I.4 THE ROTATION GROUP AND ITS FINITE SUBGROUPS

Lattices, crystals, and symmetrical molecules are invariant under rotations, but only through discrete sets of angles, thus under — finite — subgroups of the rotation group. What are these subgroups? It may seem that there are many, infinitely many — just divide 2π by any integer, no matter how large, or even any rational or real number — but these are groups, so we cannot do arbitrary things with them [Armstrong (1988), p. 37; Burn (1991), p. 57; Coxeter (1973), p. 5, 15; Ghyka (1977), p. 40; Martin (1987), p. 198; Murnaghan (1963), p. 328; Schwarzenberger (1980), p. 43; Simon (1996), p. 11].

I.4.a The finite subgroups of the two-dimensional rotation group

The two-dimensional rotation group, SO(2), is easiest to consider [Armstrong (1988), p. 104].

Problem I.4.a–1: With R(n) a rotation of $\frac{2\pi}{n}$, prove that the only finite subgroups of SO(2) are the sets

$$\mathrm{R}(n), \mathrm{R}(n)^2, \ldots, \mathrm{R}(n)^k, \ldots, \quad k = 0, 1, 2, \ldots, n-1, \qquad \text{(I.4.a–1)}$$

where n is an integer, and there is one set for each integer n. These are the cyclic groups [Mirman (1995a), sec. II.4.b, p. 49], and all. For O(2) [Mirman (1995a), sec. XII.4.b, p. 355], there is added to each one other element, the reflection; the subgroups of O(2) consist thus of all cyclic and dihedral groups, the groups of rotations of objects with two surfaces — dihedral — so having the symmetry of rotations about a central axis, plus an interchange of the two sides by a rotation of π about an axis parallel to, and halfway between, the sides; some dihedral groups contain an inversion interchanging the two sides [Mirman (1995a), sec. II.4.c, p. 50]. Construct such an object, of paper for example. An ordinary blank sheet is an example (with what symmetry group?). With a scissors other dihedral groups can be realized. Show the effect of these transformations on coordinate axes. Also find the effect of a reflection in (a plane perpendicular to) an axis, giving the realization [Mirman (1995a), sec. V.3.c, p. 157] more relevant to O(2).

I.4.b The finite subgroups for three-dimensions

The two-dimensional rotation group is quite simple and we would not expect many subgroups from it. The three-dimensional group has much more structure, but surprisingly not many more subgroups [Armstrong (1988), p. 105; Neumann, Stoy and Thompson (1994), p. 174]. This is important enough to state as a

THEOREM: A finite subgroup of SO(3) — reflections are not included — is (isomorphic to) either a cyclic group, a dihedral group or the rotational symmetry group of one of the regular polyhedra of which there are only five. Thus besides the cyclic and dihedral groups the only finite subgroups of SO(3) are the symmetric group S_4, its subgroup A_4, and alternating group A_5 [Mirman (1995a), sec. II.4.d.ii, p. 54].

That this places strong limitations on objects with symmetry can already be seen. Since crystals consist of cells with symmetry we have greatly limited the types possible — the types of crystals that we are permitting nature to have (?). Of course there may be further limitations from other conditions, and we also have to show that there are physical crystals corresponding to each allowed mathematical one — that the requirements are not inconsistent (with physics).

Why should the finite subgroups of the rotation group be so restricted? Since SO(2) is a subgroup of SO(3) its subgroups are subgroups of the latter, hence all the cyclic groups are. A cyclic group plus a rotation around an axis perpendicular to that of the group (an element of SO(3)) gives a dihedral group, so these are SO(3) subgroups. Thus we have to consider those finite subgroups that contain rotations about different axes and show that these consist only of A_5, S_4 and their subgroup A_4; not many. The objects invariant under these groups are called the Platonic solids (or polyhedra) [Holden (1991), p. 1]. Why do these groups give the Platonic polyhedra — and only them — and why, in two cases, does a single group give two polyhedra (and no more)?

Problem I.4.b–1: Cyclic (rotation) groups are Abelian, so each element is in a class by itself. Show that for dihedral groups, D_n (sec. I.7.a.i, p. 52; sec. V.2.c, p. 249), the number of classes is $\frac{1}{2}(n+6)$, for n even, and $\frac{1}{2}(n+3)$, for n odd, and that each element in the same class as its inverse [Dixon (1973), p. 5, 75; Hamermesh (1962), p. 43; Ledermann (1987), p. 65, 209; Lomont (1961), p. 32, 78; Wilson, Decius and Cross (1980), p. 316].

