$$\partial_{tt} u = \partial_{xx} u \quad \text{for} \quad 0 < x < \pi,\ t > 0$$

BC: $u(0,t) = \sin(t)$

$u(\pi,t) = 0$

IC: $u(x,0) = \sin(x)$

$\frac{du}{dx}(x,0) = \cos(x)$

$\therefore u = v + w \rightarrow \underline{v = u - w}$

$w = \sin(t) - \frac{x}{\pi}\sin(t)$

$\frac{dw}{dt} = \cos(t) - \frac{x}{\pi}\cos(t)$

$\frac{d^2w}{dt^2} = -\sin(t) + \frac{x}{\pi}\sin(t)$

$= \left(\frac{x}{\pi} - 1\right)\sin(t)$

$$\boxed{\begin{aligned}
&\underline{\frac{d^2v}{dt^2} = \frac{d^2u}{dt^2} - \frac{d^2w}{dt^2}} \\
&= \frac{d^2v}{dx^2} + \frac{d^2w}{dx^2} - \frac{d^2w}{dt^2} \\
&= \frac{d^2v}{dx^2} + \underbrace{\left(1 - \frac{x}{\pi}\right)\sin(t)}_{= Q} \\
&\text{BC: } v(0,t) = 0 \\
&\qquad v(\pi,t) = 0 \\
&\text{IC: } v(x,0) \\
&\qquad = u(x,0) - w(x,0) \\
&\qquad = \sin(x) \\
&\qquad \frac{dv}{dx}(x,0) \\
&\qquad = \frac{du}{dx}(x,0) - \frac{dw}{dx}(x,0) \\
&\qquad = \cos(x) - (1 - \cancel{\tfrac{x}{\pi}})
\end{aligned}}$$

$\phi = \sin\frac{n\pi}{L} = \sin(n)$

$\lambda_n = n^2$

Eigenfunction expansion.

$$\begin{cases} V(x,t) = \sum a_n(t)\, \phi_n(x) \\ f(x) = \sum a_n(0)\, \phi_n(x) \\ \overline{Q}(x,t) = \sum \overline{q}_n(t)\, \phi_n(x) \end{cases}$$

$$\to \partial_{tt} V = \sum a_n''(t)\, \phi_n(x)$$

$$= \partial_{xx} V + \overline{Q}$$

$$= \sum \left[-\lambda_n a_n(t) + \overline{q}_n(t) \right] \phi_n(x)$$

$$\Rightarrow a_n''(t) + \lambda_n a_n(t) = \overline{q}_n(t)$$

$a_n(0) =$ find from V_0

$a_n'(0) =$ find from V_0'

- Find $a_n(0)$

$$V(x,t) = \sum a_n(t) \sin nx$$

$$a_n(t) = \frac{2}{\pi} \int_0^{\pi} V(x,t) \sin nx \, dx$$

$$a_n(0) = \frac{2}{\pi} \int_0^{\pi} V(x,0) \sin nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \sin x \sin nx \, dx$$

$$= \boxed{\frac{-2\sin(\pi n)}{\pi(n^2-1)}}$$

- Find $a_n'(0)$

$$v'(x,t) = \sum a_n'(t) \sin nx$$

$$a_n'(t) = \frac{2}{\pi}\int_0^{\pi} v'(x,t) \sin nx \, dx$$

$$a_n'(0) = \frac{2}{\pi}\int_0^{\pi} \left[\cos x - \left(1 - \frac{x}{\pi}\right)\right] \sin nx \, dx$$

$$= \boxed{\frac{2((n^2-1)\sin(\pi n) + \pi n^3 \cos(\pi n) + \pi n)}{\pi^2 n^2 (n^2-1)}}$$

or

$$a_n'(0) = \frac{2}{\pi}\int_0^{\pi} \cos x \sin nx \, dx$$

$$= \frac{2n(\cos(\pi n)+1)}{\pi(n^2-1)}$$

- Find $q_n(t)$

$$\underbrace{\left(1 - \frac{x}{\pi}\right)\sin(t)}_{Q(x,t)} = \sum q_n(t) \sin nx$$

$$q_n(t) = \frac{2}{\pi}\int_0^{\pi} \left(1 - \frac{x}{\pi}\right)\sin(t) \sin nx \, dx$$

$$= \boxed{\frac{2\left(\frac{1}{n} - \frac{\sin(\pi n)}{\pi n^2}\right)\sin(t)}{\pi}}$$

$$a_n''(t) + n^2 a_n'(t) - \frac{2\left(\frac{1}{n} - \frac{\sin(\pi n)}{\pi n^2}\right)\sin(t)}{\pi} = 0$$

$$a_n(0) = \boxed{\frac{-2\sin(\pi n)}{\pi(n^2-1)}}$$

$$a_n'(0) =$$

$$\boxed{\frac{2((n^2-1)\sin(\pi n) + \pi n^3\cos(\pi n) + \pi n)}{\pi^2 n^2(n^2-1)}}$$