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ESTIMATING THE EIGENVALUES OF STURM–LIOUVILLE PROBLEMS BY APPROXIMATING THE DIFFERENTIAL EQUATION*

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Abstract. This paper is concerned with computing accurate approximations to the eigenvalues and eigenfunctions of regular Sturm–Liouville differential equations. The method consists of replacing the coefficient functions of the given problem by piecewise polynomial functions and then solving the resulting simplified problem. Error estimates in terms of the approximate solutions are established and numerical results are displayed. Since the asymptotic properties for Sturm–Liouville systems are preserved by the approximation, the relative error in the higher eigenvalues is much more uniform than is the case for finite difference or Rayleigh–Ritz methods.

1. Introduction. This paper is concerned with approximating the eigenvalues and eigenfunctions of the regular Sturm–Liouville problem

$$-(pv')' - qv = \lambda rv, \qquad p > 0, \quad r > 0, \quad x \in [a, b] \tag{1a}$$

with the separable linear homogeneous boundary conditions

$$B_a v = a_0 v(a) - a_1 p(a) v'(a) = 0, \qquad a_0 \geqq 0, \quad a_1 \geqq 0, \quad a_0 + a_1 \neq 0, \tag{1b}$$

$$B_b v = b_0 v(b) + b_1 p(b) v'(b) = 0, \qquad b_0 \geqq 0, \quad b_1 \geqq 0, \quad b_0 + b_1 \neq 0. \tag{1c}$$

If $p \in C^1[a, b]$ and $q, r \in C[a, b]$, then there exists a sequence of simple eigenvalues $\{\lambda_k\}$ and a corresponding sequence of orthogonal eigenfunctions $\{u_k\}$.

The most prevalent of existing methods for approximating the solutions of (1) are variational [3], [6], [11] and finite difference methods [10], [15]. Each of these techniques reduces the problem to a finite-dimensional algebraic eigenvalue problem. As a result, approximations to only the first few eigenvalues are practical. Moreover, the error for even moderately large k is considerable unless the dimension of the associated matrix is very large.

The method presented here remedies this shortcoming of Rayleigh–Ritz and finite difference methods. It is a generalization of an algorithm of Canosa and Gomes de Oliveira [5]. Their idea is to approximate the coefficient functions of (1) by step functions. The resulting simplified problem can be solved exactly to give approximations, $\hat{\lambda}_k$, to the λ_k.

The idea of replacing coefficient functions by piecewise polynomial functions has been employed before to estimate the solutions of differential equations [12, p. 154], [1], [8], but no one has discussed convergence of the approximations.

In this paper the idea of Canosa and Gomes de Oliveira is generalized to include mth degree piecewise polynomial approximations of the coefficient functions, and error bounds for the approximate eigenvalues and eigenfunctions are established. We also discuss the behavior of the relative error $(\lambda_k - \hat{\lambda}_k)/\lambda_k$ as a function of k. The paper concludes with a discussion of the numerical methods used and tables of numerical results for several sample problems.

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2. The approximate problem. By $D^k[a, b]$ we mean the space of k-times piecewise continuously differentiable functions on $[a, b]$, where we assume

(2) $$f^{(j)}(x) = \tfrac{1}{2}(f^{(j)}(x^+) + f^{(j)}(x^-)), \qquad j = 0, 1, \cdots, k.$$

The function space $\mathscr{A}$ is given by

(3) $$\mathscr{A} = \{v \in D^2[a, b] \cap C[a, b] \mid B_a v = B_b v = 0\},$$

where B_a and B_b are the boundary conditions in (1).

For $v, w \in \mathscr{A}$ we define

(4) $$(v, w)_D = \int_a^b rvw\, dx,$$

(5) $$D(v) = \sqrt{(v, v)_D},$$

(6) $$N(v) = \int_a^b (pv'^2 - qv^2)\, dx + p(a)v'(a)v(a) - p(b)v'(b)v(b).$$

Then $(\cdot, \cdot)_D$ is an inner product on $\mathscr{A}$ and D is a norm on $\mathscr{A}$. Furthermore, the Rayleigh quotient of (1) is given by

(7) $$R(v) = N(v)/D(v), \qquad v \neq 0, \quad v \in \mathscr{A};$$

the eigenvalues of (1) are characterized by

(8) $$\lambda_k = \{\inf R(v) \mid 0 \neq v \in \mathscr{A} \text{ and } (v, u_j)_D = 0, j = 1, 2, \cdots, k - 1\}$$

and

(9) $$\lambda_k = \inf\{\sup R(v) \mid 0 \neq v \in T_k\},$$

where the infimum is taken over all k-dimensional subspaces T_k of $\mathscr{A}$. The eigenfunctions $\{u_k\}$ satisfy

(10) $$(u_j, u_k)_D = 0, \qquad j \neq k.$$

To determine the eigenfunctions uniquely they are normalized to satisfy the following initial conditions (consistent with (1b)):

(11) $$u_k(a) = \begin{cases} 1, & a_1 \neq 0, \\ 0, & a_1 = 0, \end{cases} \qquad (pu_k')(a) = \begin{cases} a_0/a_1, & a_1 \neq 0, \\ 1, & a_1 = 0. \end{cases}$$