Problem I.4.b–2: In the list of rotation-group finite subgroups, subgroups of S_4 and A_5 are not listed, except for A_4. Are all others cyclic or dihedral, or direct products [Mirman (1995a), pb. IV.8.b-3, p. 138]; if not why are they not rotation-group subgroups [Lomont (1961), p. 34, 134; Streitwolf (1971), p. 62]?

I.4.c Regular polyhedra

There are thus three-dimensional figures — polyhedra — that have a special relationship to the rotation group, the regular (Platonic) polyhedra. A regular polyhedron [Armstrong (1988), p. 37; Lines (1965), p. 134; Yale (1988), p. 89] is one invariant under rotational interchange of any two vertices — the polyhedron does not change if any vertex is rotated into any other — so all its angles, and the lengths of all sides, are equal. A polyhedron — many sides — is a three-dimensional closed object. In two dimensions the analogous figure is a polygon — many angles. The generalization to arbitrary dimensions can be called a polytope [Coxeter (1973), p. vi, 126, 289; Fejes Toth (1964), p. 124, 132] — implying many places, like many vertices, from the Greek word that gives the name topology, and also utopia.

How many regular polyhedra are there? Only five [Coxeter (1973), p. 5; Holden (1991), p. 1; Ghyka (1977), p. 40; Rosen (1977), p. 52]. These are the tetrahedron [Armstrong (1988), p. 1] (with rotational symmetry group A_4), the cube and the octahedron, for both the rotational symmetry group is S_4, the dodecahedron and the icosahedron, both with group A_5. The tetrahedron (four surfaces) has four triangular faces, the cube six square ones, the octahedron (eight surfaces) has eight triangular faces, the dodecahedron has twelve pentagonal faces, the icosahedron twenty triangular ones. The dodecahedron and the icosahedron are dual figures — having the same group — so the dodecahedron (dozen surfaces) has twelve faces and twenty vertices, the icosahedron, twenty faces and twelve vertices. A diagram of the three most unfamiliar of these has been given previously [Mirman (1995a), fig. III.2.g-1, p. 78].

Problem I.4.c-1: The cube and octahedron are also dual. Check that the numbers of vertices and faces are correct.

Problem I.4.c-2: Coordinates of the vertices of the regular polyhedra are easily found [Coxeter (1973), p. 52].

I.4.c.i *Determination of the regular polyhedra*

There are various ways of proving that these are the only regular polyhedra. For them, angles between edges are almost fully determined, allowing very few such objects.

Problem I.4.c.i-1: Consider vertex V in the next diagram (which we interpret as the corner of a solid) with q lines, which we can take as rods (edges of the solid) [Coxeter (1973), p. 5], shown as the q (here three) heavy solid ones. Each pair of (heavy solid) lines defines a plane, giving for each pair an angle in their plane. What is the sum, $\angle$ (A + B + C), of the angles between the lines? To give an experimental proof take a set of rods and place their ends on a sheet of paper (containing the thin lines, the projections of the heavy ones), with V above the paper, and draw from point O directly under V (as shown by the dotted line) the thin lines to each point at which a rod touches the paper, as in

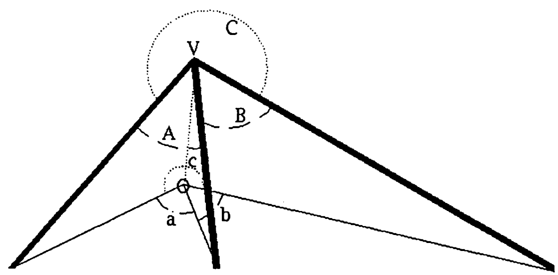

Figure I.4.c.i-1: PROJECTED ANGLES.

It is clear that angle A between two lines from V is less than its projected angle a from point O in the plane under V to the rods (and similarly for B and C). The sum of the angles between the lines from O is of course 2π. If the rods were collapsible and we moved vertex V toward the paper the angles between them would increase (and decrease moving away, becoming 0 for the vertex at infinity), while the angles in the plane would be unchanged, as we see, noting that $\angle$ A' < $\angle$ A,

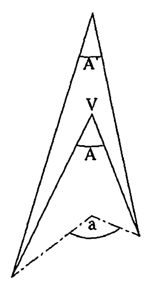

Figure I.4.c.i-2: VERTEX ANGLES.

