STABILIZATION OF INTERSECTION BETTI NUMBERS FOR MODULI SPACES OF ONE-DIMENSIONAL SHEAVES ON SURFACES

FEI SI AND FEINUO ZHANG

Abstract. In this paper, we study the intersection cohomology of the moduli space of semistable one-dimensional sheaves with fixed Euler characteristic, supported in a divisor class β on a smooth projective surface S. Assuming that this moduli space is irreducible, we prove that its intersection Betti numbers in a certain range are determined by a product formula derived from Göttsche's formula for Betti numbers of Hilbert schemes of points on S. As an application, for Enriques and bielliptic surfaces, we show the stabilization of intersection Betti numbers of this moduli space when β is sufficiently positive. In the Enriques case, we also prove a refined stabilization result concerning the perverse Hodge numbers when the moduli space is smooth.

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1. Introduction

Throughout, we work over the field $\mathbb{C}$ of complex numbers. Let S be a connected, smooth, projective surface with a fixed ample divisor H.

In recent years, moduli spaces of one-dimensional sheaves on S have attracted considerable attention. On the one hand, their connections with enumerative geometry and moduli spaces of Higgs bundles have led to a number of new conjectures and results; see, for instance, [3, 4, 9, 14, 15, 19, 21, 28, 29, 36]. On the other hand, these moduli spaces exhibit not only asymptotic behavior analogous to that of moduli spaces of torsion-free sheaves, but also pathologies [34, §5] in contrast to the torsion-free case, revealing rich geometry that remains to be understood.

Let χ be an integer and β a nonzero effective divisor of arithmetic genus $p_a(\beta)$ on S. Denote by $|\beta|^{\mathrm{int}} \subset |\beta|$ the open subscheme parametrizing integral curves. Consider the (coarse) moduli space $M_{\beta,\chi}$ parametrizing S-equivalence classes of one-dimensional semistable (with respect to H) sheaves $\mathcal{F}$ on S with fixed determinant $\det(\mathcal{F}) \cong \mathcal{O}_S(\beta)$ and Euler characteristic $\chi(\mathcal{F}) = \chi$. Here the stability is defined in terms of the slope

$$\mu(\mathcal{F}) = \frac{\chi(\mathcal{F})}{c_1(\mathcal{F}) \cdot H}.$$

There is a morphism

$$h : M_{\beta,\chi} \to |\beta|,$$

called the Hilbert–Chow morphism, which is defined by taking the Fitting support. If $|\beta|$ contains a connected smooth curve, then the morphism h is an abelian fibration.

The moduli space $M_{\beta,\chi}$ is singular in general, for which the intersection cohomology $\mathrm{IH}^*(M_{\beta,\chi})$ (with $\mathbb{Q}$-coefficients) might be a more suitable topological invariant than the ordinary cohomology. In this paper, we prove the following theorem about the intersection Betti numbers of $M_{\beta,\chi}$.

Theorem 1.1. *Fix $k \in \mathbb{Z}_{\geq 0}$ and $\chi \in \mathbb{Z}$. Suppose that $M_{\beta,\chi}$ is irreducible, and let*

$$c := \mathrm{codim}(M_{\beta,\chi} \setminus h^{-1}(|\beta|^{\mathrm{int}}), M_{\beta,\chi})$$

denote the codimension of $M_{\beta,\chi} \setminus h^{-1}(|\beta|^{\mathrm{int}})$ in $M_{\beta,\chi}$. If the following conditions hold:

(1) β is $\max\{2, 2k-2\}$-very ample (see Definition 2.2);
(2) $k \leq \min\{c-1, \frac{2}{3}\dim|\beta|\}$,

then we have

$$\dim_{\mathbb{Q}} \mathrm{IH}^k(M_{\beta,\chi}) = b_k^\infty,$$

where b_k^∞ is defined by the following identity of formal power series

$$\sum_{k\geq 0} b_k^\infty q^k = \prod_{m\geq 1} \frac{(1+q^{2m-1})^{b_1(S)}(1+q^{2m+1})^{b_1(S)}}{(1-q^{2m})^{b_2(S)+1}(1-q^{2m+2})}. \tag{1.1}$$

By Göttsche's formula [8], the number b_k^∞ is exactly the k-th Betti number $b_k(S^{[n]})$ of the Hilbert scheme $S^{[n]}$ of n points for any integer $n \geq k$. The connection between Betti numbers of moduli spaces of sheaves on S and b_k^∞ has been observed in many studies, for example, in [5, 17] for some moduli spaces of torsion-free sheaves, in [36] for moduli spaces of one-dimensional sheaves on $\mathbb{P}^2$, and in [29] for those of one-dimensional sheaves on any del Pezzo surface.

Our proof of Theorem 1.1 adapts the approach in §2 and §3 of our previous joint work with Pi and Shen [29] for a more general setting. A key new ingredient in the proof is as follows.

Theorem 1.2. *Let $\ell \geq 2$ be an integer. If β is $(2\ell-2)$-very ample, then there exists an open subscheme $U_\ell \subset |\beta|^{\mathrm{int}}$ such that*

(1) $\operatorname{codim}(|\beta|^{\mathrm{int}} \setminus U_\ell, |\beta|^{\mathrm{int}}) \geq \ell+1$*;*

(2) The open subscheme $h^{-1}(U_\ell)$ of $M_{\beta,\chi}$ is smooth.

For simplicity of exposition, now we set $\chi = 1$ and consider $M_\beta := M_{\beta,1}$ (see Remark 5.4 for the case of arbitrary χ). As an application of Theorem 1.1, we prove the stabilization of intersection Betti numbers of M_β for Enriques and bielliptic surfaces, providing new evidence for [33, Conjecture 1.3].

Theorem 1.3 (Theorem 5.3)**.** *Suppose that S is an Enriques or bielliptic surface. Fix $k \in \mathbb{Z}_{\geq 0}$. Given any ample divisor β_0 on S, there exists $d(\beta_0, k) \in \mathbb{Z}_{>0}$ (depending on β_0 and k) such that*

$$\dim_{\mathbb{Q}} \mathrm{IH}^k(M_{d\beta_0}) = b_k^\infty \quad \text{for all integer } d \geq d(\beta_0, k),$$

where b_k^∞ is defined by (1.1).

Although moduli spaces of sheaves on $K3$ and abelian surfaces have been extensively studied, much less is known about the cases of Enriques and bielliptic surfaces. In the bielliptic case, recent progress was made in [27].

When S is a generic Enriques surface and the class of β is not divisible by 2 in the group $\mathrm{Num}(S)$ of numerical equivalence classes of divisors on S, Saccà [30] proved that M_β is a smooth projective Calabi–Yau variety of dimension β^2+1. In this case, although the odd cohomology of M_β can be nonzero as shown in [30, Theorem 5.10], Theorem 1.3 implies that the k-th Betti number $b_k(M_\beta)$ for fixed odd k vanishes for sufficiently positive β. We also prove, in this setting, a refined version of Theorem 1.3 concerning the stabilization of the invariants $n_\beta^{i,j}$ (defined by (2.4) with ϕ taken to be $h : M_\beta \to |\beta|$) which are conjectured to refine the Gromov–Witten/Pandharipande–Thomas invariants of the local surface $\mathrm{Tot}(K_S)$ [10, 12, 21] and are the shifted perverse Hodge numbers considered in [28].

