1}}(F)$ is the constant sheaf *pt*.

Theorem 1.3.2 (Whitehead) *Let $f : (F,x) \longrightarrow (G,y)$ be a morphism of connected simplicial sheaves. Then the following conditions are equivalent:*

1. *f is an $\mathbf{A^1}$-weak equivalence;*
2. *for any $i \geq 0$ the morphism of the presheaves of $\mathbf{A^1}$-homotopy groups $\underline{\pi}_i^{\mathbf{A^1}}(F,x) \longrightarrow \underline{\pi}_i^{\mathbf{A^1}}(G,y)$ is an isomorphism;*
3. *for any $i \geq 0$ the morphism of the sheaves of $\mathbf{A^1}$-homotopy groups*

$$a\underline{\pi}_i^{\mathbf{A^1}}(F,x) \longrightarrow a\underline{\pi}_i^{\mathbf{A^1}}(G,y)$$

This theorem comes from directly from the fact that $L_{\mathbf{A^1}}(F)$ is a stack and therefore has the B.G-property.

In [21] the homotopy relation is defined by homotopy groups and the homotopy category is obtained by localization with respect to homotopy weak equivalence, and we obtain the homotopy category. We equip $Ho(sSh(Sm/S)_{Nis})$ with a global injective model structure where weak equivalences are global weak equivalences (defined by homotopy groups), and cofibrations are monomorphisms between simplicial sheaves. We denote this homotopy category by $Ho(sSh(Sm/S)_{Nis})_{global,inj}$.

In [13] the weak equivalence is the $\mathbf{A^1}$-weak equivalence. Let us denote the collection of morphisms $\mathcal{B} = \{X \longrightarrow X \times \mathbf{A^1}\}$ for all $X \in (Sm/S)_{Nis}$, and consider the localization of Bousfield with respect to $\mathcal{B}$, and there is a pair of adjoinctions induced by identities:

$$\mathbf{L}id : Ho(sSh(Sm/S)_{Nis})_{loc,inj} \leftrightarrow Ho(L_S((sSh(Sm/S)_{Nis})_{loc,inj})) : \mathbf{R}id$$

We know that the objects in

$$Ho(sSh(Sm/S)_{Nis})_{global,inj} \quad and \quad Ho(L_S((sSh(Sm/S)_{Nis})_{loc,inj}))$$

are the same, and so are their cofibrant objects. Moreover, in the injective structure, all objects in are cofibrant. In order to show that these two homotopy categories are equivalent, we need to show that the classes of weak equivalences coincide.

Lemma 1.3.6 *For any $X \in spc$, $Sing_*(Ex^\infty(X))$ is $\mathbf{A^1}$-local.*

proof: We will see in the second chapter that $Sing_*(Ex^\infty(X))$ is $\mathbf{A^1}$-local if it is globally fibrant for the Nisnevich topology and has the RLP with respect to all simplicial sheaf inclusions:

$$(\mathbf{A^1} \times V) \cup_V U \longrightarrow \mathbf{A^1} \times U$$

induced by $* \longrightarrow \mathbf{A^1}$ and all cofibrations $V \longrightarrow U$. This is true if and only if the diagram:

$$\begin{tikzcd} * \arrow[r] \arrow[d] & S(U, Ex^\infty(X)) \arrow[d] \\ \Delta^1 \arrow[r] & S(V, Ex^\infty(X)) \end{tikzcd}$$

solves the lifting problem. Since $Ex^\infty(X)$ has the RLP with respect to

$$U\times \mid \wedge_k^n \mid_s \cup_{V\times|\wedge_k^n|_s} V\times \mid \Delta^n \mid_s \longrightarrow U \times |\Delta^n|_s,$$

we know that $S(U, Ex^\infty(X)) \longrightarrow S(V, Ex^\infty(X))$ is a fibration. It is then clear the above diagram has a lifting arrow.

Lemma 1.3.7 *In the local model structure, if $j : Z \longrightarrow GZ$ is a fibrant model, that is to say j is a weak equivalence and GZ is fibrant, then Z has the B.G-property if and only if j induces a termwise equivalence $Z(U) \longrightarrow GZ(U)$ for any U.*

We will prove this lemma in the next section.

We know that the model category $(sPSh(Sm/S)_{Nis})_{loc,inj}$ is the obtained from the category $(sPSh(Sm/S)_{Nis})_{global,inj}$ by Bousfield localization with respect to the class of *hyper-covers*. Therefore, all the weak equivalences in $(sPSh(Sm/S)_{Nis})_{global,inj}$ are weak equivalences in $(sPSh(Sm/S)_{Nis})_{loc,inj}$.

Notice that a morphism $f : X \longrightarrow Y \in spc$ is a weak equivalence in [21] if and only if $Sing_*(Ex^\infty(X))$ and $Sing_*(Ex^\infty(Y))$ are equivalent in $Ho(sSh(Sm/S)_{Nis})_{global,inj}$. Since $Sing_*(Ex^\infty(X))$ is $\mathbf{A^1}$-local for any $X \in spc$, it has B.G-property and is fibrant in the category $L_S((sSh(Sm/S)_{Nis})_{loc,inj})$. So the morphism

$$Sing_*(Ex^\infty(X)) \longrightarrow Sing_*(Ex^\infty(Y))$$

is an equivalence in $Ho(L_S((sSh(Sm/S)_{Nis})_{loc,inj}))$ if and only if it induces a pointwise weak equivalence, and if and only if it is a global weak equivalence.

Therefore, we conclude that our two constructions are actually equivalent. This is actually the **Theorem 3.6** in [21].

1.4 Nisnevich Cohomology

Let $Ab(C_{Nis})$ denote the Abelian category of Nisnevich sheaves of Abelian groups on C. For a simplicial sheaf U, let $Z[U]$ be the free Abelian group generated by U. Suppose now $M \in Ab(C_{Nis})$. Then for any integer i, we have the canonical isomorphism $Ext^i(Z[U], M) = H^i_{Nis}(U, M)$. We know from the **Lemma 2.12** that the Abelian sheaf $Z[U]$ is a compact object, and there is a triangulated structure on $Ab(C_{Nis})$ generated by $Z[*]$ which admits small coproducts. Thus, by Brown representability theorem, we claim the cohomology presheaf is representable. More specifically, $H^i_{Nis}(U, M) = Hom_{H_s(C_{Nis})}(U, K(M, i))$. The object $K(M, i)$ is known as the Eilenberg-MacLane simplicial sheaf which has homotopy group isomorphic to M in dimension i and zero in all other dimensions.

The discussion following the **Lemma 2.1** shows that any distinguished square is a cocartesian square, and induced a Nisnevich short exact sequence:

$$0 \longrightarrow Z_{Nis}[U \times_X V] \longrightarrow Z_{Nis}[U] \oplus Z_{Nis}[V] \longrightarrow Z_{Nis}[X] \longrightarrow 0$$

Thus from the fact that the cohomology functor on a pretriangulated category is exact, we get a long exact sequence which correspond to *Mayer-Vietoris* sequence: For any simplicial sheaf F, we have:

$$\ldots \longrightarrow H^i_{Nis}(X, F) \longrightarrow H^i_{Nis}(U, F) \oplus H^i_{Nis}(V, F) \longrightarrow H^i_{Nis}(U \times_V, F)$$
$$\longrightarrow H^{i+1}_{Nis}(X, F) \longrightarrow \ldots$$

Definition 1.4.1 *Let $n \geq 0$, we say that $M \in Ab(C_{Nis})$ is n-strictly $\mathbf{A^1}$-(homotopy) invariant if and only if for any smooth k-scheme X and any integer $i \in \{0, 1, \ldots, n\}$ the homomorphism*

$$H^i_{Nis}(X, M) \longrightarrow H^i_{Nis}(X \times \mathbf{A^1}, M)$$

is an isomorphism.

Proposition 1.4.1 *Let X be a noetherian scheme of krull dimension$\leq d$, then for any sheaf of abelian groups F on Sm/S_{Nis}, we have $H^i_{Nis}(X, F) = 0$ for $i > d$.*

Theorem 1.4.1 (Nisnevich Acyclicity) *Let $M \in Ab(C_{Nis})$ be n-strictly $\mathbf{A^1}$-(homotopy) invariant, then $Ext^i_{Nis}(X, M) = 0$ for $0 \leq i \leq n$ and $X \in Sm/k$.*

We will give a *motivic* proof later.

Chapter 2

Stable $\mathbf{A^1}$-homotopy Theory

In this chapter, we want to construct the stable $\mathbf{A^1}$-homotopy Theory. In the first section, we will review some basic properties in classical stable homotopy theory without many proofs. Then in the following sections, we will use Jardine's language to define generalmotivic spectra. We will also prove many things mentioned in the first section in the motivic background. We will focus specifically on S^1-spectra and Morel-Voevodsky spectra.

2.1 A Review of Classical Stable Homotopy Theory

2.1.1 The Spanier-Whitehead Category

The Spanier-Whitehead category gives us a naive way to formally inverse the suspension functor, which makes the new category additive and triangulated.

Definition 2.1.1 *The category* $\mathbf{SW}$ *of Spanier-Whitehead is the category whose objects are pairs* (X, n)*, where* $X \in CW_*$ *and* $n \in \mathbf{Z}$*, and*

$$Hom_{\mathbf{SW}}((X,n),(Y,m)) = colim_r[\Sigma^{n+r}X, \Sigma^{m+r}Y].$$

The natural functor $st : Ho(CW_*) \longrightarrow \mathbf{SW}$ sends $X \in CW_*$ to $(X, 0) \in \mathbf{SW}$ is called the stabilization functor.

Definition 2.1.2 *Let's define:*

1. *the functor of formal suspension* $s : \mathbf{SW} \longrightarrow \mathbf{SW}$ *by* $s((X,n)) = (X, n+1)$*;*
2. *the functor of geometric suspension* $\Sigma : \mathbf{SW} \longrightarrow \mathbf{SW}$ *by* $\Sigma((X,n)) = (\Sigma X, n)$

Lemma 2.1.1

1. *The functors s and Σ are naturally isomorphic, especially $Hom_{\mathbf{SW}}(A, B) = Hom_{\mathbf{SW}}(\Sigma A, \Sigma B)$ for all $A, B \in \mathbf{SW}$;*

2. *For all pairs $(C,\ T : C \longrightarrow C)$ where T is an equivalence, and all functors $F : Ho(CW_*) \longrightarrow C$ such that $T \circ F = F \circ \Sigma$, there exists a unique functor (up to equivalence) $G : (\mathbf{SW}, s) \longrightarrow (C, T)$ makes the diagram commutative:*

$$\begin{tikzcd}
(Ho(CW_*), \Sigma) \arrow[r, "st"] \arrow[dr, "F"'] & (\mathbf{SW}, s) \arrow[d, "G"] \\
& (C, T)
\end{tikzcd}$$

The notion $G : (\mathbf{SW}, s) \longrightarrow (C, T)$ implies that $T \circ G = G \circ s$.

In particular, we notice that all (co)homological functors factorize through $\mathbf{SW}$.

Definition 2.1.3 *A sequence $(X, m) \longrightarrow (Y, n) \longrightarrow (Z, l) \longrightarrow \Sigma(X, m)$ in $\mathbf{SW}$ is called a distinguished triangle if for some integer $k \geq 0$, the sequence:*

$$\Sigma^{m+k} X \longrightarrow \sigma^{n+k} Y \longrightarrow \sigma^{l+k} Z \longrightarrow \sigma^{m+k+1} X$$

is isomorphic to a distinguished triangle in $Ho(CW_*)$.

Theorem 2.1.1 *Equipped with the class of distinguished triangles as defined above, the category $(\mathbf{SW}, s)$ is a pretriangulated category.*

Corollary 2.1.1 *If $A \longrightarrow B \longrightarrow C \longrightarrow sA$ is a distinguished triangle in $\mathbf{SW}$, then for all $D \in \mathbf{SW}$, there is a long exact sequences:*

$$\cdots \longrightarrow Hom_{\mathbf{SW}}(D, A) \longrightarrow Hom_{\mathbf{SW}}(D, B) \longrightarrow Hom_{\mathbf{SW}}(D, C) \longrightarrow Hom_{\mathbf{SW}}(D, sA) \longrightarrow \dots$$

and

$$\dots \longrightarrow Hom_{\mathbf{SW}}(sA, D) \longrightarrow Hom_{\mathbf{SW}}(C, D) \longrightarrow Hom_{\mathbf{SW}}(B, D) \longrightarrow Hom_{\mathbf{SW}}(A, D) \longrightarrow \dots$$

In order to equip the category $\mathbf{SW}$ a symmetric monoidal structure, we first define $(X, m) \wedge (Y, n) = (X \wedge Y, m + n)$. Given this definition we have:

Proposition 2.1.1

1. *The category $(\mathbf{SW}, \wedge)$ is symmetric monoidal, with unit $S = (S^0, 0)$,*

2. *There is an natural equivalence $sA \wedge B \longrightarrow s(A \wedge B)$,*

3. *The smash product is compatible with the triangulated structure: if* $A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A$ *is a distinguished triangle in* $\mathbf{SW}$*, then for all* $D \in \mathbf{SW}$*, the sequence* $(A \wedge D) \longrightarrow (B \wedge D) \longrightarrow (C \wedge D) \longrightarrow \Sigma(A \wedge D)$ *is still a distinguished triangle,*

4. *The natural morphism* $(A \wedge B) \oplus (A \wedge C) \longrightarrow A \wedge (B \oplus C)$ *is an equivalence.*

Proposition 2.1.2 *The sphere* $S = st(S^0)$ *is a graded generator of* $\mathbf{SW}$ *in the sense that* $\forall A \in \mathbf{SW}$, $Hom_{\mathbf{SW}}(\Sigma^r S, A) = 0$ *for all* $r \geq 0$, *we have* $A \simeq 0$.

2.1.2 Stable Homotopy Category

The Spanier-Whitehead category has a big drawback, which is it does not have arbitrary coproducts. Moreover, reduced cohomology theories are not always representable in SW. In these senses we say that this category is too small. In order to get a category *good enough*, the first consideration is to formally add infinite coproducts to the category $\mathbf{SW}^c$, the category of compact objects. So, intuitively, we want a category called stable homotopy category which is obtained by adding arbitrary coproducts to $\mathbf{SW}^c$.

Definition 2.1.4 *A stable homotopy category is a category* T *whose objects are called spectrum and morphisms are noted* $[X, Y] = Hom_T(X, Y)$*, which satisfy the following axioms:*

Axiom 1 *:* T *has small coproducts,*

Axiom 2 *: There exists a suspension functor* $s : T \longrightarrow T$ *and a collection* Δ *of distinguished triangles of the form* $X \longrightarrow Y \longrightarrow Z \longrightarrow sX$*, such that* (T, Δ, s) *is a triangulated category,*

Axiom 3 *: There is an additive functor* $\wedge : T \times T \longrightarrow T$*, which is called the smash product such that:*

1. $(T, \wedge)$ *is a symmetric monoidal category with unit* S*,*
2. *There exists a natural equivalence* $s(X \wedge Y) = sX \wedge Y$*,*
3. *If* $X \longrightarrow Y \longrightarrow Z \longrightarrow sX$ *is a distinguished triangle, then for all* $W \in T$*, the sequence* $X \wedge W \longrightarrow Y \wedge W \longrightarrow Z \wedge W \longrightarrow s(X \wedge W)$ *is still a distinguished triangle,*
4. *The morphism* $X \wedge \amalg Y_\alpha \longrightarrow \amalg(X \wedge Y_\alpha)$ *is an isomorphism.*

Axiom 4 *:* S *is a graded generator of* T*,*

Axiom 5 : *Let T^c be the full subcategory of T consists of the compact objects. Then T^c is a triangulated subcategory and closed under smash product. There exists an equivalence of categories between $\mathbf{SW}^c$ and T^c which preserve the triangulated and the symmetric monoidal structure.*

Definition 2.1.5 *Let R denote N or Z, and C be CW_*, Top_* or $\Delta^{op}Set$ with their natural model category structure.*

1. *An R-spectrum with value in C is a family $E = (E_k)_{k \in R}, E_k \in C$, and a collection of morphisms, which are called bonding morphisms, $\sigma : \Sigma E_k \longrightarrow E_{k+1}$ (or respectively $\sigma_* : E_k \longrightarrow \Omega \circ E_{k+1}$).*
2. *A morphism between spectra $f : E \longrightarrow F$ is given by a family of morphisms $(f_k : E_k \longrightarrow F_k)_{k \in R}$ in C, satisfying $f_{k+1} \circ \sigma = \sigma \Sigma f_k$. (or respectively $\sigma_* \circ f_k = \Omega f_{k+1} \circ \sigma_*$).*
3. *We note* **Spt** *the category of spectra as defined above.*

Definition 2.1.6 *A spectra E is called:*

- *a Σ-spectrum if $\Sigma E_k \xrightarrow{\sigma} E_{k+1}$ is an equivalence for $k \geq 0$,*
- *a Ω-spectrum if $E_k \xrightarrow{\sigma_*} \Omega E_{k+1}$ is an equivalence for $k \in R$.*

Example 2.1.1

- *Define a functor : C Spt by letting $(\Sigma^\infty X)_n = \Sigma^n X$ for $n \geq 0$, and $(\Sigma^\infty X)_n = *$ for $n < 0$. Then $(\Sigma^\infty X)$ is a Σ-spectrum with s_n identities.*
- *Let G be an Abelian group. We have the equivalence $s_{n*} : K(G, n) \xrightarrow{\sim} \Omega K(G, n+1)$.Thus we get an Ω-spectrum $HG = (K(G, n), s_n)$.*

Definition 2.1.7 *Let E be a spectrum. Let $\pi_n(E) = colim_k \pi_{k+n}(E_k)$.*

Definition 2.1.8 *A morphism of spectra $f : E \longrightarrow F$ is a stable weak equivalence if $f_n : \pi_n(E) \longrightarrow \pi_n(F)$ is an isomorphism for all $n \geq 0$.*

In the following, we set the notation by setting $R = Z$ and $C = CW_*$.

Definition 2.1.9 *An Adams spectrum is a spectrum (E, s_n) with values in CW_* such that $\sigma : \Sigma E_k \longrightarrow E_{k+1}$ are cofibrations. (so $s_n(\Sigma E_n)$ is a subcomplex of E_{n+1}).*

Definition 2.1.10 *Let E be an Adams spectrum.*

1. *A (stable) cell of E is a sequence $e = e_k, \Sigma e_k, \Sigma^2 e_k, \cdots$, where e_k is a cell of E_k which does not belong to any suspension of E_{k-1}. If e_k is of dimension d, then the dimension of the cell e is d-k. The base point of E_k is the cell of dimension $-\infty$ of E,*

2. *The n-skeleton of E is the sub-spectrum $E^{(n)}$ of E consists of cells of dimension$\leq n$,*

3. *The spectrum E is called finite if it has only finitely many of cells, is of finite type if each of its skeleton is finite, and is of finite dimension if $E = E^{(n)}$ for certain n,*

4. *A sub-spectrum of Adams $F \subset E$ is called cofinal in E if all cells of E eventually lie in F. That is, for all cells e_n of E_n, there exists an integer m such that $\Sigma^m e_n \subset F_{m+n}$.*

Definition 2.1.11 *Let (E_n, s_n) be an Adams spectrum and (F_n, t_n) be an arbitrary spectrum. Consider the collection $A = (f', E')$ of morphisms $f' : E' \longrightarrow F'$ where E' a cofinal sub-spectrum of E. We define an equivalence relation on A by setting $(f', E') \simeq (f'', E'')$ if $f'_{|B} = f''_{|B}$ where B is a cofinal spectrum of E satisfies $B \subset E \cap E'$. A such equivalent class of morphisms is called an Adams morphism from E to E'.*

Lemma 2.1.2 *Let $f : E \longrightarrow F$ be a morphisms between spectra of Adams. If $F' \subset F$ is cofinal then there exists a cofinal $E' \subset E$ such that $f(E') \subset F'$.*

Therefore, if we consider only spectra of Adams, the composition of morphisms is well defined.

Definition 2.1.12

1. *Two morphisms $f, g : E \longrightarrow F$ between spectra are said to be homotopic if there exists a morphism $H : E \wedge I_+ \longrightarrow F$ such that $H_0 = f, H_1 = g$.*

2. *Two morphisms between spectra of Adams $f, g : E \longrightarrow F$ where E is a spectrum of Adams are said to be homotopic if there exists a morphism of Adams $H : E \wedge I^+ \longrightarrow F$ such that $H_0 = f, H_1 = g$, equivalently if there exists a cofinal sub-spectrum of Adams E' of E and two morphisms homotopic $h_i : E' \longrightarrow F$ with $h_0 \in f, h_1 \in g$. This definition gives us an equivalence relation, and the equivalence classes of morphisms of Adams $E \longrightarrow F$ is noted as $[E, F]$.*

Lemma 2.1.3 *The class of morphisms $[fg]$ of two morphisms of Adams f, g depends only on the classes of morphisms $[f]$ and $[g]$.*

Definition 2.1.13 *Let S be the category of Adams spectra. Then the category $Ho(S)$ has for objects the Adams spectra and for morphisms $Hom_{HoS}(E,F)=[E,F]$. The isomorphisms in $Ho(S)$ are called the (homotopy) equivalences.*

Lemma 2.1.4 1. *Let $f : E \longrightarrow F$ is a morphism of spectra, then $Cf=(Cf,s_n)$ is called the cone of f, where:*

$$s_n : \Sigma(CF_n) = \Sigma F_n \cup C(\Sigma E_n) \overset{s_n^F \cup C s_n^E}{\longrightarrow} F_{n+1} \cup C(E_{n+1}) = Cf_{n+1},$$

2. *All morphisms of Adams $f : E \longrightarrow F$ contain a biggest element $\widetilde{f}$ for the partial order $(f',E') \leq (f'',E'')$ if $E' \subset E''$ and $f''_{|E'} = f'$. Then define the cone $Cf \in S$ by $Cf = C\,\widetilde{f}$.*

Theorem 2.1.2 *Let $f : E \longrightarrow F$ be a stable weak equivalence, then for any spectrum of Adams W, the map $f_* : [W,E] \longrightarrow [W,F]$ is an isomorphism.*

Corollary 2.1.2 (Whitehead) *A morphism of Adams of Adams spectra $f : E \longrightarrow F$ is a homotopy equivalence if and only if it is a stable weak equivalence.*

Proposition 2.1.3 *Each spectrum $E \in Spt$ is stably weakly equivalent to a spectrum of Adams, and it is unique with respect to equivalence.*

Corollary 2.1.3 *The localization of the category Spt with respect to stable weak equivalence is equivalent to $Ho(S)$.*

Proposition 2.1.4 *The spectra $E[1], \Sigma E \; and \; E \wedge S^1$ are stably equivalent. In particular if $E,F \in S$ the map $[E,F] \longrightarrow [\Sigma E, \Sigma F]$ is an isomorphism.*

Corollary 2.1.4 *The category $Ho(S)$ is additive.*

Theorem 2.1.3 *The homotopy category $Ho(S)$ is a stable homotopy category in the sense of* **Definition 3.4**.

2.1.3 Model Category Structure on Spt

We will recover $Ho(S)$ by defining a suitable model structure on Spt.

Definition 2.1.14 *A morphism of spectra $f : X \longrightarrow Y$ is:*

- *a strict weak equivalence if for any $n \geq 0$, $f_n : X_n \longrightarrow Y_n$ is a weak equivalence in base category C,*

- *a strict fibration if for any $n \geq 0$, $f_n : X_n \longrightarrow Y_n$ is a fibration in C,*
- *a strict cofibration if*
  - *$f_0 : C_0 \longrightarrow Y_0$ is a cofibration in C,*
  - *the map $(S^1 \wedge Y_n) \cup_{S^1 \wedge X_n} X_{n+1} \longrightarrow Y_{n+1}$ are cofibrations for all $n \geq 0$.*

Lemma 2.1.5 *If $i : A \hookrightarrow B$ is a strict cofibration of Spt, then:*

- *for all integers n, the map $f_n : A_n \longrightarrow B_n$ is a cofibration of C,*
- *if moreover $A \simeq B$, the for all integers n,*

$$(S^1 \wedge B_n) \cup_{S^1 \wedge A_n} B_{n+1} \xrightarrow{\sim} B_{n+1}.$$

Proposition 2.1.5 *Equipped with the strict structure, the category Spt^{strict} is a proper model category.*

Definition 2.1.15 *A morphism between spectra $f : X \longrightarrow Y$ is said to be:*

- *a stable weak equivalence if $f_* : \pi_n(X) \longrightarrow \pi_n(y)$ for all $n \in Z$,*
- *a stable cofibration if it is a strict cofibration,*
- *a stable fibration if it has the RLP with respect to all stable (strict) cofibrations who are stable weak equivalences.*

Theorem 2.1.4 *The above three classes of morphisms make the category Spt^{stable} a model category.*

In order to prove this theorem, we introduce an important functor, the stabilization functor.

Definition 2.1.16 *Let $j : X \longrightarrow FX \in Spt$ be the functor fibrant replacement term by term. We define the Ω-spectrum $\Omega^\infty X$ by*

$$\Omega^\infty X_n = colim_n(X_n \longrightarrow \Omega X_{n+1} \longrightarrow \Omega^2 X_{n+2} \longrightarrow \cdots)$$

The morphisms $(\Omega^\infty X)_n \xrightarrow{\sigma_*} \Omega(\Omega^\infty X_{n+1})$ is given by the colimit of the diagram:

$$\begin{tikzcd}
X_n \arrow[r] \arrow[d] & \Omega X_{n+1} \arrow[r] \arrow[d] & \Omega^2 X_{n+2} \arrow[r] \arrow[d] & \cdots \\
\Omega X_{n+1} \arrow[r] & \Omega^2 X_{n+2} \arrow[r] & \Omega^3 X_{n+3} \arrow[r] & \cdots
\end{tikzcd}$$

We note $\omega : Id_{Spt} \longrightarrow \Omega^\infty$ the natural transformation.

Definition 2.1.17 (Stabilization Functor) *We note $Q : Spt \longrightarrow Spt$ the stabilization functor, which is defined by $QX = \Omega^\infty FX$ and $\eta : Id_{Spt} \longrightarrow Q$ the natural transformation composed by ω and j. in this case, each $(QY)_n \in C$ is fibrant and QY is a Ω-spectrum.*

Lemma 2.1.6 *The morphism $f : X \longrightarrow Y \in Spt$ is a stable weak equivalence if and only if the induced morphism $Qf : QX \longrightarrow QY$ is a strict weak equivalence.*

proof:

$$\pi_k((QY)_n) = \pi_k(colim(FY_n \xrightarrow{\sigma_*} \Omega FY_{n+1} \xrightarrow{\Omega\sigma_*} \ldots))$$
$$= colim(\pi_k(FY_n) \longrightarrow \pi_{k+1}(FY_{n+1} \longrightarrow \pi_{k+2}(FY_{n+2})))$$
$$= \pi_{k-n}(Y)$$

Therefore, $f : X \longrightarrow Y$ is a weak equivalence if and only if $f_* : \pi_*((QX)_n) \simeq \pi_*((QY)_n)$ for any integer n, that is if and only if $Qf : QX \longrightarrow QY$ is a strict equivalence.

Definition 2.1.18 (Q-model structure) *Let M be a right proper model category (that is to say the weak equivalences are stable by pulling back along fibrations). Suppose there is a functor $Q : M \longrightarrow M$ and a natural transformation $\eta : Id_M \longrightarrow Q$. We call $f : X \longrightarrow Y \in M$ is a:*

- *Q-equivalence if $Qf : QX \xrightarrow{\sim} QY \in M$,*
- *Q-cofibration if it is a cofibration in M,*
- *Q-fibration if it has the RLP with respect to Q-cofibrations who are also Q-equivalences.*

Theorem 2.1.5 (Bousfield-Friedlander) *Let M be a right proper model category. Suppose there is a functor $Q : M \longrightarrow M$ and a natural transformation $\eta : Id_M \longrightarrow Q$, and moreover suppose:*

A4 $Q(W_M) \subset W_M$,

A5 *for all $X \in M$, the map $\eta Q_X : QX \longrightarrow QQX$ and $Q\eta X : QX \longrightarrow QQX$ lie in W_M,*

A6 *the Q-equivalences are stable under pull-backs along Q-fibrations.*

Then M equipped with Q is a model category (right proper).

Therefore, let $M = Spt$, Q to be the localization functor, and $\eta : Id_{Spt} \longrightarrow Q$ the natural transformation. Then M satisfies the condition A4, A5 and A6 which makes the category Spt^{stable} a model category.

Some properties of this model category structure: If we denote the category M equipped with a functor Q and a natural transformation $\eta : Id_M \longrightarrow Q$ by M^Q, then

- A map $p : X \longrightarrow Y \in M$ is a Q-fibration and Q-equivalence if and only if it is an acyclic fibration in C.
- Let $p : X \longrightarrow Y \in M$ be a fibration. Suppose that $\eta_X : X \longrightarrow QX$ and $\eta_Y : Y \longrightarrow QY$ are weak equivalences, then p is a Q-fibration.
- Each map $f : QX \longrightarrow QY \in M$ admits a factorization $f = q \circ j$ where j is a cofibration and a Q-equivalence, and q is a Q-fibration,
- Each map $f : X \longrightarrow Y \in M^Q$ admits a factorization $f = q \circ j$ where j is a cofibration and a Q-equivalence, and q is a Q-fibration,
- Each map $f : X \longrightarrow Y \in M^Q$ admits a factorization $f = p \circ i$ where i is a cofibration, and p is a Q-fibration and a Q-equivalence.

We have seen in stable model structure, weak equivalences and cofibrations are described, so we discuss the fibrations in the following.

