JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

ELSEVIER Journal of Computational and Applied Mathematics 125 (2000) 367–383

www.elsevier.nl/locate/cam

Numerical methods for higher order Sturm–Liouville problems

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Received 1 December 1999; received in revised form 21 February 2000

Abstract

We review some numerical methods for self-adjoint and non-self-adjoint boundary eigenvalue problems.

MSC: 34A12; 34B; 65L

Keywords: Self-adjoint; Eigenvalue; Oscillation theory; Miss-distance function

1. Introduction

Spectral problems for differential equations arise in many different physical applications. Perhaps quantum mechanics is the richest source of self-adjoint problems, while non-self-adjoint problems arise in hydrodynamic and magnetohydrodynamic stability theory. The problems in hydrodynamic and MHD stability are almost always of 'higher order', either because they involve a coupled system of ordinary differential equations, or because they have been reduced to a single equation of differential order $2m$, $m > 1$. Self-adjoint problems may also be of higher order: in particular, as mentioned in [21], certain quantum-mechanical partial differential eigenproblems can be reduced to systems of ordinary differential eigenproblems.

The solution of ODE eigenproblems presents particular difficulties to the numerical analyst who wants to construct library quality software. General purpose boundary value problem codes do not generally cope well with eigenproblems. Fortunately an increasing number of pure spectral theorists have brought their skills to bear on the numerical solution of these problems. Because of the sheer size of the literature, in this paper we restrict ourselves to a very brief summary of our own

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PII: S0377-0427(00)00480-5

work. A larger bibliography, which gives more (but still inadequate) credit to some of the other mathematicians involved in this area, may be found in [20].

2. Self-adjoint problems

In this section we shall consider a $2m$th order, nonsingular, self-adjoint problem of the form:

$$\begin{aligned}&(-1)^m(p_m(x)y^{(m)})^{(m)} + (-1)^{m-1}(p_{m-1}(x)y^{(m-1)})^{(m-1)}\\ &\quad + \cdots + (p_2(x)y'')'' - (p_1(x)y')' + p_0(x)y = \lambda w(x)y, \quad a < x < b,\end{aligned} \tag{2.1}$$

together with separated, self-adjoint boundary conditions. (The precise form of the boundary conditions will be given below.) We assume that all coefficient functions are real valued. The technical conditions for the problem to be nonsingular are: the interval (a,b) is finite; the coefficient functions p_k $(0 \leqslant k \leqslant m-1)$, w and $1/p_m$ are in $L^1(a,b)$; and the essential infima of p_m and w are both positive. Under these assumptions, the eigenvalues are bounded below. (This is proved, for example, in [11], where the proof shows that the Rayleigh quotient is bounded below.) For good numerical performance however, the coefficients need to be piecewise smooth (where the degree of smoothness depends on the order of the numerical method used).

The eigenvalues can be ordered: $\lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots$, where $\lim_{k\to\infty} \lambda_k = +\infty$ and where each eigenvalue has multiplicity at most m (so $\lambda_{k+m} > \lambda_k$ for all k). The restriction on the multiplicity arises from the fact that for each λ there are at most m linearly independent solutions of the differential equation satisfying either of the endpoint conditions which we shall describe below. The numerical methods discussed in this section are based on calculation of the following counting function:

$$N(\lambda) = \text{The number of eigenvalues of (2.1) (together with boundary conditions) that are} < \lambda. \tag{2.2}$$

We shall give two formulas for $N(\lambda)$ below, and indicate some methods to calculate it. If we can calculate $N(\lambda)$, then we can approximate eigenvalues. If $\lambda' < \lambda''$ are two values such that $N(\lambda') \leqslant j$ and $N(\lambda'') \geqslant j+1$, then the jth eigenvalue λ_j lies in the interval $\lambda' \leqslant \lambda_j < \lambda''$. Now λ_j can be approximated by applying the bisection method to $N(\lambda)$ (accelerated by an iterative rootfinder applied to various continuous functions associated with the eigenvalues).

Although the solutions of (2.1) depend on (x,λ), we shall often suppress λ in the notation. Corresponding to a solution $y(x)$ of (2.1), we define quasiderivatives:

$$\begin{aligned}u_k &= y^{(k-1)}, \quad 1 \leqslant k \leqslant m,\\ v_1 &= p_1 y' - (p_2 y'')' + (p_3 y''')'' + \cdots + (-1)^{m-1}(p_m y^{(m)})^{(m-1)},\\ v_2 &= p_2 y'' - (p_3 y''')' + (p_4 y^{(4)})'' + \cdots + (-1)^{m-2}(p_m y^{(m)})^{(m-2)},\\ &\ \vdots\\ v_k &= p_k y^{(k)} - (p_{k+1} y^{(k+1)})' + (p_{k+2} y^{(k+2)})'' + \cdots + (-1)^{m-k}(p_m y^{(m)})^{(m-k)},\\ &\ \vdots\\ v_m &= p_m y^{(m)}.\end{aligned} \tag{2.3}$$

Consider the column vector functions: $u(x)=(u_1,u_2,\dots,u_m)^{\mathrm{T}}$, $v(x)=(v_1,v_2,\dots,v_m)^{\mathrm{T}}$, $z(x)=(u_1,u_2,\dots,u_m,v_1,v_2,\dots,v_m)^{\mathrm{T}}$. Let S be the $2m\times 2m$ symmetric matrix

$$
S(x,\lambda)=\begin{pmatrix}
(\lambda w-p_0) & 0 & & & & & & 0 & & & & \\
 & -p_1 & 0 & & & & & 1 & 0 & & & \\
 & & -p_2 & & & & & & 1 & 0 & & \\
 & & & \cdot & \cdot & & & & & \cdot & \cdot & \\
 & & & & \cdot & \cdot & & & & & \cdot & \cdot \\
 & & & & & \cdot & \cdot & & & & & \cdot \quad \cdot \\
 & & & & & -p_{m-2} & 0 & & & & 1 & 0 \\
 & & & & & & -p_{m-1} & & & & & 1 \quad 0 \\
0 & 1 & & & & & & & & & & \\
 & 0 & 1 & & & & & & & & & \\
 & & 0 & 1 & & & & & & & & \\
 & & & \cdot & \cdot & & & & & & & \\
 & & & & \cdot & \cdot & & & & & & \\
 & & & & & \cdot & \cdot & & & & & \\
0 & 0 & 0 & \cdot & & 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\
0 & 0 & 0 & \cdot & & 0 & 0 & 0 & \cdot & \cdot & \cdot & 1/p_m
\end{pmatrix}. \tag{2.4}
$$

and let J be the $2m\times 2m$ symplectic matrix

$$
J=\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.
$$

Then Eq. (2.1) is equivalent to

$$
Jz' = S(x,\lambda)\,z. \tag{2.5}
$$

General, separated, self-adjoint boundary conditions for (2.1) are of the form

$$
A_1u(a)+A_2v(a)=0, \qquad B_1u(b)+B_2v(b)=0, \tag{2.6}
$$

where A_1,A_2,B_1,B_2 are $m\times m$ real matrices, such that $A_1A_2^{\mathrm{T}}=A_2A_1^{\mathrm{T}}$, $B_1B_2^{\mathrm{T}}=B_2B_1^{\mathrm{T}}$, and the $m\times 2m$ matrices (A_1A_2) and (B_1B_2) have rank m.

