THE RANK SPECTRAL SEQUENCE OF ALGEBRAIC K-THEORY

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Abstract. Bruno Kahn has constructed a rank spectral sequence by using a purely categorical approach. This spectral sequence was derived by using a filtration of the category of torsion-free modules over integral domain by ranks and hence the name: rank spectral sequence. The E^1 terms of this spectral sequence coincide with E^2 terms of Quillen's spectral sequence used to prove the finite generation of K-groups of ring of integers.

In this talk, we will show how to calculate the d^1-differential of the rank spectral sequence. We will put the differential in certain distinguished triangles of coefficients/functors over some categories, and make these functors explicit in terms of Tits building and Ash-Rudolph's modular symbols. To accomplish this, we shall use Quillen's categorical homotopy theory intensively and introduce the notion of extended (modular) symbols which is equivalent to Ash-Rudolph's via the suspension of Tits buildings.

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1. Rank Spectral Sequence

1.1. **Singular Chain Complexes.** We will study the singular homologies of categories and functors. For X a simplicial abelian group, we denote $[X] = C_*(X)$ the corresponding singular chain complex such that $C_n(X)$ is the free Abelian group generated by X_n and differentials are alternating sums of faces. If X is a simplicial set, we define $C_*(X) := [\mathbb{Z}X]$. Moreover, for a category $\mathcal{C}$, we denote by $C_*(\mathcal{C}) := [\mathbb{Z}N\mathcal{C}]$ its singular chain complex where N is the nerve functor.

Now, suppose that X is a bisimplicial set, then we denote by δX the corresponding diagonal simplicial set and define $C_*(X) := [\mathbb{Z}\delta X]$.

Definition 1. *If $\mathbb{F} : \mathcal{D} \to \mathbf{sSet}$ is a functor taking value in the category of simplicial sets, then we define $N(\mathcal{D}, \mathbb{F})$ as a bisimplicial set with*

$$N_{p,q}(\mathcal{D}, \mathbb{F}) := \coprod_{d_0 \to \cdots \to d_p} \mathbb{F}_q(d_0),$$

and $C_*(\mathcal{D}, \mathbb{F}) := [\mathbb{Z}\delta N(\mathcal{D}, \mathbb{F})]$.

We notice that this construction in definition 1 makes $C_*(\mathcal{D}, \mathbb{F})$ functorial with respect to $\mathbb{F}$. We define $C_*(\mathcal{D}, \widetilde{\mathbb{F}})$ to be the homotopy fiber of $C_*(\mathcal{D}, \mathbb{F}) \to C_*(\mathcal{D}, \star)$ in the derived category of Abelian groups $\mathbf{D}(Ab)$ so that it gives rise to a distinguished triangle

$$C_*(\mathcal{D}, \widetilde{\mathbb{F}}) \to C_*(\mathcal{D}, \mathbb{F}) \to C_*(\mathcal{D}) \to C_*(\mathcal{D}, \widetilde{\mathbb{F}})[1].$$

1.2. **Cellular Functors.**

Definition 2. *Let $T : \mathcal{C} \to \mathcal{D}$ be a functor, we call T a cellular functor if it satisfies*

- **(1):** *T is fully faithful;*
- **(2):** *$Hom_{\mathcal{D}}(d, T(c)) = \emptyset$ for any $c \in \mathcal{C}$ and $d \in \mathcal{D} - T(\mathcal{C})$*

Moreover, we say that a cellular functor $T : \mathcal{C} \to \mathcal{D}$ is connected if in addition it satisfies

- **(3):** *for any $d \in \mathcal{D}$ we have $T \downarrow d \neq \emptyset$.*

By abuse of notation, if $T : \mathcal{C} \to \mathcal{D}$ is cellular, we will write $\mathcal{D} - \mathcal{C}$ instead of $\mathcal{D} - T(\mathcal{C})$.

Definition 3 (Grothendieck Construction, [4], VI, Sections 8, 9)**.** *Let $\mathbb{F} : \mathcal{D} \to \mathbf{Cat}$ be a functor. The Grothendieck construction $\mathcal{D} \int \mathbb{F}$ is defined as a category such that:*

1. *Objects in $\mathcal{D} \int \mathbb{F}$ are pairs (d, x) such that $d \in \mathcal{D}$ and $x \in \mathbb{F}(d)$;*
2. *For two objects (d, x), $(d', x') \in \mathcal{D} \int \mathbb{F}$, a morphism between them is given by the pair $d \xrightarrow{f} d'$ and $\mathbb{F}(f)(x) \xrightarrow{g} x'$;*
3. *For three objects (d, x), (d', x') and (d'', x''), the composition of morphisms is given by $d \xrightarrow{f} d' \xrightarrow{f'} d''$ and $\mathbb{F}(f' \circ f)(x) \xrightarrow{\mathbb{F}(f')(g)} \mathbb{F}(f')(x') \xrightarrow{g'} x''$.*

Let $T : \mathcal{C} \to \mathcal{D}$ be a functor between two small categories and $\mathbb{F}_T : \mathcal{D} \to \mathbf{Cat}$ be a functor sending $d \in \mathcal{D}$ to $T \downarrow d$. The Grothendieck construction $\mathcal{D} \int \mathbb{F}_T$ has objects (d, x) such that $d \in \mathcal{D}$, $x \in \mathbb{F}_T(d) = T \downarrow d$. A morphism $(d, x) \to (d', x') \in \mathcal{D} \int \mathbb{F}_T$ is given by a morphism

$f : d \to d' \in \mathcal{D}$ and $g : \mathbb{F}_T(f)(x) \to x' \in \mathbb{F}_T(d') = T \downarrow d'$, i.e, a commutative diagram:

$$\begin{tikzcd} T(c) \arrow[r] \arrow[d, "x"'] & T(c') \arrow[d, "x'"] \\ d \arrow[r, "f"'] & d' \end{tikzcd}$$

Proposition 4 ([6], proposition 2.3.5). *If $T : \mathcal{C} \to \mathcal{D}$ is cellular, the naturally commutative diagram of categories*

$$\begin{tikzcd} (\mathcal{D} - \mathcal{C}) \int \mathbb{F}_T \arrow[r, "p"] \arrow[d, "\varepsilon"'] & \mathcal{C} \arrow[d, "T"] \\ \mathcal{D} - \mathcal{C} \arrow[r, "\iota"'] & \mathcal{D} \end{tikzcd}$$

is homotopy cocartesian (i.e, it is so after applying the nerve functor). Here, ι is inclusion, p is the projection to $\mathcal{C}$ and ε is the augmentation induced by $\mathbb{F}_T \to \star$.

1.3. Thomason's Theorem.

Theorem 5 (Thomason's Theorem, [21], theorem 1-2). *Let $\mathcal{C}$ be a category and $\mathbb{F} : \mathcal{C} \to \mathbf{Cat}$ be a functor. There is a weak equivalence:*

$$\delta N(\mathcal{C}, \mathbb{F}) \xrightarrow{\sim} N\left(\mathcal{C} \int \mathbb{F}\right),$$

sending

$$(c_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} c_n,\ x_0 \xrightarrow{g_1} \cdots \xrightarrow{g_n} x_n); \quad c_i \in \mathcal{C},\ x_i \in \mathbb{F}(c_0)$$

to

$$\Big((c_0, y_0 = x_0) \xrightarrow{h_1} \cdots \xrightarrow{h_n} (c_n, y_n)\Big)$$

with $y_i = \mathbb{F}(f_i \cdots f_1)x_i \in \mathbb{F}(c_i)$ for $0 < i \leq n$ and $h_i = (f_i, \mathbb{F}(f_i \cdots f_1)g_i)$.

According to Thomason's theorem, for any small category $\mathcal{D}$ and any functor $\mathbb{F} : \mathcal{D} \to \mathbf{Cat}$, the complex $C_*(\mathcal{D}, \mathbb{F})$ is quasi-isomorphic to the complex $C_*(\mathcal{D} \int \mathbb{F})$, so we have:

Theorem 6 ([6], theorem 2.3.6). *If $T : \mathcal{C} \to \mathcal{D}$ is a cellular functor, there exists an distinguished triangle in the derived category of abelian groups:*

$$C_*(\mathcal{D} - \mathcal{C}, \widetilde{\mathbb{F}_T}) \xrightarrow{\widetilde{p}_*} C_*(\mathcal{C}) \xrightarrow{T_*} C_*(\mathcal{D}) \xrightarrow{\partial_T} C_*(\mathcal{D} - \mathcal{C}, \widetilde{\mathbb{F}_T})[1].$$

This theorem gives us a description of the homotopy mapping cone of T_* as $C_*(\mathcal{D} - \mathcal{C}, \widetilde{\mathbb{F}_T})[1]$.

1.4. Spectral Sequence and Exact Couples.

Recall that (c.f, [6, Section 2]) the rank spectral sequence is constructed under the background of any triangulated category $\mathcal{T}$ with countable direct sums as follows. We take

$$C_0 \xrightarrow{i_1} \cdots \xrightarrow{i_n} C_n \xrightarrow{i_{n+1}} \cdots$$

being a sequence of objects in $\mathcal{T}$ and C as a homotopy colimit of the C_n. For each n, we choose a mapping cone $C_{n/n-1}$ of i_n so that there is an distinguished triangle

$$C_{n-1} \xrightarrow{i_n} C_n \xrightarrow{j_n} C_{n/n-1} \xrightarrow{k_n} C_{n-1}[1].$$

Let $H_* : \mathcal{T} \to \mathcal{A}$ be a (co)homological functor taking values in an abelian category such that H_* commutes with countable direct sums. Then the exact couple

$$D_{p,q} := H_{p+q}(C_p), \qquad E_{p,q} := H_{p+q}(C_{p/p-1})$$

gives rise to a spectral sequence $E_{p,q} = E^1_{p,q} \Rightarrow H_{p+q}(C)$ where C is the homotopy colimit of the C_n. Let $C_{p/p-2}$ be a cone of $i_p i_{p-1} : C_{p-2} \to C_p$, then we get a commutative diagram of distinguished triangles

$$\begin{tikzcd}
C_{p-2} \arrow[r, equal] \arrow[d, "i_{p-1}"'] & C_{p-2} \arrow[d, "i_p i_{p-1}"] & & \\
C_{p-1} \arrow[r, "i_p"] \arrow[d, "j_{p-1}"'] & C_p \arrow[r, "j_p"] \arrow[d] & C_{p/p-1} \arrow[r, "k_p"] \arrow[d, "="] & C_{p-1}[1] \arrow[d, "j_{p-1}[1]"] \\
C_{p-1/p-2} \arrow[r, "\bar{i}_p"'] & C_{p/p-2} \arrow[r, "\bar{j}_p"'] & C_{p/p-1} \arrow[r, "\bar{k}_p"'] & C_{p-1/p-2}[1]
\end{tikzcd} \tag{1.1}$$

The differential $d^1_{p,q}$ is the boundary map $\bar{k}_{p,n}$ with $n = p+q$ of the long distinguished sequence

$$H_{p+q}(C_{p-1/p-2}) \xrightarrow{\bar{i}_{p,n}} H_{p+q}(C_{p/p-2}) \xrightarrow{\bar{j}_{p,n}} H_{p+q}(C_{p/p-1}) \xrightarrow{\bar{k}_{p,n}} H_{p+q-1}(C_{p-1/p-2})$$

associated with the bottom distinguished triangle of the above diagram. We notice that d^1 is unique up to isomorphisms between the chosen mapping cones $C_{n/n-1}$, i.e, if we choose another mapping cone $C'_{n/n-1}$ and an isomorphism $C'_{n/n-1} \to C_{n/n-1}$ fitting into the following commutative diagram of distinguished triangles

$$\begin{tikzcd}
C_{n-1} \arrow[r, "i_n"] \arrow[d, equal] & C_n \arrow[r, "j_n"] \arrow[d, equal] & C_{n/n-1} \arrow[r, "k_n"] & C_{n-1}[1] \arrow[d, equal] \\
C_{n-1} \arrow[r, "i_n"'] & C_n \arrow[r, "j'_n"'] & C'_{n/n-1} \arrow[r, "k'_n"'] \arrow[u, "\simeq"] & C_{n-1}[1]
\end{tikzcd}$$

then the different choice of mapping cone gives us another map $\bar{k}'_p : C'_{p/p-1} \to C'_{p-1/p-2}[1]$ and a commutative diagram determined by this isomorphism of mapping cones. It follows that up to the isomorphisms of mapping cones, d^1 is unique.