Since $p, q \in C[a, b]$ there is a constant c_1 such that for all $v \in \mathscr{A}$,

$$N(v) \geqq c_1 \|v\|_{1,2}^2 \qquad \left(\|v\|_{1,2}^2 = \int_a^b (v'^2 + v^2)\, dx \right).$$

If $c_1 \leqq 0$, the problem can be trivially transformed into a positive definite Sturm–Liouville problem whose eigenvalues are $\{\lambda_k - c_1\}$. Thus it is no loss to assume there exists a $c > 0$ such that

(12) $$N(v) \geqq c\|v\|_{1,2}^2 \quad \text{for all } v \in \mathscr{A}.$$

As stated earlier we consider only approximate problems whose coefficients are piecewise polynomial functions. These approximations are described in terms of linear projectors, i.e., linear idempotent maps. The following notation is used: Π is the set of all partitions of $[a, b]$ having the form $\pi = \{a = x_1 < x_2 < \cdots < x_{N+1} = b\}$; for any $\pi \in \Pi$, let $h = \max_n (x_{n+1} - x_n)$. P_m is the space of polynomials of degree at most m.

For $p, q, r \in C^{m+1}[a, b]$, by an mth degree approximate problem we mean that of finding λ, v such that

$$v, p_\pi v' \quad \text{are in } C[a, b], \tag{13a}$$

$$a_0 v(a) - a_1 p_\pi(a) v'(a) = 0, \tag{13b}$$

$$b_0 v(b) + b_1 p_\pi(b) v'(b) = 0, \tag{13c}$$

$$-(p_\pi v')' - q_\pi v = \lambda r_\pi v, \tag{13d}$$

where p_π, q_π, r_π satisfy the following:

(i) there is a continuous linear projector Q from $C[-1, 1]$ onto P_m such that for each π on (x_n, x_{n+1}),

$$p_\pi(x) = Q(p(x(t))), \quad q_\pi(x) = Q(q(x(t))), \quad r_\pi(x) = Q(r(x(t)))$$

for $t \in (-1, 1)$, where $x(t) = \frac{1}{2}(x_{n+1} - x_n)(t + 1) + x_n$.

(ii) $p_\pi^{(j)}(a) = p^{(j)}(a), p_\pi^{(j)}(b) = p^{(j)}(b), j = 0, 1, \cdots, m$, and (2) holds for interior mesh points. The values of q_π and r_π at mesh points are defined similarly.

Note that because of (ii) the boundary conditions (13b)–(13c) are just $B_a v = B_b v = 0$ as in (1).

In what follows $\|\cdot\|$ is the sup-norm.

LEMMA 1. *If $p, q, r \in C^{m+1}[a, b]$ and p_π, q_π, r_π are as above, then there exists a positive constant K such that for each $\pi \in \Pi$,*

$$\|p^{(j)} - p_\pi^{(j)}\| \leqq K h^{m-j+1}, \tag{14a}$$

$$\|q^{(j)} - q_\pi^{(j)}\| \leqq K h^{m-j+1}, \tag{14b}$$

$$\|r^{(j)} - r_\pi^{(j)}\| \leqq K h^{m-j+1} \quad \textit{for } j = 0, 1, \cdots, m. \tag{14c}$$

Proof. It suffices to establish (14a). If Q is the given projector, then for each π on (x_n, x_{n+1}) let $F(t) = p(x(t))$, $\hat{F}(t) = p_\pi(x(t))$, $x(t) = \frac{1}{2}(x_{n+1} - x_n)(t + 1) + x_n$. Then $F \in C^{m+1}[-1, 1]$ and $\hat{F} = QF$. Thus

$$\sup_{(x_n, x_{n+1})} |p^{(j)}(x) - p_\pi^{(j)}(x)| = \left(\frac{2}{x_{n+1} - x_n}\right)^j \sup_{(-1,1)} |[(1 - Q)F]^{(j)}|. \tag{15}$$

For $j = 1, 2, \cdots, m$, Q induces on $C[-1, 1]$ a map Q_j defined as follows:

$$Q_j : C[-1, 1] \to P_{m-j} : F \to (Q(F^{(-j)}))^{(j)}.$$

It is easily seen that Q_j is a continuous linear projector from $C[-1, 1]$ onto P_{m-j}, so Lebesgue's inequality gives

$$\|(1 - Q_j)F^{(j)}\| \leqq \|1 - Q_j\| \operatorname{dist}(F^{(j)}, P_{m-j}), \quad j = 0, 1, \cdots, m. \tag{16}$$

But from Lorentz [13, p. 38] for $f \in C^{m-j+1}[-1, 1]$,

$$\operatorname{dist}(f, P_{m-j}) \leqq \frac{\|f^{(m-j+1)}\|}{2^{m-j}(m - j + 1)!}.$$

Applying this to (16) and substituting into (15) yields

$$(17)\qquad \sup_{(x_n,x_{n+1})} |p^{(j)}(x) - p_\pi^{(j)}(x)| \leqq K_j(x_{n+1} - x_n)^{m-j+1} \sup_{(x_n,x_{n+1})} |p^{(m+1)}(x)|,$$

where $K_j = \|1 - Q_j\|/(2^{2(m-j)+1}(m-j+1)!)$. The desired result then follows.

