1 Virtual Displacement Principle

From Equilibrium Equation (tensor notation):

$$\sigma_{ij,j} + f_i = 0 \text{ in } V$$

stress BC:

$$\sigma_{ij} n_j = T_i \text{ in } S_\sigma$$

Multiplying the above two equations with virtual displacement δu_i and integral in the corresponding region, we get

$$\int_V \delta u_i(\sigma_{ij,j} + f_i) dV - \int_{S_\sigma} \delta u_i(\sigma_{ij} n_j - T_i) dS = 0 \tag{1}$$

We integrate the following expression by part($\delta u_i := (\delta u)_i$):

$$\begin{aligned}\int_V \delta u_i \sigma_{ij,j} dV &= \int_V (\delta u_i \sigma_{ij})_j dV - \int_V \frac{1}{2}(\delta u_{i,j} + \delta u_{j,i})\sigma_{ij} dV \\ &= \int_{S_\sigma} \delta u_i \sigma_{ij} n_j dS - \int_V \delta\epsilon_{ij}\sigma_{ij} dV\end{aligned}$$

It follows:

$$\int_V \delta\epsilon_{ij}\sigma_{ij} dV = \int_V \delta u_i f_i dV + \int_{S_\sigma} \delta u_i T_i dS \tag{2}$$

For linear elasticity:

$$\int_V \delta\epsilon_{ij} D^e_{ijkl}\epsilon_{kl} dV = \int_V \delta u_i f_i dV + \int_{S_\sigma} \delta u_i T_i dS \tag{3}$$

$V = \cup V_i$ where $V_i \cap V_j = \emptyset$ for $i \neq j$. Each V_i is a cell which contains K nodes (vertex). The degree of freedom used on each cell equals the dimension of the problem times the number of nodes per cell. Consider a particular cell V^*, on V^* we have:

$$u_i = U_{ti}\phi_{ti}$$

Take $\delta u_i = \delta_{ii^*}\phi_{ri^*}(r, i^*$ is prescribed, corresponding to δ), we have:

$$\begin{aligned}\delta\epsilon_{ij} =& \frac{1}{2}(\delta u_{i,j} + \delta u_{j,i}) \\ =& \frac{1}{2}(\delta_{ii^*}\phi_{ri^*,j} + \delta_{ji^*}\phi_{ri^*,i})\end{aligned}$$

and

$$\begin{aligned}\epsilon_{kl} =& \frac{1}{2}(\delta u_{k,l} + \delta u_{l,k}) \\ =& \frac{1}{2}(U_{tk}\phi_{tk,l} + U_{tl}\phi_{tl,k})\end{aligned}$$

For isotropic material $D^e_{ijkl} = D^e_{ijlk}$,then

$$\begin{aligned}D^e_{ijkl}\epsilon_{kl} =& \frac{1}{2}(D^e_{ijkl}U_{tk}\phi_{tk,l} + D^e_{ijlk}U_{tk}\phi_{tk,l})\\ =& D^e_{ijkl}U_{tk}\phi_{tk,l}\end{aligned}$$

The above relation also holds for $D^{ep}_{ijkl} = D^e_{ijkl} - D^p_{ijkl}$ For given r, i^*, t, k, we have:

$$\begin{aligned}\delta\epsilon_{ij}D^e_{ijkl}\epsilon_{kl} =& \delta_{ii^*}\phi_{ri^*,j}D^e_{ijkl}\phi_{tk,l}U_{tk}\\ =& \left(\delta_{ii^*}\phi_{ri^*,j}(\lambda\delta_{ij}\phi_{tk,k} + G(\delta_{ik}\phi_{tk,j} + \delta_{jk}\phi_{tk,i}))\right)U_{tk}\\ =& (\lambda\phi_{ri^*,i^*}\phi_{tk,k} + G\phi_{ri^*,k}\phi_{tk,i^*} + G\delta_{i^*k}\phi_{ri^*,j}\phi_{tk,j})U_{tk}\end{aligned}$$

Notice the last term sums over j when $i^* = k$. For D^{ep}_{ijkl}, we revise the above by $\delta\epsilon_{ij}D^p_{ijkl}\epsilon_{kl}$ when the yielding condition is reached:

$$\delta\epsilon_{ij}D^p_{ijkl}\epsilon_{kl} = \phi_{ri^*,j}D^p_{i^*jkl}\phi_{tk,l}U_{tk} \tag{4}$$

The summation is over j and l. From (3), we get for plastic case that:

$$\int_V \delta\epsilon_{ij}(D^{ep}_{ijkl}\Delta\epsilon_{kl} + \sigma^n_{ij})dV = \int_V \delta u_i f_i dV + \int_{S_\sigma} \delta u_i T_i dS \tag{5}$$

Then we get that σ^n_{ij} contributes to the RHS:

$$-\int_V \delta\epsilon_{ij}\sigma^n_{ij} = -\int_V \phi_{ri^*,j}\sigma^n_{i^*j}dV \tag{6}$$

For axisymmetric problem, the displacement component vanishes along the $\hat{\theta}$, therefore the problem reduces to 2D. From (3) we change the integration to cylinderal representation and with $u_\theta \equiv 0$ in mind and ignoring $\boldsymbol{f}$, we get

$$\int_\Sigma \delta\epsilon_{ij}\sigma_{ij} r dr dz = \int_L \delta u_r T_r + \delta u_z T_z dz \tag{7}$$

where Σ is a section of revolution plane, and L is $\partial\Sigma$ minus the revolution axis.To solve the equivalent 2D problem, we need mesh Σ and apply Dirichlet BC on the revolution axis for u_r. For the LHS, we can further simply it using matrix form:

$$\begin{aligned}\delta\epsilon_{ij}\sigma_{ij} =& (\delta\boldsymbol{\epsilon})^T D\boldsymbol{\epsilon} \qquad (8)\\ =& (\delta\boldsymbol{u})^T(\boldsymbol{BDB})\boldsymbol{u} \qquad (9)\end{aligned}$$

where $u = (u_r, u_z)^T$ and

$$B = \begin{pmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial r} \\ \frac{1}{r} & 0 \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{pmatrix}, D = \begin{pmatrix} \lambda + 2G & \lambda & \lambda & 0 \\ \lambda & \lambda + 2G & \lambda & 0 \\ \lambda & \lambda & \lambda + 2G & 0 \\ 0 & 0 & 0 & G \end{pmatrix}$$