$$\partial_{tt} u = \partial_{xx} u \quad \text{for} \quad 0 < x < \pi, \ t > 0$$

BC: $u(0, t) = \sin(t)$

$u(\pi, t) = 0$

IC: $u(x, 0) = \sin(x)$

$\dfrac{du}{dx}(x, 0) = \cos(x)$

$\therefore u = v + w \quad \rightarrow \quad v = u - w$

$w = \sin(t) - \dfrac{x}{\pi} \sin(t)$

$\dfrac{dw}{dt} = \cos(t) - \dfrac{x}{\pi} \cos(t)$

$\dfrac{d^2 w}{dt^2} = -\sin(t) + \dfrac{x}{\pi} \sin(t)$

$= \left(\dfrac{x}{\pi} - 1 \right) \sin(t)$

$$\dfrac{d^2 v}{dt^2} = \dfrac{d^2 u}{dt^2} - \dfrac{d^2 w}{dt^2}$$

$$= \dfrac{d^2 v}{dx^2} + \dfrac{d^2 w}{dx^2} - \dfrac{d^2 w}{dt^2}$$

$$= \dfrac{d^2 v}{dx^2} + \left(1 - \dfrac{x}{\pi} \right) \sin(t)$$

$$= Q$$

BC: $v(0, t) = 0$

$v(\pi, t) = 0$

IC: $v(x, 0)$

$= u(x, 0) - w(x, 0)$

$= \sin(x)$

$\dfrac{dv}{dx}(x, 0)$

$=$

$= \cos(x) - \left(1 - \dfrac{x}{\pi} \right)$

$\phi = \sin \dfrac{n\pi}{L} = \sin(n)$

$\lambda_n = n^2$

Eigenfunction expansion.

$$V(x,t) = \sum a_n(t)\, \phi_n(x)$$
$$f(x) = \sum a_n(0)\, \phi_n(x)$$
$$\bar{Q}(x,t) = \sum \bar{q}_n(t)\, \phi_n(x)$$

$\rightarrow$

$$\partial_{tt}V = \sum a_n''(t)\, \phi_n(x)$$
$$= \partial_{xx}V + \bar{Q}$$
$$= \sum \left[-\lambda_n\, a_n(t) + \bar{q}_n(t) \right] \phi_n(x)$$
$$\Rightarrow \boxed{a_n''(t) + \lambda_n\, a_n(t) = \bar{q}_n(t)}$$

$a_n(0) =$ find from V_o
$a_n'(0) =$ find from V_o'

- Find $a_n(0)$

$$V(x,t) = \sum a_n(t)\, \sin nx$$

$$a_n(t) = \frac{2}{\pi} \int_0^\pi V(x,t)\, \sin nx\, dx$$

$$a_n(0) = \frac{2}{\pi} \int_0^\pi V(x,0)\, \sin nx\, dx$$

$$= \frac{2}{\pi} \int_0^\pi \sin x \, \sin nx \, dx$$

$$= \boxed{\frac{-2\sin(\pi n)}{\pi(n^2 - 1)}}$$

- Find $a_n'(0)$

$$v'(x,t) = \sum a_n'(t) \sin nx$$

$$a_n'(t) = \frac{2}{\pi} \int_0^{\pi} v'(x,t) \sin nx \, dx$$

$$a_n'(0) = \frac{2}{\pi} \int_0^{\pi} \left[\cos x - \left(1 - \frac{x}{\pi}\right) \right] \sin nx \, dx$$

$$= \boxed{\frac{2((n^2-1)\sin(\pi n) + \pi n^3 \cos(\pi n) + \pi n)}{\pi^2 n^2 (n^2-1)}}$$

- Find $q_n(t)$

$$\underbrace{\left(1 - \frac{x}{\pi}\right) \sin(t)}_{Q(x,t)} = \sum q_n(t) \sin nx$$

$$q_n(t) = \frac{2}{\pi} \int_0^{\pi} \left(1 - \frac{x}{\pi}\right) \sin(t) \sin nx \, dx$$

$$= \boxed{\frac{2\left(\frac{1}{n} - \frac{\sin(\pi n)}{\pi n^2}\right) \sin(t)}{\pi}}$$

$$a_n''(t) + n^2 a_n'(t) - \frac{2\left(\frac{1}{n} - \frac{\sin(\pi n)}{\pi n^2}\right)\sin(t)}{\pi} = 0$$

$$a_n(0) = \boxed{\frac{-2\sin(\pi n)}{\pi(n^2-1)}}$$

$$a_n'(0) =$$

$$\boxed{\frac{2\left((n^2-1)\sin(\pi n) + \pi n^2 \cos(\pi n) + \pi n\right)}{\pi^2 n^2(n^2-1)}}$$