If $p_\pi > 0$, $r_\pi > 0$, then (13) is a Sturm–Liouville problem and from the above lemma this is assured for sufficiently fine partitions since $p > 0$, $r > 0$.

In what follows we assume that π is so restricted, in which case (13) has a countable number of real simple eigenvalues $\{\hat{\lambda}_k\}$ with corresponding eigenfunctions $\{\hat{u}_k\}$. We define $(\cdot,\cdot)_\pi$, $D_\pi(v)$, $N_\pi(v)$, $R_\pi(v)$ analogously to (4)–(7), which yields the following characterization of the eigenvalues:

$$(18)\qquad \hat{\lambda}_k = \{\inf R_\pi(v) | 0 \neq v \in \mathscr{A} \text{ and } (v, \hat{u}_j)_\pi = 0,\ j = 1, 2, \cdots, k-1\}.$$

Note that the set of admissible functions $\mathscr{A}$ for the Rayleigh quotient is the same here as above since the boundary conditions of the approximate problem are the same as for the original.

The eigenfunctions $\{\hat{u}_k\}$ satisfy $(\hat{u}_j, \hat{u}_k)_\pi = 0$, $j \neq k$, and are normalized analogously to (11).

3. Convergence theorems for general piecewise polynomial approximation schemes. The main theorem follows from the fact that $R_\pi(v)$ uniformly approximates $R(v)$ over $\mathscr{A}$ as is proved in the following lemma.

LEMMA 2. *If $p, q, r \in C^{m+1}[a, b]$, then for m-th degree approximate problems we have that there exist positive constants δ_1 and δ_2 such that for each π and for all $v \in \mathscr{A}$, $v \neq 0$, $h_{m+1} < 1/\delta_1$ implies*

$$(19)\qquad \frac{1 - \delta_2 h^{m+1}}{1 + \delta_1 h^{m+1}} R_\pi(v) \leqq R(v) \leqq \frac{1 + \delta_2 h^{m+1}}{1 - \delta_1 h^{m+1}} R_\pi(v).$$

Proof. For $v \in \mathscr{A}$, $v \neq 0$, (14) implies

$$|D_\pi(v) - D(v)| \leqq \|(r - r_\pi)/r\| D(v) \leqq K h^{m+1} D(v) / \min_x \{r(x)\}.$$

Similarly from (14) and (12),

$$|N(v) - N_\pi(v)| \leqq \max\{\|p - p_\pi\|, \|q - q_\pi\|\} \|v\|_{1,2}^2 \leqq (K/c) h^{m+1} N(v).$$

Set $\delta_1 = K/c$, $\delta_2 = K/\min_x\{r(x)\}$. Then $N_\pi(v) \geqq (1 - \delta_1 h^{m+1}) N(v)$; hence $h^{m+1} < 1/\delta_1$ implies $R_\pi(v) > 0$. But we also have $R(v) > 0$ by (12). Finally,

$$R(v) - R_\pi(v) = R_\pi(v) \frac{D_\pi(v) - D(v)}{D(v)} + R(v) \frac{N(v) - N_\pi(v)}{N(v)},$$

and by substituting the above inequalities and rearranging one has the desired results.

THEOREM 1. *If $p, q, r \in C^{m+1}[a, b]$, then for m-th degree approximate problems there exist positive constants δ_1 and δ_2 such that for every k, $h^{m+1} < 1/\delta_1$ implies*

$$(20)\qquad \frac{1 - \delta_2 h^{m+1}}{1 + \delta_1 h^{m+1}} \hat{\lambda}_k \leqq \lambda_k \leqq \frac{1 + \delta_2 h^{m+1}}{1 - \delta_1 h^{m+1}} \hat{\lambda}_k.$$

Proof. The eigenvalues are characterized by

$$\lambda_k = \inf\{\sup R(v) | 0 \neq v \in T_k\},$$
$$\hat{\lambda}_k = \inf\{\sup R_\pi(v) | 0 \neq v \in T_k\},$$

where the infima are taken over all k-dimensional subspaces T_k of $\mathscr{A}$. The conclusion follows from (19) since δ_1 and δ_2 are independent of v.

Since δ_1 and δ_2 are independent of k, (20) says that the relative error $|(\lambda_k - \hat{\lambda}_k)/\lambda_k|$ can be bounded independently of k and is $O(h^{m+1})$ as $h \to 0$. This behavior is examined more closely in § 6.

In order to establish results on the convergence of approximate eigenfunctions we need a convenient representation of the error which is done in terms of Green's functions for systems of differential equations. In what follows the partition π is fixed and we examine the error $u_k - \hat{u}_k$, $pu'_k - p_\pi \hat{u}'_k$; since the argument is independent of k this subscript is omitted.