Theorem 2.1.6 (Bousfield-Friedlander) *Let M be a right proper model category. Suppose there is a functor $Q : M \longrightarrow M$ and a natural transformation $\eta : Id_M \longrightarrow Q$, and moreover suppose:*

A4 $Q(W_M) \subset W_M$,

A5 *for all $X \in M$, the map $\eta Q_X : QX \longrightarrow QQX$ and $Q\eta X : QX \longrightarrow QQX$ lie in W_M,*

A6 *the Q-equivalences are stable under pull-backs along Q-fibrations.*

The a morphism $f : X \longrightarrow Y \in M$ is a Q-fibration if and only if it is a fibration and the diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & QX \arrow[d] \\ Y \arrow[r] & QY \end{tikzcd}$$

is homotopic cartesian in M.

Corollary 2.1.5 *An object $X \in M$ is Q-fibrant if and only if it is fibrant and η_X : $X \longrightarrow QX$ is a weak equivalence.*

Corollary 2.1.6 *A spectrum $X \in Spt$ is stably fibrant if and only if it is strictly fibrant and all the morphisms $\sigma_* : X_n \longrightarrow \Omega X_{n+1}$ are weak equivalences in C. In this case, it is a Ω-spectrum.*

proof: We have already seen that $\pi_k((QX)_n) = \pi_{k-n}(X)$, therefore we have $\pi_k((Q(QX))_n) = \pi_{k-n}(QX)$. We also know that $QX \longrightarrow QQX$ is a stable equivalence. Observe that

$$\pi_{k-n}(QX) = \pi_{k-n}(colim(X \longrightarrow \Omega X[1] \longrightarrow \Omega^2 X[2] \longrightarrow \ldots))$$
$$= colim(\pi_{k-n}(X) \longrightarrow \pi_{k-n+1}(X[1]) \longrightarrow \pi_{k-n+2}(X[2]) \longrightarrow \ldots)$$

and

$$\pi_{k-n}(X) = colim_r \pi_{k-n+r}(X_r)$$
$$= colim(\pi_{k-n}(X_0) \longrightarrow \pi_{k-n+1}(X_1) \longrightarrow \pi_{k-n+2}(X_2) \longrightarrow \ldots)$$

The result follow from the observation.

After describing the structure of stable model category on Spt, let's see some of the properties:

- Let $X \in Spt$. Then $\sigma_* : X \longrightarrow \Omega X[1]$ is a stably weak equivalence if X is strictly fibrant.
- Let $C \in C$, then the morphism $\eta : \Sigma^\infty X \longrightarrow \Omega\Sigma(\Sigma^\infty X)$ is a stable weak equivalence.
- Let $Y \in Spt$, then the morphism $\Sigma X \longrightarrow X[1]$ is a stable weak equivalence.
- The functors $* \wedge S^1$ and Σ are stably equivalent.
- The functors $*[-1], \Omega\ and\ Hom(S^1, *)$ are stably equivalent.
- The functor Ω is right adjoint to the functor Σ.

2.2 Motivic Homotopy Theory in View of [5]

In this section, we use the language of Jardine's to construct the motivic homotopy category. We will see that this construction is equivalent to Morel-Voevodsky's construction, and we will base Jardine's methods to construct stable homotopy theory in the following sections.

2.2.1 Motivic Homotopy Theory

The Model Structures

Let $(Sm/S)_{Nis}$ be the site of smooth schemes over a scheme S equipped with the Nisnevich topology. Recall that we have equipped the category of simplicial presheaves on Nisnevich sites with an injective model structure, where the weak equivalences are local weak equivalences (or stalkwise weak equivalences), cofibrations are monomorphisms of simplicial presheaves, and fibrations the local fibrations which have RLP with respect to morphisms that are cofibrations and local weak equivalences. We should notice that a presheaf is local equivalent to its associated sheaf since they are isomorphic on stalks. We denote this category by $sPSh((Sm/S)_{Nis})_{inj}$, or $sPSh((Sm/S)_{inj})$. A locally fibrant model of a simplicial presheaf X is a local weak equivalence $X \longrightarrow W$, where W is locally fibrant.

Definition 2.2.1 *We say that a simplicial presheaf Z on the Nisnevich site is motivic fibrant if it is locally fibrant for the Nisnevich topology and has the RLP with respect to all simplicial presheaf inclusions:*

$$(f, j) : (\mathbf{A^1} \times A) \cup_A B \longrightarrow \mathbf{A^1} \times B$$

induced by $f : * \longrightarrow \mathbf{A^1}$ and all cofibrations $j : A \longrightarrow B$.

Actually, for two cofibrations $i : A \longrightarrow B$ and $j : C \longrightarrow D$, there is a cofibration:

$$(A \times D) \bigcup_{A \times C} (B \times C) \hookrightarrow B \times D$$

which is a local weak equivalence if either $i \; or \; j$ is a local weak equivalence. Therefore, if $i : A \longrightarrow B$ is a cofibration and $p : X \longrightarrow Y$ is a local fibration, the map

$$\underline{Hom}(B, X) \longrightarrow \underline{Hom} \times_{\underline{Hom}(A,Y)} \underline{Hom}(B, Y)$$

is a trivial local fibration. So, we consider $f : * \longrightarrow \mathbf{A^1}$ and cofibration $j : A \longrightarrow B$. The lifting property asserts that the induced fibration:

$$f^* : \underline{Hom}(\mathbf{A^1}, Z) \longrightarrow \underline{Hom}(*, Z) \simeq Z$$

is a trivial fibration. It follows that a simplicial presheaf Z is motivic fibrant if and only if it is locally fibrant and all inclusions $U \longrightarrow U \times \mathbf{A^1}$ induces weak equivalence of simplicial sets $Z(U \times \mathbf{A^1}) \longrightarrow Z(U)$. So, this definition is actually equivalent to our previous definition of $\mathbf{A^1}$-*local*.

Definition 2.2.2 *A simplicial presheaf map $g : X \longrightarrow Y$ is said to be a motivic weak equivalence if it induces a weak equivalence:*

$$g^* : S(Y, Z) \longrightarrow S(X, Z)$$

where $S(X, Z) = S(X, Z)_*$ is defined as $S(X, Z)_n = Hom_{sPSh}(X \times \Delta^n, Z)$ for every motivic fibrant Z.

A (motivic) cofibration is a monomorphism of simplicial presheaves. A motivic fibration is a morphism which has RLP with respect to all morphisms that are motivic weak equivalences and cofibrations.

In this case, the category $sPSh((Sm/S)_{Nis})$ of simplicial presheaves on the smooth Nisnevich site of the scheme S, together with the classes of cofibrations, motivic weak equivalences and motivic fibration, is a proper closed simplicial model category.

Actually, we could also change all the *presheaves* in the above conditions by *sheaves*. In particular, we have the following theorem:

Theorem 2.2.1

1. *The category $sSh(\mathcal{S}\Updownarrow/\mathcal{S}_{Nis})$ of simplicial sheaves on the smooth Nisnevich site over S, together with the classes of cofibrations, motivic weak equivalences and motivic fibrations, is a proper closed simplicial model category.*

2. *The forgetful functor and the associated sheafification functor together determine an adjoint equivalence of motivic homotopy categories:*

$$Ho(sPsh((Sm/S)_{Nis}) \simeq Ho(sSh((Sm/S)_{Nis})$$

The proof of this theorem is given in [5], theorem 1.2.

The B.G-property

We have already seen that if two simplicial presheaves which have the B.G-property are locally weak equivalent, then they are globally weak equivalent. The analogy of this property is:

Theorem 2.2.2 *A simplicial presheaf Z on the site $(Sm/S)_{Nis}$ has the B.G-property if and only if any locally fibrant model $j : Z \longrightarrow GZ$ for Z induces term by term weak equivalences $Z(U) \longrightarrow GZ(U)$ for all $U \in (Sm/S)_{Nis}$.*

proof: Recall that Morel and Voevodsky proved that every locally fibrant simplicial **sheaf** has the B.G-property, and that if $f : X \longrightarrow Y$ is a local weak equivalence of simplicial presheaves and both X and Y have B.G-property, then there is a pointwise weak equivalence $X(U) \longrightarrow Y(U)$.

Let $\widetilde{Z}$ denote the sheaf associated to the presheaf Z. Consider the commutative diagram:

$$\begin{tikzcd} Z \arrow[r, "\eta"] \arrow[d] & \widetilde{Z} \arrow[d] \\ GZ \arrow[r, "\eta_*"'] & G\,\widetilde{Z} \end{tikzcd}$$

in which the morphisms are local weak equivalences. Since η_* is a local weak equivalence of globally fibrant simplicial presheaves, it turns out $GZ \longrightarrow G\,\widetilde{Z}$ is a pointwise weak equivalence. So, locally fibrant simplicial presheaves have B.G-property, therefore, if Z has the B.G-property, $Z(U) \longrightarrow GZ(U)$ is a weak equivalence for all $U \in (Sm/S)_{Nis}$.

Remark 2.2.1 *The B.G-property is preserved by taking filtered colimit. Thus, if*

$$Z_1 \longrightarrow Z_2 \longrightarrow Z_3 \longrightarrow \cdots\cdots$$

is an inductive system of maps between simplicial presheaves which are locally fibrant for the Nisnevich topology, then any choice of locally fibrant model

$$j : colim Z_i \longrightarrow G(colim Z_i)$$

for the Nisnevich topology is a term-by-term weak equivalence.

Lemma 2.2.1 *Suppose given an inductive system*

$$Z_1 \longrightarrow Z_2 \longrightarrow Z_3 \longrightarrow \ldots\ldots$$

of motivic fibrant simplicial presheaves on $(Sm/S)_{Nis}$, and let:

$$j : colim Z_i \longrightarrow G(colim Z_i)$$

be a choice of locally fibrant model for the Nisnevich topology. Then the simplicial presheaf $G(colim Z_i)$ is motivic fibrant.

proof: First notice that any locally fibrant simplicial sheaf has the B.G-property. Then one observes that the morphism j is a pointwise weak equivalence, i.e a term-by-term weak equivalence. Because for each i, the morphism

$$pr^* : Z_i(U) \longrightarrow Z_i(U \times \mathbf{A^1})$$

induces a weak equivalence of simplicial sets, and so $G(colim Z_i)$ is motivic fibrant since we take a filtered colimit.

Corollary 2.2.1 *Suppose that $X_1 \longrightarrow X_2 \longrightarrow \ldots$ is an inductive system of motivic fibrant simplicial presheaves on $(Sm/S)_{Nis}$. Then any motivic fibrant model*

$$j : colim X_i \longrightarrow Z$$

is a pointwise weak equivalence.

Flasque Simplicial Presheaves

The notion of *flasque* is needed to define the *compact objects*.

Definition 2.2.3 *A simplicial presheaf X on $(Sm/S)_{Nis}$ is flasque if X is a presheaf of Kan complexes and every finite collection $U_i \hookrightarrow U,\ i = 1, 2, \ldots, n$ of subschemes of a scheme U induces a Kan fibration*

$$i^* : X(U) \cong S(U, X) \longrightarrow S(\cup_{i=1}^n U_i, X).$$

Here, the union is taken in the category of presheaves.

Notice that every locally fibrant simplicial presheaf is flasque, and the class of flasque simplicial presheaves is closed under filtered colimits. The reason that locally fibrants are flasque can be seen from the lifting diagram:

$$\begin{tikzcd}
\wedge_k^n \arrow[r] \arrow[d] & S(U, X) \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur] & S(\cup_{i=1}^n, X)
\end{tikzcd}$$

which is equivalent to the diagram:

$$\begin{tikzcd}
(\cup_{i=1}^n U_i \times \Delta^n) \cup_{\cup_{i=1}^n U_i \times \wedge_k^n} U \times \wedge_k^n \arrow[r] \arrow[d] & X \\
U \times \Delta^n \arrow[ur] &
\end{tikzcd}$$

Definition 2.2.4 *One says more generally that a map $p : X \longrightarrow Y$ of simplicial presheaves is flasque if it is a pointwise fibration and has the RLP with respect to all maps:*

$$(\cup_{i=1}^n U_i \times \Delta^n) \cup_{\cup_{i=1}^n U_i \times \wedge_k^n} U \times \wedge_k^n \hookrightarrow U \times \Delta^n$$

arising from all finite collections $U_i, i = 1, \ldots, n$ of subschemes of scheme U.

Equivalently, the map p is flasque if and only if the simplicial set map

$$S(U,X) \longrightarrow S(\cup_{i=1}^n, X) \times_{S(\cup_{i=1}^n U_i, Y)} S(U,Y)$$

is a Kan fibration. We can also say that a map is flasque if it is a pointwise fibration and has the RLP with respect to *anodyne* morphisms. Note in particular that a simplicial sheaf X is flasque if and only if the map $X \longrightarrow *$ is a flasque map.

Lemma 2.2.2 *Suppose that $p : X \longrightarrow Y$ is a flasque map of simplicial presheaves, and suppose that $j : A \hookrightarrow B$ is an inclusion of schemes. Then the induced map:*

$$\underline{Hom}(B,X) \longrightarrow \underline{Hom}(A,X) \times_{\underline{Hom}(A,Y)} \underline{Hom}(B,Y)$$

is flasque.

proof: Let the internal Hom of the two sides act on $U \in (Sm/S)_{Nis}$, then the U-section induces the map

$$X(B \times U) \longrightarrow X(A \times U) \times_{Y(A \times U)} Y(B \times U)$$

By lifting diagram, this map is a Kan fibration. The result comes from the lifting diagram of anodyne morphisms.

Corollary 2.2.2 *Suppose that X is a flasque simplicial presheaf and that B is a scheme. Then $\underline{Hom}(B,X)$ is flasque.*

proof: We conclude this by letting $A = \emptyset$ and $Y = pt$ of the previous lemma.

Corollary 2.2.3 *Suppose that X is a pointed flasque simplicial presheaf and that $j : A \hookrightarrow B$ is an inclusion of schemes. Then $\underline{Hom}_*(B/A, X)$ is flasque.*

proof: $\underline{Hom}_*(B/A,X)$ is the fiber of the flasque map $j^* : \underline{Hom}(B,X) \longrightarrow \underline{Hom}(A,X)$.

Lemma 2.2.3 *Suppose that the simplicial presheaf X is flasque, and that $j : K \hookrightarrow L$ is an inclusion of simplicial sets. Then the simplicial presheaf map*

$$j^* : X^L \longrightarrow X^K$$

is flasque. Where Y^K correspond to the adjoints in the simplicial model structure:

$$Mor_{sPSh}(K \otimes X, Y) = Mor_{sPSh}(X, Y^K) = Mor_{sSet}(K, S(X,Y)).$$

proof: We must solve the lifting problem:

$$\begin{tikzcd} \wedge^n_k \arrow[r] \arrow[d] & S(U, X^L) \arrow[d] \\ \Delta^n \arrow[r] & S(\cup_i U_i, X^L) \bigcup_{S(\cup_i U_i, X^K)} S(U, X^K) \end{tikzcd}$$

However, by adjointness, this diagram is equivalent to:

$$\begin{tikzcd} \wedge^n_k \arrow[r] \arrow[d] & S(U, X^L) \arrow[d] \\ \Delta^n \arrow[r] & S(\cup_i U_i, X)^L \bigcup_{S(\cup_i U_i, X)^K} S(U, X)^K \end{tikzcd}$$

which is moreover equivalent to:

$$\begin{tikzcd} (K \times \Delta^n) \bigcup_{K \times \wedge^n_k} (L \times \wedge^n_k) \arrow[r] \arrow[d] & S(U, X) \arrow[d] \\ L \times \Delta^n \arrow[r] & S(\cup_i U_i, X) \end{tikzcd}$$

This diagram obviously solves the lifting problem.

Lemma 2.2.4 *Suppose that $g : A \longrightarrow B$ is a map of schemes, and that X is a pointed flasque simplicial presheaf. Let M_g denote the mapping cylinder of g in the simplicial presheaf category, and let $C_g = M_g/A$ be the homotopy cofiber. Then the standard cofibration $j : A \hookrightarrow M_g$ associated to g induces a flasque map:*

$$j^* : \underline{Hom}(M_g, X) \longrightarrow \underline{Hom}(A, X)$$

The simplicial presheaves $\underline{Hom}(M_g, X)$ and $\underline{Hom}_*(C_g, X)$ are flasque.

proof: The mapping cylinder is defined by the pushout diagram:

$$\begin{tikzcd} A \sqcup A \arrow[r] \arrow[d] & B \sqcup A \arrow[d] \\ A \times \Delta^1 \arrow[r] & M_g \end{tikzcd}$$

and the map j is the composite:

$$A \longrightarrow B \sqcup A \longrightarrow M_g$$

Since X is flasque, $\underline{Hom}(A, X)$ is flasque. Then there is a flasque map:

$$\underline{Hom}(A \times \Delta^1, X) \longrightarrow \underline{Hom}(A \times \partial\Delta^1, X).$$

Since flasque maps are closed under pullback, the map

$$\underline{Hom}(M_g, X) \longrightarrow \underline{Hom}(B \sqcup A, X)$$

is flasque. Then the inclusion $A \longrightarrow A \sqcup B$ induces the projection map

$$\underline{Hom}(B, X) \times \underline{Hom}(A, X) \longrightarrow \underline{Hom}(A, X)$$

which is flasque by **Lemma 2.2.2** letting $Y = pt$ and $j : A \hookrightarrow A \sqcup B$. Since flasque maps are closed under composition, we have finished our proof.

Example 2.2.1

1. *Consider $T = S^1 \wedge G_m = \mathbf{A^1}/\mathbf{A^1} - 0$ and suppose that X is a flasque simplicial presheaf. Then the object $\underline{Hom}(T, X)$ is flasque since it is the fiber of the map*

$$\underline{Hom}(\mathbf{A^1}, X) \longrightarrow \underline{Hom}(\mathbf{A^1} - 0).$$

There is an isomorphism

$$\underline{Hom}(U, X)(V) \cong X(U \times V).$$

It follows that there is a fiber sequence:

$$\underline{Hom}_*(T, X)(U) \longrightarrow X(\mathbf{A^1} \times U) \longrightarrow X((\mathbf{A^1} - 0) \times U).$$

If X is flasque, then so the functor $\underline{Hom}_*(T, -)$ preserves pointwise weak equivalences of flasque simplicial presheaves. It follows as well that the functor $\underline{Hom}_*(T, -)$ preserves filtered colimits of simplicial presheaves.

2. *Suppose K is a finite pointed simplicial set, identified with a constant simplicial presheaf. Then there is an isomorphism:*

$$\underline{Hom}_*(K, X) \cong S_*(K, X)$$

and the functor $S_*(K, -)$ is flasque. The functor $S_*(K, -)$ preserves pointwise weak equivalences of pointed simplicial presheaves consisting of Kan complexes, so that it preserves pointwise weak equivalences of flasque simplicial presheaves. The functor $S_*(K, -)$ commutes with all filtered colimits since K is finite.

2.2.2 Motivic Stable Categories

In this subsection, we will see details and properties of motivic stable categories. The outline is first introducing *level structures* and then using the *stabilization functor* to construct stable categories.

Let T be a pointed simplicial presheaf. In this subsection, we denote the category of T-spectra by $Spt_T(Sm/S)_{Nis}$ and always placing T on the left when describing the structural bonding map $\sigma : T \wedge X_n \longrightarrow X_{n+1}$.

Level Model Structures

Definition 2.2.5 *We say that a map $f : X \longrightarrow Y$ of T-spectra is a:*

1. **level cofibration** *if all component maps $f_n : X_n \longrightarrow Y_n$ are cofibrations of simplicial presheaves, i.e monomorphisms,*
2. **level fibration** *if all component maps $f_n : X_n \longrightarrow Y_n$ are fibrations, i.e motivic fibrations,*
3. **level equivalence** *if all component maps $f_n : X_n \longrightarrow Y_n$ are motivic weak equivalences.*

Given this definition, we can display two model category structures on the category of T-spectra, which are similar to the *projective structure* and the *injective structure*, are given by:

- A cofibration (projective) is a map which has the LLP with respect to the maps that are both level fibrations and level equivalences.
- An (injective) fibration is a map which has the RLP with respect to the maps that are both level cofibrations and level equivalences.

Furthermore, we have the following lemma.

Lemma 2.2.5

1. *The category $Spt_T(Sm/S)_{Nis}$ equipped with the classes of level equivalences, level fibrations, and cofibrations satisfies the axioms of a proper, closed simplicial model category. It is called the projective structure of $Spt_T(Sm/S)_{Nis}$.*
2. *The category $Spt_T(Sm/S)_{Nis}$ equipped with the classes of level equivalences, level cofibrations, and injective fibrations satisfies the axioms of a proper, closed simplicial model category. It is called the injective structure of $Spt_T(Sm/S)_{Nis}$.*

proof: Please refer to the paper of Jardine's [5], Lemma 2.1. He uses the technique of *controlled fibrant* to prove 2.. For 1., one should notice that for $i : A \longrightarrow B$, if it satisfies:

1. $i_0 : A_0 \longrightarrow B_0$ is a cofibration of simplicial presheaves,
2. each map $T \wedge B_n \bigcup_{T\wedge A_n} A_{n+1} \longrightarrow B_{n+1}$ is a cofibration

then i is a cofibration. Therefore the model structure corresponds to 1. is actually a *strict* structure as in ordinary spectra theory.

Given the projective and injective structures, we get two fibrant models:

1. In the injective structure, there is a map of T-spectra: $i_X : X \longrightarrow IX$ such that i_X is a level cofibration and a level equivalence, and IX is injective fibrant. More generally, any level equivalence $X \longrightarrow Y$ with Y injective fibrant is called a *injective model* of X,
2. In the projective structure, there is a map of T-spectra: $j_X : X \longrightarrow JX$ such that j_X is a cofibration and a level equivalence, and JX is a level fibrant.

Note that any injective fibrant is level fibrant.

Compact Objects

We want to define *compact objects* so that the corresponding loop functors commute with filtered colimits. The class of compact simplicial presheaves is closed under finite smash products and homotopy cofiber, and includes all finite simplicial sets and schemes on the site $(Sm/S)_{Nis}$. Based on these, we could set up a stable model category structure on the category of T-spectra.

Definition 2.2.6 *Say that a simplicial presheaf X on $(Sm/S)_{Nis}$ is* **motivic flasque** *if:*

1. *X is flasque*
2. *every map $X(U) \longrightarrow X(\mathbf{A^1} \times U)$ induced by the projection $\mathbf{A^1} \times U \longrightarrow U$ is a weak equivalence of simplicial sets.*

Note that every motivic fibrant simplicial presheaf on $(Sm/S)_{Nis}$ is motivic flasque, and the class of motivic flasque simplicial presheaves is closed under filtered colimits.

Definition 2.2.7 *A pointed simplicial presheaf T on the smooth Nisnevich site is said to be* **compact** *if the following conditions hold:*

1. *All inductive systems $Y_1 \longrightarrow Y_2 \longrightarrow \dots$ of pointed simplicial presheaves induces isomorphisms*
$$\underline{Hom}_\bullet(T, colim Y_i) \cong colim \underline{Hom}_\bullet(T, Y_i)$$
2. *If X is motivic flasque, then so is $\underline{Hom}_\bullet(T, X)$*
3. *The functor $\underline{Hom}_\bullet(T, -)$ takes pointwise weak equivalences of motivic flasque simplicial presheaves to pointwise weak equivalences.*

Example 2.2.2 *Here are some examples of compact simplicial presheaves:*

1. *If $A \hookrightarrow B$ is an inclusion of schemes, then the quotient B/A is compact.*
2. *All finite simplicial sets are compact.*
3. *All pointed schemes in the underlying site $(Sm/S)_{Nis}$ are compact.*
4. *If T_1 and T_2 are compact, then $T_1 \vee T_2$ and $T_1 \wedge T_2$ are compact.*
5. *If $g : T_1 \longrightarrow T_2$ is a map of compact simplicial presheaves, then the pointed mapping cylinder M_g and the homotopy cofiber C_g are compact.*

Note that in particular, the Morel-Voevodsky object $T = \mathbf{A^1}/\mathbf{A^1} - 0$ is compact.

The Stable Model Category of T-spectra

Suppose that T is a compact pointed simplicial presheaf on the smooth Nisnevich site $(Sm/S)_{Nis}$. The T-loop functor Ω_T is defined for pointed simplicial presheaves in terms of internal hom by
$$\Omega_T(Y) = \underline{Hom}_\bullet(T, Y).$$
The T-loop functor is right adjoint to smashing with T (on the right), so the bonding map $\sigma : T \wedge X_n \longrightarrow X_{n+1}$ induces the adjoint
$$\sigma_* : X_n \longrightarrow \Omega_T(X_{n+1})$$
up to a twist. In other word, it is adjoint of the composition:
$$X_n \wedge T \xrightarrow{\sim} T \wedge X_n \xrightarrow{\sigma} X_{n+1}$$

where the first map is an isomorphism which flips the smash factors.

Then the T-loop functor defines a T-spectrum called the Ω-spectrum by letting $(\Omega_T(X))_n = \Omega_T(X_n)$, and the bonding map is denote by $\sigma : T \wedge \Omega_T(X_n) \longrightarrow \Omega_T(X_{n+1})$ which is adjoint to the composition:

$$T \wedge \Omega_T(X_n) \wedge T \overset{T\wedge ev}{\longrightarrow} T \wedge X_n \overset{\sigma}{\longrightarrow} X_{n+1}$$

More generally, there is a function complex functor $X \longmapsto \underline{Hom}_\bullet(A, X)$ for all T-spectra X and pointed simplicial presheaves A, and this functor is right adjoint to the functor $X \longmapsto X \wedge A$ defined by smashing on the right with A.

Since we have defined the loop functor adjoint to smashing on the right, there is a similar way of defining a *fake loop functor* as in ordinary stable homotopy theory. Let's define the *fake T-loop spectrum* $\Omega^l_T(X)$, with

$$(\Omega^l_T(X))_n = \Omega_T(X_n)$$

with bonding maps adjoint to the morphism:

$$\Omega_T(\sigma_*) : \Omega_T(X_n) \longrightarrow \Omega^2_T(X_{n+1})$$

The fake T-loop functor is right adjoint to the *fake suspension functor* $Y \longmapsto \Sigma^l_T(Y)$, where $(\Sigma^l_T(Y))_n = T \wedge Y_n$ and the bonding map $T \wedge \Sigma^l_T(Y_n) \longrightarrow \Sigma^l_T(Y_{n+1})$ are defined to be the morphisms

$$T \wedge \sigma : T^2 \wedge Y^n \longrightarrow T \wedge Y_{n+1}$$

Here, the subscript l denotes *smashing on the left*.

Remark 2.2.2 *The two loop functors: T-loop functor and fake T-loop functor are not isomorphic. Since the two adjoint functor of the bonding map*

$$T \wedge \Omega_T(X_n) \longrightarrow \Omega_T(X_{n+1})$$

differ by a flip of Ω_T.

The map σ_* defines a natural morphism of T-spectrum:

$$\sigma_* : X \longrightarrow \Omega_T(X)[1]$$

where the shifted spectrum $X[1]$ is defined as $X_n[1] = X_{n+1}$. The T-spectrum $Q_T X$ is defined as:

$$colim(X \overset{\sigma_*}{\longrightarrow} \Omega_T(X)[1] \overset{\Omega_T\sigma_*[1]}{\longrightarrow} (\Omega_T)^2(X)[2] \overset{(\Omega^l_T)^2\sigma_*[2]}{\longrightarrow} \ldots\ldots)$$

Write $\eta_X : X \longrightarrow Q_T X$ for the associated canonical map. We shall be particularly interested in the composite map:

$$X \xrightarrow{\sigma_*} JX \xrightarrow{\eta_{JX}} Q_T JX$$

which will be denoted by $\widetilde{\eta}$. The functor Q_T is called the stabilization functor for the object T.

Definition 2.2.8 *A map $g : X \longrightarrow Y$ of T-spectra is said to be stably equivalent if it induces a level equivalence*

$$Q_T J(g) : Q_T JX \longrightarrow Q_T JY$$

Observe that g is a stable equivalence if and only if it induces a *level equivalence*:

$$IQ_T J(g) : IQ_T JX \longrightarrow IQ_T JY$$

A stable fibration is a map which has the RLP with respect to all maps that are cofibrations and stable equivalences. A T-spectra X is called stably fibrant if $X \longrightarrow *$ is a stable fibration.

Remark 2.2.3 *$IQ_T JX$ is injective fibrant, so it is level fibrant and then has the B.G-property and is motivic flasque. Thus $g : X \longrightarrow Y$ is a stable equivalence if and only if the induced map $IQ_T J(g)$ is a pointwise weak equivalence of motivic flasque simplicial presheaves of all levels.*

One observes that if T is compact, then $\underline{Hom}_\bullet(T, -)$ sends motivic flasque object to motivic flasque object. So $\underline{Hom}_\bullet(T, JX_n) = \Omega_T JX_n$ is motivic flasque. According to **C3**, *$\underline{Hom}(T, JX_n) \longrightarrow \underline{Hom}(T, JY_n)$ is a pointwise weak equivalence. So, we furthur have: $g : X \longrightarrow Y$ is a stable equivalence if and only if the induced map $Q_T J(g)$ is a pointwise weak equivalence of motivic flasque simplicial presheaves of all levels.*

Given the definition of stable equivalences, we have the corresponding B.G-property argument for spectra:

Corollary 2.2.4 *Suppose that Z is a presheaf of spectra on the smooth Nisnevich site $(Sm/S)_{Nis}$. Then a stably fibrant model $j : Z \longrightarrow GZ$ consists of stable equivalence $Z(U) \longrightarrow GZ(U)$ in all sections if and only if the presheaf of spectra Z satisfies the (stable) B.G-property.*

proof: We say that the presheaf of spectra Z has the stable B.G-property if and only if any elementary Cartesian diagram induces a homotopy Cartesian diagram

$$\begin{tikzcd} Z(X) \arrow[r] \arrow[d] & Z(U) \arrow[d] \\ Z(V) \arrow[r] & Z(U \times_X V) \end{tikzcd}$$

of spectra with respect to stable equivalence. It follows that a presheaf of spectra has the stable B.G-property if and only if each of the simplicial presheaves $QZ_n = \Omega^\infty JZ_n$ has the B.G-property. The maps $\Omega^\infty JZ \longrightarrow GZ$ are level weak equivalences of presheaves of Ω-spectra and all simplicial presheaves GZ_n are globally fibrant. Since Z has the stable B.G-property if and only if the maps of sections $\Omega^\infty JZ_n(U) \longrightarrow GZ_n(U)$ are weak equivalences of pointed simplicial sets, and this holds if and only if all maps $Z(U) \longrightarrow GZ(U)$ are stable equivalences of spectra.

