Homework Assignment 4

Information Theory, TBSI

December 27, 2017

1. **Rate distortion function with infinite distortion.** Find the rate distortion function $R(D) = \min I(X;\hat{X})$ for $X \sim$Bernoulli($\frac{1}{2}$) and distortion

$$d(x,\hat{x}) = \begin{cases} 0, & x = \hat{x} \\ 1, & x = 1, \hat{x} = 0 \\ \infty, & x = 0, \hat{x} = 1 \end{cases}$$

2. **Bounds on the rate distortion function for squared-error distortion.** For the case of a continuous random variable X with mean zero and variance σ^2 and squared-error distortion, show that

$$h(X) - \frac{1}{2}\log 2\pi e D \leq R(D) \leq \frac{1}{2}\log\frac{\sigma^2}{D}$$

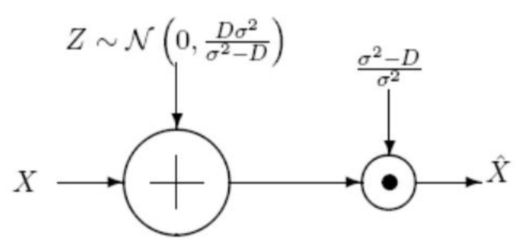
For the upper bound, consider the following joint distribution in Figure 1. Are Gaussian random variables harder or easier to describe than any other random variables with the same variance?

3. **Rate distortion.** Find and verify the rate distortion function $R(D)$ for X uniform on $\mathcal{X} = \{1, 2, \cdots, 2m\}$ and

$$d(x,\hat{x}) = \begin{cases} 1, & \text{for } x - \hat{x} \text{ odd} \\ 0, & \text{for } x - \hat{x} \text{ even} \end{cases}$$

where $\hat{X}$ is defined on $\hat{\mathcal{X}} = \{1, 2, \cdots, 2m\}$.

4. **Adding a column to the distortion matrix.** Let $R(D)$ be the rate distortion function for an i.i.d. process with probability mass function $p(x)$ and distortion function $d(x,\hat{x}), x \in \mathcal{X}, \hat{x} \in \hat{\mathcal{X}}$. Now suppose that we add a new reproduction symbol $\hat{x}_0$ to $\hat{\mathcal{X}}$ with associated distortion $d(x,\hat{x}), x \in \mathcal{X}$. Does this increase or decrease $R(D)$ and why?



$$\hat{X} = \frac{\sigma^2 - D}{\sigma^2}(X + Z)$$

Figure 1: Distribution in question 2

5. **Maximum Entropy.** Find the maximum entropy p.d.f. $f(x)$, defined for $x \geq 0$, satisfying $\mathbb{E}[X] = \alpha_1$, $\mathbb{E}[\ln X] = \alpha_2$. That is, maximize $-\int f(x) \ln f(x) dx$ subject to $\int x f(x) dx = \alpha_1$, $\int (\ln x) f(x) dx = \alpha_2$, where the integral is over $0 \leq x < \infty$. What p.d.f. $f(x)$ is this?