We now consider $2m\times m$ matrices

$$
Z(x)=\begin{pmatrix} U(x) \\ V(x) \end{pmatrix}
$$

that are solutions of the extended Hamiltonian system

$$JZ' = S(x,\lambda)Z. \tag{2.7}$$

The (linearly independent) column vectors of $Z(x)$ are solutions of (2.5).

2.1. The unitary marix $\Theta(x,\lambda)$

It can be shown that the matrix function $U^{\mathrm{T}}(x)V(x) - V^{\mathrm{T}}(x)U(x)$ is constant, and this constant equals 0 if Z satisfies either of the boundary conditions (2.6). If $U^{\mathrm{T}}(x)V(x) - V^{\mathrm{T}}(x)U(x) = 0$, (and $Z = (U^{\mathrm{T}}, V^{\mathrm{T}})^{\mathrm{T}}$ has rank m, as we suppose), then the $m \times m$ matrix $V - \mathrm{i}U$ is invertible and the matrix

$$\Theta(x) = (V + \mathrm{i}U)(V - \mathrm{i}U)^{-1} \tag{2.8}$$

is unitary. The matrix $\Theta(x)$ and its phase angles were introduced into oscillation theory by Atkinson [1] and Reid [24].

We now integrate (2.7) from the left and right endpoints toward a chosen point $c \in [a,b]$. Let

$$Z_{\mathrm{L}}(x) = \begin{pmatrix} U_{\mathrm{L}}(x) \\ V_{\mathrm{L}}(x) \end{pmatrix}, \qquad Z_{\mathrm{R}}(x) = \begin{pmatrix} U_{\mathrm{R}}(x) \\ V_{\mathrm{R}}(x) \end{pmatrix}$$

be the solutions of (2.7) with initial conditions $Z_{\mathrm{L}}(a)=(A_2,-A_1)^{\mathrm{T}}$, $Z_{\mathrm{R}}(b)=(B_2,-B_1)^{\mathrm{T}}$. Let $\Theta_{\mathrm{L}}(x)$ and $\Theta_{\mathrm{R}}(x)$ be the unitary matrices obtained from $Z_{\mathrm{L}}(x)$ and $Z_{\mathrm{R}}(x)$ by formula (2.8). The eigenvalues of $\Theta_{\mathrm{L}}(x)$ and $\Theta_{\mathrm{R}}(x)$ are $\{\exp(\mathrm{i}\theta_j^{\mathrm{L}}(x)); 1 \leqslant j \leqslant m\}$ and $\{\exp(\mathrm{i}\theta_j^{\mathrm{R}}(x)); 1 \leqslant j \leqslant m\}$, respectively. The phase angles $\theta_j^{\mathrm{L}}(x), \theta_j^{\mathrm{R}}(x)$ are uniquely determined continuous functions when normalized by the conditions:

$$\theta_1^{\mathrm{L}}(x) \leqslant \theta_2^{\mathrm{L}}(x) \leqslant \cdots \leqslant \theta_m^{\mathrm{L}}(x) \leqslant \theta_1^{\mathrm{L}}(x) + 2\pi,$$

$$\theta_1^{\mathrm{R}}(x) \leqslant \theta_2^{\mathrm{R}}(x) \leqslant \cdots \leqslant \theta_m^{\mathrm{R}}(x) \leqslant \theta_1^{\mathrm{R}}(x) + 2\pi,$$

$$0 \leqslant \theta_j^{\mathrm{L}}(a) < 2\pi, \quad 0 < \theta_j^{\mathrm{R}}(b) \leqslant 2\pi.$$

At a given point $c \in [a,b]$, let

$$\Theta_{\mathrm{LR}}(c) = \Theta_{\mathrm{L}}^{*}(c)\Theta_{\mathrm{R}}(c), \tag{2.9}$$

and let $\{\exp(\mathrm{i}\omega_j); 1 \leqslant j \leqslant m\}$ be the eigenvalues of $\Theta_{\mathrm{LR}}(c)$, where the ω_j are normalized by the condition

$$0 \leqslant \omega_j < 2\pi. \tag{2.10}$$

It is known that when $0 < \omega_j(\lambda) < 2\pi$, $\omega_j(\lambda)$ is a strictly decreasing function of λ. The normalization (2.10) ensures that $N(\lambda)$ is continuous from the left.

Recalling that all of the functions arising from (2.1) depend on (x,λ), we shall use the following notations:

$$\operatorname{Argdet} \Theta_{\mathrm{L}}(x,\lambda) = \theta_1^{\mathrm{L}}(x) + \theta_2^{\mathrm{L}}(x) + \cdots + \theta_m^{\mathrm{R}}(x),$$

$$\operatorname{Argdet} \Theta_{\mathrm{R}}(x,\lambda) = \theta_1^{\mathrm{R}}(x) + \theta_2^{\mathrm{R}}(x) + \cdots + \theta_m^{\mathrm{R}}(x),$$

$$\overline{\operatorname{Argdet}}\, \Theta_{\mathrm{LR}}(c,\lambda) = \omega_1 + \omega_2 + \cdots + \omega_m. \tag{2.11}$$

The overbar on $\overline{\text{Argdet}}\,\Theta_{\text{LR}}(c,\lambda)$ indicates that the angles are normalized to lie in the interval $[0,2\pi)$. We can now give the first formula for the function $N(\lambda)$, which is the number of eigenvalues of (2.1) and (2.6) that are less than λ. The following is proved in [11].

Theorem 1. *For any* $c \in [a,b]$,

$$N(\lambda) = \frac{1}{2\pi}\,(\text{Argdet}\,\Theta_{\text{L}}(c,\lambda) + \overline{\text{Argdet}}\,\Theta_{\text{LR}}(c,\lambda) - \text{Argdet}\,\Theta_{\text{R}}(c,\lambda)). \tag{2.12}$$

The matrix $S(x,\lambda)$ in (2.4) can be partitioned into $m \times m$ submatrices:

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}.$$

The differential equation (2.7) then translates into a differential equation for $\Theta(x,\lambda)$:

$$\Theta' = \text{i}\Theta\Omega, \quad a < x < b, \tag{2.13}$$

where Ω is the Hermitian matrix given by

$$\begin{aligned} 2\Omega = {} & (\Theta^* - I)S_{11}(\Theta - I) + \text{i}(\Theta^* - I)S_{12}(\Theta + I) \\ & - \text{i}(\Theta^* + I)S_{21}(\Theta - I) + (\Theta^* + I)S_{22}(\Theta + I). \end{aligned} \tag{2.14}$$

At the same time, $A = \text{Argdet}\,\Theta$ satisfies the equation

$$A' = \text{trace}\,\Omega. \tag{2.15}$$

There are some existing specialized codes that can integrate the system consisting of Eqs. (2.13)–(2.15). For example, the code by Dieci et al. [7] is constructed specifically for (2.13). Marletta's code [21] for solving Hamiltonian systems works by solving (2.13). One can use these or more general initial value solvers to calculate $\text{Argdet}\,\Theta_{\text{L}}(c,\lambda)$ and $\text{Argdet}\,\Theta_{\text{R}}(c,\lambda)$. $N(\lambda)$ can then be calculated by formula (2.12). Note that we need only know $\Theta_{\text{L}}(c,\lambda)$ and $\Theta_{\text{R}}(c,\lambda)$ to calculate $\overline{\text{Argdet}}\,\Theta_{\text{LR}}(c,\lambda)$, since the angles ω_j are normalized to lie in the interval $[0,2\pi)$. This is not the case for $\text{Argdet}\,\Theta_{\text{L}}(c,\lambda)$ or $\text{Argdet}\,\Theta_{\text{R}}(c,\lambda)$. This is probably the best one can do for general self-adjoint $2m$th-order problems. However for 4th and 6th-order problems, there are faster, more efficient, and more elegant methods. These will be discussed below.

