ΤΕΧΝΙΚΕΣ ΟΛΟΚΛΗΡΩΣΗΣ

Ομάδα Α.

1. Υπολογίστε τα ακόλουθα ολοκληρώματα

α) $\int \frac{2x}{x^2+2x+2}dx$ β) $\int \frac{2x^2+x+1}{(x+3)(x-1)^2}dx$ γ) $\frac{3x^2+3x+1}{x^3+2x^2+2x+1}dx$

Λύση

α) $\int \frac{2x}{x^2+2x+2}dx = \int \frac{2x}{(x+1)^2+1}dx \quad (1)$

Θέτω $\left.\begin{array}{l} y = x+1 \\ dy = dx \end{array}\right\} (2)$

$(1) \overset{(2)}{\Rightarrow} \int \frac{2x}{x^2+2x+2}dx = \int \frac{2(y-1)}{y^2+1}dy =$

$= \int \frac{2y}{y^2+1}dy - 2\underbrace{\int \frac{1}{y^2+1}dy}_{\arctan y + c} = \int \frac{d(y^2+1)}{y^2+1} - 2\arctan y + c$

$= \ln(y^2+1) + c_1 - 2\arctan y + c = \ln(y^2+1) - 2\arctan y + c'$

όπου $c' = c_1 + c$

β) $\int \frac{2x^2+x+1}{(x+3)(x-1)^2}\,dx = \int\left[\frac{a}{x+3}+\frac{b}{x-1}+\frac{c}{(x-1)^2}\right]dx$

Μετά από πράξεις βρίσκουμε $a=1, b=1, c=1$, άρα

$$\int \frac{2x^2+x+1}{(x+3)(x-1)^2}\,dx = \int \frac{1}{x+3}\,dx + \int \frac{1}{x-1}\,dx + \int \frac{1}{(x-1)^2}\,dx$$

$$= \ln|x+3| + c_1 + \ln|x-1| + c_2 + \int \frac{d(x-1)}{(x-1)^2} \quad (1)$$

Θέτω $x-1=y$ άρα $\int \frac{d(x-1)}{(x-1)^2} = \int \frac{dy}{y^2} = \int y^{-2}\,dy = \frac{y^{-2+1}}{-2+1} + c_3 = -\frac{1}{y} + c_3 \quad (2)$

$(1) \overset{(2)}{\Rightarrow} \int \frac{2x^2+x+1}{(x+3)(x-1)^2}\,dx =$

$$= \ln|x+3| + \ln|x-1| - \frac{1}{x-1} + c = \ln\left(|x+3||x-1|\right) - \frac{1}{x-1} + c$$

γ) $\int \frac{3x^2+3x+1}{x^3+2x^2+2x+1}\,dx \quad (1)$

Παρατηρούμε ότι $x^3+2x^2+2x+1 = (x+1)(x^2+x+1) \quad (\tau)$

οπότε

$(1) \overset{(\tau)}{=} \int \frac{3x^2+3x+1}{(x+1)(x^2+x+1)}\,dx$ οπότε θέτω

$$\frac{3x^2+3x+1}{(x+1)(x^2+x+1)} = \frac{A}{x+1} + \frac{Bx+\Gamma}{x^2+x+1}$$

οπότε έχουμε:

$$3x^2+3x+1 = A(x^2+x+1)+(Bx+\Gamma)(x+1)$$

$$= Ax^2+Ax+A+Bx^2+Bx+\Gamma x+\Gamma$$

$$=(A+B)x^2+(A+B+\Gamma)x+A+\Gamma$$

δηλ. $\begin{cases} A+B=3 \Leftrightarrow A=3-B \quad (1) \Rightarrow \boxed{A=1} \\ A+B+\Gamma=3 \overset{(1)(2)}{\Longrightarrow} \cancel{3}-\cancel{B}+\cancel{B}+B-2=\cancel{3} \Rightarrow \boxed{B=2} \\ A+\Gamma=1 \Rightarrow \Gamma=1-A \overset{(1)}{\Rightarrow} \Gamma=1-(3-B) \Rightarrow \Gamma=B-2 \Rightarrow \boxed{\Gamma=0} \end{cases}$

Επομένως: $\dfrac{3x^2+3x+1}{(x+1)(x^2+x+1)} = \dfrac{1}{x+1} + \dfrac{2x}{x^2+x+1}$ (3)

Είναι $\displaystyle\int \frac{1}{x+1}dx = \ln|x+1|+c_1$ (4)

$$\int\frac{2x}{x^2+x+1}dx = \int\frac{2x+1}{x^2+x+1}dx - \int\frac{1}{x^2+x+1}dx$$

$$= \int\frac{d(x^2+x+1)}{x^2+x+1} - \int\frac{1}{x^2+x+1}dx$$

$$= \ln|x^2+x+1|+c_2 - \int\frac{1}{x^2+x+1}dx \quad (5)$$

Για το $\displaystyle\int\frac{1}{x^2+x+1}dx$ έχουμε ότι $\Delta_{x^2+x+1}<0$ άρα

Γράφουμε: $x^2+x+1 = \left(x+\frac{1}{2}\right)^2+\dfrac{4\cdot1-1^2}{4} = \left(x+\frac{1}{2}\right)^2+\frac{3}{4}$ (5')

(λόγω του $x^2+bx+\gamma = \left(x+\frac{b}{2}\right)^2+\dfrac{4\gamma-b^2}{4}$) και με την αντικατάσταση $x+\frac{b}{2}=\frac{\sqrt{4\gamma-b^2}}{2}y$ δηλ. την

$x+\frac{1}{2}=\frac{\sqrt{3}}{2}y$ (6) έχουμε $dx=\frac{\sqrt{3}}{2}dy$ (7)

και $x^2+x+1\overset{(5')}{=}\left(\frac{\sqrt{3}}{2}y\right)^2+\frac{3}{4}=\frac{3}{4}(y^2+1)$ (8)

