Sturm-Liouville Eigenvalue Problem and numerical solution

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Last updated: August 1, 2021

Abstract

Summary: numerical solution to the SLEP by finite difference.

1 The Sturm-Liouville Eigenvalue problem

$$\begin{aligned}
&(p(x)\phi')' + q(x)\phi = -\lambda\sigma\phi \\
&\beta_1\phi(a) + \beta_2\phi'(a) = 0; \\
&\beta_3\phi(b) + \beta_4\phi'(b) = 0;
\end{aligned}$$

Regular SLEP:

$$\begin{aligned}
&p', q, \sigma \in C[a, b], \\
&p(x) > 0, \sigma(x) > 0, \forall x \in [a, b] \\
&\beta_1^2 + \beta_2^2 > 0, \beta_3^2 + \beta_4^2 > 0,
\end{aligned}$$

Theorem 1 (Sturm-Liouville Theorems) *A regular SLEP has eigenvalues and eigenfunctions $\{(\lambda_n, \phi_n)\}$ s.t.*

1-2 $\{\lambda_n\}_{n=1}^{\infty}$ *are real and strictly increasing to* ∞

3 ϕ_n *is the unique (up to a multiplicative factor) solution to* λ_n*;* ϕ_n *has* $n-1$ *zeros*

4 $\{\phi_n\}_{n=1}^{\infty}$ *is complete. That is, any piecewise smooth* f *can be represented by a generalized Fourier series* $f(x) \sim \sum_{n=1}^{\infty} a_n\phi_n(x) = \frac{1}{2}[f(x_-) + f(x_+)]$

5 $\{\phi_n\}_{n=1}^{\infty}$ *are orthogonal:* $\langle\phi_n, \phi_m\rangle_\sigma = 0$ *if* $n \neq m$*;* $\langle\phi_n, \phi_n\rangle_\sigma > 0$

6 Rayleigh quotient $\lambda_n = -\frac{\langle L\phi_n, \phi_n\rangle}{\langle\phi_n, \phi_n\rangle_\sigma}$*;*

where $\langle f, g\rangle := \int_a^b f(x)g(x)dx; \quad \langle f, g\rangle_\sigma := \int_a^b f(x)g(x)\sigma(x)dx.$

Example: when $p(x) = 1$ for all x and $q = 0$. The eigenvalues and eigenfunctions are $\lambda_n = (\frac{n\pi}{b-a})^2$

2 Finite difference approximations

The simplest approximation is finite difference approximation. We approximate the 1st and 2nd order derivatives by central difference:

$$\begin{aligned}
y'(x) &= \frac{y(x+h) - y(x-h)}{2h} + O(h^2), \\
y''(x) &= \frac{y(x+h) + y(x-h) - 2y(x)}{h^2} + O(h^2),
\end{aligned}$$

where the order of error follows from Taylor expansion, assuming that the function y has bounded 3rd order derivatives on $[a, b]$.

References

[1] Albert Einstein. *Zur Elektrodynamik bewegter Körper*. (German) [*On the electrodynamics of moving bodies*]. Annalen der Physik, 322(10):891âĂŞ921, 1905.

[2] Knuth: Computers and Typesetting,
`http://www-cs-faculty.stanford.edu/~uno/abcde.html`