Thus $\angle$ (A + B + C) is less than the sum of the projected angles on the paper,

$$\angle (A + B + C) < \angle (a + b + c) = 2\pi. \qquad \text{(I.4.c.i-1)}$$

Prove analytically, and with a drawing and also with a computer program that in general, for a regular polyhedron, the sum of the q (equal) angles θ between the edges at each vertex is less than 2π, so for each

$$\theta < \frac{2\pi}{q}; \qquad \text{(I.4.c.i-2)}$$

this is one reason the number of regular polyhedra is limited. Each face of a polyhedron has p edges, the sum of exterior angles is 2π, each then being $\frac{2\pi}{p}$, and the interior angle (the supplement) is

$$\theta = (1 - \frac{2}{p})\pi. \tag{I.4.c.i-3}$$

Draw these and check. Hence,

$$(1 - \frac{2}{p})\pi < \frac{2\pi}{q}, \tag{I.4.c.i-4}$$

giving an inequality relating the number of lines bounding each face and the number of lines leaving each vertex. This should give

$$\frac{1}{p} + \frac{1}{q} > \frac{1}{2}. \tag{I.4.c.i-5}$$

Note that edges and angles enter symmetrically. The number of faces and angles between edges are strongly bounded. Explain why

$$3 \leq p, q \leq 5, \tag{I.4.c.i-6}$$

and both p and q are integers. It would not be surprising if there were no regular polyhedra, thus no (other) finite subgroups of the rotation group, but (fortunately) there are.

Problem I.4.c.i-2: The reason then that the number of regular polyhedra is limited is that as the number of edges of a face increases the exterior angle decreases, and as the number of lines leaving a vertex increases so does the number of angles they make, and the sum of these is bounded; but there is one question. Explain what the number of sides of a face has to do with the number of lines at a vertex. What does this have to do with a polyhedron being regular? How would the argument break down otherwise? (What does otherwise mean here?) It is clear from this that the polyhedra are paired (dual) — the number of edges of a face of one equals the number of lines emerging from a vertex of the other. Explain why this shows that the number of faces of one equals the number of vertices of the other. Why do they have the same symmetry group?

Problem I.4.c.i-3: Check that, with V the number of vertices, F the number of faces, E the number of edges, p the number of edges per face, q the number of faces per vertex, and N the order of the symmetry group, the allowed values [Cotton (1990), p. 46; Elliott and Dawber (1987), p. 188; Ford (1972), p. 128; Gasson (1989), p. 231; Hargittai and Hargittai (1987), p. 66; Holden (1991), p. 8; Martin (1987), p. 200; Neumann, Stoy and Thompson (1994), p. 175] are

polyhedron	p	q	V	F	E	group	N
tetrahedron	3	3	4	4	6	A_4	12
hexahedron (cube)	4	3	8	6	12	S_4	24
octahedron	3	4	6	8	12	S_4	24
dodecahedron	5	3	20	12	30	A_5	60
icosahedron	3	5	12	20	30	A_5	60

Table I.4.c.i-1: REGULAR POLYHEDRA.

This gives the number of edges for each face, so their shapes. The tetrahedron is self-conjugate, the others are conjugate in pairs. Explain the other name for the cube. Check that "icosa" (from the Greek) means twenty.

I.4.c.ii *Other ways of determining these polyhedra*

It is not unusual in mathematics for there to be different proofs, often seeming to have nothing to do with each other, giving (fortunately) the same truth. Here we consider some other approaches, which besides providing useful comparisons, also provide other useful information.

Problem I.4.c.ii-1: Euler's topological equation

$$V - E + F = 2. \qquad (I.4.c.ii-1)$$

gives another proof [Coxeter (1973), p. 9, 165; Elliott and Dawber (1987), p. 186; Ford (1972), p. 127; Gasson (1989), p. 229; Hamermesh (1962), p. 48; Hilton and Pederson (1996); Lines (1965), p. 135; Martin (1987), p. 198; Meserve (1983), p. 301; Senechal (1989), p. 34; Sternberg (1994), p. 45]. Prove it. For a regular polyhedron

$$pF = 2E = qV. \qquad (I.4.c.ii-2)$$

Why? Use this to verify the table.

Problem I.4.c.ii-2: A tetrahedron has four vertices, the smallest possible (in three dimensions), a cube eight. Try to construct (say with sticks) a regular polyhedron with six vertices. Explain what goes wrong. Repeat with seven and nine.

Problem I.4.c.ii-3: Show that these polyhedra are invariant under the groups stated, and no larger ones. One way of doing this is to construct three-dimensional models and find all symmetries. Another is to use coordinates. Also it might be possible to write a computer program to do it (visually).

Problem I.4.c.ii-4: For the tetrahedron, label each vertex and verify that the permutations form S_4. However only half of these can be achieved by rigid-body rotations — without taking the object apart. Which are they? Why? What do the others do? Repeat this for the other regular polyhedra and verify that the stated symmetry groups are correct.

Problem I.4.c.ii–5: Coordinates of the vertices can be found from three-dimensional representations of their symmetry groups by having representation matrices act on a (column) vector (from which point?) giving the (arbitrary) coordinates of a single vertex. Compare with known coordinates [Coxeter (1973), p. 52].