Our main target of this talk is to give a formula of the d^1-differential of the rank spectral sequence. By proposition 4, for a cellular functor $T : \mathcal{C} \to \mathcal{D}$, we choose the mapping cone of $C_*(\mathcal{C}) \to C_*(\mathcal{D})$ as $C_*(\mathcal{D} - \mathcal{C}, \widetilde{\mathbb{F}_T})[1]$. Consider two composable cellular functors $\mathcal{B} \xrightarrow{U} \mathcal{C} \xrightarrow{T} \mathcal{D}$. It is easy to show that $T \circ U$ is still cellular. Replacing the abstract objects by chain complexes in diagram (1.1), we get the following commutative diagram:

$$\begin{tikzcd}
C_*(\mathcal{B}) \arrow[r, equal] \arrow[d] & C_*(\mathcal{B}) \arrow[d] & & \\
C_*(\mathcal{C}) \arrow[r] \arrow[d] & C_*(\mathcal{D}) \arrow[r] \arrow[d] & C_*(\mathcal{D} - \mathcal{C}, \widetilde{\mathbb{F}_T})[1] \arrow[r] \arrow[d, "="] & C_*(\mathcal{C})[1] \arrow[d] \\
C_*(\mathcal{C} - \mathcal{B}, \widetilde{\mathbb{F}_U})[1] \arrow[r] & C_*(\mathcal{D} - \mathcal{B}, \widetilde{\mathbb{F}_{TU}})[1] \arrow[r] & C_*(\mathcal{D} - \mathcal{C}, \widetilde{\mathbb{F}_T})[1] \arrow[r, "d^1[1]"'] & C_*(\mathcal{C} - \mathcal{B}, \widetilde{\mathbb{F}_U})[2]
\end{tikzcd} \tag{1.2}$$

We see that d^1 is the composition of

$$C_*(\mathcal{D}-\mathcal{C},\widetilde{\mathbb{F}_T}) \to C_*(\mathcal{C}) \to C_*(\mathcal{C}-\mathcal{B},\widetilde{\mathbb{F}_U})[1].$$

In the following sections, we will put this composition into suitable distinguished triangles so that it can be explicitly calculated.

2. Reduction using Grothendieck Constructions

Suppose that $\mathbb{F} : \mathcal{D} \to \mathbf{Cat}$ is a functor and $\mathbb{G} : \mathcal{D}\int\mathbb{F} \to \mathbf{Cat}$ is another functor. We can prove that (c.f, [20, Chapter 2])

$$(\mathcal{D}\int\mathbb{F})\int\mathbb{G} = \mathcal{D}\int(\mathbb{F}\int\mathbb{G}).$$

So we can write the above double Grothendieck constructions as $\mathcal{D}\int\mathbb{F}\int\mathbb{G}$.

2.1. **Background.** We consider the composable connected cellular functors

$$\mathcal{B} \xrightarrow{U} \mathcal{C} \xrightarrow{T} \mathcal{D}. \tag{2.1}$$

Recall that by the construction of the rank spectral sequence we have short exact sequences

$$0 \to C_*(\mathcal{B}) \xrightarrow{U} C_*(\mathcal{C}) \to C_*(\mathcal{C})/C_*(\mathcal{B}) \to 0 \tag{2.2}$$

$$0 \to C_*(\mathcal{C}) \xrightarrow{T} C_*(\mathcal{D}) \to C_*(\mathcal{D})/C_*(\mathcal{C}) \to 0, \tag{2.3}$$

which induce morphisms in the derived category of abelian groups $\mathbf{Ab}$:

$$C_*(\mathcal{D})/C_*(\mathcal{C}) \to C_*(\mathcal{C})[1] \to C_*(\mathcal{C})/C_*(\mathcal{B})[1]. \tag{2.4}$$

By theorem 6, we choose the following mapping cones

$$C_*(\mathcal{D})/C_*(\mathcal{C}) = C_*(\mathcal{D}-\mathcal{C},\widetilde{\mathbb{F}_T})[1], \qquad C_*(\mathcal{C})/C_*(\mathcal{B}) = C_*(\mathcal{C}-\mathcal{B},\widetilde{\mathbb{F}_U})[1].$$

Together with diagram (1.2), we see that in order to calculate d^1 of the rank spectral sequence, we calculate the composition of (2.4).

In this section, we put (via quasi-isomorphisms) the composition of (2.4) into certain distinguished triangles that come from distinguished triangles of functors (coefficients) defined over different categories: firstly $\mathcal{D}$, then $\mathcal{D}-\mathcal{B}$. Notice that every short exact sequence is naturally a distinguished triangle, so we say that two short exact sequences are quasi-isomorphic if corresponding distinguished triangles are. In this case, we say that one short exact sequence can be replaced (up to quasi-isomorphism) by the other. It turns out that d^1 of rank spectral sequence can be calculated via the distinguished triangles of functors we constructed.

2.2. **Pass to $\mathcal{D}$-chains.** We define a functor

$$U_* : \mathcal{B} \to \mathcal{D}\int\mathbb{F}_T; \qquad b \mapsto (b = b).$$

Then there exists a Grothendieck construction $\mathcal{D}\int\mathbb{F}_T\int\mathbb{F}_{U_*}$ whose objects can be written as $b \to c \to d$ with $b \in \mathcal{B}$ and $c \to d \in \mathcal{D}\int\mathbb{F}_T$. Notice that for a fixed $c \to d \in \mathbb{F}_T(d)$, the categories $\mathbb{F}_{U_*}(c \to d)$ and $\mathbb{F}_U(c)$ are canonically isomorphic.

Lemma 7. *The categories $\mathcal{D}\int\mathbb{F}_T\int\mathbb{F}_{U_*}$ and $\mathcal{D}\int\mathbb{F}_{TU}$ are homotopy equivalent.*

Proof. We define two functors

$$p : \mathcal{D}\int \mathbb{F}_T \int \mathbb{F}_{U_*} \to \mathcal{D}\int \mathbb{F}_{TU}; \qquad (b \to c \to d) \mapsto (b \to d)$$

and

$$s : \mathcal{D}\int \mathbb{F}_{TU} \to \mathcal{D}\int \mathbb{F}_T \int \mathbb{F}_{U_*}; \qquad (b \to d) \mapsto (b = b \to d)$$

Since s is left adjoint to p, they induce homotopy equivalences between two corresponding categories.

□

We notice that this lemma actually says that p and s make the functors $\mathbb{F}_T \int \mathbb{F}_{U_*}$ and $\mathbb{F}_{TU}$ naturally homotopy equivalent over the category $\mathcal{D}$, i.e, after applying to $\mathcal{D}$ the resulting categories are homotopy equivalent.

It is easy to see that the projection functor $\mathcal{D}\int \mathbb{F}_T \to \mathcal{C}$ (resp. $\mathcal{D}\int \mathbb{F}_{TU} \to \mathcal{B}$) admits a left adjoint $c \to [c = c]$ (resp. $b \to [b = b]$), so there exist a pair of homotopy equivalences between the categories $\mathcal{D}\int \mathbb{F}_T$ and $\mathcal{C}$ (between $\mathcal{D}\int \mathbb{F}_{TU}$ and $\mathcal{B}$). Together with Thomason's theorem (theorem 5), we obtain canonical homotopy equivalences

$$N(\mathcal{D}, \mathbb{F}_T) \approx N\mathcal{C}, \qquad N(\mathcal{D}, \mathbb{F}_{TU}) \approx N\mathcal{B}. \tag{2.5}$$

Then, up to canonical homotopy equivalences, (2.1) becomes

$$\mathcal{D}\int \mathbb{F}_{TU} \to \mathcal{D}\int \mathbb{F}_T \to \mathcal{D}, \tag{2.6}$$

and (2.3) can be replaced by (notice that T is connected cellular)

$$0 \to C_*(\mathcal{D}, \widetilde{\mathbb{F}_T}) \to C_*(\mathcal{D}, \mathbb{F}_T) \to C_*(\mathcal{D}) \to 0. \tag{2.7}$$

Definition 8 (Compare to [6], 2.2.2)**.** *Suppose that $U : \mathcal{B} \to \mathcal{C}$ is connected cellular. We define the relative reduced chain complex*

$$C_*(\mathcal{D}, \mathbb{F}_T \int \widetilde{\mathbb{F}_{U_*}}) = Ker(C_*(\mathcal{D}, \mathbb{F}_T \int \mathbb{F}_{U_*}) \to C_*(\mathcal{D}, \mathbb{F}_T)).$$

Therefore, use lemma 7, we see that (2.2) and (2.4) can be replaced by

$$0 \to C_*(\mathcal{D}, \mathbb{F}_T \int \widetilde{\mathbb{F}_{U_*}}) \to C_*(\mathcal{D}, \mathbb{F}_T \int \mathbb{F}_{U_*}) \to C_*(\mathcal{D}, \mathbb{F}_T) \to 0. \tag{2.8}$$

and

$$C_*(\mathcal{D}, \widetilde{\mathbb{F}_T}) \to C_*(\mathcal{D}, \mathbb{F}_T) \to C_*(\mathcal{D}, \mathbb{F}_T \int \widetilde{\mathbb{F}_{U_*}})[1] \xrightarrow{+1} \tag{2.9}$$

In particular, the above triangle is induced by the distinguished triangle of their coefficients:

$$C_*(\widetilde{\mathbb{F}_T}) \to C_*(\mathbb{F}_T) \to C_*(\mathbb{F}_T \int \widetilde{\mathbb{F}_{U_*}})[1] \xrightarrow{+1} \tag{2.10}$$

2.3. **distinguished Triangles of Functors/Coefficients.** There exists a commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & C_*(\mathcal{D}, \mathbb{F}_T \int \widetilde{\mathbb{F}_{U_*}}) \arrow[r] \arrow[d, equal] & C_*(\mathcal{D}, \widetilde{\mathbb{F}_T \int \mathbb{F}_{U_*}}) \arrow[r] \arrow[d] & C_*(\mathcal{D}, \widetilde{\mathbb{F}_T}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_*(\mathcal{D}, \mathbb{F}_T \int \widetilde{\mathbb{F}_{U_*}}) \arrow[r] \arrow[d] & C_*(\mathcal{D}, \mathbb{F}_T \int \mathbb{F}_{U_*}) \arrow[r] \arrow[d] & C_*(\mathcal{D}, \mathbb{F}_T) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r, equal] & 0 \arrow[r] & C_*(\mathcal{D}) \arrow[r, equal] & C_*(\mathcal{D}) \arrow[r] & 0
\end{tikzcd} \tag{2.11}$$

such that all sequences are induced by the short exact sequences of their coefficients. Thus, the composition of (2.9) fits into the distinguished triangle

$$C_*(\mathcal{D}, \mathbb{F}_T \int \widetilde{\mathbb{F}_{U_*}}) \to C_*(\mathcal{D}, \widetilde{\mathbb{F}_T \int \mathbb{F}_{U_*}}) \to C_*(\mathcal{D}, \widetilde{\mathbb{F}_T}) \xrightarrow{+1} C_*(\mathcal{D}, \mathbb{F}_T \int \widetilde{\mathbb{F}_{U_*}})[1]$$

and the composition of (2.10) fits into the distinguished triangle of coefficients

$$C_*(\mathbb{F}_T \int \widetilde{\mathbb{F}_{U_*}}) \to C_*(\widetilde{\mathbb{F}_T \int \mathbb{F}_{U_*}}) \to C_*(\widetilde{\mathbb{F}_T}) \xrightarrow{+1} C_*(\mathbb{F}_T \int \widetilde{\mathbb{F}_{U_*}})[1]. \tag{2.12}$$

This triangle is distinguished in the sense that it gives a term-wise distinguished triangle, i.e, after applying to any object $d \in \mathcal{D}$ the resulting triangle is distinguished. For simplicity, we will just say that this sequence is a distinguished triangle of coefficients or functors without mentioning which category it applies.

2.4. **Pass to $(\mathcal{D} - \mathcal{B})$-chains.** There exists a commutative diagram of distinguished triangles

$$\begin{tikzcd}
C_*(\mathcal{D} - \mathcal{B}, \mathbb{F}_T \int \widetilde{\mathbb{F}_{U_*}}) \arrow[r] \arrow[d] & C_*(\mathcal{D} - \mathcal{B}, \widetilde{\mathbb{F}_T \int \mathbb{F}_{U_*}}) \arrow[r] \arrow[d] & C_*(\mathcal{D} - \mathcal{B}, \widetilde{\mathbb{F}_T}) \arrow[r, "+1"] \arrow[d] & {} \\
C_*(\mathcal{D}, \mathbb{F}_T \int \widetilde{\mathbb{F}_{U_*}}) \arrow[r] & C_*(\mathcal{D}, \widetilde{\mathbb{F}_T \int \mathbb{F}_{U_*}}) \arrow[r] & C_*(\mathcal{D}, \widetilde{\mathbb{F}_T}) \arrow[r, "+1"] & {}
\end{tikzcd} \tag{2.13}$$

where all vertical morphisms are quasi-isomorphisms (we use [6, Prop. 2.3.4] to the case $\phi : \mathbb{F} \Rightarrow *$ and $TU : \mathcal{B} \to \mathcal{D}$, where we replace $\mathbb{F}$ by corresponding functors appearing in the above diagram). It follows that d^1 can be calculated by the top distinguished triangle of diagram (2.13) which is induced by the distinguished triangle of coefficients (2.12).