Let

$$A = \begin{pmatrix} 0 & 1/p \\ -q - \lambda r & 0 \end{pmatrix}, \quad A_\pi = \begin{pmatrix} 0 & 1/p_\pi \\ -q_\pi - \hat{\lambda} r_\pi & 0 \end{pmatrix}.$$

Then

$$\begin{pmatrix} u \\ pu' \end{pmatrix}' = A \begin{pmatrix} u \\ pu' \end{pmatrix}, \quad \begin{pmatrix} \hat{u} \\ p_\pi \hat{u}' \end{pmatrix}' = A_\pi \begin{pmatrix} \hat{u} \\ p_\pi \hat{u}' \end{pmatrix},$$
$$\begin{pmatrix} u \\ pu' \end{pmatrix}(a) = \begin{pmatrix} \hat{u} \\ p_\pi \hat{u}' \end{pmatrix}(a).$$

If $V = \begin{pmatrix} v_1 & v_3 \\ v_2 & v_4 \end{pmatrix}$ is a fundamental system for $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}' = A \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$, i.e., $V' = AV$ with the additional restriction that $V(a) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then for any $f_1, f_2 \in D[a, b]$ the solution of

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}' = A \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

is known [2, p. 12] to be

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}(x) = V(x) \int_a^x V^{-1}(t) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} dt.$$

Set

$$X = \left\{ \bar{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \,\middle|\, w_1, w_2 \in C[a, b], \|w_1 - u\| \leqq 1, \|w_2 - pu'\| \leqq 1 \right\}$$

and define

$$T(\bar{w}) = \begin{pmatrix} u \\ pu' \end{pmatrix}(x) + V(x) \int_a^x V^{-1}(t)(A_\pi - A)\bar{w}\, dt.$$

From Lemma 1 and Theorem 1 for mth degree approximations, the entries of A_π converge uniformly to those of A as $h \to 0$. Thus for h sufficiently small, T is a contraction mapping on X and has a unique fixed point, say z, in X. But

$$z' - Az = (A_\pi - A)z, \quad \text{i.e.,} \quad z' = A_\pi z,$$

and $z(a) = \begin{pmatrix} \hat{u} \\ p_\pi \hat{u}' \end{pmatrix}(a)$, so by uniqueness, $\begin{pmatrix} \hat{u} \\ p_\pi \hat{u}' \end{pmatrix} = z$, which is in X. More importantly we have the representation

$$(21) \quad \begin{pmatrix} u \\ pu' \end{pmatrix} - \begin{pmatrix} \hat{u} \\ p_\pi \hat{u}' \end{pmatrix} = \begin{pmatrix} v_1 & v_3 \\ v_2 & v_4 \end{pmatrix}(x) \int_a^x \begin{pmatrix} v_4 & -v_3 \\ -v^2 & v_1 \end{pmatrix}(t)(A - A_\pi)\begin{pmatrix} \hat{u} \\ p_\pi \hat{u}' \end{pmatrix} dt.$$

It is now an easy matter to prove the following theorem.

THEOREM 2. *If $p, q, r \in C^{m+1}[a, b]$, then for m-th degree approximate problems, for h sufficiently small, we have as $h \to 0$,*

$$(22) \qquad \|u_k - \hat{u}_k\| = O(h^{m+1}), \quad \|pu_k' - p_\pi \hat{u}_k'\| = O(h^{m+1}) \quad \textit{for each } k.$$

Proof. Omitting subscripts we have for h sufficiently small, $\begin{pmatrix} \hat{u} \\ p_\pi \hat{u}' \end{pmatrix} \in X$, which guarantees that $\hat{u}$, $\hat{u}'$ can be bounded independently of π. But (21) says that

$$(23) \qquad (u - \hat{u})(x) = v_1(x)\int_a^x (v_4 f_1 - v_3 f_2)\,dt + v_3(x)\int_a^x (v_1 f_2 - v_2 f_1)\,dt,$$

where $f_1 = (p_\pi - p)\hat{u}'/p$, $f_2 = (q_\pi + \hat{\lambda} r_\pi - q - \lambda r)\hat{u}$. For m-th degree approximate problems we have $\|f_1\| = O(h^{m+1})$, $\|f_2\| = O(h^{m+1})$, which suffices to prove convergence of the approximate eigenfunctions. The argument for the derivatives is similar.

COROLLARY. *As $h \to 0$ for each k we have*

$$(24) \qquad \|u_k^{(j+1)} - \hat{u}_k^{(j+1)}\| = O(h^{m-j+1}), \qquad j = 1, 2, \cdots, m.$$

Proof. From the differential equation which $\hat{u}$ satisfies, $\|\hat{u}\|$, $\|\hat{u}'\|$ bounded independently of π implies $\|\hat{u}^{(j)}\|$ is also, for $j = 2, 3, \cdots, m + 2$. Now

$$u^{(j+1)} - \hat{u}^{(j+1)} = \{[(-q - \lambda r)(u - \hat{u}) + (q_\pi + \hat{\lambda} r_\pi - q - \lambda r)\hat{u} + (p_\pi - p)\hat{u}'' - p'(u - \hat{u}') + (p' - p_\pi')\hat{u}']/p\}^{(j-1)}$$

for $j = 1, 2, \cdots, m$ so the result is established by induction; the error at each step is limited by the last term $(p^{(j)} - p_\pi^{(j)})\hat{u}'/p$.

Before considering higher order results, let us mention several projectors which yield mth degree approximation schemes.

