Motion and Dynamics of a Rigid Body

Bin Yao

[1] School of Mechanical Engineering, Purdue University, West Lafayette, Indiana 47907-2088, USA. byao@purdue.edu
[2] Chang Jiang Chair Professor, The State Key Laboratory of Fluid Power and Mechatronic Systems, Zhejiang University, Hangzhou 310027, China

1 Introduction

This document clarifies the definition of posture and rotation of a rigid body via Euler Angles or Row-Pitch-Yaw Angles and their uses in describing the motion and dynamics of a rigid body.

2 Rotation of a Rigid Body

This section considers the rotation of a rigid body with respect to a point O on the body which is fixed in space and presents two different perspectives in describing such a rotation operation.

2.1 Rotation Matrix

With respect to the fixed point O, any other point P on the rigid body is described by the vector $\overrightarrow{OP}$. Let $\mathbf{r}$ and $\mathbf{r'}$ be the vector $\overrightarrow{OP}$ before and after the rotation respectively, as shown in Fig.1 and Fig.2. A rotation operation of the rigid body with respect to O can thus be described by a mapping $\mathbf{R}$ from $\mathbf{r}$ to $\mathbf{r'}$ as

$$\mathbf{R}: \ \mathbf{r} \rightarrow \mathbf{r}' = \mathbf{R}\mathbf{r} \tag{1}$$

It is easy to verify that the rotation operator $\mathbf{R}$ is linear and norm (or length) preserving.

The mathematical expression of a linear operator in a finite dimensional space is a matrix. With the position vectors $\mathbf{r}$ and $\mathbf{r'}$ in three dimensional space, the rotational operator in (1) is described by a 3×3 matrix for any chosen set of basis vectors for $\mathbf{r}$ and $\mathbf{r'}$. To make it concrete, let us attach a Cartesian coordinate frame to the rigid body at the fixed point O (i.e., a body-fixed frame) and denote such a coordinate frame before the rotation as $Oxyz$ shown in Fig.1 and after the rotation as $Ox'y'z'$ shown in Fig.2.

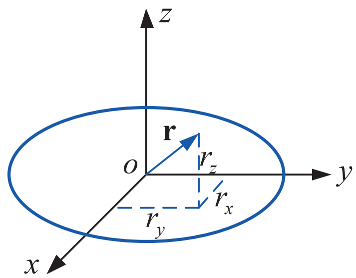

Fig. 1. Before Rotation

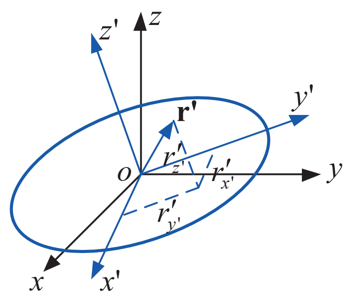

Fig. 2. Before Rotation

Then, the axis unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ of $Oxyz$ and the axis unit vectors $\mathbf{i'}, \mathbf{j'}, \mathbf{k'}$ of $Ox'y'z'$ represent two different sets of bases for the three dimensional position vector $\mathbf{r}$ and $\mathbf{r'}$. Denote the coordinates of $\mathbf{r}$ and $\mathbf{r'}$ in the frame $Oxyz$ as $\mathbf{r}_{xyz} = [r_x, r_y, r_z]^T$ and $\mathbf{r'}_{xyz} = [r'_x, r'_y, r'_z]^T$ respectively, i.e.,

$$\mathbf{r} = [\mathbf{i}\ \mathbf{j}\ \mathbf{k}] \underbrace{\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}}_{\mathbf{r}_{xyz}}, \qquad \mathbf{r}' = [\mathbf{i}\ \mathbf{j}\ \mathbf{k}] \underbrace{\begin{bmatrix} r'_x \\ r'_y \\ r'_z \end{bmatrix}}_{\mathbf{r'}_{xyz}} \tag{2}$$

Similarly let $\mathbf{r}_{x'y'z'} = [r_{x'}, r_{y'}, r_{z'}]^T$ and $\mathbf{r'}_{x'y'z'} = [r'_x, r'_y, r'_z]^T$ be the coordinates of $\mathbf{r}$ and $\mathbf{r'}$ in the frame $Ox'y'z'$ respectively, i.e.,

$$\mathbf{r} = [\mathbf{i'}\ \mathbf{j'}\ \mathbf{k'}] \underbrace{\begin{bmatrix} r_{x'} \\ r_{y'} \\ r_{z'} \end{bmatrix}}_{\mathbf{r}_{x'y'z'}}, \qquad \mathbf{r}' = [\mathbf{i'}\ \mathbf{j'}\ \mathbf{k'}] \underbrace{\begin{bmatrix} r'_{x'} \\ r'_{y'} \\ r'_{z'} \end{bmatrix}}_{\mathbf{r'}_{x'y'z'}} \tag{3}$$

With these notations, it is clear that $\mathbf{r'}_{x'y'z'} = \mathbf{r}_{xyz}$ as $Oxyz$ and $Ox'y'z'$ represent the same body-fixed coordinate frame before and after the rotation. Denote the coordinates of the unit vectors $\mathbf{i'}, \mathbf{j'}, \mathbf{k'}$ in the frame $Oxyz$ as $\mathbf{i'}_{xyz} = [i'_x, i'_y, i'_z]^T, \mathbf{j'}_{xyz} = [j'_x, j'_y, j'_z]^T$, and $\mathbf{k'}_{xyz} = [k'_x, k'_y, k'_z]^T$ respectively. Then,

$$[\mathbf{i'}\ \mathbf{j'}\ \mathbf{k'}] = [\mathbf{i}\ \mathbf{j}\ \mathbf{k}] \underbrace{\begin{bmatrix} i'_x & j'_x & k'_x \\ i'_y\,, & j'_y\,, & k'_y \\ i'_z & j'_z & k'_z \end{bmatrix}}_{\mathbf{R}_{xyz}} \tag{4}$$

From (3),

$$
\begin{aligned}
\mathbf{r} = [\mathbf{i}'\ \mathbf{j}'\ \mathbf{k}'] \underbrace{\begin{bmatrix} r_{x'} \\ r_{y'} \\ r_{z'} \end{bmatrix}}_{\mathbf{r}_{x'y'z'}} &= [\mathbf{i}\ \mathbf{j}\ \mathbf{k}] \underbrace{\begin{bmatrix} i'_x & j'_x & k'_x \\ i'_y\,, & j'_y\,, & k'_y \\ i'_z & j'_z & k'_z \end{bmatrix}}_{\mathbf{R}_{xyz}} \underbrace{\begin{bmatrix} r_{x'} \\ r_{y'} \\ r_{z'} \end{bmatrix}}_{\mathbf{r}_{x'y'z'}} \\
\mathbf{r}' = [\mathbf{i}'\ \mathbf{j}'\ \mathbf{k}'] \underbrace{\begin{bmatrix} r'_{x'} \\ r'_{y'} \\ r'_{z'} \end{bmatrix}}_{\mathbf{r'}_{x'y'z'}} &= [\mathbf{i}\ \mathbf{j}\ \mathbf{k}] \underbrace{\begin{bmatrix} i'_x & j'_x & k'_x \\ i'_y\,, & j'_y\,, & k'_y \\ i'_z & j'_z & k'_z \end{bmatrix}}_{\mathbf{R}_{xyz}} \underbrace{\begin{bmatrix} r'_{x'} \\ r'_{y'} \\ r'_{z'} \end{bmatrix}}_{\mathbf{r'}_{x'y'z'}}
\end{aligned}
\tag{5}
$$

Noting (2),

$$
\begin{aligned}
\underbrace{\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}}_{\mathbf{r}_{xyz}} &= \underbrace{\begin{bmatrix} i'_x & j'_x & k'_x \\ i'_y\,, & j'_y\,, & k'_y \\ i'_z & j'_z & k'_z \end{bmatrix}}_{\mathbf{R}_{xyz}} \underbrace{\begin{bmatrix} r_{x'} \\ r_{y'} \\ r_{z'} \end{bmatrix}}_{\mathbf{r}_{x'y'z'}} \\
\underbrace{\begin{bmatrix} r'_x \\ r'_y \\ r'_z \end{bmatrix}}_{\mathbf{r'}_{xyz}} &= \underbrace{\begin{bmatrix} i'_x & j'_x & k'_x \\ i'_y\,, & j'_y\,, & k'_y \\ i'_z & j'_z & k'_z \end{bmatrix}}_{\mathbf{R}_{xyz}} \underbrace{\begin{bmatrix} r'_{x'} \\ r'_{y'} \\ r'_{z'} \end{bmatrix}}_{\mathbf{r'}_{x'y'z'}}
\end{aligned}
\tag{6}
$$

Noting that $\mathbf{r'}_{x'y'z'} = \mathbf{r}_{xyz}$, the coordinates of $\mathbf{r'}$ and $\mathbf{r}$ in the frame $Oxyz$ are related by

$$
\underbrace{\begin{bmatrix} r'_x \\ r'_y \\ r'_z \end{bmatrix}}_{\mathbf{r'}_{xyz}} = \underbrace{\begin{bmatrix} i'_x & j'_x & k'_x \\ i'_y\,, & j'_y\,, & k'_y \\ i'_z & j'_z & k'_z \end{bmatrix}}_{\mathbf{R}_{xyz}} \underbrace{\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}}_{\mathbf{r}_{xyz}}
\tag{7}
$$

and in the frame $Ox'y'z'$ by

$$
\underbrace{\begin{bmatrix} r'_{x'} \\ r'_{y'} \\ r'_{z'} \end{bmatrix}}_{\mathbf{r'}_{x'y'z'}} = \underbrace{\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}}_{\mathbf{r}_{xyz}} = \underbrace{\begin{bmatrix} i'_x & j'_x & k'_x \\ i'_y\,, & j'_y\,, & k'_y \\ i'_z & j'_z & k'_z \end{bmatrix}}_{\mathbf{R}_{xyz}} \underbrace{\begin{bmatrix} r_{x'} \\ r_{y'} \\ r_{z'} \end{bmatrix}}_{\mathbf{r}_{x'y'z'}}
\tag{8}
$$

It is thus clear that $\mathbf{R}_{xyz}$ is the matrix representation of the rotation operation (1) either in the frame $Oxyz$ or $Ox'y'z'$, which is referred to as the rotation matrix in the literature. Note that the rotation matrix relates the coordinates of any point fixed to the rigid body before and after the rotation expressed in the same coordinate frame. In other words, it captures the rotational motion of the rigid body relative to (or seen from) a fixed reference frame.