Theorem 1.4 (Theorem 5.7)**.** *Suppose that S is a generic Enriques surface. Fix $i, j \in \mathbb{Z}_{\geq 0}$. Given any ample divisor β_0 on S such that $2 \nmid \beta_0$ in $\mathrm{Num}(S)$, there exists $d(\beta_0, i, j) \in \mathbb{Z}_{>0}$ (depending on β_0, i and j) such that*

$$n_{d\beta_0}^{i,j} = n_\infty^{i,j} \quad \text{for all integer } d \geq d(\beta_0, i, j) \text{ with } 2 \nmid d,$$

where $n_\infty^{i,j}$ is defined by the following identity of formal power series

$$\sum_{i,j \geq 0} n_\infty^{i,j} q^i t^j = (1-qt) \prod_{m \geq 1} \frac{1}{(1-q^{m+1}t^{m-1})(1-q^m t^m)^{10}(1-q^{m-1}t^{m+1})}. \tag{1.2}$$

Remark 1.5. Under the assumption of [28, Conjecture A], Oberdieck proved a stronger stabilization result [28, Proposition 1.1] for the shifted perverse Hodge numbers. We note that the product formula (1.2) coincides with the one in [28, Proposition 1.1]. It remains open whether our approach, which does not rely on [28, Conjecture A], can be modified to show $n_\beta^{i,j} = n_\infty^{i,j}$ if $i < \beta^2/4$ and $j < \beta^2/4 - 1$. An explicit expression for $d(\beta_0, i, j)$ in Theorem 1.4 is given in (5.27).

The rest of this paper is organized as follows. In Section 2, we collect some notation, definitions, and facts that will be useful later. Section 3 is devoted to the local versality of the universal family of curves in high codimension and thus to a proof of Theorem 1.2. In Section 4, we prove Theorem 1.1 and discuss some difficulties arising in the general case. Finally, in Section 5, we study in detail the cases of Enriques and bielliptic surfaces and prove Theorems 1.3 and 1.4.

Conventions. A variety is an integral and separated scheme of finite type. A curve is a projective scheme of pure dimension 1. By a point of a scheme, we always mean a closed point. All sheaves are assumed to be coherent. For a sheaf $\mathcal{E}$ on a projective scheme X, we denote by $h^i(\mathcal{E})$ (resp. $\mathrm{ext}^i(\mathcal{E},\mathcal{E})$) the dimension of the i-th sheaf cohomology $H^i(X,\mathcal{E})$ (resp. i-th Ext group $\mathrm{Ext}^i(\mathcal{E},\mathcal{E})$) as a $\mathbb{C}$-vector space.

Acknowledgments. The authors thank Weite Pi and Junliang Shen for helpful discussions. The second author is grateful to her advisor Jun Li for his constant support, to János Kollár and Claire Voisin for inspiring discussions on the deformation theory of singularities of planar curves, and to Georg Oberdieck for suggesting considering bielliptic surfaces. The first author is partially supported by the Fundamental Research Funds for the Central Universities. The second author is supported by the NSFC grants (No. 12121001 and No. 12425105) and LMNS.

2. Preliminaries

This section gathers the notation and background material that will be essential later.

2.1. **Relative Hilbert schemes of points.** Let $\pi : \mathcal{C} \to |\beta|$ be the universal family of curves in $|\beta|$. For $n \in \mathbb{Z}_{\geq 0}$, denote by

$$\pi^{[n]} : \mathcal{C}^{[n]} \to |\beta|$$

the relative Hilbert scheme of n points on the fibers of π. For $C \in |\beta|$, the fiber of $\pi^{[n]}$ over C is the Hilbert scheme $C^{[n]}$ parametrizing 0-dimensional, length n closed subschemes of C. Note that $\pi^{[0]} : |\beta| \to |\beta|$ is the identity and $\pi = \pi^{[1]} : \mathcal{C} \to |\beta|$. For any open subscheme $U \subset |\beta|$, write $\mathcal{C}_U^{[n]}$ for $(\pi^{[n]})^{-1}(U)$. The following property of $\pi^{[n]}$ is a corollary of [18, Theorem 1.1] (see also [34, Proposition 2.7]).

Proposition 2.1. *The morphism $\pi^{[n]} : \mathcal{C}^{[n]} \to |\beta|$ has fibers of the same dimension n.*

There is a natural morphism $\sigma_n : \mathcal{C}^{[n+1]} \to S^{[n+1]}$ defined by

$$[Z \subset C] \mapsto [Z \subset S], \tag{2.3}$$

where $C \in |\beta|$ and $Z \subset C$ is a 0-dimensional closed subscheme of length $n+1$. The fiber of σ_n over $[Z \subset S] \in S^{[n+1]}$ is the projective space $\mathbb{P}(H^0(S, \mathcal{I}_Z \otimes \mathcal{O}_S(\beta))^\vee)$, where $\mathcal{I}_Z$ is the ideal sheaf of $Z \subset S$.

2.2. **Positivity conditions.** We recall the following notion of k-very ampleness, which generalizes base-point-freeness (0-very ampleness) and very ampleness (1-very ampleness).

Definition 2.2. An effective divisor β on S is called *k-very ample* ($k \in \mathbb{Z}_{\geq 0}$) *if for any* 0*-dimensional closed subscheme* $Z \subset S$ *of length* $k+1$*, the restriction map*

$$r_Z : H^0(S, \mathcal{O}_S(\beta)) \to H^0(Z, \mathcal{O}_S(\beta)|_Z)$$

is surjective. We adopt the convention that every divisor is (-1)*-very ample.*

The definition immediately implies two properties, which will be used in the sequel:

(1) If β is k-very ample and $k' \leq k$ ($k' \in \mathbb{Z}_{\geq 0}$), then β is also k'-very ample.
(2) If β is ample, then for any $k \in \mathbb{Z}_{\geq 0}$, there exists $d_k \in \mathbb{Z}_{>0}$ such that the multiple $d\beta$ is k-very ample for every integer $d \geq d_k$.

A well-known connection between k-very ampleness and the structure of $\mathcal{C}^{[k+1]}$ as a scheme over $S^{[k+1]}$ via σ_k is as follows (e.g., [34, Proposition 2.6]).

Proposition 2.3. *If β is k-very ample, then $\mathcal{C}^{[k+1]}$ is a projective bundle over $S^{[k+1]}$.*

2.3. **Perverse filtrations.** Next we recall some generalities of perverse filtrations. Let $\phi : X \to Y$ be a proper morphism between smooth quasiprojective varieties. We assume that $\dim X = a$, $\dim Y = b$, and all fibers of ϕ have dimension $a - b$. The perverse filtration

$$P_0 H^m(X) \subset P_1 H^m(X) \subset \cdots \subset H^m(X)$$

induced by ϕ is an increasing filtration on the rational cohomology $H^*(X)$, governed by the topology of the morphism ϕ. It is defined to be

$$P_k H^m(X) := \mathrm{Im}\left\{ H^{m-b}(Y, {}^{\mathbf{p}}\tau_{\leq k}(R\phi_* \mathbb{Q}_X[b])) \to H^{m-b}(Y, R\phi_* \mathbb{Q}_X[b]) \right\} \subset H^m(X),$$

where ${}^{\mathbf{p}}\tau_{\leq \bullet}$ is the perverse truncation functor [2]. If we apply the decomposition theorem to $\phi : X \to Y$, we obtain

$$R\phi_* \mathbb{Q}_X[b] \simeq \bigoplus_{i=0}^{2a-2b} \mathcal{P}_i[-i] \in D^b_c(Y)$$

with $\mathcal{P}_i$ a semisimple perverse sheaf on Y. There is an identity

$$P_k H^m(X) = \mathrm{Im}\Big\{ H^{m-b}(Y, \bigoplus_{i=0}^{k} \mathcal{P}_i[-i]) \to H^m(X) \Big\}.$$

The dimension of the graded piece $\mathrm{Gr}^P_i H^{i+j}(X) := P_i H^{i+j}(X)/P_{i-1} H^{i+j}(X)$ is denoted by

$$n_\phi^{i,j} := \dim_{\mathbb{Q}} \mathrm{Gr}^P_i H^{i+j}(X) = \dim_{\mathbb{Q}} H^{j-b}(Y, \mathcal{P}_i). \tag{2.4}$$

Note that for the Hilbert–Chow morphism $h : M_{\beta,\chi} \to |\beta|$, the fibers of h over integral curves in $|\beta|$ are the compactified Jacobians of degree $p_a(\beta) - 1 + \chi$ and have the same dimension $p_a(\beta)$. Later, we will use the following result of Maulik–Yun [22, Theorem 2.13] and Migliorini–Shende [24, Corollary 2].