Επομένως,

$$\int\frac{1}{x^2+x+1}dx\overset{(7)(8)}{=\!=}\int\frac{4}{3}\cdot\frac{1}{y^2+1}\cdot\frac{\sqrt{3}}{2}dy=$$

$$=\frac{\sqrt{3}}{2}\int\frac{1}{y^2+1}dy=\frac{\sqrt{3}}{2}\arctan y+c \quad (9)$$

Άρα, από (3), (4), (5), (9), (6) έχουμε ότι

$$\int\frac{3x^2+3x+1}{(x+1)(x^2+x+1)}dx=\ln|x+1|+\ln|x^2+x+1|-\frac{\sqrt{3}}{2}\arctan\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)+c$$

$$=\ln|(x+1)(x^2+x+1)|-\frac{\sqrt{3}}{2}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)+c$$

---

2) Υπολογίστε τα ακόλουθα ολοκληρώματα

α) $\int \frac{dx}{x^4+1}$ β) $\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$ γ) $\int \frac{dx}{x\sqrt{x^2-1}}$ δ) $\int \frac{dx}{\sqrt{1+e^x}}$

Λύση

α) Είναι: $x^4+1=(x^2+1)^2-2x^2=(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)$

Θέτω $\frac{1}{x^4+1}=\frac{Ax+B}{x^2+\sqrt{2}x+1}+\frac{\Gamma x+\Delta}{x^2-\sqrt{2}x+1} \Leftrightarrow$

$\Leftrightarrow 1=(Ax+B)(x^2-\sqrt{2}x+1)+(\Gamma x+\Delta)(x^2+\sqrt{2}x+1)$

$\Leftrightarrow \dots \Leftrightarrow \boxed{B=\Delta=1/2,\ A=\frac{1}{2\sqrt{2}},\ \Gamma=-\frac{1}{2\sqrt{2}}}$ (*)

οπότε

$$I_A=\int\frac{Ax+B}{x^2+\sqrt{2}x+1}dx=\int\frac{\frac{1}{2\sqrt{2}}x+\frac{1}{2}}{x^2+\sqrt{2}x+1}dx=\frac{1}{2\sqrt{2}}\int\frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1}dx \quad (1)$$

Από τη θεωρία, για το $\int\frac{Bx+\Gamma}{(x^2+bx+\gamma)^k}dx$, όπου το $x^2+bx+\gamma$ έχει $\Delta<0$ λύνεται θέτοντας $Bx+\Gamma=\frac{B}{2}(2x+b)+\left(\Gamma-\frac{Bb}{2}\right)$ (2) και τελικά ανάγεται στη λύση των ολοκληρωμάτων

$$\int\frac{2x+b}{(x^2+bx+\gamma)^k}dx \quad @ \quad \int\frac{1}{(x^2+bx+\gamma)^k}$$

Επομένως, λόγω της (2) έχουμε ότι: $(B\leftarrow 1,\ \Gamma\leftarrow\sqrt{2},\ b\leftarrow\sqrt{2},\ \gamma\leftarrow 1)$

οπότε $x+\sqrt{2}=\frac{1}{2}(2x+\sqrt{2})+\left(\sqrt{2}-\frac{1\cdot\sqrt{2}}{2}\right)=\frac{1}{2}(2x+\sqrt{2})+\frac{\sqrt{2}}{2}$ (2')

Οποτε η (1) $\xRightarrow{(2')}$ $I_A = \frac{1}{2\sqrt{2}}\Big[\frac{1}{2}\underbrace{\int\frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1}dx}_{I_1} + \frac{\sqrt{2}}{2}\underbrace{\int\frac{1}{x^2+\sqrt{2}x+1}dx}_{I_2}\Big]$ (3)

Για το I_1 θετουμε:

$y = x^2+\sqrt{2}x+1$ αρα $dy = (2x+\sqrt{2})dx$ (4)

οποτε $I_1 = \int\frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1}dx \overset{(4)}{=} \int\frac{dy}{y} = \ln|y| + c_1$
$= \ln|x^2+\sqrt{2}x+1| + c_1$ (5)

Για το I_2 από τη θεωρια εχουμε ότι γραφουμε το

$$x^2+bx+\gamma = \left(x+\frac{b}{2}\right)^2 + \frac{4\gamma-b^2}{4}$$

και κανουμε την αντικατάσταση

$$x+\frac{b}{2} = \frac{\sqrt{4\gamma-b^2}}{2}y$$

οποτε εχουμε οτι το $x^2+\sqrt{2}x+1 = \left(x+\frac{\sqrt{2}}{2}\right)^2 + \frac{4\cdot 1-\sqrt{2}^2}{4} =$

$= \left(x+\frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2}$ (5α)

και κάνουμε την αντικαταστασι

$x+\frac{\sqrt{2}}{2} = \frac{\sqrt{4\cdot 1-\sqrt{2}^2}}{2}y \iff x+\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}y$ (5b)

οποτε απο (5b) εχουμε $dx = \frac{\sqrt{2}}{2}dy$ (5γ)

Αρα $I_2 = \int \frac{1}{x^2+\sqrt{2}x+1}\,dx \overset{(5a)(5b)}{\underset{(5c)}{=}} \int \frac{\frac{\sqrt{2}}{2}dy}{\left(\frac{\sqrt{2}}{2}y\right)^2+\frac{1}{2}} =$

$= \frac{\sqrt{2}}{2}\int \frac{dy}{\frac{1}{2}y^2+\frac{1}{2}} = \frac{\sqrt{2}\cancel{2}}{\cancel{2}}\int \frac{dy}{y^2+1} = \sqrt{2}\cdot\arctan y + C_2$
$= \sqrt{2}\cdot\arctan\left[\frac{2}{\sqrt{2}}\left(x+\frac{\sqrt{2}}{2}\right)\right] + C_2$ (6)