In the following, when no notation confusion will be caused, the subscript representing the coordinate frame that the rotation operation is expressed will be omitted from the rotation matrix, i.e., $\mathbf{R}$ will be used to represent the rotation matrix when there is no confusion with the rotation operator in (1).

As rotation operator is norm preserving, i.e., $\|\mathbf{r'}_{xyz}\| = \|\mathbf{r}_{xyz}\|$, noting (7), $\mathbf{r}_{xyz}^T\mathbf{R}^T\mathbf{R}\mathbf{r}_{xyz} = \|\mathbf{R}\mathbf{r}_{xyz}\| = \|\mathbf{r'}_{xyz}\| = \|\mathbf{r}_{xyz}\| = \mathbf{r}_{xyz}^T\mathbf{r}_{xyz}$, $\forall \mathbf{r}_{xyz}$. Thus $\mathbf{R}^T\mathbf{R} = I$ or $\mathbf{R}^{-1} = \mathbf{R}^T$, i.e., $\mathbf{R}$ is an unitary matrix.

Example 1. Consider the rotation of a rigid body about the z-axis of the frame $Oxyz$ by an angle of θ as shown in Fig.3. Such a rotation is referred to as one of the principle rotations and denoted by such a rotation operation as $\mathbf{R}_z(\theta)$.

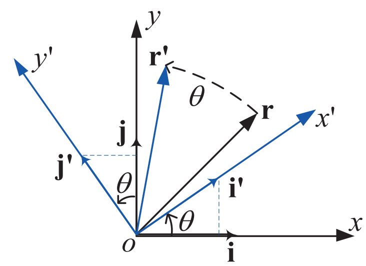

Fig. 3. Rotation

With the rotation $\mathbf{R}_z(\theta)$,

$$\begin{aligned}
\mathbf{i'} &= \cos(\theta)\,\mathbf{i} + \sin(\theta)\,\mathbf{j} = [\mathbf{i}\ \mathbf{j}\ \mathbf{k}]\underbrace{\begin{bmatrix} C_\theta \\ S_\theta \\ 0 \end{bmatrix}}_{\mathbf{i'}_{xyz}} \\
\mathbf{j'} &= -\sin(\theta)\,\mathbf{i} + \cos(\theta)\,\mathbf{j} = [\mathbf{i}\ \mathbf{j}\ \mathbf{k}]\underbrace{\begin{bmatrix} -S_\theta \\ C_\theta \\ 0 \end{bmatrix}}_{\mathbf{j'}_{xyz}} \\
\mathbf{k'} &= \mathbf{k}
\end{aligned} \tag{9}$$

where C_θ and S_θ are the short hand notation for $\cos(\theta)$ and $\sin(\theta)$ respectively. The rotation matrix in (7) is thus obtained as

$$\mathbf{R}_{z\theta} = \begin{bmatrix} C_\theta & -S_\theta & 0 \\ S_\theta & C_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{10}$$

Similarly, the principle rotation of rotating a rigid body about the x-axis of the frame $Oxyz$ by an angle of α is denoted as $\mathbf{R}_x(\alpha)$ and the associated rotation matrix is given by

$$\mathbf{R}_{x\alpha} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\alpha & -S_\alpha \\ 0 & S_\alpha & C_\alpha \end{bmatrix} \tag{11}$$

where C_α and S_α are the short hand notation for $\cos(\alpha)$ and $\sin(\alpha)$ respectively. The principle rotation of rotating a rigid body about the y-axis of the frame $Oxyz$ by an angle of β is denoted as $\mathbf{R}_y(\beta)$ and the associated rotation matrix is given by

$$\mathbf{R}_{y\beta} = \begin{bmatrix} C_\beta & 0 & S_\beta \\ 0 & 1 & 0 \\ -S_\beta & 0 & C_\beta \end{bmatrix} \tag{12}$$

where $C_\beta = \cos(\beta)$ and $S_\beta = \sin(\beta)$. △

2.2 Orientation Matrix

In the above subsection, the rotation of a rigid body is described from the viewpoint of observing the motion of the rigid body (i.e., the motion of a body fixed point P) from a space fixed reference frame (e.g., the frame $Oxyz$). If the same rotation is observed by a person sitting on the rigid body (i.e., the body fixed frame), then, the body will appear to be stationary while the space is moving and the rotation could be indirectly described by the motion of a space fixed point seen from the body fixed frame. Specifically, for a fixed vector $\mathbf{r}$ (i.e., a point fixed in the space) as shown in Fig.4, its coordinates seen from the body fixed frame before and after the rotation would be $\mathbf{r}_{xyz}$ and $\mathbf{r}_{x'y'z'}$ respectively and the rotation motion is indirectly described by the relationship between $\mathbf{r}_{x'y'z'}$ and $\mathbf{r}_{xyz}$ as

$$\mathbf{r}_{x'y'z'} = \mathbf{\Omega}\mathbf{r}_{xyz} \tag{13}$$

where $\mathbf{\Omega}$ is a 3×3 matrix commonly referred to as the *orientation matrix*. Note that the orientation matrix relates the coordinates of the same vector in two different coordinate frames – the second frame $Ox'y'z'$ can be thought as the result of the first frame $Oxyz$ through the rotation.

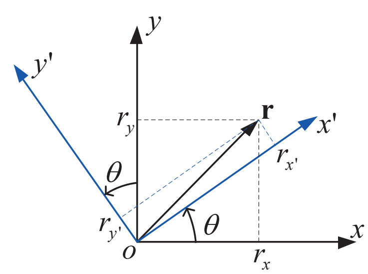

Fig. 4. Orientation

Noting (6), it is clear that the orientation matrix $\mathbf{\Omega}$ and the rotation matrix $\mathbf{R}$ for the same rotation is related by

$$\mathbf{\Omega} = \mathbf{R}^{-1} = \mathbf{R}^T \tag{14}$$

Thus, the orientation matrices for the three principle rotations introduced in *Example 1* are given by

$$\mathbf{\Omega}_{x\alpha} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\alpha & S_\alpha \\ 0 & -S_\alpha & C_\alpha \end{bmatrix}, \ \mathbf{\Omega}_{y\beta} = \begin{bmatrix} C_\beta & 0 & -S_\beta \\ 0 & 1 & 0 \\ S_\beta & 0 & C_\beta \end{bmatrix}, \ \mathbf{\Omega}_{z\theta} = \begin{bmatrix} C_\theta & S_\theta & 0 \\ -S_\theta & C_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{15}$$

3 Posture of a Rigid Body

Posture of a rigid body in a reference coordinate frame $Oxyz$ can be defined by fixing a body frame $O\tilde{x}\tilde{y}\tilde{z}$ to it and describing the body frame $O\tilde{x}\tilde{y}\tilde{z}$ relative to the reference frame $Oxyz$ in terms of three consecutive rotations with respect to certain axes. The angles of the three consecutive rotations are referred to as the *Euler angles*. Depending on the axes about which the three consecutive rotations are carried out, various conventions on the definition of Euler angles exist. In the following subsections, two most widely used conventions are presented.

3.1 Euler Angles for $z - x - z$ Rotations

In the $z - x - z$ convention, the three Euler angles are defined as shown in Fig.5. Specifically, the intersection of the $\tilde{x}\tilde{y}$ plane of the body frame and the xy plane of the reference frame is called the line of nodes (N). Then, α is the angle between the x-axis of reference frame and the line of nodes. β is the angle between the z-axis of the reference frame and the $\tilde{z}$-axis of the body frame. γ is the angle between the line of nodes and the $\tilde{x}$-axis of the body frame. All angles are considered equivalent modulo 2π. In some contexts such as the motion description of a spinning gyroscope shown in Fig.7, $\alpha = \psi$ is called the precession angle, $\beta = \theta$ called the nutation angle, and $\gamma = \phi$ called the spin angle.

With the above definition of Euler angles, the body frame $O\tilde{x}\tilde{y}\tilde{z}$ can be thought as the result of the reference frame $Oxyz$ after three consecutive rotations as detailed below.

Body-fixed Axis Rotations

Rotations about the axes of body-fixed frames are normally called intrinsic rotations or body-fixed rotations. With the body-fixed rotation specification, the frame $O\tilde{x}\tilde{y}\tilde{z}$ is obtained by first aligning the body-fixed frame with the reference frame $Oxyz$ and then rotating the rigid body through the the three consecutive rotations as shown in Fig.8. Namely, first rotate the rigid body in terms of the z-axis of its body-fixed frame by an angle of α and denote the

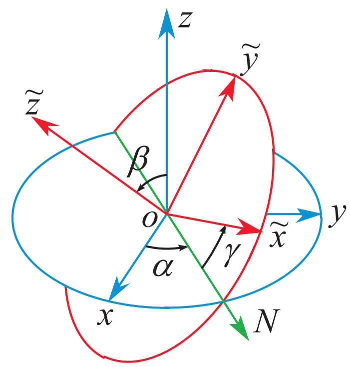

Fig. 5. Euler Angles in $z - x - z$ Convention

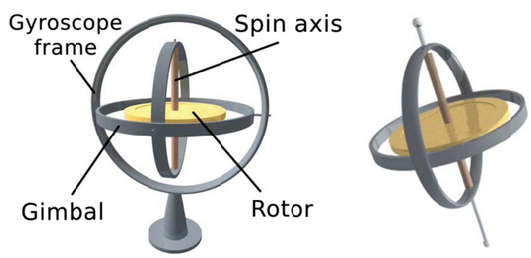

Fig. 6. Gyroscopes

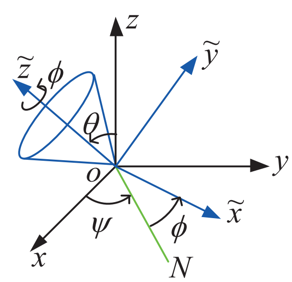

Fig. 7. Euler Angles

resulting body-fixed frame as $Ox'y'z'$. Then rotate the rigid body in terms of the x-axis of its body-fixed frame (i.e., the x'-axis of the intermediate frame $Ox'y'z'$) by an angle of β and denote the resulting body-fixed frame as $Ox''y''z''$. Finally, rotate the rigid body in terms of the z-axis of its body-fixed frame (i.e., the z''-axis of the second intermediate frame $Ox''y''z''$) by an angle of γ to obtain the final body frame $O\tilde{x}\tilde{y}\tilde{z}$. In other words, the final posture of the rigid body is obtained by the following rotations:

$$\mathbf{R}_{z''}(\gamma) \circ \mathbf{R}_{x'}(\beta) \circ \mathbf{R}_z(\alpha) : \ \mathbf{r} \to \tilde{\mathbf{r}} = \mathbf{R}_{z''}(\gamma) \left[\mathbf{R}_{x'}(\beta) \left[\mathbf{R}_z(\alpha)\mathbf{r}\right]\right] \tag{16}$$

where $\mathbf{r}$ and $\tilde{\mathbf{r}}$ represent the position vectors of a body-fixed point before and after the rotations.