Theorem 2.4 ([22, 24][1])**.** *Suppose that $|\beta|$ contains a connected smooth curve. Let $U \subset |\beta|^{\mathrm{int}}$ be an open subscheme such that $h^{-1}(U)$ is smooth, and let $P_\bullet H^*(h^{-1}(U))$ be the perverse filtration induced by the restriction $h_U : h^{-1}(U) \to U$ of h. For any $\ell, m \in \mathbb{Z}_{\geq 0}$, we have an isomorphism of vector spaces*

$$H^m\left(\mathcal{C}_U^{[\ell]}\right) \cong \sum_{i+j\leq \ell;\, i,j\geq 0} \mathrm{Gr}_i^P H^{m-2j}\left(h^{-1}(U)\right). \tag{2.5}$$

Remark 2.5. When $\ell = 0$, the isomorphism (2.5) reads

$$H^m(U) \cong \mathrm{Gr}_0^P H^m\left(h^{-1}(U)\right).$$

This isomorphism can be obtained as a consequence of Ngô's support theorem [26, Théorème 7.2.1]. This theorem shows that $h_U : h^{-1}(U) \to U$ has full support; see [22, Theorem 2.4, §2.5] or [24, Corollary 9]. More precisely, there is an isomorphism

$$Rh_{U*}\mathbb{Q}_{h^{-1}(U)}[\dim U] \cong \bigoplus_{i=0}^{2p_a(\beta)} \mathrm{IC}(\wedge^i R^1\pi^s_*\mathbb{Q}_{\mathcal{C}^{\mathrm{sm}}})[-i],$$

where $\pi^s : \mathcal{C}^{\mathrm{sm}} \to |\beta|^{\mathrm{sm}}$ is the universal family of smooth curves in $|\beta|$.

3. Singular locus of relative compactified Jacobians

The goal of this section is to prove Theorem 1.2 about the smoothness of a big open subscheme of $h^{-1}(|\beta|^{\mathrm{int}})$. We begin by reviewing some basic notions of plane curve singularities and deformation theory.

3.1. Singularities and deformations of planar curves.

3.1.1. *Singularity invariants.* We first recall some basic invariants of isolated singularities in planar curves.

Definition 3.1. Let $f \in \mathbb{C}[[x, y]]$ be a convergent power series.

(1) The numbers

$$\mu(f) := \dim_{\mathbb{C}} \mathbb{C}[[x, y]]/\left(\partial_x f, \partial_y f\right), \quad \tau(f) := \dim_{\mathbb{C}} \mathbb{C}[[x, y]]/\left(f, \partial_x f, \partial_y f\right)$$

are called the *Milnor number* and *Tjurina number* of f, respectively.

(2) If f is reduced, let

$$\mathcal{O} = \mathbb{C}[[x, y]]/(f) \hookrightarrow \overline{\mathcal{O}}$$

be the normalization of $\mathcal{O}$. Then the number

$$\delta(f) := \dim_{\mathbb{C}} \overline{\mathcal{O}}/\mathcal{O}$$

is called the *δ-invariant of f.*

The Milnor number and the δ-invariant are related by the following formula of Milnor.

[1]The result of [22, 24] can be applied here since the smoothness of $h^{-1}(U)$ implies that all relative compactified Jacobians $\overline{J}^e_U$ ($e \in \mathbb{Z}$) are smooth by the existence of a section of the universal curve (étale locally on U, avoiding the singularities of fibers) and so are all relative Hilbert schemes $\mathcal{C}_U^{[\ell]}$ ($\ell \in \mathbb{Z}_{\geq 0}$) by [32, Proposition 14].

Proposition 3.2 ([25, §10])**.** *Let $f \in (x,y) \subset \mathbb{C}[[x,y]]$ be a reduced, convergent power series. Denote by $r(f)$ the number of irreducible factors of f. Then*

$$\mu(f) = 2\delta(f) - r(f) + 1. \tag{3.6}$$

If C is a reduced curve on S, then the singular locus C_{sing} of C is discrete. For any point $p \in C_{\mathrm{sing}}$, there exists a polynomial $f \in (x,y) \subset \mathbb{C}[x,y]$ without multiple factors such that the singularity of C at p is analytically isomorphic to that of $\{f = 0\}$ at $(0,0)$, i.e.,

$$\hat{\mathcal{O}}_{C,p} \cong \mathbb{C}[[x,y]]/(f).$$

Thus, the Milnor and Tjurina numbers, and the δ-invariant of C at p can be defined as those of f in Definition 3.1, denoted by $\mu_p(C)$, $\tau_p(C)$, and $\delta_p(C)$, respectively.

For completeness, we give a proof of the following standard interpretation of the total Tjurina number of C in terms of the Ext sheaf $\mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)$, where Ω_C is the cotangent sheaf of C.

Lemma 3.3. *If C is a reduced curve on S, then*

$$\sum_{p \in C_{\mathrm{sing}}} \tau_p(C) = h^0(\mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)).$$

Proof. The sheaf Ω_C is locally free over the smooth locus $C \setminus C_{\mathrm{sing}}$ of C, so $\mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)$ is a torsion sheaf supported on C_{sing}. Let $p \in C_{\mathrm{sing}}$ and assume that the singularity of C at p is analytically isomorphic to that of $\{f = 0\}$ at $(0,0)$ for $f \in (x,y) \subset \mathbb{C}[x,y]$ without multiple factors. The lemma can be proved locally around p. For $R := \mathbb{C}[x,y]/(f)$, we have $\Omega_R = (Rdx \oplus Rdy)/(\partial_x f dx + \partial_y f dy)$. By the conormal exact sequence

$$0 \to (f)/(f^2) \to R^{\oplus 2} \to \Omega_R \to 0,$$

we conclude that $\mathrm{Ext}^1(\Omega_R, R) = R/(\partial_x f, \partial_y f)$, from which the result follows. □

3.1.2. *Smoothness from local versality.* Let $\pi_B : \mathcal{C}_B \to B$ be a family of reduced planar curves, i.e., a proper and flat morphism whose fibers are reduced curves with at worst planar singularities. Assume that B is a smooth variety. There is a natural map, called the local Kodaira–Spencer map:

$$k^{\mathrm{loc}}_{\pi_B,b} : T_b B \to \prod_{p \in (\mathcal{C}_b)_{\mathrm{sing}}} T\mathrm{Def}_{\mathcal{C}_b,p} = H^0(\mathcal{C}_b, \mathcal{E}xt^1(\Omega_{\mathcal{C}_b}, \mathcal{O}_{\mathcal{C}_b})), \tag{3.7}$$

where $(\mathcal{C}_b)_{\mathrm{sing}}$ is the singular locus of $\mathcal{C}_b := \pi_B^{-1}(b)$, and $T\mathrm{Def}_{\mathcal{C}_b,p}$ is the tangent space of the deformation functor of the complete local $\mathbb{C}$-algebra $\hat{\mathcal{O}}_{\mathcal{C}_b,p}$.

Definition 3.4. The family $\pi_B : \mathcal{C}_B \to B$ is called *locally versal* if for any point $b \in B$, the local Kodaira–Spencer map $k^{\mathrm{loc}}_{\pi_B,b}$ in (3.7) is surjective.

If $\pi_B : \mathcal{C}_B \to B$ is a family of integral planar curves, then for any $e \in \mathbb{Z}$, denote by

$$p_{B,e} : \overline{J}^e_B \to B$$

the relative compactified Jacobian of degree e associated with π_B, whose fiber over a point $b \in B$ is the moduli space of rank 1, torsion-free sheaves of degree e (or equivalently, Euler characteristic $e + 1 - p_a(\mathcal{C}_b)$) on $\mathcal{C}_b$. Since π_B admits, étale locally on B, a section which avoids the singularities of fibers of π_B, the smoothness of $\overline{J}^e_B$ for any e is equivalent to the smoothness of $\overline{J}^0_B$. Hence, the following result follows directly from [7, Corollary B.3].