(a) Let Q be the map which takes $F \in C[-1, 1]$ into the mth degree polynomial which interpolates F on some arbitrary set $\{t_j\}_{j=1}^{m+1}$ in $[-1, 1]$. As an example, for $m = 1$, $t = -t_2 = 1$ on each (x_n, x_{n+1}) of a given partition π, we would have

$$(25) \qquad p_\pi(x) = p(x_n) + \frac{p(x_{n+1}) - p(x_n)}{x_{n+1} - x_n}(x - x_n), \quad \text{etc.}$$

(b) Let Q be the map which takes $F \in C[-1, 1]$ into its mth degree polynomial best approximation in the least squares sense. For example, when $m = 0$ we have for each π on (x_n, x_{n+1}),

$$(26) \qquad p_\pi(x) = \frac{1}{x_{n+1} - x_n}\int_{x_n}^{x_{n+1}} p(z)\,dz, \quad \text{etc.}$$

4. Higher order results for specific approximation schemes. The above results are based on the fact that the Rayleigh quotient of the given problem can be approximated uniformly over $\mathscr{A}$ by the Rayleigh quotient of an approximate problem. In this section we show that for the same degree of approximation the rates of convergence for the approximate eigenvalues can be doubled by a propitious choice of projectors. For such schemes the Rayleigh quotient is no longer approximated uniformly but only locally to the eigenfunction. By this we mean that hypothesis (ii) of the following lemma is satisfied; this leads in a straightforward manner to the inequalities on eigenvalue approximation in Theorem 3.

LEMMA 3. *If* (i) $p, q, r \in C^{m+1}[a, b]$, (ii) *for each* k *there exists* $K_1 > 0$ *such that for each* π, *when* $v = u_k$ *or* $v = \hat{u}_k$,

$$|N(v) - N_\pi(v)| \leqq K_1 h^{2m+2},$$

$$|D(v) - D_\pi(v)| \leqq K_1 h^{2m+2};$$

then for m-*th degree approximate problems we have for each* k:

$$|\lambda_k - \hat{\lambda}_k| = O(h^{2m+2}) \quad \text{as } h \to 0.$$

Proof. Set

$$v_k = u_k - \sum_{j=1}^{k-1} (u_k, \hat{u}_j)_\pi/(\hat{u}_j, \hat{u}_j)_\pi \hat{u}_j,$$

$$\hat{v}_k = \hat{u}_k - \sum_{j=1}^{k-1} (\hat{u}_k, u_j)_D/(u_j, u_j)_D u_j.$$

It can be shown (using Theorem 2) that for $j \neq k$,

$$(u_k, \hat{u}_j)_\pi^2/(\hat{u}_j, \hat{u}_j)_\pi = O(h^{2m+2}) \quad \text{as } h \to 0.$$

As a result there exists a positive constant C_1 such that

$$\begin{aligned} N_\pi(v_k) &= N_\pi(u_k) - \sum_{j=1}^{k-1} \lambda_j (u_k, \hat{u}_j)_\pi^2/(\hat{u}_j, \hat{u}_j) \\ &\leqq N_\pi(u_k) + C_1 h^{2m+2} \\ &\leqq N(u_k) + (K_1 + C_1) h^{2m+2}. \end{aligned}$$

Similarly there exists $C_2 > 0$ such that

$$D_\pi(v_k) \geqq D(u_k) - (K_1 + C_2) h^{2m+2}.$$

By construction, $(v_k, \hat{u}_j)_\pi = 0, j = 1, 2, \cdots, k-1$, so (18) implies (for h sufficiently small)

$$\hat{\lambda}_k \leqq R_\pi(v_k) \leqq \lambda_k + C_3 h^{2m+2} \quad \text{for some } C_3 > 0.$$

By a similar argument, $(\hat{v}_k, u_j)_D = 0$, $j = 1, 2, \cdots, k-1$, implies $\lambda_k \leqq R(\hat{v}_k)$ and there exists $C_4 > 0$ such that

$$\lambda_k \leqq \hat{\lambda}_k + C_4 h^{2m+2}.$$

THEOREM 3. *If* (i) $p, q, r \in C^{2m+2}[a, b]$, (ii) *for the linear projector* Q *there exists* K_1 *such that for* $F \in C^{2m+2}[-1, 1]$,

$$\left| \int_{-1}^{1} t^i (1 - Q) F \, dt \right| \leqq K_1 \|F^{(2m+2-i)}\|, \qquad i = 0, 1, \cdots, m; \tag{27}$$

then for m-th degree approximate problems as $h \to 0$,

(28) $$|\lambda_k - \hat{\lambda}_k| = O(h^{2m+2}) \quad \textit{for each } k.$$

Proof. Let $v = u_k$ or $\hat{u}_k$, $w = v^2$ and $z_n = \frac{1}{2}(x_n + x_{n+1})$. Then by expanding w in a Taylor series about z_n in each subinterval we have

(29) $$\int_{x_n}^{x_{n+1}} (r - r_\pi) w\, dx = \sum_{i=0}^{m} \left\{ \frac{w^{(i)}(z_n)}{i!} \int_{x_n}^{x_{n+1}} (x - z_n)^i (r - r_\pi)(x)\, dx \right\} + \frac{1}{(m+1)!} \int_{x_n}^{x_{n+1}} (x - z_n)^{m+1} (r - r_\pi)(x) w^{(m+1)}(\xi_x)\, dx$$

for some $\xi_x \in (x_n, x_{n+1})$. With $F(t) = r(x(t))$, $\hat{F}(t) = r_\pi(x(t))$, $x(t) = \frac{1}{2}(x_{n+1} - x_n) \cdot (t + 1) + x_n$, we have

$$\left| \int_{x_n}^{x_{n+1}} (x - z_n)^i (r - r_\pi)(x)\, dx \right| = \left(\frac{x_{n+1} - x_n}{2} \right)^{i+1} \left| \int_{-1}^{1} t^i (1 - Q) F\, dt \right|$$
$$\leqq \left(\frac{x_{n+1} - x_n}{2} \right)^{i+1} K_1 \|F^{(2m+2-i)}\|$$
$$= K_1 \left(\frac{x_{n+1} - x_n}{2} \right)^{2m+3} \|r^{(2m+2-i)}\|.$$