Επομένως, η (3) $\overset{(5)(6)}{\Longrightarrow} I_A = \frac{1}{2\sqrt{2}}\left[\frac{1}{2}\left(\ln|x^2+\sqrt{2}x+1| + C_1\right) + \frac{\sqrt{2}}{2}\left(\sqrt{2}\arctan\left(\frac{2}{\sqrt{2}}\left(x+\frac{\sqrt{2}}{2}\right)\right) + C_2\right)\right]$

$= \frac{1}{4\sqrt{2}}\left[\ln|x^2+\sqrt{2}x+1| + 2\arctan\left(\frac{\sqrt{2}}{2}\left(x+\frac{\sqrt{2}}{2}\right)\right)\right] + C_A$ (7) $\quad C_A = C_1 + C_2$

$I_B = \int \frac{\Gamma x+\Delta}{x^2-\sqrt{2}x+1}\,dx \overset{(*)}{=} \int \frac{-\frac{1}{2\sqrt{2}}x+\frac{1}{2}}{x^2-\sqrt{2}x+1}\,dx = \frac{1}{2\sqrt{2}}\int \frac{-x+\sqrt{2}}{x^2-\sqrt{2}x+1}\,dx$ (8)

Λόγω της (2) έχουμε:

$-x+\sqrt{2} = -\frac{1}{2}(2x-\sqrt{2}) + \left(\sqrt{2} - \frac{-1(-\sqrt{2})}{2}\right)$

$= -\frac{1}{2}(2x-\sqrt{2}) + \left(\sqrt{2}-\frac{\sqrt{2}}{2}\right) = -\frac{1}{2}(2x-\sqrt{2}) + \frac{\sqrt{2}}{2}$ (8')

Οπότε, η (8) λόγω της (8') γράφεται:

$I_B = \frac{1}{2\sqrt{2}}\left[-\frac{1}{2}\underbrace{\int\frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1}\,dx}_{I_3} + \frac{\sqrt{2}}{2}\underbrace{\int\frac{1}{x^2-\sqrt{2}x+1}\,dx}_{I_4}\right]$ (9)

Για το I_3 θετουμε:

$$y = x^2 - \sqrt{2}x + 1 \quad \text{αρα} \quad dy = (2x - \sqrt{2})dx \quad (10)$$

οποτε $I_3 = \int \frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1}dx = \int \frac{dy}{y} = \ln|y| + c_3$
$= \ln|x^2 - \sqrt{2}x + 1| + c_3$ (11)

Για το I_4 ανάλογα με το I_2 εχουμε:

$$x^2 - \sqrt{2}x + 1 = \left(x - \frac{\sqrt{2}}{2}\right)^2 + \frac{4\cdot 1 - (-\sqrt{2})^2}{4}$$

$$= \left(x - \frac{\sqrt{2}}{2}\right)^2 + \frac{4-2}{4} = \left(x - \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2} \quad (12\alpha)$$

και κανουμε την αντικατάσταση

$$x - \frac{\sqrt{2}}{2} = \frac{\sqrt{4\cdot 1 - (-\sqrt{2})^2}}{2}y \Leftrightarrow x - \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}y \quad (12b)$$

οποτε από (12b) εχουμε $dx = \frac{\sqrt{2}}{2}dy$ (12c)

$$\text{Αρα} \quad I_4 = \int \frac{1}{x^2 - \sqrt{2}x + 1}dx \overset{(12a)(12b)}{\underset{(12c)}{=}} \int \frac{\frac{\sqrt{2}}{2}dy}{\left(\frac{\sqrt{2}}{2}y\right)^2 + \frac{1}{2}} =$$

$$= \frac{\sqrt{2}}{2}\int \frac{dy}{\frac{1}{2}y^2 + \frac{1}{2}} = \frac{\sqrt{2}}{\cancel{2}}\cancel{2}\int \frac{dy}{y^2+1} = \sqrt{2}\arctan y + c_4$$

$$= \sqrt{2}\arctan\left[\frac{2}{\sqrt{2}}\left(x - \frac{\sqrt{2}}{2}\right)\right] + c_4 \quad (12d)$$

Άρα η (9) $\xrightarrow[(12d)]{(11)}$ $I_B = \frac{1}{2\sqrt{2}}\left[-\frac{1}{2}\left(\ln|y| + C_3\right) + \frac{\sqrt{2}}{2}\left(\sqrt{2}\arctan y + C_4\right)\right]$

$= \frac{1}{4\sqrt{2}}\left(-\ln|y| + 2\arctan y + C_B\right)$, $C_B = C_3 + C_4$

$= \frac{1}{4\sqrt{2}}\left(-\ln|x^2 - \sqrt{2}x + 1| + 2\arctan\left(\frac{2}{\sqrt{2}}\left(x - \frac{\sqrt{2}}{2}\right)\right) + C_B\right)$ (13)

Επομένως

$$\int \frac{dx}{x^4+1} = I_A + I_B$$

$$= \frac{1}{4\sqrt{2}}\left[\ln|x^2 + \sqrt{2}x + 1| + 2\arctan\left(\frac{2}{\sqrt{2}}\left(x + \frac{\sqrt{2}}{2}\right)\right) - \ln|x^2 - \sqrt{2}x + 1| + 2\arctan\left(\frac{2}{\sqrt{2}}\left(x - \frac{\sqrt{2}}{2}\right)\right)\right] + C.$$

β) $\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$

Θέτω $u=\sqrt[6]{x} \Leftrightarrow u^6=x$

$\Leftrightarrow dx=6u^5\,du.$

Άρα $u^2=\sqrt[6]{x}^2=\sqrt[3]{x}$, $u^3=\sqrt[6]{x^3}=\sqrt{x}$. (*)

οπότε $\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}=\int \frac{6u^5\,du}{u^3+u^2}=6\int \frac{u^3\,du}{u+1}$ (1)