The stable equivalences satisfy the following statements:

A4 Every level equivalence is a stable equivalence.

A5 The maps:

$$\widetilde{\eta}_{(Q_TJX)}, Q_TJ(\widetilde{\eta}_X) : Q_TJX \longrightarrow (Q_TJ)^2X$$

are stable equivalences.

A6 Stable equivalences are closed under pullback along stable fibrations, and stable equivalences are closed under pushout along cofibrations

The following two steps prove the statements **A4**-**A6**:

1. **Lemma 2.2.6** *The statements $A4$ and $A5$ hold for T-spectra.*

   proof: If $g : X \longrightarrow Y$ is a level equivalence between $T-spectra$ such that X and Y are level fibrant, then g is a pointwise weak equivalence between motivic flasque objects in all levels. Therefore, all $\Omega^n_T g$ and $Q_T g$ are level pointwise equivalences. This proves A4. The map $Q_TJ(j_X) : Q_TJX \longrightarrow Q_TJ^2X$ is a level equivalence. There is a commutative map

   $$\begin{tikzcd} Q_TJ^2X \arrow[r] & Q_TJQ_TJX \\ Q_TJX \arrow[r] \arrow[u] & Q_TQ_TJX \arrow[u] \end{tikzcd}$$

   Where the vertical lines are given by $Q_T(j_{JX}) : Q_TJX \longrightarrow Q_TJ^2X$ and $Q_T(j_{Q_TJX}) : Q_TQ_TJX \longrightarrow Q_TJQ_TJX$, which are level equivalence since j_{JX} and j_{Q_TJX} are pointwise weak equivalences of motivic flasque objects. The horizontal lines are given by $Q_T(\eta_{JX}) : Q_TJX \longrightarrow Q_TJQ_TJX$, and all maps like $Q_T(\eta_Z)$ are isomorphisms, so $Q_TJ(\eta_{JX}) : Q_TJ^2X \longrightarrow Q_TJQ_TJX$ are level equivalences.

There is a commutative diagram

$$\begin{tikzcd} JQ_TJX_n \arrow[r] & \Omega_T JQ_TJX_{n+1} \\ Q_TJX_n \arrow[r] \arrow[u] & \Omega_T Q_TJX_{n+1} \arrow[u] \end{tikzcd}$$

The vertical line j_{Q_TJX} is a level pointwise weak equivalence, and the map $\sigma_* : Q_TJX_n \longrightarrow \Omega_T Q_TJX_{n+1}$ is an isomorphism. The other vertical line $\Omega_T(j_{Q_TJX})$ is a pointwise weak equivalence of motivic flasque simplicial presheaves. It follows that all maps $\sigma_* : JQ_TJX_n \longrightarrow \Omega_T JQ_TJX_{n+1}$ are pointwise weak equivalences, and so the map

$$\eta_{JQ_TJX} : JQ_TJX \longrightarrow Q_TJQ_TJX$$

is a pointwise level equivalence. In particular, the composition

$$Q_TJX \overset{j_{Q_TJX}}{\longrightarrow} JQ_TJX \overset{\eta_{JQ_TJX}}{\longrightarrow} Q_TJQ_TJX$$

is a pointwise level equivalence. This finishes **A5**.

Note that this proof **A5** also makes one observes that

$$X \longrightarrow Q_TJX$$

is stable equivalent.

2. **Lemma 2.2.7** *The class of stable equivalences is closed under pullback along level fibrations.*

   proof: Consider a pull-back diagram:

$$\begin{tikzcd} A \times_Y X \arrow[r, "g_*"] \arrow[d] & X \arrow[d, "p"] \\ A \arrow[r, "g"] & Y \end{tikzcd}$$

in which g is a stable equivalence and p is a level fibration. We have to show that g_* is a stable equivalence. Let us first assume that all objects are level fibrant. Since $Q_Tg : Q_TA \longrightarrow Q_TY$ is a pointwise equivalence of motivic flasque simplicial presheaves, all maps $Q_TA_n \longrightarrow Q_TY_n$ are pointwise weak equivalences. The maps $p_* : Q_TX_n \longrightarrow Q_TY_n$ are filtered colimits of pointed Kan fibrations, and are therefore pointwise Kan fibrations. Finally, Q_T preserves pullbakcs (*because filtered inductive limit and finite projective limit are interchangeable)* and the ordinary simplicial set category is proper, so the maps

$$Q_T(g_*) : Q_T(A \times_Y X)_n \longrightarrow Q_T X_n$$

are pointwise weak equivalences of simplicial presheaves. Every stable fibration is a level fibration, because every level equivalence is a stable equivalence. Then the Lemma is therefore implies A6.

Together the properties and discussions we displayed in **Section 2.1**, especially **Corollary 2.1.5, 2.1.6** we have:

Lemma 2.2.8 *A map $p : X \longrightarrow Y$ is a stable fibration if p is a level fibration and the diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Q_T JX \arrow[d] \\ Y \arrow[r] & Q_T JY \end{tikzcd}$$

is level homotopy Cartesian.

As a consequence, a T-spectrum X is stably fibrant if X is level fibrant and $X \longrightarrow Q_T JX$ is a level equivalence, that is to say, the maps $\sigma_* : X_n \longrightarrow \Omega_T X_{n+1}$ are pointwise weak equivalences.

Remark 2.2.4

1. *We can concretely construct a stably fibrant model $IQ_T JX$ for X. Firstly, this object is obviously level fibrant. Secondly, it satisfies the condition that $IQ_T JX_n \longrightarrow \Omega_T IQ_T JX_{n+1}$ is a weak equivalence for all n. This comes from the compactness conditions C1, C2 and the B.G-property. More specifically, C1 makes sure that we can move the Ω_T functor into the inductive system, and C2 makes sure that each object of this inductive system is motivic flasque. Without the condition of compactness of T, this selection of stably fibrant can not be sure.*

2. *According to the lifting diagram, a map is a stable fibration and a stable equivalence if and only if it is a level fibration and a level equivalence, since they all have the RLP with respect to all cofibrations.*

3. *A level fibration between stably fibrant objects must be a stable fibration.*

4. *The reason that we emphasize both injective and projective structures is that the original model category of simplicial presheaves is defined by injective structure, but in stable category, we use theprojective structure instead. (Recall that we use cofibrations rather than level cofibrations). But the injective structure is still useful if we want to describe the stable category by homotopies.*

Lemma 2.2.9 *Suppose that X is stably fibrant. Then X is level fibrant, and all maps $\sigma_* : X_n \longrightarrow \Omega_T X_{n+1}$ are pointwise weak equivalence.*

proof: Let IQ_TJX be a stably fibrant model for X. Consider the diagram of factorization:

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "\alpha"'] & & IQ_TJX \\ & Z \arrow[ur, "\pi"'] & \end{tikzcd}$$

where π level fibration and a level equivalence, and α is a cofibration. Then we know that π is a stable equivalence and a stable fibration. It follows that Z is stably fibrant, and $Z_n \longrightarrow \Omega_T Z_{n+1}$ are pointwise weak equivalences. So, α is a cofibration and a stable equivalence, then we conclude the result since X is a retract of Z.

Theorem 2.2.3 *Suppose that T is a compact object on the smooth Nisnevich site $(Sm/S)_{Nis}$. Then the category of T-spectra on this site, together with the classes of cofibrations, stable fibrations, and stable equivalences satisfies the axioms of a proper closed simplicial model category.*

proof: Let us focus on the proof of CM4 and CM5. We know that a map p is a level fibration and a level equivalence if and only if it is a stable fibration and a stable equivalence. Then the existence of the cofibration-acyclic fibration decomposition of CM5 comes from the level structure, and so does the corresponding part of CM4. To prove the remaining part of CM5, let us first notice that a level fibration between two stably fibrant objects must be a stable fibration. Suppose given a map $g : X \longrightarrow Y$ of T-spectra. There is a diagram:

$$\begin{tikzcd} X \arrow[rr, "\mu_X"] \arrow[dd] \arrow[dr, "\alpha_*"'] & & IQ_TJX \arrow[dd, "g_*"' near end] \arrow[dr, "\alpha"] & \\ & Y \times_{IQ_TJY} Z \arrow[rr, "\mu_*" near start] \arrow[dl, "p_*"] & & Z \arrow[dl, "p"] \\ Y \arrow[rr, "\mu_Y"'] & & IQ_TJY & \end{tikzcd}$$

where p is a level fibration and α is a cofibration and a level equivalence. Because IQ_TJY is injective fibrant, then it is a level fibrant, and so Z is level fibrant. According to the B.G-property, $\alpha : IQ_TJX_n \longrightarrow Z_n$ are pointwise weak equivalences of motivic flasque simplicial presheaves, then Z is stably fibrant. Thus p is a stable fibration.

The map μ_* is a stable equivalence (stable equivalence is closed under pull-back along level fibrations), so that α_* is a stable equivalence. Factorize α_* as:

$$\begin{tikzcd} X \arrow[r, "\alpha'"] \arrow[dr] & W \arrow[d, "\pi"] \\ & Y \times_{IQ_TJY} Z \end{tikzcd}$$

where α' is a cofibration and π is a level fibration and a level equivalence. Then π is a stable fibration and a stable equivalence, and it follows that α' is a stable equivalence. So, $g = (p_*\pi) \circ \alpha'$ is a factorization of g as a stable fibration following a cofibration which is a stable equivalence, giving CM5.

Remark 2.2.5 *For any map $f : A \longrightarrow B$ between cofibrant objects in a closed model category, we can factorise f by the mapping cylinder M_f as*

$$A \xrightarrow{j} M_f \xrightarrow{\pi} B$$

It turns out that j is a cofibration and π is a left inverse of a trivial cofibration. Therefore, equipped an injective structure, the model category of simplicial sheaves has a closed model structure and all objects are cofibrant. If W is fibrant and $g : A \longrightarrow B$ is a weak equivalence between cofibrant objects, the induced map

$$g^* : S(B, W) \longrightarrow S(A, W)$$

is a weak equivalence of Kan complexes.

According to this remark, in the category of T-spectra, let W be both stably fibrant and injective. Since all spectra are cofibrant in the injective model structure, we have an identification

$$\pi_0 S(X, W) = [X, W]$$

in the stable homotopy category, where $[X, W]$ means the stable homotopy class.

Lemma 2.2.10 *A map $g : X \longrightarrow Y$ is a stable equivalence if and only if it induces bijections*

$$g^* : [Y, W] \xrightarrow{\sim} [X, W]$$

of morphisms in the stable (equivalently, level) homotopy category of all stably fibrant, injective objects W.

proof: Every stable equivalence clearly induces a bijection as above. For the converse, assume the map g^* is bijective. The injective stably fibrant model $X \longrightarrow IQ_T JX$ is a stable equivalence, so let us assume that both X *and* Y are injective stably fibrant. Then g is a homotopy equivalence, since the homotopy inverse of g is a preimage under g^* of the class of 1_X for the case $W = X$.

Corollary 2.2.5 *A map $g : X \longrightarrow Y$ of T-spectra is a stable equivalence if and only if it induces a weak equivalence*

$$g^* : S(Y, W) \longrightarrow S(X, W)$$

of Kan complexes for all stably fibrant injective objects W.

Change of Suspension

Any map $\theta : T_1 \longrightarrow T_2$ of pointed simplicial presheaves on the site $(Sm/S)_{Nis}$ induces a functor

$$\theta^* : Spt_{T_2}(Sm/S)_{Nis} \longrightarrow Spt_{T_1}(Sm/S)_{Nis}$$

by precomposing the bonding maps with θ. For any T_2-spectrum X, $\theta^*(X)$ is a T_1-spectrum with $(\theta^* X)_n = X_n$, and with bonding map given by the composition

$$T_1 \wedge X_n \xrightarrow{\theta \wedge 1} T_2 \wedge X_n \xrightarrow{\sigma} X_{n+1}.$$

There is a homotopical content to this construction when T_1 *and* T_2 are compact and θ is an equivalence:

Theorem 2.2.4 *Suppose that $\theta : T_1 \longrightarrow T_2$ is a motivic weak equivalence of compact objects on the site $(Sm/S)_{Nis}$. The the functor θ^* induces an equivalence of motivic stable homotopy categories*

$$\theta^* : Ho(Spt_{T_2}(Sm/S)_{Nis}) \longrightarrow Ho(Spt_{T_2}(Sm/S)_{Nis}).$$

proof: Write σ_θ for the induced map corresponding to $\sigma : X_n \longrightarrow \Omega_{T_2} X_{n+1}$. The functor θ^* clearly preserves level equivalences, level fibrations and level cofibrations. If X is level fibrant, there is a diagram:

$$\begin{tikzcd}
X_n \arrow[r, "\sigma"] \arrow[dr, "\sigma_\theta"'] & \Omega_{T_2} X_{n+1} \arrow[r, "\Omega_{T_2}\sigma"] \arrow[d, "\theta^*"] & \Omega^2_{T_2} X_{n+2} \arrow[d, "\theta^*"] \\
& \Omega_{T_1} X_{n+1} \arrow[r, "\Omega_{T_1}\sigma"] \arrow[dr, "\Omega_{T_1}\sigma_\theta"'] & \Omega_{T_1}\Omega^2_{T_2} X_{n+2} \arrow[d, "\Omega^2_{T_1}\theta^*"] \\
& & \Omega^2_{T_1} X_{n+2}
\end{tikzcd}$$

All vertical maps are pointwise weak equivalences, so there are induced natural pointwise weak equivalences $\theta^* : Q_{T_2}X_n \longrightarrow Q_{T_1}\theta^* X_n$ for level fibrant objects X. From the commutative diagram:

$$\begin{tikzcd} Q_{T_2}X_n \arrow[r] \arrow[d] & Q_{T_2}Y_n \arrow[d] \\ Q_{T_1}\theta^* X_n \arrow[r] & Q_{T_2}\theta^* Y_n \end{tikzcd}$$

It follows that $g : X \longrightarrow Y$ is a stable equivalence of T_2-spectra if and only if $\theta^* g : \theta^* X \longrightarrow \theta^* Y$ is a stable equivalence of presheaves of T_1-spectra. In particular, θ^* induces a functor:

$$\theta^* : Ho(Spt_{T_2}(Sm/S)_{Nis}) \longrightarrow Ho(Spt_{T_1}(Sm/S)_{Nis})$$

on stable homotopy categories. By the criterion of stable fibrations, we conclude that θ^* preserves stable fibrations. Then, we assume that θ is a cofibration as well as an equivalence. Actually, according to **Remark 3.7**, this can always be realized. Under this assumption, let $i : A \longrightarrow B$ be a cofibration of T_2-spectra, and there is a pushout disgram:

$$\begin{tikzcd} (T_1 \wedge B_n)\bigcup_{T_1\wedge A_n}(T_2 \wedge B_n) \arrow[r] \arrow[d, "(\theta,i)_*"'] & (T_1 \wedge B_n)\bigcup_{T_1\wedge A_n} A_{n+1} \arrow[d, "\theta_*"] \\ T_2 \wedge B_n \arrow[r] & (T_2 \wedge B_n)\bigcup_{T_2\wedge A_n} A_{n+1} \end{tikzcd}$$

in which $(\theta, i)_*$ is a cofibration. Note that $(T_2 \wedge B_n)\bigcup_{T_2\wedge A_n} A_{n+1}$ is the pushout of the diagram

$$\begin{tikzcd} T_2 \wedge A_n \arrow[r] \arrow[d] & A_{n+1} \\ T_2 \wedge B_n & \end{tikzcd}$$

Therefore, the canonical map

$$(T_1 \wedge B_n)\bigcup_{T_1\wedge A_n} A_{n+1} \longrightarrow B_{n+1}$$

for $\theta^* i$ is the composite of

$$(T_1 \wedge B_n)\bigcup_{T_1\wedge A_n} A_{n+1} \xrightarrow{\theta_*} (T_2 \wedge B_n)\bigcup_{T_2\wedge A_n} A_{n+1} \xrightarrow{i} B_{n+1}$$

so $\theta^* i$ is a cofibration of T_1-spectra if i is a cofibration of T_2-spectra. We then know that θ^* preserves cofibration. Every stably fibrant T_1-spectrum X is of the form $X = \theta^* \overline{X}$ for some stably fibrant T_2-spectrum $\overline{X}$. To see this, let $\overline{X}_n = X_n$, and choose bonding maps $\overline{\sigma} : T_2 \wedge X_n \longrightarrow X_{n+1}$ making the following diagram commute:

$$\begin{tikzcd} T_1 \wedge X_n \arrow[r] \arrow[d, "\theta \wedge 1"'] & X_{n+1} \\ T_2 \wedge X_n \arrow[ur, "\overline{\sigma}"] & \end{tikzcd}$$

One gets away with this because $\theta \wedge 1$ is a trivial cofibration.

2.2.3 T-suspensions and T-loops

We will investigate T-suspensions and T-loops this section, and prove some theorems correspond to ordinary stable homotopy theory. By Voevodsky's observation, the compactness of T and the triviality of the action of the cyclic permutation on $T^{\wedge 3}$ are the minimum requirement to set up the machinery of spectra. Based on this observation, we will see that the T-suspension is invertible up to stably fibrant model and the various standard flavors of suspensions (right, fake and shift) are equivalent.

Let $j_X : X \longrightarrow X_s$ be a natural choice of stably fibrant model for a T-spectrum X, where j_X is a cofibration and a stable equivalence. The main aim of this subsection is to show:

Theorem 2.2.5 *The composition*

$$X \xrightarrow{\eta_X} \Omega_T(X \wedge T) \xrightarrow{\Omega j_{X \wedge T}} \Omega_T(X \wedge T)_s$$

arising from the adjunction map η_X is a stable equivalence for all T-spectra. That is to say, in motivic stable homotopy category, the T-suspension is invertible.

Lemma 2.2.11 *Suppose that the comparison diagram of inductive system*

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & \dots \\ Y_0 \arrow[r] & Y_1 \arrow[r] & Y_2 \arrow[r] & \dots \end{tikzcd}$$

consists of stable equivalences $f_i : X_i \longrightarrow Y_i$. Then the induced map

$$colim f_i : colim X_i \longrightarrow colim Y_i$$

is a stable equivalence.

The main idea is to show that we can assume that the spectra $X_i \; and \; Y_i$ are stably fibrant, and the result comes from the corresponding results between fibrant objects.

Recall that the notion $c_{p,q}$ in the symmetric group $\Sigma_{p,q}$ is the shuffle which moves the first p elements past the last q elements, in order. So, $c_{p,q}(i) = q + i \; for \; i \leq p$ and $c_{p,q}(i) = i - p \; for \; i \geq p+1$.

Lemma 2.2.12 (Voevodsky) *The cyclic permutation $c_{1,2} = (3,2,1) \in \Sigma_3$ induces the identity morphism on $T^3 = T^{\wedge 3}$ in the pointed motivic homotopy category, where T is the Morel-Voevodsky object.*

The idea of the proof is to construct a pointed algebraic homotopy

$$\mathbf{A^1} \times T^3 \stackrel{1\times\omega}{\longrightarrow} Gl_3 \times T^3 \longrightarrow T^3$$

from $C_{1,2}$ to the identity on T^3. Then the map $C_{1,2}$ and the identity coincide in the motivic homotopy category.

Layer filtrations

Let $L_n X$ be the spectrum

$$X_0, X_1, \ldots, X_n, T \wedge X_n, T^{\wedge 2} \wedge X_n, \ldots$$

and there is a natural filtration for a T-spectrum

$$X \simeq colim(L_n X).$$

There is a natural pushout diagram

$$\begin{tikzcd} \Sigma_T^\infty (T \wedge X_n)[-(n+1)] \arrow[r] \arrow[d] & L_n X \arrow[d] \\ \Sigma_T^\infty X_{n+1}[-(n+1)] \arrow[r] & L_{n+1} X \end{tikzcd}$$

Note further that the canonical map $\Sigma_T^\infty X_n[-n] \longrightarrow L_n X$ is a stable equivalence. The filtration $\{L_n X\}$ is called the *layer filtration* of X.

Main Results

Assume that the cyclic permutation on $T^{\wedge 3}$ is trivial in the pointed motivic homotopy category.

Lemma 2.2.13 *Suppose that K is a pointed simplicial presheaf. Then the composition*

$$\Sigma_T^\infty K \xrightarrow{\eta} \Omega_T((\Sigma_T^\infty K) \wedge T) \xrightarrow{\Omega j} \Omega((\Sigma_T^\infty K) \wedge T)_s$$

is a stable equivalence.

proof: Recall that if Y is a spectrum, then the homotopy group presheaves $\pi_r Y_s^n(U)$ of the selection of a stably firant model $Y_s = IQ_T JY$ are computed by the filtered colimits

$$[S^r, Y^n]_U \xrightarrow{\Sigma} [T \wedge S^r, Y^{n+1}]_U \xrightarrow{\Sigma} \dots$$

where $[K, X]_U = [K|_U, X|_U]$ means homotopy classes of maps of the restriction to the site over U. The suspension homomorphism Σ takes a morphism $\theta : T^k \wedge S^r \longrightarrow Y^{n+k}$ to the composition

$$T \wedge T^k \wedge S^r \xrightarrow{T\wedge\theta} T \wedge Y^{n+k} \xrightarrow{\sigma} Y^{n+k+1}.$$

Practically, the suspension morphism is induced by smashing with T on the left.

Observe as well that if Y is level fibrant, then the adjunction isomorphisms

$$[T^k \wedge S^r, \Omega_T Y^{n+k}]_U \simeq [T^k \wedge S^r \wedge T, Y^{n+k}]_U$$

which fits into the commutative diagram:

$$\begin{tikzcd}
{[T^k \wedge S^r, \Omega_T Y^{n+k}]_U} \arrow[r, "\sim"] \arrow[d, "\Sigma"'] & {[T^k \wedge S^r \wedge T, Y^{n+k}]_U} \arrow[d, "\Sigma"'] \\
{[T^{k+1} \wedge S^r, \Omega_T Y^{n+k+1}]_U} \arrow[r, "\sim"] & {[T^{k+1} \wedge S^r \wedge T, Y^{n+k+1}]_U}
\end{tikzcd}$$

It follows that the map in presheaves of stable homotopy groups induced by the composition

$$\Sigma_T^\infty K \xrightarrow{\eta} \Omega_T((\Sigma_T^\infty K) \wedge T) \xrightarrow{\Omega j} \Omega((\Sigma_T^\infty K) \wedge T)_s$$

is isomorphic to the filtered colimit of the maps

$$[T^k \wedge S^r, T^{n+k} \wedge K]_U \xrightarrow{\wedge T} [T^k \wedge S^r \wedge T, T^{n+k} \wedge K \wedge T]_U$$

which are induced by smashing with T on the right. Suppose that $\phi : K \wedge T \longrightarrow X \wedge T$ is a map of pointed simplicial presheaves, and write $c_t(\phi)$ for the map $T \wedge K \longrightarrow T \wedge X$ arising from ϕ by conjugation with the twist of smash factors. There is a commutative diagram

$$\begin{tikzcd}
K \wedge T \arrow[r] \arrow[d, "\phi"'] & T \wedge K \arrow[d, "c_t(\phi)"] \\
X \wedge T \arrow[r] & T \wedge X
\end{tikzcd}$$

Then there is a diagram

$$\begin{tikzcd}
T \wedge T^2 \wedge K \arrow[d, "c_t(T^2 \wedge \phi)"'] & T^2 \wedge K \wedge T \arrow[l] \arrow[r] \arrow[d, "T^2 \wedge \phi"] & T^2 \wedge T \wedge K \arrow[d, "T^2 \wedge c_t(\phi)"] \\
T \wedge T^2 \wedge X & T^2 \wedge X \wedge T \arrow[l] \arrow[r] & T^2 \wedge T \wedge X
\end{tikzcd}$$

and hence a diagram:

$$\begin{tikzcd}
T^3 \wedge K \arrow[r, "c_{1,2} \wedge K"] \arrow[d, "c_t(T^2 \wedge \phi)"'] & T^3 \wedge K \arrow[d, "T^2 \wedge c_t(\phi)"] \\
T^3 \wedge X \arrow[r, "c_{1,2} \wedge X"] & T^3 \wedge X
\end{tikzcd}$$

Thus the maps in the homotopy category represented by $T^2 \wedge c_t(\phi)$ and $c_t(T^2 \wedge \phi)$ coincide.

As a consequence, there are commutative diagrams

$$\begin{tikzcd}
{[T \wedge S^r, T^{n+k} \wedge K]_U} \arrow[r] \arrow[d, "\wedge T"'] & {[T^2 \wedge T^k \wedge S^r, T^2 \wedge T^{n+k} \wedge K]_U} \arrow[d, "\wedge T"'] \\
{[T^k \wedge S^r \wedge T, T^{n+k} \wedge K \wedge T]_U} \arrow[r] \arrow[d, "c_t"', "\sim"] & {[T^2 \wedge T^k \wedge S^r \wedge T, T^2 \wedge T^{n+k} \wedge K \wedge T]_U} \arrow[d, "c_t"', "\sim"] \\
{[T \wedge T^k \wedge S^r, T \wedge T^{n+k} \wedge K]_U} \arrow[r] & {[T^3 \wedge T^k \wedge S^r, T^3 \wedge T^{n+k} \wedge K]_U}
\end{tikzcd}$$

The vertical composition coincide with the map $T \wedge$ induced by smashing on the left, so a cofinality argument says that the induced map on the filtered colimits is an isomorphism.

proof of the **Theorem 3.16**:

Notice that the functor $X \longrightarrow X \wedge T$ preserves stable equivalences. It follows that the functors $X \longrightarrow \Omega_T(X \wedge T)_s$ preserves stable equivalences. The T-spectrum X is a filtered colimit of its layers L_nX, and there is a stable equivalence

$$\Sigma_T^\infty X_n[-n] \longrightarrow L_nX$$

for $n \geq 0$. Consider $\eta_* : X \longrightarrow \Omega(X \wedge T)_s$, the proof consists of showing that all maps:

$$\Sigma_T^\infty K[-n] \xrightarrow{\eta_*} \Omega_T(\Sigma_T^\infty K[-n] \wedge T)_s$$

are stable equivalences. Then show that these equivalences pass appropriately to filtered colimits. Since shifts preserve stable equivalence, it suffices to consider the case of $n = 0$, but this is **Lemma 2.2.16**.

Suppose given a system

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \ldots$$

of T-spectra such that all maps

$$\eta_* : X_i \longrightarrow \Omega_T(X_i \wedge T)_s$$

are stable equivalences. We want to prove that the induced map

$$colimX_i \longrightarrow \Omega_T((colimX_i) \wedge T)_s$$

is a stable equivalence. The composition

$$colimX_i \longrightarrow colim\Omega_T(X_i \wedge T) \stackrel{colim\Omega_{Tj}}{\longrightarrow} colim\Omega_T(X_i \wedge T)_s$$

stable equivalence by **Lemma 2.2.14**. There is a commutative diagram:

$$\begin{tikzcd}
colim(X_i \wedge T) \arrow[r, "colim(j)"] \arrow[d, "\sim"] & colim(X_i \wedge T)_s \arrow[d, "c"] \\
(colimX_i) \wedge T \arrow[r, "j"'] & ((colimX_i) \wedge T)_s
\end{tikzcd}$$

The map $colim(j)$ is a stable equivalence by **Lemma 2.2.14**, and so the map c is a pointwise weak equivalence of motivic flasque objects in all levels by B.G-property's argument. Applying the functor Ω_T to this diagram we have the following commutative diagram:

$$\begin{tikzcd}
\Omega_T(colim(X_i)\wedge T) \arrow[r, "\Omega_T j"] & \Omega_T(colim(X_i)\wedge T)_s \\
\Omega_T(colim(X_i\wedge T)) \arrow[u, "\sim"] \arrow[r, "\Omega_T j"'] & \Omega_T(colim(X_i\wedge T)_s) \arrow[u, "\Omega_T c"'] \\
colim\Omega_T(X_i\wedge T) \arrow[u, "\sim"] \arrow[r, "colim\Omega_T j"'] & colim\Omega_T(X_i\wedge T)_s \arrow[u, "\sim"']
\end{tikzcd}$$

The map $\Omega_T c$ is a pointwise weak equivalence in all levels, so the composition

$$colim X_i \xrightarrow{\eta} \Omega_T((colim X_i)\wedge T) \xrightarrow{\Omega_T j} \Omega_T((colim X_i)\wedge T)_s$$

is a stable equivalence.

