State Space Modeling

Topics Covered

- Inverted pendulum modeling.
- Introduction to state-space models.
- Model validation.

Prerequisites

- Integration laboratory experiment.
- First Principles Modeling laboratory experiment.
- Pendulum Moment of Inertia laboratory experiment.
- Rotary pendulum module is attached to the QUBE-Servo 2.

1 Background

1.1 Pendulum Model

The rotary pendulum model is shown in Figure 1.1. The rotary arm pivot is attached to the QUBE-Servo 2 system and is actuated. The arm has a length of L_r, a moment of inertia of J_r, and its angle θ increases positively when it rotates counter-clockwise (CCW). The servo (and thus the arm) should turn in the CCW direction when the control voltage is positive ($V_m > 0$).

The pendulum link is connected to the end of the rotary arm. It has a total length of L_p and it center of mass is at $L_p/2$. The moment of inertia about its center of mass is J_p. The inverted pendulum angle α is zero when it is hanging downward and increases positively when rotated CCW.

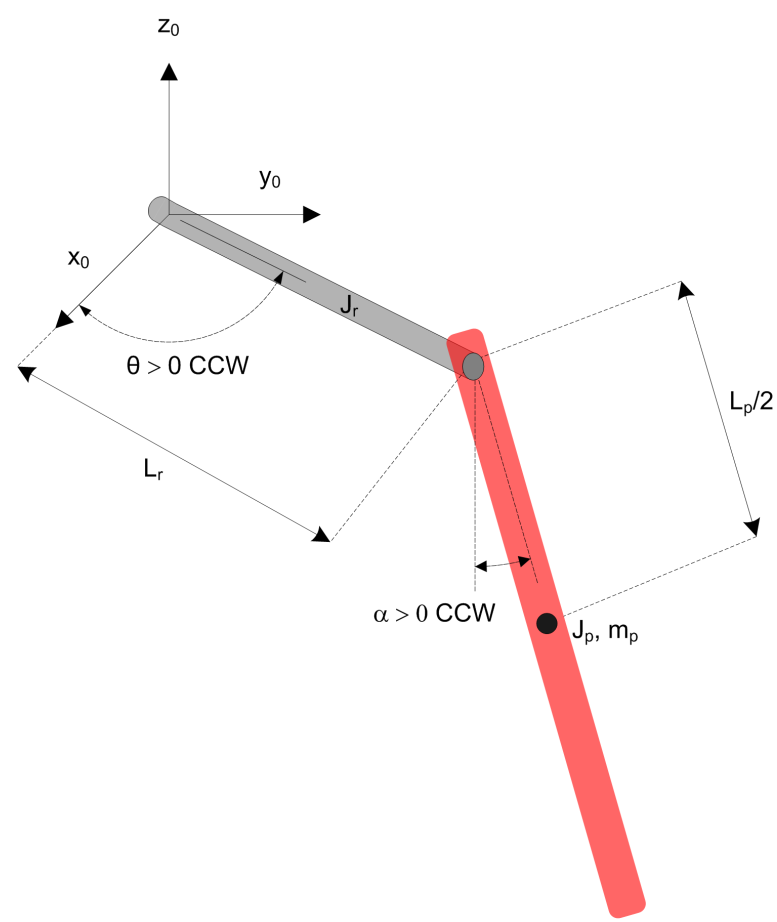


Figure 1.1: Rotary inverted pendulum model

The equations of motion (EOM) for the pendulum system were developed using the Euler-Lagrange method. This systematic method is often used to model complicated systems such as robot manipulators with multiple joints. The total kinetic and potential energy of the system is obtained, then the Lagrangian can be found. A number of derivatives are then computed to yield the EOMs. The complete derivation of the EOM for the pendulum system are presented in the *Rotary Pendulum Modeling Summary* and Maple workbook.