Θέτω $y=u+1 \Rightarrow dy=du$ (2)

$\int \frac{u^3\,du}{u+1}=\int \frac{(y-1)^3\,dy}{y}=\int \frac{y^3-3y^2+3y-1}{y}\,dy=$

$=\int \left(y^2-3y+3-\frac{1}{y}\right)dy=\int y^2dy-3\int y\,dy+3\int dy-\int \frac{dy}{y}$

$=\frac{y^3}{3}-3\frac{y^2}{2}+3y-\ln|y|+c$

$\overset{(2)}{=}\frac{(u+1)^3}{3}-3\frac{(u+1)^2}{2}+3(u+1)-\ln|u+1|+c$ (3)

(1) $\overset{(3)(*)}{\Longrightarrow} \int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}=6\cdot\left[\frac{(\sqrt[6]{x}+1)^3}{3}-\frac{3}{2}(\sqrt[6]{x}+1)^2+3(\sqrt[6]{x}+1)-\ln|\sqrt[6]{x}+1|\right]+c.$

γ) $\int \frac{dx}{x\sqrt{x^2-1}}$

Θέτω $u=\sqrt{x^2-1}$ (1) $\Rightarrow u^2=x^2-1 \Leftrightarrow x^2=u^2+1$ (2)

άρα $du=\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2-1}}dx \Rightarrow \frac{\sqrt{x^2-1}}{x}du=dx \Leftrightarrow$

$\Leftrightarrow \frac{\sqrt{x^2-1}}{x^2}du=\frac{dx}{x} \overset{(1)(2)}{\Leftrightarrow} \frac{dx}{x}=\frac{u\,du}{u^2+1}$ (3)

οπότε $\int\frac{dx}{x\sqrt{x^2-1}} \overset{(3)}{=} \int\frac{1}{u}\cdot\frac{u\,du}{u^2+1}=\int\frac{du}{u^2+1}=$

$=\arctan u+c \overset{(1)}{=} \arctan(\sqrt{x^2-1})+c.$

δ) $\int\frac{dx}{\sqrt{1+e^x}}$

Θέτω $u=\sqrt{1+e^x} \Rightarrow du=\frac{e^x}{2\sqrt{1+e^x}}dx \Rightarrow \frac{2}{e^x}du=\frac{dx}{\sqrt{1+e^x}} \Rightarrow$

$\Downarrow$

$u^2=1+e^x \Rightarrow e^x=u^2-1$ (1)

$\overset{(1)}{\Rightarrow} \frac{2du}{u^2-1}=\frac{dx}{\sqrt{1+e^x}}$

άρα $\int\frac{dx}{\sqrt{1+e^x}}=2\int\frac{du}{u^2-1}$ (2)

$\frac{1}{u^2-1}=\frac{A}{u-1}+\frac{B}{u+1} \Leftrightarrow A(u+1)+B(u-1)=1 \Leftrightarrow Au+A+Bu-B=1$

$\Leftrightarrow (A+B)u+(A-B)=1 \Leftrightarrow$

$\begin{cases}A+B=0\end{cases}$ $A=-B$ $A=-B$ $A=-B$ $A=\frac{1}{2}$

Άρα, η (2) $\Rightarrow \displaystyle\int \frac{dx}{\sqrt{1+e^x}} = 2\left[\int \frac{1}{u-1}\,du + \int \frac{1}{u+1}\,du\right] =$

$= 2\left(\ln|u-1| + \ln|u+1| + c\right)$

$= 2\left(\ln|u^2-1| + c\right) \overset{(1)}{=} 2\ln e^x + c = 2x + c$

---

3) Υπολογίστε τα ολοκληρώματα

α) $\int \cos^3 x\,dx$ β) $\int \cos^2 x \cdot \sin^3 x\,dx$ γ) $\int \tan^2 x\,dx$ δ) $\int \sqrt{\tan x}\,dx$

ε) $\displaystyle\int \frac{dx}{\cos^4 x}$

Λύση

α) $\int \cos^3 x\,dx = \int \cos^2 x \cdot \cos x\,dx = \int (1-\sin^2 x)(\sin x)'\,dx$ (1)

Θέτω $u = \sin x \Rightarrow du = (\sin x)'\,dx$ (2)

$(1) \overset{(2)}{=} \int (1-u^2)\,du = \int du - \int u^2\,du = u - \frac{u^3}{3} + c =$

$= \sin x - \frac{\sin^3 x}{3} + c$

β) $\int \cos^2 x \cdot \sin^3 x\,dx = \int \cos^2 x \cdot \sin^2 x \cdot \sin x\,dx =$

$= \int \cos^2 x \cdot (1-\cos^2 x) \cdot (-1)(\cos x)'\,dx$ (1)

Θέτω $u = \cos x \Rightarrow du = (\cos x)'\,dx$ (2)

$(1) \overset{(2)}{=} \int u(1-u)\,du = -\int u\,du + \int u^2\,du = -\frac{u^2}{2} + \frac{u^3}{3} + c = -\frac{\cos^2 x}{2} + \frac{\cos^3 x}{3} + c$

ε) $I = \int \frac{1}{\cos^4 x}\,dx = \int (\tan x)' \frac{1}{\cos^2 x}\,dx =$

$= \frac{\tan x}{\cos^2 x} - \int \tan x \cdot \left(\frac{1}{\cos^2 x}\right)' dx =$

$\left(\frac{1}{\cos^2 x}\right)' = (\cos^{-2} x)' = -2\cos^{-3} x\,(\cos x)' = \frac{2\sin x}{\cos^3 x}$

$= \frac{\tan x}{\cos^2 x} - \int \tan x \cdot \frac{2\sin x}{\cos^3 x}\,dx =$

$= \frac{\tan x}{\cos^2 x} - \int 2\,\frac{\sin x}{\cos x} \cdot \frac{\sin x}{\cos^3 x}\,dx =$