Remark 9. *Use the proof of lemma 7, we can show that p and s give homotopy equivalences between $(\mathcal{D} - \mathcal{B}) \int \mathbb{F}_T \int \mathbb{F}_{U_*}$ and $(\mathcal{D} - \mathcal{B}) \int \mathbb{F}_{TU}$. So $\mathbb{F}_T \int \mathbb{F}_{U_*}$ and $\mathbb{F}_{TU}$ are naturally homotopy equivalent over the category $\mathcal{D} - \mathcal{B}$.*

We define $\mathcal{C}' := \mathcal{C} - \mathcal{B}$, $\mathcal{D}' := \mathcal{D} - \mathcal{B}$ and $T' = T|_{\mathcal{C}'} : \mathcal{C}' \to \mathcal{D}'$. Notice that if T is cellular then so does T'.

2.5. **Replace T by T'.** Suppose that $T : \mathcal{C} \to \mathcal{D}$ and $T' : \mathcal{C}' \to \mathcal{D}'$ are connected cellular. We first notice that for any $d \in \mathcal{D}'$ there exists a functor

$$\mathbb{F}_{U_*}|_{\mathbb{F}_{T'}(d)} : \mathbb{F}_{T'}(d) \to \mathbf{Cat}$$

so that we have a Grothendieck construction $\mathbb{F}_{T'}(d) \int \mathbb{F}_{U_*}$ whose objects are $b \to c \to d$ with $b \in \mathcal{B}$ and $c \in \mathcal{C} - \mathcal{B}$.

Let $d \in \mathcal{D}'$. The inclusion functor $i : \mathbb{F}_{TU}(d) \hookrightarrow \mathbb{F}_T(d)$ is connected cellular and

$$\mathbb{F}_T(d) - \mathbb{F}_{TU}(d) = \mathbb{F}_{T'}(d).$$

Applying [6, Proposition 2.3.4] to the cellular functor i and the natural transformation $\mathbb{F}_{U_*} \Rightarrow \star$ and use Thomason's formula, we can construct a homotopy cocartesian diagram

$$\begin{tikzcd} \mathbb{F}_{T'}(d) \int \mathbb{F}_{U_*} \arrow[r] \arrow[d] & \mathbb{F}_T(d) \int \mathbb{F}_{U_*} \arrow[d] \\ \mathbb{F}_{T'}(d) \arrow[r] & \mathbb{F}_T(d) \end{tikzcd} \tag{2.14}$$

which gives a canonical quasi-isomorphism

$$C_*(\mathbb{F}_{T'}(d) \int \widetilde{\mathbb{F}_{U_*}}) \xrightarrow{\sim} C_*(\mathbb{F}_T(d) \int \widetilde{\mathbb{F}_{U_*}}), \quad \forall d \in \mathcal{D}'. \tag{2.15}$$

Combined with the top distinguished triangle of diagram (2.13), we obtain an distinguished triangle

$$C_*(\mathcal{D} - \mathcal{B}, \mathbb{F}_{T'} \int \widetilde{\mathbb{F}_{U_*}}) \to C_*(\mathcal{D} - \mathcal{B}, \widetilde{\mathbb{F}_T \int \mathbb{F}_{U_*}}) \to C_*(\mathcal{D} - \mathcal{B}, \widetilde{\mathbb{F}_T}) \xrightarrow{+1} \tag{2.16}$$

which is induced by the distinguished triangle of coefficients/functors

$$C_*(\mathbb{F}_{T'} \int \widetilde{\mathbb{F}_{U_*}}) \to C_*(\widetilde{\mathbb{F}_T \int \mathbb{F}_{U_*}}) \to C_*(\widetilde{\mathbb{F}_T}) \xrightarrow{+1} C_*(\mathbb{F}_{T'} \int \widetilde{\mathbb{F}_{U_*}})[1]. \tag{2.17}$$

Moreover, according to remark 9, (2.16) is isomorphic to the distinguished triangle

$$C_*(\mathcal{D} - \mathcal{B}, \mathbb{F}_{T'} \int \widetilde{\mathbb{F}_{U_*}}) \to C_*(\mathcal{D} - \mathcal{B}, \widetilde{\mathbb{F}_{TU}}) \to C_*(\mathcal{D} - \mathcal{B}, \widetilde{\mathbb{F}_T}) \xrightarrow{+1} \tag{2.18}$$

which is induced by the distinguished triangle of coefficients/functors

$$C_*(\mathbb{F}_{T'} \int \widetilde{\mathbb{F}_{U_*}}) \to C_*(\widetilde{\mathbb{F}_{TU}}) \to C_*(\widetilde{\mathbb{F}_T}) \xrightarrow{+1} C_*(\mathbb{F}_{T'} \int \widetilde{\mathbb{F}_{U_*}})[1]. \tag{2.19}$$

We consider Quillen's Q-construction: let A be an integral Noetherian domain and $\mathcal{D} = Q_n$ be the full sub-category of $Q^{\mathrm{tf}}(A)$ consisting of torsion-free modules of rank smaller or equal to n. Similarly, we define $\mathcal{B} = Q_{n-2}$ and $\mathcal{C} = Q_{n-1}$. Let $T' : \mathcal{C} - \mathcal{B} \hookrightarrow \mathcal{D} - \mathcal{B}$ be the inclusion functor. We notice that for any $d \in \mathcal{D} - \mathcal{C}$ the category $\mathbb{F}_{T'}(d)$ has only objects the admissible monomorphisms $c \rightarrowtail d$ and admissible epimorphisms $d \twoheadrightarrow c$, hence it is discrete. On the other hand, if $d \in \mathcal{C} - \mathcal{B}$ then $\mathbb{F}_{T'}(d)$ is contractible since this category has a terminal object $d = d$. In particular, applying (2.19) to any $d \in \mathcal{D} - \mathcal{C}$, we get a distinguished triangle

$$\begin{aligned} \bigoplus_{c \to d \in \mathbb{F}_{T'}(d)} C_*(\widetilde{\mathbb{F}_U}(c)) \to C_*(\widetilde{\mathbb{F}_{TU}}(d)) \to C_*(\widetilde{\mathbb{F}_T}(d)) \\ \xrightarrow{+1} \bigoplus_{c \to d \in \mathbb{F}_{T'}(d)} C_*(\widetilde{\mathbb{F}_U}(c))[1]. \end{aligned} \tag{2.20}$$

Therefore, d^1 on coefficients is given by

$$
\begin{aligned}
\widetilde{St}(d) &\simeq H_{n-1}(\widetilde{\mathbb{F}_T}(d)) \to H_{n-2}\Big(\mathbb{F}_{T'}(d)\int \widetilde{\mathbb{F}_{U_*}}\Big) \\
&= \bigoplus_{c\to d\in\mathbb{F}_{T'}(d)} H_{n-2}(\widetilde{\mathbb{F}_{U_*}}(c\to d)) \simeq \bigoplus_{c\to d\in\mathbb{F}_{T'}(d)} \widetilde{St}(c).
\end{aligned}
\tag{2.21}
$$

3. Tits Buildings, (Extended) Modular Symbols and the Rank Spectral Sequence in Algebraic K-Theory

3.1. **Tits Buildings and Rank Spectral Sequence.** Suppose that A is an integral domain and $Q^{\mathrm{tf}}(A)$ be Quillen's Q-construction over the the category of finitely generated torsion-free modules. For an A-module M we define its rank as

$$rank(A) := \dim(M \otimes_A K)$$

where $K = Frac(A)$. Then we denote $Q_n^{\mathrm{tf}}(A)$ for $n \geq 0$ the full subcategory of $Q^{\mathrm{tf}}(A)$ of modules with rank less or equal to n. If d is a torsion-free Noetherian A-module, a submodule $c \subset d$ is said to be pure if d/c is torsion-free. According to [6, Prop. 4.2.4], there is a bijection between the poset of proper pure submodules of d and the poset of proper subspaces of $V := d \otimes K$. Let V be an n-dimensional vector space over a field K. According to [16], the Tits building $T(V)$ is defined to be the simplicial complex whose p-simplices are

$$V_0 < \cdots < V_p; \qquad V_0 > 0,\ V_p < V.$$

We use Quillen's model of the suspension of the Tits building, $\Sigma T(V)$, which is a simplicial complex where a p-simplex is a flag of subspaces

$$W_0 < \cdots < W_p$$

with either $0 < W_0$ or $W_p < V$. This does give a model of suspension of $T(V)$ for $\dim(V) > 1$, and when $\dim(V) = 1$, $\Sigma T(V)$ consists of two points, whereas $T(V) = \emptyset$. We denote by $J(V)$ the ordered set of proper layers in V. This consists of pairs (W_0, W_1) of subspaces of V, with $W_0 \subset W_1$, excluding the pair $(0, V)$, and where $(W_0, W_1) \leq (W_0', W_1')$ if $W_0' \subset W_0 \subset W_1 \subset W_1'$. $J(V)$ has the homotopy type of the suspension of the Tits building, $\Sigma T(V)$.

Lemma 10. *For any $d \in Q^{\mathrm{tf}}(A)$ of rank n, the functor $Q_{n-1}^{\mathrm{tf}}(A) \downarrow d \to J(d \otimes K) = J(V)$ sending*

$$
\begin{tikzcd}
 & c' \arrow[dl, "\phi"'] \arrow[dr] & \\
c & & d
\end{tikzcd}
$$

to $(ker(\phi) \otimes K, c' \otimes K)$ is an isomorphism of categories.

Proof. Since c' and $ker(\phi)$ are pure submodules of d, we see that this functor is fully faithful and bijective on objects.

□

The Solomon-Tits theorem ([16, Theorem 2]) says that $T(V)$ has the homotopy type of a bouquet of $(n-2)$-spheres for $n \geq 2$. The Steinberg module $St(V)$ will be defined as

$$St(V) := \begin{cases} H_{n-2}(T(V)) & ; \quad n \geq 2 \\ \mathbb{Z} & ; \quad n = 1 \end{cases}$$

Definition 11. *[[7], (1)] The reduced Steinberg module, $\widetilde{St}(V)$, is defined as*

$$\widetilde{St}(V) := \begin{cases} St(V) & ; \ \dim(V) > 2 \\ Ker(St(V) \to \mathbb{Z}) & ; \ \dim(V) = 2 \\ \mathbb{Z} & ; \ \dim(V) = 1 \\ \mathbb{Z} & ; \ \dim(V) = 0 \end{cases}$$

Let us write $Q_{n-1} := Q^{\text{tf}}_{n-1}$, $Q_n := Q^{\text{tf}}_n$ for short and denote $T_n : Q_{n-1} \hookrightarrow Q_n$ the inclusion functor (which is connected cellular by definition). By [6, Thm 2.3.6], we are able to describe the homotopy mapping cone of $C_*(Q_{n-1}) \hookrightarrow C_*(Q_n)$ as $C_*(Q_n - Q_{n-1}, \widetilde{\mathbb{F}_{T_n}})[1]$. If we denote $D^1_{p,q} = H_{p+q}(Q_p)$ and $E^1_{p,q} = H_{p+q-1}(Q_p, \widetilde{\mathbb{F}_{T_p}})$, we get an exact couple and hence a spectral sequence ([6, theorem 2.4.1]) converging to the homology groups of $BQ^{\text{tf}}(A)$

$$E^1_{p,q} = H_{p+q-1}(Q_p(A) - Q_{p-1}(A), \widetilde{\mathbb{F}_p}) \Rightarrow H_{p+q}(BQ^{\text{tf}}(A)).$$

This spectral sequence is called the rank spectral sequence. Moreover, since for any $d \in Q_n - Q_{n-1}$, $\mathbb{F}_{T_n}(d) = T_n \downarrow d \simeq J(V)$, the rank spectral sequence becomes

$$E^1_{n,i-n+1} = H_i(Q_n - Q_{n-1}, \widetilde{\mathbb{F}_{T_n}}) \simeq \bigoplus_d H_{i-n+1}(Aut(d), \widetilde{St}(V)) \Rightarrow H_{i+1}(BQ^{\text{tf}}(A))$$

where d runs over the isomorphism classes of torsion-free A-modules of rank n. We denote by $QP(A)$ the Quillen's Q-construction over the category of finitely generated projective A-modules. By Quillen's resolution theorem, if A is moreover regular, the inclusion $Q^{\text{tf}}(A) \hookrightarrow QP(A)$ is a weak equivalence. In this case, the rank spectral sequence converges to $H_*(BQP(A))$.

We notice that, in particular, if A is a Dedekind domain, then all projective modules are torsion-free and hence $Q^{\text{tf}}(A) = QP(A)$ in this case. So our construction above generalizes Quillen's setup in [16, Section 3]. The idea above applies to the case of an integral scheme X and the Q-construction over the category of finitely generated coherent sheaves $Q^{\text{coh}}(X)$ (resp. finitely generated torsion-free sheaves $Q^{\text{tf}}(X)$). Please go to [6, Section 4] for more details.

3.2. **Some Homotopy Properties.** Let X be a set and $E(X)$ be the simplicial complex of finite non-empty subsets of X. Let $\mathcal{P}_f(X) := \mathbf{Simpl}(E(X))$ be the ordered set of simplices of $E(X)$. By [7, Lemma 1], $\mathcal{P}_f(X)$ and $E(X)$ are contractible.

Let V be a finite dimensional vector space over a field K such that $\dim(V) \geq 1$. If $X \subset V$ is a non-empty finite subset, we denote $\langle X \rangle$ the subspace of V generated by X.