Moreover,

$$\left| \int_{x_n}^{x_{n+1}} (x - z_n)^{m+1} (r - r_\pi)(x) w^{(m+1)}(\xi_x)\, dx \right| \leqq \left(\frac{x_{n+1} - x_n}{2} \right)^{m+2} \|r - r_\pi\|\, \|w^{(m+1)}\|.$$

Substituting these results into (29) and summing one has

$$|D(v) - D_\pi(v)| = \left| \int_a^b (r - r_\pi) w\, dx \right| \leqq K_1 h^{2m+2}$$

for some constant K_1.

An analogous argument gives a similar bound for $|N(v) - N_\pi(v)|$ so the conclusion follows from Lemma 3.

A higher order result can also be established for eigenfunction approximation. Unfortunately one does not have the error bounded uniformly by h^{2m+2} but only the error at points of the partition π. The uniform rate of convergence is improved, however, by one power of h.

THEOREM 4. *If the hypotheses of Theorem* 3 *hold, then for each k there exists* $C > 0$ *such that for h sufficiently small, and every x in* $[a, b]$,

(30) $$|u_k(x) - \hat{u}_k(x)| \leqq Ch^{2m+2}, \qquad x \in \pi,$$

(31) $$|u_k(x) - \hat{u}_k(x)| \leqq Ch^{m+2}, \qquad x \notin \pi,$$

(32) $$|pu_k'(x) - p_\pi \hat{u}_k'(x)| \leqq Ch^{2m+2}, \qquad x \in \pi,$$

(33) $$|pu_k'(x) - p_\pi \hat{u}_k'(x)| \leqq Ch^{m+2}, \qquad x \notin \pi.$$

Proof. The procedure is similar to that of the previous theorem except that here we are concerned with the eigenfunction error as given by (23), i.e. (omitting subscripts),

$$(u - \hat{u})(x) = v_1(x)\int_a^x (v_4 f_1 - v_3 f_2)\,dt + v_3(x)\int_a^x (v_1 f_2 - v_2 f_1)\,dt.$$

The v_i are defined in the proof of Theorem 2 and here are in $C^{2m+2}[a, b]$. Set $w(t) = (v_4\hat{u}'/p)(t)$ so that for $x \in (x_n, x_{n+1})$,

$$\begin{aligned}\int_a^x v_4(t)f_1(t)\,dt &= \int_a^x (p_\pi - p)(t)(v_4\hat{u}'/p)(t)\,dt \\ &= \int_a^{x_n} (p_\pi - p)(t)w(t)\,dt + \int_{x_n}^x (p_\pi - p)(t)w(t)\,dt.\end{aligned}$$

The first integral is $O(h^{2m+2})$ by an argument similar to that of Theorem 3; the second is bounded by $h\|p - p_\pi\| \cdot \|w\|$ which is $O(h^{m+2})$. Note that the lower order term does not appear when $x \in \pi$, for then the integration is only over (a, x_n). The other terms can be approximated similarly to give the desired result; (32)–(33) follow by the same argument.

5. Examples giving higher order convergence. We seek approximation schemes whose associated projectors satisfy: there exists $K_1 > 0$ such that for each $F \in C^{2m+2}[-1, 1]$,

$$\left|\int_{-1}^{1} t^i(1 - Q)F\,dt\right| \leqq K\|F^{(2m+2-i)}\|, \qquad i = 0, 1, \cdots, m. \tag{34}$$

Several such examples are presented here.

Example 1. If Q is given by (b) of § 3, i.e., on each (x_n, x_{n+1}) the coefficient functions are replaced by their mth degree polynomial best approximations in the least squares sense, then the conclusions of Theorems 3 and 4 hold.

Proof. $1 - Q$ is Hermitian with respect to the inner product on $C^{m+1}[-1, 1]$ given by $(u, v) = \int_{-1}^{1} uv\,dt$. Since $\ker(1 - Q) = P_m$ this says that for $F \in C^{m+1}[-1, 1]$,

$$\int_{-1}^{1} t^i(1 - Q)F\,dt = 0, \qquad i = 0, 1, \cdots, m. \tag{35}$$

Example 2. If Q is given by (a) of § 3, i.e., piecewise approximation of the coefficient functions by mth degree interpolating polynomials and the set of interpolating points is the set of zeros of the $(m + 1)$st degree Legendre polynomial, then the conclusions of Theorems 3 and 4 hold.