$= \frac{\tan x}{\cos^2 x} - 2\int \frac{\sin^2 x}{\cos^4 x}\,dx = \frac{\tan x}{\cos^2 x} - 2\int \frac{1-\cos^2 x}{\cos^4 x}\,dx$

$= \frac{\tan x}{\cos^2 x} - 2\int \frac{1}{\cos^4 x}\,dx + 2\int \frac{1}{\cos^2 x}\,dx$

$= \frac{\tan x}{\cos^2 x} - 2I + 2\tan x + c$

$\Leftrightarrow 3I = \frac{\tan x}{\cos^2 x} + 2\tan x + c$

$\Leftrightarrow I = \frac{1}{3}\left(\frac{\tan x}{\cos^2 x} + 2\tan x + c\right)$

4). Χρησιμοποιώντας ολοκλήρωση κατά μέρη, ν.δ.ο. $\forall n \in \mathbb{N}$

$$\int \frac{dx}{(x^2+1)^{n+1}} = \frac{1}{2n} \cdot \frac{x}{(x^2+1)^n} + \frac{2n-1}{2n} \int \frac{dx}{(x^2+1)^n}$$

Λύση

$$I_n = \int \frac{dx}{(x^2+1)^n} = \int (x)' \cdot \frac{1}{(x^2+1)^n} dx =$$

$$= \frac{x}{(x^2+1)^n} - \int \left((x^2+1)^{-n}\right)' x \, dx =$$

$$= \frac{x}{(x^2+1)^n} - (-n) \cdot \int \frac{2x}{(x^2+1)^{n+1}} \cdot x \, dx$$

$$= \frac{x}{(x^2+1)^n} + 2n \int \frac{x^2}{(x^2+1)^{n+1}} dx = \frac{x}{(x^2+1)^n} + 2n \int \frac{x^2+1-1}{(x^2+1)^{n+1}} dx$$

$$= \frac{x}{(x^2+1)^n} + 2n \int \frac{1}{(x^2+1)^n} dx - 2n \int \frac{1}{(x^2+1)^{n+1}} dx$$

$$= \frac{x}{(x^2+1)^n} + 2n I_n - 2n I_{n+1} \Rightarrow \cdots \Rightarrow$$

$$\Rightarrow I_{n+1} = \frac{1}{2n} \frac{x}{(x^2+1)^n} + \frac{2n-1}{2n} I_n$$

5) Υπολογίστε τα ακόλουθα ολοκληρώματα

α) $\int \frac{x^2}{(x^2-4)(x^2-1)}dx$ β) $\int \frac{1}{(1+x)(1+x^2)}dx$ γ) $\int x \log x \, dx$

δ) $\int x \cos x \, dx$, ε) $\int e^x \sin x \, dx$, στ) $\int x \sin^2 x \, dx$

ζ) $\int \log(x+\sqrt{x}) dx$ η) $\int \frac{1}{x\sqrt{1-x^2}} dx$, θ) $\int \frac{x+4}{(x^2+1)(x-1)} dx$

ι) $\int \frac{x}{1+\sin x} dx$ κ) $\int \frac{\cos^3 x}{\sin^2 x} dx$ λ) $\int \frac{dx}{(x^2+2x+2)^2}$

Λύση

α) $\int \frac{x^2}{(x^2-4)(x^2-1)}dx$

$$\frac{x^2}{(x^2-4)(x^2-1)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{\Gamma}{x-1} + \frac{\Delta}{x+1}$$ Λύνεται εύκολα...

β). $\int \frac{1}{(1+x)(1+x^2)}dx$

Θέτω $\frac{1}{(1+x)(1+x^2)} = \frac{A}{1+x} + \frac{Bx+\Gamma}{1+x^2}$ λύνεται εύκολα..

→

γ) $\int x \log x \, dx$

Ολοκλήρωση κατά μέρη

$$\int x \log x \, dx = \frac{1}{2} \int (x^2)' \log x \, dx = \frac{x^2 \log x}{2} - \frac{1}{2}\int x \, dx =$$

$$= \frac{x^2 \log x}{2} - \frac{x^2}{4} + c$$

δ) $\int x \cos x \, dx$

Ολοκλήρωση κατά μέρη

$$\int x \cos x \, dx = \int x (\sin x)' dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x + c$$

ε) $\int e^x \sin x \, dx$

Ολοκλήρωση κατά μέρη

$$I = \int e^x \sin x \, dx = \int (e^x)' \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx =$$

$$= e^x \sin x - \int (e^x)' \cos x \, dx = e^x \sin x - e^x \cos x + \int e^x (\cos x)' dx =$$

$$= e^x (\sin x - \cos x) - \int e^x \sin x \, dx = e^x (\sin x - \cos x) - I$$

$$\Rightarrow 2I = e^x (\sin x - \cos x)^{+c} \Leftrightarrow I = \frac{e^x (\sin x - \cos x)}{2} + c$$

6τ). $\int x\sin^2 x\,dx$

Λύση: $\cos 2x = 1-2\sin^2 x \Leftrightarrow 2\sin^2 x = 1-\cos 2x \Leftrightarrow \sin^2 x = \dfrac{1-\cos 2x}{2}$

οπότε

$$\int x\sin^2 x\,dx = \int x\cdot\frac{1-\cos 2x}{2}\,dx = \int \frac{x}{2}\,dx - \int \frac{x\cos 2x}{2}\,dx \quad (1)$$

$$\int \frac{x}{2}\,dx = \frac{1}{2}\int x\,dx = \frac{1}{2}\,\frac{x^2}{2} + c = \frac{x^2}{4} + c \quad (2)$$