Definition 12.

(1) *We define $E^*(V)$ to be the subsimplicial complex of $E(V)$ such that*

$$Vert(E^*(V)) = Vert(E(V))$$

and if $0 \in X \in \mathbf{Simpl}(E^*(V))$ then $\langle X \rangle < V$.

(2) *$\mathcal{P}^*(V)$ is defined as the ordered set $\mathbf{Simpl}(E^*(V))$.*

Definition 13. *Let V be a finite dimensional vector space and $W_0, W_1 \subseteq V$ be two subspaces such that $W_0 \leq W_1$. We define*

$$\mathcal{P}_f(W_0, W_1) = \{X \in \mathcal{P}_f(W_1) \mid W_0 \leq \langle X \rangle\}.$$

Lemma 14. *$\mathcal{P}_f(W_0, W_1)$ is contractible.*

Proof. Let $B \subseteq W_1$ be a non-empty finite subset such that $W_0 = \langle B \rangle$ and we define $\mathcal{P}_f(W_0, W_1)_B$ the sub ordered set of $\mathcal{P}_f(W_0, W_1)$ whose elements always contain B. Then the inclusion $\mathcal{P}_f(W_0, W_1)_B \hookrightarrow \mathcal{P}_f(W_0, W_1)$ has a left adjoint $X \mapsto X \cup \{B\}$. So $\mathcal{P}_f(W_0, W_1)_B$ and $\mathcal{P}_f(W_0, W_1)$ are homotopy equivalent. Since $\mathcal{P}_f(W_0, W_1)_B$ admits a smallest element B, it is contractible. It follows that $\mathcal{P}_f(W_0, W_1)$ is also contractible.

□

Lemma 15. *The order preserving map*

$$g' : \mathbf{Simpl}(B\mathcal{P}^*(V)) \to J(V); \qquad X_0 < \cdots < X_p \mapsto (\langle X_0 \rangle, \langle X_p \rangle)$$

is a homotopy equivalence.

Proof. We will prove this lemma by using Quillen's theorem A. Let $(W, W') \in J(V)$ be a proper layer of V. We consider the category

$$g' \downarrow (W, W') = \{X_0 < \cdots < X_p \in \mathbf{Simpl}(B\mathcal{P}^*(V)) \mid W \leq \langle X_0 \rangle \leq \langle X_p \rangle \leq W'\}.$$

If $W' \neq V$, we see that $g' \downarrow (W, W') = \mathbf{Simpl}(B\mathcal{P}_f(W, W'))$ which is contractible by lemma 14.

It leaves us to prove that $g' \downarrow (W, V)$ with $(W, V) \in J(V)$ is contractible. Let $\mathcal{P}^*(V)_W$ be the sub-ordered set of $\mathcal{P}^*(V)$ whose object X satisfies $W \leq \langle X \rangle$. We denote $0 \notin B \subset V$ a finite subset such that $W = \langle B \rangle$ and $\mathcal{P}^*(V)_B$ the sub-ordered set of $\mathcal{P}^*(V)$ whose objects contain B. The natural inclusion $\mathcal{P}^*(V)_B \hookrightarrow \mathcal{P}^*(V)_W$ admits a left adjoint $X \mapsto X \cup \{B\}$ and hence is a homotopy equivalence. Since $\mathcal{P}^*(V)_B$ has a minimal object B, it is contractible which implies that $\mathcal{P}^*(V)_W$ is also contractible. We conclude by noticing that $g' \downarrow (W, V) = \mathbf{Simpl}(B\mathcal{P}^*(V)_W)$ which is contractible.

□

Proposition 16 ([16], section 2)**.** *Suppose $n \geq 2$. There is a $GL(V)$-equivariant homotopy equivalence*

$$g : \mathbf{Simpl}(\Sigma T(V)) \to J(V); \qquad (W_0 < \cdots < W_p) \mapsto (W_0, W_p).$$

Corollary 17. *The simplicial complexes $E^*(V)$ and $\Sigma T(V)$ are homotopy equivalent.*

Proof. According to Proposition 16, there is a homotopy equivalence $\mathbf{Simpl}(\Sigma T(V)) \xrightarrow{\approx} J(V)$. Since $B \circ \mathbf{Simpl} = sd$, by lemma 15, we see that $\mathbf{Simpl}(sdE^*(V))$ and $\mathbf{Simpl}(\Sigma T(V))$ are homotopy equivalent. It follows that $sdE^*(V)$ and $\Sigma T(V)$ are homotopy equivalent (c.f, argument at the beginning of Chapter 4) and hence so do $E^*(V)$ and $\Sigma T(V)$.

□

By Proposition 16 and Corollary 17, we see that $E^*(V)$ and $J(V)$ are homotopy equivalent.

3.3. **Extended Modular Symbols.** Let $n := \dim(V) \geq 2$. By definition, we have $C_*(E(V)) = C_*(E^*(V))$ for $* \leq n-1$. Since $E(V)$ is contractible ([7, Lemma 1]), the complex $C_*(E(V))$ is acyclic for $* > 0$. It follows that

$$Ker(C_{n-1}(E^*(V)) \xrightarrow{\partial} C_{n-2}(E^*(V))) = Im(C_n(E(V)) \xrightarrow{\partial} C_{n-1}(E(V)))$$

So we have

$$H_{n-1}(E^*(V)) = Im(C_n(E(V)) \xrightarrow{\partial} C_{n-1}(E(V)))/Im(C_n(E^*(V)) \xrightarrow{\partial} C_{n-1}(E^*(V)))$$

and the sequence

$$C_{n+1}(E(V)) \xrightarrow{\partial} C_n(E(V))/C_n(E^*(V)) \xrightarrow{\partial} H_{n-1}(E^*(V)) \to 0 \tag{3.1}$$

is exact. By definition 12, the symbol $(g_0, \cdots, g_i) \in C_i(E^*(V))$ if it satisfies one the following two conditions

- none of the vectors in $\{g_0, \cdots, g_i\}$ is zero.
- the collection of vectors $\{g_0, \cdots, g_i\}$ contains zero and $\langle g_0, \cdots, g_i\rangle < V$.

Thus $C_n(E(V))/C_n(E^*(V))$ is a free Abelian group on symbols $(g_0, \cdots, g_n)$ such that one of the vectors is zero and the others are linearly independent. Moreover, if $g_0, \cdots, g_{n+1} \in V$, then

$$0 = \partial \circ \partial(g_0, \cdots, g_{n+1}) = \sum_{i=0}^{n+1}(-1)^i \partial(g_0, \cdots, \widehat{g_i}, \cdots, g_{n+1}) \in C_{n-1}(E(V)) = C_{n-1}(E^*(V)).$$

So, when $n = \dim(V) \geq 2$, $H_{n-1}(E^*(V))$ is generated by symbols $\partial(g_0, g_1, \cdots, g_n)$ and presented by the relations

(a): if the collection of vectors $\{g_0, \cdots, g_n\}$ does not contain zero or it contains zero but does not generate V, then $\partial(g_0, g_1, \cdots, g_n) = 0$.

(b): the alternating sum

$$\sum_{i=0}^{n+1}(-1)^i \partial(g_0, \cdots, \widehat{g_i}, \cdots, g_{n+1})$$

equals zero.

We will call the symbols $\partial(g_0, \cdots, g_n)$ the extended (modular) symbols.

Proposition 18. *The extended symbols $\partial(g_0, \cdots, g_n)$ satisfy the relations*

(a0): *Swapping two vectors in $\partial(g_0, g_1, \cdots, g_n)$ changes the sign;*

(b0): $\partial(ag_0, g_1, \cdots, g_n) = \partial(g_0, g_1, \cdots, g_n)$ *for any $a \in K - \{0\}$.*

Proposition 19. *When $n = \dim(V) \geq 2$, The homology group $H_{n-1}(E^*(V))$ is generated by symbols $\partial(0, g_1, \cdots, g_n)$ and presented by the relations*

(1) *If $g_1, \cdots, g_n$ are linearly dependent then $\partial(0, g_1, \cdots, g_n) = 0$.*
(2) *The alternating sum*

$$\sum_{i=1}^{n+1}(-1)^i \partial(0, g_1, \cdots, \widehat{g_i}, \cdots, g_{n+1})$$

equals zero.

Proof. We have seen that $H_{n-1}(E^*(V))$ is generated by symbols $\partial(g_0, g_1, \cdots, g_n)$ and presented by relations (a) and (b) above. In particular, these symbols satisfy the properties listed in proposition 18. Since a non-zero symbol consists one zero vector, we can thus permute 0 to the beginning after a proper sign change. It follows that $H_{n-1}(E^*(V))$ is generated by symbols $\partial(0, g_1, \cdots, g_n)$ which equal zero if $g_1, \cdots .g_n$ are linearly dependent, thus the relation 1.

We notice that, by sequence (3.1), if none of $g_0, \cdots, g_{n+1}$ is zero, then each summand in the alternating sum $\partial(g_0, \cdots, g_n)$ is zero in $C_n(E(V))/C_n(E^*(V))$. If two of these vectors are zero, the symbol $C_n(E(V))/C_n(E^*(V)) \ni (g_0, \cdots, \widehat{g_i}, \cdots, g_{n+1})$ is zero if $g_i \neq 0$. Suppose $g_j = g_k = 0$, then by (a0) it is easy to see that

$$(-1)^j \partial(g_0, \cdots, \widehat{g_j}, \cdots, g_n) + (-1)^k \partial(g_0, \cdots, \widehat{g_k}, \cdots, g_n) = 0.$$

If more than three vectors are zero, we see that these vectors do not generate V, so by (a), each symbol $(g_0, \cdots, \widehat{g_i}, \cdots, g_{n+1}) = 0$.

It leaves us to consider the case where exactly one of these vectors is zero. According to the formula given in (b), if $g_0 = 0$, we get the relation 2. If, say, $g_j = 0$ for some $0 < j \leq n$, then we use (a0) of proposition 18 which implies that

$$\partial(g_0, \cdots, \widehat{g_i}, \cdots, g_n) = \begin{cases} (-1)^{j-1}\partial(0, g_0, \cdots, \widehat{g_i}, \cdots, \widehat{g_j}, \cdots, g_n); & i < j \\ (-1)^{j}\partial(0, g_0, \cdots, \widehat{g_j}, \cdots, \widehat{g_i}, \cdots, g_n); & i > j \end{cases}$$

Moreover, by (a), we have $\partial(g_0, \cdots, \widehat{g_j}, \cdots, g_n) = 0$. It follows from (b) that

$$\sum_{i=0}^{j-1}(-1)^{i+j-1}\partial(0, g_0, \cdots, \widehat{g_i}, \cdots, \widehat{g_j}, \cdots, g_n) + \sum_{i=j+1}^{n}(-1)^{i+j}\partial(0, g_0, \cdots, \widehat{g_j}, \cdots, \widehat{g_i}, \cdots, g_n)$$

equals zero which implies the relation 2.

$\square$

Remark 20. *In* [20, Theorem 34]*, we show that the correspondence*

$$\partial(0, g_1, \cdots, g_n) \leftrightarrow [g_1, \cdots, g_n]$$

gives a bijection between extended symbols and modular symbols in the sense of Ash-Rudolph in [1]*. The reason we need extended symbols is that they form a set of generators of $\widetilde{H}_{n-1}(\Sigma T(V)) \simeq \widetilde{H}_{n-1}(\mathbb{F}_{T_n}(d))$ for $d \in \mathcal{D} - \mathcal{C}$ and $V = d \otimes K$ which was used in the explicit form of rank spectral sequence (c.f, Section 3.1).*

4. Calculation of d^1 on Coefficents

In this section, we will put d^1 into distinguished triangles of singular homology groups of simplicial complexes so that we can use the extended symbols (and hence modular symbols) to calculate d^1. According to lemma 10, the category $\mathbb{F}_T(d)$ is isomorphic to the ordered set $J(V)$ of proper layers of V so that we use a proper layer of $J(V)$ to denote an object in $\mathbb{F}_T(d)$ and vice versa.

4.1. Some Homology Properties.

Definition 21 ([18], p186)**.** *We say that a commutative diagram of simplicial complexes*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "i'"'] & B \arrow[d, "j"] \\ C \arrow[r, "j'"'] & D \end{tikzcd}$$

is a Mayer-Vietoris diagram if all the maps are inclusions and

$$D = B \cup C, \qquad A = B \cap C.$$

According to [18, p186], a Mayer-Vietoris diagram induces a short exact sequence of (reduced) chain complexes

$$0 \to \widetilde{C}_*(A) \xrightarrow{(i_*, -i'_*)} \widetilde{C}_*(B) \oplus \widetilde{C}_*(C) \xrightarrow{(j_*, j'_*)} \widetilde{C}_*(D) \to 0$$

where $\widetilde{C}_*$ denotes the reduced chain complex, i.e, $\widetilde{C_*} = Ker(C_* \to \mathbb{Z})$.