To obtain the rotation matrix of the above rotations, denote the position vector of the body-fixed point after the first rotation by $\mathbf{r'}$ and the second rotation by $\mathbf{r''}$. Then,

$$\begin{aligned} \mathbf{r'} &= \mathbf{R}_z(\alpha)\mathbf{r} \\ \mathbf{r''} &= \mathbf{R}_{x'}(\beta)\mathbf{r'} \\ \tilde{\mathbf{r}} &= \mathbf{R}_{z''}(\gamma)\mathbf{r''} \end{aligned} \tag{17}$$

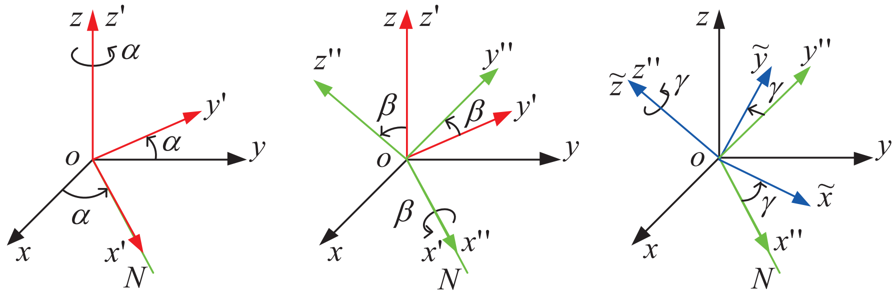

Fig. 8. Body-fixed Rotation Specification with Euler Angles

As $Ox'y'z'$ is the rotation of $Oxyz$ in terms of its z-axis by an angle of α, from (7) and (13),

$$\begin{aligned} &\mathbf{r'}_{xyz} = \mathbf{R}_{z\alpha}\mathbf{r}_{xyz} \\ &\mathbf{r'}_{x'y'z'} = \mathbf{\Omega}_{z\alpha}\mathbf{r'}_{xyz}, \qquad \tilde{\mathbf{r}}_{x'y'z'} = \mathbf{\Omega}_{z\alpha}\tilde{\mathbf{r}}_{xyz} \end{aligned} \tag{18}$$

where $\mathbf{R}_{z\alpha}$ is the rotation matrix in (10) by substituting α for θ and $\mathbf{\Omega}_{z\alpha} = \mathbf{R}_{z\alpha}^{-1} = \mathbf{R}_{z\alpha}^{T}$ is the orientation matrix of the rotation. Similarly, as $Ox''y''z''$ is the rotation of $Ox'y'z'$ in terms of its x-axis by an angle of β,

$$\begin{aligned} &\mathbf{r''}_{x'y'z'} = \mathbf{R}_{x\beta}\mathbf{r'}_{x'y'z'} \\ &\mathbf{r''}_{x''y''z''} = \mathbf{\Omega}_{x\beta}\mathbf{r''}_{x'y'z'}, \qquad \tilde{\mathbf{r}}_{x''y''z''} = \mathbf{\Omega}_{x\beta}\tilde{\mathbf{r}}_{x'y'z'} \end{aligned} \tag{19}$$

where $\mathbf{R}_{x\beta}$ is the rotation matrix in (11) by substituting β for α and $\mathbf{\Omega}_{x\beta} = \mathbf{R}_{x\beta}^{-1} = \mathbf{R}_{x\beta}^{T}$ is the orientation matrix of the rotation. With the final body-fixed frame $O\tilde{x}\tilde{y}\tilde{z}$ being the rotation of $Ox''y''z''$ in terms of its z-axis by an angle of γ,

$$\begin{aligned} &\tilde{\mathbf{r}}_{x''y''z''} = \mathbf{R}_{z\gamma}\mathbf{r''}_{x''y''z''} \\ &\tilde{\mathbf{r}}_{\tilde{x}\tilde{y}\tilde{z}} = \mathbf{\Omega}_{z\gamma}\tilde{\mathbf{r}}_{x''y''z''} \end{aligned} \tag{20}$$

where $\mathbf{R}_{z\gamma}$ is the rotation matrix in (10) by substituting γ for θ and $\mathbf{\Omega}_{z\gamma} = \mathbf{R}_{z\gamma}^{-1} = \mathbf{R}_{z\gamma}^{T}$ is the orientation matrix of the rotation. Noting that

$$\tilde{\mathbf{r}}_{\tilde{x}\tilde{y}\tilde{z}} = \mathbf{\Omega}_{z\gamma}\mathbf{R}_{z\gamma}\mathbf{r''}_{x''y''z''} = \mathbf{r''}_{x''y''z''} = \mathbf{r'}_{x'y'z'} = \mathbf{r}_{xyz} \tag{21}$$

from (18) to (20),

$$\begin{aligned} \tilde{\mathbf{r}}_{xyz} &= \mathbf{\Omega}_{z\alpha}^{-1}\tilde{\mathbf{r}}_{x'y'z'} = \mathbf{\Omega}_{z\alpha}^{-1}\mathbf{\Omega}_{x\beta}^{-1}\tilde{\mathbf{r}}_{x''y''z''} = \mathbf{\Omega}_{z\alpha}^{-1}\mathbf{\Omega}_{x\beta}^{-1}\mathbf{\Omega}_{z\gamma}^{-1}\tilde{\mathbf{r}}_{\tilde{x}\tilde{y}\tilde{z}} \\ &= \underbrace{\mathbf{R}_{z\alpha}\mathbf{R}_{x\beta}\mathbf{R}_{z\gamma}}_{\mathbf{R}}\mathbf{r}_{xyz} \end{aligned} \tag{22}$$

Thus, the rotation matrix from $Oxyz$ to $O\tilde{x}\tilde{y}\tilde{z}$ is given by

$$\begin{aligned}\mathbf{R} &= \mathbf{R}_{z\alpha}\mathbf{R}_{x\beta}\mathbf{R}_{z\gamma} \\ &= \begin{bmatrix} C_\alpha C_\gamma - S_\alpha C_\beta S_\gamma & -C_\alpha S_\gamma - S_\alpha C_\beta C_\gamma & S_\alpha S_\beta \\ S_\alpha C_\gamma + C_\alpha C_\beta S_\gamma & -S_\alpha S_\gamma + C_\alpha C_\beta C_\gamma & -C_\alpha S_\beta \\ S_\beta S_\gamma & S_\beta C_\gamma & C_\beta \end{bmatrix}\end{aligned} \tag{23}$$

Note that with the body-fixed rotation specifications, the rotation matrix is obtained by multiplying the rotation matrix of each subsequent principle rotation from the right. In terms of the orientation matrix, it is from the left as

$$\mathbf{\Omega} = \mathbf{R}^{-1} = (\mathbf{R}_{z\alpha}\mathbf{R}_{x\beta}\mathbf{R}_{z\gamma})^{-1} = \mathbf{R}_{z\gamma}^{-1}\mathbf{R}_{x\beta}^{-1}\mathbf{R}_{z\alpha}^{-1} = \mathbf{\Omega}_{z\gamma}\mathbf{\Omega}_{x\beta}\mathbf{\Omega}_{z\alpha} \tag{24}$$

Space-fixed Axis Rotations

Rotations about the axes of the reference frame are normally called extrinsic rotations or space-fixed rotations. With the space-fixed rotation specification, the frame $O\tilde{x}\tilde{y}\tilde{z}$ is obtained by first aligning the body-fixed frame with the reference frame $Oxyz$ and then rotating the rigid body through the the three consecutive rotations as shown in Fig.9. Namely, first rotate the rigid body in terms of the z-axis of the reference frame $Oxyz$ by an angle of γ and denote the resulting body-fixed frame as $Ox'y'z'$. Then rotate the rigid body in terms of the x-axis of the reference frame $Oxyz$ by an angle of β and denote the resulting body-fixed frame as $Ox''y''z''$. Finally, rotate the rigid body in terms of the z-axis of the reference frame $Oxyz$ by an angle of α to obtain the final body frame $O\tilde{x}\tilde{y}\tilde{z}$. In other words, the final posture of the rigid body is obtained by the following rotations:

$$\mathbf{R}_z(\alpha) \circ \mathbf{R}_x(\beta) \circ \mathbf{R}_z(\gamma) : \ \mathbf{r} \to \tilde{\mathbf{r}} = \mathbf{R}_z(\alpha)\left[\mathbf{R}_x(\beta)\left[\mathbf{R}_z(\gamma)\mathbf{r}\right]\right] \tag{25}$$

where $\mathbf{r}$ and $\tilde{\mathbf{r}}$ represent the position vectors of a body-fixed point before and after the rotations. Note that the order of the rotation angles is from γ to β to α, opposite to that in the body-fixed specification. It is shown below that such a space-fixed rotation sequence leads to the same posture of the rigid body as in the body-fixed specification.

