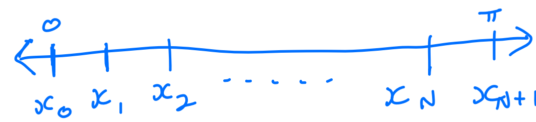
General SLEP

$0 \le x \le \pi$

$$-(p(x)y') - q(x)y = \lambda r(x) y$$



BC: $y(0) = y(L) = 0$

$$y'(x_i) \approx \frac{y(x_{i+1}) - y(x_i)}{\Delta x}$$

$$y''(x_i) \approx \frac{y(x_{i+1}) + y(x_{i-1}) - 2y(x_i)}{\Delta x^2}$$

$\Downarrow$ $h = \Delta x$

$$-\boxed{p(x_i)h^{-2}\left[y(x_{i+1}) + y(x_{i-1}) - 2y(x_i)\right]} - \boxed{p'(x_i)h^{-1}\left[y(x_{i+1}) - y(x_i)\right]}$$
$$-\boxed{q(x_i)y(x_i)} = \boxed{\lambda_i r(x_i) y(x_i)}$$

- Dirichlet BC

part.

$$-\boxed{p(x_i)h^{-2}\left[y(x_{i+1}) + y(x_{i-1}) - 2y(x_i)\right]}$$

Say $N = 5$ $\quad y_1 = y(x_1)$. $\quad y_0 = y_6 = 0$ (BC)

$(2h^{-2}y_1 - h^{-2}y_2) \times P_1$

$(-h^{-2}y_1 + 2h^{-2}y_2 - h^{-2}y_3) \times P_2$

$(-h^{-2}y_2 + 2h^{-2}y_3 - h^{-2}y_4) \times P_3$

$(-h^{-2}y_3 + 2h^{-2}y_4 - h^{-2}y_5) \times P_4$

$(-h^{-2}y_4 + 2h^{-2}y_5) \times P_5$

$$\Rightarrow \underbrace{h^{-2}\begin{bmatrix} P_1 & P_1 & P_1 & P_1 & P_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ P_5 & P_5 & P_5 & P_5 & P_5 \end{bmatrix} \cancel{\times} \bullet \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}}_{\to A} \begin{bmatrix} y_1 \\ \vdots \\ \vdots \\ y_5 \end{bmatrix}$$

part

$$-\boxed{p'(x_i)h^{-1}\left[y(x_{i+1})-y(x_i)\right]}$$

say $N=5$

$$\begin{aligned}
&(h^{-1}y_2 - h^{-1}y_1)p'_1\\
&(h^{-1}y_3 - h^{-1}{}_2)p'_2\\
-&(h^{-1}y_4 - h^{-1}{}_3)p'_3\\
&(h^{-1}y_5 - h^{-1}{}_4)p'_4\\
&(h^{-1}\underset{0}{\cancel{y_6}} - h^{-1}{}_5)p'_5
\end{aligned}
\Rightarrow -h^{-1}\underbrace{\begin{bmatrix} p'_1 & p'_1 & p'_1 & p'_1 & p'_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p'_5 & p'_5 & p'_5 & p'_5 & p'_5 \end{bmatrix} * \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}}_{B} \begin{bmatrix} y_1 \\ \vdots \\ y_5 \end{bmatrix}$$

part

$$-q(x_i)y(x_i)$$

say $N=5$

$$-\begin{bmatrix} q_1 & 0 & 0 & 0 & 0 \\ 0 & q_2 & 0 & 0 & 0 \\ 0 & 0 & q_3 & 0 & 0 \\ 0 & 0 & 0 & q_4 & 0 \\ 0 & 0 & 0 & 0 & q_5 \end{bmatrix}\begin{bmatrix} y_1 \\ \vdots \\ y_5 \end{bmatrix}$$

part

$$\lambda_i r(x_i) y(x_i)$$

$$\begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 \end{bmatrix}\begin{bmatrix} r_1 & 0 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 & 0 \\ 0 & 0 & r_3 & 0 & 0 \\ 0 & 0 & 0 & r_4 & 0 \\ 0 & 0 & 0 & 0 & r_5 \end{bmatrix}\begin{bmatrix} y_1 \\ \vdots \\ y_5 \end{bmatrix}$$

$$A + B + C = D \longrightarrow Dy = r\lambda y.$$

Need to append BC to eigenvectors.