We define the simplicial complex

$$E^{<n}(V) := \bigcup_{W<V, \dim(W)=n-1} E(W).$$

Let $E^{n-1}(V)$ be the subset of $\mathbf{Simpl}(E^*(V))$ with elements $(0, X)$ such that $\dim(\langle X \rangle) = n-1$. We take $E^{**}(V) := E^*(V) - E^{n-1}(V)$ which is a sub-simplicial complex of $E^*(V)$. Thus, we have a Mayer-Vietoris diagram

$$\begin{tikzcd} E^{<n}(V) \cap E^{**}(V) \arrow[r] \arrow[d] & E^{**}(V) \arrow[d] \\ E^{<n}(V) \arrow[r] & E^*(V) \end{tikzcd} \tag{4.1}$$

which induces a short exact sequence

$$\begin{aligned} 0 \to \widetilde{C}_*(E^{<n}(V) \cap E^{**}(V)) \to \widetilde{C}_*(E^{**}(V)) \oplus \widetilde{C}_*(E^{<n}(V)) \\ \to \widetilde{C}_*(E^*(V)) \to 0. \end{aligned} \tag{4.2}$$

Lemma 22. *Suppose that $(X, 0)$ is a pointed simplicial complex (0 be the base point) which is covered by a family of pointed sub-simplicial complexes $\{X_i\}_{i\in I}$. If the intersection $\bigcap_{j\in J} X_j$ is contractible for all non-empty subsets $J \subseteq I$, then the natural map*

$$\bigvee_{i\in I} X_i \to X$$

is a homology equivalence.

Proof. Let us define

$$X' := \bigvee_{i\in I} X_i$$

which is also covered by $X'_i = X_i,\ i \in I$ such that $X'_i \cap X'_{i'} = 0$ (the base point) for all $i \neq i' \in I$. So we have a map between E^1-terms of Mayer-Vietoris spectral sequences

$$\begin{aligned} E'^1_{p,q} = H_p\left(\coprod_{|J|=q, j_i\neq j_k} X'_{j_0} \cap \cdots \cap X'_{j_q}\right) = \bigoplus_{|J|=q, j_i\neq j_k} H_p\left(X'_{j_0} \cap \cdots \cap X'_{j_q}\right) \\ \longrightarrow E^1_{p,q} = H_p\left(\coprod_{|J|=q, j_i\neq j_k} X_{j_0} \cap \cdots \cap X_{j_q}\right) = \bigoplus_{|J|=q, j_i\neq j_k} H_p\left(X_{j_0} \cap \cdots \cap X_{j_q}\right). \end{aligned}$$

Here, $|J|$ denotes the cardinality of the set J. If the indices in $J \subseteq I$ are not identical then intersections of both sides are contractible and if the indices in J are identical then the corresponding map $H_p(\bigcap_J X'_{j_k}) \to H_p(\bigcap_J X_{j_k})$ is the identity map. So above map is an isomorphism. Since E^1 converges to $H_{p+q}\left(\bigcup_{i\in K} X_i\right)$ and E'^1 converges to $H_{p+q}(X')$, we get an homology isomorphism

$$H_n(X') \xrightarrow{\simeq} H_n\left(\bigcup_{i\in K} X_i\right).$$

□

Corollary 23. *$E^{<n}(V) \cap E^{**}(V)$ (resp. $E^{<n}(V)$) is homology equivalent to $\bigvee E^*(W)$ (resp. $\bigvee E(W)$) where the wedge sum is indexed by the set*

$$\{W \mid W < V, \dim(W) = n-1\}.$$

Proof. In the category of simplicial complexes, we have

$$E^{<n}(V) \cap E^{**}(V) = \bigcup_{W<V,\dim(W)=n-1} E^*(W), \qquad E^{<n}(V) = \bigcup_{W<V,\dim(W)=n-1} E(W)$$

which are covered by $E^*(W)$ and $E(W)$ respectively and have 0 as base points. Since the W's do not contain each other, any proper intersection satisfies

$$\bigcap_i E^*(W_i) = E\left(\bigcap_i W_i\right)$$

and is contractible ([7, Lemma 1]). It suffices to apply lemma 22. $\square$

This corollary implies that the distinguished triangle (4.2) can be replaced by (up to quasi-isomorphism)

$$\bigoplus_{\dim(W)=n-1} \widetilde{C}_*(E^*(W)) \to \widetilde{C}_*(E^{**}(V)) \to \widetilde{C}_*(E^*(V)) \xrightarrow{+1} \tag{4.3}$$

since $E^{<n}(V)$ has trivial reduced homology.

4.2. **d^1 in Terms of Singular Homologies of Simplicial Complexes.** Since $\mathbb{F}_{T'}(d)$ is discrete and consists of admissible monomorphisms and admissible epimorphisms, we shall distinguish these two cases and get the formula of d^1 on coefficients by combining them.

4.2.1. *The Formula for d^1 Having Image in Direct Sums of Reduced Steinberg Modules Indexed by Admissible Monomorphisms.*

Some Calculation of Singular Chains

Lemma 24. *The simplicial map*

$$AR' : sdE^*(V) \to \Sigma T(V), \qquad X \mapsto \langle X \rangle.$$

is a homotopy equivalence.

Proof. According to lemma 15, the functor

$$g' : \mathbf{Simpl}(sdE^*(V)) = \mathbf{Simpl}(B\mathcal{P}^*(V)) \to J(V), \qquad (X_0 < \cdots < X_p) \mapsto (\langle X_0 \rangle, \langle X_p \rangle)$$

is a homotopy equivalence. So, combined with proposition 16, we get the following commutative diagram

$$\begin{tikzcd}
\mathbf{Simpl}(sdE^*(V)) \arrow[r, "\mathbf{Simpl}\circ AR'"] \arrow[dr, "g'"'] & \mathbf{Simpl}(\Sigma T(V)) \arrow[d, "g", "\approx"'] \\
 & J(V)
\end{tikzcd}$$

It follows that $\mathbf{Simpl} \circ AR'$ (and hence $B \circ \mathbf{Simpl} \circ AR' = sd \circ AR'$) is a homotopy equivalence. According to ([7, (3)]), for any simplicial complex C there is a homotopy equivalence $\varepsilon_C : |sdC| \xrightarrow{\approx} |C|$ which is natural on C. So, $|AR'|$ is a homotopy equivalence and so is AR'.

$\square$

Moreover, for any sub simplicial complex $C \subseteq E^*(V)$, the map AR' sends sdC to a sub simplicial complex of $\Sigma T(V)$. In particular, $AR'(sdE^*(W)) = \Sigma T(W)$ for W a subspace of V. Applying the functor $AR' \circ sd$ to (4.1) we get a Mayer-Vietoris diagram

$$\begin{tikzcd} \bigcup_{\dim(W)=n-1} \Sigma T(W) \arrow[r] \arrow[d] & AR'(sdE^{**}(V)) \arrow[d] \\ AR'(sdE^{<n}(V)) \arrow[r] & \Sigma T(V) \end{tikzcd} \tag{4.4}$$

where $AR'(sdE^{<n}(V)) = \bigcup_W Ct(W)$ such that $Ct(W)$ is the simplicial complex (c.f, [16, Section 2]) whose p-simplices are

$$W_0 < \cdots < W_p$$

with W_i subspaces (may equals 0 or W) for each $0 \leq i \leq p$. It follows that, by lemma 22, $AR'(sdE^{<n}(V))$ has trivial reduced homology since the $Ct(W)$'s and their intersections are contractible. Lemma 22 also gives us the following canonical quasi-isomorphisms

$$\bigoplus_{\dim(W)=n-1} \widetilde{C}_*(E^*(W)) \xrightarrow{\sim} \widetilde{C}_*\left(\bigcup_{\dim(W)=n-1} E^*(W)\right)$$

and

$$\bigoplus_{\dim(W)=n-1} \widetilde{C}_*(\Sigma T(W)) \xrightarrow{\sim} \widetilde{C}_*\left(\bigcup_{\dim(W)=n-1} \Sigma T(W)\right).$$

Thus, the functor $AR' \circ sd$ from diagram (4.1) to diagram (4.4) gives a morphism between distinguished triangles

$$\begin{tikzcd} \bigoplus_{\dim(W)=n-1} \widetilde{C}_*(E^*(W)) \arrow[r] \arrow[d] & \widetilde{C}_*(E^{**}(V)) \arrow[r] \arrow[d] & \widetilde{C}_*(E^*(V)) \arrow[r, "+1"] \arrow[d] & {} \\ \bigoplus_{\dim(W)=n-1} \widetilde{C}_*(\Sigma T(W)) \arrow[r] & \widetilde{C}_*(AR'(sdE^{**}(V))) \arrow[r] & \widetilde{C}_*(\Sigma T(V)) \arrow[r, "+1"] & {} \end{tikzcd} \tag{4.5}$$

such that all vertical morphisms are quasi-isomorphisms.

d^1 in Terms of the Boundary Map of Mayer-Vietoris Sequence

Recall that there is a homotopy equivalence (c.f, [16, Prop. p10])

$$g : \mathbf{Simpl}(\Sigma T(V)) \to J(V), \qquad (W_0 < \cdots < W_p) \mapsto (W_0, W_p).$$

We define $J_1 := g \circ \mathbf{Simpl}(AR'(sdE^{**}(V))) \subset J(V)$ which is an ordered set consisting of proper layers (X, Y) such that if $Y \neq V$ then $\dim(Y/X) \leq n-2$ and (L, V) with $\dim(L) = 1$. In particular, the above order preserving map g gives a homotopy equivalence between $\mathbf{Simpl}(AR'(sdE^{**}(V)))$

and J_1. Therefore, we obtain a morphism of distinguished triangles

$$
\begin{tikzcd}
\bigoplus_{\dim(W)=n-1}\widetilde{C}_*(\Sigma T(W)) \arrow[r] \arrow[d] & \widetilde{C}_*(AR'(sdE^{**}(V))) \arrow[r] \arrow[d] & \widetilde{C}_*(\Sigma T(V)) \arrow[r, "+1"] \arrow[d] & {} \\
\bigoplus_{\dim(W)=n-1}\widetilde{C}_*(J(W)) \arrow[r] & \widetilde{C}_*(J_1) \arrow[r] & \widetilde{C}_*(J(V)) \arrow[r, "+1"'] & {}
\end{tikzcd}
\tag{4.6}
$$

such that all vertical morphisms are induced by the functor $g \circ \mathbf{Simpl}$ and hence are quasi-isomorphisms.

For any $d \in \mathcal{D} - \mathcal{C}$ such that $V = d \otimes K$, the inclusion functor $i' : \mathbb{F}_{TU}(d) \hookrightarrow J_1$ is connected cellular which gives rise to a homotopy cocartesian diagram

$$
\begin{tikzcd}
(J_1 - \mathbb{F}_{TU}(d)) \int \mathbb{F}_{i'} \arrow[r] \arrow[d] & \mathbb{F}_{TU}(d) \arrow[d, "i'"] \\
J_1 - \mathbb{F}_{TU}(d) \arrow[r] & J_1
\end{tikzcd}
$$

Since $\mathbb{F}_{TU}(d) \subset J(V)$ consists of proper layers (X, Y) such that $\dim(Y/X) \leq n-2$, by our description of J_1, we see that $J_1 - \mathbb{F}_{TU}(d)$ is a set (i.e, discrete category) consisting of proper layers (L, V) with $\dim(L) = 1$. Since $(L, V) \int \mathbb{F}_{i'} \simeq \mathbb{F}_{U_*}(L, V)$, the homotopy cocartesian diagram above gives us a distinguished triangle

$$
\bigoplus_{d \twoheadrightarrow c} C_*(\widetilde{\mathbb{F}_U}(c)) \to C_*(\widetilde{\mathbb{F}_{TU}}(d)) \to \widetilde{C}_*(J_1) \xrightarrow{+1}
\tag{4.7}
$$

Moreover, the lower triangle of (4.6) fits into the commutative diagram of distinguished triangles

$$
\begin{tikzcd}
\bigoplus_{d \twoheadrightarrow c \in \mathbb{F}_{T'}(d)} C_*(\widetilde{\mathbb{F}_U}(c)) \arrow[r, "="] \arrow[d] & \bigoplus_{d \twoheadrightarrow c} C_*(\widetilde{\mathbb{F}_U}(c)) \arrow[d] & & \\
\bigoplus_{c \rightarrow d \in \mathbb{F}_{T'}(d)} C_*(\widetilde{\mathbb{F}_U}(c)) \arrow[r] \arrow[d] & C_*(\widetilde{\mathbb{F}_{TU}}(d)) \arrow[r] \arrow[d] & C_*(\widetilde{\mathbb{F}_T}(d)) \arrow[r, "d^1"] \arrow[d, "="] & {} \\
\bigoplus_{\dim(W)=n-1}\widetilde{C}_*(J(W)) \arrow[r] & \widetilde{C}_*(J_1) \arrow[r] & \widetilde{C}_*(J(V)) \arrow[r, "+1"'] & {}
\end{tikzcd}
\tag{4.8}
$$

where the left vertical sequence is split short exact, the middle vertical distinguished triangle is the triangle (4.7) and the middle horizontal one is the triangle (2.20). Hence, there exists a commutative diagram

$$
\begin{tikzcd}
H_{n-1}(\widetilde{\mathbb{F}_T}(d)) \arrow[r, "d^1"] \arrow[d, "="] & \bigoplus_{c \rightarrow d \in \mathbb{F}_{T'}(d)} H_{n-2}(\widetilde{\mathbb{F}_U}(c)) \arrow[d] \\
\widetilde{H}_{n-1}(J(V)) \arrow[r, "+1"'] & \bigoplus_{\dim(W)=n-1}\widetilde{H}_{n-2}(J(W))
\end{tikzcd}
$$

The lower map of the above diagram is calculated as the boundary map of the lower triangle of diagram (4.8) which gives part of d^1. We will give its formula in this subsection.