Theorem 3.5 ([7])**.** *If $\pi_B : \mathcal{C}_B \to B$ is a locally versal family of integral planar curves, then $\overline{J}^e_B$ is smooth for any $e \in \mathbb{Z}$.*

3.2. **Local versality and smoothness in high codimension.** Now we show that the universal family $\pi : \mathcal{C} \to |\beta|$ of curves in $|\beta|$ is locally versal in sufficiently high codimension if β is positive enough.

Proposition 3.6. *Let $\ell \geq 2$ be an integer. If β is $(2\ell - 2)$-very ample, then there exists an open subscheme $V_\ell \subset |\beta|$ such that*

(1) Every point of V_ℓ corresponds to a reduced curve;
(2) $\operatorname{codim}(|\beta| \setminus V_\ell, |\beta|) \geq \ell + 1$;
(3) The universal family $\pi|_{\pi^{-1}(V_\ell)} : \pi^{-1}(V_\ell) \to V_\ell$ of curves in V_ℓ is locally versal.

Proof. We use an argument in [16] to control the locus of curves that are not ℓ-nodal. Here, a curve is called ℓ-nodal if it is reduced and has no singularities other than ℓ nodes. Since β is also ℓ-very ample, by the proof of [16, Proposition 2.1], there is an open subscheme $V_\ell \subset |\beta|$ such that

- $\operatorname{codim}(|\beta| \setminus V_\ell, |\beta|) > \ell$.
- Curves in V_ℓ are either ℓ-nodal, or reduced of geometric genus $> p_a(\beta) - \ell$.

Note that for a curve $C \in V_\ell$ which is not ℓ-nodal, its geometric genus is

$$p_a(\beta) - \sum_{p \in C_{\mathrm{sing}}} \delta_p(C) > p_a(\beta) - \ell. \tag{3.8}$$

By (3.8) and Milnor's formula (3.6), we have

$$\sum_{p \in C_{\mathrm{sing}}} \tau_p(C) \leq \sum_{p \in C_{\mathrm{sing}}} \mu_p(C) \leq 2 \sum_{p \in C_{\mathrm{sing}}} \delta_p(C) < 2\ell.$$

It remains to show Property (3). For any curve $D \in V_\ell$, the map $k^{\mathrm{loc}}_{\pi,D}$ in (3.7) can be identified with

$$\bar{r}_W : H^0(\mathcal{O}_S(\beta))/\langle s_D \rangle \to H^0(\mathcal{O}_S(\beta)|_W)$$

induced by the restriction to a 0-dimensional closed subscheme $W \subset S$ of length

$$\sum_{p \in D_{\mathrm{sing}}} \tau_p(D) < 2\ell$$

by Lemma 3.3, where $s_D \in H^0(\mathcal{O}_S(\beta))$ is a defining section of D. Hence, $\bar{r}_W$ is surjective by the $(2\ell - 2)$-very ampleness of β. □

Combining the above proposition with Theorem 3.5, we can prove Theorem 1.2.

Proof of Theorem 1.2. Let V_ℓ be as in Proposition 3.6. Take $U_\ell = V_\ell \cap |\beta|^{\mathrm{int}}$. By Property (2) of V_ℓ in Proposition 3.6, we have

$$\operatorname{codim}(|\beta|^{\mathrm{int}} \setminus U_\ell, |\beta|^{\mathrm{int}}) \geq \ell + 1.$$

It remains to show that $h^{-1}(U_\ell)$ is smooth. Note that $h^{-1}(U_\ell)$ is the relative compactified Jacobian of degree $p_a(\beta) - 1 + \chi$ associated with the universal family of curves in U_ℓ.

This universal family is locally versal by Property (3) of V_ℓ in Proposition 3.6. Thus, the smoothness of $h^{-1}(U_\ell)$ follows from Theorem 3.5. $\square$

4. Intersection Betti numbers

In this section, we generalize the approach in [29] to prove Theorem 1.1. For any open subscheme $U \subset |\beta|$, let

$$N(U) := 2\,\mathrm{codim}(|\beta| \setminus U, |\beta|) - 2.$$

We simply write $N(\beta)$ for $N(|\beta|^{\mathrm{int}})$.

First, we focus on a smooth open subscheme $h^{-1}(U)$ of $M_{\beta,\chi}$ and prove the following result concerning the dimensions of graded pieces of the perverse filtration induced by the restriction of $h : M_{\beta,\chi} \to |\beta|$ (see (2.4)).

Theorem 4.1. *Suppose that $|\beta|$ contains a connected smooth curve. Let $U \subset |\beta|^{\mathrm{int}}$ be an open subscheme such that $h^{-1}(U)$ is smooth, and let $h_U : h^{-1}(U) \to U$ be the restriction of h. For given $i, j \in \mathbb{Z}_{\geq 0}$, if $i + j \leq N(U)$, the divisor β is $(i-1)$-very ample, and $2\dim|\beta| \geq 3i + j$, then*

$$n^{i,j}_{h_U} = n^{i,j}_{\infty}, \tag{4.9}$$

where $n^{i,j}_\infty$ is the coefficient of $q^i t^j$ in the expansion of

$$H(q,t) := (1-qt)\prod_{m\geq 1} \frac{(1+q^m t^{m-1})^{b_1(S)}(1+q^m t^{m+1})^{b_1(S)}}{(1-q^{m+1}t^{m-1})(1-q^m t^m)^{b_2(S)}(1-q^{m-1}t^{m+1})}. \tag{4.10}$$

Remark 4.2. Recall Göttsche's formula [8] for $\sum_{n\geq 0}\sum_{i\geq 0} b_i(S^{[n]})z^i w^n$:

$$G(z,w) := \prod_{m\geq 1} \frac{(1+z^{2m-1}w^m)^{b_1(S)}(1+z^{2m+1}w^m)^{b_1(S)}}{(1-z^{2m-2}w^m)(1-z^{2m}w^m)^{b_2(S)}(1-z^{2m+2}w^m)}. \tag{4.11}$$

After the change of variables

$$z = t, \quad w = \frac{q}{t},$$

the product formula $H(q,t)$ in (4.10) satisfies the following equality

$$\frac{H(q,t)}{1-qt} = G(z,w)\cdot\frac{1-w}{1-z^2}.$$

By (2.4), the k-th Betti number of $h^{-1}(U)$ is

$$b_k(h^{-1}(U)) = \sum_{i=0}^{k} n^{i,k-i}_{h_U}. \tag{4.12}$$

Since the right-hand side of (1.1) is exactly $H(q,q)$, the next corollary follows immediately from Theorem 4.1 and (4.12).

Corollary 4.3. *Let $k \in \mathbb{Z}_{\geq 0}$. In the setting of Theorem 4.1, if $k \leq \min\{N(U), 2\dim|\beta|/3\}$ and β is $(k-1)$-very ample, then*

$$b_k(h^{-1}(U)) = b^\infty_k,$$

where b^∞_k is defined by (1.1).

4.1. **Strategy of the proof.** The proof of Theorem 4.1 consists of the following steps.
Step 1. Relate $h^{-1}(U)$ to the relative Hilbert scheme $\mathcal{C}_U^{[n]}$. By (2.5), for any $\ell, m \in \mathbb{Z}_{\geq 0}$,

$$b_m(\mathcal{C}_U^{[\ell]}) = \sum_{i+j\leq \ell;\, i,j\geq 0} n_{h_U}^{i,m-i-2j}. \tag{4.13}$$

Step 2. Calculate the Betti numbers of $\mathcal{C}_U^{[\ell]}$ using Proposition 2.3.

