Problem Set 6

Issued: Tuesday 5th December, 2017 **Due:** Monday 11th December, 2017

6.1. Let $\underline{\mathsf{y}} = [\mathsf{y}_1\ \mathsf{y}_2]^{\mathrm{T}}$ be a vector random variable whose components are i.i.d. Bernoulli random variables with parameter x, $0 < x < 1$, i.e., $\mathbb{P}(\mathsf{y}_i = 1) = x, i = 1, 2$.

(a) Show that $t(\underline{y}) = y_1 + y_2$ is a sufficient statistic.

(b) Let $\hat{x}(\underline{y}) = y_1$ be an estimator of the parameter x from the observation $\underline{y}$. Find $\mathrm{MSE}_{\hat{x}}(x)$, the mean-square error of this estimator.

(c) Let $\hat{x}'(t) = \mathbb{E}[\hat{x}(\underline{\mathsf{y}})|\mathsf{t} = t]$ be an estimator of the parameter x that uses the sufficient statistic t instead of the observations $\underline{y}$.

i. Show that $\hat{x}'(t)$ is a valid estimator, i.e., it does not depend on x.

ii. Show that $\mathrm{MSE}_{\hat{x}'}(x) = \gamma\,\mathrm{MSE}_{\hat{x}}(x)$ and find the constant γ.

(d) We now consider a generalization of this problem. Let $\underline{\mathsf{y}}$ be a random variable generated by a distribution $p_{\underline{\mathsf{y}}}(\cdot; x)$ and $\underline{t}(\underline{y})$ be a sufficient statistic. Let $\hat{x}(\underline{y})$ be an estimator of the parameter x based on the observation $\underline{y}$. We define an alternate estimator $\hat{x}'(\underline{t}) = \mathbb{E}[\hat{x}(\underline{\mathsf{y}})|\underline{\mathsf{t}} = \underline{t}]$.

i. Show that $\hat{x}'(\underline{t})$ is a valid estimator, i.e., it does not depend on x.

ii. Show that for any cost function $C(x, \hat{x})$ that is convex in $\hat{x}$, the following inequality holds:

$$\mathbb{E}[C(x, \hat{x}'(\underline{\mathsf{t}}))] \leq \mathbb{E}[C(x, \hat{x}(\underline{\mathsf{y}}))].$$

6.2. You are given a coin that might contain gold. You are asked to test for the presence of gold by using a spectrometer. Let x be a Bernoulli random variable with parameter p that indicates the presence of gold in the coin, i.e., $\mathsf{x} = 1$ means the coin contains gold. Let y be the reading of the spectrometer, distributed according to $p_{\mathsf{y}|\mathsf{x}}(\cdot|\cdot)$.

After observing the reading of the spectrometer y, you are asked to report a *soft* decision b_y that allows you to hedge your belief about the presence of gold in the coin. b_y is a real number between 0 and 1; it represents your belief that the coin contains gold. Naturally then, $1 - b_y$ represents your belief that there is no gold in the coin.

Note that in the binary hypothesis testing, we made *hard* decisions by completely committing to one of the two hypotheses. These hard decisions represent special, limiting cases of soft decisions. In the case of a hard decision, belief b_y is constrained to 0 (completely accepting the null hypothesis and completely rejecting the alternative hypothesis) or 1 (completely accepting the alternative hypothesis while completely rejecting the null hypothesis).

The reward for your work is a function of how well you guessed the correct answer, i.e.,

$$R(x, b_y) = \begin{cases} r(b_y), & x = 1 \\ r(1 - b_y), & x = 0 \end{cases}$$

where x is the true value of x, and $r : [0, 1] \mapsto \mathbb{R}$ is a function that maps belief values to reward values.

(a) Suppose the reward function is logarithmic: $r(z) = 1 + \ln(z)$. Note that under this reward function, you lose money if you assigned a belief of less than 1/e to the correct answer! Given that you observed $\mathsf{y} = y$, find the belief value, $b_y^{*\ln}$, that maximizes the expected reward.

(b) Now suppose the reward function is linear: $r(z) = z$. Given that you observed $\mathsf{y} = y$, find the belief value, $b_y^{*\text{linear}}$, that maximizes the expected reward.

(c) A reward function is called *proper* if it forces the players to report the posterior probability of the hidden variable as a belief.

i. Is the log reward function in (a) proper?

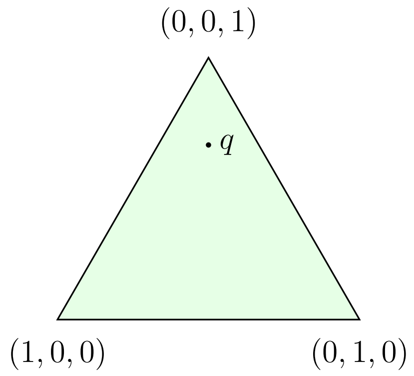
ii. Is the linear reward function in (b) proper?

iii. Do your answers to c(i) and c(ii) depend on the prior $p_{\mathsf{x}}(\cdot)$ and the likelihood $p_{\mathsf{y}|\mathsf{x}}(\cdot|\cdot)$? Explain.

6.3. Consider the set of distributions on $\Omega = \{0, 1, 2\}$ and note that they lie on the 2-simplex

$$\{p = (p_0, p_1, p_2) : p_0 + p_1 + p_2 = 1, p_0 \geq 0, p_1 \geq 0, p_2 \geq 0\}$$

represented by the triangular figure. Let y be a random variable such that $p_{\mathsf{y}}(i) = p_i, i \in \{0, 1, 2\}$. Let $q = \left(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right)$ be a particular probability mass function.



(a) Draw on the simplex the linear family corresponding to the expectation $\mathbb{E}[\mathsf{y}] = 0$, i.e. draw $\mathcal{L}_0 = \{p : \mathbb{E}_p[\mathsf{y}] = 0\}$.

(b) Draw $\mathcal{L}_{\frac{1}{2}} = \left\{p : \mathbb{E}_p[\mathsf{y}] = \frac{1}{2}\right\}$.

(c) Specify the exponential family $\mathcal{E}$ that passes through q and is orthogonal to $\mathcal{L}_{\frac{1}{2}}$, and draw the entire family on the 2-simplex.

(d) Calculate the I-projection p^* of q onto $\mathcal{L}_{\frac{1}{2}}$ and mark it on the simplex.

(e) Draw $\mathcal{P} = \left\{p : \mathbb{E}_p[\mathsf{y}] \leq \frac{1}{2}\right\}$.

(f) Calculate the I-projection $p^\star$ of q onto $\mathcal{P}$ and mark it. *Hint*: $D(\cdot\|q)$ is convex in its first argument.