2.2. The symmetric matrix W and correction parameter σ

In order to develop another formula for $N(\lambda)$, we return to the matrices $U_{\text{L}}(x)$ and $U_{\text{R}}(x)$, and we define the following integer-valued functions:

$$\nu_{\text{L}}(x) = \text{nullity}\,U_{\text{L}}(x) = m - \text{rank}\,U_{\text{L}}(x) \quad \text{for } a < x < c,$$

$$\nu_{\text{R}}(x) = \text{nullity}\,U_{\text{R}}(x) = m - \text{rank}\,U_{\text{R}}(x) \quad \text{for } c < x < b,$$

$$N_{\text{L}}(c,\lambda) = \sum_{a<x<c} \nu_{\text{L}}(x), \qquad N_{\text{R}}(c,\lambda) = \sum_{c<x<b} \nu_{\text{R}}(x). \tag{2.16}$$

It is shown in [11] that $v_{\mathrm{L}}(x)$ and $v_{\mathrm{R}}(x)$ can differ from zero at only finitely many points x; therefore the sums in (2.16) are finite. If $U_{\mathrm{L}}(x)$ and $U_{\mathrm{R}}(x)$ are nonsingular, we define

$$W_{\mathrm{L}}(x) = V_{\mathrm{L}}(x)U_{\mathrm{L}}(x)^{-1}, \quad W_{\mathrm{R}}(x) = V_{\mathrm{R}}(x)U_{\mathrm{R}}(x)^{-1}. \tag{2.17}$$

It is known that $W_{\mathrm{L}}(x)$ and $W_{\mathrm{R}}(x)$ are symmetric matrices. (This follows from the fact that $U^{\mathrm{T}}(x)V(x) = V^{\mathrm{T}}(x)U(x)$). For any symmetric matrix W, let $v(W)$ be the negative index of inertia (number of negative eigenvalues) of W. We can now give a second formula for $N(\lambda)$. The following theorem is proved in [11].

Theorem 2. *If $U_{\mathrm{L}}(c,\lambda)$ and $U_{\mathrm{R}}(c,\lambda)$ are nonsingular, then*

$$N(\lambda) = N_{\mathrm{L}}(c,\lambda) + N_{\mathrm{R}}(c,\lambda) + v(W_{\mathrm{L}}(c,\lambda) - W_{\mathrm{R}}(c,\lambda)). \tag{2.18}$$

There is a more general formula:

$$N(\lambda) = N_{\mathrm{L}}(c,\lambda) + N_{\mathrm{R}}(c,\lambda) + \sigma(c,\lambda). \tag{2.19}$$

If $\det U_{\mathrm{L}}(c,\lambda) \neq 0 \neq \det U_{\mathrm{R}}(c,\lambda)$, Eq. (2.18) implies $\sigma(c,\lambda) = v(W_{\mathrm{L}}(c,\lambda) - W_{\mathrm{R}}(c,\lambda))$. More generally,

$$\sigma(c,\lambda) = \frac{1}{2\pi}(\overline{\mathrm{Argdet}}\, \Theta_{\mathrm{L}}(c,\lambda) + \overline{\mathrm{Argdet}}\, \Theta_{\mathrm{LR}}(c,\lambda) - \overline{\mathrm{Argdet}}\, \Theta_{\mathrm{R}}(c,\lambda)), \tag{2.20}$$

where the overbars indicate normalized angles:

$$\overline{\mathrm{Argdet}}\, \Theta_{\mathrm{L}}(c,\lambda) = \sum_{i=1}^{m} \bar{\theta}_i^{\mathrm{L}}, \qquad \overline{\mathrm{Argdet}}\, \Theta_{\mathrm{R}}(c,\lambda) = \sum_{i=1}^{m} \bar{\theta}_i^{\mathrm{R}},$$

$$\theta_i^{\mathrm{L}} = 2\pi n_i^{\mathrm{L}} + \bar{\theta}_i^{\mathrm{L}}, \qquad \theta_i^{\mathrm{R}} = -2\pi n_i^{\mathrm{R}} + \bar{\theta}_i^{\mathrm{R}},$$

where n_i^{L} and n_i^{R} are nonnegative integers, and

$$0 \leqslant \bar{\theta}_i^{\mathrm{L}} < 2\pi, \qquad 0 < \bar{\theta}_i^{\mathrm{R}} \leqslant 2\pi.$$

Numerical methods for problems of order 4 and 6 are given in [13–15], using coefficient approximation. The coefficient functions are approximated by piecewise-constant functions (equal to their values at the centers of the mesh intervals). This gives an $O(h^2)$ approximation to the original problem. It turns out that for orders 4 and 6, $N(\lambda)$ can be calculated exactly for the approximate problems, using formulas (2.18)–(2.20). On each mesh interval, the approximate ODE has constant coefficients, and the exact solutions can be found. Nevertheless, it is still difficult to calculate the contribution $N(x_{i-1},x_i)$ of a mesh interval to $N_{\mathrm{L}}(c,\lambda)$ or $N_{\mathrm{R}}(c,\lambda)$. Fortunately, it turns out that there is a simple relation between $N(x_{i-1},x_i)$ and the oscillation number $N_0(x_{i-1},x_i)$ corresponding to any other solution $Z_0(x) = (U_0^{\mathrm{T}}(x), V_0^{\mathrm{T}}(x))^{\mathrm{T}}$ of the approximate problem on $[x_{i-1},x_i]$. For 4th-order problems, $N_0(x_{i-1},x_i)$ can be calculated for the solution $Z_0(x)$ satisfying Dirichlet conditions at x_i: $U_0(x_i) = 0$, $V_0(x_i) = I$. For 6th-order problems, a special solution $Z_0(x)$ is devised for each case, depending on the number of real and purely imaginary roots of the characteristic equation. The case with 6 purely imaginary roots is still too difficult to calculate directly, and requires a homotopy theorem to show that they all have the same behavior. In these problems, the integration of the extended Hamiltonian system (2.7) is stabilized by using Ricatti variables, and the error is controlled by Richardson extrapolation.

