Optimal LQR Control

Topics Covered

- Introduction to state-space models.
- State-feedback control.
- Linear Quadratic Regulator (LQR) optimization.

Prerequisites

- Filtering laboratory experiment.
- Rotary Pendulum Modeling laboratory experiment.
- Rotary pendulum module is attached to the QUBE-Servo 2.

1 Background

Linear Quadratic Regulator (LQR) theory is a technique that is ideally suited for finding the parameters of the pendulum balance controller in the Rotary Pendulum Modeling laboratory experiment. Given that the equations of motion of the system can be described in the form

$$\dot{x} = Ax + Bu,$$

where A and B are the state and input system matrices, respectively, the LQR algorithm computes a control law u such that the performance criterion or cost function

$$J = \int_0^\infty (x_{ref} - x(t))^T Q (x_{ref} - x(t)) + u(t)^T Ru(t)dt \tag{1.1}$$

is minimized. The design matrices Q and R hold the penalties on the deviations of state variables from their setpoint and the control actions, respectively. When an element of Q is increased, therefore, the cost function increases the penalty associated with any deviations from the desired setpoint of that state variable, and thus the specific control gain will be larger. When the values of the R matrix are increased, a larger penalty is applied to the aggressiveness of the control action, and the control gains are uniformly decreased.

In our case the state vector x is defined

$$x = \begin{bmatrix}\theta & \alpha & \dot{\theta} & \dot{\alpha}\end{bmatrix}^T. \tag{1.2}$$

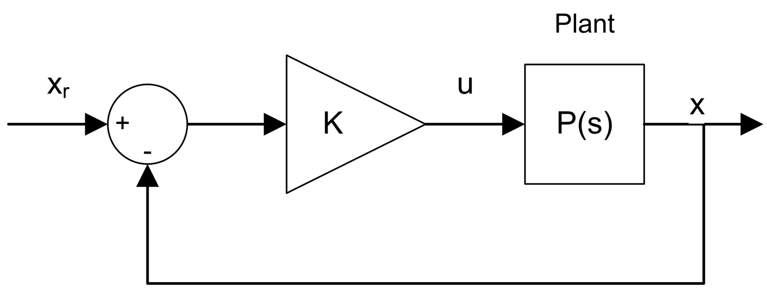

Figure 1.1: Block diagram of balance state-feedback control for rotary pendulum

Since there is only one control variable, R is a scalar. The reference signal x_{ref} is set to $\begin{bmatrix}\theta_r & 0 & 0 & 0\end{bmatrix}^T$, and the control strategy used to minimize cost function J is thus given by

$$u = K(x_{ref} - x) = k_{p,\theta}(\theta_r - \theta) - k_{p,\alpha}\alpha - k_{d,\theta}\dot{\theta} - k_{d,\alpha}\dot{\alpha}. \tag{1.3}$$

This control law is a state-feedback control and is illustrated in Figure 1.1. It is equivalent to the PD control explained in the Rotary Pendulum Modeling laboratory experiment.

2 In-Lab Exercises

Construct a QUARC® controller similarly to Figure 2.1 that balances the pendulum on the QUBE-Servo 2 rotary pendulum system using a generated control gain K.