Problem I.4.c.ii–6: Explain why the symmetry group of the cube [Mirman (1995a), sec. II.2.h, p. 40] has odd permutations, like transpositions, but that of the tetrahedron does not. Also construct a tetrahedron and perform transpositions on it. Repeat for a cube. How about the other regular polyhedra?

Problem I.4.c.ii–7: That these polyhedra actually exist is clear but it is useful to construct models of them to check. It should be obvious (why?) that an object invariant under a subgroup of the rotation group is a regular solid, and conversely. What objects are invariant under the (various) cyclic and dihedral subgroups, and why is there a difference (in that they are — presumably — not regular) between objects invariant under these groups? Verify that the relevant subgroups of the symmetric groups do give the correct objects. This requires an extra step beyond showing that the groups are correct.

Problem I.4.c.ii–8: For each regular polyhedron find the numbers of two-fold, three-fold, four-fold and five-fold symmetry axes and the numbers and positions of all mirrors (planes of reflection symmetry) [Holden (1991), p. 13].

Problem I.4.c.ii–9: A regular polyhedron is one for which there is a set of finite rotations about axes through every corner (all corners are equivalent) that leave it unchanged. So its sides are invariant under rotations through a set of discrete angles — the same for all sides. Thus all edges are the same length and all angles are equal. For each such polyhedron there is a finite subgroup of SO(3). Since the latter are limited the regular polyhedra are. Thus it is impossible to take, say, a figure of sixty equal sides and rotate it so that each side generates another similar figure. Rotating a regular polygon (out of its plane) so that each side generates a similar polygon is hard and there are only three cases for which it can be done (which three?). A regular figure cannot be obtained using a six-sided polygon. Rotate a regular hexagon. What goes wrong? Repeat for other polygons; notice, and explain, the differences between ones that give regular polyhedra, and those that do not.

Problem I.4.c.ii–10: Would there be a difference if instead of SO(3), we considered O(3) [Mirman (1995a), sec. XII.4.b, p. 355]? There is a slight ambiguity here. Why? What? Does it matter?

I.4.c.iii *Dual polyhedra*

Regular polyhedra, as we see, come in pairs dual to each other (the tetrahedron is self-dual); thus there are only three of these finite subgroups, though there are five regular polyhedra. This can be shown pictorially by drawing lines connecting the centers of the faces of a solid; these lines form the edges of faces of the dual solid [Gasson (1989), p. 247; Ghyka (1977), p. 43; Holden (1991), p. 4]. The dual of the dual is the polyhedron. A polyhedron and its dual clearly have the same symmetry axes and planes, thus the same symmetry group.

The dual of a cube is an octahedron; perpendiculars through the centers of its eight faces pass through the corners of the cube. Its six diagonals go through the centers of the six faces of the cube. While the axes of maximum symmetry of a cube go through its faces, for a octahedron they lie along the diagonals.

Problem I.4.c.iii-1: A tetrahedron, with four faces and four vertices, is self-dual. Its symmetry group is a subgroup of S_4, but it has fewer symmetries than a cube, so it is proper subgroup — it is neither the group itself, nor the identity. However it is not Abelian (why?), so it must be (?) A_4. In a way, a tetrahedron is half a cube, so it has only half the symmetry operations. Is the last sentence correct?

Problem I.4.c.iii-2: Draw (or construct) a cube and an octahedron and check that they are duals [Armstrong (1988), p. 39] — either can be placed in the other with (all) vertices of one touching the centers of (all) faces of the other — so they have the same symmetry group. Why?

Problem I.4.c.iii-3: The other dual polyhedra, the dodecahedron with twelve faces and twenty vertices, and the icosahedron, with twenty faces and twelve vertices, have sixty rotational symmetries, which is the order of A_5. Rotations through vertices of the dodecahedron produce all the three-cycles of S_5 (why?), and these generate A_5 [Mirman (1995a), pb. VIII.2.c-5, p. 218], so this is the symmetry group of the two objects. Check (diagrammatically) that these are dual. Why is the symmetry group A_5 rather than S_5; why are the other permutations not symmetries?

Problem I.4.c.iii-4: Obviously the dual of the dual is the original polyhedron. However an analytic proof would be interesting.

I.4.c.iv *Extensions and generalizations*

Three dimensions, and regular figures, are familiar. To what extent do these considerations depend on such limitations?

Problem I.4.c.iv–1: There are many polyhedra with different numbers of faces and vertices, but these are not regular, their angles are not all equal and their faces not all congruent. It might be interesting, say with balls and sticks, to try various possibilities, and see why they do not work. A computer graphics program that tries to draw them, in color, can prove enlightening.