3. Non-self-adjoint problems

While our numerical methods for self-adjoint problems have all been based on the well-developed oscillation theory for such problems, no such theory exists for non-self-adjoint problems. Numerical methods for such problems have tended to be more ad-hoc: one typical approach is to adjoin to the differential equation an additional equation $\mathrm{d}\lambda/\mathrm{d}x = 0$ plus an additional boundary condition determining the normalization and sign of the eigenfunction; this gives a boundary value problem which can be solved with a boundary value code. Finite difference and finite element methods have also been used, but perhaps the most popular method involving the representation of the eigenfunctions by a finite basis set has been the Chebychev τ method, which has been extensively developed and used by many authors including Straughan and Walker [26].

Although there is no oscillation theory for non-self-adjoint problems there is nevertheless a rich literature on the analytical aspects of these problems, including the classical works of Naimark [22] and Gohberg and Krein [10]. Many non-self-adjoint operators which arise in applications (see, e.g., all of the examples of Chandrasekhar [5]) are relatively compact perturbations of self-adjoint operators and are therefore unlikely to exhibit the extreme ill-conditioning of eigenvalues observed by Davies [6] and Trefethen [27]. Birkhoff [3] was perhaps the first person to obtain the asymptotic distribution of the eigenvalues for a general class of nth-order problems with this property, which we term *Birkhoff regularity*. For numerical methods based on shooting, Birkhoff regularity has important consequences: for example, it allows one to develop very efficient methods for counting the number of eigenvalues of a problem in a half-plane $\mathrm{Re}\,\lambda < s$.

3.1. Asymptotics and Birkhoff regularity

We consider a differential equation of even order $n = 2m$ of the form

$$y^{(n)} + p_{n-2}(x)y^{(n-2)} + \cdots + p_0(x)y = \lambda y, \quad x \in [0,1], \tag{3.1}$$

in which the coefficients p_k are smooth, together with $2m$ evenly separated boundary conditions normalized to the form

$$U_{0\nu}(y) := y^{(j_\nu)}_{(0)} + \sum_{i=0}^{j_\nu - 1} \alpha_{i\nu} y^{(i)}(0) = 0 \quad (\nu = 1, 2, \ldots, m), \tag{3.2}$$

$$U_{1\nu}(y) := y^{(k_\nu)}_{(1)} + \sum_{i=0}^{k_\nu - 1} \beta_{i\nu} y^{(i)}(1) = 0 \quad (\nu = 1, 2, \ldots, m). \tag{3.3}$$

Here the integers j_ν and k_ν satisfy $2m-1 \geqslant j_1 > j_2 > \cdots > j_m \geqslant 0$ and $2m-1 \geqslant k_1 > k_2 > \cdots > k_m \geqslant 0$. We require asymptotic information about the behavior of the solutions of (3.1) for large $|\lambda|$. Put $\lambda = -\rho^n$ in (3.1) and consider the sectors $S_k = \{\rho \in \mathbb{C} \mid k\pi/n \leqslant \arg\rho \leqslant (k+1)\pi/n\}$, $k = 0, 1, \ldots, 2n-1$. Let $\omega_1, \ldots, \omega_n$ be the nth roots of unity.

Theorem 3 (Birkhoff [2]). *Suppose that the coefficients in* (3.1) *are continuous in* $[0,1]$. *Then in each sector* S_k *the equation*

$$y^{(n)} + p_{n-2}(x)y^{(n-2)} + \cdots + p_0(x)y = -\rho^n y \tag{3.4}$$

has n linearly independent solutions $y_1(x,\rho),\ldots,y_n(x,\rho)$ *which are analytic functions of* $\rho\in S_k$ *for all sufficiently large* $|\rho|$ *and which have the asymptotic properties*

$$y_k = \mathrm{e}^{\rho\omega_k x}(1+\mathrm{O}(1/\rho)), \tag{3.5}$$

$$\frac{\mathrm{d}^j y_k}{\mathrm{d}x^j} = \rho^j \mathrm{e}^{\rho\omega_k x}(\omega_k^j + \mathrm{O}(1/\rho)), \quad j=1,\ldots,n-1. \tag{3.6}$$

Now consider the sector S_0, and suppose $\omega_1,\ldots,\omega_n$ are ordered so that

$$\mathrm{Re}\,(\rho\omega_1) \leqslant \mathrm{Re}\,(\rho\omega_2) \leqslant \cdots \leqslant \mathrm{Re}\,(\rho\omega_n), \quad \rho\in S_0. \tag{3.7}$$

Let $j_1,\ldots,j_m$ and $k_1,\ldots,k_m$ be the integers in (3.2) and (3.3) and consider

$$\begin{vmatrix} \omega_1^{j_1} & \cdots & \omega_{m-1}^{j_1} & \omega_m^{j_1} & \omega_{m+1}^{j_1} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ \omega_1^{j_m} & \cdots & \omega_{m-1}^{j_m} & \omega_m^{j_m} & \omega_{m+1}^{j_m} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & s\omega_m^{k_1} & \frac{1}{s}\omega_{m+1}^{k_1} & \omega_{m+2}^{k_1} & \cdots & \omega_n^{k_1} \\ \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & s\omega_m^{k_m} & \frac{1}{s}\omega_{m+1}^{k_m} & \omega_{m+2}^{k_m} & \cdots & \omega_n^{k_m} \end{vmatrix} = \frac{\theta_{-1}}{s} - s\theta_1, \tag{3.8}$$

where

$$\theta_{-1} = \begin{vmatrix} \omega_1^{j_1} & \cdots & \omega_m^{j_1} \\ \vdots & & \vdots \\ \omega_1^{j_m} & \cdots & \omega_m^{j_m} \end{vmatrix} \begin{vmatrix} \omega_{m+1}^{k_1} & \cdots & \omega_n^{k_1} \\ \vdots & & \vdots \\ \omega_{m+1}^{k_m} & \cdots & \omega_n^{k_m} \end{vmatrix}, \quad \theta_1 = \begin{vmatrix} \omega_1^{j_1} & \cdots & \omega_{m-1}^{j_1} & \omega_{m+1}^{j_1} \\ \vdots & & & \vdots \\ \omega_1^{j_m} & \cdots & \omega_{m-1}^{j_m} & \omega_{m+1}^{j_m} \end{vmatrix} \begin{vmatrix} \omega_m^{k_1} & \omega_{m+2}^{k_1} & \cdots & \omega_n^{k_1} \\ \vdots & & & \vdots \\ \omega_m^{k_m} & \omega_{m+2}^{k_m} & \cdots & \omega_n^{k_m} \end{vmatrix}. \tag{3.9}$$

Definition 4. The boundary conditions are Birkhoff regular if $\theta_{-1}\theta_1 \neq 0$.

Although we have stated this definition for the ordering (3.7) for $\rho\in S_0$, it is easily seen that the definition does not depend on the sector chosen. Moreover, the following result has been proved recently in [16].

Theorem 5. *For even order* $n=2m$ *all evenly divided, separated,* λ*-independent boundary conditions* (3.2) *and* (3.3) *are Birkhoff regular.*

Birkhoff regularity has two important consequences. Firstly, asymptotic expressions for the eigenvalues were proved by Birkhoff [3] (see Theorem 6); secondly, an asymptotic expression can be obtained for a certain analytic *miss-distance function* $f(\lambda)$ whose zeros are the eigenvalues (see Section 3.2 below).

