MODELING DOCUMENTATION

1 QUBE-Servo 2 Rotary Inverted Pendulum Modeling

1.1 Model Convention

The rotary inverted pendulum model is shown in Figure 1.1. The rotary arm pivot is attached to the QUBE-Servo 2 base and is actuated. The arm has a length of L_r, a moment of inertia of J_r, and its angle, θ, increases positively when it rotates counter-clockwise (CCW). The servo (and thus the arm) should turn in the CCW direction when the control voltage is positive, i.e., $V_m > 0$.

The pendulum link is connected to the end of the rotary arm. It has a total length of L_p and it center of mass is $\frac{L_p}{2}$. The moment of inertia about its center of mass is J_p. The inverted pendulum angle, α, is zero when it is perfectly upright in the vertical position and increases positively when rotated CCW.

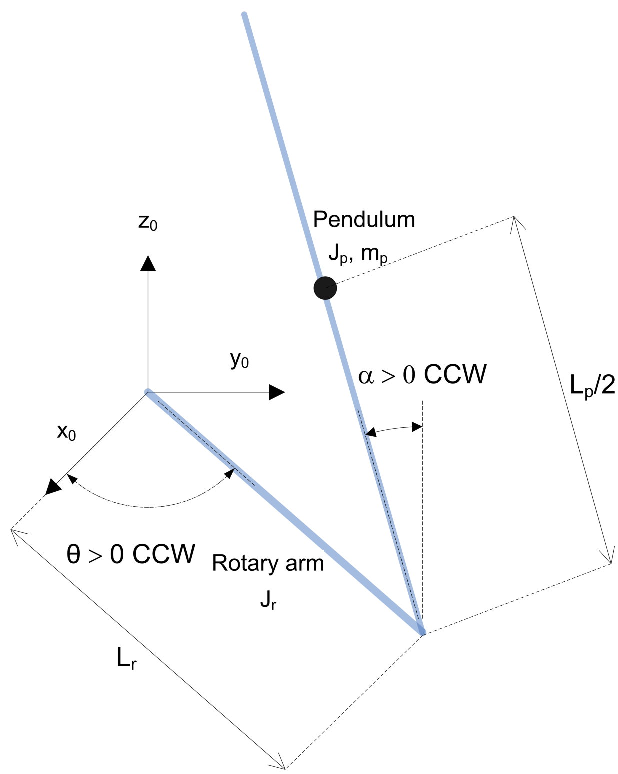

Figure 1.1: Rotary inverted pendulum conventions

1.2 Nonlinear Equations of Motion

Instead of using classical mechanics, the Lagrange method is used to find the equations of motion of the system. This systematic method is often used for more complicated systems such as robot manipulators with multiple joints.

More specifically, the equations that describe the motions of the rotary arm and the pendulum with respect to the servo motor voltage, i.e. the dynamics, will be obtained using the Euler-Lagrange equation:

$$\frac{\partial^2 L}{\partial t \partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i$$

The variables q_i are called *generalized coordinates*. For this system let

$$q(t)^\top = [\theta(t)\ \alpha(t)] \tag{1.1}$$

where, as shown in Figure 1.1, $\theta(t)$ is the rotary arm angle and $\alpha(t)$ is the inverted pendulum angle. The corresponding velocities are

$$\dot{q}(t)^\top = \left[\frac{\partial\theta(t)}{\partial t}\ \frac{\partial\alpha(t)}{\partial t}\right]$$

Note: The dot convention for the time derivative will be used throughout this document, i.e., $\dot{\theta} = \frac{d\theta}{dt}$. The time variable t will also be dropped from θ and α, i.e., $\theta = \theta(t)$ and $\alpha = \alpha(t)$.

With the generalized coordinates defined, the Euler-Lagrange equations for the rotary pendulum system are

$$\begin{aligned} \frac{\partial^2 L}{\partial t \partial \dot{\theta}} - \frac{\partial L}{\partial \theta} &= Q_1 \\ \frac{\partial^2 L}{\partial t \partial \dot{\alpha}} - \frac{\partial L}{\partial \alpha} &= Q_2 \end{aligned}$$

The Lagrangian of the system is described

$$L = T - V$$

where T is the total kinetic energy of the system and V is the total potential energy of the system. Thus the Lagrangian is the difference between a system's kinetic and potential energies.