The Formula

Together with (4.5) and (4.6), up to explicit quasi-isomorphisms, we are able to calculate one part of d^1 on coefficients through the boundary map of (4.3) and hence of (4.2) which fits into a Mayer-Vietoris sequence after applying the functor $H_*(-)$. More precisely, we take $[a] \in \widetilde{Z}_{n-1}(E^*(V))$ and lift it to $[a] = [a_1] + [a_2] \in \widetilde{C}_{n-1}(E^{<n}(V)) \oplus \widetilde{C}_{n-1}(E^{**}(V))$. Then we apply the differential map to $[a_2]$ to get $\partial([a_2]) = -\partial([a_1])$. The boundary map is then given by $[a] \mapsto -\partial([a_1])$.

We write $\widetilde{H}_{n-1}(E^*(V)) = \widetilde{St}(d)$ and $\widetilde{H}_{n-2}(E^*(W)) = \widetilde{St}(c)$ for $d \in \mathcal{D} - \mathcal{C}$ of rank n with $V = d \otimes K$ and $c \subset d$ a sub-module of rank $n-1$ with $W = c \otimes K$.

Proposition 25. *The boundary map of (4.2) is given by*

$$\widetilde{St}(d) \to \bigoplus_{c \rightarrowtail d \in \mathbb{F}_{T'}(d)} \widetilde{St}(c), \qquad \partial(0, g_1, \cdots, g_n) \mapsto -\sum_{i=1}^{n} (-1)^i ((0, W_i), \partial(0, g_1, \cdots, \widehat{g_i}, \cdots, g_n))$$

where $d \in \mathcal{D} - \mathcal{C}$, $W_i := \langle g_1, \cdots, \widehat{g_i}, \cdots, g_n \rangle$ for $1 \le i \le n$ and $(0, W_i)$ stands for the index of the summand.

Proof. We will use extended symbols. Consider

$$[a] := \partial(0, g_1, \cdots, g_n) = \sum_{i=1}^{n} (-1)^i (0, g_1, \cdots, \widehat{g_i}, \cdots, g_n) + (g_1, \cdots, g_n) \in \widetilde{Z}_{n-1}(E^*(V))$$

such that

$$(0, g_1, \cdots, \widehat{g_i}, \cdots, g_n) \in C_{n-1}(E(W_i)), \qquad (g_1, \cdots, g_n) \in C_{n-1}(E^{**}(V)).$$

It follows that we can lift $[a]$ to

$$[a'] = [a_1] + [a_2] = \sum_{i=1}^{n} (-1)^i (0, g_1, \cdots, \widehat{g_i}, \cdots, g_n) + (g_1, \cdots, g_n) \in C_*(E^{<n}(V)) \oplus C_*(E^{**}(V))$$

where $[a_1] = \sum (-1)^i (0, g_1, \cdots, \widehat{g_i}, \cdots, g_n) \in C_{n-1}(E^{<n}(V))$ and $[a_2] = (g_1, \cdots, g_n) \in C_{n-1}(E^{**}(V))$. Apply ∂ to $[a_2]$, we have

$$\partial[a_2] = -\partial[a_1] = -\sum_{i=1}^{n} (-1)^i \partial(0, g_1, \cdots, \widehat{g_i}, \cdots, g_n).$$

So we get the formula we are looking for

$$\partial(0, g_1, \cdots, g_n) \longmapsto -\sum_{i=1}^{n} (-1)^i ((0, W_i), \partial(0, g_1, \cdots, \widehat{g_i}, \cdots, g_n)).$$

□

4.2.2. *The Formula for d^1 Having Image in Direct Sums of Reduced Steinberg Modules Indexed by Admissible Epimorphisms.* In this section, we will calculate the other part of d^1 than proposition 25.

Some distinguished Triangles

Suppose $\dim(V) \geq 3$. We define J_2 as the sub-ordered set of $J(V)$ obtained by removing proper layers (L, V) with $\dim(L) = 1$. Thus, $J(V) - J_2 = \{(L, V) \mid \dim(L) = 1\}$ and the inclusion $i'' : J_2 \hookrightarrow J(V)$ is connected cellular. We have the following homotopy cocartesian diagram

$$\begin{tikzcd} \coprod_{(L,V)} J(L,V) \arrow[r] \arrow[d] & J_2 \arrow[d, "i''"] \\ \coprod_{(L,V)} \star \arrow[r] & J(V) \end{tikzcd}$$

where $J(L, V) = i'' \downarrow (L, V)$ denotes the ordered set of proper layers $\{(W, W') \mid (W, W') < (L, V)\}$. This diagram induces a distinguished triangle

$$\bigoplus_{\dim(L)=1} \widetilde{C}_*(J(L, V)) \to \widetilde{C}_*(J_2) \to \widetilde{C}_*(J(V)) \xrightarrow{+1}$$

We write X as the sub-simplicial complex of $B\mathcal{P}^*(V) = sdE^*(V)$ obtained by deleting the simplices

$$\{K_0 < \cdots < K_p \mid \dim(\langle K_0 \rangle) = 1, \ \langle K_p \rangle = V\}.$$

Lemma 26. *The order preserving map*

$$g' : \mathbf{Simpl}(X) \to J_2, \qquad (K_0 < \cdots < K_p) \mapsto (\langle K_0 \rangle, \langle K_p \rangle)$$

is a homotopy equivalence.

Proof. It suffices to restrict g' of lemma 15 to $\mathbf{Simpl}(X)$.

□

Suppose $\dim(L) = 1$. Let $B\mathcal{P}(V - \{0\})_L$ be the sub-simplicial complex of $B\mathcal{P}(V - \{0\}) = sdE(V - \{0\})$ whose vertices K satisfy $L \leq \langle K \rangle$ and we define X_L as the sub-simplicial complex of $B\mathcal{P}(V - \{0\})_L$ obtained by deleting the set of simplices

$$\{K_0 < \cdots < K_p \mid \langle K_0 \rangle = L, \ \langle K_p \rangle = V\}.$$

For two subspaces $W \subseteq W'$ of V, we denote

$$\mathcal{P}'_f(W, W') = \{K \in \mathcal{P}_f(W' - \{0\}) \mid W \leq \langle K \rangle\}.$$

Lemma 27.

(1) *The ordered set $\mathcal{P}'_f(W, W')$ is contractible.*
(2) *The functor g' restricted on X_L induces a homotopy equivalence*

$$g' : \mathbf{Simpl}(X_L) \to J(L, V).$$

Proof.

(1) Let $A \subset W' - \{0\}$ be a finite subset such that $\langle A \rangle = W$ and $\mathcal{P}'_f(W, W')_A$ be the sub-simplicial complex of $\mathcal{P}'_f(W, W')$ whose elements contain A. The natural inclusion $\mathcal{P}'_f(W, W')_A \hookrightarrow \mathcal{P}_f(W, W')$ admits a left adjoint $B \mapsto B \cup A$, so the two ordered sets are homotopy equivalent. Since $\mathcal{P}'_f(W, W')_A$ has a minimal element A, it is contractible. It follows that $\mathcal{P}'_f(W, W')$ is contractible.
(2) For any proper layer $(W, W') < (L, V)$, we consider the comma category

$$g' \downarrow (W, W') =$$
$$\{K_0 \leq \cdots \leq K_p \in \mathbf{Simpl}(B\mathcal{P}_f(V - \{0\})) \mid W \leq \langle K_0 \rangle \leq \langle K_p \rangle \leq W' - \{0\}\}$$

$$= \mathbf{Simpl}(B\mathcal{P}'_f(W,W')).$$

We conclude by 1. and Quillen's theorem A.

□

Corollary 28. *The natural chain map*

$$\bigoplus_{\dim(L)=1} \widetilde{C}_*(X_L) \to \widetilde{C}_*\left(\bigcup_{\dim(L)=1} X_L\right)$$

is a quasi-isomorphism.

Proof. By the proof of the previous lemma, it is easy to show that $\bigcup_L sdX_L \to \bigcup_L BJ(L,V)$ is a homotopy equivalence by using Quillen's theorem A. Moreover, $BJ(L,V)$ is a simplicial complex pointed at (V,V). If $L \neq L' \subset V$ we have $BJ(L,V) \cap BJ(L',V) = BJ'(L \oplus L',V)$ where $J'(L \oplus L',V)$ denotes the ordered set consisting of proper layers $(W,W') \leq (L \oplus L',V)$. Since $J'(L \oplus L',V)$ has a maximal element $(L \oplus L',V)$ it is contractible. Similarly, we can prove that $\bigcap_{L\in S} BJ(L,V)$ is also contractible for any subset S of $\{L \mid \dim(L) = 1\}$. Then, lemma 22 shows that $\bigvee_L BJ(L,V) \to \bigcup_L BJ(L,V)$ is a homology equivalence. Thus, we have a commutative diagram

$$\begin{tikzcd}
\bigoplus_L \widetilde{C}_*(sdX_L) \arrow[r, dashed] \arrow[d] & \widetilde{C}_*\left(\bigcup_L sdX_L\right) \arrow[d] \\
\bigoplus_L \widetilde{C}_*(BJ(L,V)) \arrow[r] & \widetilde{C}_*\left(\bigcup_L BJ(L,V)\right)
\end{tikzcd}$$

where all solid morphisms are quasi-isomorphisms, it follows that the dotted one is so. Hence the map in question is a quasi-isomorphism.

□

Since the diagram

$$\begin{tikzcd}
\bigcup_{\dim(L)=1} X_L \arrow[r] \arrow[d] & X \arrow[d] \\
sdE(V-\{0\}) \arrow[r] & sdE^*(V)
\end{tikzcd}$$

is Mayer-Vietoris, together with corollary 28, we get a short exact sequence fitting into the commutative diagram of distinguished triangles

$$\begin{tikzcd}
\bigoplus_{\dim(L)=1} \widetilde{C}_*(X_L) \arrow[r] \arrow[d] & \widetilde{C}_*(X) \oplus \widetilde{C}_*(sdE(V-\{0\})) \arrow[r] \arrow[d, "(g'\circ\mathbf{Simpl},0)"] & \widetilde{C}_*(sdE^*(V)) \arrow[d] \\
\bigoplus_{\dim(L)=1} \widetilde{C}_*(J(L,V)) \arrow[r] & \widetilde{C}_*(J_2) \arrow[r] & \widetilde{C}_*(J(V))
\end{tikzcd}$$

$$(4.9) \qquad \begin{tikzcd}
\arrow[r] & \bigoplus_L \widetilde{C}_*(X_L)[1] \arrow[d] \\
\arrow[r] & \bigoplus_L \widetilde{C}_*(J(L,V))[1]
\end{tikzcd}$$

where all vertical morphisms are quasi-isomorphisms.

As mentioned at the beginning of this section, we identify $\mathbb{F}_{T'}(d)$ with $J(V)$ and use a proper layer to denote an object of $\mathbb{F}_{T'}(d)$ and vice versa. Notice that J_2 consists of proper layers (W, W') with $\dim(W'/W) \leq n-2$ and $(0, W)$ with $\dim(W) = n-1$. Thus $\mathbb{F}_{TU}(d) \hookrightarrow J_2$ is connected cellular and we have a homotopy cocartesian diagram

$$\begin{tikzcd}
\coprod_{(0,W)} \mathbb{F}_U(W) \simeq \coprod_{(0,W)} \mathbb{F}_{U_*}(0,W) \arrow[r] \arrow[d] & \mathbb{F}_{TU}(d) \arrow[d, hook] \\
\coprod_{(0,W)} \star \arrow[r] & J_2
\end{tikzcd}$$

which gives rise to a distinguished triangle

$$\bigoplus_{c \rightarrowtail d \in \mathbb{F}_{T'}(d)} C_*(\widetilde{\mathbb{F}_U}(c)) \to C_*(\widetilde{\mathbb{F}_{TU}}(d)) \to C_*(J_2) \xrightarrow{+1} \tag{4.10}$$

Moreover, the lower triangle of (4.9) fits into the commutative diagram of distinguished triangles

$$\begin{tikzcd}
\bigoplus_{c \rightarrowtail d \in \mathbb{F}_{T'}(d)} C_*(\widetilde{\mathbb{F}_U}(c)) \arrow[r, "="] \arrow[d] & \bigoplus_{c \rightarrowtail d \in \mathbb{F}_{T'}(d)} C_*(\widetilde{\mathbb{F}_U}(c)) \arrow[d] & & \\
\bigoplus_{c \to d \in \mathbb{F}_{T'}(d)} C_*(\widetilde{\mathbb{F}_U}(c)) \arrow[r] \arrow[d] & C_*(\widetilde{\mathbb{F}_{TU}}(d)) \arrow[r] \arrow[d] & C_*(\widetilde{\mathbb{F}_T}(d)) \arrow[r, "d^1"] \arrow[d, "="] & {} \\
\bigoplus_{\dim(L)=1} \widetilde{C}_*(J(L,V)) \arrow[r] & \widetilde{C}_*(J_2) \arrow[r] & \widetilde{C}_*(J(V)) \arrow[r, "+1"'] & {}
\end{tikzcd} \tag{4.11}$$

where the left vertical sequence is split short exact, the middle vertical distinguished triangle is the triangle (4.10) and the middle horizontal one is the triangle (2.20).

d^1 in Terms of the Boundary Map of Mayer-Vietoris Sequence

Our discussion above implies that we have a commutative diagram

$$\begin{tikzcd}
H_{n-1}(\widetilde{\mathbb{F}_T}(d)) \arrow[r, "d^1"] \arrow[d, "="] & \bigoplus_{c \to d \in \mathbb{F}_{T'}(d)} H_{n-2}(\widetilde{\mathbb{F}_U}(c)) \arrow[d] \\
ai\widetilde{H}_{n-1}(J(V)) \arrow[r, "+1"'] & \bigoplus_{\dim(L)=1} \widetilde{H}_{n-2}(J(L,V))
\end{tikzcd}$$

The lower map of the above diagram is calculated as the boundary map of the lower triangle of diagram (4.11) which gives the other part of d^1. We will give its formula in this subsection. So, together with (4.9), we see that the remaining part of d^1 can be calculated via the boundary map of the top distinguished triangle of (4.9).