Lemma 2.2.14 *Suppose that X is level fibrant. Then there is an isomorphism*

$$Q_T(\Omega_T X)_n \simeq \Omega_T(Q_T X)_n.$$

In particular, the loop functor $X \longrightarrow \Omega_T X$ preserves stable equivalences of level fibrant objects.

proof: Recall that $\Omega_T X$ has the bonding map $\sigma : T\wedge \Omega_T X_n \longrightarrow \Omega_T X_{n+1}$, which adjoint to

$$T\wedge \Omega_T X_n \wedge T \longrightarrow T\wedge X_n \longrightarrow X_{n+1}.$$

It follows that there is a commutative diagram:

$$\begin{tikzcd}
\Omega_T X_n \arrow[r, "\sigma_*"] \arrow[dr, "\Omega_T(\sigma_*)"'] & \Omega_T^2 X_{n+1} \arrow[d, "\sim"] \\
 & \Omega_T^2 X_{n+1}
\end{tikzcd}$$

induced by the loop and the fake loop functor. Therefore, one finds a diagram

$$\begin{tikzcd}
\Omega_T^{k+1} X_{n+k} \arrow[r] \arrow[d, "c^*_{k,1}"'] & \Omega_T^{k+2} X_{n+k+1} \arrow[d, "c^*_{k+1,1}"] \\
\Omega_T^{k+1} X_{n+k} \arrow[r] & \Omega_T^{k+2} X_{n+k+1}
\end{tikzcd}$$

where the upper line corresponds to the *right loop* and the lower line corresponds to the *left loop*, i.e, the *fake loop*, and $c^*_{k,1}$ is precomposition with the map which is induced by the shuffle $c_{k,1}$ in the loop functors. The map $c^*_{k,1}$ is therefore the desired isomorphism.

Corollary 2.2.6 *Suppose that Y is level fibrant. Then the evaluation map:*

$$ev : \Omega_T Y \wedge T \longrightarrow Y$$

is a stable equivalence.

Corollary 2.2.7 *Let $j : Y \longrightarrow Y_s$ be a choice of stably fibrant model for Y. Then a map $g : X \wedge T \longrightarrow Y$ is a stable equivalence if and only if the composition*

$$X \xrightarrow{g_*} \Omega_T Y \xrightarrow{\Omega_T j} \Omega_T Y_s$$

is a stable equivalence, where g_* is the adjoint of g.

proof: Consider the diagram:

$$\begin{tikzcd} X \wedge T \arrow[r] \arrow[d, "g"] & (X \wedge T)_s \arrow[d, "\widetilde{g}"] \\ Y \arrow[r] & Y_s \end{tikzcd}$$

We know that g is a stable equivalent if and only if $\widetilde{g}$ is a stable equivalent, if and only if

$$X \longrightarrow \Omega_T(X \wedge T)_s \xrightarrow{\Omega_T \widetilde{g}} \Omega_T Y_s$$

is stable equivalent.

Corollary 2.2.8 *A map $g : X \longrightarrow Y$ is a stable equivalence if and only if the suspension $g \wedge T : X \wedge T \longrightarrow Y \wedge T$ is a stable equivalence.*

Lemma 2.2.15 *The canonical map $\sigma : \Sigma_T^l X \longrightarrow X[1]$ from the fake suspension Σ_T^l to the shift $X[1]$ is a natural stable equivalence.*

proof: Let K be a pointed simplicial presheaf. The map

$$\sigma : \Sigma_T^l(\Sigma_T^\infty K[-n]) \longrightarrow (\Sigma_T^\infty K[-n])$$

is an isomorphism in level p for $p \geq n$, for all $n \geq 0$. The fake suspension $S \longmapsto \Sigma_T^\infty X$ and shift $X \longmapsto X[1]$ functors preserve colimits, so we can argue along the layer filtration. It therefore suffice to show that both functors preserve stable equivalence.

In order to see the shift functor preserves stable equivalences, it suffices to show that the shift $X[1] \longrightarrow (IQ_T JX)[1]$ of the canonical stable equivalence is a stable equivalence. For this, it is enough to show that the shift map $(JX)[1] \longrightarrow (Q_T JX)[1]$

is a stable equivalence, but this is a consequence of the isomorphism $(Q_TJX)[1] \simeq Q_T(JX[1])$.

The fake loop functor $X \longmapsto \Omega_T^l X$ preserves stably fibrant. The fake suspension functor $X \longmapsto \Sigma_T^l Y$ preserves level cofibrations and level weak equivalences, so that the fake loop functor preserves injective fibrations by lifting property. It follows that the fake loop functor preserves the class of stably fibrant injective objects.

We have known that a map $f : X \longrightarrow Y$ is a stable equivalence if and only if induces a weak equivalence

$$f^* : \underline{Hom}(Y, W) \longrightarrow \underline{Hom}(X, W)$$

for all stably fibrant injective object W. If $f : X \longrightarrow Y$ is a stable equivalence of T-spectra and W is stably fibrant and injective, then the map

$$(\Sigma_T^l f)^* : \underline{Hom}(\Sigma_T^l Y, W) \longrightarrow \underline{Hom}(\Sigma_T^l X, W)$$

is isomorphic to the map

$$f^* : \underline{Hom}(Y, \Omega_T^l W) \longrightarrow \underline{Hom}(X, \Omega_T^l W).$$

Since $\Omega_T^l W$ is still stably fibrant and injective, the above map f^* is therefore a weak equivalence. Thus, $\Sigma_T^l f : \Sigma_T^l X \longrightarrow \Sigma_T^l Y$ is a stable equivalence.

Lemma 2.2.16 *The fake suspension functor is naturally Staley equivalent to the functor $X \longrightarrow X \wedge T$.*

Corollary 2.2.9 *Suppose that X is a level fibrant spectrum. Then the spectra $\Omega_T^l X, \Omega_T X,$ and $X[-1]$ are naturally stably equivalent.*

2.2.4 Stable Homotopy Groups, Fiber and Cofiber Sequences

In this section, we discuss the fiber and cofiber sequences. The results include the studies of weighted stable homotopy groups. We will also see that fiber and cofiber sequences are indistinguishable in the motivic stable category. As a result, we assert that the motivic stable category is a triangulated category, and fiber and cofiber sequences are distinguished triangles.

Stable Homotopy Groups of S^1-spectra

Since S^1 is a finite pointed simplicial set, it is compact. Then there is a proper closed model category structure on

$$Spt(Sm/S)_{Nis} = Spt_{S^1}(Sm/S)_{Nis}.$$

So, we first investigate the sequences of S^1-spectra.

We call the stable structure we constructed in this paper the motivic stable structure and the one defined by using stable homotopy groups the ordinary stable structure. The level structures are not changed, so we still have injective model and level fibrant model. We call a map $g : X \longrightarrow Y$ of presheaves of S^1-spectra an ordinary equivalence if it induces an isomorphism on all sheaves of ordinary stable homotopy groups $\pi_n^s(X) = \pi_n^s(Y)$. Notice that since S^1 is compact, $\pi_n^s(X) = \pi_n(QX)$.

Lemma 2.2.17 *Suppose that a map $g : X \longrightarrow Y$ of S^1-spectra is an ordinary local stable equivalence. Then g is a motivic stable equivalence.*

proof: Suppose that every object is level fibrant. According to ordinary spectra theory, we know that a T-spectrum is ordinarily stable fibrant if and only if it is level fibrant and $X_n \longrightarrow \Omega_T X_{n+1}$ is weak equivalent for all n. So, we get if an S^1-spectrum is motivic injective and stably fibrant, it must be injective and stably fibrant for the ordinary theory. Then, it is true that the ordinary homotopy classes $[X, W]$ coincide with the naive homotopy classes $\pi(QX, QW)$ and therefore coincide with the level homotopy classes $[X, W]^s = [QX, QW]$ in the motivic theory for any injective stably fibrant spectrum W. So every ordinary stable equivalent spectra $X \longrightarrow Y$ induces a bijection in level homotopy classes $[QY, QW] \longrightarrow [QX, QW]$ if W is injective and stably fibrant. So, $X \longrightarrow Y$ must be a motivic stable equivalence.

Recall that in stable homotopy category of S^1-spectra, a map $g : X \longrightarrow Y$ is a stable equivalence of S^1-spectra if and only if it induces a pointwise level equivalence $g_* : QJX \longrightarrow QJY$, and g_* is a pointwise level equivalence if and only if it induces pointwise isomorphisms

$$\pi_n QJX(U) \longrightarrow \pi_n QJY(U)$$

in all homotopy groups. Here, the group $\pi_n IQJX(U)$ is defined up to isomorphisms as the filtered colimit for the system

$$[S^{n+r}, X_r|_U] \longrightarrow [S^{n+r+1}, X_{r+1}|_U] \longrightarrow \dots$$

These morphisms are in the *motivic homotopy category* and be computed over the scheme U. We define a presheaf $\pi_n X$ of stable homotopy groups in U-section to be the filtered colimit of this system.

A map $g : X \longrightarrow Y$ is a motivic stable equivalence if and only if it induces presheaf isomorphisms $\pi_n X \simeq \pi_n Y$ for all $n \in Z$. But one should notice that the groups $\pi_n X$ are defined by morphisms in the motivic homotopy category. Despite the notion, they do not coincide with the stable homotopy group presheaves of X, but rather with the stable homotopy group presheaves of a motivic stably fibrant model of X.

Exact Sequences for S^1-spectra

Corollary 2.2.10 *Suppose that*

$$F \xrightarrow{i} X \xrightarrow{p} Y$$

is a level motivic fiber sequence of $S^1 - spectra$. Then the induced map $p_* : X/F \longrightarrow Y$ is a motivic stable equivalence.

From now on, all weak equivalences, stable equivalences, fibrations and so on will be tacitly assumed to be *motivic*. We shall not mention the term motivic except when we need to distinct it from others.

Lemma 2.2.18 *Suppose given a commutative diagram of $S^1 - spectra$*

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d] & B_1 \arrow[r] \arrow[d] & C_1 \arrow[d] \\ A_2 \arrow[r] & B_2 \arrow[r] & C_2 \end{tikzcd}$$

in which the horizontal sequences are level cofiber sequences. Then if any two of $f_1,\ f_2$ or f_3 are stable equivalences, then so is the third one.

proof: we will show that f_1 is a stable equivalence if $f_2\ and\ f_3$ are stable equivalences.We consider the induced morphism

$$f_1^* : S(A_2, W) \longrightarrow S(A_1, W)$$

for any stably fibrant, injective object W. The map of cofiber sequence induces a corresponding diagram of fiber sequence:

$$\begin{tikzcd} S(C_2, W) \arrow[r] \arrow[d] & S(B_2, W) \arrow[r] \arrow[d] & S(A_2, W) \arrow[d] \\ S(C_1, W) \arrow[r] & S(B_1, W) \arrow[r] & S(A_1, W) \end{tikzcd}$$

The level equivalences $W \longrightarrow \Omega W[1]$ of stably fibrant injective objects give all spaces in this diagram the structure of infinite loop spaces, and $f_2^*,\ f_3^*$ are the maps at level 0 for stable equivalences of spectra. The map f_1^* is therefore the level 0 part of a stable equivalence of stably fibrant spectra, and so it is a weak equivalence of simplicial sets.

We now have the following immediate consequences of the previous two lemmas:

Corollary 2.2.11 *suppose given a commutative diagram of S^1-spectra*

$$\begin{tikzcd} F_1 \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[d] & Y_1 \arrow[d] \\ F_2 \arrow[r] & X_2 \arrow[r] & Y_2 \end{tikzcd}$$

in which the horizontal sequences are level fiber sequences. Then if any two of $f_1,\ f_2$ or f_3 are stable equivalences, then so is the third one.

Any level fiber sequence

$$F \xrightarrow{i} X \xrightarrow{p} Y$$

can be functorially replaced up to level equivalence by a fiber sequence in which all objects are level fibrant. Suppose that this has been done, and then it induces maps of S^1-spectra

$$QF \xrightarrow{Qi} QX \xrightarrow{Qp} QY$$

forms a level fiber sequence of spectra

$$QF(U) \xrightarrow{Qi} QX(U) \xrightarrow{Qp} QY(U)$$

in each section, and therefore determines a long exact sequence:

$$\dots \xrightarrow{p_*} \pi_{n+1}QY(U) \xrightarrow{\partial} \pi_n QF(U) \xrightarrow{i_*} \pi_n QX(U) \xrightarrow{p_*} \pi_n QY(U) \xrightarrow{\partial} \dots$$

of presheaves of stable homotopy groups. It follows that there is a natural long exact sequence

$$\dots \xrightarrow{p_*} \pi_{n+1}Y \xrightarrow{\partial} \pi_n F \xrightarrow{i_*} \pi_n X \xrightarrow{p_*} \pi_n Y \xrightarrow{\partial} \dots$$

of presheaves of groups associated to a level fiber sequence.

Suppose given a level cofiber sequence

$$A \xrightarrow{i} B \xrightarrow{\pi} B/A$$

and replace the map π up to motivic weak equivalence by a level motivic fibration by taking factorization

$$B \xrightarrow{j} X \xrightarrow{q} B/A$$

where q is a level motivic fibration and j is a cofibration and a level motivic equivalence. Let F_q be the fiber of q. Then the above cofiber sequence is a fiber sequence in the standard way in the motivic background, in the sense that:

Lemma 2.2.19 *The cofibration j induces a motivic stable equivalence $j' : A \longrightarrow F_q$.*

proof: There is a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j'"] & B \arrow[r] \arrow[d, "j"] & B/A \arrow[d, "j_*"] \\ F_q \arrow[r] & X \arrow[r] & X/F \end{tikzcd}$$

The map $q : X \longrightarrow B/A$ factors through $\pi : X \longrightarrow X/F$ in that there is a map $q_* : X/F \longrightarrow B/A$ such that $q_* \circ \pi = q$. The map q_* is a stable equivalence. It is easy to see that $q_* j_* \pi = \pi$ so that $q_* j_* = 1$ on B/A, and so j_* is a stable equivalence. Then according to **Lemma 2.2.5**, j' is a stable equivalence.

Corollary 2.2.12 *Any cofiber sequence*

$$A \xrightarrow{i} B \xrightarrow{\pi} B/A$$

induces a natural long exact sequence

$$\dots \xrightarrow{\pi_*} \pi_{i+1} B/A \xrightarrow{\partial} \pi_i A \xrightarrow{i_*} \pi_i B \xrightarrow{\pi_*} \pi_i B/A \xrightarrow{\partial} \dots$$

The Morel-Voevodsky Spectra

After discussing the S^1-spectra, we continue to talk about T-spectra when T is the Morel-Voevodsky object.

First, we notice that there is a sequence of motivic equivalence

$$T = \mathbf{A^1}/\mathbf{A^1} - 0 \xleftarrow{\sim} M_i/\mathbf{A^1} - 0 \xrightarrow{\sim} S^1 \wedge (\mathbf{A^1} - 0)$$

These objects are all compact, and so according to **Theorem 2.2.4**, these motivic equivalences induce equivalences of corresponding motivic stable categories.

We proceed as we have done in the category of S^1-spectra. A map $g : X \longrightarrow Y$ of T-spectra is a stable equivalence if and only if the induced map $g_* : Q_T JX \longrightarrow Q_T JY$ is a pointwise level equivalence. For the spectrum $Q_T Y$, its n-level object is given by the filtered colimit

$$Y_n \xrightarrow{\sigma_*} \Omega_T Y_{n+1} \xrightarrow{\Omega_T \sigma_*} \Omega_T^2 Y_{n+2} \longrightarrow \dots$$

The homotopy group $\pi_r Q_T Y_n(U)$ in U-section is isomorphic to the filtered colimit of the system

$$\pi_r Y_n \xrightarrow{\sigma_*} \pi_r \Omega_T Y_{n+1} \xrightarrow{\Omega_T \sigma_*} \pi_r \Omega_T^2 Y_{n+2} \longrightarrow \dots$$

which can be identified with a filtered colimit of maps in the motivic homotopy category over the scheme U of the form

$$[S^r, Y^n|_U] \longrightarrow [S^r \wedge T, Y^{n+1}|_U] \longrightarrow [S^r \wedge T^2, Y^{n+2}|_U] \longrightarrow \dots$$

Here, since $T \simeq S^1 \wedge \mathbf{G}_m$, we can rewrite the above system as

$$[S^r, Y^n|_U] \longrightarrow [S^{r+1} \wedge \mathbf{G}_m, Y^{n+1}|_U] \longrightarrow [S^{r+2} \wedge \mathbf{G}_m^2, Y^{n+2}|_U] \longrightarrow \dots$$

Therefore, here comes a definition:

Definition 2.2.9 *Let $\pi_{t,s}Y(U)$ be the colimit of the sequence*

$$[S^{t+n} \wedge \mathbf{G}_m^{s+n}, Y^n|_U] \longrightarrow [S^{t+n+1} \wedge \mathbf{G}_m^{s+n+1}, Y^{n+1}|_U] \longrightarrow \dots$$

The variable t is usually called the **degree**, *and s is called the* **weight**. *$\pi_{t,s}Y$ is called the weighted stable homotopy groups.*

The definition of the weighted stable homotopy group makes us notice that

$$\pi_r Q_T J Y_n(U) \simeq \pi_{r-n,-n} Y(U).$$

From a different point of view, if $t \leq s$, there are isomorphisms

$$colim_n[S^{t+n} \wedge \mathbf{G}_m^{s+n}, Y^n|_U] \simeq colim_n[S^n \wedge \mathbf{G}_m^{s-t+n}, Y[-t]^n|_U]$$

$$\simeq colim_n[S^n \wedge \mathbf{G}_m^n, \Omega_{\mathbf{G}_m}^{s-t} JY[-t]^n|_U]$$

It follows that there is an isomorphism

$$\pi_{t,s}Y \simeq \pi_0 \Omega_{\mathbf{G}_m}^{s-t} Q_T(JY[-t])^0$$

Similarly, if $t \geq s$, there is an isomorphism

$$\pi_{t,s}Y \simeq \pi_0 \Omega_{\mathbf{G}_m}^{t-s} Q_T(JY[-s])^0.$$

If $g : X \longrightarrow Y$ is a stable equivalence, then $g_* : Q_T(JX[k]) \longrightarrow Q_T(JY[k])$ is a pointwise level equivalence for all integers k, so that all induced maps

$$g_* : \pi_{t,s}X \longrightarrow \pi_{t,s}Y$$

are isomorphisms of presheaves. Conversely, if g induces isomorphisms in all bigraded stable homotopy group presheaves, then g induces isomorphisms $g_* : \pi_{t,s}X \longrightarrow \pi_{t,s}Y$ for all $s \leq 0$ and $t \geq s$, so we have

$$\pi_{t,s}Y \simeq \pi_{(t-s+s),s}Y \simeq \pi_{t-s}Q_T Y_{-s},$$

so that $g_* : Q_T JX \longrightarrow Q_T JY$ is a pointwise level equivalence. Therefore, we have proven:

Theorem 2.2.6 *A map $g : X \longrightarrow Y$ of T-spectra is a stable equivalence if and only if g induces isomorphisms*

$$\pi_{t,s}X \simeq \pi_{t,s}Y$$

of presheaves of groups for all $t, s \in Z$.

We can now describe a *bispectrum* consists pointed simplicial presheaves $X^{m,n}, m, n \geq 0$, together with bonding maps *vertically* $\sigma_v : \mathbf{G}_m \wedge X^{m,n} \longrightarrow X^{m,n+1}$, and *horizontally* $\sigma_h : S^1 \wedge X^{m,n} \longrightarrow X^{m+1,n}$, such that the diagram:

$$\begin{tikzcd}
S^1 \wedge X^{m,n+1} \arrow[r] & X^{m+1,n+1} \\
S^1 \wedge \mathbf{G}_m \wedge X^{m,n} \arrow[u] \arrow[d] & \\
\mathbf{G}_m \wedge S^1 \wedge X^{m,n} \arrow[r] & \mathbf{G}_m \wedge X^{m+1,n} \arrow[uu]
\end{tikzcd}$$

commutes involving a flip functor $t : S^1 \wedge \mathbf{G}_m \longrightarrow \mathbf{G}_m \wedge S^1$ which is a canonical isomorphism. Thus alternatively, we have a collection of S^1-spectra

$$X_n = X_{*,n}$$

together with maps of S^1-spectra $X_n \wedge \mathbf{G}_m \longrightarrow X_{n+1}$ induced by vertical bonding maps.

Actually, the *bigraded homotopy groups* $\pi_{t,s}X$ are defined in bidgree (t, s) and in U-sections to be the colimit of the system:

$$
\begin{tikzcd}
\cdots & \cdots \\
{[S^{t+k} \wedge \mathbf{G}_m^{s+l+1}, X^{k,l+1}|_U]} \arrow[u] \arrow[r] & {[S^{t+k+1} \wedge \mathbf{G}_m^{s+l+1}, X^{k+1,L=1}|_U]} \arrow[u] \arrow[r] & \cdots \\
{[S^{t+k} \wedge \mathbf{G}_m^{s+l}, X^{k,l}|_U]} \arrow[u] \arrow[r] & {[S^{t+k+1} \wedge \mathbf{G}_m^{s+l}, X^{k+1,l}]} \arrow[u] \arrow[r] & \cdots
\end{tikzcd}
$$

The bispectrum X determines a sequence of maps of S^1-spectra

$$X_0 \xrightarrow{\sigma_{v,*}} \Omega_{\mathbf{G}_m} X_1 \xrightarrow{\Omega_{\mathbf{G}_m}(\sigma_{v,*})} \Omega^2_{\mathbf{G}_m} X^2 \longrightarrow \ldots$$

where $\Omega_{\mathbf{G}_m}$ is the functor $\underline{Hom}_\bullet(\mathbf{G}_m, -)$. Then the presheaf $\pi_{t,s}X$ is the filtered colimit of the presheaves of stable homotopy groups

$$\pi_t \Omega^{s+l}_{\mathbf{G}_m} JX_l \longrightarrow \pi_t \Omega^{s+l+1}_{\mathbf{G}_m} JX_{l+1} \longrightarrow \ldots$$

when X has been replaced up to levelwise equivalence by a levelwise filtrant object JX so that the *loop* constructions make sense.

Given the above discussion, we have the analogues of fiber and cofiber sequences for bigraded stable homotopy groups. A map $g : X \longrightarrow Y$ is said to be a level fibration if it consists of fibrations $f : X_{m,n} \longrightarrow Y_{m,n}$ for all $m, n \geq 0$. Level equivalences and level cofibrations have analogous definitions. One can show that any map $g : X \longrightarrow Y$ between the bigraded spectra admits a factorization $X \xrightarrow{j} Z \xrightarrow{p} Y$ where p is a level fibration and j is a level cofibration and a level equivalence.

Suppose that

$$F \xrightarrow{i} X \xrightarrow{p} Y$$

is a level fiber sequence of bispectra, and suppose that Y (and hence X) is level fibrant. Then there are fiber sequences of S^1-spectra

$$\Omega^{s+r}_{\mathbf{G}_m} F_r \xrightarrow{i_*} \Omega^{s+r}_{\mathbf{G}_m} X_r \xrightarrow{p_*} \Omega^{s+r}_{\mathbf{G}_m} Y_r$$

and hence long exact sequences in stable homotopy groups presheaves

$$\ldots \xrightarrow{p_*} \pi_{t+1}\Omega^{s+r}_{\mathbf{G}_m} Y_r \xrightarrow{\partial} \pi_t \Omega^{s+r}_{\mathbf{G}_m} F_r \xrightarrow{i_*} \pi_t \Omega^{s+r}_{\mathbf{G}_m} X_r \xrightarrow{p_*} \pi_t \Omega^{s+r}_{\mathbf{G}_m} Y_r \xrightarrow{\partial} \ldots$$

Taking a filtered colimit in r gives a long exact sequence

$$\ldots \xrightarrow{p_*} \pi_{t+1} Y \xrightarrow{\partial} \pi_t F \xrightarrow{i_*} \pi_t X \xrightarrow{p_*} \pi_t Y \xrightarrow{\partial} \ldots$$

for each s.

If

$$A \xrightarrow{i} B \xrightarrow{\pi} B/A$$

is a level cofiber sequence of bispectra, then replacing the map π up to level equivalence by a fibration p as above gives us a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B/A \arrow[d] \\ F \arrow[r] & X \arrow[r] & B/A \end{tikzcd}$$

in which $p : X \longrightarrow B/A$ is a level fibration and $j : B \longrightarrow X$ is a level equivalence. From **Lemma 2.2.13**, we know the map $j_* : A_n \longrightarrow F_n$ are stable equivalences of S^1-spectra. But then the induced maps

$$\pi_{t,s}A \xrightarrow{j_*} \pi_{t,s}F$$

are isomorphisms in all bidgrees. This implies that there is a natural long exact sequence

$$\ldots \xrightarrow{\pi_*} \pi_{t+1,s}B/A \xrightarrow{\partial} \pi_{t,s}A \xrightarrow{i_*} \pi_{t,s}B \xrightarrow{\pi_*} \pi_{t,s}B/A \xrightarrow{\partial} \ldots$$

associated to a cofiber sequence of bispectra in each s.

As a corollary, we have

Corollary 2.2.13 *Suppose that*

$$F \xrightarrow{i} X \xrightarrow{p} Y$$

is a level fiber sequence of T-spectra. Then the induced map $X/F \longrightarrow Y$ is a stable equivalence.

Corollary 2.2.14 *Suppose that*

$$A \xrightarrow{i} B \xrightarrow{\pi} B/A$$

is a level cofiber sequence of T-spectra, and take a factorization of π:

$$B \xrightarrow{j} X \xrightarrow{p} B/A$$

such that j is a level equivalence and p is a level fibration. Let F be the fiber of the map p. Then the induced map $j_* : A \longrightarrow F$ is a stable equivalence.

2.3 Stable $\mathbf{A}^1$-Homotopy Theory of S^1-spectra

In this section, we focus on S^1-spectra, since S^1 is compact. We will ignore the foot note so that Ω and Σ mean the S^1-loop functor and the S^1-suspension functor.

Let $S = speck(k)$ where k is a field. Then we get the site $(Sm/k)_{Nis}$ of smooth schemes over the field k with the Nisnevich topology. We denote $SH^{S^1}(k)$ the motivic stable model category of S^1-spectra.

2.3.1 Basic Definitions

Let us first forget the motivic structure, which means the level structure is defined by level local weak equivalences, level cofibrations (level monomorphisms) and level local fibrations. As we know in ordinary spectra theory, we get the simplicial stable model category of S^1-spectra $SH_s^{S^1}(k)$.

There are some standard corollaries from ordinary spectra theory. Notice that if an S^1-spectrum E is fibrant, if and only if it is level fibrant and the morphisms $E_n \longrightarrow \Omega E_{n+1}$ are weak equivalent.

Corollary 2.3.1 *Let E be a fibrant S^1-spectrum. For any $U \in (Sm/k)_{Nis}$, the canonical map $Hom_{SH_s^{S^1}(k)}(\Sigma^\infty(U_+), E) \longrightarrow Hom_{H_\bullet(k)}(U, E_0)$ is bijective.*

It follows that the canonical map

$$Hom_{SH_s^{S^1}(k)}(\Sigma^\infty(U_+)[n], E) \longrightarrow Hom_{H_\bullet(k)}(U \wedge S^n, E_0)$$

is bijective.

Corollary 2.3.2 *Let E be a fibrant S^1-spectrum. Then the sheaf $\pi_0(E)$ is the sheaf associated to the presheaf*

$$U \longrightarrow Hom_{SH_s^{S^1}(k)}(\Sigma^\infty(U_+), E)$$

The sheaf $\pi_n(E)$ is the sheaf associated to the presheaf

$$U \longrightarrow Hom_{SH_s^{S^1}(k)}(\Sigma^\infty(U_+)[n], E)$$

2.3.2 The Triangulated Structure on $SH_s^{S^1}(S)$

The category $SH_s^{S^1}(k)$ admits arbitrary coproducts: the coproduct of a collection of S^1-spectra is a S^1-spectra whose n-th term is the wedge of the n-th term in that collection.

Lemma 2.3.1 *The smash-product by S^1:*

$$SH_s^{S^1}(k) \longrightarrow SH_s^{S^1}(k), E \longmapsto S^1 \wedge E$$

is an equivalence of categories.

proof: First, we choose a stable fibrant model $E \longrightarrow E_s$ for E. Then from the adjoinction of Σ and Ω, we have two stable weak equivalences:

$$E \xrightarrow{\sim} \Omega((S^1 \wedge E)_s)$$

and

$$S^1 \wedge \Omega(E_s) \xrightarrow{\sim} E_s$$

Therefore, the lemma comes from the fact that the funtor $E \longmapsto \Omega(E_s)$ is an inverse to $E \longmapsto S^1 \wedge E$.

In fact, we know that the category $SH_s^{S^1}(k)$ is additive and the shift functor $-[1]$ is isomorphic to $S^1 \wedge -$. So a distinguished triangle in $SH_s^{S^1}(k)$ is a triangle isomorphic to the form:

$$E \longrightarrow F \longrightarrow Cone(f) \longrightarrow S^1 \wedge E$$

where the S^1-spectrum $Cone(f)$ is defined as $Cone(f)_n = Cone(f_n)$. This is actually the *cofiber sequence* in the stable homotopy category of S^1-spectra which induces long exact sequence of stable homotopy groups.

Remark 2.3.1

1. *We notice that the triangulated category $SH_s^{S^1}(k)$ is generated by spectra of the form $\Sigma^\infty(U_+)$ in the sense that $E \in SH_s^{S^1}(k)$ is trivial if and only if any morphism in $SH_s^{S^1}(k)$ from any shift of those spectra to E is trivial. This is an easy consequence the* **Corollary 2.3.1** *and* **Corollary 2.3.2**.

2. *For each distinguished diagram as we defined in* **Definition 1.1.1**, *the following triangle:*

   $$\Sigma^\infty((p^{-1}(U))_+) \longrightarrow \Sigma^\infty(U_+) \oplus \Sigma^\infty(V_+) \longrightarrow \Sigma^\infty(X_+) \longrightarrow \Sigma^\infty((p^{-1}(U))_+)[1]$$

   where the first morphism is the difference of the two obvious morphisms and the second is the sum of the two obvious morphisms is a distinguished triangle in the triangulated category $SH_s^{S^1}(k)$. Actually, this sequence is still a cofiber sequence.