$$\begin{array}{r|c}
 & \text{for } \lambda_1 \quad \text{for } \lambda_2 \\
\text{for } y_0 & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
\text{for } y_1 &
\end{array}\; V$$

$V = [\text{zeros}(1,N); V; \text{zeros}(1,N)];$

Neumann BC

$$-\boxed{p(x_i)h^{-2}\left[y(x_{i+1})+y(x_{i-1})-2y(x_i)\right]}-\boxed{p'(x_i)h^{-1}\left[y(x_{i+1})-y(x_i)\right]}$$

$$-\boxed{q(x_i)y(x_i)}=\boxed{\lambda_i r(x_i)y(x_i)}$$

$$\left.\begin{array}{l} y'(a)=0 \\ y'(b)=0 \end{array}\right] BC$$

part.

$$-\boxed{p(x_i)h^{-2}\left[y(x_{i+1})+y(x_{i-1})-2y(x_i)\right]}$$

say $N=5$ (inner points)

$$-h^{-2}y_0$$

$$(-h^{-2}y_1-\cancel{h^{-2}y_{-1}}+2h^{-2}y_0)p_0$$

$$(-h^{-2}y_2-h^{-2}y_0+2h^{-2}y_1)p_1$$

$$(-h^{-2}y_3-h^{-2}y_1+2h^{-2}y_2)p_2$$

$$(-h^{-2}y_4-h^{-2}y_2+2h^{-2}y_3)p_3$$

$$(-h^{-2}y_5-h^{-2}y_3+2h^{-2}y_4)p_4$$

$$(-h^{-2}y_6-h^{-2}y_4+2h^{-2}y_5)p_5$$

$$(-\cancel{h^{-2}y_7}-h^{-2}y_5+2h^{-2}y_6)p_6$$

$$-h^{-2}y_6$$

$$y'(x_i)\approx\frac{y(x_{i+1})-y(x_i)}{\Delta x} \longrightarrow \Delta x\, y_6' \approx y_7-y_6$$

$$\rightarrow y_7 \approx y_6+\Delta x \underbrace{\cancel{y_6'}}_{BC}^{0}$$

$$\Rightarrow y_7 \sim y_6$$

$$\rightarrow h^{-2}\begin{bmatrix} p_0 & \cdots & p_0 \\ \vdots & & \vdots \\ p_6 & \cdots & p_6 \end{bmatrix} * \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

part

$$-\boxed{p'(x_i)h^{-1}\left[y(x_{i+1})-y(x_i)\right]}$$

$$(-h^{-1}y_1+h^{-1}y_0)p'_0$$

$$(-h^{-1}y_2+h^{-1}y_1)p'_1$$

$$(-h^{-1}y_3+h^{-1}y_2)p'_2$$

$$(-h^{-1}y_4+h^{-1}y_3)p'_3$$

$$(-h^{-1}y_5+h^{-1}y_4)p'_4$$

$$(-h^{-1}y_6+h^{-1}y_5)p'_5$$

$$\cancel{(-h^{-1}y_7+h^{-1}y_6)p'_6} \quad 0$$

$$\rightarrow h^{-1}\begin{bmatrix} p'_0 & \cdots & p'_0 \\ \vdots & & \vdots \\ p'_6 & \cdots & p'_6 \end{bmatrix} \overset{*}{\bullet} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$-\boxed{q(x_i)y(x_i)}$$

$$-\begin{bmatrix} q_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q_6 \end{bmatrix}$$

$$\boxed{\lambda_i r(x_i)y(x_i)}$$

$$\begin{bmatrix} r_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & \ddots & & & & & \\ & & & & & & r_6 \end{bmatrix} \begin{bmatrix} \lambda_0 & & \\ & \ddots & \\ & & \lambda_6 \end{bmatrix}$$

Mixed BC

$$-\boxed{p(x_i)\,h^{-2}\left[y(x_{i+1})+y(x_{i-1})-2y(x_i)\right]}-\boxed{p'(x_i)\,h^{-1}\left[y(x_{i+1})-y(x_i)\right]}$$
$$-\boxed{q(x_i)\,y(x_i)}=\boxed{\lambda_i\, r(x_i)\, y(x_i)}$$

part.

$$-\boxed{p(x_i)\,h^{-2}\left[y(x_{i+1})+y(x_{i-1})-2y(x_i)\right]}$$

Say $N=5$ $\quad y_1 = y(x_1)$. $\quad y_0 = 0$ (BC)