To obtain the rotation matrix of the above space-fixed rotations, denote the position vector of the body-fixed point after the first rotation by $\mathbf{r'}$ and the second rotation by $\mathbf{r''}$. Then,

$$\begin{aligned}\mathbf{r'} &= \mathbf{R}_z(\gamma)\mathbf{r} \\ \mathbf{r''} &= \mathbf{R}_x(\beta)\mathbf{r'} \\ \tilde{\mathbf{r}} &= \mathbf{R}_z(\alpha)\mathbf{r''}\end{aligned} \tag{26}$$

As $Ox'y'z'$ is the rotation of $Oxyz$ in terms of its z-axis by an angle of γ, from (7),

$$\mathbf{r'}_{xyz} = \mathbf{R}_{z\gamma}\mathbf{r}_{xyz} \tag{27}$$

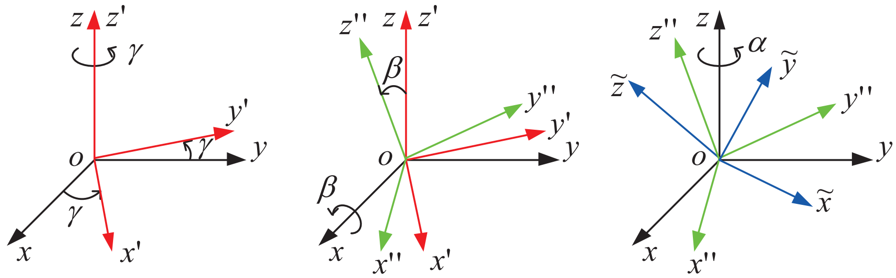

Fig. 9. Space-fixed Rotation Specification with Euler Angles

where $\mathbf{R}_{z\gamma}$ is the rotation matrix in (10) by substituting γ for θ. Similarly, as $\mathbf{r}''$ is the rotation of $\mathbf{r}'$ in terms of the x-axis of the fixed reference frame $Oxyz$ by an angle of β,

$$\mathbf{r}''_{xyz} = \mathbf{R}_{x\beta}\mathbf{r}'_{xyz} \tag{28}$$

where $\mathbf{R}_{x\beta}$ is the rotation matrix in (11) by substituting β for α. With $\tilde{\mathbf{r}}$ being the rotation of $\mathbf{r}''$ in terms of the z-axis of the fixed reference frame $Oxyz$ by an angle of α,

$$\tilde{\mathbf{r}}_{xyz} = \mathbf{R}_{z\alpha}\mathbf{r}''_{xyz} \tag{29}$$

where $\mathbf{R}_{z\alpha}$ is the rotation matrix in (10) by substituting α for θ. Thus,

$$\begin{aligned}\tilde{\mathbf{r}}_{xyz} &= \mathbf{R}_{z\alpha}\mathbf{R}_{x\beta}\mathbf{r}'_{xyz} \\ &= \underbrace{\mathbf{R}_{z\alpha}\mathbf{R}_{x\beta}\mathbf{R}_{z\gamma}}_{\mathbf{R}}\mathbf{r}_{xyz}\end{aligned} \tag{30}$$

which has the same rotation matrix from $Oxyz$ to $O\tilde{x}\tilde{y}\tilde{z}$ as in the body-fixed specification, i.e., (23). Note that with the space-fixed rotation specifications, the rotation matrix is obtained by multiplying the rotation matrix of each subsequent principle rotations from the left.

3.2 Row-Pitch-Yaw Angles

Another set of widely used Euler angles are the so-called roll, pitch, and yaw angles or the Euler angles in the $x-y-z$ convention (also known as Tait-Bryan angles). This set of angles are often used to describe the attitude of a vehicle with respect to a chosen reference frame. The common practice is to choose the center of the mass of the vehicle as the origin of the coordinate systems with the x-axis of the body-fixed frame (roll axes) pointing forward, regardless of whether they are cars, ships, airplanes, or space vehicles. In case of land vehicles like cars, tanks etc., which use the ENU-system (East-North-Up) as external reference (world frame), as shown in Fig.10, the y-axis of the vehicle body-fixed frame always points to the left of the vehicle so that the resulting

z-axis of the body-fixed frame by the right hand rule always points up. In contrast, in case of air and sea vehicles like submarines, ships, and airplanes, which use the NED-system (North-East-Down) as external reference (world frame), as shown in Fig.11, the y-axis of the vehicle body-fixed frame points to the right of the vehicle (starboard) so that the resulting z-axis of the body-fixed frame by the right hand rule always points down.

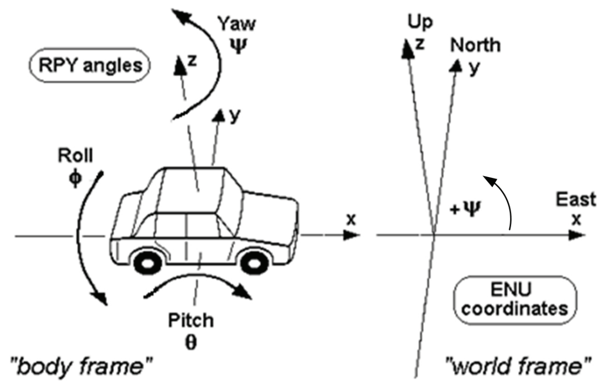

Fig. 10. Roll-Pitch-Yaw Axes of Land Vehicles

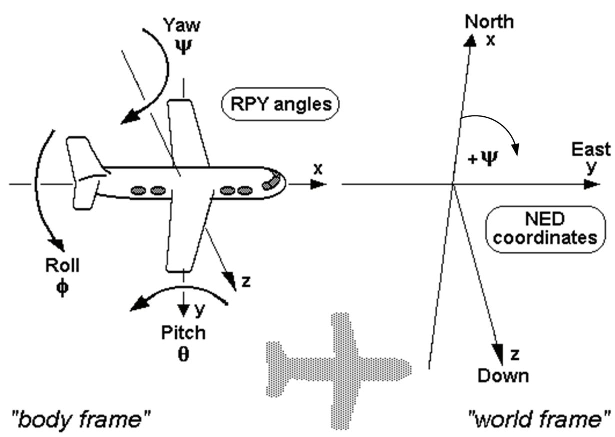

Fig. 11. Roll-Pitch-Yaw Axes of Aerial Vehicles

Let $Oxyz$ (the reference frame) and $O\tilde{x}\tilde{y}\tilde{z}$ be the body-fixed frame before and after rotations. The roll, pitch, and yaw angles are then defined as shown in Fig.12. Specifically, the intersection of the $\tilde{y}\tilde{z}$ plane and the xy plane is called the line of nodes (N). The row angle $\phi \in (-180^o, 180^o]$ is then the angle from the line of nodes to the $\tilde{y}$-axis, the pitch angle $\theta \in [-90^o, 90^o]$ is the angle between the $\tilde{x}$-axis and the xy plane, and the yaw angle $\psi \in (-180^o, 180^o]$ is the angle from the y-axis to the line of nodes.

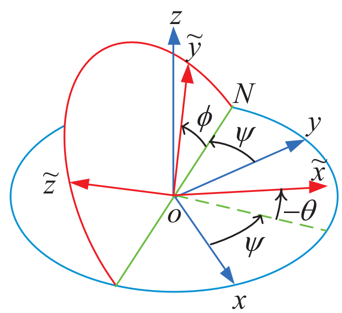

Fig. 12. Roll, Pitch, and Yaw Angles

With the above definition of Row-Pitch-Yaw angles, the body frame can be obtained by rotating the reference frame $Oxyz$ in three consecutive axis rotations using either the body-fixed specification or the space-fixed specification. The body-fixed specification is shown in Fig.13. Namely, first rotate the reference frame $Oxyz$ in terms of its z-axis by the yaw angle of ψ to obtain the first intermediate frame $Ox'y'z'$. Then rotate the intermediate frame $Ox'y'z'$ in terms of its y'-axis by the pitch angle of θ to obtain the second intermediate frame $Ox''y''z''$. Finally, rotate the second intermediate frame $Ox''y''z''$ in terms of its x''-axis ($\tilde{x}$-axis of the body frame $O\tilde{x}\tilde{y}\tilde{z}$) by the row angle of ϕ to obtain the body frame $O\tilde{x}\tilde{y}\tilde{z}$.

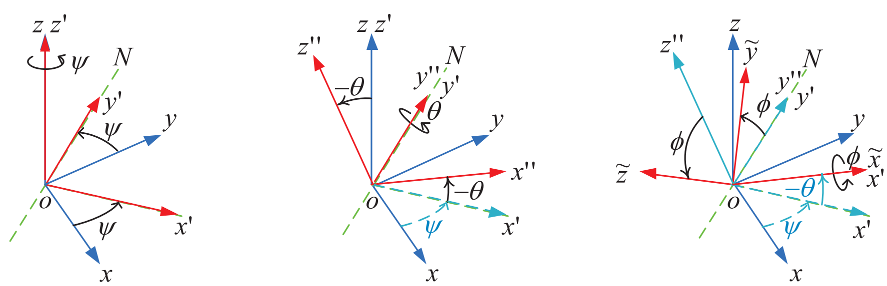

Fig. 13. Body-fixed Specification with RPY Angles

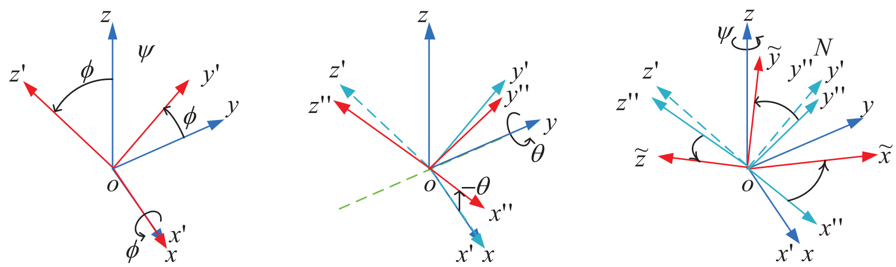

Fig. 14. Space-fixed Specification with RPY Angles

The space-fixed specification is shown in Fig.14. Namely, first rotate the reference frame $Oxyz$ in terms of its x-axis by the row angle of ϕ to obtain the first intermediate frame $Ox'y'z'$. Then rotate the intermediate frame $Ox'y'z'$ in terms of the y-axis of the reference frame $Oxyz$ by the pitch angle of θ to obtain the second intermediate frame $Ox''y''z''$. Finally, rotate the second intermediate frame $Ox''y''z''$ in terms of the z-axis of the reference frame $Oxyz$ by the yaw angle of ψ to obtain the body frame $O\tilde{x}\tilde{y}\tilde{z}$. Note that with the body-fixed specification, the rotation sequence is $z-y-x$ with yaw-pitch-row angles respectively while the space-fixed specification uses the reverse rotation sequence of $x-y-z$ with row-pitch-yaw angles respectively. As shown in the previous section, both specifications produce the same final body frame $O\tilde{x}\tilde{y}\tilde{z}$, and the rotation matrix from $Oxyz$ to $O\tilde{x}\tilde{y}\tilde{z}$ is given by

$$\begin{aligned}\mathbf{R} &= \mathbf{R}_{z\psi}\mathbf{R}_{y\theta}\mathbf{R}_{x\phi}\\ &= \begin{bmatrix} C_\psi C_\theta & -S_\psi C_\phi + C_\psi S_\theta S_\phi & S_\psi S_\phi + C_\psi S_\theta C_\phi \\ S_\psi C_\theta & C_\psi C_\phi + S_\psi S_\theta S_\phi & -C_\psi S_\phi + S_\psi S_\theta C_\phi \\ -S_\theta & C_\theta S_\phi & C_\theta C_\phi \end{bmatrix}\end{aligned} \tag{31}$$

where $C_\psi = \cos(\psi)$, $S_\psi = \sin(\psi)$, $C_\theta = \cos(\theta)$, $S_\theta = \sin(\theta)$, $C_\phi = \cos(\phi)$, and $S_\phi = \sin(\phi)$.