Proof. If the $(m + 1)$st degree Legendre polynomial is

$$L_{m+1}(t) = \prod_{j=1}^{m+1} (t - z_i)$$

for $f_i \in P_{2m+1-i}$, set $g_i = Qf_i$, $i = 0, 1, \cdots, m$. Then

$$(f_i - g_i)(z_j) = 0, \qquad j = 1, 2, \cdots, m + 1, \tag{36}$$

and $f_i - g_i \in P_{2m+1-i}$, so there exists $h_i \in P_{m-i}$ such that $f_i(t) - g_i(t) = h_i(t)L_{m+1}(t)$. But then

$$\int_{-1}^{1} t^i(1 - Q)f_i\,dt = \int_{-1}^{1} (t^i h_i)L_{m+1}\,dt = 0$$

since $t^i h_i \in P_m$ and L_{m+1} is orthogonal to all polynomials of degree m. Thus $\int_{-1}^{1} t^i(1 - Q)F\,dt$ is a continuous linear functional on $C[-1, 1]$ which vanishes for $F \in P_{2m+1-i}$, so the desired inequality (34) is a consequence of the Peano kernel theorem [14, p. 14].

The following two examples are presented without proof: the arguments in each case are elementary.

Example 3. If $(QF)(t) = \frac{1}{2}(F(-1) + F(1))$, then the conclusions of Theorems 3 and 4 hold for $m = 0$.

Example 4. If

$$(QF)(t) = \tfrac{1}{2}(F(-1) + F(1))(t + 1) + \tfrac{1}{8}(5F(-1) + 3F(-\tfrac{1}{3}) + 3F(\tfrac{1}{3}) - 3F(1)), \tag{37}$$

then the conclusions of Theorems 3 and 4 hold for $m = 1$.

6. The relative error as a function of k. In § 3 it was shown that the relative error $|(\lambda_k - \hat{\lambda}_k)/\lambda_k|$ can be bounded independently of k. This behavior is examined in more detail in this section by use of the asymptotic formulas for Sturm–Liouville eigenvalues.

From [7, p. 120] we have

$$\sqrt{\lambda_k} = k\pi \Big/ \int_a^b \sqrt{r/p}\,dx + O(1/k), \tag{38}$$

but for each partition π the approximate problem is also Sturm–Liouville so there is a similar expression for $\hat{\lambda}_k$. The two imply

$$(\lambda_k - \hat{\lambda}_k)/\lambda_k = \left\{1 - \left(\int_a^b \sqrt{r/p}\,dx \Big/ \int_a^b \sqrt{r_\pi/p_\pi}\,dx\right)^2\right\} + O(1/k^2). \tag{39}$$

Thus for large k we expect the relative error to be uniform and to be dominated by the first term of (39).

A more interesting result is applicable to problems which have been transformed into Liouville normal form [4, p. 296]:

$$-u'' - qu = \lambda u \quad \text{on } [0, \pi]. \tag{40}$$

THEOREM 5. *For m-th degree approximate problems of* (40), *for each* π *as* $k \to \infty$,

$$(\lambda_k - \hat{\lambda}_k)/\lambda_k = \frac{1}{k^2\pi} \int_0^\pi (q_\pi(t) - q(t))\,dt + O(1/k^4). \tag{41}$$

Proof. The conclusion follows from the asymptotic expansions for Sturm–Liouville eigenvalues found in [9].

Note that for any of the methods involving least squares approximations as in (b) of § 3, (41) becomes

$$(\lambda_k - \hat{\lambda}_k)/\lambda_k = O(1/k^4).$$

7. Numerical results. There are several ways of solving the approximate Sturm–Liouville problems. The first was used by Canosa and Gomes de Oliveira [5]: they assume that the approximation scheme is chosen so that the resulting differential equation can be solved exactly, e.g., piecewise constant approximations have a basis of solutions consisting of piecewise sine and cosine functions. The boundary conditions together with the continuity conditions ($\hat{u}_k, p_\pi \hat{u}_k'$ are continuous) give rise to a homogeneous linear system which depends transcendentally on $\hat{\lambda}$; the eigenvalues are just those values of $\hat{\lambda}$ for which the system is singular.

The method used in generating the following data consists of normalizing the solution as in (11) and then integrating the resulting initial value problem. The eigenvalues are just those values of $\hat{\lambda}$ for which $B_b\hat{u} = 0$. This integration can be done exactly if a basis of solutions can be computed in closed form, but the chief advantage of this approach is that the integration can be performed numerically, obviating the need for knowing a basis of solutions. Since the coefficient functions of the approximate problem are piecewise polynomial functions an obvious candidate for a numerical procedure is to use Taylor series expansions. For the problems worked here at least fifteen terms of the series are used.

Several test problems have been tried and a representative sample of the results is presented. The data were computed on the CDC 6500 at Purdue University with single precision arithmetic (14 decimal digits). In all cases the $N + 1$ mesh points are equally spaced.

The first example, from [7, p. 12], is an equation which can be solved in closed form:

$$u'' + \frac{3}{4x^2}u = -\frac{1}{x^6}\lambda u \quad \text{in } (1, 2),$$

$$u(1) = u(2) = 0.$$

The solutions are

$$\lambda_k = \frac{64}{9}k^2\pi^2, \quad u_k(x) = \frac{3}{8k\pi}x^{3/2} \sin\frac{4k\pi}{3}\left(1 - \frac{1}{x^2}\right).$$

The relative error in the approximate eigenvalues is presented in Table 1 for approximation schemes based on piecewise interpolation at the zeros of the Legendre polynomials. The $O(h^{2m+2})$ rate of convergence is fairly apparent, especially for the smaller eigenvalues. The relative error is reasonably uniform and this is seen more closely in Table 2 which gives the relative error in $\hat{\lambda}_1, \hat{\lambda}_{10}, \hat{\lambda}_{40}$ and $\hat{\lambda}_{100}$ for several approximation schemes.