The generalized forces Q_i are used to describe the non-conservative forces (e.g., friction) applied to a system with respect to the generalized coordinates. In this case, the generalized force acting on the rotary arm is

$$Q_1 = \tau - B_r\dot{\theta}$$

and acting on the pendulum is

$$Q_2 = -B_p\dot{\alpha}.$$

See the QUBE-Servo 2 User Manual for a description of the corresponding QUBE parameters (e.g. such as the back-emf constant, k_m). Our control variable is the input servo motor voltage, V_m. Opposing the applied torque is the viscous friction torque, or viscous damping, corresponding to the term B_r. Since the pendulum is not actuated, the only force acting on the link is the damping. The viscous damping coefficient of the pendulum is denoted by B_p.

The Euler-Lagrange equations is a systematic method of finding the equations of motion, i.e., EOMs, of a system. Once the kinetic and potential energy are obtained and the Lagrangian is found, then the task is to compute various derivatives to get the EOMs. After going through this process, the nonlinear equations of motion for the QUBE-Servo rotary inverted pendulum are:

$$\begin{aligned} \left(m_p L_r^2 + \frac{1}{4}m_p L_p^2 - \frac{1}{4}m_p L_p^2 \cos(\alpha)^2 + J_r\right)\ddot{\theta} - \left(\frac{1}{2}m_p L_p L_r \cos(\alpha)\right)\ddot{\alpha} & \\ + \left(\frac{1}{2}m_p L_p^2 \sin(\alpha)\cos(\alpha)\right)\dot{\theta}\dot{\alpha} + \left(\frac{1}{2}m_p L_p L_r \sin(\alpha)\right)\dot{\alpha}^2 &= \tau - B_r\dot{\theta} \qquad (1.2) \\ -\frac{1}{2}m_p L_p L_r \cos(\alpha)\ddot{\theta} + \left(J_p + \frac{1}{4}m_p L_p^2\right)\ddot{\alpha} - \frac{1}{4}m_p L_p^2 \cos(\alpha)\sin(\alpha)\dot{\theta}^2 & \\ -\frac{1}{2}m_p L_p g \sin(\alpha) &= -B_p\dot{\alpha}. \qquad (1.3) \end{aligned}$$

The torque applied at the base of the rotary arm (i.e., at the load gear) is generated by the servo motor as described by the equation

$$\tau = \frac{\eta_g K_g \eta_m k_t (V_m - K_g k_m \dot{\theta})}{R_m}. \tag{1.4}$$

Both the equations match the typical form of an EOM for a single body:

$$J\ddot{x} + b\dot{x} + g(x) = \tau_1$$

where x is an angular position, J is the moment of inertia, b is the damping, $g(x)$ is the gravitational function, and τ_1 is the applied torque (scalar value).

For a generalized coordinate vector q, this can be generalized into the matrix form

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau \tag{1.5}$$

where D is the inertial matrix, C is the damping matrix, $g(q)$ is the gravitational vector, and τ is the applied torque vector.

The nonlinear equations of motion given in 1.2 and 1.3 can be placed into this matrix format.

1.3 Linearizing

Here is an example of how to linearize a two-variable nonlinear function called $f(z)$. Variable z is defined

$$z^\top = [z_1 \; z_2]$$

and $f(z)$ is to be linearized about the operating point

$$z_0{}^\top = [a \; b]$$

The linearized function is

$$f_{lin} = f(z_0) + \left(\frac{\partial f(z)}{\partial z_1}\right)\Bigg|_{z=z_0}(z_1 - a) + \left(\frac{\partial f(z)}{\partial z_2}\right)\Bigg|_{z=z_0}(z_2 - b)$$

1.4 Linear State-Space Model

The linear state-space equations are

$$\dot{x} = Ax + Bu \tag{1.6}$$

and

$$y = Cx + Du \tag{1.7}$$

where x is the state, u is the control input, A, B, C, and D are state-space matrices. For the rotary pendulum system, the state and output are defined

$$x^\top = [\theta \; \alpha \; \dot{\theta} \; \dot{\alpha}] \tag{1.8}$$

and

$$y^\top = [x_1 \; x_2]. \tag{1.9}$$

In the output equation, only the position of the servo and link angles are being measured. Based on this, the C and D matrices in the output equation are

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{1.10}$$

and

$$D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{1.11}$$

The velocities of the servo and pendulum angles can be computed in the digital controller, e.g., by taking the derivative and filtering the result though a high-pass filter.