Proposition 4.4. *In the setting of Theorem 4.1, if $m \leq \min\{N(U), 2\dim|\beta| - 2\ell\}$ and β is $(\ell-1)$-very ample, then*

$$b_m(\mathcal{C}_U^{[\ell]}) = \sum_{n=0}^{\lfloor m/2 \rfloor} b_{m-2n}(S^{[\ell]}). \tag{4.14}$$

Proof. Since $\pi^{[\ell]} : \mathcal{C}^{[\ell]} \to |\beta|$ has fibers of dimension ℓ by Proposition 2.1, we have

$$\operatorname{codim}(\mathcal{C}^{[\ell]} \setminus \mathcal{C}_U^{[\ell]}, \mathcal{C}^{[\ell]}) = \operatorname{codim}(|\beta| \setminus U, |\beta|).$$

By Proposition 2.3, the $(\ell-1)$-very ampleness of β implies that $\mathcal{C}^{[\ell]}$ is a projective bundle over $S^{[\ell]}$. In particular, $\mathcal{C}^{[\ell]}$ is irreducible and smooth. Then it follows from [29, Lemma 2.4] (or the proof of Lemma 5.6) that for $m \leq N(U)$,

$$b_m(\mathcal{C}_U^{[\ell]}) = b_m(\mathcal{C}^{[\ell]}).$$

Combining with

$$\dim \mathcal{C}^{[\ell]} - \dim S^{[\ell]} = \dim|\beta| - \ell \geq m/2$$

by our assumption, we conclude

$$b_m(\mathcal{C}_U^{[\ell]}) = b_m(\mathcal{C}^{[\ell]}) = b_m(S^{[\ell]} \times \mathbb{P}^{\dim|\beta|-\ell}) = \sum_{n=0}^{\lfloor m/2 \rfloor} b_{m-2n}(S^{[\ell]}),$$

which proves (4.14). □

Since all Betti numbers of $S^{[\ell]}$ are determined by Göttsche's formula (4.11), we can prove Theorem 4.1 by induction on i, combining (4.13) and (4.14).

Proof of Theorem 4.1. We proceed by induction on i to prove (4.9). For $i = 0$ and $j \leq \min\{N(U), 2\dim|\beta|\}$,

$$\begin{aligned} n_{h_U}^{0,j} &= \dim P_0 H^j(h^{-1}(U)) \\ &= \dim H^j(U) && \text{(by Remark 2.5)} \\ &= \dim H^j(|\beta|) && \text{(by [29, Lemma 2.4])} \\ &= \begin{cases} 1, & (j \text{ is even}) \\ 0, & (j \text{ is odd}) \end{cases} \end{aligned}$$

which proves the induction base of (4.9). Suppose the result is true for (i', j') with $i' < i$ and $i' + j' \leq i + j$. For $m \leq i + j$, it follows by (4.13), our induction hypothesis, and (4.14) that

$$\begin{aligned}
n_{h_U}^{i,m-i} &= \sum_{i'+j'=i;\, i',j'\geq 0} n_{h_U}^{i',m-i'-2j'} - \sum_{i'<i;\, i'+j'=i;\, i',j'\geq 0} n_{h_U}^{i',m-i'-2j'} \\
&= \Big(b_m(\mathcal{C}_U^{[i]}) - b_m(\mathcal{C}_U^{[i-1]})\Big) - \sum_{i'<i;\, i'+j'=i;\, i',j'\geq 0} n_{\infty}^{i',m-i'-2j'} \\
&= \sum_{n=0}^{\lfloor m/2\rfloor} \Big(b_{m-2n}(S^{[i]}) - b_{m-2n}(S^{[i-1]})\Big) - \sum_{i'<i;\, i'+j'=i;\, i',j'\geq 0} n_{\infty}^{i',m-i'-2j'} \\
&= n_{\infty}^{i,m-i},
\end{aligned}$$

where the last identity is essentially the same as [29, (31)] by Göttsche's formula [8] (see also Remark 4.2). □

Finally, we prove our first main theorem stated in the introduction.

Proof of Theorem 1.1. By the upper semicontinuity of fiber dimensions, we have

$$N(\beta) \geq 2c-2. \tag{4.15}$$

Since β is $\max\{2, 2k-2\}$-very ample, let $U_k \subset |\beta|^{\mathrm{int}}$ be as in Theorem 1.2 (set $U_k := U_2$ if $k \leq 1$). Using (4.15), the assumption $k \leq c-1$, and Theorem 1.2, we obtain the following properties:

- $N(U_k) \geq \min\{N(\beta), 2k\} = 2k$;
- $h^{-1}(U_k)$ is smooth.

By Corollary 4.3, we have

$$b_k(h^{-1}(U_k)) = b_k^{\infty}. \tag{4.16}$$

Since fibers of h over integral curves have the same dimension, Theorem 1.2 implies

$$\operatorname{codim}(h^{-1}(|\beta|^{\mathrm{int}}) \setminus h^{-1}(U_k), h^{-1}(|\beta|^{\mathrm{int}})) = \operatorname{codim}(|\beta|^{\mathrm{int}} \setminus U_k, |\beta|^{\mathrm{int}}) \geq k+1.$$

This, together with the assumption on $c = \operatorname{codim}(M_{\beta,\chi} \setminus h^{-1}(|\beta|^{\mathrm{int}}), M_{\beta,\chi})$, yields

$$\operatorname{codim}(M_{\beta,\chi} \setminus h^{-1}(U_k), M_{\beta,\chi}) \geq \min\{c, k+1\} > k.$$

Then by [23, Theorem 6.7.4] and the smoothness of $h^{-1}(U_k)$, we have isomorphisms

$$\mathrm{IH}^k(M_{\beta,\chi}) \cong \mathrm{IH}^k(h^{-1}(U_k)) \cong H^k(h^{-1}(U_k)),$$

and therefore

$$\dim_{\mathbb{Q}} \mathrm{IH}^k(M_{\beta,\chi}) = b_k(h^{-1}(U_k)) = b_k^{\infty}$$

by (4.16), which completes the proof. □

4.2. **Discussion on the general case.** In general, the moduli space $M_{\beta,\chi}$ can be reducible even if β is sufficiently positive; see [34, §5]. Moreover, even if we assume that $M_{\beta,\chi}$ is irreducible, it is usually hard to compute the codimension c in Theorem 1.1. It is not necessarily true that c is the same as $\operatorname{codim}(|\beta| \setminus |\beta|^{\mathrm{int}}, |\beta|)$. In some nice cases where $c = \operatorname{codim}(|\beta| \setminus |\beta|^{\mathrm{int}}, |\beta|)$, the asymptotic behavior of $\mathrm{IH}^k(M_{\beta,\chi})$ can be described more precisely, provided that $\operatorname{codim}(|\beta| \setminus |\beta|^{\mathrm{int}}, |\beta|)$, or equivalently $N(\beta)$, can be calculated.

The following lemma allows us to estimate $N(\beta)$ by analyzing the decompositions of β.

Lemma 4.5. *Fix an ample divisor β on S. There are finitely many pairs of classes (θ_1, θ_2) in $\mathrm{Num}(S)$ such that θ_i ($i = 1, 2$) are the classes of nonzero effective divisors C_i with*

$$C_1 + C_2 = \beta.$$

Proof. Let $D_1, \cdots, D_\rho$ be an orthogonal basis for $\mathrm{Num}(S)_{\mathbb{Q}}$, where D_1 is the class of an ample divisor. By Hodge index theorem, $D_\ell^2 < 0$ for $1 < \ell \le \rho$. Given any C_1, C_2 as in the statement, write

$$\beta \equiv \sum_{j=1}^{\rho} a_j D_j, \quad C_i \equiv \sum_{j=1}^{\rho} a_{i,j} D_j$$

for some $a_j, a_{i,j} \in \mathbb{Q}$ ($i = 1, 2$), where $\equiv$ denotes the numerical equivalence. For $1 < \ell \le \rho$, let $n_\ell > 0$ be the integer such that $A_\ell := n_\ell D_1 + D_\ell$ is ample. Then $D_1 \cdot C_i > 0$ and $A_\ell \cdot C_i > 0$, which implies

$$a_{i,1} > 0 \quad \text{and} \quad n_\ell a_{i,1} D_1^2 + a_{i,\ell} D_\ell^2 > 0.$$

Combined with $D_\ell^2 < 0$ and $C_1 + C_2 = \beta$, this shows that the possible values of $a_{i,j}$ ($i = 1, 2$; $j = 1, \cdots, \rho$) are bounded since β is fixed, and thus the result follows. □

By the calculations in Section 5, we ask the following question.