Problem I.4.c.iv–2: Regular polyhedra can be inscribed in a sphere (obviously — why?). In fact, they can be inscribed within each other [Bishop (1993), p. 40; Ghyka (1977), p. 43; Holden (1991), p. 29]. Why? However these are not the only ones that can be inscribed in a sphere. There are others, the semi-regular (Archimedean) polyhedra (sec. IV.3.b, p. 199). For these there are sets of faces, all identical with all sides of a face equal, but unlike the Platonic polyhedra, they have more than one set of faces [Coxeter (1973), p. 30; Gasson (1989), p. 228; Ghyka (1977), p. 50; Hargittai and Hargittai (1987), p. 70; Holden (1991), p. 46; Lines (1965), p. 159; Martin (1987), p. 206]. They are semi-regular because the number of faces meeting at each vertex is the same, although these faces need not be the same (in what way?). It would be interesting to derive these, finding the number, and types of faces, and the number of vertices, with the angle(s) at each. Are there relationships between the regular and semi-regular polyhedra? What (if any) are the symmetry groups of the Archimedean polyhedra? How are they related to the rotation-group subgroups? Why? To those of the Platonic polyhedra? Could such relationships be generalized to get other comparable figures (and with what properties)? Why?

Problem I.4.c.iv–3: Euler's equation can be generalized to any dimension giving Schläfli's equation [Coxeter (1973), p. 118; Ghyka (1977), p. 69]. Use this to find the number of regular polyhedra, so the finite subgroups of the rotation group, for each dimension, in particular for four dimensions [Ghyka (1977), p. 68]. Consider various projections of these into three-space [Coxeter (1973), p. 236; Senechal (1995), p. 54]. Do they give anything interesting? Could materials be constructed with these as unit cells? Might they have (economically) useful properties?

Problem I.4.c.iv–4: These arguments are firmly rooted in Euclidean geometry, emphasizing the close connection between a geometry, the objects in it, and the properties of the transformation group under which it is (and they are) invariant. Thus could there be other types of geometries, giving different sets of (analogs to) regular polyhedra? What would their transformation groups be? What are the finite subgroups (these presumably still being relevant)? It might be expected that such geometries are limited — there are only a few (sets of semisimple) Lie groups [Mirman (1995a), sec. XIII.5, p. 383]. Or are there ways of avoiding this problem, if it is relevant?

Problem I.4.c.iv-5: What postulates [Blumenthal (1980), p. 50, 149; Fejes Toth (1964), p. 124; Meserve (1983), p. 185; Mirman (1995a), sec. P.5.b, p. xviii; Neumann, Stoy and Thompson (1994), p. 116; Nikulin and Shafarevich (1987), p. 45] of Euclidean geometry lead to there being five regular solids, only five, these five? Do the same postulates give the Archimedean polyhedra? Which postulates, if any, can be changed to give other results? What would these be? What would the resultant geometries be like?

I.4.d How the rotation group determines its finite subgroups

The preceding determination of the finite rotation groups uses objects; it is interesting because, among other reasons, objects are what physics studies. There are other proofs [Armstrong (1988), p. 104; Coxeter (1973), p. 53; Fejes Toth (1964), p. 58; Heine (1993), p. 132; Lyubarskii (1960), p. 18; Sternberg (1994), p. 27; Yale (1988), p. 89], some perhaps more clearly related to SO(3). The result is important enough to make these interesting, and helpful in the understanding, so application, of crystals and molecules — and groups. And it is always possible that some objects are different from what the proof expects — all the conditions may be needed to obtain a class of materials that we have suitably limited, but may not be for a material that nature likes. Understanding the result, including different proofs, could help in finding and mastering such objects. Thus, and because of its intrinsic interest for group theory, including the rotation group and its generalizations, we consider a different proof in depth [Weyl (1989), p. 77, 149].

I.4.d.i *Orbits and poles and why there are different types*

A rotation leaves fixed all points on a line, its axis (Euler's theorem [Goldstein (1953), p. 118, 132; Jansen and Boon (1967), p. 335.]). These points include two called poles, at which it pierces the surface of a (say, unit) sphere [Murnaghan (1963), p. 329]. Some rotations leave the poles fixed, others move them, causing a pole to trace a curve on the sphere's surface — the orbit of the pole (for a finite group the orbit consists of discrete points). Each pole is obtained from any other one by a transformation of the point group, R. And each transformation takes that given one into a different pole, except for p transformations forming the cyclic group of rotations around the pole, C_p. The transformations that take a pole into a different one form a group, a subgroup of finite group R, specifically a factor group of R with respect to the cyclic subgroup, R/C_p. There is a one-to-one correspondence between poles and transformations of this subgroup.