Για το: $\displaystyle\int \frac{x\cos 2x}{2}\,dx \quad (3)$

Θέτω: $\left.\begin{array}{l} u = 2x \Rightarrow du = 2dx \Rightarrow dx = \dfrac{du}{2} \\ x = \dfrac{u}{2} \end{array}\right\} \quad (4)$

$$(3) \overset{(4)}{\Rightarrow} \int \frac{x\cos 2x}{2}\,dx = \int \frac{u}{2}\,\frac{\cos u}{2}\,\frac{du}{2} = \frac{1}{8}\int u\cos u\,du \overset{(5)}{=} \ldots =$$

$$= \frac{1}{8}\left(u\sin u + \cos u + c\right) = \frac{1}{8}\left(2x\sin 2x + \cos 2x + c\right)$$

f) $\int \log(x+\sqrt{x})\,dx = \int (x)' \log(x+\sqrt{x})\,dx =$

$= x\log(x+\sqrt{x}) - \int \frac{x}{x+\sqrt{x}}\left(1+\frac{1}{2\sqrt{x}}\right)dx$ (1)

Για το $\int \frac{x}{x+\sqrt{x}}\left(1+\frac{1}{2\sqrt{x}}\right)dx$ (2) θέτω

$\left.\begin{aligned} u=\sqrt{x} &\Rightarrow u^2 = x \\ &\Rightarrow 2u\,du = dx \end{aligned}\right\}$ (3)

$(2) \overset{(3)}{=} \int \frac{u^2}{u^2+u}\left(1+\frac{1}{2u}\right)2u\,du =$

$= \int \frac{\cancel{2}u^3(2u+1)}{u(u+1)\cancel{2}u}\,du = \int \frac{u(2u+1)}{u+1}\,du$

$= \int \frac{2u^2+u}{u+1}\,du = 2\int \frac{u^2}{u+1}\,du + \int \frac{u}{u+1}\,du$ (4)

Όμως: $\int \frac{u}{u+1}\,du = \int u\cdot(\ln|u+1|)'\,du = u\ln|u+1| - \int u'\ln(u+1)\,du =$

$u\ln|u+1| - \int \ln|u+1|\,du = u\ln(u+1) - \int (u+1)'\ln(u+1)\,du =$

$= u\ln|u+1| - (u+1)\ln(u+1) + \int \frac{u+1}{u+1}\,du = -\ln|u+1| + \int du =$

$= -\ln|u+1| + u + c$ (5)

$$\int \frac{u^2}{u+1}\,du = \int u^2 (\ln u)'\,du = u^2 \ln u - 2\int u \ln u\,du \overset{(\gamma)}{=}$$

$$= u^2 \ln u - 2\left(\frac{u^2 \ln u}{2} - \frac{u^2}{4} + c\right) \quad (6)$$

$$(4) \overset{(5)(6)}{=} -2\ln|u+1| + 2u + \cancel{u^2 \ln u} - \cancel{u^2 \ln u} + \frac{u^2}{2} + c$$

$$= -2\ln|u+1| + 2u + \frac{u^2}{2} + c$$

$$\overset{(3)}{=} -2\ln|\sqrt{x}+1| + 2\sqrt{x} + \frac{x}{2} + c$$

η) $\int \frac{1}{x\sqrt{1-x^2}}\,dx$. Θέτω $u = \sqrt{1-x^2} \Rightarrow u^2 = 1-x^2 \Leftrightarrow x^2 = 1-u^2 \quad (*)$

άρα $du = \frac{-2x}{2\sqrt{1-x^2}}\,dx \Leftrightarrow du = \frac{-x}{\sqrt{1-x^2}}\,dx$

$$\Leftrightarrow \frac{du}{x^2} = \frac{-x\,dx}{x^2\sqrt{1-x^2}} \Leftrightarrow \frac{du}{x^2} = \frac{-dx}{x\sqrt{1-x^2}}$$

$$\overset{(*)}{\Leftrightarrow} -\frac{du}{1-u^2} = \frac{dx}{x\sqrt{1-x^2}} \quad (1)$$

Άρα $\int \frac{1}{x\sqrt{1-x^2}}\,dx \overset{(1)}{=} -\int \frac{du}{1-u^2} = -\int \frac{du}{(1-u)(1+u)} \quad (2)$

Θέτω

$$\frac{1}{(1-u)(1+u)} = \frac{A}{1-u} + \frac{B}{1+u} \Leftrightarrow A(1+u) + B(1-u) = 1 \Leftrightarrow$$

$$\Leftrightarrow A + Au + B - Bu = 1 \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} A+B=1 \end{cases} \Leftrightarrow \begin{cases} 2A=1 \end{cases} \Leftrightarrow \begin{cases} A=1/2 \end{cases}$$

Άρα,

$$(2) = -\frac{1}{2}\int \frac{1}{1+u}\,du - \frac{1}{2}\int \frac{1}{1-u}\,du = -\frac{1}{2}\left(\ln|1+u| + \ln|1-u|\right) + c$$

$$= -\frac{1}{2}\ln|1-u^2| + c$$

θ) $\int \frac{x+4}{(x^2+1)(x-1)}\,dx$

Θέλω $\frac{x+4}{(x^2+1)(x-1)} = \frac{A}{x-1} + \frac{Bx+\Gamma}{x^2+1}$ λύνεται εύκολα

ι) $\int \frac{x}{1+\sin x}\,dx$

$\frac{x}{2} = \arctan y \Leftrightarrow x = 2\arctan y \quad (1')$

Θέτω $y = \tan\frac{x}{2}$ ① $\Rightarrow dy = \frac{1}{\cos^2\frac{x}{2}} \cdot \left(\frac{x}{2}\right)' dx \Leftrightarrow dy = \frac{1}{2\cos^2\frac{x}{2}}\,dx$

$\Leftrightarrow 2\cos^2\frac{x}{2}\,dy = dx \Leftrightarrow 2\cdot\frac{1}{1+\tan^2\frac{x}{2}}\,dy = dx \overset{①}{\Leftrightarrow}$