The resultant nonlinear EOM are:

$$\left(m_p L_r^2 + \frac{1}{4} m_p L_p^2 - \frac{1}{4} m_p L_p^2 \cos(\alpha)^2 + J_r\right) \ddot{\theta} - \left(\frac{1}{2} m_p L_p L_r \cos(\alpha)\right) \ddot{\alpha}$$
$$+ \left(\frac{1}{2} m_p L_p^2 \sin(\alpha)\cos(\alpha)\right) \dot{\theta}\dot{\alpha} + \left(\frac{1}{2} m_p L_p L_r \sin(\alpha)\right) \dot{\alpha}^2 = \tau - D_r \dot{\theta} \quad (1.1)$$

$$\frac{1}{2} m_p L_p L_r \cos(\alpha)\ddot{\theta} + \left(J_p + \frac{1}{4} m_p L_p^2\right) \ddot{\alpha} - \frac{1}{4} m_p L_p^2 \cos(\alpha)\sin(\alpha)\dot{\theta}^2 + \frac{1}{2} m_p L_p g \sin(\alpha) = -D_p \dot{\alpha}. \quad (1.2)$$

with an applied torque at the base of the rotary arm generated by the servo motor as described by the equation:

$$\tau = \frac{k_m \left(V_m - k_m \dot{\theta}\right)}{R_m} \quad (1.3)$$

When the nonlinear EOM are linearized about the operating point, the resultant linear EOM for the inverted pendulum are defined as:

$$\left(m_p L_r^2 + J_r\right) \ddot{\theta} - \frac{1}{2} m_p L_p L_r \ddot{\alpha} = \tau - D_r \dot{\theta}. \quad (1.4)$$

and

$$\frac{1}{2} m_p L_p L_r \ddot{\theta} + \left(J_p + \frac{1}{4} m_p L_p^2\right) \ddot{\alpha} + \frac{1}{2} m_p L_p g \alpha = -D_p \dot{\alpha}. \quad (1.5)$$

Solving for the acceleration terms yields:

$$\ddot{\theta} = \frac{1}{J_T}\left(-\left(J_p + \frac{1}{4} m_p L_p^2\right) D_r \dot{\theta} + \frac{1}{2} m_p L_p L_r D_p \dot{\alpha} + \frac{1}{4} m_p^2 L_p^2 L_r g \alpha + \left(J_p + \frac{1}{4} m_p L_p^2\right)\tau\right). \quad (1.6)$$

and

$$\ddot{\alpha} = \frac{1}{J_T}\left(\frac{1}{2} m_p L_p L_r D_r \dot{\theta} - \left(J_r + m_p L_r^2\right) D_p \dot{\alpha} - \frac{1}{2} m_p L_p g \left(J_r + m_p L_r^2\right)\alpha - \frac{1}{2} m_p L_p L_r \tau\right). \quad (1.7)$$

where

$$J_T = J_p m_p L_r^2 + J_r J_p + \frac{1}{4} J_r m_p L_p^2. \quad (1.8)$$

1.1.1 Linear State-Space Model

The linear state-space equations are

$$\dot{x} = Ax + Bu \quad (1.9)$$

and

$$y = Cx + Du \quad (1.10)$$

where x is the state, u is the control input, A, B, C and D are state-space matrices. For the rotary pendulum system, the state and output are defined

$$x = \begin{bmatrix} \theta & \alpha & \dot{\theta} & \dot{\alpha} \end{bmatrix}^T \quad (1.11)$$

and

$$y = \begin{bmatrix} \theta & \alpha \end{bmatrix}^T. \quad (1.12)$$

2 In-Lab Exercises

2.1 Pendulum State-Space Model

1. A-1 Based on the sensors available on the pendulum system, find the C and D matrices in Equation 1.10.

Answer 2.1

Outcome	Solution
A-1	In the output equation, only the position of the servo and link angles are being measured. Based on this, the C and D matrices in the output equation are

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \qquad \text{(Ans.2.1)}$$

□□□

2. A-1, A-2 Using Equation 1.6 and Equation 1.7 and the defined state in Equation 1.11, derive the linear state-space model of the pendulum system.

Answer 2.2

Outcome	Solution
A-1	From the defined state in Equation 1.11, it is given that $\dot{x}_1 = x_3$ and $\dot{x}_2 = x_4$. Substitute state x into the equations of motion found, where we have $\theta = x_1, \alpha = x_2, \dot{\theta} = x_3, \dot{\alpha} = x_4$. The A and B matrices for $\dot{x} = Ax + Bu$ can then be found.
A-2	Substituting x into Equation 1.6 and Equation 1.7 gives

$$\begin{aligned} \dot{x}_3 = &\frac{1}{J_T}\left(-\left(J_p + \frac{1}{4}m_p L_p^2\right) D_r x_3 + \frac{1}{2} m_p L_p L_r D_p x_4 \right. \\ &\left. + \frac{1}{4} m_p^2 L_p^2 L_r g x_2 + \left(J_p + \frac{1}{4} m_p L_p^2\right) u\right) \end{aligned} \qquad \text{(Ans.2.2)}$$