Question 4.6. *For any ample divisor β on an arbitrary smooth projective surface S, is it true that*

$$\lim_{d \to +\infty} N(d\beta) = +\infty?$$

5. Stabilization in the Enriques and bielliptic cases

In this section, let S be an Enriques or bielliptic surface. In particular, its canonical divisor K_S is numerically trivial. When β is a base-point-free, ample divisor on S, $|\beta|$ contains a connected smooth curve by Bertini's theorem. Denote by $|\beta|^{\mathrm{sm}} \subset |\beta|$ the open subscheme parametrizing smooth curves.

To apply Theorem 1.1, we first prove the irreducibility of the moduli spaces of one-dimensional sheaves on these surfaces.

Proposition 5.1. *Suppose that S is an Enriques or bielliptic surface, and β is a base-point-free, ample divisor satisfying $\gcd(\beta \cdot H, \chi) = 1$. Then $M_{\beta,\chi}$ is irreducible.*

Proof. Since the open subscheme $h^{-1}(|\beta|^{\mathrm{int}})$ is irreducible and contains the smooth open subscheme $h^{-1}(|\beta|^{\mathrm{sm}})$ by [34, Proposition 2.2], there is a unique irreducible component $Y \subset M_{\beta,\chi}$ containing $h^{-1}(|\beta|^{\mathrm{int}})$. We prove the irreducibility of $M_{\beta,\chi}$ by contradiction. If $Y \neq M_{\beta,\chi}$, then take another irreducible component $Y' \neq Y$. It follows from $Y' \cap h^{-1}(|\beta|^{\mathrm{sm}}) = \emptyset$ that $Y' \subset h^{-1}(|\beta| \setminus |\beta|^{\mathrm{sm}})$. By [36, Corollary 1.3] and the fact that K_S is numerically trivial, the fibers of $h : M_{\beta,\chi} \to |\beta|$ have the same dimension $p_a(\beta)$; therefore

$$\dim Y' \le \dim h^{-1}(|\beta| \setminus |\beta|^{\mathrm{sm}}) < \dim |\beta| + p_a(\beta) = \beta^2 + \chi(\mathcal{O}_S), \tag{5.17}$$

where the last equality follows from the Kodaira vanishing theorem and the Riemann–Roch formula. However, using the morphism $\det : M \to \mathrm{Pic}(X)$ in [11, Theorem 4.5.4] [2] and the Riemann–Roch formula, for any point $\mathcal{F} \in Y'$, we have

$$\begin{aligned}\dim Y' &\geq \mathrm{ext}^1(\mathcal{F},\mathcal{F}) - \mathrm{ext}^2(\mathcal{F},\mathcal{F}) - h^1(\mathcal{O}_S)\\ &= \beta^2 + \chi(\mathcal{O}_S),\end{aligned}$$

which contradicts (5.17). □

Then we assume the following property of $N(\beta)$, whose proof will be given in §5.1.2 and §5.2.2 by analyzing the Enriques and bielliptic cases separately.

Proposition 5.2. *If β is an ample divisor on an Enriques or bielliptic surface, then*

$$\lim_{d\to+\infty} N(d\beta) = +\infty.$$

Combining the above results with Theorem 1.1, we can prove the stabilization of intersection Betti numbers of $M_\beta = M_{\beta,1}$.

Theorem 5.3. *Suppose that S is an Enriques or bielliptic surface. Fix $k \in \mathbb{Z}_{\geq 0}$. Given any ample divisor β_0 on S, there exists $d(\beta_0, k) \in \mathbb{Z}_{>0}$ (depending on β_0 and k) such that*

$$\dim_{\mathbb{Q}} \mathrm{IH}^k(M_{d\beta_0}) = b_k^\infty \quad \textit{for all integer } d \geq d(\beta_0, k), \tag{5.18}$$

where b_k^∞ is defined by (1.1).

Proof. Choose $d(\beta_0, k) \in \mathbb{Z}_{>0}$ such that for all $d \geq d(\beta_0, k)$, the following properties hold:

(1) $N(d\beta_0) \geq 2k$ (guaranteed by Proposition 5.2);
(2) $d\beta_0$ is $\max\{2, 2k-2\}$-very ample;
(3) $2\dim|d\beta_0| \geq 3k$.

Since the fibers of $h : M_{d\beta_0} \to |d\beta_0|$ have the same dimension by [36, Corollary 1.3], we have

$$\begin{aligned}\mathrm{codim}(M_{d\beta_0} \setminus h^{-1}(|d\beta_0|^{\mathrm{int}}), M_{d\beta_0}) &= \mathrm{codim}(|d\beta_0| \setminus |d\beta_0|^{\mathrm{int}}, |d\beta_0|)\\ &= 1 + \frac{1}{2}N(d\beta_0) \geq k+1.\end{aligned} \tag{5.19}$$

Hence, the result follows from Proposition 5.1 and Theorem 1.1 by taking $\beta = d\beta_0$. □

Remark 5.4. When $\beta \cdot H$ and χ are not necessarily coprime, there can be strictly semistable sheaves in $M_{\beta,\chi}$ and thus $h : M_{\beta,\chi} \to |\beta|$ may not have constant fiber dimension by [36]. In this case, one could possibly prove the irreducibility of $M_{\beta,\chi}$ by showing that the corresponding moduli stack is irreducible (cf. [19, §2.6]), but the first equality of (5.19) may fail. Therefore, to ensure the equality in (5.18) holds for $M_{d\beta_0,\chi}$, we need an additional assumption that $d\beta_0 \cdot H$ is coprime with χ.

5.1. **The case of Enriques surfaces.**

[2]Although [11, Theorem 4.5.4] is stated for moduli spaces of positive rank sheaves, the proof of $\dim_{[F]} M(\mathcal{Q}) \geq \dim_{[F]} M - \dim_{[\det F]} \mathrm{Pic}(X)$ does not rely on the positive rank assumption.

5.1.1. *Basic properties.* Recall that an Enriques surface S is a minimal surface of Kodaira dimension 0 with $h^1(\mathcal{O}_S) = h^2(\mathcal{O}_S) = 0$. Its second Betti number is $b_2(S) = 10$. The following property of linear systems on S is well-known (e.g., [13, Theorem]).

Lemma 5.5. *Let D be a nonzero, nef, effective divisor on an Enriques surface. We have the following properties:*

(i) If $D^2 > 0$, then

$$\dim |D| = \chi(\mathcal{O}_S(D)) - 1 = \frac{D^2}{2}.$$

(ii) If $D^2 = 0$, then D is numerically equivalent to kE for some $k \in \mathbb{Z}_{>0}$ and a primitive, effective divisor E with $E^2 = 0$. Moreover,

$$\dim |D| = \begin{cases} \lfloor \frac{k}{2} \rfloor & (\textit{if } D = kE) \\ \lfloor \frac{k-1}{2} \rfloor & (\textit{if } D = kE + K_S). \end{cases}$$

5.1.2. *Proof of Proposition 5.2 in the Enriques case.* Any non-integral curves in $|d\beta|$ are in the image of $|C_1| \times |C_2|$ for finitely many decompositions $d\beta = C_1 + C_2$ (up to linear or numerical equivalence) by Lemma 4.5, where C_1 and C_2 are nonzero effective divisors on S.

