Homework Assignment 3

Information Theory, TBSI

November 30, 2017

1. **Channel Capacity.** Consider the DMC $Y = X + Z \mod 11$, where

$$Z = \begin{pmatrix} 1, 2, 3 \\ \\ \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \end{pmatrix}$$

and $X \in \{0, 1, 2, 3, \cdots, 10\}$. Assume that Z is independent of X.

   (a) Find the capacity.

   (b) What is the maximizing $p^*(x)$?

2. **Cascade of binary symmetric channels.** Show that a cascade of n identical independent binary symmetric channels,

$$X_0 \to BSC \to X_1 \to \cdots \to X_{n-1} \to BSC \to X_n$$

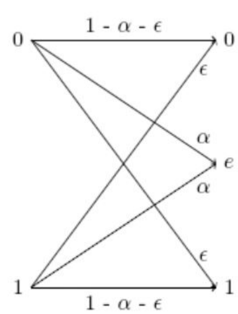
each with raw error probability p, is equivalent to a single BSC with error probability $\frac{1}{2}(1-(1-2p)^n)$ and hence that $\lim_{n\to\infty} I(X_0; X_n) = 0$ if $p \neq 0, 1$. No encoding or decoding takes place at the intermediate terminals $X_1, \cdots, X_{n-1}$. Thus, the capacity of the cascade tends to zero.

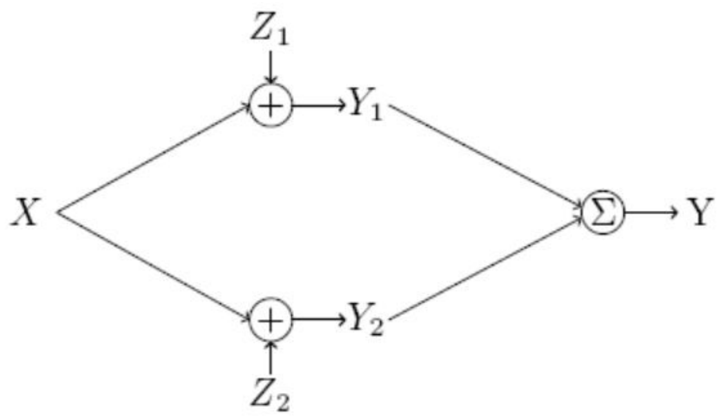
3. **Erasures and errors in a binary channel.** Consider a channel with binary inputs that has both erasures and errors. Let the probability of error be ϵ and the probability of erasure be α

   (a) Find the capacity of this channel.

   (b) Specialize to the case of the binary symmetric channel ($\alpha = 0$).

   (c) Specialize to the case of the binary erasure channel ($\epsilon = 0$).





4. **Multipath Gaussian channel.** Consider a Gaussian noise channel with power constraint P, where the signal takes two different paths and the received noisy signals are added together at the antenna, shown as above.

   (a) Find the capacity of this channel if Z_1 and Z_2 are jointly normal with covariance matrix

   $$K_z = \begin{bmatrix} \rho\sigma^2 & \sigma^2 \\ \sigma^2 & \rho\sigma^2 \end{bmatrix}$$

   (b) What is the capacity for $\rho = 0; \rho = 1; \rho = -1$?

5. **A parametric form for channel capacity.** Consider m parallel Gaussian channels, $Y_i = X_i + Z_i$, where $Z_i \sim N(0, \lambda_i)$ and the noises X_i are independent random variables. Thus $C = \sum_{i=1}^{m} \frac{1}{2} \log(1 + \frac{(\lambda - \lambda_i)^+}{\lambda_i})$ where λ is chosen to satisfy $\sum_{i=1}^{m} (\lambda - \lambda_i)^+ = P$. Show that this can be rewritten in the form

   $$P(\lambda) = \sum_{i:\lambda_i <= \lambda} (\lambda - \lambda_i), C(\lambda) = \sum_{i:\lambda_i <= \lambda} \frac{1}{2} \log \frac{\lambda}{\lambda_i}$$

   Here $P(\lambda)$ is piecewise linear and $C(\lambda)$ is piecewise logarithmic in λ.

6. **Converse of channel coding theorem.** Proof that for a DMC, any sequence of $(2^{nR}, n)$ codes with maximum probability of error $\lambda^{(n)} \to 0$ must have $R \leq C$.

7. **Telephone channel power.** Assume a time discrete and amplitude continuous memory-less channel's input is a zero-mean gaussian random variable with variance E. Channel noise is additive gaussian noise, variance $\sigma = 1\mu W$, channel bandwidth $r = 8000Hz$. If a telephone signal pass through this channel, and its message rate is $64kbps$. What is the minimum value of input channel power E?

8. **Z channel.** The Z-channel has binary input and output alphabets $\mathcal{X}, \mathcal{Y} \in \{0, 1\}$ and transition probabilities $p(y|x)$ given by the following matrix:

   $$Q = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

   Find the capacity of the Z-channel and the maximizing input probability distribution.

9. **Joint typicality.** Let X_i, Y_i, Z_i be i.i.d. according to $p(x, y, z)$. We will say that (X^n, Y^n, Z^n) is jointly typical(denoted as $(X^n, Y^n, Z^n) \in A_\epsilon^{(n)}$) if

   - $p(x^n) \in 2^{-n(H(X)\pm\epsilon)}$
   - $p(y^n) \in 2^{-n(H(Y)\pm\epsilon)}$
   - $p(z^n) \in 2^{-n(H(Z)\pm\epsilon)}$
   - $p(x^n, y^n) \in 2^{-n(H(X,Y)\pm\epsilon)}$
   - $p(x^n, z^n) \in 2^{-n(H(X,Z)\pm\epsilon)}$
   - $p(y^n, z^n) \in 2^{-n(H(Y,Z)\pm\epsilon)}$
   - $p(x^n, y^n, z^n) \in 2^{-n(H(X,Y,Z)\pm\epsilon)}$

   Now suppose $(\widetilde{X}^n, \widetilde{Y}^n, \widetilde{Z}^n)$ is drawn according to $p(x^n)p(y^n)p(z^n)$. Thus $\widetilde{X}^n, \widetilde{Y}^n, \widetilde{Z}^n$ have the same marginals as $p(x^n, y^n, z^n)$ but are independent. Find bounds on $Pr\{(X^n, Y^n, Z^n) \in A_\epsilon^{(n)}\}$ in terms of the entropies $H(X), H(Y), H(Z), H(X,Y), H(X,Z), H(Y,Z), H(X,Y,Z)$.

10. **Fano's inequality.** Message W with alphabet cardinality J, code length N, goes through a channel with capacity C, $W \to X^n \to Y^n \to \hat{X}^n \to \hat{W}$, proof that error probability P_e satisfies

$$P_e \geq \frac{1}{\log J}(R - C - \frac{1}{N})$$

   in which R is the input information rate of channel.