The groups of symmetry of polyhedra have rotations that form disjoint subsets, determined by the different types of axes, each giving a different orbit [Inui, Tanabe and Onodera (1990), p. 179; Janssen (1973), p. 74; Miller (1972), p. 27; Schwarzenberger (1980), p. 45; Senechal (1990), p. 28; Sternberg (1994), p. 16; Weyl (1989), p. 149; Yale (1988), p. 91]. So the set of poles splits into subsets — orbits — with the members of each set intermixed by the rotations of C_p, but not mixed with members of other sets. How many sets — orbits — are there? The limitation on this number is a reason for that on the number of finite subgroups of the rotation group.

Problem I.4.d.i-1: For a three-sided pyramid (not a regular tetrahedron, but with a regular base so one axis of symmetry), there are two orbits; each contains just one of the two points at the opposite ends of this axis. Draw and check this — inscribe the pyramid in a sphere; the poles are the points at which it is pierced by the axis. Find and describe the orbits. Note that no symmetry rotation of the object takes one pole into the other.

Problem I.4.d.i-2: A cube has three types of rotations [Ford (1972), p. 124; Inui, Tanabe, and Onodera (1967), p. 172; Mirman (1995a), sec. II.2.h, p. 40], with axes through opposite vertices, through midpoints of opposite faces, and through the midpoints of edges diagonally opposite each other (drawn more heavily in the rightmost figure below). We draw these (for guidance with extra lines and different widths — destroying the symmetry) with one pole, P, marked; there is another at the second point at which the axis pierces the (undrawn) sphere. The lines going into each other under a rotation around the labeled $\frac{\pi}{3}$ axis are emphasized. Also the other three equivalent axes are shown as dashed lines. The rotations in each set are equivalent — they are obtained from each other by a rotation. Check the correctness of these axes,

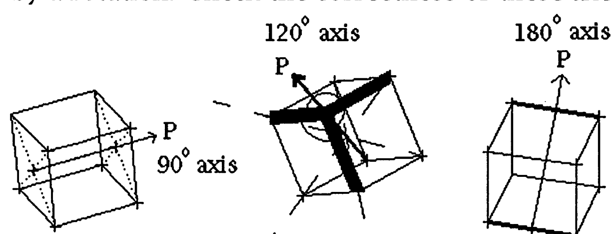

Figure I.4.d.i-1: SYMMETRY AXES OF A CUBE.

To emphasize the meaning of different types of axes, in the next diagram all three equivalent axes of the same type are labeled, and the cube, clearly invariant, is drawn before and after rotations of $\frac{2\pi}{3}$ about the heavy dashed line,

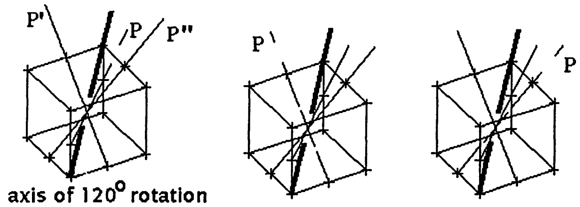

Figure I.4.d.i-2: ROTATION OF A CUBE BY $\frac{2\pi}{3}$.

The poles are given by P, P' and P"; these three points (at which the lines pierce a sphere) form the orbit. Finish the labeling for the two incomplete cases. Find the orbit of the light dashed line when the cube is rotated about each of the solid lines. Are the orbits the same? Rotations in different sets are not equivalent (a $\frac{\pi}{2}$ axis cannot be converted into a $\frac{2\pi}{3}$ one; these also pierce the sphere at different points — different poles). There are three types of orbits. Each contains only a finite number of points, and their numbers of points are different. Check that these are correct, and also the equivalence, and inequivalence, of the sets of rotations. Why do inequivalent rotations give different types of orbits? Take the poles given by an axis through the midpoints of the faces and apply to this axis, and so the poles, all rotations about the other two sets of axes. Find the orbits so generated. Compare with those for the other two types of axes. How do they differ? How many points are in each? Find each set of poles by drawing (as exemplified in the figure). It can be seen that there are three sets, the members taken into those of the same set by rotations, but the sets are not mixed. This should be clear, but checked using this as an example.

I.4.d.ii *The number of orbits and the finite subgroups*

In these examples there are two or three sets of orbits, and this is true in general — a finite subgroup of the (three-dimensional) rotation group can have only two or three types of orbits: for a nontrivial, finite group of rotations (with n elements) and a sphere centered at the point the group leaves fixed, the number k of orbits of poles is either 2 or 3 [Senechal (1990), p. 33]. Why should this be?