First we assume that C_1 and C_2 have nef irreducible components. To prove the lemma, we estimate all possible values of

$$\dim |d\beta| - \dim |C_1| - \dim |C_2|.$$

Case 1.1: *$C_1^2 > 0$ and $C_2^2 > 0$.* In this case, $C_1^2 \geq 2$ and $C_2^2 \geq 2$. By Lemma 5.5 (i) and the Hodge index theorem, we have

$$\begin{aligned} \dim |d\beta| - \dim(|C_1| \times |C_2|) = C_1 \cdot C_2 &\geq \sqrt{C_1^2 C_2^2} \\ &\geq \sqrt{2(d^2\beta^2 - 2C_1 \cdot C_2 - 2)}, \end{aligned}$$

where the last inequality follows from $d^2\beta^2 = C_1^2 + C_2^2 + 2C_1 \cdot C_2$. Therefore,

$$\dim |d\beta| - \dim(|C_1| \times |C_2|) = C_1 \cdot C_2 \geq d\sqrt{2\beta^2} - 2. \tag{5.20}$$

Case 1.2: *$C_1^2 = 0$ and $C_2^2 > 0$.* By Lemma 5.5 (ii), $C_1 = sE_1$ or $sE_1 + K_S$ for some $s \in \mathbb{Z}_{>0}$ and a primitive effective divisor E_1 with $E_1^2 = 0$. Moreover,

$$\begin{aligned} \dim |d\beta| - \dim(|C_1| \times |C_2|) &\geq C_1 \cdot C_2 - \lfloor \frac{s}{2} \rfloor \\ &\geq s\left(d\beta \cdot E_1 - \frac{1}{2}\right) \geq d - \frac{1}{2}. \end{aligned} \tag{5.21}$$

Case 1.3: *$C_1^2 = C_2^2 = 0$.* Let s and E_1 be as in Case 1.2. By Lemma 5.5 (ii), $C_2 = tE_2$ or $tE_2 + K_S$ for some $t \in \mathbb{Z}_{>0}$ and a primitive effective divisor E_2 with $E_2^2 = 0$. Then we have $C_1 \cdot C_2 = stE_1 \cdot E_2 = d^2\beta^2/2 > 0$ and

$$\begin{aligned} \dim |d\beta| - \dim(|C_1| \times |C_2|) \geq C_1 \cdot C_2 - \lfloor \frac{s}{2} \rfloor - \lfloor \frac{t}{2} \rfloor &\geq \frac{d^2\beta^2 - (s+t)}{2} \\ &\geq \frac{d^2\beta^2 - (1 + d^2\beta^2/2)}{2} = \frac{d^2\beta^2 - 2}{4}. \end{aligned} \tag{5.22}$$

It remains to consider the case where every irreducible component of C_1 (resp. C_2) is nef (resp. a (-2)-curve). By Lemma 4.5, for a fixed d, there are finitely many such C_2. In this case, it suffices to estimate

$$\dim|d\beta| - \dim|C_1|.$$

<u>*Case 2.1:*</u> $C_1^2 > 0$. By Lemma 5.5 (i),

$$\dim|d\beta| - \dim|C_1| = \frac{d\beta\cdot C_2 + C_1\cdot C_2}{2} \geq \frac{d}{2}. \tag{5.23}$$

<u>*Case 2.2:*</u> $C_1^2 = 0$. Let s and E_1 be as in Case 1.2. By Lemma 5.5 (ii),

$$\dim|d\beta| - \dim|C_1| \geq \frac{d^2\beta^2}{2} - \lfloor\frac{s}{2}\rfloor \geq \frac{s(d\beta\cdot E_1 - 1) + d\beta\cdot C_2}{2} \geq d - \frac{1}{2}. \tag{5.24}$$

Combining (5.20)–(5.24) concludes the proof. □

5.1.3. *Refined invariants.* Assume that S is a generic Enriques surface and that the class of β in $\mathrm{Num}(S)$ is not divisible by 2 so that the moduli space $M_\beta = M_{\beta,1}$ is a smooth projective Calabi–Yau variety by [30]. Denote by $n_\beta^{i,j}$ the number defined by (2.4) with ϕ taken to be $h : M_\beta \to |\beta|$.

In order to apply Theorem 4.1 to compute $n_\beta^{i,j}$, we need the following lemma which is parallel to [20, Corollary 5.3]. For the reader's convenience, we provide a proof below.

Lemma 5.6. *Let β be a base-point-free, ample divisor on a generic Enriques surface S such that $2 \nmid \beta$ in $\mathrm{Num}(S)$. When $i + j \leq N(\beta)$, we have*

$$n_\beta^{i,j} = n_{h^\circ}^{i,j},$$

where $h^\circ = h_{|\beta|^{\mathrm{int}}} : h^{-1}(|\beta|^{\mathrm{int}}) \to |\beta|^{\mathrm{int}}$.

Proof. Let $W = M_\beta \setminus h^{-1}(|\beta|^{\mathrm{int}})$. Denote by $\iota : W \hookrightarrow M_\beta$ the closed embedding, and by $\jmath : h^{-1}(|\beta|^{\mathrm{int}}) \hookrightarrow M_\beta$, $\jmath^\circ : |\beta|^{\mathrm{int}} \to |\beta|$ the open embeddings. Since the fibers of h have the same dimension, we have

$$\mathrm{codim}(W, M_\beta) = \mathrm{codim}(|\beta| \setminus |\beta|^{\mathrm{int}}, |\beta|) = 1 + \frac{1}{2}N(\beta).$$

Consider the exact triangle

$$\iota_*\iota^!\mathbb{Q}_{M_\beta} \longrightarrow \mathbb{Q}_{M_\beta} \longrightarrow \jmath_*\jmath^*\mathbb{Q}_{M_\beta} \xrightarrow{+1}$$

which yields another exact triangle

$$h_*\iota_*\iota^!\mathbb{Q}_{M_\beta} \longrightarrow h_*\mathbb{Q}_{M_\beta} \longrightarrow \jmath^\circ_*h^\circ_*\mathbb{Q}_{h^{-1}(|\beta|^{\mathrm{int}})} \xrightarrow{+1} .$$

By the property of Verdier duals ([23, Proposition 5.3.9]) and the smoothness of M_β,

$$\iota_*\iota^!\mathbb{Q}_{M_\beta} \cong (D_{M_\beta}\iota_*\mathbb{Q}_W)[-2\dim M_\beta],$$

and therefore for $k \leq N(\beta) + 1$,

$$H^k(\iota_*\iota^!\mathbb{Q}_{M_\beta}) = H^{2\dim M_\beta - k}(W)^\vee = 0.$$

By taking the long exact sequence of cohomologies, the vanishing of $H^k(\iota_*\iota^!\mathbb{Q}_{M_\beta})$ shows that for $\ell \leq N(\beta)$, the restriction map induces an isomorphism

$$H^\ell(M_\beta) \cong H^\ell(h^{-1}(|\beta|^{\mathrm{int}})). \tag{5.25}$$

Since the restriction to $h^{-1}(|\beta|^{\mathrm{int}})$ clearly preserves a relatively ample class (relative to h), it follows from [6, Lemma 3.3] that $n_\beta^{i,j} \leq n_{h^\circ}^{i,j}$ for $i + j \leq N(\beta)$. Then the result follows by (4.12) and (5.25). □

Now we can prove the stabilization of $n_\beta^{i,j}$ for the moduli space of one-dimensional sheaves on a generic Enriques surface.

Theorem 5.7. *Fix $i, j \in \mathbb{Z}_{\geq 0}$. Given any ample divisor β_0 on a generic Enriques surface S such that $2 \nmid \beta_0$ in $\mathrm{Num}(S)$, there exists $d(\beta_0, i, j) \in \mathbb{Z}_{>0}$ (depending on β_0, i and j) such that*

$$n_{d\beta_0}^{i,j} = n_\infty^{i,j} \quad \text{for all integer } d \geq d(\beta_0, i, j) \text{ with } 2 \nmid d, \tag{5.26}$$

where $n_\infty^{i,j}$ is determined by

$$\sum_{i,j\geq 0} n_\infty^{i,j} q^i t^j = (1 - qt) \prod_{m \geq 1} \frac{1}{(1 - q^{m+1}t^{m-1})(1 - q^m t^m)^{10}(1 - q^{m-1}t^{m+1})}.$$

Proof. Choose the integer $d(\beta_0, i, j) \in \mathbb{Z}_{>0}$ such that for all $d \geq d(\beta_0, i, j)$, the following properties hold:

(1) $N(d\beta_0) \geq i + j$ (guaranteed by Proposition 5.2);
(2) $d\beta_0$ is $\max\{0, i-1\}$-very ample;
(3) $d^2\beta_0^2 \geq 3i + j$.