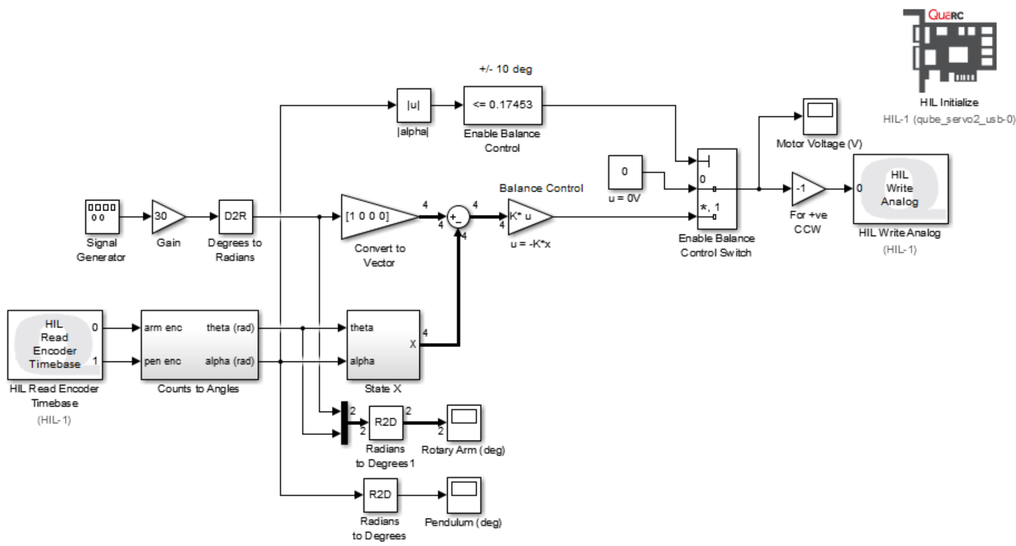

Figure 2.1: Simulink® model used with QUARC® run optimized balance controller

The LQR theory has been packaged in the Matlab® *Control Design Module*. Given the model of the system,in the form of the state-space matrices A and B, and the weighting matrices Q and R, the LQR function computes the feedback control gain automatically.

In this experiment, the state-space model is already available. In the laboratory, the effect of changing the Q weighting matrix while R is fixed to 1 on the cost function J will be explored.

2.1 LQR Control Design

1. In Matlab®, run the `setup_qube2_rotpen.m` script. This loads the QUBE-Servo 2 rotary pendulum state-space model matrices A, B, C, and D. The A and B matrices should be displayed in the Command Window:

```
A =

         0         0    1.0000         0
         0         0         0    1.0000
         0  149.2751   -0.0104         0
         0  261.6091   -0.0103         0


B =

         0
         0
   49.7275
   49.1493
```

2. B-5, B-7 Use the `eig` command to find the open-loop poles of the system. What do you notice about the

location of the open-loop poles? How does that affect the system?

Answer 2.1

Outcome	Solution
B-5	If the script was ran and the `eig` command were used correctly, they should be able to draw some conclusions based on the pole locations.
B-7	The inverted rotary pendulum system is unstable because there is one pole in the right-hand plane.

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3. **K-3** Using the `lqr` function with the loaded model and the weighting matrices

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad R = 1,$$

generate gain K. Give the value of the control gain generated.

Answer 2.2

Outcome	Solution
K-3	Using the following Matlab® commands:

```
Q = eye(4,4);
R = 1;
K = lqr(A,B,Q,R)
```

generates the gain

$$K = \begin{bmatrix} -1.00 & 34.24 & -1.23 & 3.08 \end{bmatrix}.$$

☐☐☐

4. **B-7, B-9** Change the LQR weighting matrix to the following and generate a new gain control gain:

$$Q = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad R = 1.$$

Record the gain generated. How does changing q_{11} affect the generated control gain? Based on the description of LQR in 2.1, is this what you expected?

Answer 2.3

Outcome	Solution
B-7	Using the following Matlab® commands:

```
Q = diag([5 1 1 1]);
K = lqr(A,B,Q,R)
```

generates the gain

$$K = \begin{bmatrix} -2.24 & 37.62 & -1.50 & 3.38 \end{bmatrix}.$$

Changing element q_{11} primarily increases the rotary arm proportional gain, $k_{p,\theta}$, but it also affect all the other gains.

B-9 This makes sense when looking at the cost function in Equation 1.1. Increasing q_{11} makes the LQR algorithm work harder to reduce the $x_1^2 = \theta^2$ term (as well as other states affected by this change), which augments gain $k_{p,\theta}$.

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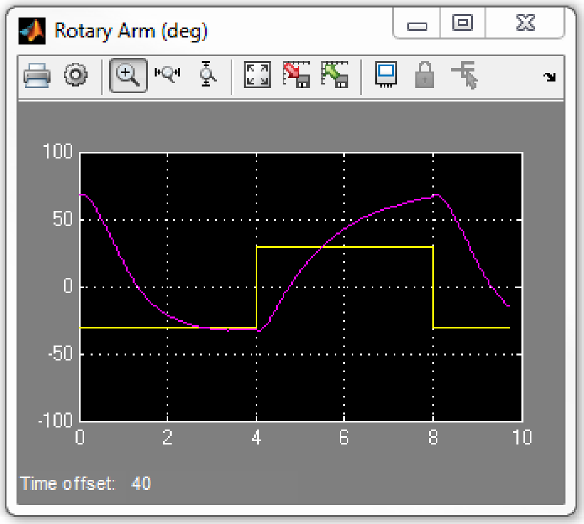
2.2 LQR-Based Balance Control