and

$$\begin{aligned} \dot{x}_4 = &\frac{1}{J_T}\left(\frac{1}{2} m_p L_p L_r D_r x_3 - \left(J_r + m_p L_r^2\right) D_p x_4 \right. \\ &\left. - \frac{1}{2} m_p L_p g \left(J_r + m_p L_r^2\right) x_2 - \frac{1}{2} m_p L_p L_r u\right). \end{aligned} \qquad \text{(Ans.2.3)}$$

The A and B matrices in the $\dot{x} = Ax + Bu$ equation are

$$A = \frac{1}{J_T}\begin{bmatrix} 0 & 0 & J_T & 0 \\ 0 & 0 & 0 & J_T \\ 0 & \frac{1}{4}m_p^2 L_p^2 L_r g & -\left(J_p + \frac{1}{4}m_p L_p^2\right) D_r & \frac{1}{2}m_p L_p L_r D_p \\ 0 & -\frac{1}{2}m_p L_p g \left(J_r + m_p L_r^2\right) & \frac{1}{2}m_p L_p L_r D_r & -\left(J_r + m_p L_r^2\right) D_p \end{bmatrix} \quad \text{(Ans.2.4)}$$

and

$$B = \frac{1}{J_T}\begin{bmatrix} 0 \\ 0 \\ J_p + \frac{1}{4}m_p L_p^2 \\ -\frac{1}{2}m_p L_p L_r \end{bmatrix}. \quad \text{(Ans.2.5)}$$

□□□

3. Based on the state space model derived in Step 2 and the Matlab® script `rotpen_ABCD_eqns.m` provided, create the appropriate matrices that correspond to the linear state space model of the pendulum.
4. Based on the Simulink® model already designed in the Pendulum Moment of Inertia laboratory experiment, design a similar model to the one shown in Figure 2.1 that applies a $0-1$ V, 1 Hz square wave to the pendulum system and state-space model.

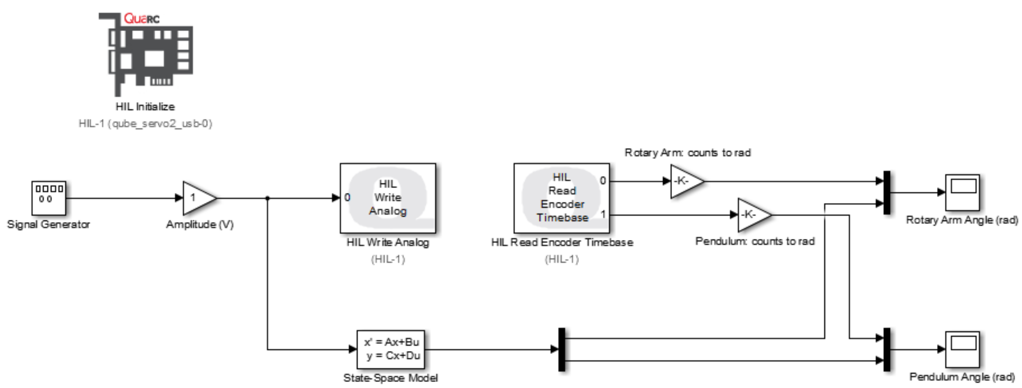
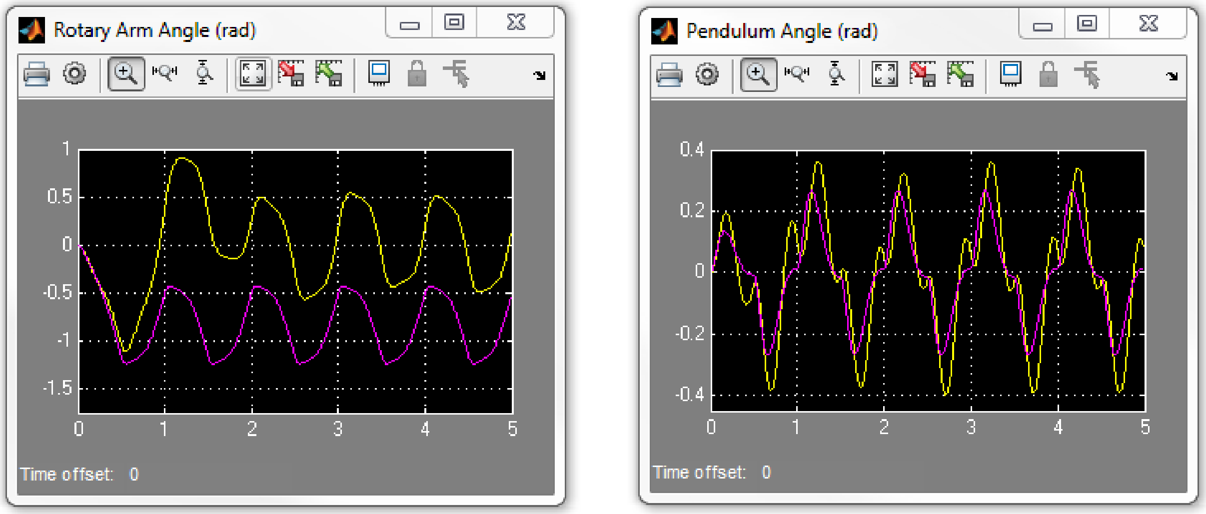