Condition (3) is equivalent to $2 \dim |d\beta_0| \geq 3i + j$ by Lemma 5.5 (i). By Lemma 5.6 and Theorem 4.1, taking $U = |d\beta_0|^{\mathrm{int}}$, we have

$$n_{d\beta_0}^{i,j} = n_{h_U}^{i,j} = n_\infty^{i,j},$$

which proves (5.26). □

Again by (4.12), the next corollary follows immediately.

Corollary 5.8. *Fix $k \in \mathbb{Z}_{\geq 0}$. Given any ample divisor β_0 on a generic Enriques surface S such that $2 \nmid \beta_0$ in $\mathrm{Num}(S)$, there exists $d(\beta_0, k) \in \mathbb{Z}_{>0}$ (depending on β_0 and k) such that*

$$b_k(M_{d\beta_0}) = b_k^\infty \quad \text{for all } d \geq d(\beta_0, k) \text{ with } 2 \nmid d,$$

where b_k^∞ is determined by (1.1).

Remark 5.9. By (5.20)–(5.22) and [35, Proposition 2.5], we can take

$$d(\beta_0, i, j) = \max\left\{2,\ i+1,\ \left\lceil \frac{i+j+2}{2} \right\rceil,\ \left\lceil \frac{i+j+6}{2\sqrt{2\beta_0^2}} \right\rceil,\ \left\lceil \sqrt{\frac{2i+2j+6}{\beta_0^2}} \right\rceil \right\} \tag{5.27}$$

in Theorem 1.4 (Theorem 5.7).

5.2. **The case of bielliptic surfaces.**

5.2.1. *Basic properties.* Recall that a bielliptic surface S is a minimal surface of Kodaira dimension 0 with $h^1(\mathcal{O}_S) = 1$ and $h^2(\mathcal{O}_S) = 0$. In particular, the exponential sequence reads

$$0 \to \mathrm{Pic}^0(S) \to \mathrm{Pic}(S) \to H^2(S, \mathbb{Z}) \to 0,$$

where $\mathrm{Pic}^0(S) = H^1(\mathcal{O}_S)/H^1(S, \mathbb{Z})$ is the group of algebraically trivial divisors modulo linear equivalence and has the structure of an elliptic curve. There exist two elliptic curves A, B, and an abelian group G acting on A and B, such that ([31, Proposition 1.1])

(1) A/G is elliptic and $B/G \cong \mathbb{P}^1$;
(2) $S \cong (A \times B)/G$, where G acts on $A \times B$ componentwise;

see [1]. We follow the convention in [31] by abuse of notation to denote by A (resp. B) the class (in $\mathrm{Num}(S)$, $H^2(S,\mathbb{Z})$, or $H^2(S,\mathbb{Q})$) of a smooth fiber of the natural projection $\Psi : S \to B/G$ (resp. $\Phi : S \to A/G$). These classes satisfy

$$A^2 = B^2 = 0, \quad A \cdot B = \gamma,$$

where γ is the order of G. Let λA, μB be a basis for $\mathrm{Num}(S)$ ($\lambda, \mu \in \mathbb{Q}$); see [31, Theorem 1.4] for the values of λ and μ.

We need the following lemma to compute the dimension of linear systems.

Lemma 5.10 ([31, Lemma 1.3])**.** *Let D be a divisor of numerical class $sA+tB$ with $s,t \in \mathbb{Q}$ on a bielliptic surface. The following properties hold:*

(i) $\chi(\mathcal{O}_S(D)) = st\gamma$.
(ii) *D is ample if and only if $s > 0$, $t > 0$.*
(iii) *If D is ample, then $h^0(\mathcal{O}_S(D)) = \chi(\mathcal{O}_S(D))$.*
(iv) *If $H^0(\mathcal{O}_S(D)) \neq 0$, then $s \geq 0$, $t \geq 0$.*

5.2.2. *Proof of Proposition 5.2 in the bielliptic case.* The numerical class of β in $\mathrm{Num}(S)$ is of the form $a\lambda A + b\mu B$ with $a,b \in \mathbb{Z}_{>0}$. By Lemma 5.10 (i) and (iii),

$$\dim |d\beta| = \chi(\mathcal{O}_S(d\beta)) - 1 = d^2ab\lambda\mu\gamma - 1.$$

Any non-integral curve in $|d\beta|$ is in the image of a morphism from a scheme P to $|d\beta|$, where P parametrizes the triples (C,D,N) satisfying $N \in \mathrm{Pic}^0(S)$, $C \in |C_1 + N|$, and $D \in |C_2 - N|$ for two nonzero effective divisors C_1, C_2 such that $C_1 + C_2 = d\beta$. By Lemma 4.5 (or Lemma 5.10), the images of finitely many such morphisms cover $|d\beta| \setminus |d\beta|^{\mathrm{int}}$. Thus, it suffices to estimate all possible values of

$$\dim |d\beta| - \dim |C_1| \times |C_2| \times \mathrm{Pic}^0(S) = \dim |d\beta| - \dim |C_1| - \dim |C_2| - 1.$$

Suppose the numerical class of C_i ($i = 1,2$) is $a_i\lambda A + b_i\mu B$ ($a_i, b_i \in \mathbb{Z}$). Then

$$a_1 + a_2 = da, \quad b_1 + b_2 = db.$$

It follows from Lemma 5.10 (iv) that $a_i \geq 0$ and $b_i \geq 0$.
<u>Case 1</u>: *$a_i > 0$ and $b_i > 0$ for $i = 1,2$.* By Lemma 5.10 (ii) and (iii),

$$\dim |C_1| + \dim |C_2| = \chi(\mathcal{O}_S(C_1)) + \chi(\mathcal{O}_S(C_2)) - 2.$$

Therefore, by the proof of [29, (13)] (or Case 1.1 in §5.1.2), we have

$$\dim |d\beta| - \dim |C_1| - \dim |C_2| - 1 = C_1 \cdot C_2 \geq d\sqrt{\beta^2} - 1. \tag{5.28}$$

<u>Case 2</u>: *At least one of a_1, a_2, b_1, b_2 is zero.* We may assume $a_1 = 0$. Then $b_1 > 0$ since otherwise C_1 would be numerically trivial. By Lemma 5.10 (i)–(iii),

$$\dim |C_1| \leq \dim |A + C_1| = b_1\mu\gamma - 1. \tag{5.29}$$

If $b_2 > 0$, then it follows from $b_1 > 0$ and (5.29) that

$$\begin{aligned} \dim |d\beta| - \dim |C_1| - \dim |C_2| - 1 &= C_1 \cdot C_2 - \dim |C_1| \\ &\geq dab_1\lambda\mu\gamma - (b_1\mu\gamma - 1) \\ &\geq (da\lambda - 1)\mu\gamma + 1 \quad (\text{if } da\lambda > 1). \end{aligned} \tag{5.30}$$

If $b_2 = 0$, then

$$\dim |C_2| \leq \dim |C_2 + B| = da\lambda\gamma - 1,$$

which, together with (5.29), implies

$$\begin{aligned} \dim |d\beta| - \dim |C_1| - \dim |C_2| - 1 &\geq d^2 ab\lambda\mu\gamma - 1 - (db\mu\gamma - 1) - (da\lambda\gamma - 1) - 1 \\ &= d^2 ab\lambda\mu\gamma - db\mu\gamma - da\lambda\gamma. \end{aligned} \tag{5.31}$$

Combining (5.28), (5.30), and (5.31) completes the proof. □

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The School of Mathematics and Statistics, Xi'an Jiaotong University, 28 West Xianning Road, Xi'an, Shaanxi, P.R.China 710049

Email address: `sifei@xjtu.edu.cn`

Shanghai Center for Mathematical Sciences, Fudan University, Jiangwan Campus, Shanghai, 200438, China

Email address: `fnzhang17@fudan.edu.cn`