4 Angular Velocity of a Rigid Body Rotating at a Fixed Point

This section considers the angular velocity of a rigid body rotating at a fixed point O in the reference coordinate frame $Oxyz$. Using a body-fixed frame $O\tilde{x}\tilde{y}\tilde{z}$, at each time t, one can describe the posture of the rigid body by three Euler angles introduced in the previous section. The following gives the angular velocity of the rigid body with respect to the reference frame $Oxyz$ when the time-history of the three Euler angles are known.

4.1 Rotations using $z - x - z$ Euler Angles

In this case, at each time t, the body-fixed frame $O\tilde{x}\tilde{y}\tilde{z}$ is specified by the three Euler angles introduced in subsection 3.1 which are denoted as $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ respectively. Let $Ox'y'z'(t)$ and $Ox''y''z''(t)$ be the two intermediate coordinate frames when describing the body frame $O\tilde{x}\tilde{y}\tilde{z}(t)$ using the body-fixed specification in Fig.8. Then, the relative motion between the frame $Ox'y'z'(t)$ and the reference frame $Oxyz$ is by the rotation about the fixed z-axis with an angle of $\alpha(t)$. Thus, the relative angular velocity of the frame $Ox'y'z'(t)$ to $Oxyz$ is $\boldsymbol{\omega}_{xyz}^{x'y'z'} = \dot{\alpha}(t)\mathbf{k}$. Similarly, the relative angular velocity of the frame $Ox''y''z''(t)$ to $Ox'y'z'(t)$ is $\boldsymbol{\omega}_{x'y'z'}^{x''y''z''} = \dot{\beta}(t)\mathbf{i'}(t)$ as, when $Ox'y'z'$ is freezed for relative velocity calculation, the relative motion of $Ox''y''z''(t)$ to $Ox'y'z'(t)$ is a rotation about an "instantaneously fixed" x'-axis with an angle of $\beta(t)$. In the same way, the relative angular velocity of the frame $O\tilde{x}\tilde{y}\tilde{z}(t)$ to $Ox''y''z''(t)$ is $\boldsymbol{\omega}_{x''y''z''}^{\tilde{x}\tilde{y}\tilde{z}} = \dot{\gamma}(t)\mathbf{k''}(t)$. The angular velocity $\boldsymbol{\omega}$ of the body frame $O\tilde{x}\tilde{y}\tilde{z}(t)$ with respect to the reference frame can thus be obtained as

$$\begin{aligned} \boldsymbol{\omega} &= \boldsymbol{\omega}_{x''y''z''}^{\tilde{x}\tilde{y}\tilde{z}} + \boldsymbol{\omega}_{x'y'z'}^{x''y''z''} + \boldsymbol{\omega}_{xyz}^{x'y'z'} \\ &= \dot{\gamma}(t)\ \mathbf{k''}(t) + \dot{\beta}(t)\ \mathbf{i'}(t) + \dot{\alpha}(t)\mathbf{k} \end{aligned} \tag{32}$$

Noting that $\mathbf{k''}_{xyz} = \mathbf{R}_{z\alpha}\mathbf{k''}_{x'y'z'} = \mathbf{R}_{z\alpha}\mathbf{R}_{x\beta}\mathbf{k''}_{x''y''z''}$, $\mathbf{i'}_{xyz} = \mathbf{R}_{z\alpha}\mathbf{i'}_{x'y'z'}$, $\mathbf{k''}_{x''y''z''} = [0,0,1]^T$, $\mathbf{i'}_{x'y'z'} = [1,0,0]^T$, and $\mathbf{k}_{xyz} = [0,0,1]^T$, the angular velocity $\boldsymbol{\omega}$ expressed in the reference frame can be obtained as

$$\begin{aligned} \boldsymbol{\omega}_{xyz} &= \dot{\gamma}\ \mathbf{R}_{z\alpha}\mathbf{R}_{x\beta}\begin{bmatrix}0\\0\\1\end{bmatrix} + \dot{\beta}\ \mathbf{R}_{z\alpha}\begin{bmatrix}1\\0\\0\end{bmatrix} + \dot{\alpha}\ \begin{bmatrix}0\\0\\1\end{bmatrix} \\ &= \begin{bmatrix}0 & C_\alpha & S_\alpha S_\beta\\0 & S_\alpha & -C_\alpha S_\beta\\1 & 0 & C_\beta\end{bmatrix}\begin{bmatrix}\dot{\alpha}\\\dot{\beta}\\\dot{\gamma}\end{bmatrix} \end{aligned} \tag{33}$$

Similarly, noting that $\mathbf{k''}_{\tilde{x}\tilde{y}\tilde{z}} = [0,0,1]^T$, $\mathbf{i'}_{x''y''z''} = \mathbf{i''}_{x''y''z''} = [1,0,0]^T$, $\mathbf{i'}_{\tilde{x}\tilde{y}\tilde{z}} = \boldsymbol{\Omega}_{z\gamma}\mathbf{i'}_{x''y''z''} = \mathbf{R}_{z\gamma}^T\mathbf{i'}_{x''y''z''}$, $\mathbf{k}_{x'y'z'} = \mathbf{k}_{xyz} = [0,0,1]^T$, and $\mathbf{k}_{\tilde{x}\tilde{y}\tilde{z}} = \mathbf{R}_{z\gamma}^T\mathbf{k}_{x''y''z''} = \mathbf{R}_{z\gamma}^T\mathbf{R}_{x\beta}^T\mathbf{k}_{x'y'z'}$, the angular velocity $\boldsymbol{\omega}$ expressed in the body frame is thus given by

$$\begin{aligned} \boldsymbol{\omega}_{\tilde{x}\tilde{y}\tilde{z}} &= \dot{\gamma}\ \begin{bmatrix}0\\0\\1\end{bmatrix} + \dot{\beta}\ \mathbf{R}_{z\gamma}^T\begin{bmatrix}1\\0\\0\end{bmatrix} + \dot{\alpha}\ \mathbf{R}_{z\gamma}^T\mathbf{R}_{x\beta}^T\begin{bmatrix}0\\0\\1\end{bmatrix} \\ &= \begin{bmatrix}S_\beta S_\gamma & C_\gamma & 0\\S_\beta C_\gamma & -S_\gamma & 0\\C_\beta & 0 & 1\end{bmatrix}\begin{bmatrix}\dot{\alpha}\\\dot{\beta}\\\dot{\gamma}\end{bmatrix} \end{aligned} \tag{34}$$

4.2 Rotations using Row-Pitch-Yaw Angles

In this case, at each time t, the body-fixed frame $O\tilde{x}\tilde{y}\tilde{z}$ is specified by the Row-Pitch-Yaw angles introduced in subsection 3.2 which are denoted as $\phi(t)$, $\theta(t)$, and $\psi(t)$ respectively. Let $Ox'y'z'(t)$ and $Ox''y''z''(t)$ be the two intermediate coordinate frames when describing the body frame $O\tilde{x}\tilde{y}\tilde{z}(t)$ using the body-fixed specification in Fig.13. Then, the relative motion between the frame $Ox'y'z'(t)$ and the reference frame $Oxyz$ is by the rotation about the fixed z-axis with an angle of $\psi(t)$. Thus, the relative angular velocity of the frame $Ox'y'z'(t)$ to $Oxyz$ is $\boldsymbol{\omega}_{xyz}^{x'y'z'} = \dot{\psi}(t)\mathbf{k}$. Similarly, the relative angular velocity of the frame $Ox''y''z''(t)$ to $Ox'y'z'(t)$ is $\boldsymbol{\omega}_{x'y'z'}^{x''y''z''} = \dot{\theta}(t)\mathbf{j'}(t)$ as, when $Ox'y'z'$ is freezed for relative velocity calculation, the relative motion of $Ox''y''z''(t)$ to $Ox'y'z'(t)$ is a rotation about an "instantaneously fixed" j'-axis with an angle of $\theta(t)$. In the same way, the relative angular velocity of the frame $O\tilde{x}\tilde{y}\tilde{z}(t)$ to $Ox''y''z''(t)$ is $\boldsymbol{\omega}_{x''y''z''}^{\tilde{x}\tilde{y}\tilde{z}} = \dot{\phi}(t)\mathbf{i''}(t)$. The angular velocity $\boldsymbol{\omega}$ of the body frame $O\tilde{x}\tilde{y}\tilde{z}(t)$ with respect to the reference frame can thus be obtained as

$$\begin{aligned}\boldsymbol{\omega} &= \boldsymbol{\omega}_{x''y''z''}^{\tilde{x}\tilde{y}\tilde{z}} + \boldsymbol{\omega}_{x'y'z'}^{x''y''z''} + \boldsymbol{\omega}_{xyz}^{x'y'z'} \\ &= \dot{\phi}(t)\ \mathbf{i''}(t) + \dot{\theta}(t)\ \mathbf{j'}(t) + \dot{\psi}(t)\mathbf{k}\end{aligned} \tag{35}$$