2.3.3 The t-structure on $SH_s^{S^1}(k)$

Definition 2.3.1 *A t-category D is a triangulated category, equipped with two full subcategories $D_{\geq 0}$ and $D_{\leq 0}$, such that if let $D_{\geq n} = D[-n]_{\geq 0}$ and $D_{\leq n} = D[-n]_{\leq 0}$, we have:*

1. *For X in $D_{\geq 0}$ and Y in $D_{\leq -1}$, $Hom(X,Y) = 0$,*
2. *We have $D_{\geq 0} \subset D_{\geq -1}$ and $D_{\leq 0} \supset D_{\leq -1}$,*
3. *For any $X \in D$, there is a distinguished triangle (A, X, B) such that $A \in D_{\geq 0}$ and $B \in D_{\leq -1}$.*

We say that $(D_{\leq 0}, D_{\geq 0})$ is a t-structure on D. Its heart is the full subcategory $D_{\leq 0} \cap D_{\geq 0}$.

Definition 2.3.2

1. *An S^1-spectrum is said to be non-negative if and only if for any integer $n \leq 0$, one has:*
$$\pi_n(E) = 0$$
We denote $SH_s^{S^1}(k)_{\geq 0} \subset SH_s^{S^1}(k)$ the full subcategory whose objects are non-negative.
2. *An S^1-spectrum is said to be non-positive if and only if for any integer $n \geq 0$, one has:*
$$\pi_n(E) = 0$$
We denote $SH_s^{S^1}(k)_{\leq 0} \subset SH_s^{S^1}(k)$ the full subcategory whose objects are non-positive.

Lemma 2.3.2 *An S^1-spectrum F is non-positive if and only if for any $n \geq 0$ and $U \in Sm/k$ the group*
$$[\Sigma^\infty(U_+)[n], F]_s^{S^1}$$
vanishes.

Example 2.3.1 *Spectra of the form $\Sigma^\infty(U_+)[n] = \Sigma^\infty(U_+) \wedge S^n, n \geq 0$ and $U \in Sm/k$ are non-negative.*

Theorem 2.3.1 *The triple $(SH_s^{S^1}(k), SH_s^{S^1}(k)_{\geq 0}, SH_s^{S^1}(k)_{\leq 0})$ is a t-structure on $SH_s^{S^1}(k)$.*

The inclusion $SH_s^{S^1}(k)_{\geq 0} \subset SH_s^{S^1}(k)$ has a left adjoint $E \longmapsto E_{\geq 0}$, and the inclusion $SH_s^{S^1}(k)_{\leq 0} \subset SH_s^{S^1}(k)$ has a right adjoint $E \longmapsto E_{\leq 0}$. Moreover, we have the following properties:

1. $\forall E \in SH_s^{S^1}(k)_{\geq 0}$, $F \in SH_s^{S^1}(k)_{\leq 0}$, we have $[E[-1], F]_s^{S^1} = 0$,

2. For any spectrum E there is a distinguished triangle:

$$E_{\geq 0} \longrightarrow E \longrightarrow E_{\leq -1} \longrightarrow E_{\geq 0}[1]$$

3. In general, we have distinguished triangles in the form:

$$E_{\geq n} \longrightarrow E \longrightarrow E_{\leq (n-1)} \longrightarrow E_{\geq n}[1]$$

Notice that the sequences 2. and 3. are fiber sequences, and since fiber sequences and cofiber sequences are indistinguishable in simplicial stable category (we could replace *motivic* by *simplicial* in our previous discussions of cofiber and fiber sequences), we see that sequences 2. and 3. are distinguished triangles.

Proposition 2.3.1

- *The $t-structure$ on $SH_s^{S^1}(k)$ is non-degenerate in the sense that for any $U \in (SM/k)_{Nis}$ and for any $E \in SH_s^{S^1}(k)$, the morphism:*

$$[\Sigma^\infty(U_+), E_{\geq n}]_s^{S^1} \longrightarrow [\Sigma^\infty(U_+), E]_s^{S^1}$$

  is an isomorphism for $n \leq 0$.

  The group

$$[\Sigma^\infty(U_+), E_{\geq n}]_s^{S^1}$$

  vanishes for $n > dim(U)$ because by cohomology dimension argument.

- *According to the triangulated structure and the distinguished triangle given above, the morphism:*

$$[\Sigma^\infty(U_+), E]_s^{S^1} \longrightarrow [\Sigma^\infty(U_+), E_{\leq n}]_s^{S^1}$$

  is an isomorphism for $n > dim(U)$.

The Heart of the t-structure

The functor

$$SH_s^{S^1}(k) \longrightarrow Ab((Sm/k)_{Nis}), E \longrightarrow \pi_0(E)$$

induces an equivalence of categories from the heart of the $t-structure$ to the category $Ab((Sm/k)_{Nis})$ of sheaves of Abelian groups on the Nisnevich site $(Sm/k)_{Nis}$. The inverse of this functor is given by:

$$H : Ab((Sm/k)_{Nis}) \longrightarrow SH_s^{S^1}(k)$$

which sends M to the following S^1-spectrum HM: its n-th term is the simplicial sheaf of Abelian group $K(M,n)$ which has only one non-trivial homotopy sheaf isomorphism to M in degree n.

2.3.4 $\mathbf{A^1}$-localization of S^1-spectra and stable $\mathbf{A^1}$-homotopy category of S^1-spectra

The Model Structure

When we consider $\mathbf{A^1}$ as a pointed scheme, we will always consider 0 as its base point.

Definition 2.3.3 *Let E be an S^1-spectrum. The following conditions are equivalent:*

(i) *for any $X \in Sp^{S^1}(k)$, the projection:*

$$X \wedge \Sigma^\infty(\mathbf{A^1}_+) \longrightarrow X \wedge \Sigma^\infty(spec(k)_+) = X$$

induces a bijection:

$$[X, E]_s^{S^1} \longrightarrow [X \wedge \Sigma^\infty(\mathbf{A^1}_+), E]_s^{S^1}$$

(ii) *for any $X \in Sp^{S^1}(k)$, the group*

$$[X \wedge \Sigma^\infty(\mathbf{A^1}), E]_s^{S^1}$$

vanishes;

(iii) *the S^1-spectrum*

$$\underline{Hom}_\bullet(\mathbf{A^1}, E)$$

is trivial;

(iv) *for any smooth k-scheme U, any integer n, the group homomorphism induced by evaluation at 1 $ev_1(E)$:*

$$[\Sigma^\infty(U_+)[n], \underline{Hom}_\bullet(\mathbf{A^1}, E)]_s^{S1} \longrightarrow [\Sigma^\infty(U_+)[n], E]_s^{S1}$$

is trivial

Suppose moreover that E is a Ω-spectrum (for example it is a stably fibrant). Then the previous conditions are also equivalent to:

(v) *each of the pointed simplicial sheaf E_n is $\mathbf{A^1}$-local.*

Definition 2.3.4

1. *An S^1-spectrum E is called $\mathbf{A^1}$-local if it satisfies the equivalent conditions of the previous proposition.*
2. *A morphism $f : X \longrightarrow Y$ in $Sp^{S^1}(k)$ is called a stable $\mathbf{A^1}$-weak equivalence if and only if for any $\mathbf{A^1}$-local E, the map:*

$$[Y, E]_s^{S1} \longrightarrow [X, E]_s^{S1}$$

is bijective.

Given this definition, we are able to define the motivic stable homotopy category of S^1-spectra. That is, the level structure is the *motivic* level structure, and the notion of stable weak equivalence and stable fibrations are what we defined in the last section. The resulting homotopy category is denoted by $SH^{S^1}(k)$. Given two S^1-spectra E, F, denote by $[E, F]^{S^1}$ the Abelian group of morphisms between E and F in $SH^{S^1}(k)$.

The triangulated structure of $SH^{S^1}(k)$ is given by fiber and cofiber sequences since they are indistinguishable.

Since the left adjoint of the inclusion $H_s^{\mathbf{A^1}}(k) \subset H_s(k)$ identifies $\mathbf{A^1}$-homotopy category with the full subcategory of $H_s(k)$ whose objects are $\mathbf{A^1}$-local, one observes that the left adjoint of the inclusion $SH^{S^1}(k) \subset SH_s^{S^1}(k)$ identifies the category $SH^{S^1}(k)$ with the full subcategory of $SH_s^{S^1}(k)$ whose objects are stable $\mathbf{A^1}$-local.

The $\mathbf{A^1}$-localization functor

Definition 2.3.5 *Let $F \longrightarrow F_s$ be the functorial fibrant replacement. Suppose that E is an S^1-spectrum, and $L^{(1)}(E)$ be the cone of the morphism:*

$$ev_1 : \underline{Hom}_\bullet(\mathbf{A^1}, E_s) \longrightarrow E_s$$

Let $L_s^{(1)}(E)$ be $L^{(1)}(E)_s$. We let $E \longrightarrow L_s^{(1)}(E)$ be the obvious morphism of pointed simplicial sheaves. Then define by induction for $n \geq 0$, $L^{(n)} = L_s^{(1)} \circ L_s^{(n-1)}$. So we have a natural morphism for any E, $E \longrightarrow L^{(n)}(E)$ and we set $L^\infty(E) = colim_n L^{(n)}(E)$.

Lemma 2.3.3 *Let E be an S^1-spectrum. Then the S^1-spectrum $L^\infty(E)$ is $\mathbf{A^1}$-local and the morphism:*

$$E \longrightarrow L^\infty(E)$$

is a stable $\mathbf{A^1}$-weak equivalence.

We note that the localization functor L^∞ is exactly the functor that identifies the category $SH^{S^1}(k)$ with $SH^{S^1}_{s,\mathbf{A^1}}(k)$, the full subcategory of $SH^{S^1}_s(k)$ whose objects are stable $\mathbf{A^1}$-local. This functor is called the $\mathbf{A^1}$-localization functor, and we will denote it by $L_{\mathbf{A^1}}$ instead.

2.3.5 The Homotopy t-structure on $SH^{S^1}(k)$

Definition 2.3.6 *An S^1-spectrum F is said to be $\mathbf{A^1}$ non-positive if and only if for any integer $n > 0$, one has:*

$$[\Sigma^\infty(U_+)[n], F]^{S^1}$$

is trivial. We denote $SH^{S^1}_{\leq 0}(k) \subset SH^{S^1}(k)$ the full subcategory whose objects are $\mathbf{A^1}$ non-positive..

An S^1-spectrum E is said to be $\mathbf{A^1}$ non-negative if and only if for any $F \in SH^{S^1}_{\leq 0}(k)$, one has:

$$[E, F[1]] = 0$$

We denote $SH^{S^1}_{\geq 0}(k) \subset SH^{S^1}(k)$ the full subcategory whose objects are $\mathbf{A^1}$ non-negative.

Note that spectra of the form $\Sigma^\infty(U_+)[n]$ with $n \geq 0$ and $U \in Sm/k$ are obviously $\mathbf{A^1}$ non-negative.

Lemma 2.3.4

1. *An S^1-spectrum F is $\mathbf{A^1}$ non-positive if and only if its $\mathbf{A^1}$-localization $L^\infty(F) = L_{\mathbf{A^1}}(F)$ is non-positive.*

2. *An S^1-spectrum F is $\mathbf{A^1}$ non-negative if and only if its $\mathbf{A^1}$-localization $L^\infty(F) = L_{\mathbf{A^1}}(F)$ is non-negative.*

proof: This comes easily from the isomorphism:

$$[\Sigma^\infty(U_+)[n], F]^{S^1} \cong [\Sigma^\infty(U_+)[n], L_{\mathbf{A^1}}(F)]^{S^1}_s$$

Theorem 2.3.2

1. *Let E be an $\mathbf{A^1}$-local S^1-spectrum, then is non-negative part $E_{\geq 0}$ is still an $\mathbf{A^1}$-local S^1-spectrum. As a consequence, the triangle of S^1-spectra in $SH_s^{S^1}(k)$*

$$E_{\geq 0} \longrightarrow E \longrightarrow E_{\leq -1}$$

consists of $\mathbf{A^1}$-local S^1-spectra

2. *The pair $(SH^{S^1}_{\geq 0}(k), SH^{S^1}_{\leq 0}(k))$ is a t-structure on $SH^{S^1}(k)$.*

This t-structure is called the homotopy t-structure on $SH^{S^1}(k)$.

The $t-structure$ on $SH^{S^1}(k)$ is non-degenerate in the sense that for any $U \in (Sm/k)_{Nis}$ and for any $E \in SH^{S^1}(k)$, the morphism:

$$[\Sigma^\infty(U_+), E_{\geq n}]^{S^1} \longrightarrow [\Sigma^\infty(U_+), E]^{S^1}$$

is an isomorphism for $n \leq 0$, and the group

$$[\Sigma^\infty(U_+), E_{\geq n}]^{S^1}$$

vanishes for $n > dim(U)$.

According to the triangulated structure, the morphism:

$$[\Sigma^\infty(U_+), E]^{S^1} \longrightarrow [\Sigma^\infty(U_+), E_{\leq n}]^{S^1}$$

is an isomorphism for $n \geq dim(U)$.

The Heart of the Homotopy t-structure

The heart of the t-structure on $SH^{S^1}(k)$ is denoted by $\pi^{\mathbf{A^1}}(k)$. There is an obvious morphism

$$\pi^{\mathbf{A^1}}(k) \longrightarrow SH^{S^1}_{s,\geq 0}(k) \cap SH^{S^1}_{s,\leq 0}(k).$$

is an exact full embedding.

We have already noticed that $SH^{S^1}_{s,\geq 0}(k) \cap SH^{S^1}_{s,\leq 0}(k)$ is canonically equivalent to $Ab((Sm/k)_{Nis})$ by the functor which maps E to the associated sheaf to the presheaf

$$U \longmapsto [\Sigma^\infty(U_+), E]_s^{S^1}$$

thus it is an immediate thought to discuss the relation of $\pi^{\mathbf{A^1}}(k)$ between $Ab((Sm/k)_{Nis})$, and we will see soon we can identify $\pi^{\mathbf{A^1}}(k)$ with a subcategory of $Ab((Sm/k)_{Nis})$.

Definition 2.3.7 *We say that a sheaf of Abelian groups $M \in Ab((Sm/k)_{Nis})$ is strictly $\mathbf{A^1}$-homotopy invariant if and only if for any smooth k-scheme U, the homomorphism*

$$H^n_{Nis}(U, M) \longrightarrow H^n_{Nis}(U \times \mathbf{A^1}, M)$$

is an isomorphism. We denote by $Ab_{st\mathbf{A^1}}((Sm/k)_{Nis})$ the full subcategory of $Ab((Sm/k)_{Nis})$ consisting of strictly $\mathbf{A^1}$-homotopy invariant sheaves.

Lemma 2.3.5 *Given a sheaf of Abelian groups $M \in Ab((Sm/k)_{Nis})$, the S^1-spectrum HM is $\mathbf{A^1}$-local if and only if M is strictly $\mathbf{A^1}$-invariant.*

proof: This comes from the isomorphism

$$Hom_{H_s}(U, K(M, n)) \cong H^n_{Nis}(U, M)$$

for any $U \in (Sm/k)_{Nis}$.

Theorem 2.3.3

1. *For any S^1-spectrum E the associated Nisnevich sheaf to the presheaf*

$$U \longmapsto [\Sigma^\infty(U_+), E]^{S^1} = [\Sigma^\infty(U_+), L_{\mathbf{A^1}}(E)]^{S^1}_s$$

   is a strictly $\mathbf{A^1}$-homotopy invariant sheaf which we denote by $\pi_0^{\mathbf{A^1}}(E)$.

2. *the functor*

$$\pi_0^{\mathbf{A^1}} : SH^{S^1}(k) \longrightarrow Ab_{st\mathbf{A^1}}((Sm/k)_{Nis}), E \longmapsto \pi_0^{\mathbf{A^1}}(E)$$

   induces an equivalence of categories:

$$\pi^{\mathbf{A^1}}(k) \cong Ab_{st\mathbf{A^1}}((Sm/k)_{Nis})$$

proof: We know that if $E = E_{\geq 1}$, the associated sheaf will be trivial, and so it is if $E = E_{\leq -1}$. We thus can reduce to the case that $E = HM$.

Remark 2.3.2 *Since we can identify the category $SH^{S^1}(k)$ with the category $SH^{S^1}_{s,\mathbf{A^1}}(k)$, the category $SH^{S^1}(k)$ inherits the triangulated structure in which the distinguished triangles are isomorphic to fiber and cofiber sequences.*

Homotopy Groups of $\underline{Hom}_\bullet(\mathbf{G}_m, E)$

Definition 2.3.8 *Let $F : (Sm/k)^{op} \longrightarrow Set_\bullet$ be a presheaf of pointed sets. Denote by $F_{-1} : (Sm/k)^{op} \longrightarrow Set_\bullet$ the presheaf of sets which sends $U \in (Sm/k)^{op}$ to the kernel (in the category of pointed sets) of the evaluation at 1: $F(U \times \mathbf{G}_m) \longrightarrow F(U)$.*

One observes that if F is a sheaf of pointed sets so is F_{-1} and in fact F_{-1} is the pointed function sheaf: $\underline{Hom}_\bullet(\mathbf{G}_m, F)$.

Recall that for an S^1-spectra E, the sheaf $\pi_0(E)$ is the sheaf associated to the presheaf $U \longrightarrow Hom_{SH_s^{S^1}(k)}(\Sigma^\infty(U_+), E)$. Therefore, there is a natural transformation:

$$[\Sigma^\infty(U_+) \wedge \Sigma^\infty(\mathbf{G}_m), E]_s^{S^1} \wedge Hom_{Sh^{Nis}}(U, \mathbf{G}_m) \longrightarrow \pi_0(E)(U)$$

induces a morphism of sheaves of pointed sets

$$\pi_0(\underline{Hom}_\bullet(\mathbf{G}_m, E)) \wedge \mathbf{G}_m \longrightarrow \pi_0(E)$$

and thus a morphism of sheaves of abelian groups

$$\pi_0(\underline{Hom}_\bullet(\mathbf{G}_m, E)) \longrightarrow \pi_0(E)_{-1}$$

More generally, for any integer n the above construction induces a canonical morphism

$$\pi_n(\underline{Hom}_\bullet(\mathbf{G}_m, E)) \longrightarrow \pi_n(E)_{-1}$$

Proposition 2.3.2 *Let $E \in Spt^{S^1}(k)$ be an $\mathbf{A^1}$-local S^1-spectrum. Then for any integer n, the canonical morphism*

$$\pi_n(\underline{Hom}_\bullet(\mathbf{G}_m, E)) \longrightarrow \pi_n(E)_{-1}$$

is an isomorphism.

The *proof* of this proposition consists of the following two steps:

1. **Lemma 2.3.6** *If $E \in \pi(k)$ is an $\mathbf{A^1}$-local S^1-spectrum which is in the heart of the homotopy t-structure on $SH^{S^1}(k)$, and consider $\mathbf{G}_m$ be pointed at 1, then the function spectrum $\underline{Hom}_\bullet(\mathbf{G}_m, E)$ is still in the heart. More precisely, if $E = HM$, then $\underline{Hom}_\bullet(\mathbf{G}_m, HM) \simeq H(M_{-1})$.*

   proof of the lemma: In order to prove this lemma, we want to establish an equivalence of S^1-spectra

   $$\underline{Hom}_\bullet(\mathbf{G}_m, HM) \longrightarrow H(M_{-1})$$

   for $M \in Ab_{st\mathbf{A^1}}(Sm/k)_{Nis}$. Consider S^0 as an S^1-spectrum which means $S^0 = \Sigma^\infty(S^0)$. Let us check whether the morphism:

   $$[S^0, \underline{Hom}_\bullet(\mathbf{G}_m, HM)[n]]_s^{S^1} \longrightarrow [S^0, H(M_{-1})[n]]_s^{S^1}$$

   is an isomorphism for each integer n. The morphism can be reformulated as:

(a) $[\Sigma^\infty \mathbf{G}_m, HM[n]]_s^{S^1} = 0 \ for \ n \neq 0$

(b) $[\Sigma^\infty \mathbf{G}_m, HM]_s^{S^1} \longrightarrow [S^0, H(M_{-1})]_s^{S^1}$.

The second one comes directly from the construction. For the first one, one observes that $\Sigma^\infty(\mathbf{G}_m) \simeq \Sigma^\infty(\mathbf{P^1})[-1]$. Moreover, for any smooth scheme X, the group $[\Sigma^\infty(X_+), HM[n]]_s^{S^1}$ is canonically isomorphic to $H^n_{Nis}(X, M)$. Since $\mathbf{P^1}$ has cohomological dimension≤ 1, we have done.

2. Given this lemma, let $M = \pi_n(E)$. Then we have $\pi_k(HM) = \pi_n(E)$ if $k = 0$, and $\pi_k(HM) = 0$ if $k \neq 0$. Notice that S^0 and $\mathbf{G_m}$ are both compact, we have:

$$\pi_n(\underline{Hom}_\bullet(\mathbf{G}_m, HM)) = colim_r[S^{n+r}, \underline{Hom}_\bullet(\mathbf{G}_m, HM)_r]_s = colim_r[S^{n+r} \wedge \mathbf{G}_m, HM_r]_s$$

$$= colim_r[\mathbf{G}_m, \underline{Hom}_\bullet(S^{n+r}, HM)_r]_s = [\mathbf{G}_m, colim_r \underline{Hom}((S^{n+r}, HM_r))]_s = [\mathbf{G}_m, \pi_n(HM)]_s$$

Thus, according to the lemma, we have:

$$\pi_n(\underline{Hom}_\bullet(\mathbf{G}_m, E)) \xrightarrow{\sim} \pi_n(E)_{-1}$$

2.4 Stable $\mathbf{A^1}$-Homotopy Category of T-Spectra

2.4.1 $\mathbf{P^1}$-spectra

We actually have $T \simeq P^1$. So, we consider now the Morel-Voevodsky P^1-spectra. We will denote $[X, Y]^{\mathbf{P^1}}$ the homotopy class in the motivic stable homotopy category of $\mathbf{P^1}$-spectra.

Definition 2.4.1 *Note that*

$$\pi_n((\Omega_T^\infty(X))_m) = [S^n, colim_r \underline{Hom}_\bullet((P^1)^m, X_{m+r})]^{\mathbf{A^1}} = colim_r[S^n \wedge (P^1)^m, X_{m+r}]^{\mathbf{A^1}}$$

and this leads us to the definition of

$$\overline{\pi}_n(E)_m(U) = colim_{r>0} Hom_{H_\bullet^{\mathbf{A^1}}(k)}(S^{n+m} \wedge (U_+) \wedge (P^1)^{r-m}, E_r)$$

for $U \in Sm/k, n, m \in Z$.

Definition 2.4.2 *A morphism $f : E \longrightarrow F$ of P^1-spectra is called a $\mathbf{A^1}$-stable weak equivalence if and only if for any $U \in Sm/k$ and any pair of integers (n, m) the homomorphism:*

$$\overline{\pi}_n(E)_m(U) \longrightarrow \overline{\pi}_n(F)_m(U)$$

is an isomorphism.

One notices that the above definitions coincide with the definition of motivic stable homotopy category of T-spectra.

Remark 2.4.1

1. *Let $S^1 = S^1_s$ and $\mathbf{G}_m = S^1_t$, then*
$$S^{n,i} = (S^1_s)^{\wedge(n-2i)} \wedge (S^{1,1})^{\wedge i} = S_s^{\wedge(n-i)} \wedge S_t^{\wedge i},$$
where $S^{1,1} = S^1 \wedge G_m$ and let $E(i)[n] := E \wedge S^{n,i}$. Then by the definition of stable homotopy presheaf, the presheaf $\overline{\pi}_n(E)_m$ is isomorphic to the presheaf
$$U \longmapsto [\Sigma^\infty_{P^1}(U_+) \wedge S^n \wedge G_m^{\wedge(-m)}, E]^{\mathbf{P^1}} \cong [\Sigma^\infty_{P^1}(U_+)[n], E \wedge G_m^{\wedge(m)}]^{\mathbf{P^1}}$$
$$\cong [\Sigma^\infty_{P^1}(U_+)[n], E(m)[m]]^{\mathbf{P^1}}.$$
up to the motivic fibrant model.

2. *For any spectrum E and $X \in \Delta^{op}Sh_\bullet((Sm/k)_{Nis})$, set*
$$\tilde{E}^{n,i}(X) = [\Sigma^\infty(X), E(i)[n]]^{\mathbf{P^1}}$$
and for any $X \in \Delta^{op}Sh((Sm/k)_{Nis})$, set:
$$E^{n,i}(X) = [\Sigma^\infty(X_+), E(i)[n]]^{\mathbf{P^1}} \simeq \tilde{E}^{n,i}(X_+).$$
The functor $\Delta^{op}Sh((Sm/k)_{Nis}) \longrightarrow Ab^{*,*},\ X \longmapsto E^{*,*}(X)$ is called the cohomology theory on the category of simplicial smooth k-schemes associated to E.

The reason that we define the seemingly strange bi-graded objects is hidden in the theory of motivic cohomology theory. We will see it in the next chapter.

Remark 2.4.2

For any pointed simplicial sheaf T one can define the category $Sp^T(k)$ of T-spectra over k, the corresponding presheaves

$$U \longmapsto \overline{\pi}^T_n(E)_m(U); = colim_{r>0}[S^n \wedge (U_+) \wedge (T)^{r-m}, E_r]^{\mathbf{A^1}}$$

and the corresponding notion of stable $\mathbf{A^1}$-weak equivalence of T-spectra.

If T is reasonable in sense of

1. *T is compact,*
2. *T is isomorphic in $H_\bullet(k)$ to a suspension,*
3. *the cyclic permutation on the three variables*
$$T \wedge T \wedge T \longrightarrow T \wedge T \wedge T$$
is the identity in $H_\bullet(k)$.

Then $SH^T(k)$ has a structure of triangulated, symmetric monoidal category in which the suspension T-spectrum of T is invertible.

Moreover, for a morphism between two reasonable pointed simplicial sheaves $S \longrightarrow S'$, there is a triangulated monoidal functor
$$SH^{T'}(k) \longrightarrow SH^T(k)$$
which is an equivalent of categories if $T \longrightarrow T'$ is an $\mathbf{A^1}$-weak equivalence.

Finally, given these two reasonable objects, the smash-product defines a canonical triangulated and symmetric monoidal functor
$$\sigma^{T'} : SH^T(k) \longrightarrow SH^{T \wedge T'}(k).$$

The basic examples are which we have seen:

1. The canonical isomorphisms
$$P^1 \cong T \cong \mathbf{A^1}/\mathbf{A^1} - \{0\} \cong S^1 \wedge G_m \cong S^1_s \wedge S^1_t$$
give equivalences
$$SH^{P^1}(k) \cong SH^T(k) \cong SH^{S^1 \wedge G_m}(k)$$
2. The canonical functor induced by the smash-product by G_m gives us:
$$\sigma^{G_m} : SH^{S^1}(k) \longrightarrow SH^{S^1 \wedge G_m}(k) \cong SH^T(k) \cong SH^{P^1}(k)$$

We notice that the previous functor σ^{G_m} admits a right adjoint:
$$\omega^{G_m} : SH^{S^1 \wedge G_m}(k) \cong SH^T(k) \cong SH^{P^1} \longrightarrow SH^{S^1}(k)$$

As a consequence, for any P^1-spectrum E and any pair of integers (n, m) the presheaf $\overline{\pi}_n(E)_m$ is isomorphic to the presheaf
$$U \longmapsto [\Sigma^\infty_{S^1}(U_+)[n], \omega^{G_m}(E(m)[m])]^{S^1}$$

Definition 2.4.3 *For any P^1-spectrum E and any pair of integers (n, m). We denote by $\pi_n(E)_m$ the sheaf associated to the presheaf $\overline{\pi}_n(E)_m$.*

Proposition 2.4.1 *We have $\pi_n(E)_m = \pi_n(\omega^{G_m}(E(m)[m])$. Moreover, the sheaves $\pi_n(E)_m$ are each strictly $\mathbf{A^1}$-homotopy invariant.*

2.4.2 The Homotopy t-Structure

In this section, we use $SH(k)$ for short to denote the category $SH^{\mathbf{P^1}}(k)$.

Definition 2.4.4

1. *We denote $SH_{\geq 0}$ the full subcategory of $SH(k)$ consisting of $\mathbf{P^1}$-spectra E with*

$$\pi_n(E)_m = 0$$

   for each integer m and $n < 0$.

2. *Similarly, we denote $SH_{\leq 0}$ the full subcategory of $SH(k)$ consisting of $\mathbf{P^1}$-spectra F with*

$$\pi_n(F)_m = 0$$

   for each integer m and $n > 0$.

Example 2.4.1 *For $U \in Sm/k$, spectra of the form*

$$\Sigma^{\infty}_{\mathbf{P^1}}(U_+)(i)[m] \simeq \Sigma^{\infty}_{\mathbf{P^1}}(U_+) \wedge S^{m-i} \wedge \mathbf{G_m}^m$$

are non-negative, which means it is in $SH_{\geq 0}$ if $m - i \geq 0$.

The pair of adjoint functors $(\sigma^{\mathbf{G_m}}, \omega^{\mathbf{G_m}})$ allows one to prove

Theorem 2.4.1 *The triple $(SH(k), SH_{\geq 0}, SH_{<0})$ is a t-structure on $SH(k)$, which is called the homotopy t-structure on $SH(k)$.*

The structure is non-degenerate in the sense that for any $E \in SH(k)$ and any $U \in Sm/k$, the morphism:

$$[\Sigma^{\infty}_{\mathbf{P^1}}(U_+), E_{\geq n}] \longrightarrow [\Sigma^{\infty}_{\mathbf{P^1}}(U_+), E]$$

is an isomorphism for $n \leq 0$ and the morphism:

$$[\Sigma^{\infty}_{\mathbf{P^1}}(U_+), E] \longrightarrow [\Sigma^{\infty}_{\mathbf{P^1}}, E_{\leq n}]$$

is an isomorphism for $n > dim(U)$.