Let $X'_L := sdE^*(V/L)$. The quotient map $V - \{0\} \to V/L$ induces a simplicial map $q_L : X_L \to X'_L$. By lemma 15, the functor

$$g' : \mathbf{Simpl}(X'_L) \to J(V/L), \qquad (K'_0, \cdots, K'_p) \mapsto (\langle K'_0 \rangle, \langle K'_p \rangle)$$

is a homotopy equivalence. Moreover, the functor

$$J(L,V) \to J(V/L), \qquad (W, W') \mapsto (W/L = \overline{W}, W'/L = \overline{W'})$$

is an isomorphism. So there exists a commutative diagram

$$\begin{tikzcd}
X_L \arrow[r, "q_L"] \arrow[d, "g'", "\approx"'] & X'_L \arrow[d, "g'"', "\approx"] \\
J(L,V) \arrow[r, "\simeq"'] & J(V/L)
\end{tikzcd}$$

It follows that q_L is a homotopy equivalence. Moreover, in the following diagram

$$\begin{tikzcd}
\bigoplus_{\dim(L)=1} \widetilde{C}_*(X_L) \arrow[r] \arrow[d, "\bigoplus q_{L*}"] & \widetilde{C}_*(X) \oplus \widetilde{C}_*(sdE(V-\{0\})) \arrow[r] \arrow[d, "="] & \widetilde{C}_*(sdE^*(V)) \arrow[d, "="] \\
\bigoplus_{\dim(L)=1} \widetilde{C}_*(sdE^*(V/L)) \arrow[r] \arrow[d, "\varepsilon_*"] & \widetilde{C}_*(X) \oplus \widetilde{C}_*(sdE(V-\{0\})) \arrow[r] \arrow[d, "(1,\varepsilon_*)"] & \widetilde{C}_*(sdE^*(V)) \arrow[d, "\varepsilon_*"] \\
\bigoplus_{\dim(L)=1} \widetilde{C}_*(E^*(V/L)) \arrow[r] & \widetilde{C}_*(X) \oplus \widetilde{C}_*(E(V-\{0\})) \arrow[r] & \widetilde{C}_*(E^*(V))
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & \bigoplus_{\dim(L)=1} \widetilde{C}_*(X_L)[1] \arrow[d] \\
{} \arrow[r] & \bigoplus_{\dim(L)=1} \widetilde{C}_*(sdE^*(V/L))[1] \arrow[d] \\
{} \arrow[r] & \bigoplus_{\dim(L)=1} \widetilde{C}_*(E^*(V/L))[1]
\end{tikzcd} \tag{4.12}$$

the middle and lower triangles are quasi-isomorphic to the top one, so they are distinguished triangles. It suggests that up to explicit quasi-isomorphisms we can use the bottom triangle to calculate our formula. By the naturality of ε (c.f, [7, (3)]), we first calculate the boundary map of the second horizontal distinguished triangle. Since the top triangle is Mayer-Vietoris, we take $[z] \in \widetilde{Z}_{n-1}(sdE^*(V))$ and lift it as $[z] = [z_1] + [z_2]$ such that $[z_1] \in \widetilde{C}_{n-1}(X)$ and $[z_2] \in \widetilde{C}_{n-1}(sdE(V - \{0\}))$. Then we apply the differential map to get the homology class

$$\partial([z_1]) = -\partial([z_2]) \in \widetilde{H}_{n-2}\left(\bigcup_L X_L\right) \simeq \bigoplus_L \widetilde{H}_{n-2}(X_L).$$

We need to calculate the image of $\partial([z_1]) = -\partial([z_2])$ in $\bigoplus \widetilde{H}_{n-2}(sdE^*(V/L))$ under $\bigoplus q_{L*}$.

There exists a commutative diagram for each $L \subset V$ with $\dim(L) = 1$

$$\begin{tikzcd}
\widetilde{C}_{n-1}\left(\bigcup_L B\mathcal{P}(V-\{0\})_L\right) \arrow[r, equal] & \widetilde{C}_{n-1}(sdE(V-\{0\})) \arrow[r, "q'_L"] \arrow[d, "\partial"] & \widetilde{C}_{n-1}(sdE(V/L)) \arrow[d, "\partial"] \\
\widetilde{Z}_{n-2}\left(\bigcup_L X_L\right) \arrow[r, hook] & \widetilde{Z}_{n-2}(sdE(V-\{0\})) \arrow[r, "q'_L"'] & \widetilde{Z}_{n-2}(sdE^*(V/L))
\end{tikzcd}$$

where q'_L is induced by the simplicial map $E(V-\{0\}) \to E(V/L)$ sending a vertex v to the quotient class $\bar{v}$ represented by v. Notice that since

$$sdE^*(V-\{0\}) = B\mathcal{P}(V-\{0\}) = \bigcup_{\dim(L)=1} B\mathcal{P}(V-\{0\})_L,$$

we may then write

$$[z_2] = \sum_L n_L[y_L], \qquad [y_L] \in \widetilde{C}_{n-1}(B\mathcal{P}(V-\{0\})_L).$$

In particular, if $[y_L]$ also belong to $\widetilde{C}_{n-1}(B\mathcal{P}(V-\{0\})_{L'})$ with $L \neq L'$, then it must lie in $\widetilde{C}_{n-1}(X_L)$ (and in $\widetilde{C}_{n-1}(X_{L'})$) and hence

$$0 = \partial[y_L] \in \widetilde{H}_{n-2}\left(\bigcup_L X_L\right).$$

Therefore, we may write

$$\partial[z_1] = -\partial[z_2] = \sum n_L \partial[y_L] \in \bigoplus_L \widetilde{H}_{n-2}(X_L)$$

such that each $[y_L]$ in a unique $\widetilde{C}_{n-1}(B\mathcal{P}(V-\{0\})_L)$. Moreover, if $[y_{L'}] \in \widetilde{C}_{n-1}(B\mathcal{P}(V-\{0\})_{L'})$ and $L \neq L' \subset V$, then

$$0 = q'_L(\partial[y_{L'}]) = \partial q'_L([y_{L'}]) \in \widetilde{H}_{n-2}(sdE^*(V/L))$$

since $q'_L[y_{L'}] \in \widetilde{C}_{n-1}(sdE^*(V/L))$. Thus we find that

$$\sum_L q'_L\left(\sum n_L \partial[y_L]\right) = \sum n_L q'_L(\partial[y_L]) \in \bigoplus_L \widetilde{H}_{n-2}(sdE^*(V/L))$$

and it suffices to calculate the image of $-\partial([z_2])$ under the morphisms

$$\widetilde{Z}_{n-2}(sdE(V-\{0\})) \xrightarrow{q'_L} \widetilde{Z}_{n-2}(sdE^*(V/L)) \to \widetilde{H}_{n-2}(sdE^*(V/L))$$

for all lines $L \subset V$.

Proposition 29. *We write $\widetilde{St}(V/L) := \widetilde{H}_{n-2}(E^*(V/L))$. The second part of d^1 on coefficients*

$$\widetilde{St}(d) \to \bigoplus_{(L,V)\in\mathbb{F}_{T'}(d)} \widetilde{St}(V/L)$$

is given by

$$\partial(0, g_1, \cdots, g_n) \mapsto \sum_{i=1}^{n} (-1)^i((L_i, V), \partial(0, \bar{g}_1, \cdots, \widehat{\bar{g}_i}, \cdots, \bar{g}_n))$$

where $L_i := \langle g_i \rangle$ and $\bar{g}_j$ denotes the image of g_j under the quotient map $V \to V/L_i$ for all $1 \leq i \neq j \leq n$. The proper layers (L_i, V) stand for the index of the summands.

Proof. By our discussion above, the boundary map of the bottom triangle can be calculated as follows. Let $[a] \in \widetilde{Z}_{n-1}(E^*(V))$ and we lift $[a]$ to

$$[a'] = [a'_1] + [a'_2] \in \widetilde{C}_{n-1}(X) \oplus \widetilde{C}_{n-1}(E(V-\{0\})).$$

We have $\partial([a'_1]) \in \widetilde{Z}_{n-2}(X)$ and $\partial([a'_2]) \in \widetilde{Z}_{n-2}(E(V-\{0\}))$. Then, we calculate the image of $-\partial([a'_2])$ under the morphism

$$\widetilde{Z}_{n-2}(E(V-\{0\})) \to \widetilde{Z}_{n-2}(E^*(V/L)) \to \widetilde{H}_{n-2}(E^*(V/L))$$

to find the image of $[a]$ in $\bigoplus_L \widetilde{St}(V/L)$. Here, the first map is induced by the simplicial map $E(V-\{0\}) \to E(V/L)$ sending a vertex v to the quotient class $\bar{v}$ represented by v.

Let us take

$$0 \neq [a] = \partial(0, g_1, \cdots, g_n) = \sum_{i=1}^{n}(-1)^i(0, g_1, \cdots, \widehat{g_i}, \cdots, g_n) + (g_1, \cdots, g_n) \in \widetilde{Z}_{n-1}(E^*(V))$$

and write $[a] = [a_1] + [a_2]$ such that

$$[a_1] = \sum_{i=1}^{n}(-1)^i(0, g_1, \cdots, \widehat{g_i}, \cdots, g_n), \qquad [a_2] = (g_1, \cdots, g_n).$$

We lift $[a] = [a_1] + [a_2]$ to $[a'] = [a'_1] + [a'_2]$ with $[a'_1] \in \widetilde{C}_*(X)$ and

$$[a'_2] = [a_2] = (g_1, \cdots, g_n) \in \widetilde{C}_*(E(V - \{0\})).$$

By our discussion in section 4.3, $\widetilde{St}(V/L)$ is generated by the extended symbols $\partial(\bar{v}_0, \cdots, \bar{v}_{n-1})$ such that if none of $\bar{v}_0, \cdots, \bar{v}_{n-1}$ is zero or some $\bar{v}_i = 0$ but they do not span V/L then $\partial(\bar{v}_0, \cdots, \bar{v}_{n-1}) = 0$. So the image of $-\partial([a_2])$ under the morphism

$$\widetilde{Z}_{n-2}(E(V - \{0\})) \to \widetilde{H}_{n-2}(E^*(V/L)) = \widetilde{St}(V/L)$$

equals zero if none of $g_i, 1 \leq i \leq n$ generates L, and equals

$$-\partial(\bar{g}_1, \cdots, \bar{g}_{i-1}, 0, \bar{g}_{i+1}, \cdots, \bar{g}_n) = (-1)^i(0, \bar{g}_1, \cdots, \widehat{g_i}, \cdots, \bar{g}_n) \in \widetilde{St}(V/L_i)$$

for each $\langle g_i \rangle = L_i$ with $1 \leq i \leq n$. Thus we get the formula

$$\partial(0, g_1, \cdots, g_n) \mapsto \sum_{i=1}^{n}(-1)^i((L_i, V), \partial(0, \bar{g}_1, \cdots, \widehat{g_i}, \cdots, \bar{g}_n)).$$

□

4.2.3. *The Formula of d^1 on Coefficients.*

Theorem 30. *The formula of d^1 is given by*

$$\partial(0, g_1, \cdots, g_n) \longmapsto$$
$$-\sum_{i=1}^{n}(-1)^i((0, W_i), \partial(0, g_1, \cdots, \widehat{g_i}, \cdots, g_n)) + \sum_{i=1}^{n}(-1)^i((L_i, V), \partial(0, \bar{g}_1, \cdots, \widehat{g_i}, \cdots, \bar{g}_n)).$$

Proof. It suffices to notice that, by our discussion in previous sections, there exists a commutative diagram

$$\begin{tikzcd}
H_{n-1}(\widetilde{\mathbb{F}_T}(d)) \arrow[r, "d^1"] \arrow[d, "\simeq"'] & \bigoplus_{c\to d\in\mathbb{F}_{T'}(d)} H_{n-2}(\widetilde{\mathbb{F}_U}(c)) \arrow[d, "\simeq"] \\
\widetilde{St}(d) \arrow[r] & \Big(\bigoplus_{(0,W)\in\mathbb{F}_{T'}(d)} \widetilde{St}(c)\Big) \oplus \Big(\bigoplus_{(L,V)\in\mathbb{F}_{T'}(d)} \widetilde{St}(V/L)\Big)
\end{tikzcd}$$

where $W = c \otimes K$.