Noting that $\mathbf{i''}_{xyz} = \mathbf{R}_{z\psi}\mathbf{i''}_{x'y'z'} = \mathbf{R}_{z\psi}\mathbf{R}_{y\theta}\mathbf{i''}_{x''y''z''}$, $\mathbf{j'}_{xyz} = \mathbf{R}_{z\psi}\mathbf{j'}_{x'y'z'}$, $\mathbf{i''}_{x''y''z''} = [1,0,0]^T$, $\mathbf{j'}_{x'y'z'} = [0,1,0]^T$, and $\mathbf{k}_{xyz} = [0,0,1]^T$, the angular velocity $\boldsymbol{\omega}$ expressed in the reference frame can be obtained as

$$\begin{aligned}\boldsymbol{\omega}_{xyz} &= \dot{\phi}\ \mathbf{R}_{z\psi}\mathbf{R}_{y\theta}\begin{bmatrix}1\\0\\0\end{bmatrix} + \dot{\theta}\ \mathbf{R}_{z\psi}\begin{bmatrix}0\\1\\0\end{bmatrix} + \dot{\psi}\ \begin{bmatrix}0\\0\\1\end{bmatrix} \\ &= \begin{bmatrix}C_\theta C_\psi & -S_\psi & 0\\ C_\theta S_\psi & C_\psi & 0\\ -S_\theta & 0 & 1\end{bmatrix}\begin{bmatrix}\dot{\phi}\\ \dot{\theta}\\ \dot{\psi}\end{bmatrix}\end{aligned} \tag{36}$$

Similarly, noting that $\mathbf{i''}_{\tilde{x}\tilde{y}\tilde{z}} = [1,0,0]^T$, $\mathbf{j'}_{x''y''z''} = \mathbf{j''}_{x''y''z''} = [0,1,0]^T$, $\mathbf{j'}_{\tilde{x}\tilde{y}\tilde{z}} = \boldsymbol{\Omega}_{x\phi}\mathbf{j'}_{x''y''z''} = \mathbf{R}_{x\phi}^T\mathbf{j'}_{x''y''z''}$, $\mathbf{k}_{x'y'z'} = \mathbf{k}_{xyz} = [0,0,1]^T$, and $\mathbf{k}_{\tilde{x}\tilde{y}\tilde{z}} = \mathbf{R}_{x\phi}^T\mathbf{k}_{x''y''z''} = \mathbf{R}_{x\phi}^T\mathbf{R}_{y\theta}^T\mathbf{k}_{x'y'z'}$, the angular velocity $\boldsymbol{\omega}$ expressed in the body frame is thus given by

$$\begin{aligned}\boldsymbol{\omega}_{\tilde{x}\tilde{y}\tilde{z}} &= \dot{\phi}\ \begin{bmatrix}1\\0\\0\end{bmatrix} + \dot{\theta}\ \mathbf{R}_{x\phi}^T\begin{bmatrix}0\\1\\0\end{bmatrix} + \dot{\psi}\ \mathbf{R}_{x\phi}^T\mathbf{R}_{y\theta}^T\begin{bmatrix}0\\0\\1\end{bmatrix} \\ &= \underbrace{\begin{bmatrix}1 & 0 & -S_\theta\\ 0 & C_\phi & S_\phi C_\theta\\ 0 & -S_\phi & C_\phi C_\theta\end{bmatrix}}_{\boldsymbol{\Omega}_{\phi\theta\psi}}\begin{bmatrix}\dot{\phi}\\ \dot{\theta}\\ \dot{\psi}\end{bmatrix}\end{aligned} \tag{37}$$

5 Motion Description of a Rigid Body

Pick up any point on the rigid body and fix a coordinate frame to the body at the point as shown by the frame $o\tilde{x}\tilde{y}\tilde{z}$ in Fig.15, which is referred to as the body-frame. Let $OXYZ$ be an inertial frame (i.e., the fixed frame). Then, the motion of a rigid body can be completely described by the motion of the point o in the inertial frame $OXYZ$, which is represented by the time-history of the position vector $\mathbf{r_o}$ in Fig.15, and the relative motion of other points of the rigid body to the point o (e.g., the relative motion of the point P to the point o is represented by the time-history of the position vector $\mathbf{r_b}$ in Fig.15). Noting that the relative motion is the same as the rotation of the rigid body to the point o, it can be completely characterized by the time-history of the posture of the body-frame $o\tilde{x}\tilde{y}\tilde{z}$ in a reference frame $oxyz$ having the same origin and being parallel to the inertial frame $OXYZ$. Thus, in the following, it is assumed that the motion of a rigid body is described by $\{\mathbf{r_o}(t),\ t \in \mathcal{R}\}$ and $\{[\phi(t), \theta(t), \psi(t)]^T,\ t \in \mathcal{R}\}$, in which $\phi(t)$, $\theta(t)$, and $\psi(t)$ are the three Euler angles describing the posture of the body-frame $o\tilde{x}\tilde{y}\tilde{z}$ in the reference frame $oxyz$ at time t. How to obtain the position, velocity, and acceleration of any point on the rigid body based on such a motion description of the rigid body is detailed below.

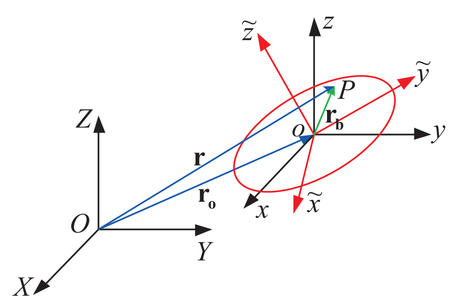

Fig. 15. Motion of a rigid body

5.1 Homogenous Representation and Transformation

The position of any point on a rigid body can be concisely determined through homogenous transformation introduced below. As seen in Fig.15,

$$\mathbf{r} = \mathbf{r_o} + \mathbf{r_b} \tag{38}$$

where $\mathbf{r}$ and $\mathbf{r_b}$ represent the position vector of the same point P in the inertia frame $OXYZ$ and the body frame $o\tilde{x}\tilde{y}\tilde{z}$ respectively. As before, let $\mathbf{r}_{b\tilde{x}\tilde{y}\tilde{z}} = [r_{b\tilde{x}},\ r_{b\tilde{y}},\ r_{b\tilde{z}}]^T$ be the coordinate values of the vector $\mathbf{r}_b$ in the body frame $o\tilde{x}\tilde{y}\tilde{z}$. Then the coordinate values of the vector $\mathbf{r}_b$ in the reference frame $oxyz$ can be obtained from (13) as

$$\mathbf{r}_{\mathbf{b}xyz} = \mathbf{\Omega}^{-1}\mathbf{r}_{\mathbf{b}\tilde{x}\tilde{y}\tilde{z}} = \mathbf{R}\mathbf{r}_{\mathbf{b}\tilde{x}\tilde{y}\tilde{z}} \tag{39}$$

where $\mathbf{R}$ is the rotation matrix from $oxyz$ to $o\tilde{x}\tilde{y}\tilde{z}$ given by (23) or (31) depending on the Euler angles used. Thus, noting that $\mathbf{r_b}_{XYZ} = \mathbf{r_b}_{xyz}$ as $oxyz$ is always parallel to $OXYZ$, the coordinate values of the position vector $\mathbf{r}$ in $OXYZ$ frame are obtained from (38) and (39) as

$$\underbrace{\begin{bmatrix} r_X \\ r_Y \\ r_Z \end{bmatrix}}_{\mathbf{r}_{XYZ}} = \mathbf{r_o}_{XYZ} + \mathbf{r_b}_{XYZ} = \underbrace{\begin{bmatrix} r_{oX} \\ r_{oY} \\ r_{oZ} \end{bmatrix}}_{\mathbf{r_o}_{XYZ}} + \mathbf{R} \underbrace{\begin{bmatrix} r_{b\tilde{x}} \\ r_{b\tilde{y}} \\ r_{b\tilde{z}} \end{bmatrix}}_{\mathbf{r_b}_{\tilde{x}\tilde{y}\tilde{z}}} \tag{40}$$

Eq.(40) can also be expressed in a concise matrix form as

$$\underbrace{\begin{bmatrix} r_X \\ r_Y \\ r_Z \\ 1 \end{bmatrix}}_{\bar{\mathbf{r}}_{XYZ}} = \underbrace{\left[\begin{array}{c:c} \mathbf{R} & \begin{matrix} r_{oX} \\ r_{oY} \\ r_{oZ} \end{matrix} \\ \hdashline 0 & 1 \end{array}\right]}_{\mathbf{T}} \underbrace{\begin{bmatrix} r_{b\tilde{x}} \\ r_{b\tilde{y}} \\ r_{b\tilde{z}} \\ 1 \end{bmatrix}}_{\bar{\mathbf{r}}_{\mathbf{b}\tilde{x}\tilde{y}\tilde{z}}} \tag{41}$$

in which $\bar{\mathbf{r}}_{XYZ}$ and $\bar{\mathbf{r}}_{\mathbf{b}\tilde{x}\tilde{y}\tilde{z}}$ essentially represent the coordinate values of the same point P in the inertial frame $OXYZ$ and the body frame $o\tilde{x}\tilde{y}\tilde{z}$ respectively, and the transformation matrix $\mathbf{T}$ is purely determined by the relative position of the two frames, i.e., the translation vector $\mathbf{r_o}_{XYZ}$ and the rotation matrix $\mathbf{R}$ from $OXYZ$ to $o\tilde{x}\tilde{y}\tilde{z}$. In the literature, the 4×1 vectors $\bar{\mathbf{r}}_{XYZ}$ and $\bar{\mathbf{r}}_{\mathbf{b}\tilde{x}\tilde{y}\tilde{z}}$ are called the *homogeneous coordinates* of the point P in the inertia frame and the body-frame respectively, and the 4×4 matrix $\mathbf{T}$ is normally referred to as the homogenous transformation matrix.