$y'_6 = 0$ (BC)

$$(2h^{-2}y_1 - h^{-2}y_2)\times p_1$$
$$(-h^{-2}y_1 + 2h^{-2}y_2 - h^{-2}y_3)\times p_2$$
$$(-h^{-2}y_2 + 2h^{-2}y_3 - h^{-2}y_4)\times p_3$$
$$(-h^{-2}y_3 + 2h^{-2}y_4 - h^{-2}y_5)\times p_4$$
$$(-h^{-2}y_4 + 2h^{-2}y_5 - h^{-2}y_6)\times p_5$$
$$\underline{(-h^{-2}y_5 + 2h^{-2}y_6 - h^{-2}y_7)\times p_6}$$

$$\longrightarrow \quad y'(x_i) \approx \frac{y(x_{i+1}) - y(x_i)}{\Delta x} \quad \longrightarrow \quad \Delta x\, y'_6 \approx y_7 - y_6$$

$$\rightarrow \quad y_7 \approx y_6 + \Delta x \underbrace{\cancel{y'_6}}_{BC}{}^{0}$$

$$\Rightarrow y_7 \sim y_6$$

~

$$\underbrace{\begin{bmatrix} P_1 & P_1 & P_1 & P_1 & P_1 & P_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ P_6 & P_6 & P_6 & P_6 & P_6 & P_6 \end{bmatrix} \odot \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}}_{A} h^{-2} \begin{bmatrix} y_1 \\ \vdots \\ y_6 \end{bmatrix}$$

part

$$-\boxed{p'(x_i)h^{-1}\left[y(x_{i+1})-y(x_i)\right]}$$

say $N=5$

$$-\begin{matrix} (h^{-1}y_2 - h^{-1}y_1)p'_1 \\ (h^{-1}y_3 - h^{-1}y_2)p'_2 \\ (h^{-1}y_4 - h^{-1}y_3)p'_3 \\ (h^{-1}y_5 - h^{-1}y_4)p'_4 \\ (h^{-1}y_6 - h^{-1}y_5)p'_5 \\ (h^{-1}\cancel{y_7 - h^{-1}}y_6)p'_6 \\ 0 \end{matrix} \Rightarrow -h^{-1} \begin{bmatrix} P'_1 & P'_1 & P'_1 & P'_1 & P'_1 & P'_1 \\ \vdots & \vdots & \vdots & & & \\ P'_6 & P'_6 & P'_6 & P'_6 & P'_6 & P'_6 \end{bmatrix} \odot \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_6 \end{bmatrix}$$

Part

$$\boxed{-q(x_i)\,y(x_i)}$$

say $N=5$

$$-\underbrace{\begin{bmatrix} q_1 & 0 & 0 & 0 & 0 \\ 0 & q_2 & 0 & 0 & 0 \\ 0 & 0 & q_3 & 0 & 0 \\ 0 & 0 & 0 & q_4 & 0 \\ 0 & 0 & 0 & 0 & q_5 \end{bmatrix}}_{C} \begin{bmatrix} y_1 \\ \vdots \\ y_5 \end{bmatrix}$$

q_6 y_6

part

$$\boxed{\lambda_i\, r(x_i)\, y(x_i)}$$

$$\begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 \end{bmatrix} \begin{bmatrix} r_1 & 0 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 & 0 \\ 0 & 0 & r_3 & 0 & 0 \\ 0 & 0 & 0 & r_4 & 0 \\ 0 & 0 & 0 & 0 & r_5 \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_5 \end{bmatrix}$$

λ_6 r_6 y_6

Central Diff.

Dirichlet

part

$$-\boxed{p'(x_i)h^{-1}\left[y(x_{i+1})-y(x_i)\right]}$$

central

$$-p'(x_i)\frac{y(x_{i+1})-y(x_{i-1})}{2h}$$

say $N=5$

$$\begin{aligned}
&(h^{-1}y_2 - h^{-1}y_1)p'_1\\
&(h^{-1}y_3 - h^{-1}{}_2)p'_2\\
-&(h^{-1}y_4 - h^{-1}{}_3)p'_3\\
&(h^{-1}y_5 - h^{-1}{}_4)p'_4\\
&(\underset{0}{\cancel{h^{-1}y_6}} - h^{-1}{}_5)p'_5
\end{aligned}
\Rightarrow -h^{-1}\underbrace{\begin{bmatrix} p'_1 & p'_1 & p'_1 & p'_1 & p'_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p'_5 & p'_5 & p'_5 & p'_5 & p'_5 \end{bmatrix} * \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}}_{\hookrightarrow B}\begin{bmatrix} y_1 \\ \vdots \\ y_5 \end{bmatrix}$$