Theorem 6. *For* $n=2m$, *Eq.* (3.1) *with evenly separated* λ*-independent boundary conditions has precisely two sequences of eigenvalues* λ_k^+ *and* λ_k^- *given for large k by*

$$\lambda_k^{\pm} = (-1)^m(2k\pi)^n[1-(-1)^m m\log\xi^{\pm}/(k\pi\mathrm{i}) + \mathrm{O}(1/k^2)],$$

where ξ^+ *and* ξ^- *are the distinct roots of the equation* $\theta_1\xi^2 = \theta_{-1}$ *for the sector* S_0 *and* log *is any fixed branch of the natural logarithm.*

3.2. The miss-distance function and the characteristic determinant

Eq. (3.1) can be transformed to a 1st-order equation in n variables by many methods. If the coefficients are sufficiently smooth then we can first write it in the form

$$y^{(n)} + (q_{n-2}(x)y^{(m-1)})^{(m-1)} + (q_{n-3}(x)y^{(m-1)})^{(m-2)} + (q_{n-4}(x)y^{(m-2)})^{(m-2)}$$
$$+ \cdots + (q_2(x)y')' + q_1(x)y' + q_0(x)y = \lambda y, \quad x \in (0,1). \tag{3.10}$$

We then consider new variables defined by

$$u_k = y^{(k-1)}, \quad k = 1,\ldots,m, \tag{3.11}$$

$$v_k = (-1)^{k-1}[y^{(n-k)} + (q_{n-2}y^{(m-1)})^{(m-k-1)} + (q_{n-3}y^{(m-1)})^{(m-k-2)}$$
$$+ \cdots + (q_{2k+2}y^{(k+1)})' + q_{2k+1}y^{(k+1)} + q_{2k}y^{(k)}], \quad k = 1,\ldots,m. \tag{3.12}$$

Let $u = (u_1,\ldots,u_m)^{\mathrm{T}}$, $v = (v_1,\ldots,v_m)^{\mathrm{T}}$ and $z = (u^{\mathrm{T}}, v^{\mathrm{T}})^{\mathrm{T}}$. Eq. (3.1) becomes

$$Jz' = S(x,\lambda)z, \tag{3.13}$$

where

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \qquad S(x,\lambda) = \begin{pmatrix} S_{11}(x,\lambda) & S_{12} \\ S_{21} & S_{22} \end{pmatrix},$$

the $m \times m$ matrices S_{12}, S_{21} and S_{22} being independent of x and λ. Likewise the boundary conditions (3.2) and (3.3) can be expressed in the form

$$A_1u(0) + A_2v(0) = \mathbf{0} = B_1u(1) + B_2v(1). \tag{3.14}$$

Now let $Z_{\mathrm{L}} = (U_{\mathrm{L}}^{\mathrm{T}}, V_{\mathrm{L}}^{\mathrm{T}})^{\mathrm{T}}$ and $Z_{\mathrm{R}} = (U_{\mathrm{R}}^{\mathrm{T}}, V_{\mathrm{R}}^{\mathrm{T}})^{\mathrm{T}}$ be $2m \times m$ solution matrices of (3.13) of full rank m, such that each column of Z_{L} satisfies the boundary condition at $x = 0$ and each column of Z_{R} satisfies the boundary condition at $x = 1$: in particular,

$$A_1U_{\mathrm{L}}(0) + A_2V_{\mathrm{L}}(0) = \mathbf{0} = B_1U_{\mathrm{R}}(1) + B_2V_{\mathrm{R}}(1). \tag{3.15}$$

Fix $c \in [0,1]$. Then λ is an eigenvalue if and only if there exist nonzero vectors ξ and ζ such that $Z_{\mathrm{L}}(c,\lambda)\xi = Z_{\mathrm{R}}(c,\lambda)\zeta$; the corresponding eigenfunction z of (3.13) is then given by

$$z(x) = Z_{\mathrm{L}}(x,\lambda)\xi, \quad 0 \leqslant x \leqslant c; \quad Z_{\mathrm{R}}(x,\lambda)\zeta, \quad c \leqslant x \leqslant 1.$$

The existence of ξ and ζ to satisfy $Z_{\mathrm{L}}(c,\lambda)\xi = Z_{\mathrm{R}}(c,\lambda)\zeta$ is equivalent to the condition

$$f(\lambda) := \det(Z_{\mathrm{L}}(c,\lambda), Z_{\mathrm{R}}(c,\lambda)) = 0. \tag{3.16}$$

This equation defines our miss-distance function $f(\lambda)$.

The more commonly used miss distance is the *characteristic determinant* (see [22]) defined in terms of the boundary operators U_{0v} and U_{1v}. Let $y_1(x,\lambda),\ldots,y_{2m}(x,\lambda)$ be any $2m = n$ linearly independent solutions of (3.10) which are also analytic functions of λ in some domain $\Omega \subseteq \mathbb{C}$. Then the characteristic determinant is

$$\Delta(\lambda) = \begin{vmatrix} U_{01}(y_1) & U_{01}(y_2) & \cdots & U_{01}(y_n) \\ \vdots & & & \vdots \\ U_{0m}(y_1) & U_{0m}(y_2) & \cdots & U_{0m}(y_n) \\ U_{11}(y_1) & U_{11}(y_2) & \cdots & U_{11}(y_n) \\ \vdots & & & \vdots \\ U_{1m}(y_1) & U_{1m}(y_2) & \cdots & U_{1m}(y_n) \end{vmatrix}. \tag{3.17}$$

It is known that for $\lambda \in \Omega$, the zeros of $\Delta(\lambda)$ are precisely the eigenvalues in Ω; moreover Keldysh has shown that the order of a zero λ_* of Δ is precisely the algebraic multiplicity [1] of λ_* as an eigenvalue [22]. Since so much is known about $\Delta(\lambda)$ it is obviously important to know the relationship between $\Delta(\lambda)$ and $f(\lambda)$: the following result is proved in [16].

Theorem 7. *Let $u_{1i},\dots,u_{mi}$, $v_{1i},\dots,v_{mi}$ be the quasiderivatives for the solution $y_i(x,\lambda)$, for $i = 1,\dots,n$. Let*

$$Y_{12}(x,\lambda) = \begin{pmatrix} u_{11} & \cdots & u_{1n} \\ \vdots & & \vdots \\ u_{m1} & \cdots & u_{mn} \\ v_{11} & \cdots & v_{1n} \\ \vdots & & \vdots \\ v_{m1} & \cdots & v_{mn} \end{pmatrix}, \tag{3.18}$$

which is a fundamental matrix for (3.13). *Let $W_{\mathrm{L}} = Z_{\mathrm{L}}^* Z_{\mathrm{L}}$ and $W_{\mathrm{R}} = Z_{\mathrm{R}}^* Z_{\mathrm{R}}$, which are Gram matrices and nonsingular. Let*

$$A = \begin{pmatrix} A_1 & A_2 \\ U_{\mathrm{L}}^*(0) & V_{\mathrm{L}}^*(0) \end{pmatrix}, \qquad B = \begin{pmatrix} U_{\mathrm{R}}^*(1) & V_{\mathrm{R}}^*(1) \\ B_1 & B_2 \end{pmatrix}. \tag{3.19}$$