The second example is the Mathieu equation:

$$u'' - (100 \cos^2 x)u = -\lambda u \quad \text{in } (0, \pi),$$

$$u(0) = u(\pi) = 0.$$

Since this example is in Liouville normal form, (41) says that

$$(\lambda_k - \hat{\lambda}_k)/\lambda_k = \frac{1}{k^2\pi}\int_0^\pi (q_\pi(x) - q(x))\,dx + O(1/k^4). \tag{42}$$

For the piecewise linear approximation scheme based on interpolation at the zeros of the Legendre polynomials it is easily seen that the integral vanishes, so we have $(\lambda_k - \hat{\lambda}_k)/\lambda_k = O(1/k^4)$. This behavior is clearly seen in Table 3.

One of the limitations of this approach for approximating eigenvalues is the computation time required. For example, Table 4 gives machine times in the first test problem for finding $\hat{\lambda}_1$ with $N = 32$ and an initial error of 10%. All methods are based on least squares type approximation ((b) of § 3): the first method

TABLE 1

Relative error in the approximate eigenvalues of Example 1 based on interpolation at the zeros of Legendre polynomials

Notation: 1.0 − n means 1.0 × 10^{-n}

N	$(\lambda_1 - \hat{\lambda}_1)/\lambda_1$	$(\lambda_4 - \hat{\lambda}_4)/\lambda_4$	$(\lambda_{10} - \hat{\lambda}_{10})/\lambda_{10}$
Piecewise Constant ($m = 0$)			
16	4.34 − 3	6.63 − 3	1.26 − 2
32	1.07 − 3	1.19 − 3	2.52 − 3
64	2.66 − 4	2.74 − 4	3.18 − 4
128	6.65 − 5	6.73 − 5	6.94 − 5
Piecewise Linear ($m = 1$)			
8	−1.14 − 4	−5.29 − 3	−4.69 − 4
16	−1.17 − 5	3.42 − 5	−2.68 − 5
32	−7.98 − 7	−1.36 − 6	2.14 − 5
64	−5.09 − 8	−4.88 − 8	−1.14 − 9
Piecewise Quadratic ($m = 2$)			
8	−7.99 − 6	1.48 − 4	2.01 − 4
16	−1.23 − 7	−3.47 − 6	−4.60 − 6
32	−1.91 − 9	−4.20 − 8	−5.70 − 7

TABLE 2

The relative error as a function of k for Example 1

$(\lambda_k - \hat{\lambda}_k)/\lambda_k$

Least Squares − $O(h^{2m+2})$	N	$k = 1$	$k = 10$	$k = 40$	$k = 100$
m					
0	64	5.31 − 4	5.84 − 4	−7.41 − 4	9.92 − 4
	128	1.33 − 4	1.36 − 4	2.54 − 4	9.74 − 5
1	128	−7.13 − 9	−6.36 − 9	1.51 − 7	4.15 − 6
2	32	−1.49 − 9	−5.76 − 7	−7.13 − 7	5.50 − 7
Meshpoint Interpolation $O(h^2)$					
1	64	5.31 − 4	5.34 − 4	6.04 − 4	6.10 − 4
	128	1.32 − 4	1.34 − 4	1.34 − 4	2.09 − 4

($m = 0$) uses exact integration, the last two numerical integration by Taylor series. Most of the effort is expended in integrating the differential equation so the time required is roughly proportional to the number of subintervals. In spite of the large amount of computation, the process was quite stable numerically in all cases tested.

TABLE 3
Relative error in the approximate eigenvalues to Mathieu's equation ($N = 64$)

k	$(\lambda_k - \hat{\lambda}_k)/\lambda_k$	$k^4(\lambda_k - \hat{\lambda}_k)/\lambda_k$
1	−2.96 − 7	−2.96 − 7
5	−3.46 − 9	−2.16 − 6
10	2.81 − 9	2.81 − 5
12	1.53 − 9	3.17 − 5
13	1.18 − 9	3.36 − 5
14	8.90 − 10	3.40 − 5
15	6.90 − 10	3.50 − 5

TABLE 4
Machine times for Example 1

m	Time (seconds)	$\lvert\lambda_1 - \hat{\lambda}_1\rvert/\lambda_1$
0	.11	2.13 − 3
1	1.3	1.62 − 6
2	1.6	1.49 − 9

8. Conclusions. The chief advantage of the method presented here is that the relative error of the approximate eigenvalues is much more uniform as a function of k than is true for other methods, such as Rayleigh–Ritz or finite differences. Thus, if approximations are desired to more than the first few eigenvalues, the method given here is competitive; for the smaller eigenvalues, the Rayleigh–Ritz method is probably faster, though this gap can be lessened somewhat through the use of extrapolation.

It is clear that several extensions of this method are possible. First, more general boundary conditions, e.g., periodic conditions, can be handled with little change in the arguments. These techniques can also be applied to higher order problems or to general linear boundary value problems. In the latter case the argument for convergence proceeds as in Theorem 4.

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