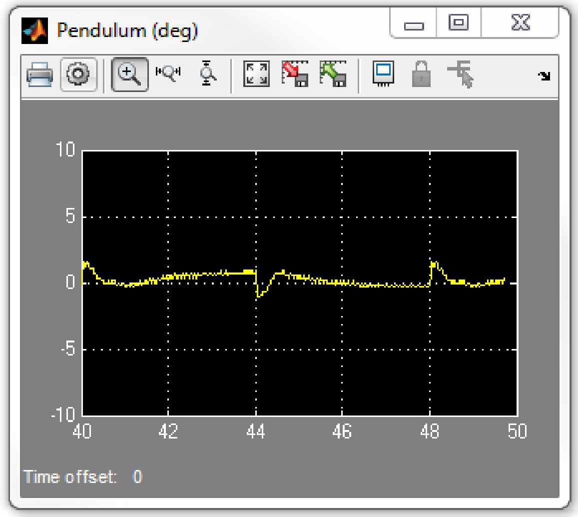
1. Run the `setup_qube2_rotpen.m` script in Matlab.
2. Using the Simulink® model you made in the Rotary Pendulum Modeling laboratory experiment, construct the controller shown in Figure 2.1:
   - Using the angles from the Counts to Angles subsystem you designed in the Rotary Pendulum Modeling laboratory experiment (which converts encoder counts to radians), build state x given in Equation 1.2. In Figure 2.1, it is bundled in the subsystem called *State X*. Use high-pass filters $50s/(s+50)$ to compute the velocities $\dot{\theta}$ and $\dot{\alpha}$.
   - Add the necessary Sum and Gain blocks to implement the state-feedback control given in Equation 1.3. Since the control gain is a vector, make sure the gain block is configured to do matrix type multiplication.
   - Add the Signal Generator block in order to generate a varying, desired arm angle. To generate a reference state, make sure you include a Gain block of $\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$.
3. Load the gain designed in Step 3. Make sure it is set as variable `K` in the Matlab® workspace.
4. Set the Signal Generator block to the following:
   - Type = Square
   - Amplitude = 1
   - Frequency = 0.125 Hz
5. Set the Gain block that is connected to the Signal Generator to 0.
6. Build and run the QUARC® controller.
7. Manually rotate the pendulum in the upright position until the controller engages.
8. B-5, K-2 Once the pendulum is balanced, set the Gain to 30 to make the arm angle go between $\pm 30°$. The scopes should read something similar as shown in Figure Ans.2.1. Attach your response of the rotary arm, pendulum, and controller voltage.

Answer 2.4

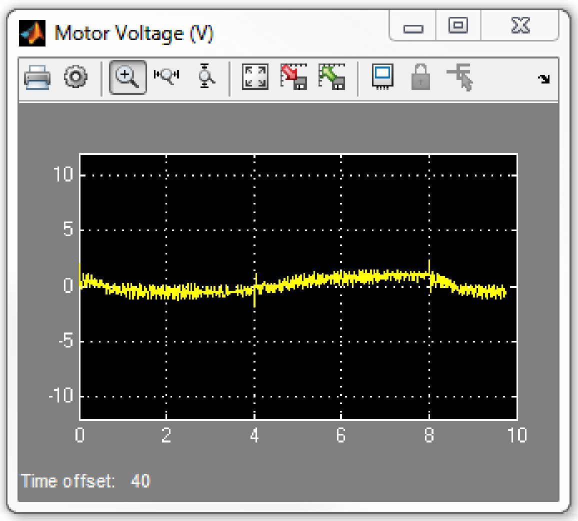
Outcome	Solution
B-5	If the Simulink® model was constructed and ran correctly and the pendulum is being balanced, then they can generate the scopes response below.
K-2	The scopes responses are given in Figure Ans.2.1.