Then A and B are invertible and

$$f(\lambda) = (-1)^m \frac{\det W_{\mathrm{L}}(0) \det W_{\mathrm{R}}(1)}{\det A \det B} \frac{\Delta(\lambda) \det Y_{12}(c,\lambda)}{\det Y_{12}(0,\lambda) \det Y_{12}(1,\lambda)}. \tag{3.20}$$

Since $f(\lambda)$ is an entire function this result, combined with the known properties of $\Delta(\lambda)$, imply that the order of a point λ_* as a zero of f is the algebraic multiplicity of λ_* as an eigenvalue of the problem. Moreover, by choosing for $y_1,\dots,y_n$ the n solutions whose asymptotics are described in

[1] For non-self-adjoint problems the algebraic and geometric multiplicities of an eigenvalue may be different. An eigenvalue λ_* of a general non-self-adjoint operator L has not only eigenfunctions, but additional *associated functions* which are elements of the null-spaces of the operators $(L-\lambda_*)^p$, $p = 1,2,3,\dots$. The algebraic multiplicity is the dimension of the sum of all these null spaces.

Theorem 3, we can obtain the asymptotics for $f(\lambda)$ for large $|\lambda|$. We shall see in the next section how important this can be.

3.3. λ-Integration

For self-adjoint problems all the eigenvalues are real, and there is a monotone increasing miss-distance function which takes prescribed values at the eigenvalues. For non-self-adjoint problems one has the harder problem of finding the zeros of an entire function $f(\lambda)$ in the complex plane, already addressed by many authors, e.g., [18,28]. An often used approach is based on the argument principle: the number of zeros of f inside a closed contour Γ is $(1/2\pi i)\int_\Gamma f'(\lambda)/f(\lambda)\,d\lambda$. The integral is computed by splitting up Γ into a number of segments $[z_j, z_{j+1}]$ such that for each j, for $z \in [z_j, z_{j+1}]$, $w_j(z) := f(z)/f(z_j)$ traces out a curve which lies entirely in the right half-plane $\mathrm{Re}(w_j) > 0$. The integral is then equal to $\sum_j \log(f(z_{j+1})/f(z_j))$. In practice it is usually impossible to verify the condition $\mathrm{Re}(w_j(z)) > 0$ for all $z \in [z_j, z_{j+1}]$, and so one replaces this by a heuristic such as $|\arg w_j(z_j)| < \pi/4$, where arg is the branch of the argument taking values in $(-\pi, \pi]$. Various strategies have been proposed for choosing the points z_j.

Knowing the number of zeros of f in, say, a rectangle in $\mathbb{C}$, one can set up a recursive bisection procedure to home in on individual zeros. For simple zeros it is usually possible, when the rectangles become sufficiently small, to switch to a quasi-Newton method based on finite difference approximation of the derivative, and converge rapidly to the zero.

In applications related to linear stability analysis it is often important to know whether or not any eigenvalues of a problem lie in a half-plane. From Theorem 3 there will be infinitely many eigenvalues in the half-plane $(-1)^m\,\mathrm{Re}(\lambda) > 0$, so the question is: how many eigenvalues lie in the half-plane $(-1)^m\,\mathrm{Re}(\lambda) < 0$? Ideally one would like the answer to be given by the integral $\int_{-i\infty}^{+i\infty} f'(\lambda)/f(\lambda)\,d\lambda$, but for the function f defined by (3.16) this integral does not converge. To circumvent this we use (3.20). The asymptotics of $\Delta(\lambda)$ are known [22, p. 60], as are those of the solutions $y_1, \ldots, y_n$, so the asymptotics of the terms $\det Y_{12}(0,\lambda)$, $\det Y_{12}(1,\lambda)$ and $\det Y_{12}(c,\lambda)$ can also be computed, all in terms of analytic functions of λ. One is then able to find a function $g(\lambda)$ which is (a) analytic in the half-plane $(-1)^m\,\mathrm{Re}(\lambda) \leqslant 0$, with no zeros there, (b) such that as $|\lambda| \to \infty$ in this half-plane, $f(\lambda)/g(\lambda) \to 1$. Defining a new miss-distance $\hat{f}$ by $\hat{f} = f/g$, the number of eigenvalues in the half-plane $(-1)^m\,\mathrm{Re}(\lambda) < 0$, counted according to algebraic multiplicity, is given by $\int_{-i\infty}^{+i\infty} \hat{f}'(\lambda)/\hat{f}(\lambda)\,d\lambda$.

3.4. x-Integration

Evaluating $f(\lambda)$ defined by (3.16) involves integrating the differential system in some form. Because λ may be large for evaluating some of the integrals mentioned at the end of Section 3.3, one should perhaps reformulate the system in a more stable set of variables; ideally one should also use a special numerical method capable of integrating the system for large $|\lambda|$ at a reasonable cost. One method of achieving these ends is to use the *compound matrix method*, described in Drazin and Reid [8, p. 311]. This involves using variables closely related to Riccati variables but satisfying a linear system of ODEs instead of the usual nonlinear system. The linearity can be exploited by using a special integrator for linear ODEs, e.g., a method [4,17] based on the Magnus series [19].

Unfortunately the compound matrix method involves an ODE in binomial $(2n, n)$ variables and is therefore impractical for equations of order >6. However, many high-order problems actually originated from *systems* of equations of order 2. (This is true of the Orr–Sommerfeld equation, for example.) In terms of the matrices $Z_L = (U_L^T, V_L^T)^T$ and $Z_R = (U_R^T, V_R^T)^T$, let $U = U_L$ (or U_R) and let $V = V_L$ (resp. V_R); then these equations may be written as

$$-U'' + Q(x, \lambda)U = \mathbf{0}, \tag{3.21}$$

with $V = U'$. Eq. (3.21) can be solved for each fixed λ by replacing the $m \times m$ coefficient matrix $Q(x, \lambda)$ by a matrix $\hat{Q}(x, \lambda)$ which is piecewise constant on mesh intervals $(x_{j-1}, x_j]$, $j=0, \ldots, N$, $x_0=0$, $x_N = 1$. On each mesh interval one can solve this approximate equation 'exactly' (i.e., symbolically) and hence obtain a symbolic expression for the Riccati variables associated with the system. Evaluated in the correct way, this symbolic expression gives a stable way of finding the Riccati variables for the approximated system. This method has the disadvantage that the error is at best $O(h^2)$, where h is a typical steplength; however it has the advantage that for a given mesh, the relative accuracy of $f(\lambda)$ often does not deteriorate as $|\lambda|$ increases. The $O(h^2)$ can be improved to higher order by Richardson extrapolation.

4. Numerical examples

We shall give two examples each of self-adjoint and non-self-adjoint problems. We begin with the self-adjoint examples.