A Description of the Heart

Definition 2.4.5 *A homotopy module over k is a pair (M_*, μ_*) consisting of a Z-graded strictly homotopy invariant sheaf M_* together with, for each integer n, an isomorphism of abelian shaves:*

$$M_n \simeq (M_{n+1})_{-1}.$$

Lemma 2.4.1 *For any $\mathbf{P^1}$-spectrum F the canonical morphism*

$$\omega^{\mathbf{G_m}}(F) \longrightarrow \underline{Hom}_\bullet(\mathbf{G_m}, \omega^{\mathbf{G_m}}(F \wedge \mathbf{G_m}))$$

is an isomorphism.

proof: If F is an S^1-spectrum and G is a $\mathbf{P^1}$-spectrum, one obtains by adjunction and invertibility of the smash-product by $\mathbf{G_m}$ the following sequence of isomorphisms:

$$[F \wedge \mathbf{G_m}, \omega^{\mathbf{G_m}}(G \wedge \mathbf{G_m})]^{S^1} \simeq [\sigma^{\mathbf{G_m}}(F \wedge \mathbf{G_m}), G \wedge \mathbf{G_m}]$$
$$\simeq [\sigma^{\mathbf{G_m}}(F) \wedge \mathbf{G_m}, G \wedge \mathbf{G_m}] \simeq [\sigma^{\mathbf{G_m}}(F), G] \simeq [F, \omega^{\mathbf{G_m}}(G)]^{S^1}$$

Let E be a $\mathbf{P^1}$-spectrum. We have seen in the previous section that there are natural isomorphisms of sheaves:

$$\pi_n(E)_m \simeq \pi_n(\omega^{\mathbf{G_m}}(E(m)[m])).$$

By the preceding lemma the canonical morphisms

$$\omega^{\mathbf{G_m}}(E(m)[m]) \longrightarrow \underline{Hom}_\bullet(\mathbf{G_m}, \omega^{\mathbf{G_m}}(E(m+1)[m+1]))$$

are isomorphisms. By **Proposition 2.3.2**, one gets canonical isomorphisms:

$$\pi_n(E)_m \simeq \pi_n(\omega^{\mathbf{G_m}}(E(m)[m])) \simeq \pi_n(\underline{Hom}_\bullet(\mathbf{G_m}, \omega^{\mathbf{G_m}}(E(m+1)m+1)))$$
$$\simeq (\pi_n(\omega^{\mathbf{G_m}}(E(m+1)[m+1])))_{-1} \simeq (\pi_n(E)_{m+1})_{-1}.$$

Thus for a fixed integer n, the collection of $\pi_n(E)_m$, for m an integer, forms a homotopy module, which is denoted by $\pi_n(E)_*$ and is called the n-homotopy module of E.

Conversely, let M_* be a homotopy module. We construct a $\mathbf{P^1}$-spectrum denoted HM_* as follows. Its n-th term is the simplicial sheaf $K(M_n, n)$. The structure morphism is the obvious composition

$$K(M_n, n) \wedge S^1 \wedge \mathbf{G_m} \longrightarrow K(M_n, n+1) \wedge \mathbf{G_m} \longrightarrow K(M_{n+1}, n+1)$$

Using these results, it is easy to see

Theorem 2.4.2 *The functor from the category of homotopy modules to the stable homotopy category of $\mathbf{P^1}$-spectra*

$$M_* \longmapsto HM_*$$

is fully faithful and induces an equivalence between the category of homotopy modules and the heart of the homotopy t-structure. Its inverse is induced by the functor

$$E \longmapsto \pi_0(E)_*$$

Example 2.4.2 (Motivic cohomology) *We have already encountered a bi-graded object before, and the object $\pi_n(E)_m$ gives us another example of bi-graded object. These two objects all contribute to the theory of motivic cohomology, which will be further discussed in the next chapter. Here, we just simply mention the relations, the proof can be found in the next section.*

Let H_Z be a $\mathbf{P^1}$-spectrum named Eilenberg-MacLane spectrum which will be defined later, the value of the sheaf $\pi_n(H_Z)_m$ on $spec(k)$ is the group

$$\pi_n(H_Z)_m(spec(k)) \simeq \pi_n(\omega^{\mathbf{G_m}}(H_Z(m)[m]))(spec(k)) \simeq [\Sigma^\infty_{\mathbf{P^1}}(S^n), H_Z(m)[m]]^{\mathbf{P^1}}$$

which is equivalent to the motivic cohomology group $H^{m-n}(spec(k), Z(m))$.

Chapter 3

Motives

3.1 Triangulated Categories of Motives

Definition 3.1.1 *Let X, Y be in Sch_k. The group $c(X, Y)$ is the subgroup of $z(X \times Y)$ generated by integral closed subschemes $W \subset X \times_k Y$ such that*

1. *the projection $p_1 : W \longrightarrow X$ is finite,*
2. *the image $p_1(W) \subset X$ is an irreducible component of X.*

The elements of $c(X, Y)$ are called the **finite correspondences** *from X to Y.*

Given this definition, it is easy to prove:

Lemma 3.1.1 *Take X and Y in Sm_k and Z in Sch_k, $W \subset c(X, Y), W' \subset c(Y, Z)$. Suppose that X and Y are irreducible. Then each irreducible component C of $Supp(W) \times Z \cap X \times Supp(W')$ is finite over X and $p_X(C) = X$.*

Therefore, for $W \subset c(X, Y), W' \subset c(Y, Z)$, and $X, Y, Z \in Sm_k$, we have the composition:

$$W' \circ W = p^S_{XZ*}(p^*_{XY}(W) \cdot p^*_{YZ}(W'))$$

where $S = Supp(W) \cap Supp(W')$ and $p^S_{XY} : S \longrightarrow X \times Z$ is the morphism induced from the projection p_{XZ}. This operation yields an associated bilinear composition law

$$\circ : c(Y, Z) \times c(X, Y) \longrightarrow c(X, Z).$$

Remark 3.1.1 *There is a correspondence of cycles. However, the traditional definition of composition can only apply to the cases of smooth projective schemes. In order to extend the composition formula to smooth and non-projective schemes, we have to introduce the definition of the finite correspondences.*

Definition 3.1.2 *The category $Cor(k)$ is the category with the same objects as Sm/k, with*

$$Hom_{Cor(k)}(X,Y) = c(X,Y)$$

and with the composition as defined above.

We then have the functor $Sm/k \longrightarrow Cor(k)$ sending a morphism $f : X \longrightarrow Y$ in Sm/k to the graph $\Gamma_f \subset X \times_k Y$. We write the morphism corresponding to Γ_f as f_* and the object corresponding to $X \in Sm/k$ as $[X]$. Notice that the operation $\times_k$ (on smooth k-schemes and on cycles) makes $Cor(k)$ a tensor category. Thus, the bounded homotopy category $K^b(Cor(k))$ is a triangulated tensor category.

Definition 3.1.3 *The category $\widehat{DM}^{eff}_{gm}(k)$ is the localization of $K^b(Cor(k))$, as a triangulated tensor category, by*

1. **Homotopy** *For $X \in Sm/k$, invert $p_* : [X \times \mathbf{A^1}] \longrightarrow [X]$*
2. **Mayer-Vietoris** *Let X be in Sm/k. Write X as a union of Zariski open subschemes U, V and $X = U \cup V$. We have the canonical map*

$$Cone([U \cup V] \stackrel{(J_{U,U\cup V*}, -j_{V,U\cup V*})}{\longrightarrow} [U] \oplus [V]) \stackrel{(j_{U*}+j_{V*})}{\longrightarrow} [X]$$

since $(j_{U*} + j_{V*}) \circ (j_{U,U\cup V*}, -j_{V,U\cup V*}) = 0$ invert this map.

The category $DM^{eff}_{gm}(k)$ of *effective geometric motives* is the pseudo-abelian hull of $\widehat{DM}^{eff}_{gm}(k)$.

Note that the morphisms inverted to form $\widehat{DM}^{eff}_{gm}(k)$ are closed under $\otimes$, so $DM^{eff}_{gm}(k)$ inherits the tensor structure $\otimes$ from $K^b(Cor(k))$. Actually, the pseudo-abelian hull of a triangulated category is a triangulated category, so $DM^{eff}_{gm}(k)$ is still a triangulated category.

Definition 3.1.4 *For $X \in Sm/k$, by the triangulated structure, the reduced motive is*

$$[\tilde{X}] = Cone(p_* : [X] \longrightarrow [speck])[-1].$$

That is to say, $[\tilde{X}]\ [1]$ is represented in $K^b(Cor(k))$ by the complex

$$[X] \longrightarrow [speck],$$

which means the sequence

$$[\tilde{X}] \longrightarrow [X] \longrightarrow [speck] \longrightarrow \tilde{X}\ [1]$$

is a distinguished triangle in the triangulated category $K^b(Cor(k))$.

We set $\mathbf{Z}(1) = [\widetilde{\mathbf{P}^1}]\ [-2]$, and set $\mathbf{Z}(n) = \mathbf{Z}(1)^{\otimes n}$ for $n \geq 0$.

Definition 3.1.5 *The category of* **geometry motives**, $DM_{gm}(k)$, *is defined by inverting the functor* $\mathbf{Z}(1)$ *on* $DM^{eff}_{gm}(k)$, *i.e, one has objects* $X(n)$ *for* $X \in DM^{eff}_{gm}(k), n \in \mathbf{Z}$ *and*

$$Hom_{DM_{gm}(k)}(X(n), Y(m)) = colim_N Hom_{DM^{eff}_{gm}(k)}(X \otimes \mathbf{Z}(n+N), Y \otimes \mathbf{Z}(m+N)).$$

Remark 3.1.2

1. *Sending* X *to* $X(0)$ *and using the canonical map to the limit*
$$Hom_{DM_{gm}(k)}(X, Y) = colim_N Hom_{DM^{eff}_{gm}(k)}(X \otimes \mathbf{Z}(N), Y \otimes \mathbf{Z}(N))$$
defines the functor $i : DM^{eff}_{gm}(k) \longrightarrow DM_{gm}(k)$. *For* $n \geq 0$, *we have the evident map* $i(X \otimes \mathbf{Z}(\mathbf{n})) \longrightarrow \mathbf{X}(n)$, *which is an isomorphism.*

2. *In order that* $DM_{gm}(k)$ *be again a triangulated tensor category, it suffices that the commutativity involution* $\mathbf{Z}(1) \otimes \mathbf{Z}(1) \longrightarrow \mathbf{Z}(1) \otimes \mathbf{Z}(1)$ *be the identity, which is in fact the case*

3. *Setting* $\mathbf{Z}(n) = \mathbf{1}(n)$ *for* $n \in \mathbf{Z}$, *we have* $X(n) \simeq X \otimes \mathbf{Z}(n)$ *and* $\mathbf{Z}(n) \otimes \mathbf{Z}(m) \simeq \mathbf{Z}(n+m)$.

4. *We have the functor* $M_{gm} : Sm/k \longrightarrow DM^{eff}_{gm}(k)$ *sending* X *to the image of* $[X]$ *and* f *to the image of the graph* Γ_f.

There is a **cancellation theorem** stating that the functor $i : DM^{eff}_{gm}(k) \longrightarrow DM_{gm}(k)$ is a fully faithful faithful embedding.

3.2 Motivic Complexes

Definition 3.2.1 *Let* $Sh^{Nis}(X)$ *be the Nisnevich sheaves of abelian groups on* X, *and* $Sh^{Nis}(k)$ *be the Nisnevich sheaves of abelian groups on* Sm/k. *For a presheaf* F *on* Sm/k *or* X_{Nis}, *we let* F_{Nis} *be the associated sheaf.*

For $X \in Sch_k$, $\mathbf{Z}(X)$ denotes the (representable) presheaf of abelian groups on Sm/k freely generated by $Hom_{Sch_k}(-, X)$, $\mathbf{Z}_{Nis}(X)$ the Nisnevich sheaf. $PSh^{Nis}(Sm/k)$ has a tensor product

$$(F \otimes G)(X) = F(X) \otimes_{\mathbf{Z}} G(X)$$

and internal Hom $\underline{Hom}(F, G)(X) = Hom_{PSh^{Nis}(Sm/k)}(F \otimes \mathbf{Z}(X), G)$. $Sh^{Nis}(Sm/k)$ has the tensor product by sheafifying the presheaf $\otimes$. The internal Hom in $Sh^{Nis}(Sm/k)$ is given by

$$\underline{Hom}(F, G)(X) = Hom_{Sh^{Nis}(Sm/k)}(F \otimes \mathbf{Z}_{Nis}(X), G)$$

Definition 3.2.2

1. *The category $PST(k)$ of presheaves with transfers is the category of presheaves of abelian groups on $Cor(k)$.*

2. *The category of Nisnevich sheaves with transfer on Sm/k, $Sh^{Nis}(Cor(k))$ is the full subcategory of $PST(k)$ with objects those F such that for each $X \in Sm/k$ the restriction of F to X_{Nis} is a sheaf. We have the sheafification functor $F \longmapsto F_{Nis}$.*

Remark 3.2.1 *A PST F is a presheaf on SM/k together with transfer maps*

$$Tr(a) : F(Y) \longrightarrow F(X)$$

for every finite correspondence $a \in c(X, Y)$, satisfying:

$$Tr(\Gamma_f) = f^*, Tr(a \circ b) = Tr(b) \circ Tr(a), Tr(a \pm b) = Tr(a) \pm Tr(b).$$

Definition 3.2.3 *Let F be a presheaf of abelian groups on Sm/k. We call F homotopy invariant if for all $X \in Sm/k$, the map:*

$$p^* : F(X) \longrightarrow F(\mathbf{A^1} \times X)$$

is an isomorphism. We call F strictly homotopy invariant if for all $q \geq 0$, the cohomology presheaf $X \longmapsto H^q(X_{Nis}, F_{Nis})$ is homotopy invariant.

Theorem 3.2.1 (PST) *Let F be a homotopy invariant PST on Sm/k, then*

1. *The cohomology presheaves $X \longmapsto H^q(X_{Nis}, F_{Nis})$ are PST's,*

2. *F_{Nis} is strictly homotopy invariant,*

3. *$F_{Zar} = F_{Nis}$ and $H^q(X_{Zar,F_Z}) = H^q(X_{Nis}, F_{Nis})$.*

The proof can be found in [23], chapter 3,theorem 4.27 and 5.7.

Lemma 3.2.1 (Voevodsky's moving lemma, [23], chapter 3, lemma 4.5) *Let X be in Sm/k, S a finite set of points of X, $j_U : U \longrightarrow X$ an open subscheme. Then there is an open neighborhood $j_V : V \longrightarrow X$ of S in X and a finite correspondence $a \in Cor(X, Y)$ such that, for all homotopy invariant $PST's$ F, the diagram:*

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & F(U) \arrow[dl] \\ F(V) & \end{tikzcd}$$

commutes.

There are some consequences of the moving lemma:

1. If X is semi-local, then $F(X) \longrightarrow F(U)$ is a split injection,
2. If X is semi-local and smooth then $F(X) = F_{Zar}(X)$ and $H^n(X_{Zar}, F_{Zar}) = 0$ for $n > 0$,
3. If U is an open subset of $\mathbf{A^1}_k$, then $F_{Zar}(U) = F(U)$ and $H^n(U, F_{Zar}) = 0$ for $n > 0$,
4. If $j : U \longrightarrow X$ has complement a smooth k-scheme $i : Z \longrightarrow X$, the $coker F(Z_{Zar}) \longrightarrow j_* F(U_{Zar})$ (as a sheaf on Z_{Zar}) depends only on the Nisnevich neighborhood of Z in X.

Definition 3.2.4 *Inside the derived category $D^-(Sh^{Nis}(Cor(k)))$, we have the full subcategory $DM_-^{eff}(k)$ consisting of complexes whose cohomology sheaves are homotopy invariant.*

Lemma 3.2.2 *Let $HI(k) \subset D^-(Sh^{Nis}(Cor(k)))$ be the full subcategory of homotopy invariant sheaves. Then $HI(k)$ is a abelian subcategory of $D^-(Sh^{Nis}(Cor(k)))$, closed under extension.*

proof: Given $f : F \longrightarrow G$ in $HI(k)$, $ker(f)$ is the sheaf kernel in $HI(k)$. The presheaf $coker(f)$ is homotopy invariant, so by the PST theorem $coker(f)_{Nis}$ is homotopy invariant. Given a short exact sequence $0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0$ in $Sh^{Nis}(Cor(k))$ with $A, B \in HI(k)$. Consider $p : X \times \mathbf{A^1} \longrightarrow X$ and *five lemma* to the induced long exact cohomology sequence, we can get $p_* E = E$, so E is homotopy invariant.

Given this lemma, we can prove:

Proposition 3.2.1 *$DM_-^{eff}(k)$ is a triangulated subcategory of $D^-(Sh^{Nis}(Cor(k)))$.*

In order to understand $DM_-^{eff}(k)$ better, we realize it as a localization of $D^-(Sh^{Nis}(Cor(k)))$ rather than just a subcategory.

We have already seen before that given the definition of

$$\Delta_s^n = speck[t_0, \dots, t_n]/(\Sigma_0^n t_i - 1),$$

there is an associated simplicial presheaf to a presheaf F, $Sing_*(F)$, with the definition $Sing_*(F)(U) = S(U, X)_*$. By Dold-Kan theorem, there is a complex of abelian groups associated to each simplicial abelian groups. We write the complex of abelian groups associated to $Sing_*(F)$ as $C_*(F)$, and it is clear that $C_n(F)(U) = F(U \times \Delta_s^n)$.

For $X \in Sm/k$, we first let $L(X)$ denote the representable presheaf $Cor(-, X)$ and then let $C_*(X) = C_*(L(X))$. More specifically, $C_n(X)(U) = Cor(U \times \Delta_s^n, X)$.

Remark 3.2.2

1. *If F is a PST (or sheaf with transfers), then $C_*(F)$ is a complex of PST's*

2. *The homology presheaves $h_i(F) = H^{-i}(C_*(F))$ are homotopy invariant for any presheaf F ([24]). Thus by the PST theorem, the associated Nisnevich sheaves $h_i^{Nis}(F)$ are strictly homotopy invariant for F a PST. We thus have the functor*

$$C_* : Sh^{Nis}(Cor(k)) \longrightarrow DM_-^{eff}(k).$$

3. *For $X \in Sch/k$, we have the* **sheaf** *with transfers $L(X)(Y) = Cor(Y, X)$ for $Y \in Sm/k$.*

For $X \in Sch/k$, $L(X)$ is the free sheaf with transfers generated by the representable sheaf of sets $Cor(-, X)$, and we have the canonical isomorphism $Hom(L(X), F) = F(X)$. In fact, for $F \in Sh^{Nis}(Cor(k))$ there is a canonical isomorphism

$$Ext^n_{Sh^{Nis}(Cor(k))}(L(X), F) \simeq H^n(X_{Nis}, F).$$

This should remind us of our classical example

$$Ext^n_{Sh^{Top}(X)}(Z[X], F) \simeq H^n(X, F).$$

for F a sheaf of abelian groups on a topological space X, and $Z[X]$ the constant sheaf.

As we saw in **subsection 1.3.1**, two maps of presheaves $f, g : F \longrightarrow G$ are said to be $\mathbf{A^1}$-homotopic if there is a map

$$h : F \otimes Z(\mathbf{A^1}) \longrightarrow G$$

with $f = h \circ (id \otimes i_0),\ g = h \circ (id \otimes i_1)$, where $i_0, i_0 : spec(k) \longrightarrow \mathbf{A^1}$ are inclusions. We say $f : F \longrightarrow G$ is strict $\mathbf{A^1}$-homotopic if there is a map $g : G \longrightarrow F$ such that $fg \simeq_{\mathbf{A^1}} id_G,\ gf \simeq_{\mathbf{A^1}} id_F$. We also know that **(**Lemma 2.8) any strict $\mathbf{A^1}$-homotopy equivalence is an $\mathbf{A^1}$-weak equivalence.

We can prove the followings:

1. $p^* : F \longrightarrow C_n(F)$ is a strict $\mathbf{A^1}$-homotopy equivalence, and therefore an $\mathbf{A^1}$-equivalence,

2. the inclusion $F = C_0(F) \longrightarrow C_*(F)$ is a strict $\mathbf{A^1}$-homotopy equivalence, and therefore an $\mathbf{A^1}$-equivalence.

Lemma 3.2.3 *Let F be in $Sh^{Nis}(Sm/k)$, $G \in HI(k)$. Then $id \otimes p_* : F \otimes Z_{Nis}(\mathbf{A^1}) \longrightarrow F$ induces an isomorphism*

$$Ext^n_{Nis}(F, G) \longrightarrow Ext^n_{Nis}(F \otimes Z_{Nis}(\mathbf{A^1}), G).$$

An immediate consequence of this lemma is the functor $Ext^n_{Nis}(-, G)$ sends $\mathbf{A^1}$-homotopy equivalences into isomorphisms, for $G \in HI(k)$. In the derived category, this induces:

Proposition 3.2.2 *Let $f : F_* \longrightarrow F'_*$ be an $\mathbf{A^1}$-homotopy equivalence in $C^-(Sh^{Nis}(Sm/k))$. Then*

$$Hom_{D^-(Sh^{Nis}(Sm/k))}(F_*, G[n]) \xrightarrow{f^*} Hom_{D^-(Sh^{Nis}(Sm/k))}(F'_*, G[n])$$

is an isomorphism for all $G \in HI(k)$.

Theorem 3.2.2 (Nisnevich-acyclicity) *For $G \in HI(k)$, F a presheaf on Sm/k, we have:*

$$Ext^n_{Nis}(F_{Nis}, G) \simeq Hom_{D^-(Sh^{Nis}(Sm/k))}(C_*(F)_{Nis}, G[n])$$

for all n. If F is a PST, then:

$$Ext^i_{Nis}(F_{Nis}, G) = 0 \ for \ 0 \leq i \leq n \ and \ all \ G \in HI(k)$$

$$\Longleftrightarrow h^{Nis}_i(F) = 0 \ for \ all \ 0 \leq i \leq n$$

$$\Longleftrightarrow h^{Zar}_i(F) = 0 \ for \ all \ 0 \leq i \leq n$$

proof: The $\mathbf{A^1}$-homotopy equivalence $F \longrightarrow C_*(F)$ induces an $\mathbf{A^1}$-homotopy equivalence $F_{Nis} \longrightarrow C_*(F)_{Nis}$. If F is a PST, then $h^{Nis}_i(F) = h^{Zar}_i(F)$ by Voevodsky's PST theorem. If $h^{Nis}_n(F) \neq 0$, but $h^{Nis}_i(F) = 0$ for $i < n$, taking $G = h^{Nis}_n(F)$, we have the canonical non-zero map $C_*(F)_{Nis} \longrightarrow G[n]$ in $D^-(Sh^{Nis}(Sm/k))$. Note that his also gives a proof of **Theorem 2.6**.

Corollary 3.2.1 *Let F be a PST such that $F_{Nis} = 0$. Then $C_*(F)_{Zar}$ is acyclic, i.e, $h^{Zar}_i(F) = 0$ for all i.*

Theorem 3.2.3 *The functor C_* extends to an* **exact functor**

$$\mathbf{R}C_* : D^-(Sh^{Nis}(Cor(k))) \longrightarrow DM^{eff}_-(k),$$

left adjoint to the inclusion $DM^{eff}_-(k) \longrightarrow D^-(Sh^{Nis}(Cor(k)))$. $\mathbf{R}C_*$ identifies $DM^{eff}_-(k)$ with the localization $D^-(Sh^{Nis}(Cor(k)))/\mathcal{A}$, where $\mathcal{A}$ is the localizing subcategory of $D^-(Sh^{Nis}(Cor(k)))$ generated by complexes

$$L(X \times \mathbf{A^1}) \xrightarrow{L(p_1)} L(X); \ X \in Sm/k.$$

proof: We have to prove:

1. For each $F \in Sh^{Nis}(Cor(k))$, $F \longrightarrow C_*(F)$ is an isomorphism in $D^-(Sh^{Nis}(Cor(k)))$,

2. For each $T \in DM_-^{eff}(k),\ B \in \mathcal{A},\ Hom(B,T) = 0.$

Because 1. implies $DM_-^{eff}(k) \longrightarrow D^-(Sh^{Nis}(Cor(k)))/\mathcal{A}$ is surjective on isomorphism classes, and 2. implies $DM_-^{eff}(k) \longrightarrow D^-(Sh^{Nis}(Cor(k)))/\mathcal{A}$ is fully faithful.

To prove 2.: since $\mathcal{A}$ is generated by $I(X) = L(X \times \mathcal{A}^\infty) \longrightarrow L(X)$. We know that $Hom(L(Y),T) \simeq H^0(Y_{Nis},T)$ for $T \in D^-(Sh^{Nis}(Cor(k)))$ and

$$H^*(X,T) \simeq H^*(X \times \mathbf{A^1}, T)$$

for $T \in DM_-^{eff}(k)$, so $Hom(I(X),T) = 0$.

To prove 1.:

Consider the functor

$$L : Cor(k) \longrightarrow Sh^{Nis}(Cor(k))$$

sending X to the representable sheaf $L(X)$. L extends to the homotopy category of bounded complexes

$$L : K^b(Cor(k)) \longrightarrow D^-(Sh^{Nis}(Cor(k))).$$

Theorem 3.2.4 *There is a commutative diagram of* **exact tensor funtors**

$$\begin{tikzcd} K^b(Cor(k)) \arrow[r, "L"] \arrow[d] & D^-(Sh^{Nis}(Cor(k))) \arrow[d, "\mathbf{R}C_*"] \\ DM_{gm}^{eff}(k) \arrow[r, "i"] & DM_-^{eff}(k) \end{tikzcd}$$

such that

1. *i is a full embedding with dense image*

2. *$\mathbf{R}C_*(L(X)) \simeq C_*(X)$*

An important consequence of this theorem is the following corollary:

Corollary 3.2.2 *For $X, Y \in Sm/k$,we have the isomorphisms:*

$$Hom_{DM_{gm}^{eff}(k)}(M_{gm}(Y)[n], M_{gm}(X)) \simeq H_n(Y, NZ[Sing_*(X)])$$

$$\simeq H_n(Y, C_*(X)) \simeq Hom_{H^{\mathbf{A^1}}(k)}(S^n \wedge Y, Sing_*(X))$$

where the functor N is the **Dold-Kan** *realization of equivalences of categories*

$$N : \{The\ category\ of\ simplicial\ abelian\ groups\}$$

$$\leftrightarrow \{The\ category\ of\ chain\ complexes\ of\ abelian\ groups\} : K$$

and the functor K is the right adjoint to N.

proof of the Corollary:

The first part of the lemma is a direct consequence of the last theorem, and the fact that the complexes NA and $\widetilde{A}$ are homotopically equivalent for a simplicial abelian group A.

For the second part, we use the **theorem 2.3**, and since we have proved the two constructions of [13] and [21] are equivalent, thus we abuse the notions in these two papers. Therefore,

$$\pi_i(Sing_*(X)(Y)) = H_i(N(Z[Sing_*(X)(Y)])) = H_i(Y, C_*(X)) = Hom_{H^{\mathbf{A^1}}(k)}(S^i \wedge Y, Sing_*(X))$$

and the result comes from the *Dold-Kan* realization applied to the simplicial sheaves $Y \longmapsto Sing_*(X)(Y)$.

Remark 3.2.3 *The functor $Sing_*$ sends the sheaf X to the representable simplicial presheaf $Sing_*(X)$. If we identify sheaves with corresponding representable sheaves, we can ignore the functor $Sing_*$ and get*

$$Hom_{DM_{gm}^{eff}(k)}(M_{gm}(Y)[n], M_{gm}(X)) \simeq H_n(Y, C_*(X)) \simeq Hom_{H^{\mathbf{A^1}}(k)}(S^n \wedge Y, X).$$

3.3 Motivic Eilenberg-MacLane Spaces and Motivic Cohomology

As stated in **Remark 2.4.1**, for an object $E \in SH_T$, we assign a bi-graded object, which is a cohomology theory $E^{p,q}(-)$. In this subsection, we will define the motivic Eilenberg-MacLane spectra H_Z, and the motivic cohomology theory $H_Z^{p,q}(-) = H^p(-, Z(q))$.

Definition 3.3.1 *Let X be a pointed set, and let $Symm^\infty X$ be the infinite symmetric product of X, that is to say, $colim(X^n/\Sigma_n)$. The map $Symm^n \longrightarrow Symm^{n+1}$ send $(x_1, \ldots, x_n)$ to $(x_1, \ldots, x_n, *)$. $Symm^\infty X$ has a commutative monoid structure and we note $(Symm^\infty(X))^+$ the symmetric abelian group associated. This definition can be extended to the category of simplicial sets.*

Theorem 3.3.1 *Let $X \in \Delta^{op}Set_*$. Then there exists an isomorphism (non-canonic) in Ho_*^{top}*

$$(Symm^\infty(X))^+ \simeq \Pi_n K(H_n(X; Z), n)$$

Lemma 3.3.1 *Given a pointed set (X, x), there exists an isomorphism of abelian groups*

$$(Symm^\infty(X, x))^+ = Z[X]/Z[x].$$

Remark 3.3.1 *Given this lemma, when we considering the category of* **finite correspondences**, *we can see:*

when $X_+ \in Cor(k)_\bullet$, $L(X_+)$ plays the role of $(Symm^\infty(X_+))^+$.

Actually, let us consider the pointed scheme $X_+ = X/x$, and $Z[(X_+)] = Z[(X/x)] = \{\Sigma_i n_i[x_i - x] | x_i \in X, n_i \in Z\}$, which is a linear combination generated by $\{[x_i] - [x],\ x_i \in X \backslash x\}$. A point in $L(X_+)$ is an element in $Hom(pt, L(X_+)) = Cor(spec(k), X_+)$ is by definition a formal linear combination of $\{[x_i] - [x],\ x_i \in X \setminus x\}$, which is exactly one could expect from points of the infinite symmetric products according to our lemma.