(a) Rotary Arm



(b) Pendulum



(c) Motor Voltage

Figure Ans.2.1: QUBE-Servo 2 rotary pendulum response

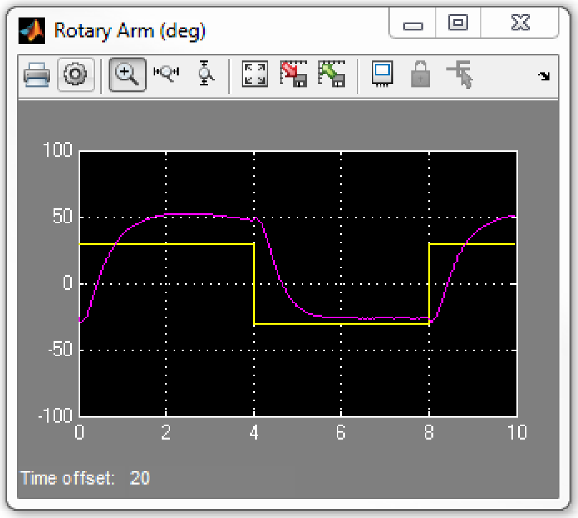
9. In Matlab®, generate the gain using $Q = \mathtt{diag}(\begin{bmatrix}5 & 1 & 1 & 1\end{bmatrix})$ performed in Step 4 in the first part of this laboratory experiment. The `diag` command specifies the diagonal elements in a square matrix.

10. To apply the newly designed gain to the running QUARC controller, go to the Simulink model and select *Edit | Update Diagram* (or press CTRL-D).

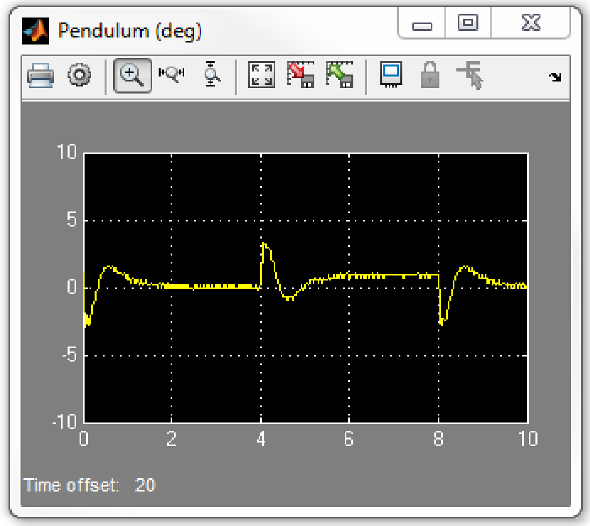
11. B-7 Examine and describe the change in the *Rotary Arm (deg)* and *Pendulum (deg)* scopes.

Answer 2.5

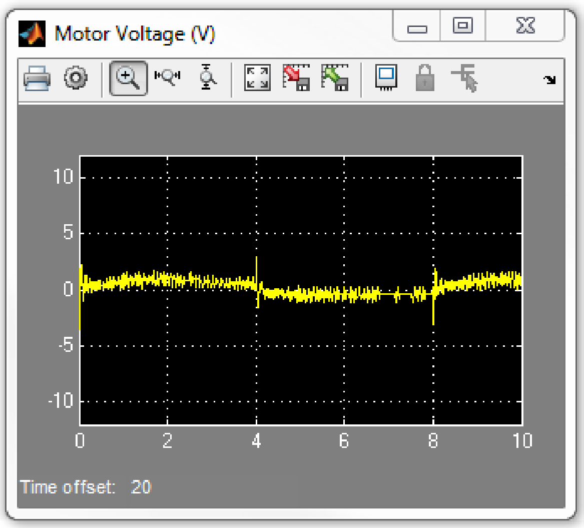
Outcome	**Solution**
B-7	As shown in Figure Ans.2.2, the arm response becomes faster, i.e. peak time decreases, mainly due to the increased arm proportional gain. Because the rotary arm responds faster, the pendulum tends to deflect more from its vertical position.



(a) Rotary Arm



(b) Pendulum



(c) Motor Voltage

Figure Ans.2.2: QUBE-Servo 2 rotary pendulum response when increasing q_{11} to 5

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12. B-2, B-4 Adjust the diagonal elements of Q matrix to reduce how much the pendulum angle deflects (overshoots) when the arm angle changes. Describe your experimental procedure to find the necessary control gain.

Answer 2.6

Outcome	**Solution**
B-2	Based on the analysis done in 2.1, the elements in Q that are most likely to affect the pendulum gains are q_{22} and q_{44}. When going through the procedure outlined below, it will be observed that changing q_{22} has little affect on the generated gain K. Leaving q_{44} as the main variable to adjust.

B-4 The procedure to find out which Q element affect the pendulum the most effectively is as follows:

(a) Adjust Q matrix element.

(b) Generate gain K.

(c) Verify if the new gain has changed significantly. If not, then adjust another element or increase more. If it has changed, go to next step.