To find the number of inequivalent orbits we let α_i be the number of rotations leaving pole i fixed, and β_i the number of poles in orbit i (labeled by any pole i on it), the number of poles each is taken into by the group transformations. Then, with n the order of the group,

$$n = \alpha_i \beta_i, \tag{I.4.d.ii-1}$$

for each orbit; a rotation either takes a pole to another of the same orbit, there are β_i of these, or leaves it fixed, there being α_i that do. This, if not obvious, can be shown rigorously [Weyl (1989), p. 150]. Summing over the k orbits gives

$$\sum_{i=1}^{k} \beta_i(\alpha_i - 1) = \sum_{i=1}^{k} (n - \beta_i). \tag{I.4.d.ii-2}$$

Each element is a rotation so there are on the sphere two poles that it does not move. Thus (not including the identity)

$$2(n-1) = \sum_{i=1}^{k} (n - \beta_i); \tag{I.4.d.ii-3}$$

dividing the equation by n and using eq. I.4.d.ii-1, gives

$$2 - \frac{2}{n} = \sum_{i=1}^{k} (1 - \frac{1}{\alpha_i}). \tag{I.4.d.ii-4}$$

As $n \geq 2$,

$$1 \leq 2 - \frac{2}{n} < 2. \tag{I.4.d.ii-5}$$

Also $\alpha_i \geq 2$ so

$$\frac{1}{2} \leq 1 - \frac{1}{\alpha_i} < 1, \tag{I.4.d.ii-6}$$

for each i. Summing over the orbits we get

$$\frac{k}{2} \leq 2 - \frac{2}{n} \leq k. \tag{I.4.d.ii-7}$$

Problem I.4.d.ii-1: Verify that these two inequalities are consistent only for $k = 2$ or 3. Carry out the algebra explicitly for both the pyramid (pb. I.4.d.i-1, p. 18) and the cube.

Problem I.4.d.ii-2: Since the poles in each pair are not mixed, if $k = 2$ there are two poles, thus a single axis for all rotations, and the finite subgroup is isomorphic to a finite subgroup of O(2). The number of poles fixed by the identity equals the number of poles — the number obtained from one by using all group transformations — and here that equals 2. Why? This is the case discussed above (sec. I.4.a, p. 8). More interesting is $k = 3$. There are four cases. In the first the finite subgroup is a cyclic group of order n with rotations through angle $\frac{2\pi}{n}$, $n = 1, 2, 3, \ldots$. This is C_n. Show that this also gives dihedral groups D_n, $n = 2, 3, \ldots$. Why is $n = 1$ not included? For the second case each orbit consists of four points, and rotations preserve distance so any three are at the same distance from the fourth and are vertices of an equilateral triangle. Taking all sets of three-at-a-time shows that the four points are the vertices of a (regular) tetrahedron (a four-surfaced figure, with each surface an equilateral triangle). Check that this is the group of rotations of the tetrahedron, and that it has the right order. Verify that for case 3 there are six points on an orbit and the same argument gives a (regular) octahedron (an eight-surfaced figure). Show that this is also the symmetry group of a cube. How are the cube and octahedron related geometrically? The last case has twelve points on an orbit. The polyhedron is a (regular) icosahedron (with twenty surfaces). This is also the group of the dodecahedron. Why? What is the geometrical relationship between the dodecahedron and the icosahedron? Also show that these are the only possibilities, so we get [Neumann, Stoy and Thompson (1994), p. 179; Yale (1988), p. 93],

case	α_1	α_2	α_3	β_1	β_2	β_3	n	group symbol
							n	C_n
1	2	2	q	q	q	2	$2q$	D_n
2	2	3	3	6	4	4	12	T
3	2	3	4	12	8	6	24	O
4	2	3	5	30	20	12	60	I

Table I.4.d.ii-1: FINITE SUBGROUPS OF SO(3).

Problem I.4.d.ii-3: Of course no pair of these groups should be isomorphic.

Problem I.4.d.ii-4: There is a slightly different version of the algebra [Armstrong (1988), p. 106] that uses the counting theorem [Armstrong (1988), p. 98]: the number of distinct orbits equals the average number of points left fixed by a group element. Prove this. Since the elements are rotations, each leaves two points fixed, except the identity leaves more (all). The average number of points must be at least 2, and it cannot be much more than 2. We might expect the counting theorem even without a rigorous proof. Suppose that each element left fixed the same number of points. How many poles are there? The answer is the number of poles generated from some set (the orbit) by the transformations of the group times the number of orbits; this is the number of orbits times the number on each, which equals the number generated by an element. However this is not the same for each element, so the number of orbits equals the average number so generated. Each rotation leaves two poles fixed, except that the identity leaves all. From this show that $k = 2$ or 3.