(1) Consider the so-called modified harmonic oscillator, which consists of the equation

$$\ell(y) = -y'' + (x^2 + x^4)y = \lambda y$$

on the interval $(-\infty, \infty)$. No boundary conditions are needed because the problem is of limit-point type: the requirement that the eigenfunctions be square integrable suffices as a boundary condition. We truncate this problem to the interval $(-100, 100)$, and impose the boundary conditions $y(-100) = 0 = y(100)$. Now consider the square $L = \ell^2$ of the above operator on the interval $(-100, 100)$. Thus the fourth-order problem is

$$L(y) = y^{(iv)} - 2((x^2 + x^4)y')' + (x^8 + 2x^6 + x^4 - 12x - 2)y = \lambda y,$$

with boundary conditions $y(c) = y''(c) = 0$, for $c = \pm 100$. The eigenvalues of L are the squares of the eigenvalues of ℓ. Clearly the coefficients become quite large at the endpoints, so this problem tests how well the code SLEUTH can cope with stiffness.

(2) Self-adjoint fourth-order problems often arise in the study of vibration and buckling of beams. For example, Roseau [25, p. 141] analyzes vibrations in a turbine blade. By looking for normal modes of transverse vibration in the associated wave equation, he obtains the eigenproblem consisting of the differential equation

$$(EIy'')'' - ((F - \omega^2 I\rho)y')' - \omega^2 \rho y = 0, \quad 0 < x < \ell,$$

subject to the boundary conditions

$$y(0) = y'(0) = 0, \qquad EIy''(\ell) = (EIy'')'(\ell) - (F - \omega^2 I\rho)y(\ell) = 0.$$

Table 1
Eigenvalues found by SLEUTH at $TOL = 10^{-6}$

Problem number	Eigenvalue index	Eigenvalue approximation	Code relative error estimate	"True" relative error	CPU (secs)	Number of extrapolations, number of mesh points
	0	1.9386467	1E − 6	2E − 6	8.4	4,320
1	9	2205.7105	5E − 8	9E − 6	12.7	5,176
	99	1044329.235	9E − 9	1E − 9	113.7	3,504
	0	1.8115460	∗ ∗ ∗	2E − 7	∗ ∗ ∗	2,320
2	1	5.9067512	∗ ∗ ∗	3E − 8	∗ ∗ ∗	3,640
	2	10.8786209	∗ ∗ ∗	2E − 9	∗ ∗ ∗	5,2560

Here ω is the vibrational frequency; y is the displacement perpendicular to the blade; E is the Young's modulus; I is the moment of inertia of a cross-section of the blade; ρ is the linear density of the blade; and F is the (variable) centrifugal force:

$$F(x) = \Omega^2 \int_x^{\ell} \rho A(s)(r + s)\,\mathrm{d}s,$$

where Ω is the angular velocity, $A(\cdot)$ is the cross-sectional area of the blade, and r is the radius of the turbine.

We took $E = I = A(x) = \Omega = \ell = 1$ and $r = 2/3$. With the cross-sectional area constant we chose $\rho(x) = x$, corresponding to a blade made of a material of nonuniform density. Then $F(x) = (1/3)(2 + 2x + x^2)(1 - x)$. We converted the problem to a standard eigenproblem by introducing a new eigenparameter λ:

$$y^{(\mathrm{iv})} - (((1/3)(2 + 2x + x^2)(1 - x) - \omega^2 x)y')' - \omega^2 xy = \lambda y, \quad 0 < x < 1;$$

the boundary conditions are actually just Dirichlet conditions $u_1 = u_2 = 0$ at $x = 0$ and Neumann conditions $v_1 = v_2 = 0$ at $x = 1$ (see (2.3) for definitions of u_1, u_2, v_1 and v_2). For each value $\omega > 0$ this problem has an infinite sequence of λ-eigenvalues

$$\lambda_0(\omega) \leqslant \lambda_1(\omega) \leqslant \lambda_2(\omega) \leqslant \cdots.$$

The results in Greenberg [12] imply that $\lambda_k(\omega)$ is a strictly decreasing function of ω; the kth eigenvalue ω_k of the original nonlinear problem is defined by $\lambda_k(\omega_k) = 0$. Using a simple rootfinding process, we determined ω_0, ω_1 and ω_2.

The results are shown in Table 1. (We do not quote CPU times for Problem 2 as these depend very strongly on the rootfinding method used and the quality of the initial approximation.)

The two non-self-adjoint problems we shall consider both involve the Orr–Sommerfeld equation for plane laminar flow:

$$(-D^2 + \alpha^2)^2 y + \mathrm{i}\alpha R(U(x)(-D^2 + \alpha^2)y + U''(x)y) = \lambda(-D^2 + \alpha^2)y, \tag{4.1}$$

where $D = \mathrm{d}/\mathrm{d}x$, and $U(x)$ is a flow profile whose stability is in question. The parameters α and R are the wave number and Reynolds number, respectively.

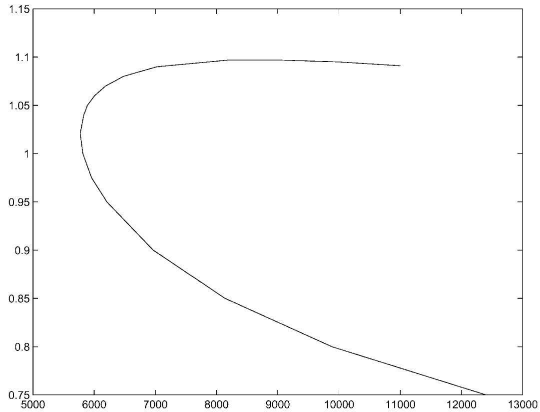

Fig. 1. Marginal curve for plane Poiseuille flow

(3) In this first example, we sketch the marginal curve for the Poiseuille profile: $U(x)=1-x^2$ on the interval $[-1,1]$, with Dirichlet boundary conditions $y(c)=y'(c)=0$ for $c=\pm 1$. (By symmetry, this reduces to the equivalent problem on $[0,1]$, with boundary conditions $y'(0)=y'''(0)=0$ and Dirichlet conditions at $x=1$.) If for some (R,α), the problem has an eigenvalue λ with $\mathrm{Re}(\lambda)<0$, then the flow is unstable under a perturbation with wave number α and Reynolds number R. If all eigenvalues have $\mathrm{Re}(\lambda)>0$, then the perturbed flow is stable. The pair (R,α) is *marginal* if all eigenvalues satisfy $\mathrm{Re}(\lambda)\geqslant 0$ and there is at least one eigenvalue with $\mathrm{Re}(\lambda)=0$. The minimum R on the marginal curve is the *critical Reynolds number*. This value R_{crit} has the property that any pair (R,α) is stable if $R<R_{\mathrm{crit}}$. Using the code SLNSA [16], a nonlinear solver and a path-following procedure, we sketched the marginal curve for plane Poiseuille flow (see Fig. 1). We used only 100 meshpoints, yet found the critical Reynolds number $R_{\mathrm{crit}}=5771.8$, which compares well with the result of Orszag [23] of $R_{\mathrm{crit}}=5772.2$.