$\Leftrightarrow \frac{2}{1+y^2}\,dy = dx$ ②

και $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$ (γιατί $\sin 2x = 2\sin x\cos x$)

$= 2\cdot\sqrt{1-\cos^2\frac{x}{2}}\cdot\sqrt{\frac{1}{1+\tan^2\frac{x}{2}}}$ (γιατί $\sin^2x+\cos^2x=1$, $1+\tan^2x = \frac{1}{\cos^2x}$)

$$= 2 \cdot \sqrt{1 - \frac{1}{1+\tan^2\frac{x}{2}}} \cdot \sqrt{\frac{1}{1+\tan^2\frac{x}{2}}}$$

$$= 2 \sqrt{\frac{\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}} \cdot \sqrt{\frac{1}{1+\tan^2\frac{x}{2}}}$$

$$= 2 \sqrt{\frac{\tan^2\frac{x}{2}}{\left(1+\tan^2\frac{x}{2}\right)^2}} = 2 \frac{\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} \overset{(1)}{=} \frac{2y}{1+y^2} \quad (3)$$

Άρα,

$$\int \frac{x}{1+\sin x}\,dx \overset{(1')(2)(3)}{=\!=\!=} \int 2\arctan y \cdot \frac{1}{1+\frac{2y}{1+y^2}} \cdot \frac{2}{1+y^2}\,dy$$

$$= 4\int \arctan y \cdot \frac{\cancel{1+y^2}}{1+2y+y^2} \cdot \frac{1}{\cancel{1+y^2}}\,dy$$

$$= 4\int \arctan y \cdot \frac{1}{(y+1)^2}\,dy = 4\int \arctan y \cdot \left(-\frac{1}{1+y}\right)' dy$$

$$= -4\,\frac{\arctan y}{1+y} + 4 \cdot \int (\arctan y)' \frac{1}{1+y}\,dy =$$

$$= -\frac{4\arctan y}{1+y} + 4\int \frac{1}{(1+y^2)} \cdot \frac{1}{1+y}\,dy \quad (4)$$

Θέτω $\frac{1}{(1+y^2)(1+y)} = \frac{A}{1+y} + \frac{B\cancel{x}+\Gamma}{1+y^2} \Leftrightarrow$

$\Leftrightarrow A(1+y^2) + (By+\Gamma)(1+y) = 1 \Leftrightarrow \begin{cases} A= \\ B= \quad \text{--} \\ \Gamma= \quad \text{--} \end{cases}$ (5)

$\text{η } (4) \overset{(5)}{\Longrightarrow}$ λύνεται εύκολα

κ) $\int \frac{\cos^3 x}{\sin^2 x}\,dx = \int \frac{1-\sin^2 x}{\sin^2 x}(\sin x)'dx$ (1)

Θέτω $u=\sin x \Rightarrow du=(\sin x)'dx$ (2)

$(1) \overset{(2)}{=} \int \frac{1-u^2}{u^2}\,du = \int \frac{1}{u^2}\,du - \int du = \frac{u^{-2+1}}{-2+1} - u + c$

$= -\frac{1}{u} - u + c = -\frac{1}{\sin x} - \sin x + c$

λ) $\int \frac{dx}{(x^2+2x+2)^2}$

Αντικατάσταση $y=x+1$ ο ο ο ο ο

---

6) Υπολογίστε τα ολοκληρώματα.

$\alpha = \int \sin(\log x)\,dx$, $\beta)\ \int \frac{1}{x\sqrt{x}} \log(1-x)\,dx$

Λύση

α) $\int \sin(\log x)\,dx$

Θέτω $u = \log x \Rightarrow x = e^u \Rightarrow dx = e^u du$

άρα $\int \sin(\log x)\,dx = \int \sin u \cdot e^u\,du \overset{\text{πχ(γ) σελ.260}}{=}$

$$= \frac{e^u(\sin u - \cos u)}{2} + c$$

$$= \frac{x\cdot(\sin(\log x) - \cos(\log x))}{2} + c$$

β) $\int \frac{1}{x\sqrt{x}} \log(1-x)\,dx = -2\int \left(\frac{1}{\sqrt{x}}\right)' \log(1-x)\,dx =$

$$= -\frac{2\log(1-x)}{\sqrt{x}} - 2\int \frac{1}{\sqrt{x}}\,\frac{1}{1-x}\,dx \quad (1)$$

Θέτω $u = \sqrt{x} \Rightarrow x = u^2 \Rightarrow dx = 2u\,du$ (2)

άρα $\int \frac{1}{\sqrt{x}}\,\frac{1}{1-x}\,dx = \int \frac{1}{\cancel{u}} \cdot \frac{1}{1-u^2}\, 2\cancel{u}\,du = 2\int \frac{du}{1-u^2}$ (3)

$A(1-u)+B(1+u)=1 \Leftrightarrow \{A+B=1\} \; A=B=1/2$

Άρα $(3) = \int \frac{1}{1+u}\,du + \int \frac{1}{1-u}\,du = \ln|1+u| + \ln|1-u| + c$

$= \ln|1-u^2| + c$ (4)

$(1) \overset{(4)}{\Longrightarrow} \int \frac{1}{x\sqrt{x}} \log(1-x)\,dx = -\frac{2\log(1-x)}{\sqrt{x}} - 2\cdot\log|1-x| + c$

---

7) Υπολογίστε τα ολοκληρώματα:

α) $\int \frac{x \arctan x}{(1+x^2)^2}\,dx$ β) $\int \frac{x e^x}{(1+x)^2}\,dx$

<u>Λύση</u>

α) $\int \frac{x \arctan x}{(1+x^2)^2}\,dx = -\frac{1}{2}\int \left(\frac{1}{1+x^2}\right)' \arctan x\,dx =$

$= -\frac{1}{2}\,\frac{\arctan x}{1+x^2} + \frac{1}{2}\int \frac{1}{(1+x^2)^2}\,dx$

Για το τελευταίο ολοκλήρωμα χρησιμοποιούμε τον αναγωγικό τύπο της άσκ. 4.

