We give the constitutive model for non-linear material behaviour in small deformation with von-Mises criterion yield function. The flow potential F is given by:

$$F = f - k$$

where

$$f = \frac{1}{2} s_{ij} s_{ij}$$

in tensor nototation, and

$$k = \frac{1}{3}\sigma_y^2$$

The consistency condition (in plastic deformation period) requires that: $F = 0$. Therefore the total derivative $dF = 0$ and we can get (in tensor notation)

$$\frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} - \frac{2}{3}\sigma_y \frac{d\sigma_y}{d\bar{\epsilon}_p} d\bar{\epsilon}_p = 0 \tag{1}$$

where $\bar{\epsilon}_p$ is the acumulated plastic strain. By flow rule we can get

$$d\epsilon^p = d\lambda \frac{\partial f}{\partial \boldsymbol{\sigma}}$$

we also have

$$d\bar{\epsilon}_p = \sqrt{\frac{2}{3}} ||d\epsilon^p||_f$$

where $||\cdot||_f$ denotes the Frobinious norm of a rank 2 tensor. By the expresion of f, we have

$$\frac{\partial f}{\partial \sigma_{ij}} = s_{ij}$$

therefore,

$$d\bar{\epsilon}_p = \sqrt{\frac{2}{3}} d\lambda ||\boldsymbol{s}||_f$$

Since $f - k = 0$ we have $||\boldsymbol{s}||_f = \sqrt{\frac{2}{3}}\sigma_y$ In conclusion we have:

$$d\bar{\epsilon}_p = \frac{2}{3}\sigma_y d\lambda \tag{2}$$

Define the hardening modulus: $E_p = \frac{d\sigma_y}{d\bar{\epsilon}_p}$ From (1) we get:

$$s_{ij} d\sigma_{ij} - \frac{4}{9}\sigma_y^2 E_p d\lambda = 0 \tag{3}$$

In small deformation, the strain increment can be decomposed additively as:

$$d\epsilon_{ij} = d\epsilon_{ij}^e + d\epsilon_{ij}^p$$

By generalized Hook's Law:

$$
\begin{aligned}
d\sigma_{ij} =& D^e_{ijkl}(d\epsilon_{kl} - d\epsilon^p_{kl}) \\
=& D^e_{ijkl}(d\epsilon_{kl} - s_{kl}d\lambda)
\end{aligned}
$$

Combining this equation with (3), we can solve out:

$$
d\lambda = \frac{s_{ij}D^e_{ijkl}d\epsilon_{kl}}{s_{ij}s_{kl}D^e_{ijkl} + \frac{4}{9}\sigma_y^2 E_p}
$$

We replace $d\lambda$ in $d\sigma_{ij} = D^e_{ijkl}(d\epsilon_{kl} - s_{kl}d\lambda)$ with the above expression and get the non-linear relation between stress and strain in elastoplastic deformation:

$$
d\sigma_{ij} = D^{ep}_{ijkl}d\epsilon_{kl}
$$

where $D^{ep}_{ijkl} = D^e_{ijkl} - D^p_{ijkl}$. $\boldsymbol{D}^{ep}$ is called elastoplastic tangent modulus and

$$
D^p_{ijkl} = \frac{s_{mn}D^e_{ijmn}s_{qr}D^e_{qrkl}}{s_{mn}s_{qr}D^e_{qrmn} + \frac{4}{9}\sigma_y^2 E_p}
$$

Since (in 3D)

$$
D^e_{qrmn} = \lambda\delta_{qr}\delta_{mn} + G(\delta_{qm}\delta_{rn} + \delta_{qn}\delta_{rm})
$$

The denominator can be simplified to (use $\frac{1}{2}s_{ij}s_{ij} = \frac{\sigma_y^2}{3}$):

$$
\begin{aligned}
s_{mn}s_{qr}D^e_{qrmn} + \frac{4}{9}\sigma_y^2 E_p =& \lambda(s_{mm}s_{qq}) + 2Gs_{mn}s_{mn} + \frac{4}{9}\sigma_y^2 E_p \\
=& \frac{4\sigma_y^2}{9}(3G + E_p)
\end{aligned}
$$

Similarly, the numerator can be simplified to

$$
s_{mn}D^e_{ijmn}s_{qr}D^e_{qrkl} = 4G^2 s_{ij}s_{kl}
$$

As a result, we get:

$$
D^p_{ijkl} = \frac{9G^2 s_{ij}s_{kl}}{\sigma_y^2(3G + E_p)} \tag{4}
$$