$$\begin{aligned}
&\left(\frac{h^{-1}}{2}y_2 - \cancel{\frac{h^{-1}}{2}y_0}\right)p'_1\\
-&\left(\frac{h^{-1}}{2}y_3 - \frac{h^{-1}}{2}y_1\right)p'_2\\
&\left(\frac{h^{-1}}{2}y_4 - \frac{h^{-1}}{2}y_2\right)p'_3\\
&\left(\frac{h^{-1}}{2}y_5 - \frac{h^{-1}}{2}y_3\right)p'_4\\
&\left(\cancel{\frac{h^{-1}}{2}y_6} - \frac{h^{-1}}{2}y_4\right)p'_5
\end{aligned}
\Rightarrow [P]\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ \vdots \\ y_5 \end{bmatrix}\left(-\frac{h^{-1}}{2}\right)$$

Neumann

part

$$-\boxed{p'(x_i)\,h^{-1}\,[y(x_{i+1})-y(x_i)]}$$

$(-h^{-1}y_1 + h^{-1}y_0)\,p'_0$

$(-h^{-1}y_2 + h^{-1}y_1)\,p'_1$

$(-h^{-1}y_3 + h^{-1}y_2)\,p'_2$

$(-h^{-1}y_4 + h^{-1}y_3)\,p'_3$

$(-h^{-1}y_5 + h^{-1}y_4)\,p'_4$

$(-h^{-1}y_6 + h^{-1}y_5)\,p'_5$

$\cancel{(-h^{-1}y_7 + h^{-1}y_6)\,p'_6}$ 0

$$\to h^{-1}\begin{bmatrix} p'_0 & \cdots & p'_0 \\ \vdots & & \vdots \\ p'_6 & \cdots & p'_6 \end{bmatrix} *\cdot \underbrace{\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$$

central difference

$\to \cancel{(-h^{-1}y_1 + h^{-1}y_{-1})}\,p'_0$ 0

$(-h^{-1}y_2 + h^{-1}y_0)\,p'_1$

$(-h^{-1}y_3 + h^{-1}_1)\,p'_2$

$(-h^{-1}y_4 + h^{-1}_2)\,p'_3$

$(-h^{-1}y_5 + h^{-1}_3)\,p'_4$

$(-h^{-1}y_6 + h^{-1}_4)\,p'_5$

$\cancel{(-h^{-1}y_7 + h^{-1}y_5)\,p'_6}$ 0

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix}$$
$$0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0$$

mixed

part

$$-\boxed{p'(x_i)\,h^{-1}\,[y(x_{i+1}) - y(x_i)]}$$

say $N=5$

$$(h^{-1}y_2 - h^{-1}y_1)\,p'_1$$
$$(h^{-1}y_3 - h^{-1}y_2)\,p'_2$$
$$-\ (h^{-1}y_4 - h^{-1}y_3)\,p'_3$$
$$(h^{-1}y_5 - h^{-1}y_4)\,p'_4$$
$$(h^{-1}y_6 - h^{-1}y_5)\,p'_5$$
$$(\underset{0}{\cancel{h^{-1}y_7 - h^{-1}}}y_6)\,p'_6$$

$$\Rightarrow -h^{-1}\begin{bmatrix} p'_1 & p'_1 & p'_1 & p'_1 & p'_1 & p'_1 \\ \vdots & \vdots & \vdots & & & \\ p'_6 & p'_6 & p'_6 & p'_6 & p'_6 & p'_6 \end{bmatrix} * \underbrace{\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}\begin{bmatrix} y_1 \\ \vdots \\ y_6 \end{bmatrix}$$

$$\downarrow$$

$$(h^{-1}y_2 - \overset{0}{\cancel{h^{-1}y_0}})\,p'_1$$
$$(h^{-1}y_3 - h^{-1}y_1)\,p'_2$$
$$(h^{-1}y_4 - h^{-1}y_2)\,p'_3$$
$$(h^{-1}y_5 - h^{-1}y_3)\,p'_4$$
$$(h^{-1}y_6 - h^{-1}y_4)\,p'_5$$
$$\underset{0}{\cancel{(h^{-1}y_7 - h^{-1}y_5)}}\,p'_6$$

$$-\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$