(4) Gheorghiu and Pop [9] considered the Orr–Sommerfeld equation for a liquid film flowing over an inclined plane, with a surface tension gradient. In Eq. (4.1) they replace λ by $\alpha R\lambda$ on the right-hand side, a rescaling which does not change the stability criteria. The problem then consists of the differential equation on the interval $[0,1]$ and the following λ-dependent boundary conditions:

$$(-\mathrm{i}\lambda - U(0))(y''(0)+\alpha^2 y(0)) + U''(0)y(0) = 0,$$

$$U''(0)y'''(0) + \mathrm{i}\alpha\{R(-\mathrm{i}\lambda - U(0)) + 3\mathrm{i}\alpha\}U''(0)y'(0)$$

$$-\mathrm{i}\alpha\{2\cot\beta + \alpha^2 C_a + (-\mathrm{i}\lambda - U(0))RU'(0)\}\{y''(0)+\alpha^2 y(0)\} = 0,$$

$$y(1)=0, \quad y'(1)=0.$$

Using the flow profile

$$U(x) = (1-x)(1+x+\tau),$$

Gheorghiu and Pop calculate the critical Reynolds number for $\tau=\pm 1.75$ and $\cot\beta = 1.19175$. We took $C_a=1$ (as the results seem independent of C_a). For this problem, R_{crit} is obtained as the

Table 2
Leftmost eigenvalue for Problem (4)

α	R	τ	Magnus λ	Vector SL λ
40 *meshpoints* (20/40 *extrap. for Vector* SL)				
10^{-2}	0.7947	1.75	$(5.7 \times 10^{-7}, 3.754980)$	$(5.7 \times 10^{-7}, 3.754980)$
10^{-2}	0.7949	1.75	$(-1.4 \times 10^{-6}, 3.754980)$	$(5.7 \times 10^{-7}, 3.754980)$
80 *meshpoints* (40/80 *extrap. for Vector* SL)				
10^{-2}	0.7947	1.75	$(5.7 \times 10^{-7}, 3.754980)$	$(5.7 \times 10^{-7}, 3.754980)$
10^{-2}	0.7949	1.75	$(-1.4 \times 10^{-6}, 3.754980)$	$(5.7 \times 10^{-7}, 3.754980)$
40 *meshpoints* (20/40 *extrap. for Vector* SL)				
10^{-4}	0.7945	1.75	$(1.2 \times 10^{-11}, 3.750000)$	$(1.4 \times 10^{-8}, 3.750000)$
10^{-4}	0.7946	1.75	$(-1.0 \times 10^{-8}, 3.750000)$	$(-3.2 \times 10^{-9}, 3.750000)$
80 *meshpoints* (40/80 *extrap. for Vector* SL)				
10^{-4}	0.7944	1.75		$(8.6 \times 10^{-9}, 3.750000)$
10^{-4}	0.7945	1.75	$(1.6 \times 10^{-11}, 3.750000)$	$(-1.0 \times 10^{-8}, 3.750000)$
10^{-4}	0.7946	1.75	$(-1.0 \times 10^{-8}, 3.750000)$	
40 *meshpoints* (20/40 *extrap. for Vector* SL)				
10^{-2}	11.9300	−1.75	$(2.8 \times 10^{-6}, 0.249792)$	$(2.8 \times 10^{-6}, 0.249792)$
10^{-2}	11.9400	−1.75	$(-3.8 \times 10^{-6}, 0.249792)$	$(-3.8 \times 10^{-6}, 0.249792)$
80 *meshpoints* (40/80 *extrap. for Vector* SL)				
10^{-2}	11.9300	−1.75	$(2.8 \times 10^{-6}, 0.249792)$	$(2.8 \times 10^{-6}, 0.249792)$
10^{-2}	11.9400	−1.75	$(-3.8 \times 10^{-6}, 0.249792)$	$(-3.8 \times 10^{-6}, 0.249792)$
40 *meshpoints* (20/40 *extrap. for Vector* SL)				
10^{-4}	11.9173	−1.75		$(2.4 \times 10^{-8}, 0.250000)$
10^{-4}	11.9174	−1.75	$(6.6 \times 10^{-10}, 0.250000)$	$(-2.0 \times 10^{-8}, 0.250000)$
10^{-4}	11.9175	−1.75	$(-3.9 \times 10^{-12}, 0.250000)$	
80 *meshpoints* (40/80 *extrap. for Vector* SL)				
10^{-4}	11.9174	−1.75	$(6.6 \times 10^{-10}, 0.250000)$	
10^{-4}	11.9175	−1.75	$(-2.9 \times 10^{-13}, 0.250000)$	$(5.3 \times 10^{-9}, 0.250000)$
10^{-4}	11.9176	−1.75		$(-8.3 \times 10^{-9}, 0.250000)$

limiting case as $\alpha \searrow 0$. In Table 2 we show the left-most eigenvalue in the complex plane for various values of R and α, and for the two different values of τ. We compare the Magnus method with the coefficient approximation vector Sturm–Liouville method. The values of R are chosen close to the stability/instability boundary predicted by Gheorgiu and Pop for the case $\alpha \searrow 0$, which are $R = 0.7945$ in the case $\tau = 1.75$ and $R = 11.9175$ in the case $\tau = -1.75$. Both methods show the sign of the real part of the left-most eigenvalue changing at values of the Reynolds number close to these predicted values, for small α, even though the number of meshpoints used is very modest. It is particularly interesting to note the exceptional accuracy of both methods when $\alpha = 10^{-2}$: they agree to all digits quoted, even using just 40 mesh intervals.

5. Conclusions

We have discussed some numerical methods for self-adjoint and non-self-adjoint Sturm–Liouville problems. We have concentrated on our own work because of space limitations, and we apologize to the many authors whose important contributions have not been included. The methods discussed here for self-adjoint problems not only approximate the eigenvalues and eigenvectors, but by approximating the counting function $N(\lambda)$, they also find the eigenvalue index (and in fact can aim for an eigenvalue with given index). For high eigenvalues, the ODE methods discussed here are usually more accurate and less costly than Galerkin or finite difference methods.

For self-adjoint problems of orders 4 and 6, coefficient approximation together with the W-matrix method (as discussed in [13–15]) is the cheapest method we know with high accuracy. Self-adjoint problems of order greater than 6 require Θ-matrices, and solution of the equation $\Theta' = \mathrm{i}\Theta\Omega$. Two methods for this are Marletta's method [21] using the Magnus series (which keeps Θ unitary) and the method of Dieci et al. [7] (which projects to unitary matrices). The computational costs of these methods seem to be remarkably similar (see [21] for a comparison). These methods can be quite expensive for high-order problems; and finding new, accurate methods with lower cost is an important and challenging problem.

For non-self-adjoint problems we have discussed the methods given in [16], using the argument principle. The code described in [16] can find the eigenvalues in a rectangle, left half-plane, or vertical strip. It can find the kth eigenvalue as ordered by the real part. The x-integration is carried out using compound matrices (which can be quite expensive) or, when possible, by transformation to a 2nd-order vector Sturm–Liouville problem (which is considerably cheaper). Some further problems and future directions are:

- methods for singular problems, including the approximation of essential spectra,
- analysis and codes for systems of mixed order (or block operators), and the associated problems with rational coefficients,
- applications of the various codes discussed here to physical problems, especially in hydrodynamics and magnetohydrodynamics.

Acknowledgements

The authors are grateful for the support of the EPSRC (through grants GR/J99438, GR/K59620, GR/L74354) which enabled them to work closely together on these projects.

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