β) $\displaystyle\int \frac{xe^x}{(1+x)^2}\,dx = -\int \left(\frac{1}{1+x}\right)' xe^x\,dx =$

$\displaystyle= -\frac{xe^x}{1+x} + \int \frac{1}{1+x}(xe^x)'\,dx =$

$\displaystyle= -\frac{xe^x}{1+x} + \int \frac{1}{1+x}(1+x)e^x\,dx = \frac{-xe^x}{1+x} + e^x + c$

---

8) Υπολογίστε τα ολοκληρώματα

α) $\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx$ β) $\displaystyle\int \frac{\log(\tan x)}{\cos^2 x}\,dx$

Λύση

α) $\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx \quad (1)$

Θέτω $u=e^x \Rightarrow du=e^x dx \quad (2)$

$\displaystyle (1) \overset{(2)}{=} \int \frac{du}{1+u^2} = \arctan u + c = \arctan e^x + c$

$\longrightarrow$

β) $\int \frac{\log(\tan x)}{\cos^2 x}\,dx \quad (1)$

Θέτω $\left.\begin{aligned} & u = \tan x \\ & du = (\tan x)'\,dx \Leftrightarrow du = \frac{1}{\cos^2 x}\,dx \end{aligned}\right\} (2)$

$$(1) \overset{(2)}{=} \int \log u \, du = \int u' \log u \, du = u \log u - \int u \cdot \frac{1}{u}\, du$$

$$= u \log u - u + c$$

$$= \tan x \cdot \log(\tan x) - \tan x + c$$

9) Υπολογίστε τα ολοκληρώματα

α) $\displaystyle\int_0^{\pi/4} \frac{x}{\cos^2 x}\,dx$ β) $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{\tan^3 x}{\cos^3 x}\,dx$

γ) $\displaystyle\int_0^5 x \log\left(\sqrt{1+x^2}\right) dx$ δ) $\displaystyle\int_0^{\pi/4} x \tan^2 x\,dx$

Λύση (Υπολογίζουμε πρώτα τα αόριστα ολοκληρώματα)

α) $\displaystyle\int \frac{x}{\cos^2 x}\,dx = x\tan x - \int \tan x\,dx = x\tan x + \log(\cos x) + c$

οπότε $\displaystyle\int_0^{\pi/4} \frac{x}{\cos^2 x}\,dx = x\tan x\Big|_0^{\pi/4} + \log(\cos x)\Big|_0^{\pi/4}$

$$= \frac{\pi}{4}\tan\frac{\pi}{4} - 0 + \log\left(\cos\frac{\pi}{4}\right) - \log(\cos 0)$$

$$= \frac{\pi}{4}\cdot\frac{\sqrt{2}}{2} + \log\frac{\sqrt{2}}{2} - \cancelto{0}{\log 1} = \frac{\pi\sqrt{2}}{8} + \log\frac{\sqrt{2}}{2}$$

---

8) $\displaystyle\int \frac{\tan^3 x}{\cos^3 x}\,dx = \int \frac{\frac{\sin^3 x}{\cos^3 x}}{\cos^3 x}\,dx = \int \frac{\sin^3 x}{\cos^6 x}\,dx$

$$= \int \frac{(1-\cos^2 x)\sin x\,dx}{\cos^6 x} \quad (1)$$

Θέτω $u = \cos x \Rightarrow du = -\sin x\,dx$ (2)

$$(1) \stackrel{(2)}{=} \int \frac{(1-u^2)(-1)\,du}{u^6} = \int \frac{u^2-1}{u^6}\,du =$$

$$= \int \frac{1}{u^4}\,du - \int \frac{1}{u^6}\,du = \frac{u^{-4+1}}{-4+1} - \frac{u^{-6+1}}{-6+1} + C$$

$$= -\frac{1}{3u^3} + \frac{1}{5u^5} + C = -\frac{1}{3\cos^3 x} + \frac{1}{5\cos^5 x} + C$$

$$\int_{-\pi/4}^{\pi/4} \frac{\tan^3 x}{\cos^3 x}\,dx = -\frac{1}{3\cos^3 x}\Bigg|_{-\pi/4}^{\pi/4} + \frac{1}{5\cos^2 x}\Bigg|_{-\pi/4}^{\pi/4} = \ldots$$

γ) $\int x \log\sqrt{1+x^2}\,dx$ ①

Θέτω $u=\sqrt{1+x^2} \Leftrightarrow u^2=1+x^2 \Leftrightarrow 2u\,du=2x\,dx$

$\Leftrightarrow u\,du = x\,du$ ②

$(1) \overset{(2)}{=} \int \log u \cdot u\,du \underset{\text{Πχ 1 σελ. 260}}{=}$

$= \frac{u^2 \log u}{2} - \frac{u^2}{4} + c = \frac{(1+x^2)\log\sqrt{1+x^2}}{2} - \frac{1+x^2}{4} + c$ ③

και $\int_0^5 x\log\sqrt{1+x^2}\,dx = \frac{(1+x^2)\log\sqrt{1+x^2}}{2}\Big|_0^5 - \frac{1+x^2}{4}\Big|_0^5$.

δ) $\int x \tan^2 x\,dx = \int x\,\frac{1}{\cos^2 x}\,dx - \int x\,dx \overset{(\alpha)}{=}$

$= x\tan x + \log(\cos x) + c_1 - \frac{x^2}{2} + c_2$

άρα $\int_0^{\pi/4} x\tan^2 x = x\tan x\Big|_0^{\pi/4} + \log(\cos x)\Big|_0^{\pi/4} - \frac{x^2}{2}\Big|_0^{\pi/4} = \dots$