□

Remark 31. *According to remark 20, the formula for d^1 on coefficients theorem 30 can be written as*

$$[g_1, \cdots, g_n] \longmapsto$$
$$-\sum_{i=1}^{n}(-1)^i((0,W_i),[g_1,\cdots,\widehat{g_i},\cdots,g_n]) + \sum_{i=1}^{n}(-1)^i((L_i,V),[\bar{g}_1,\cdots,\widehat{g_i},\cdots,\bar{g}_n]).$$

5. The Formula for $d^1|_{E^1_{n\geq 3,0}}$

5.1. The Induced Functor.

Definition 32. *Suppose that $f : \mathcal{C}_1 \to \mathcal{C}_2$ and $\mathbb{F} : \mathcal{C}_1 \to$ **Cat** are two functors and for each $y_2 \in \mathcal{C}_2$ the category $\mathbb{F}_f(y_2) \neq \emptyset$. We define*

$$\mathrm{Ind}_f\mathbb{F} : \mathcal{C}_2 \to \mathbf{Cat}, \qquad y_2 \mapsto \mathbb{F}_f(y_2) \int \mathbb{F} \circ \pi_{y_2}$$

where $\pi_{y_2} : \mathbb{F}_f(y_2) \to \mathcal{C}_1$ is the projection.

Lemma 33. *There is a canonical isomorphism*

$$H_*(\mathcal{C}_2, \mathrm{Ind}_f\mathbb{F}) \simeq H_*(\mathcal{C}_1, \mathbb{F}).$$

Proof. We define the projection functor

$$p_1 : \mathcal{C}_2 \int \mathrm{Ind}_f\mathbb{F} \to \mathcal{C}_1 \int \mathbb{F}, \qquad \{(y_1 \to y_2, x) \mid x \in \mathbb{F}(y_1)\} \mapsto \{(y_1, x) \mid x \in \mathbb{F}(y_1)\}$$

and

$$s_1 : \mathcal{C}_1 \int \mathbb{F} \to \mathcal{C}_2 \int \mathrm{Ind}_f\mathbb{F}, \qquad \{(y_1, x) \mid x \in \mathbb{F}(y_1)\} \mapsto \{(y_1 = y_1, x) \mid x \in \mathbb{F}(y_1)\}.$$

It is easy to verify that s_1 is left adjoint to p_1 and hence the categories $\mathcal{C}_2 \int \mathrm{Ind}_f\mathbb{F}$ and $\mathcal{C}_1 \int \mathbb{F}$ are homotopy equivalent and hence the isomorphism we are looking for.

□

This lemma tells us that we have a pair of canonical isomorphisms

$$H_*(\mathcal{C}_1, \mathbb{F}) \simeq H_*(\mathcal{C}_2, \mathrm{Ind}_f\mathbb{F}).$$

Combined with Eilenberg-Zilber-Cartier theorem, we obtain

$$H_q(\mathcal{C}_1, H_*(\mathbb{F})) \simeq H_q(\mathcal{C}_2, H_*(\mathrm{Ind}(_f\mathbb{F}))), \qquad q \geq 0.$$

Suppose that we are under the condition given at the beginning of section 5.1. We apply our discussion to

$$\begin{tikzcd}
\mathcal{C}-\mathcal{B} \arrow[r, "*"] \arrow[d, "T'"'] & \mathbf{Cat} \\
\mathcal{D}-\mathcal{B} \arrow[ur, "\mathrm{Ind}_{T'}*"'] &
\end{tikzcd}$$

Notice that $\mathrm{Ind}_{T'}* = \mathbb{F}_{T'}$, so the projection functor $(\mathcal{D}-\mathcal{B}) \int \mathbb{F}_{T'} \xrightarrow{p_1} C-\mathcal{B}$ given by $(c \to d) \to c$ admits a left adjoint $c \to (c = c)$ and hence is a homotopy equivalence. So we have

$$C_*\left((\mathcal{D}-\mathcal{B}) \int \mathbb{F}_{T'}, \widetilde{\mathbb{F}_{U_*}}\right)$$

$$= Ker\left(C_*\left((\mathcal{D}-\mathcal{B})\int \mathbb{F}_{T'},\mathbb{F}_{U_*}\right)\to C_*\left((\mathcal{D}-\mathcal{B})\int \mathbb{F}_{T'}\right)\xrightarrow{\simeq} C_*(\mathcal{C}-\mathcal{B})\right)$$

$$= Ker\left(C_*\left((\mathcal{D}-\mathcal{B})\int \mathbb{F}_{T'},\mathbb{F}_{U_*}\right)\xrightarrow{p_{1*}} C_*(\mathcal{C}-\mathcal{B},\mathbb{F}_U)\to C_*(\mathcal{C}-\mathcal{B})\right).$$

It follows that there is a canonical quasi-isomorphism

$$C_*\left((\mathcal{D}-\mathcal{B})\int \mathbb{F}_{T'},\widetilde{\mathbb{F}_{U_*}}\right)\xrightarrow{\simeq} C_*(\mathcal{C}-\mathcal{B},\widetilde{\mathbb{F}_U})$$

and hence by Thomason's theorem (theorem 5) we get canonical isomorphisms

$$p_{1*}: H_q\left(\mathcal{D}-\mathcal{B},H_{n-2}(\mathbb{F}_{T'}\int\widetilde{\mathbb{F}_{U_*}})\right)\simeq H_q\left((\mathcal{D}-\mathcal{B})\int\mathbb{F}_{T'},H_{n-2}(\widetilde{\mathbb{F}_{U_*}})\right)$$

$$\xrightarrow{\simeq} H_q(\mathcal{C}-\mathcal{B},H_{n-2}(\widetilde{\mathbb{F}_U})). \tag{5.1}$$

5.2. **The Formula for d^1.** We notice that the canonical quasi-isomorphism $C_*(\mathcal{D}-\mathcal{B},\mathbb{F}_{T'})\to C_*((\mathcal{D}-\mathcal{B})\int\mathbb{F}_{T'})$ induced by Thomason's theorem 5 sends the simple chain

$$\begin{array}{ccccccc} d_0 & \longrightarrow & d_1 & \longrightarrow & \cdots & \longrightarrow & d_p \\ \uparrow & & & & & & \\ c_0 & = & c_0 & = & \cdots & = & c_0 \end{array}$$

to

$$\begin{array}{ccccccc} d_0 & \longrightarrow & d_1 & \longrightarrow & \cdots & \longrightarrow & d_p \\ \uparrow & & \uparrow & & & & \uparrow \\ c_0 & = & c_0 & = & \cdots & = & c_0 \end{array}$$

Since $\mathbb{F}_{T'}(d)$ is discrete for $d\in\mathcal{D}-\mathcal{C}$ and if $d\in\mathcal{C}$ then $H_{n-1}(\widetilde{\mathbb{F}_T(d)})=0$. Apply the functor $C_q(\mathcal{D}-\mathcal{B},-)$ to the formula obtained in proposition 25, we get

$$C_q(\mathcal{D}-\mathcal{B},H_{n-1}(\widetilde{\mathbb{F}_T}))\to C_q\left(\mathcal{D}-\mathcal{B},H_{n-2}(\mathbb{F}_{T'}\int\widetilde{\mathbb{F}_{U_*}})\right)$$

sending $0\neq\sum_d n_d(d_0\to\cdots\to d_q,\sum_g n_g[g_1,\cdots,g_n])$ with $d_j\in\mathcal{D}-\mathcal{C}$ for $0\le j\le q$ to

$$\sum_d n_d\left(d_0\to\cdots\to d_q,\sum_g n_g\left(-\sum_{i=1}^n(-1)^i((0,W_i),[g_1,\cdots,\widehat{g_i},\cdots,g_n])\right.\right.$$
$$\left.\left.+((L_i,V),[\bar{g}_1,\cdots,\widehat{\bar{g}_i},\cdots,\bar{g}_n])\right)\right).$$

Combined with (5.1), we have proved that

Theorem 34. *Let A be an integral Noetherian ring and $Q^{\mathrm{tf}}(A)$ be Quillen's Q-construction over the category of finitely generated torsion-free modules. Then the differential $E^1_{n,q}\xrightarrow{d^1_{n,q}}E^1_{n-1,q}$ is given by*

$$\sum_d n_d\left(d_0\xrightarrow{f_1}\cdots\xrightarrow{f_q}d_q,\sum_g n_g[g_1,\cdots,g_n]\right)$$

$$\longmapsto \sum_{d,g} n_d n_g \left(\sum_i (-1)^i \left(-(c_i^0 \to \cdots \to c_i^q, [g_1, \cdots, \widehat{g_i}, \cdots, g_n]) \right.\right.$$
$$\left.\left. +(c_i^{'0} \to \cdots \to c_i^{'q}, [\bar{g}_1, \cdots, \widehat{g_i}, \cdots, \bar{g}_n]) \right) \right).$$

Here, $(0, W_i) = (0, c_i^0)$ *is an admissible monomorphism and* $c_i^j \to d_j = (f_j \circ \cdots \circ f_1)(0, W_i)$*, meanwhile* $(L_i, V) = d_0 \twoheadrightarrow c_i^{'0}$ *is an admissible epimorphism and* $(c_i^{'j} \to d_j) = (f_j \circ \cdots \circ f_1)(L_i, V)$*.*

Remark 35. *For any* $1 \leq i \neq j \leq n$*, the map* d^1 *sends the simple chain*

$$(d_0 \to \cdots \to d_q, \sum_g n_g [g_1, \cdots, g_n])$$

to

$$\sum_i (-1)^i \left(-(c_i^0 \to \cdots \to c_i^q, [g_1, \cdots, \widehat{g_i}, \cdots, g_n]) + (c_i^{'0} \to \cdots \to c_i^{'q}, [\bar{g}_1, \cdots, \widehat{g_i}, \cdots, \bar{g}_n]) \right).$$

According to [15, p17, diagram (4)]*, any morphism under Q-construction can be factored as an injective- followed-by-surjective map or a surjective-followed-by-injective map, and the factorizations are unique up to unique isomorphisms. In the language of proper layers, this can be written as*

$$(L_i, W_j) \circ (0, W_j) = (L_i, W_j) \circ (L_i, V) = (L_i, W_j).$$

So if we apply d^1 *again, the above chain will further be sent to*

$$\sum_{i=1}^{n} (-1)^i \left(\sum_{k=1}^{n-1} (-1)^k \left((b_{ik}^0 \to \cdots \to b_{ik}^q, [g_1, \cdots, \widehat{g_i}, \cdots, \widehat{g_k}, \cdots, g_n]) \right.\right.$$
$$\left.\left. -(b_{ik}^{'0} \to \cdots \to b_{ik}^{'q}, [\bar{g}_1, \cdots, \widehat{g_i}, \cdots, \widehat{g_k}, \cdots, \bar{g}_n]) \right) \right) +$$
$$\sum_{i=1}^{n} (-1)^i \left(\sum_{k=1}^{n-1} (-1)^k \left(-(b_{ik}^{'0} \to \cdots \to b_{ik}^{'q}, [\bar{g}_1, \cdots, \widehat{g_i}, \cdots, \widehat{g_k}, \cdots, \bar{g}_n]) \right.\right.$$
$$\left.\left. +(b_{ik}^{''0} \to \cdots \to b_{ik}^{''q}, [\bar{g}_1, \cdots, \widehat{g_i}, \cdots, \widehat{g_k}, \cdots, \bar{g}_n]) \right) \right)$$

such that $k = j$ *if* $1 \leq j < i$ *and* $k = j - 1$ *if* $i < j \leq n$*. Here,* $b_{ik}^0 = \langle g_1, \cdots, \widehat{g_i}, \cdots \widehat{g_k}, \cdots, g_n \rangle \cap c_i^0$*,* $b_{ik}^{'0} \to c_i^0 = (L_k, W_i)$*,* $b_{ik}^{'0} \to c_i^{'0} = (L_i, W_k)$ *and* $b_{ik}^{''0} \to c_0' = (L_k, V/L_i) = (L_k \oplus L_i, V)$*. Thus we find that in the above formula, a summand with same two entries omitted appears twice with opposite signs, so the resulting alternating sum is zero which verifies that* $d^1 \circ d^1 = 0$*.*

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