To define motivic Eilenberg-MacLane spaces we apply the *infinite symmetric products* to the Morel-Voevodsky object $T = S^1 \wedge \mathbf{G}_m$. We say that a sheaf $X \in Spc_*$ is *scheme-like* if it is isomorphic to a pointed sheaf of the form $X/(\bigcup_{i=1}^n Z_i)$ where X is a smooth scheme and Z_i's are smooth sub-schemes in X such that all the intersections of Z_i's are still smooth. The class of scheme-like sheaves includes in particular the spaces $S^n \wedge \mathbf{G}_m^m$. We then set

$$L(X/(\bigcup_{i=1}^n Z_i)) = (L(X)/\Sigma_{i=1}^n L(Z_i))$$

Definition 3.3.2 (Motivic Eilenberg-MacLane Spaces) *We pose*

$$K(Z(n), 2n) = L((\mathbf{P^1}, \infty)^{\wedge n}),$$

and define the Eilenberg-MacLane spectrum H_Z to be the $(\mathbf{P^1}, \infty)$-spectra such that $(H_Z)_n = K(Z(n), 2n)$. Let $i : (\mathbf{P^1}, \infty) \longrightarrow L((\mathbf{P^1}, \infty))$ be the canonical morphism, we get the bonding map of the Eilenberg-MacLane spectrum

$$e_n : (\mathbf{P^1}, \infty) \wedge K(Z(n), 2n) \longrightarrow K(Z(n+1), 2n+2)$$

According to **Definition 3.41**, we have $Z(n) = [\widetilde{(\mathbf{P^1})^{\wedge n}}]\ [-2n]$.

Definition 3.3.3 (Motivic Cohomology) *For $X \in Sm/k$, $q \geq 0$, set*

$$H_Z^{p,q}(X) = H^p(X, Z(q)) = Hom_{DM_{gm}^{eff}(k)}(M_{gm}(X), Z(q)[p])$$

Remark 3.3.2 *: Notice that $C_*(X) = C_*(L(X)),\ for\ X \in Sm/k$, and we abuse the notation by identifying $X \in Sm/k$ and $L(X) \in Sh^{Nis}(Sm/k)$, and denote $i \circ M_{gm}(X)$*

after composing the embedding functor i. So, according to the **Corollary 4.2**, *we have: for $X, Y \in Sm/k$*

$$H^n(Y_{Nis}, C_*(L(X))) = Hom_{DM_-^{eff}(k)}(i \circ M_{gm}(Y), i \circ M_{gm}(L(X))[n]).$$

By using this observation to motivic cohomologies, and consider pointed schemes, we have:

$$H^p(X, Z(q)) = Hom_{DM_{gm}^{eff}(k)}(M_{gm}(X), Z(q)[p])$$

$$= Hom_{DM_-^{eff}(k)}(i \circ M_{gm}(X), i \circ M_{gm}(L(\mathbf{P^1})^q)[p-2q])$$

$$= Hom_{H\mathbf{A^1}(k)}(S_s^{2q-p} \wedge X, L(\mathbf{P^1})^q)) = Hom_{H\mathbf{A^1}(k)}(S_s^{2q-p} \wedge X, (H_Z)_q)$$

and if the $\mathbf{P^1}$-spectrum H_Z is an $\Omega_{\mathbf{P^1}}$-spectra, by applying stabilization functor, we have further equalities:

$$= Hom_{SH^{\mathbf{P^1}}(k)}(S_s^{2q-p} \wedge (\Sigma_{\mathbf{P^1}}^{\infty} X), H_Z[q])$$

$$= Hom_{SH^{\mathbf{P^1}}(k)}(\Sigma_{\mathbf{P^1}}^{\infty} X, H_Z \wedge S_s^{p-q} \wedge S_t^q).$$

Lemma 3.3.2 *If k is a perfect field, the $\mathbf{P^1}$-spectrum H_Z is an $\Omega_{\mathbf{P^1}}$-spectrum.*

proof: It is enough to show that

$$Hom_{H\mathbf{A^1}(k)}(S^i \wedge X, K(Z(n)), 2n) \longrightarrow Hom_{H\mathbf{A^1}(k)}(\mathbf{P^1} \wedge S^i \wedge X, K(Z(n+1)), 2n+2)$$

is bijective. According to **Corollary 3.4**, this morphism is equivalent to:

$$Hom_{DM_{gm}^{eff}(k)}(M_{gm}(X), Z(n)[2n-k]) \longrightarrow Hom_{DM_{gm}^{eff}(k)}(M_{gm}(\mathbf{P^1} \wedge X), Z(n+1)[2n-k+2])$$

which is equivalent to:

$$Hom_{DM_{gm}^{eff}(k)}(M_{gm}(X), Z(n)[2n-k]) \longrightarrow Hom_{DM_{gm}^{eff}(k)}(M_{gm}(\mathbf{P^1} \wedge X)[-2], Z(n+1)[2n-k]).$$

If we let $Y = Z(n)[2n-k]$ and notice that $Z[1] \overset{\sim}{=} \mathbf{P^1}[-2]$, the above morphism is further equivalent to:

$$Hom_{DM_{gm}^{eff}(k)}(M_{gm}(X), Y) \overset{-\otimes Z(1)}{\longrightarrow} Hom_{DM_{gm}^{eff}(k)}(M_{gm}(X) \otimes Z[1], Y \otimes Z[1]).$$

This is actually a bijection according to the **cancellation theorem**.

Proposition 3.3.1 *If k is a perfect field, $X \in Sm/k$ there are canonical isomorphisms:*

$$H_Z^{p,q}(X) \simeq Hom_{SH^{\mathbf{P^1}}(k)}(\Sigma_{\mathbf{P^1}}^{\infty}(X), H_Z \wedge S_s^{p-q} \wedge S_t^q) \simeq Hom_{DM_{gm}^{eff}(k)}(M_{gm}(X), Z(q)[p]).$$

Chapter 4

Appendix

Appendix A

Model Categories

Here I will review simplicial techniques which will be used in the rest of the article. These include categories and simplicial construction, model category and abstract homotopy theory.

A.1 Model Category

Definition A.1.1 *Let $f : A \longrightarrow B$ and $g : X \longrightarrow Y$ be two morphisms, if there is a lifting diagram:*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & Y \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

then we say that f has the Left Lifting Property (LLP) with respect to g, and g has the Right Lifting Property (RLP) with respect to f.

Definition A.1.2 *Let $\mathcal{C}$ a category equipped with three classes of morphisms $\mathcal{W} = weak\ equivalences$, $Fib = fibrations$, $Cof = cofibrations$. We usually write $\{\mathcal{W}\} = \{\simeq\}, \{Fib\} = \{\Longrightarrow\}\ and\ \{Cof\} = \{\hookrightarrow\}$. We call the class $\mathcal{W} \cap Fib$ the acyclic fibrations and the class $\mathcal{W} \cap Cof$ the acyclic cofibrations. Then $\mathcal{C}$ is called a model category if $(\mathcal{C}, \mathcal{W}, Fib, Cof)$ satisfy:*

CM1 *$\mathcal{C}$ has all small limits and colimits*

CM2 *$\mathcal{W}$ satisfies the condition of two of three, that is: when the composition of two morphisms $f\ and\ gf$ is defined and any two of the three morphisms $f,\ g,\ and\ f \circ g$ are weak equivalences then the third one is*

CM3 *$\mathcal{W}, Fib\ and\ Cof$ are closed under retraction,*

CM4 *Any fibration has the Right Lifting Property (RLP) with respect to acyclic cofibrations, and any acyclic fibration has the RLP with respect to cofibrations,*

CM5 *Any morphism can be factorised into the composition of a cofibration and an acyclic fibration, which can be denoted by* $\hookrightarrow \cdot \overset{\sim}{\Longrightarrow}$, *and an acyclic cofibration and a fibration, which can be denoted by* $\overset{\sim}{\hookrightarrow} \cdot \Longrightarrow$.

One of the advantages of model category is that it gives us a way of inverting the morphisms in $\mathcal{W}$.

Definition A.1.3
We say that $f, g : X \longrightarrow Y$ *are left homotopic if there is a cylinder object Cyl(X):* $X \amalg X \hookrightarrow Cyl(X) \overset{\sim}{\longrightarrow} X$ *and there exists* $H : Cyl(X) \longrightarrow Y$ *such as the following diagram:*

$$\begin{tikzcd} X \arrow[dr, "i_0"'] \arrow[drr, "f"] & & \\ & Cyl(X) \arrow[r, "H"] & Y \\ X \arrow[ur, "i_1"] \arrow[urr, "g"'] & & \end{tikzcd}$$

commutes.

2. *We say that* $f, g : X \longrightarrow Y$ *are right homotopic if there is a path object* $\alpha(Y)$*:* $Y \overset{\sim}{\hookrightarrow} Path(Y) \Longrightarrow Y \times Y$ *and there exists* $B : X \longrightarrow Pat(Y)$ *such that the diagram:*

$$\begin{tikzcd} & & Y \\ X \arrow[urr, "f"] \arrow[r, "B"] \arrow[drr, "g"'] & Pat(Y) \arrow[ur] \arrow[dr] & \\ & & Y \end{tikzcd}$$

commutes.

Proposition A.1.1 *Let* $\mathcal{C}$ *be a model category,* $f,\ g : A \to X$ *be two morphisms in* $\mathcal{C}$. *Suppose that* A *is cofibrant and* X *is fibrant, and*

$$A \coprod A \hookrightarrow Cyl(X) \sim X; \qquad (resp.\ \ X \overset{\sim}{\hookrightarrow} Path(X) \longrightarrow X \times X)$$

are chosen cylinder object of A (resp. path object of X). Then the following conditions are equivalent:

Remark A.1.1 *According to* **CM5**, *there are always special kinds of factorization. Let 0 the initial object and $*$ the final object in the model category. Then: $0 \hookrightarrow QX \overset{\sim}{\Longrightarrow} X$ gives us a QX which is called the cofibrant replacement of X, $Y \overset{\sim}{\hookrightarrow} RY \Longrightarrow *$ gives us a RY which is called the fibration replacement of Y.*

Further let us denote by $\mathcal{C}_{c,f}$ the full subcategory of $\mathcal{C}$ whose objects are both fibrant and cofibrant. Then the $\pi(\mathcal{C}_{c,f})$ is a category whose objects are objects of $\mathcal{C}_{c,f}$, and morphisms are:

$$Mor_{\pi(\mathcal{C}_{c,f})}(X,Y) = Mor_{\mathcal{C}_{c,f}}(X,Y)/\sim$$

where $\sim$ is the weak equivalence relation. The homotopy category $Ho(\mathcal{C})$ is the category whose objects are objects of $\mathcal{C}$ and $Mor_{Ho(\mathcal{C})}(X,Y) = Mor_{\mathcal{C}}(RQX, RQY)/\sim = Mor_{C_{c,f}}(X,Y)/\sim = Mor_{\mathcal{C}}(QX, RY)/\sim$. Then denote $[X,Y] = Mor_{Ho(\mathcal{C})}(X,Y)$.

A.2 Simplicial Sets

Definition A.2.1 *Let Δ be a category with finite ordinals as objects, thought of as totally ordered sets which are denoted by $[n]$, $n \in Z$, and the morphisms in Δ are order preserving functions. A simplicial set is a contravariant functor from Δ to the category of sets denoted by $\mathcal{S}$.*

We define the standard simplicial n-simplex by $\Delta[n]_p = Hom_{\Delta}([p],[n])$. By Yoneda Lemma $X_n = Hom_{set}(\Delta[n], X)$ for any X a simplicial set. Note that $\Delta[n]$ is a simplicial set, but the assignment $[n] \longrightarrow \Delta[n]$ is cosimplicial.

The model category structure on $\mathcal{S}$ is given by: $\{Cof\} = \{monomorphisms\}$ and $f \in \mathcal{W}$ if and only if $\mid f \mid : \mid X \mid \longrightarrow \mid Y \mid$ is a weak equivalence. Moreover, we can conclude that all objects are cofibrants.

We denote $\Delta^{op} \longrightarrow Set$ by $Set^{\Delta^{op}}$. Let c the constant functor from $\mathcal{S} \longrightarrow Set^{\Delta^{op}}$: $c(X)_n = X$ for any n. Then there exist two functors which are adjoint to this constant functor:

- $()_0 : X \longrightarrow X_0$ is right adjoint to c,
- $\pi_0 : X \longrightarrow \pi_0(X)$ is left adjoint to c.

So according to Quillen, if $X \in C$ is fibrant, $\pi_0(X) = X_0/\sim$ where the equivalence relation is defined as: $X \sim Y$ if and only if there exists $Z \in X_1$ such that $d_0 Z = X, d_1 Z = Y$. $\pi_n(X) = X_n^{'}/\sim$ where $X_n^{'} = \{\alpha \in X_n$ who send $\partial\Delta[n]$ to $\upsilon : \Delta[0] \longrightarrow X$. If X is fibrant, then $\pi_0(X) = \pi_0(|X|)$ and for any $\upsilon \in X_0, |\upsilon| \in |X|$, and $\pi_n(X,\nu) = \pi_n(|X|,|\upsilon|)$. So we have : $f : X \longrightarrow Y$ between simplicial sets are equivalent if and only if $\pi_0(X) = \pi_0(|X|)\ and\ \pi_n(X,\nu) = \pi_n(|X|,|\upsilon|)$ for any $\upsilon \in X_0$.

Definition A.2.2 *$F : \mathcal{C} \longrightarrow \mathcal{D}$ between model categories is called a left Quillen functor if*

1. *F has a right adjoint U*
2. *F preserves cofibrations and acyclic cofibrations*

Respectively, $U : \mathcal{D} \longrightarrow \mathcal{C}$ is a right Quillen functor if

1. *U has a left adjoint F,*
2. *U preserves fibrations and acyclic fibrations.*

A pair (F, U) satisfying the above definition is called a Quillen pair.

Definition A.2.3 *Let $\mathcal{C}$ a model category, $p : \mathcal{C} \longrightarrow Ho(\mathcal{C})$ the canonical projection and $F : \mathcal{C} \longrightarrow \mathcal{D}$ a functor from C to any category D:*

1. *A left derived functor of F is a pair $(LF : Ho(\mathcal{C}) \longrightarrow \mathcal{D}, t : LF \circ p \longrightarrow F)$ has the* **left** *universal property: for all the other pairs (G, s) there exists a unique transformation functor $s^{'} : G \longrightarrow LF$ such that s factorises through t,$s = (s^{'} \circ p) \circ t$*
2. *A right derived functor of F is a pair $(RF : Ho(\mathcal{C}) \longrightarrow \mathcal{D}, t : F \longrightarrow RF \circ p)$ has the* **right** *universal property: for all other pairs (G, s) there exists a unique transformation functor $s^{'} : RF \longrightarrow G$ such that s factorises through t, $s = t \circ (s^{'} \circ p)$*

Definition A.2.4 *A left total derived functor is a functor $\mathbf{L}F : Ho(\mathcal{C}) \longrightarrow Ho(\mathcal{D})$ such that $\mathbf{L}F(X) = F(QX)$ A right total derived functor is a functor $\mathbf{R}U : Ho(\mathcal{D}) \longrightarrow Ho(\mathcal{C})$ such that $\mathbf{R}U(X) = U(RX)$*

If (F, U) is a pair of Quillen adjoinction, then $(\mathbf{L}F, \mathbf{R}U)$ is a pair of adjoinction.

Definition A.2.5 *A Quillen pair (F, U) is called a Quillen equivalence if the pair $(\mathbf{L}F, \mathbf{R}U)$ defines an equivalence of categories. Equivalently, (F, U) is a Quillen equivalence if and only if for any $X \in \mathcal{C}_{Cof}$ and $Y \in \mathcal{D}_{Fib}$, $FX \simeq Y \in \mathcal{D} \Longleftrightarrow X \simeq UY \in \mathcal{C}$.*

A.3 Homotopy Limits and Homotopy Colimits

Lets recall *the definition of limits and colimits*: Let F be a functor in $\mathcal{C}^{\mathcal{D}}$ where $\mathcal{C}$ is a category and $\mathcal{D}$ is a small category. Denote by c the constant functor $c : \mathcal{C} \longrightarrow \mathcal{C}^{\mathcal{D}}$ such that $c(X)$ sends objects in $\mathcal{D}$ to X and all morphisms in $\mathcal{D}$ to Id_X.

1. A colimit of F is an object $colimF \in \mathcal{C}$ and a natural transformation $t : F \longrightarrow c(colimF)$ such that for all $X \in \mathcal{C}$ and all natural transformation $s : F \longrightarrow c(X)$ there exists an unique morphism $s^{'} : colimF \longrightarrow X$ in $\mathcal{C}$ and a commutative diagram $s = t \circ c(s^{'})$.

2. A limit of F is an object $limF \in \mathcal{C}$ and a natural transformation $t : c(limF) \longrightarrow F$ such that for all $X \in \mathcal{C}$ and all natural transformation $s : c(X) \longrightarrow F$ there exists an unique morphism $s^{'} : X \longrightarrow limF$ in $\mathcal{C}$ and a commutative diagram $s = c(s^{'}) \circ t$.

When $\mathcal{C}$ is cocomplete, the universal property of and $colimF$ give us a left adjoint of c:

$$colim : \mathcal{C}^{\mathcal{D}} \leftrightarrow \mathcal{C} : c$$

and when C is complete, the universal property of $limF$ give us a right adjoint:

$$c : \mathcal{C} \leftrightarrow \mathcal{C}^{\mathcal{D}} : limF$$

Notice that in general the functors lim and $colim$ do not preserve weak equivalences, so by previous subsection we introduce the notion of *homotopy limits and homotopy colimits*. Of course we can define according to the above adjoinction between c and $lim, colim$ as Quillen functors and induce adjoinctions of total derived functors. We can also describe them as follows:

A.4 Simplicial Model Category

Definition A.4.1 *A model category $\mathcal{C}$ is called a simplicial model category if there exists*

1. $- \otimes - : \mathcal{S} \times \mathcal{C} \longrightarrow \mathcal{C}$

2. $S : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathcal{S}$

3. $\mathcal{S}^{op} \times \mathcal{C} \longrightarrow \mathcal{C}$ *sends* (K, Y) *to* Y^K

such that:

$$Mor_{\mathcal{C}}(K \otimes X, Y) \simeq Mor_{\mathcal{C}}(X, Y^K) \simeq Mor_{\mathcal{S}}(K, S(X,Y))$$

plus the compatibility with the model structure: if

$$f : K \hookrightarrow L \in \mathcal{S}, g : X \hookrightarrow Y \in \mathcal{C}, \text{ then } f \diamond g : L \otimes X \amalg_{K \otimes X} K \otimes Y \hookrightarrow L \otimes Y$$

If moreover either f or g is acyclic then $f \diamond g$ is acyclic.

Passing to the homotopy category and using the total derived functors, we get:

1. $- \overset{\mathbf{L}}{\otimes} - : Ho(\mathcal{S}) \times Ho(\mathcal{C}) \longrightarrow Ho(\mathcal{C})$
2. $\mathbf{R}\underline{Hom} : Ho(\mathcal{C})^{op} \times Ho(\mathcal{C}) \longrightarrow Ho(\mathcal{S})$
3. $Ho(\mathcal{S})^{op} \times Ho(\mathcal{C}) \longrightarrow Ho(\mathcal{C})$ sends (K, Y) to $Y^{\mathbf{R}K}$

Remark A.4.1 *if $\mathcal{C}$ is a simplicial model category, one has: $Map(X,Y) = \mathbf{R}\underline{Hom}(X,Y) = \underline{Hom}(QX, RY)$. We have $\pi_0(Map(X,Y)) = [X,Y] = Mor_{Ho(\mathcal{C})}(X,Y)$*

A.5 Model Categories with Cofibrantly Generators

Definition A.5.1 *Let C be a category and I a collection of morphisms in C. We say that $X \in C$ is small with respect to I, if for any chain of morphisms*

$$Y_1 \to Y_2 \to \dots$$

in I, we have a bijection $colim Hom_C(X, Y_i) = Hom_C(X, colim Y_i)$.

Definition A.5.2 *Let C be a cocomplete category and I a class of morphisms in C.*

1. *A morphism $p : X \to Y$ is called I-injective if it has the RLP with respect to all the morphisms in I. The set of I-injective morphisms will be noted as $r(I)$.*
2. *A morphism $i : A \to B$ is called an I-cofibration if it has the LLP with respect to all I-injective morphisms.*
3. *$f : A \to B$ is I-cellular if*

$$A = B_0 \to B_1 \to \dots \to colim B_i \simeq B$$

and

$$\begin{tikzcd} \amalg_\alpha U_{\alpha,i} \arrow[r] \arrow[d] & B_i \arrow[d] \\ \amalg_\alpha V_{\alpha,i} \arrow[r] & B_{i+1} \end{tikzcd}$$

is a pushout, with $U_{\alpha,i} \to V_{\alpha,i} \in I$.

Proposition A.5.1 (Argument of Small Objects) *Let C be a cocomplete category, I be a set of morphisms in C. Suppose that the source of the morphisms in I are small with respect to the class of I-cellular morphisms. Then all the morphisms $f : X \to Y$ in C factorizes functorially as:*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[rr, bend right, "f"'] & Z \arrow[r, "p"] & Y \end{tikzcd}$$

where i is cellular and p is I-injective.

proof:

Let $f : X \to Y \in C$. Let S_0 be the set (cause I is a set) of diagrams of the form:

$$S_0 = \left\{ \begin{tikzcd} A_i \arrow[r] \arrow[d, "f_i"] & X = Z_0 \arrow[d, "f"] \\ B_i \arrow[r] & Y \end{tikzcd} , f_i \in I \right\}$$

Construct the pushout Z_1 of the diagram:

$$\begin{tikzcd} \amalg_{s\in S_0} A_s \arrow[r] \arrow[d] & X \arrow[d, "i_0"'] \arrow[ddr] & \\ \amalg_{s\in S_0} B_s \arrow[r] \arrow[drr] & Z_1 \arrow[dr, dashed] & \\ & & Y \end{tikzcd}$$

We repeat this procedure by replacing X by Z_1m Z_1 by Z_2, etc... Then consider the diagram:

$$\begin{tikzcd} X \arrow[r, "i_0"] \arrow[rrr, bend left, "i"] \arrow[d, "f"] & Z_1 \arrow[r, "i_1"] \arrow[d] & \cdots \arrow[r] & colim Z_i =: Z . \arrow[d, "p"] \\ Y \arrow[r, equal] & Y \arrow[r, equal] & \cdots \arrow[r, equal] & Y \end{tikzcd}$$

Since by definition each morphism $Z_i \to Z_{i+1}$ is I-cellular, $i : X \to Z$ is I-cellular as well. It leaves us to prove that p is I-injective. Thus we consider the diagram:

$$\begin{tikzcd} A_i \arrow[r] \arrow[d, "f_i"] & Z \arrow[d, "p"] \\ B_i \arrow[r] & Y \end{tikzcd} , with\ f_i \in I.$$

Since A_i is small with respect to the class of I-cellular morphisms, the morphism $A_i \to Z$ factorizes by certain Z_k as $A \to Z_k \to Z$. However, the diagram

$$\begin{tikzcd} A_i \arrow[r] \arrow[d] & Z_k \arrow[d] \\ B_i \arrow[r] & Y \end{tikzcd}$$

belongs to S_k and by definition belongs to S_{k+1}, so there exists a lifting diagram:

$$\begin{tikzcd} A_i \arrow[r] \arrow[d] & Z_k \arrow[r] \arrow[d] & Z_{k+1} \arrow[r] \arrow[d] & Z \arrow[d] \\ B_i \arrow[r] \arrow[urr, dashed] & Y \arrow[r, equal] & Y \arrow[r, equal] & Y \end{tikzcd} .$$

Lemma A.5.1 *Let C be a cocomplete category and I a class of morphisms in C.*

1. *All I-cellular morphisms are I-cofibrations.*
2. *Under the hypothesis of the argument of small objects, each I-cofibration is a retract of an I-cellular morphism.*

proof:

For 1., note that by definition, we have $I \subset I$-cofibration. Since the class I-cofibration is defined by LLP, it is stable by compositions and pushouts, and then I-cell$\subset I$-cof.

For 2., let $f : X \to Y \in I$-cof. By the argument of small objects, there exists a factorization:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[rr, bend right, "f"'] & Z \arrow[r, "p"] & Y \end{tikzcd}$$

with i an I-cellular morphism and p an I-injective morphism. The diagram:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"] & Z \arrow[d, "p"] \\ Y \arrow[r, equal] \arrow[ur, dashed, "\phi"] & Y \end{tikzcd}$$

admits a lifting $\phi \in I$-cof and $p \in I$-inj. We thus have a diagram of retract:

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "f"] & X \arrow[r, equal] \arrow[d, "i"] & X \arrow[d, "f"] \\
Y \arrow[r, "\phi"] & Z \arrow[r, "p"] & Y
\end{tikzcd} .$$

Definition A.5.3 *We say a model category C is cofibrantly generated, if there exists collections of morphisms $I,\ J$, such that*

1. *The sources of I are small with respect to all I-cells*
2. *The sources of J are small with respect to all J-cells*
3. *The class of fibrations in C is the class of J-injective morphisms*
4. *The class of morphisms which are both fibration and weak equivalence in C is the class of I-injective morphisms.*

The morphisms in I is called the cofibration generators, and the morphisms in J are called acyclic cofibration generators.

Example A.5.1 *Let $\mathcal{C} = \mathcal{S}$, then $I = \{\partial\Delta[n] \longrightarrow \Delta[n]\}$ and $J = \{\wedge^r[n] \longrightarrow \Delta[n]\}$.*

Let us consider the general case of localization: $\mathcal{C}$ is a model category and $\mathcal{W}$ is the class of equivalence. Let S be a class of morphisms. We want to equip $\mathcal{C}$ with a new model structure $(\mathcal{C}, \mathcal{W}')$ such that $\mathcal{W}' \supset \{\mathcal{W}, S\}$.

Definition A.5.4

1. *Let $X \in \mathcal{C}$, we say that X is S-local if and only if*
   - *X is fibrant*
   - *$\forall\ Y \longrightarrow Y' \in S, Mor(Y', X) \longrightarrow Mor(Y, X)$ is a weak equivalence.*
2. *$f : U \longrightarrow V$ is an S-equivalence if $\forall\ X\ which\ is\ S - local, Mor(V, X) \longrightarrow Mor(U, X)$ is a weak equivalence.*

Theorem A.5.1 (Bousfield) *The three classes of morphisms*

- $\mathcal{W}_S = \{S - equivalences\}$
- $Cof_S = Cof = \{cofibrations\ in\ \mathcal{C}\}$
- $Fib_S = \{morphisms\ having\ the\ RLP\ with\ respect\ to\ acyclic\ cofibrations\}$

define a new model structure on $\mathcal{C}$.

Definition A.5.5 *Let us denote $L_S\mathcal{C}$ the localization of Bousfield with respect to the full subcategory S. Then:*

$$Id : \mathcal{C} \leftrightarrow L_S\mathcal{C} : Id$$

is a pair of Quillen adjoinction which gives us a pair of total derived functors:

$$a = \mathbf{L}id : Ho(\mathcal{C}) \longrightarrow Ho(L_S\mathcal{C}) : j = \mathbf{R}id$$

Remark A.5.1 *:*

1. $Cof(\mathcal{C}) = Cof(L_S\mathcal{C})$, $\mathcal{W}(\mathcal{C}) \cap Cof(\mathcal{C}) \subset \mathcal{W}(L_S\mathcal{C}) \cap Cof(L_S\mathcal{C})$, $\mathcal{W}(\mathcal{C}) \subset \mathcal{W}(L_S\mathcal{C})$, $Fib(\mathcal{C}) \supset Fib(L_S\mathcal{C})$, $\mathcal{W}(\mathcal{C}) \cap Fib(\mathcal{C}) = \mathcal{W}(L_S\mathcal{C}) \cap Fib(L_S\mathcal{C})$.

2. $\{fibrant\ objects\ in\ L_S\mathcal{C}\} = \{objects\ which\ are\ S - local\}$.

A.6 Simplicial Presheaves

Let $\mathcal{C}$ be a category, the category of simplicial presheaves is the category of contravariant functors: $\mathcal{C}^{op} \longrightarrow \mathcal{S}$ and denote by $sPsh(\mathcal{C}) = \mathcal{S}^{\mathcal{C}^{op}}$.

Let $(\mathcal{M}, I, J)$ be a cofibrantly generated model category, we want to equip $\mathcal{M}^{\mathcal{C}}$ with a model structure.

Projective Structure on $\mathcal{M}^{\mathcal{C}}$:

let $f : X \longrightarrow Y$ be an equivalence(resp. fibration) if it is an equivalence(resp. fibration) term by term. We have the following theorem:

Theorem A.6.1 *For any $c \in \mathcal{C}$, $ev_c : \mathcal{M}^{\mathcal{C}} \longrightarrow \mathcal{M}$ sends X to $X(c)$. Then $i_c : \{0\} \longrightarrow \mathcal{C}$ defines a morphism $i_c^* : \mathcal{M}^{\mathcal{C}} \longrightarrow \mathcal{M}^{\{0\}} = \mathcal{M}$ equals to ev_c has a left adjoint*

$$i_c! : i_c!(A) = F_A^c : \mathcal{C} \longrightarrow \mathcal{M} : i_c^*$$

where $F_A^c(d) = \amalg_{Mor_{\mathcal{C}}(d,c)} A$. If I (resp. J) are cofibration generators (resp. acyclic cofibration generators) of the model category $\mathcal{M}$ and $u : A \longrightarrow B$ is a morphism in $\mathcal{M}$, then let $i_c!(u) = F_u^c : F_A^c \longrightarrow F_B^c$ then $\{F_u^c\ such\ that\ c \in \mathcal{C}, u \in I\}$ is the set of cofibration generators of the projective structure of the model category of $\mathcal{M}^{\mathcal{C}}$ (resp. J).