5.2 Velocity of a Point on a Rigid Body

Taking the time derivative of (38), the velocity of a point on the rigid body can be obtained as

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{\mathbf{r}}_\mathbf{o} + \dot{\mathbf{r}}_\mathbf{b} \tag{42}$$

To obtain the specific expression of $\dot{\mathbf{r}}_\mathbf{b}$, let us look at the coordinate representation of two vector cross-product first. Namely, let $\mathbf{u}$ and $\mathbf{w}$ be two vectors and $\mathbf{u} \times \mathbf{w}$ their cross-product. For any coordinate frame, say the frame $oxyz$, denote the coordinate values of the two vectors $\mathbf{u}$ and $\mathbf{w}$ as $\mathbf{u}_{xyz} = [u_x, u_y, u_z]^T$ and $\mathbf{w}_{xyz} = [w_x, w_y, w_z]^T$. Then,

$$\begin{aligned} \mathbf{u} \times \mathbf{w} &= (u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}) \times (w_x\mathbf{i} + w_y\mathbf{j} + w_z\mathbf{k}) \\ &= (u_y w_z - u_z w_y)\mathbf{i} + (u_z w_x - u_x w_z)\mathbf{j} + (u_x w_y - u_y w_x)\mathbf{k} \\ &= [\mathbf{i}\ \ \mathbf{j}\ \ \mathbf{k}] \underbrace{\begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix}}_{\mathbf{S}_{\mathbf{u}_{xyz}}} \underbrace{\begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix}}_{\mathbf{w}_{xyz}} \end{aligned} \tag{43}$$

In other words, the coordinate values of the cross-product of any two vectors in a frame can be obtained from the coordinate values of the two vectors in the same frame as

$$(\mathbf{u} \times \mathbf{w})_{xyz} = \mathbf{S}_{\mathbf{u}_{xyz}} \mathbf{w}_{xyz} \tag{44}$$

where $\mathbf{S}_{\mathbf{u}_{xyz}}$ is a skew-symmetric matrix (i.e., $\mathbf{S}^T_{\mathbf{u}_{xyz}} = -\mathbf{S}_{\mathbf{u}_{xyz}}$) formed by the coordinate values of the first vector $\mathbf{u}$ given in (43).

For any body-fixed vector such as the unit axis vector $\tilde{\mathbf{i}}$, its time-derivative due to the rotational motion of the body is given by

$$\dot{\tilde{\mathbf{i}}} = \boldsymbol{\omega} \times \tilde{\mathbf{i}} \tag{45}$$

where $\boldsymbol{\omega}$ is the angular velocity of the body. Using the notations in (44), the coordinate values of $\dot{\tilde{\mathbf{i}}}$ in the frame $oxyz$ are then given by

$$\dot{\tilde{\mathbf{i}}}_{xyz} = \mathbf{S}_{\boldsymbol{\omega}_{xyz}} \tilde{\mathbf{i}}_{xyz} \tag{46}$$

where $\boldsymbol{\omega}_{xyz}$ is related to the Euler angles of the body frame by (33) or (36) depending on the Euler angles used, and $\mathbf{S}_{\boldsymbol{\omega}_{xyz}}$ is the skew-symmetric matrix given by

$$\mathbf{S}_{\boldsymbol{\omega}_{xyz}} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \tag{47}$$

In the same way, the time derivatives of the other axis unit vectors of the body frame are

$$\begin{aligned} \dot{\tilde{\mathbf{j}}}_{xyz} &= \mathbf{S}_{\boldsymbol{\omega}_{xyz}} \tilde{\mathbf{j}}_{xyz} \\ \dot{\tilde{\mathbf{k}}}_{xyz} &= \mathbf{S}_{\boldsymbol{\omega}_{xyz}} \tilde{\mathbf{k}}_{xyz} \end{aligned} \tag{48}$$

Noting (4), $\mathbf{R}$ in (40) is the same as $[\tilde{\mathbf{i}}_{xyz},\ \tilde{\mathbf{j}}_{xyz},\ \tilde{\mathbf{k}}_{xyz}]$. Thus, from (47) and (48), the time derivative of the rotation matrix $\mathbf{R}$ is

$$\begin{aligned} \dot{\mathbf{R}} &= \left[\dot{\tilde{\mathbf{i}}}_{xyz},\ \dot{\tilde{\mathbf{j}}}_{xyz},\ \dot{\tilde{\mathbf{k}}}_{xyz}\right] \\ &= \mathbf{S}_{\boldsymbol{\omega}_{xyz}} \left[\tilde{\mathbf{i}}_{xyz},\ \tilde{\mathbf{j}}_{xyz},\ \tilde{\mathbf{k}}_{xyz}\right] \\ &= \mathbf{S}_{\boldsymbol{\omega}_{xyz}} \mathbf{R} \end{aligned} \tag{49}$$

Noting (40) and (49), the expression of the velocity vector (42) in the inertial frame $OXYZ$ is obtained as

$$\begin{aligned} \mathbf{v}_{XYZ} = \dot{\mathbf{r}}_{XYZ} &= \dot{\mathbf{r}}_{\mathbf{o}XYZ} + \dot{\mathbf{R}} \mathbf{r}_{\mathbf{b}\tilde{x}\tilde{y}\tilde{z}} + \mathbf{R} \underbrace{\frac{d\mathbf{r}_{\mathbf{b}\tilde{x}\tilde{y}\tilde{z}}}{dt}}_{\mathbf{v}_{\mathbf{r}\tilde{x}\tilde{y}\tilde{z}}} \\ &= \dot{\mathbf{r}}_{\mathbf{o}XYZ} + \mathbf{S}_{\boldsymbol{\omega}_{xyz}} \underbrace{\mathbf{R} \mathbf{r}_{\mathbf{b}\tilde{x}\tilde{y}\tilde{z}}}_{\mathbf{r}_{\mathbf{b}xyz}} + \underbrace{\mathbf{R} \mathbf{v}_{\mathbf{r}\tilde{x}\tilde{y}\tilde{z}}}_{\mathbf{v}_{\mathbf{r}xyz}} \end{aligned} \tag{50}$$

in which $\mathbf{v}_{\mathbf{r}}$ represents the relative velocity of the point P seen in the body-frame $o\tilde{x}\tilde{y}\tilde{z}$. Noting that the second term represents the coordinate values of the cross product vector $\boldsymbol{\omega} \times \mathbf{r_b}$ in the frame $oxyz$ or $OXYZ$, Eq.(50) can be rewritten in a coordinate-free vector form as

$$\underbrace{\mathbf{v} = \dot{\mathbf{r}}}_{\substack{\text{absolute velocity}\\ \text{of any point}}} = \underbrace{\dot{\mathbf{r}}_\mathbf{o}}_{\substack{\text{velocity due to}\\ \text{translational movement}\\ \text{of body-frame}}} + \underbrace{\boldsymbol{\omega} \times \mathbf{r_b}}_{\substack{\text{velocity due to}\\ \text{rotational movement}\\ \text{of body-frame}}} + \underbrace{\mathbf{v_r}}_{\substack{\text{relative velocity}\\ \text{seen}\\ \text{in body-frame}}} \tag{51}$$

For any point on the rigid body, the relative velocity seen in the body-frame is zero, as $\mathbf{r}_{\mathbf{b}\tilde{x}\tilde{y}\tilde{z}}$ is constant over time. Thus, the velocity of a point on a rigid body is given by

$$\underbrace{\mathbf{v} = \dot{\mathbf{r}}}_{\substack{\text{absolute velocity}\\ \text{of a point}\\ \text{on a rigid body}}} = \underbrace{\dot{\mathbf{r}}_\mathbf{o}}_{\substack{\text{velocity due to}\\ \text{translational movement}\\ \text{of body-frame}}} + \underbrace{\boldsymbol{\omega} \times \mathbf{r_b}}_{\substack{\text{velocity due to}\\ \text{rotational movement}\\ \text{of body-frame}}} \tag{52}$$

6 Dynamics of a Rigid Body

This section first obtains the linear and angular momentum as well as the kinetic energy of a rigid body. Dynamic modeling based on two distinct modeling approaches, the Newtonian and the Lagrange methods, will then be detailed for the rigid body.

6.1 Linear Momentum

For an infinitesimal point mass dm of the rigid body located at the point P shown in Fig.15, its linear momentum is $\mathbf{v}dm$ where $\mathbf{v}$ is given by (52). Thus, the linear momentum of the rigid body can be obtained by integrating it over the entire space B of the rigid body as

$$\mathbf{L} = \int_B \mathbf{v}dm = \int \left(\dot{\mathbf{r}}_\mathbf{o} + \boldsymbol{\omega} \times \mathbf{r_b}\right) dm = m\dot{\mathbf{r}}_\mathbf{o} + \boldsymbol{\omega} \times \underbrace{\int_B \mathbf{r_b} dm}_{m\mathbf{r_{cb}}} \tag{53}$$

where m is the total mass of the rigid body and $\mathbf{r_{cb}}$ is the vector from the body-frame origin o to the center of the gravity of the rigid body.