(d) Implement gain on QUARC® controller (CTRL-D).

(e) Examine pendulum response.

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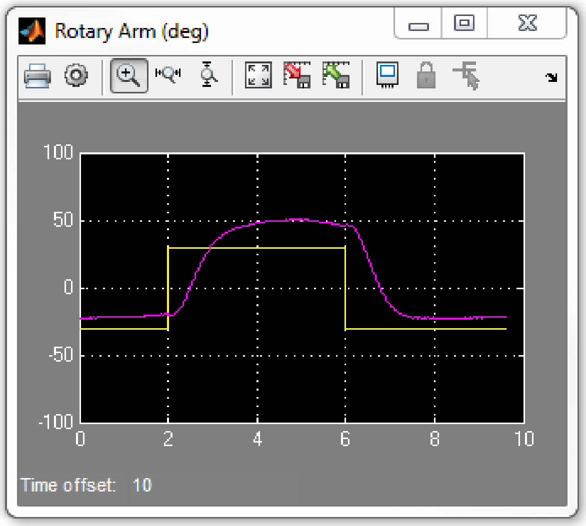
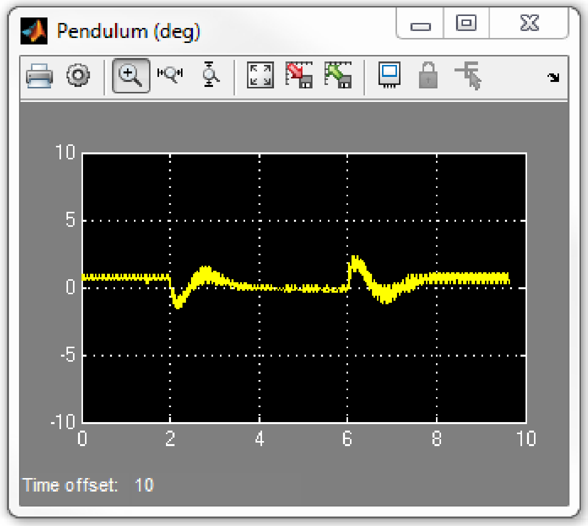
13. K-1, K-2 List the resulting LQR Q matrix and control gain K used to yield the desired results. Attach the responses using this new control gain and briefly outline how the response changed.

Answer 2.7

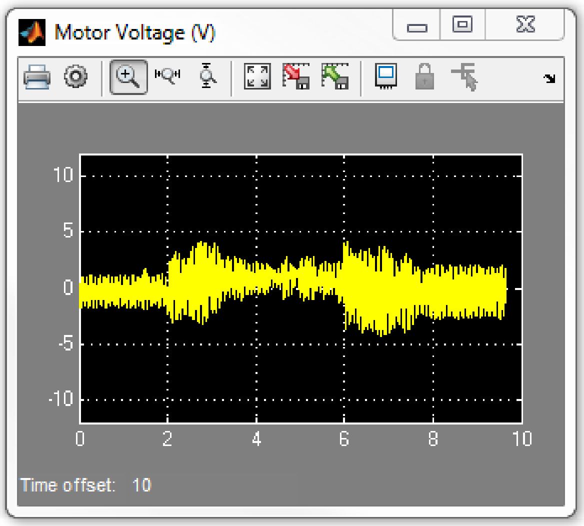
Outcome	Solution
K-1	The gain generated using $Q = \texttt{diag}\left(\begin{bmatrix} 5 & 1 & 1 & 20 \end{bmatrix}\right)$ is $$K = \begin{bmatrix} -2.24 & 53.86 & -1.70 & 6.53 \end{bmatrix}.$$ The response when increasing the LQR gain element $q_{44} = 20$ is shown in Figure Ans.2.3. As shown, the pendulum deflection is decreased from $\pm 4.5°$, as depicted in Figure Ans.2.2, down to $\pm 1.9°$. Having larger proportional and derivative gains acting on the pendulum decreases the amount it deflects when the rotary arm rotates back-and-forth.
K-2	Response when increasing the LQR gain element $q_{44} = 20$ is shown in Figure Ans.2.3.



(a) Rotary Arm



(b) Pendulum



(c) Motor Voltage

Figure Ans.2.3: QUBE-Servo 2 rotary pendulum response with less pendulum deflection

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14. Stop the QUARC® controller.
15. Power *OFF* the QUBE-Servo 2.

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