Respectively, we can define an injective structure by letting $f : X \longrightarrow Y$ be an weak equivalence (resp. cofibration) if and only if for any $c \in \mathcal{C}, X(c) \longrightarrow Y(c)$ is a weak equivalence (resp. cofibration).

Proposition A.6.1 *If $f : X \longrightarrow Y$ is a cofibration in M^C with the projective structure, then it is a cofibration term by term.*

Proposition A.6.2 *If*

$$F : \mathcal{M} \leftrightarrow \mathcal{N} : U$$

is a pair of Quillen adjoinction or Quillen equivalence, then the pair of induced functors

$$F^{\mathcal{C}} : \mathcal{M}^{\mathcal{C}} \longrightarrow \mathcal{N}^{\mathcal{C}} : G^{\mathcal{C}}$$

is a pair of Quillen adjoinction or Quillen equivalence.

Let $\mathcal{C}$ a category, by Yoneda lemma we can identify any $X \in \mathcal{C}$ with a presheaf $h_X = Hom_{\mathcal{C}}(*, X)$. Therefore, all functors $\alpha : \mathcal{C} \longrightarrow \mathcal{D}$ where $\mathcal{D}$ is cocomplete can be factorised by $\beta : Psh(\mathcal{C}) \longrightarrow \mathcal{D}$ who preserves colimits.

We have known the constant functor $c : Set \longrightarrow \mathcal{S}$ has two adjoint functors $()_0$ and π_0. Then by passing to the level of simplicial presheaves we get two pairs of adjoint functors:

$$\pi_0 : sPsh(\mathcal{C}) \leftrightarrow Psh(\mathcal{C}) : c$$

and:

$$c : Psh(\mathcal{C}) \leftrightarrow sPsh(\mathcal{C}) : ()_0$$

Now, by Yoneda funtor and the constant functor c we can identify any $X \in \mathcal{C}$ with a simplicial sheaf h_X. So all functors $\mathcal{C} \longrightarrow M$ where M is a model category may be factorised by $\mathcal{C} \longrightarrow sPsh(\mathcal{C})_{proj} = U\mathcal{C}$ and makes the diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[dr] & U\mathcal{C} \arrow[d, shift right] \\ & \mathrm{M} \arrow[u, shift right] \end{tikzcd}$$

Commutative.

A.7 Model Category with Monoidal Structure

Definition A.7.1 *A symmetric monoidal structure on a model category C is the data of two functors:*

1. $- \otimes - : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$
2. $\underline{Hom} : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathcal{C}$

satisfying functorially:

$$Mor_{\mathcal{C}}(K \otimes X, Y) \simeq Mor_{\mathcal{C}}(K, \underline{Hom}(X, Y))$$

plus the compatibility: if

$f : K \hookrightarrow L \, and \, g : X \hookrightarrow Y are \, cofibrations, \, then \, f \diamond g : L \otimes X \amalg_{K \otimes X} K \otimes Y \hookrightarrow L \otimes Y$

Moreover if either f or g is acyclic then $f \diamond g$ is acyclic.

Passing to the homotopy category we obtain:

- $- \overset{\mathbf{L}}{\otimes} - : Ho(\mathcal{C}) \times Ho(\mathcal{C}) \longrightarrow Ho(\mathcal{C})$
- $\mathbf{R}\underline{Hom} : Ho(\mathcal{C})^{op} \times Ho(\mathcal{C}) \longrightarrow Ho(\mathcal{C})$

Thus $(Ho(\mathcal{S}), - \overset{\mathbf{L}}{\otimes} -)$ is a symmetric monoidal category, and $\mathbf{R}\underline{Hom}(X, Y) = \underline{Hom}(QX, RY)$ and $X \overset{\mathbf{L}}{\otimes} Y = QX \otimes QY$.

Definition A.7.2 *Let $(\mathcal{M}, \otimes)$ be a symmetric monoidal model category and $\mathcal{N}$ be a model category. We say that $\mathcal{N}$ is a $\mathcal{M}$-category if:*

1. $- \otimes - : \mathcal{M} \times \mathcal{N} \longrightarrow \mathcal{N}$
2. $\underline{Hom} : \mathcal{N}^{op} \times \mathcal{N} \longrightarrow \mathcal{M}$
3. $\mathcal{M}^{op} \times \mathcal{N} \longrightarrow \mathcal{N}$ *sends* (K, Y) *to* Y^K

and there are adjoinctions:

$$Mor_{\mathcal{N}}(K \otimes X, Y) \simeq Mor_{\mathcal{N}}(X, Y^K) \simeq Mor_{\mathcal{M}}(K, \underline{Hom}(X, Y))$$

plus the compatibility: if

$$f : K \longrightarrow L \in \mathcal{M}, g : X \longrightarrow Y \in \mathcal{N} f \diamond g : L \otimes X \amalg_{K \otimes X} K \otimes Y \longrightarrow L \otimes Y$$

then f or g is a cofibration $\Longrightarrow$ $f \diamond g$ is a cofibration, moreover if f or g is acyclic $\Longrightarrow$ $f \diamond g$ is acyclic.

Moreover, we could again pass the definition to the level of homotopy category.

Proposition A.7.1 *Let $\mathcal{C}$ be a category and $\mathcal{M}$ a symmetric monoidal model category which is cofibrantly generated, then $\mathcal{M}^{\mathcal{C}}$ has a projective structure and is an $\mathcal{M}$-model category.*

Example A.7.1 *As $\mathcal{S}$ is a model category with symmetric monoidal structure and is cofibrantly generated, so the category of simplicial sheaves $sPsh\mathcal{S}$ has a projective structure and is a $\mathcal{S}$-category with model structure,that is, a simplicial model category. The projective structure on $sPsh(\mathcal{C})$ is given by:*

- $i_X^* = ev_X : sPsh(\mathcal{C}) \longrightarrow \mathcal{S}$ *realized by* $i_X^*(\mathcal{F}) = ev_X(\mathcal{F}) = \mathcal{F}(X) = \underline{Hom}(h_X, \mathcal{F})$
- *the adjoint of* $i_X^* = i_X! : \mathcal{S} \longrightarrow sPsh(\mathcal{C})$ *realized by* $i_X!(K) = K \otimes h_X \simeq K \otimes X$

and $\mathcal{F} \longrightarrow \mathcal{G}$ is a fibration in $sPsh(\mathcal{C})$ if and only if it is a fibration term by term, and if and only if the diagram:

$$\begin{tikzcd} \wedge^r[n] \arrow[r] \arrow[d] & \mathcal{F}(X) \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur] & \mathcal{G}(X) \end{tikzcd}$$

has a lifting $\forall X \in \mathcal{C}$ and $n \geq 1, r \geq 0$.

As we showed previously, the Yoneda functor and the constant functor identify a category $\mathcal{C}$ as a fully subcategory of the category of simplicial sheaves. We call the simplicial sheaf h_X for $X \in \mathcal{C}$ the constant simplicial sheaf, and it is fibrant. (Use the above diagram to see this fact.)

Let $\mathcal{F} \in sPsh(\mathcal{C})$. According to the simplicial model category structure $\mathcal{S} : sPsh(\mathcal{C})^{op} \times sPsh(\mathcal{C}) \longrightarrow \mathcal{S}$, we have:

Lemma A.7.1 (Yoneda) $\forall X \in \mathcal{X}, and \, \forall \mathcal{F} \in sPsh(\mathcal{C})$, *we have* $F(X) = S(h_X, F)$.

proof: We know that $S(h_X, \mathcal{F})_n = Mor_{\mathcal{S}}(\Delta[n], S(h_X, \mathcal{F})) = Mor_{sPsh(\mathcal{C})}(\Delta[n] \otimes h_X, \mathcal{F}) = Mor_{sPsh(\mathcal{C})}(h_X, \mathcal{F}^{\Delta[n]})$. Consider the adjoinction

$$c : Psh(\mathcal{C}) \leftrightarrow sPsh(\mathcal{C}) : ()_0$$

we have: $Mor_{sPsh(\mathcal{C})}(h_X, \mathcal{F}^{\Delta[n]}) = Mor_{sPsh(\mathcal{C})}(ch_X, \mathcal{F}^{\Delta[n]}) \simeq Mor_{Psh(\mathcal{C})}(h_X, (\mathcal{F}^{\Delta[n]})_0) = (\mathcal{F}^{\Delta[n]})_0(X) = F_n(X)$.

A.8 Model Category Structures on Simplicial Presheaves, Prestacks and Stacks

By the previous subsection, we know that there is a projective (or injective) model structure on the category of simplicial presheaves. More precisely,:

Theorem A.8.1 (Bousfield-Kan) *The following classes of morphisms:*

1. $\mathcal{W}_{proj} = \{f : \mathcal{F} \longrightarrow \mathcal{G} \in sPsh(\mathcal{C}) \text{ such that } \forall X \in \mathcal{C}, \mathcal{F}(X) \simeq \mathcal{G}(X) \in \mathcal{S}\}$;
2. $I_{proj} = \{\partial\Delta[n] \otimes X \longrightarrow \Delta[n] \otimes X, n \geq 0, X \in \mathcal{C}\}$;
3. $J_{proj} = \{\wedge^k[n] \otimes X \longrightarrow \Delta[n] \otimes X, n \geq 0, 0 \leq k \leq n, X \in \mathcal{C}\}$

define a cofibrantly generated model structure on $sPSh(C)$, which is called the projective model structure. The weak equivalences and fibrations are defined term by term.

Actually, this theorem can be derived from **Example A 7.1**.

One can similarly define an injective structure, with weak equivalences and cofibrations defined term by term.

Next, we want to define the local structure on the category of simplicial presheaves, where the weak equivalences are defined stalk-wise.

Definition A.8.1 (Local Structure)

$\mathcal{F} \longrightarrow \mathcal{G} \in sPsh(\mathcal{C})$ *is called a local fibration if for any diagram:*

$$\begin{tikzcd} \wedge^r[n] \arrow[r] \arrow[d] & \mathcal{F}(X) \arrow[d] \\ \Delta[n] \arrow[r] & \mathcal{G}(X) \end{tikzcd}$$

there exists a covering family $\mathcal{U} = \{U_.\} \longrightarrow X$ such that for each U_i, the diagram:

$$\begin{tikzcd} \wedge^r[n] \arrow[r] \arrow[d] & \mathcal{F}(X) \arrow[r] \arrow[d] & \mathcal{F}(U_i) \arrow[d] \\ \Delta[n] \arrow[r] & \mathcal{G}(X) \arrow[r] & \mathcal{G}(U_i) \end{tikzcd}$$

admits a lifting $\Delta[n] \longrightarrow \mathcal{F}(U_i)$.

2. $\mathcal{F} \longrightarrow \mathcal{G} \in sPsh(\mathcal{C})$ *is called a local weak equivalence if they are stalk-wise weak equivalence.*

Proposition A.8.1 $\mathcal{F} \longrightarrow \mathcal{G}$ *is a local acyclic fibration (local fibration and local equivalence) if for any diagram:*

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d] & \mathcal{F}(X) \arrow[d] \\ \Delta[n] \arrow[r] & \mathcal{G}(X) \end{tikzcd}$$

there exists a covering family $\mathcal{U} = U_{\cdot} \longrightarrow X$ such that for each U_i, the diagram:

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d] & \mathcal{F}(U_i) \arrow[d] \\ \Delta[n] \arrow[r] & \mathcal{G}(U_i) \end{tikzcd}$$

admits a lifting $\Delta[n] \longrightarrow \mathcal{F}(U_i)$.

Definition A.8.2 *Let $f : E \longrightarrow B$ be a morphism of presheaves of sets. Then f is called a local epimorphism if $\forall U \in \mathcal{C}, h_U \longrightarrow B$, there exists $U_i \longrightarrow U$, such that:*

$$\begin{tikzcd} h_{U_i} \arrow[r] \arrow[d] & E \arrow[d] \\ h_U \arrow[r] & B \end{tikzcd}$$

Theorem A.8.2 *The category $sPsh(\mathcal{C})$ equipped with local weak equivalences and respectively:*

- *globally projective cofibrations*
- *globally injective cofibrations*

is a simplicial model category, which is called the projective (resp. injective) local structure.

Definition A.8.3 *Let X be a representable simplicial presheaf (i.e constant) and U a simplicial presheaf. $\mathcal{U} \longrightarrow X$ in the category of simplicial presheaves is called a hypercover, if for any n, U_n is a coproduct of representable sheaves and $\mathcal{U} \longrightarrow X$ is a local acyclic fibration.*

Definition A.8.4 *Let $\mathcal{F}$ a simplicial presheaf. We say that $\mathcal{F}$ satisfies the condition of descent with respect to the hypercover $\mathcal{U} \longrightarrow X$ if $\mathcal{F}(X) \longrightarrow holim_n \mathcal{F}(\mathcal{U}_\backslash)$ is isomorphic in $Ho(sSet)$.*

Theorem A.8.3 *The model categories $sPsh(\mathcal{C})_{proj,loc}$, $sPsh(\mathcal{C})_{inj,loc}$ are the Bousfield's localization with respect to all the hypercovers of the categories $sPsh(\mathcal{C})_{proj}$ and $sPsh(\mathcal{C})_{inj}$.*

Therefore, if we denote S the class of all the hypercovers, we could get a local projective structure on $L_S(sPsh(\mathcal{C})_{proj})$.

Definition A.8.5 *A simplicial presheaf $\mathcal{F}$ is called a stack if it satisfies the condition of descent with respect to all the hypercovers.*

Theorem A.8.4 *The fibrant objects in the local structure of $sPsh(\mathcal{C})$ are the fibrant objects for the global structure which are stacks.*

So, local fibrants objects are the stacks who are global fibrants.

The homotopy category $Ho(sPsh(\mathcal{C})_{proj.loc})$ is called the homotopy category of (pre)stacks.

Remark A.8.1 *From* **Theorem 1.1** *and* **Definition 1.12**, *if S is the set of all the hypercovers, we have a Quillen pair:*

$$Id : sPsh(\mathcal{C})_{proj} \longleftrightarrow sPsh(\mathcal{C})_{proj,loc} = L_S(sPsh(\mathcal{C})_{proj}) : Id$$

which induces adjoint functors:

$$\mathbf{L}id : Ho(sPsh(\mathcal{C})_{proj}) \longleftrightarrow Ho(sPsh(mathcalC)_{proj,loc}) : \mathbf{R}id$$

Appendix B

The Dold-Kan realization

Let G be a simplicial abelian group, we define a complex NG as follows: $(NG)_n = \bigcap_{i=0}^{n-1}(ker(d_i : G_n \longrightarrow G_{n-1}))$. The boundary map $\partial : (NG)_n \longrightarrow (NG)_{n-1}$ is defined by $\partial = (-1)^n d_n$. We let $\widetilde{\pi}_n\ (G) = Z_n(NG)/B_n(NG) = H_n(NG)$.

Proposition B.0.2 $\widetilde{\pi}_n\ (G) = \pi_n(G, e)$ *for* $n \geq 1$ *and* $\widetilde{\pi}_0\ (G) = \pi_0(G)$.

proof: Let

$[X] \in \pi_n(G, e) = \{x \in G | \partial x = \{e, \ldots, e\}\} = \{x | d_i x = e, 0 \leq i < n\} \cap \{x | d_n x = e\}$

and notice that the last formula indicates that $x \in (NG)_n\ and\ \partial x = 0$ which means $x \in Z_n(NG)$. Moreover,

$$x \sim e \Longleftrightarrow \exists w \in G_{n+1}\ such\ that\ \partial w = \{e, \ldots, e, x\}$$

which means $w \in (NG)_{n+1}$, and $\partial w = x$ for $x \in B(NG)$.

Definition B.0.6 *The homology groups of a simplicial set* X *are defined by* $H_n(X, Z) = \widetilde{\pi}_n\ (Z[X]) = H_n(NZ[X])$. *Where* $Z[X] = \{\Sigma_i n_i[x_i] | x_i \in X, n_i \in Z\}$. *The reduced homology of pointed simplicial set is defined as* $\widetilde{H}_n\ (X, Z) = \widetilde{\pi}_n\ (Z^+[X])$, *where* $Z^+[X] = \{\Sigma_i n_i[x_i] | x_i \in X, n_i \in Z, \Sigma_i n_i = 0\}$.

Definition B.0.7

1. *Let* A *be a simplicial abelian group. We associate a complex of abelian group* $\widetilde{A}$ *to* A *by letting* $\widetilde{A}_n = A_n$, *and* $\partial = \Sigma(-1)^i d_i : \widetilde{A}_n \longrightarrow \widetilde{A}_{n-1}$.

2. *Let* DA *be a sub-complex of* $\widetilde{A}$ *defined by* $(DA)_n = \oplus_{i=0}^{n-1} Im(S_i : A_{n-1} \longrightarrow A_n)$.

Theorem B.0.5

1. $NA \oplus DA \simeq \widetilde{A}$
2. *The complex* DA *is homotopic to zero.*

Corollary B.0.1

1. *The composition*
$$NA \hookrightarrow \widetilde{A} \longrightarrow \widetilde{A} / DA$$
is an isomorphism of complexes.
2. *The inclusion* $NA \hookrightarrow \widetilde{A}$ *is a quasi-isomorphism, with a retraction* $r : \widetilde{A} \longrightarrow NA$*, such that* $i \circ r = Id_{NA} \simeq Id_{\widetilde{A}}$*.*

Theorem B.0.6 (Dold-Kan) *There exists an equivalence of categories between the category of simplicial abelian groups and the category of complexes of abelian groups:*

$$N : \{Simplicial\ Ab\} \leftrightarrow \{Complexes\ Ab\} : K$$

where K *is defined as* $K(C_*)_n = \oplus_{[n] \xrightarrow{\sigma} [k]} C_k$ *such that* $k \leq n$*. Moreover, that* $NK(C_*) \simeq C_*,\ and\ KN(A) \simeq A.$

This theorem allows us to construct *Eilenberg-MacLane* spaces. Let us denote $A[n]$ the complex concentrated in degree n. Consider $C_* = A[0]$. Notice that there is a unique surjective morphism $[n] \longrightarrow [0]$, therefore we have $K(A[0])_n = A, \forall n$, and the d_i, s_i's are all identities. Further, by definition, we could also get $K(A[1])$ is the simplex satisfies:

$$K(A[n])_1 = A,\ K(A[n])_2 = A^2,\ \ldots,\ K(A[n])_m = A^m$$

That is to say, we get the *nerve* of the abelian group A. More generally, for $n \geq 0$, we define the simplicial abelian group $K(A,n) = K(A[n])$. By definition, we have

$$\pi_n(K(A,n)) = H_i(NK(A[n])) \simeq H_i(A[n])$$

Therefore, we call $K(A,n)$ the Eilenberg-MacLane spaces.

Recall that for an abelian group A, we have the cohomology group

$$H^n(X,A) = H^n(C^*(X,A)) = H^n(Hom(C_*(X),A)) = H^n(Hom(Z[X]^{\sim},A))$$

Proposition B.0.3 *Let* X *be a simplicial set, and* A *an abelian group, for* $n \geq 0$*:*

1. *Giving a morphism of simplicial sets* $X \longrightarrow K(A, n)$ *is equivalent of giving an element of* $Z^n(X, A)$.
2. *Two maps* $u, v : X \longrightarrow K(A, n)$ *are homotopic if and only if their images in* $C^n(X, A)$ *different by a boundary.*
3. $[X, K(A, n)] \simeq H^n(X, A)$.

Appendix C

Grothendieck Topology

A Grothendieck topology decodes the information about covering without any reference to the space itself. A **sieve** is a subfunctor of a functor of the form $Hom(-, X)$ for some object X. In other word, if S is a sieve, then $S(X') \subset Hom(X', X)$ for some object X, and for morphism f, $S(f)$ is the restriction of $Hom(f, X)$, the pull-back by f. A Grothendieck topology J on a category C is defined by giving, for each object c of C, a collection $J(c)$ of sieves on c, subject to certain conditions. These sieves are called **covering sieves**. The conditions we impose on a **Grothendieck topology** are:

T1 (Base Change) Let S be a covering sieve on X, and let $f : Y \longrightarrow X$. Let f^*S be the pullback of S, the fiber product $S \times_{Hom(-,X)} Hom(-, Y) \subset Hom(-, Y)$. Equivalently, for each object $Z \in C$, $f^*(SZ) = \{f : Z \longrightarrow Y | fg \in S(X)\}$. Then the pullback of S along f is a covering sieve on Y.

T2 (Local Character) Let S be a covering sieve on X, and let T be any sieve on X. Suppose that for each object $Y \in C$ and each morphism $f : X \longrightarrow Y \in S(X)$, the pullback sieve f^*T is a covering sieve on Y. Then T is a covering sieve on X.

T3 (Identity) $Hom(-, X)$ is a covering sieve on X for any object $X \in C$.

In fact, it is possible to put these axioms in another form where their geometric character is more apparent, assuming that the underlying category C contains certain fiber products. In this case, instead of specifying sieves, we can specify that certain collections of maps with a common codomain should cover their codomain. These collections are called **covering families**. If the collection of all covering families satisfies certain axioms, then we say that they form a **Grothendieck pretopology**. These axioms are:

PT 0 (Existence of fibered products) For all objects $X \in C$, and for all morphisms $X_0 \longrightarrow X$ which appear in some covering family of X, and for all morphisms $Y \longrightarrow X$, the fiber product $X_0 \times_X Y$ exists,

PT 1 (Stability under base change) For all objects $X \in C$, all morphisms $Y \longrightarrow X$, and all covering families $\{X_n \longrightarrow X\}$, the families $\{X_n \times_X Y \longrightarrow X\}$ is a covering family,

PT 2 (Local character) If $\{X_n \longrightarrow X\}$ is a covering family, and if for all n, $\{X_m n \longrightarrow X\}$ is a covering family, then the family of compositions $\{X_m n \longrightarrow X_n \longrightarrow X\}$ is a covering family,

PT3 (Isomorphisms) If $f : Y \longrightarrow X$ is an isomorphism, then $\{f\}$ is a covering family.

For any pretopology, the collection of all sieves that contain a covering family from the pretopology is always a Grothendieck topology. For categories with fibered products, there is a converse. Given a collection of arrows $\{X_n \longrightarrow X\}$, we construct a sieve S by letting $S(Y)$ be the set of all morphisms $Y \longrightarrow X$ that factor through some arrow $X_n \longrightarrow X$. This is called the sieve generated by $\{X_n \longrightarrow X\}$. Now choose a topology, say that $\{X_n \longrightarrow X\}$ is a covering family if and only if the sieve that it generates is a covering sieve for the given topology. It is easy to check that this defines a pretopology.

PT 3' (Identity) If $I_X : X \longrightarrow X$ is the identity arrow, then I_X is a covering family.

(PT 3) implies (PT 3'), but not conversely. However, suppose that we have a collection of covering families that satisfies (PT 0) to (PT 2) and (PT 3'), but not (PT 3). These families generate a pretopology. The topology generated by the original collection of covering families is then the same as the topology generated by the pretopology, because the sieve generated by an isomorphism $Y \longrightarrow X$ is $Hom(-, X)$. Consequently, if we restrict our attention to topologies, (PT 3) and (PT 3') are equivalent.

Let C be a category and J a Grothendieck topology on C. The pair (C, J) is called a **site**. A **presheaf** on a category is a contravariant functor from C to the category of all sets. Note that for this definition C is not required to have a topology. A sheaf on a site, however, should allow gluing, just like sheaves in classical topology. Consequently, we define a sheaf on a site to be a presheaf F such that for all objects X and all covering sieves S on X, the natural map $Hom(Hom(-, X), F) \longrightarrow Hom(S, F)$ induced by the inclusion of S into $Hom(-, X)$ is a bijection. Halfway in between a presheaf and a sheaf is the notion of a **separated presheaf**, where the natural map above is required to be only an injection, not a bijection, for all sieves S. Sheaves on a pretopology have a particularly simple description: For each covering family $\{X_n \longrightarrow X\}$, the diagram:

$$F(X) \longrightarrow \prod_{n \in I} F(X_n) \rightrightarrows \prod_{n,m \in I} F(X_n \times_X X_m)$$

must be an equalizer. For a separated presheaf, the first arrow need only to be an injection.

Similarly, one can define presheaves and sheaves of abelian groups, rings, modules, and so on. One can require either that a presheaf F is a contravariant functor to the category of abelian groups (or rings, or modules, etc.), or that F be an abelian group (ring, module, etc.) object in the category of all contravariant functors from C to the category of sets. These two definitions are equivalent.

Let's consider a category A satisfying A *admits small projective limits and small inductive limits*. Consider a morphism of presites (only consider the category structure) $f : X \longrightarrow Y$, that is, a functor $f^t : C_Y \longrightarrow C_X$.

Definition C.0.8 *Consider a morphism of small presites:*

1. *Let $F \in PSh(X, A)$. One defines $f_*F \in PSh(Y, A)$, be the direct image of F by f, which is defined as for $V \in C_Y$, $f_*F(V) = F(^t(V))$.*
2. *Let $G \in PSh(Y, A)$. One defines $f^\dagger(G)$ by setting for $U \in C_X$*

$$f^\dagger(G)(U) = colim_{(U \longrightarrow f^t(V))} G(V).$$

3. *Let $G \in PSh(Y, A)$. One defines $f^\ddagger(G)$ by setting for $U \in C_X$*

$$f^\ddagger(G)(U) = lim_{(f^t(V) \longrightarrow U)} G(V).$$

Theorem C.0.7 *Let $f : X \longrightarrow Y$ be a morphism of small sites*

1. *The functor $f^\dagger : PSh(Y, A) \longrightarrow PSh(X, A)$ is left adjoint to the functor $f_* : PSh(X, A) \longrightarrow PSh(Y, A)$. More specifically, we have an isomorphism, functorial through $F \in PSh(X, A)$ and $G \in PSh(Y, A)$:*

$$Hom_{PSh(X,A)}(f^\dagger G, F) \simeq Hom_{PSh(Y,A)}(G, f_*F)$$

2. *The functor $f^\ddagger : PSh(Y, A) \longrightarrow PSh(X, A)$ is right adjoint to the functor $f_* : PSh(X, A) \longrightarrow PSh(Y, A)$. More specifically, we have an isomorphism, functorial through $F \in PSh(X, A)$ and $G \in PSh(Y, A)$:*

$$Hom_{PSh(X,A)}(F, f^\ddagger G) \simeq Hom_{PSh(Y,A)}(f_*F, G).$$

Appendix D

Nisnevich Topology

Definition D.0.9 *Let X be a Noetherian scheme. Let X_{Nis} be the category of etale morphisms $\{Y \longrightarrow X\}$ which are of finite type and separated. If $Y \in X_{Nis}$, let $Cov_{Nis}(X)$ the class of family $\mathcal{U} = \{U_i \longrightarrow Y\}_{i \in I}$ such that for each $y \in Y$ and $i \in I$, there exists a $u \in U$ with $f_i(u) = y$ and $\kappa(u) \simeq \kappa(y)$, where κ denotes the residual field.*

One easily check this defines a pretopology, which generates a topology called the *Nisnevich topology*.

Example D.0.1 *An elementary distinguished square if given by:*

$$\begin{tikzcd} & V \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

in which $p : V \longrightarrow X$ is etale, $U \hookrightarrow X$ is an open immersion, and $p^{-1}(X \setminus U)_{red} \xrightarrow{\sim} (X \setminus U)_{red}$.

Theorem D.0.8 *Let X be a Noetherian scheme, $\mathcal{U} \in Cov_{Nis}(X)$. Then there exists a sub-family of finite number $(U_j \longrightarrow X)_{j \in J} \in Cov_{Nis}(X)$*

proof:

1. **Definition D.0.10** *Let $p : W \longrightarrow X$ a morphism of schemes. A splitting sequence of p is $\emptyset = Z_{n+1} \hookrightarrow Z_n \hookrightarrow \cdots Z_0 = X$ such that $W \times_X (Z_i - Z_{i+1}) \longrightarrow Z_i - Z_{i+1}$,where each inclusion is a closed immersion, admits a section.*

2. **Lemma D.0.1** *If X is a Noetherian scheme, and $\mathcal{U} \in Cov_{Nis}(X)$, then $W = \coprod U_i \longrightarrow X \in Cov_{Nis}(X)$ admits a splitting sequence.*

Let us admit the previous lemma and finish the proof of **Theorem 1.10**:

Let $\emptyset = Z_{n+1} \hookrightarrow Z_n \hookrightarrow \cdots Z_0 = X$ a splitting sequence for $W \times_X (Z_i - Z_{i+1}) \longrightarrow Z_i \longrightarrow Z_{i+1}$. If $S_i : Z_i - Z_{i+1} \longrightarrow W$ is a section of $p \mid_{W \times_X (Z_i - Z_{i+1})}$. Then because $Z_i - Z_{i+1}$ is Noetherian, so for any $i \in \{0, 1, \cdots, n\} \in \coprod_{j \in J, finite} U_j$. Therefore exists a finite number of $(U_j \longrightarrow X)_{j \in J} \in Cov_{Nis}(X)$.

In order to prove **Lemma 1.1**, we introduce the following lemma who implies **Lemma 1.1**.

Lemma D.0.2 *Let X be a Noetherian scheme and $f : Y \longrightarrow X$ a morphism locally of finite type. Suppose for any $x \in X$, there exist $y \in Y$ with $f(y) = x$ and $\kappa(x) \simeq \kappa(y)$. If moreover X is reduced or f is etale, then there exists an open dense $U \in X$ such that $f \mid_U$ splits.*

proof:

1. if X is reduced: We can suppose that X is integral affine. Then f induces a birational morphism between X and an integral closed sub-scheme of Y, which proves the lemma.

2. If f is etale: Let $X_{red} \xrightarrow{\varphi} X$. Then $\varphi^* : X - etale\ schemes \longrightarrow X - reduced\ etale\ schemes$ induces an equivalence of category(Topological invariance of etale morphisms). So we can reduce to the case 1.

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