Problem I.4.d.ii-5: The important point is that the number of poles is finite, since the number of rotations is, so the number of orbits is.

This strictly limits the number of orbits. It is not surprising that it is limited to 3 since there are only three different sets of rotations, the rotations around three different axes, for the three-dimensional rotation group. We might expect the number of distinct orbits to be determined by and likely equal to this (as it is) so restricting the number of finite subgroups. But does the number of distinct orbits equal the number of axes in other dimensions?

I.4.d.iii *Implications and extensions*

These results are of interest in themselves, and as a foundation for deeper thought. Limitations on the number of poles can be viewed in another way. The set of rotations leaving a pole invariant forms a subgroup of finite subgroup G of SO(3). Thus the points on the orbit correspond to equivalent subgroups, and the rotations taking one orbit point to another, a subgroup, give the cosets of G [Mirman (1995a), sec. IV.6, p. 128]. Different orbits result from cosets based on different subgroups. The number of orbits equals the number of inequivalent subgroups of G, and this is either 2 or 3.

Problem I.4.d.iii-1: For the pyramid (pb. I.4.d.i-1, p. 18), and likewise the cube (pb. I.4.d.i-2, p. 18), what are the rotation groups that leave them invariant? What are the inequivalent subgroups? Find the cosets for all, and check that they give the poles in each orbit.

Problem I.4.d.iii-2: There are several derivations of the finite subgroups of the rotation group. How are they related? Can one be obtained from others? Do they all use the same assumptions? Which is most easily generalized to higher dimensions [Brown, *et al*, (1978)]? How do these proofs and results come from the postulates of Euclidean geometry (which proofs, which results, which postulates)? Can they be generalized to other geometries? If a material is found that does not fit into the present scheme (it is not quite a crystal) which proof best shows the conditions that such material could (or must) violate?

Problem I.4.d.iii-3: It may not be intuitively obvious that there are only a small number of finite subgroups of SO(3). After all, we can partition the surface of a sphere into a set of identical regions and perform a — finite — rotation from the center of one to the center of another. And there are only a finite number (but one as large as we wish) of such regions. Start with a ring (any great circle of such a partitioned sphere), divided into equal strips. Rotate it through some angle about an axis through the center of the sphere. Notice that the distance between the center of a strip and the point to which it goes under the rotation differs for a strip near a pole (the point at which the axis pierces the sphere) and for one near the equator, thus so does the partition of the surface these strips generate. Is it possible to find a finite rotation taking one partition of a spherical surface, partitioned into identical regions, into another? How about a set of rotations about different axes?

Problem I.4.d.iii-4: Crystals are much limited. There are not many types. Of course, we cannot prevent nature from producing solids that still have symmetry, including especially either translational or rotational symmetry. The result just would not be as esthetically enjoyable — although this does not say anything about the scientific, technological or economic value. Are there such solids? Are there limitations on them? Do they have value? How do the violations of these requirements affect their value?

Problem I.4.d.iii-5: It is interesting to extend this [Coxeter (1973); Fejes Toth (1964), p. 124; Schwarzenberger (1980), p. 49; Senechal (1990), p. 103], and find the finite subgroups of the n-dimensional rotation group (and even the finite subgroups of the pseudo-rotation groups). For $n = 4$, the group, and its Lie algebra, are not simple, only semisimple [Mirman (1995a), pb. X.7.e-4, p. 300; (1995b), sec. 7.1.2, p. 124]. Does this make a difference? Are there more, or fewer, subgroups in higher dimensions? Why? What happens to these subgroups upon restriction to three dimensions? Do these provide any (useful) information about three-dimensional space, or the symmetrical objects that can exist within it? What are the (generalizations of the) regular polyhedra in higher dimensions? Find their (various) three-dimensional projections. Are these regular (in any way)? Might solids be constructed with these as unit cells? Would they be (technologically) useful? Why?

I.5 HOW TRANSLATIONS LIMIT ROTATION GROUPS OF CRYSTALS

The finite subgroups of SO(3), so crystals and molecules with symmetry, are limited. But a lattice has an additional symmetry, invariance under translations. How does this effect which rotation groups can be crystallographic symmetry groups — symmetry groups of crystals? A rotation giving a symmetry of a crystal (but not of a molecule) is limited to orders 1,2,3,4 or 6, only, that is only with angle (less than 2π),

$$\theta = 0, \frac{2\pi}{2}, \frac{2\pi}{3}, \frac{2\pi}{4}, \frac{2\pi}{6}. \tag{I.5-1}$$

(The order of element r is the power n for which

$$r^n = \text{the identity}, \tag{I.5-2}$$