Solution to wave equation (String with fixed BC)

(PDE) $\partial_{tt} u = \partial_{xx} u \qquad x\in(0,\pi),\ t\geq 0$

① should be $\partial_t u$, NOT $\partial_x u$

BC: $u(0,t) = \underbrace{\sin t}_{F(t)}$; $\quad u(\pi,t)=0$

② $\partial_t u(\pi,0)=0$ should be 0

IC: $u(x,0) = \underbrace{\sin x}_{f(x)}$; $\quad \partial_t u(x,0) = \underbrace{\cos x}_{g(x)}$

(so as to be compatible w/ $u(\pi,t)=0$).

1>. Homogenize BC: $\quad u = v + w \qquad w = (1-\frac{x}{\pi})\sin t \quad F(t)$

$v = u - w \quad \rightarrow \quad$ BC: $v(0,t) = u(0,t) - w(0,t) = 0$;

$v(\pi,t) = u(\pi,t) - w(\pi,t) = 0$

$\partial_{tt} v = \partial_{tt} u - \partial_{tt} w \overset{(PDE)}{=} \partial_{xx} u - \partial_{tt} w \qquad \partial_{tt} w = (1-\frac{x}{\pi})\underbrace{(-1)\sin t}_{F''(t)}$

$= \partial_{xx} v + \partial_{xx} w - \partial_{tt} w \qquad \partial_{xx} w = 0$

(WE) $\begin{cases} \partial_{tt} v = \partial_{xx} v + (1-\frac{x}{\pi})\sin t \ \ (-F''(t)) \\ \text{BC: } v(0,t) = v(\pi,t) = 0 \\ \text{IC: } v(x,0) = \sin x ; \quad \partial_t v(x,0) = \cos x - (1-\frac{x}{\pi})\ F(0) \end{cases}$

$\partial_t v(x,0) = \partial_t u(x,0) - \partial_t w(x,0)$

$= \underbrace{\cos x}_{g(x)} - (1-\frac{x}{\pi})\cos 0 \quad F'(t)$

$f(x) - (1-\frac{x}{\pi})F(0) = \tilde{f}(x)$

$= g(x) - (1-\frac{x}{\pi})F'(0) = \tilde{g}(x)$

$Q(x,t) = -(1-\frac{x}{\pi})F''(t) = \sum_{n=1}^{\infty} q_n(t)\sin nx \quad \Rightarrow \quad q_n(t) = -F''(t)\ \frac{1}{\pi}\int_0^{\pi}(1-\frac{x}{\pi})\sin nx\, dx$

2> Solve wave Eq. of v by eigenfunction expansion. $\quad$ BC & (WE) $\Rightarrow \quad \phi_n(x) = \sin nx$

$v(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin nx \xrightarrow{\text{TBTD}} a_n''(t) = -n^2 a_n(t) + q_n(t)$

$v(x,0) = \sum_{n=1}^{\infty} a_n(0)\sin nx = \underbrace{\sin x}_{\tilde{f}(x)} \Rightarrow \quad a_1(0)=1;\quad a_n(0)=0,\ n\geq 1 \qquad a_n(0) = \frac{1}{\pi}\int_0^{\pi}\tilde{f}(x)\sin nx\, dx$

$\partial_t v(x,0) = \sum_{n=1}^{\infty} a_n'(0)\sin nx = \underbrace{\cos x - (1-\frac{x}{\pi})}_{\tilde{g}(x)} \Rightarrow \quad a_n'(0) = \frac{1}{\pi}\int_0^{\pi}\square\sin nx\, dx; \quad a_n'(0) = \frac{1}{\pi}\int_0^{\pi}\tilde{g}(x)\sin nx\, dx$

use FSsine.m.

solve the 2nd-order ODE $\begin{cases} a_n''(t) = -n^2 a_n(t) + q_n(t) \\ a_n(0);\quad a_n'(0) \end{cases}$