Figure 2.1: Applies a step voltage and displays measured and simulated pendulum response.

5. Run `setup_ss_model.m` to create the state space model parameters in the Matlab® workspace. Ensure that the generated matrices match your solution in Step 2.
6. In `setup_ss_model.m`, set the rotary arm viscous damping coefficient D_r to 0.0015 N.m.s/rad, and the pendulum damping coefficient D_p to 0.0005 N.m.s/rad. These parameters were found experimentally to reasonably accurctly reflect the viscous damping of the system due to effects such as friction, when subject to a step response.
7. B-5, B-9 Build and run the model. The scope response should be similar to Figure 2.2. Attach a screen capture of your scopes. Does your model represent the actual pendulum well? If not, explain why there might be discrepancies.



(a) Rotary Arm Angle (rad)

(b) Pendulum Angle (rad)

Figure 2.2: Step response of the pendulum system

Answer 2.3

Outcome	**Solution**
B-5	If the experimental procedure was followed correctly, the user should be able to run the model and obtain a response similar to Figure Ans.2.1.
B-9	The pendulum response of the model represents the actual pendulum system accurately because in the simulated response (blue) matches the measured response (red) quite well in Figure Ans.2.1. The model of the arm response displays the same characteristics of the measured arm response, but an offset is observed due to un-modelled dynamics such as the disturbance introduced by the encoder cable.

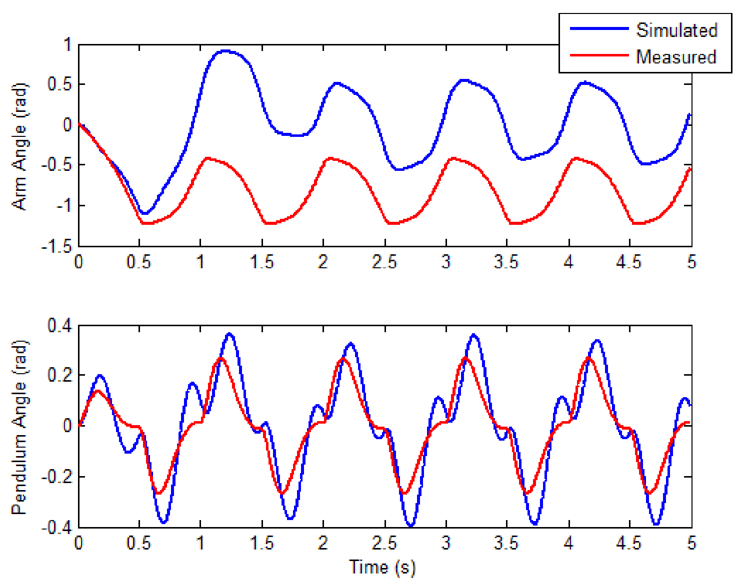


Figure Ans.2.1: Step response of the pendulum system.

8. The viscous damping of each inverted pendulum can vary slightly from system to system. If your model does not accuratly represent your specific pendulum system, try modifying the damping coefficients D_r and D_p to obtain a more accurate model.
9. Stop the QUARC® controller.
10. Power *OFF* the QUBE-Servo 2.

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