6.2 Angular Momentum

Similarly, the angular momentum of the rigid body with respect to the inertia frame $OXYZ$ can be obtained by integrating the angular momentum of the infinitesimal point mass dm of the rigid body with respect to the origin O of the inertia frame over the entire space of the rigid body as

$$\begin{aligned}\mathbf{H} &= \int_B \mathbf{r}\times\mathbf{v}dm = \int_B (\mathbf{r_o}+\mathbf{r_b})\times(\dot{\mathbf{r}}_\mathbf{o}+\boldsymbol{\omega}\times\mathbf{r_b})\,dm\\ &= \int_B [\mathbf{r_o}\times\dot{\mathbf{r}}_\mathbf{o}+\mathbf{r_o}\times(\boldsymbol{\omega}\times\mathbf{r_b})+\mathbf{r_b}\times\dot{\mathbf{r}}_\mathbf{o}+\mathbf{r_b}\times(\boldsymbol{\omega}\times\mathbf{r_b})]\,dm\\ &= \mathbf{r_o}\times m\dot{\mathbf{r}}_\mathbf{o}+\mathbf{r_o}\times(\boldsymbol{\omega}\times\int_B\mathbf{r_b}dm)+\int_B\mathbf{r_b}dm\times\dot{\mathbf{r}}_\mathbf{o}+\int_B\mathbf{r_b}\times(\boldsymbol{\omega}\times\mathbf{r_b})dm\\ &= \mathbf{r_o}\times m\dot{\mathbf{r}}_\mathbf{o}+\mathbf{r_o}\times(\boldsymbol{\omega}\times m\mathbf{r_{cb}})+m\mathbf{r_{cb}}\times\dot{\mathbf{r}}_\mathbf{o}+\int_B\mathbf{r_b}\times(\boldsymbol{\omega}\times\mathbf{r_b})dm\end{aligned} \tag{54}$$

For simplicity, *assuming that the origin o of the body-frame is chosen to be the center of the gravity of the rigid body, i.e.,* $\mathbf{r_{cb}} = 0$, then, the angular momentum of (54) becomes

$$\mathbf{H} = \underbrace{\mathbf{r_o}\times m\dot{\mathbf{r}}_\mathbf{o}}_{\mathbf{H_o}} + \underbrace{\int_B \mathbf{r_b}\times(\boldsymbol{\omega}\times\mathbf{r_b})dm}_{\mathbf{H_b}} \tag{55}$$

where $\mathbf{H_o}$ represents the angular momentum of the rigid body when its entire mass is at the center of gravity (i.e., a point mass) and $\mathbf{H_b}$ is the additional angular momentum due to the distributed mass nature of the rigid body when it is not a point mass. It is thus obvious that $\mathbf{H_b}$ depends on the physical shape and mass distribution of the rigid body. To obtain the specific expression of such a quantity, the following mathematical fact on the cross product of three vectors will be used

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}, \quad \forall \mathbf{a},\mathbf{b},\mathbf{c}\in\mathcal{R}^3 \tag{56}$$

Using the above triple-vector product formula, the integrant in $\mathbf{H_b}$ becomes

$$\mathbf{r_b}\times(\boldsymbol{\omega}\times\mathbf{r_b}) = (\mathbf{r_b}\cdot\mathbf{r_b})\boldsymbol{\omega} - (\mathbf{r_b}\cdot\boldsymbol{\omega})\mathbf{r_b} \tag{57}$$

Thus, noting that $\mathbf{r_b} = [\tilde{\mathbf{i}}\ \tilde{\mathbf{j}}\ \tilde{\mathbf{k}}]\mathbf{r}_{\mathbf{b}\tilde{x}\tilde{y}\tilde{z}} = r_{b\tilde{x}}\tilde{\mathbf{i}}+r_{b\tilde{y}}\tilde{\mathbf{j}}+r_{b\tilde{z}}\tilde{\mathbf{k}}$ and $\boldsymbol{\omega} = [\tilde{\mathbf{i}}\ \tilde{\mathbf{j}}\ \tilde{\mathbf{k}}]\boldsymbol{\omega}_{\tilde{x}\tilde{y}\tilde{z}} = \omega_{\tilde{x}}\tilde{\mathbf{i}}+\omega_{\tilde{y}}\tilde{\mathbf{j}}+\omega_{\tilde{z}}\tilde{\mathbf{k}}$, $\mathbf{H_b}$ can be expressed in the body-frame $o\tilde{x}\tilde{y}\tilde{z}$ as

$$\begin{aligned}\mathbf{H_b} &= \int_B \{(\mathbf{r_b}\cdot\mathbf{r_b})\boldsymbol{\omega} - (\mathbf{r_b}\cdot\boldsymbol{\omega})\mathbf{r_b}\}xdm\\ &= \int_B \left\{(r_{b\tilde{x}}^2+r_{b\tilde{y}}^2+r_{b\tilde{z}}^2)[\tilde{\mathbf{i}}\ \tilde{\mathbf{j}}\ \tilde{\mathbf{k}}]\boldsymbol{\omega}_{\tilde{x}\tilde{y}\tilde{z}} - (r_{b\tilde{x}}\omega_{\tilde{x}}+r_{b\tilde{y}}\omega_{\tilde{y}}+r_{b\tilde{z}}\omega_{\tilde{z}})[\tilde{\mathbf{i}}\ \tilde{\mathbf{j}}\ \tilde{\mathbf{k}}]\mathbf{r}_{\mathbf{b}\tilde{x}\tilde{y}\tilde{z}}\right\}dm\\ &= [\tilde{\mathbf{i}}\ \tilde{\mathbf{j}}\ \tilde{\mathbf{k}}]\int_B \left\{(r_{b\tilde{x}}^2+r_{b\tilde{y}}^2+r_{b\tilde{z}}^2)\begin{bmatrix}\omega_{\tilde{x}}\\ \omega_{\tilde{y}}\\ \omega_{\tilde{z}}\end{bmatrix} - (r_{b\tilde{x}}\omega_{\tilde{x}}+r_{b\tilde{y}}\omega_{\tilde{y}}+r_{b\tilde{z}}\omega_{\tilde{z}})\begin{bmatrix}r_{b\tilde{x}}\\ r_{b\tilde{y}}\\ r_{b\tilde{z}}\end{bmatrix}\right\}dm\\ &= [\tilde{\mathbf{i}}\ \tilde{\mathbf{j}}\ \tilde{\mathbf{k}}]\int_B \left\{(r_{b\tilde{x}}^2+r_{b\tilde{y}}^2+r_{b\tilde{z}}^2)\begin{bmatrix}\omega_{\tilde{x}}\\ \omega_{\tilde{y}}\\ \omega_{\tilde{z}}\end{bmatrix} - \begin{bmatrix}r_{b\tilde{x}}^2 & r_{b\tilde{x}}r_{b\tilde{y}} & r_{b\tilde{x}}r_{b\tilde{z}}\\ r_{b\tilde{y}}r_{b\tilde{x}} & r_{b\tilde{y}}^2 & r_{b\tilde{y}}r_{b\tilde{z}}\\ r_{b\tilde{z}}r_{b\tilde{x}} & r_{b\tilde{z}}r_{b\tilde{y}} & r_{b\tilde{z}}^2\end{bmatrix}\begin{bmatrix}\omega_{\tilde{x}}\\ \omega_{\tilde{y}}\\ \omega_{\tilde{z}}\end{bmatrix}\right\}dm\\ &= [\tilde{\mathbf{i}}\ \tilde{\mathbf{j}}\ \tilde{\mathbf{k}}]\underbrace{\int_B \begin{bmatrix}r_{b\tilde{y}}^2+r_{b\tilde{z}}^2 & -r_{b\tilde{x}}r_{b\tilde{y}} & -r_{b\tilde{x}}r_{b\tilde{z}}\\ -r_{b\tilde{y}}r_{b\tilde{x}} & r_{b\tilde{x}}^2+r_{b\tilde{z}}^2 & -r_{b\tilde{y}}r_{b\tilde{z}}\\ -r_{b\tilde{z}}r_{b\tilde{x}} & -r_{b\tilde{z}}r_{b\tilde{y}} & r_{b\tilde{x}}^2+r_{b\tilde{y}}^2\end{bmatrix}dm}_{\mathbf{I}_{\tilde{x}\tilde{y}\tilde{z}}}\underbrace{\begin{bmatrix}\omega_{\tilde{x}}\\ \omega_{\tilde{y}}\\ \omega_{\tilde{z}}\end{bmatrix}}_{\boldsymbol{\omega}_{\tilde{x}\tilde{y}\tilde{z}}}\end{aligned} \tag{58}$$

where $\mathbf{I}_{\tilde{x}\tilde{y}\tilde{z}}$ is the moment of inertia matrix of the rigid body with respect to its body-frame, which is a constant matrix as $\mathbf{r}_{\mathbf{b}\tilde{x}\tilde{y}\tilde{z}}$ is independent of motion and time-invariant for a rigid body. Traditionally, $\mathbf{I}_{\tilde{x}\tilde{y}\tilde{z}}$ is written as

$$\mathbf{I}_{\tilde{x}\tilde{y}\tilde{z}} = \begin{bmatrix} I_{\tilde{x}\tilde{x}} & I_{\tilde{x}\tilde{y}} & I_{\tilde{x}\tilde{z}} \\ I_{\tilde{y}\tilde{x}} & I_{\tilde{y}\tilde{y}} & I_{\tilde{y}\tilde{z}} \\ I_{\tilde{z}\tilde{x}} & I_{\tilde{z}\tilde{y}} & I_{\tilde{z}\tilde{z}} \end{bmatrix} \tag{59}$$

where $I_{\tilde{x}\tilde{x}} = \int_B \left(r_{b\tilde{y}}^2 + r_{b\tilde{z}}^2\right) dm$, $I_{\tilde{y}\tilde{y}} = \int_B \left(r_{b\tilde{x}}^2 + r_{b\tilde{z}}^2\right) dm$, and $I_{\tilde{z}\tilde{z}} = \int_B \left(r_{b\tilde{x}}^2 + r_{b\tilde{y}}^2\right) dm$ are the constant moment of inertia of the rigid body with respect to each axis of its body frame, and $I_{\tilde{x}\tilde{y}} = I_{\tilde{y}\tilde{x}} = -\int_B r_{b\tilde{x}} r_{b\tilde{y}} dm$, $I_{\tilde{x}\tilde{z}} = I_{\tilde{z}\tilde{x}} = -\int_B r_{b\tilde{x}} r_{b\tilde{z}} dm$, and $I_{\tilde{y}\tilde{z}} = I_{\tilde{z}\tilde{y}} = -\int_B r_{b\tilde{z}} r_{b\tilde{y}} dm$. Since the matrix $\mathbf{I}_{\tilde{x}\tilde{y}\tilde{z}}$ is positive definite, it can always be diagonalized with a real orthogonal matrix $\mathbf{U} = [\mathbf{u_1} \;\; \mathbf{u_2} \;\; \mathbf{u_3}]$ as

$$\mathbf{U^T I_{\tilde{x}\tilde{y}\tilde{z}} U} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \tag{60}$$

where $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ are the three independent unity eigenvectors of $\mathbf{I}_{\tilde{x}\tilde{y}\tilde{z}}$ with I_1, I_2, and I_3 being the corresponding eigenvalue respectively. Thus, by choosing a new body-fixed frame $o\tilde{x}'\tilde{y}'\tilde{z}'$ such that its three axes are aligned with $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$, the constant moment of inertia matrix of the rigid body with respect to this new body frame would be diagonal and given by

$$\mathbf{I}_{\tilde{x}'\tilde{y}'\tilde{z}'} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \tag{61}$$

The axes of such a new body frame are called the *principle rotational axes of the rigid body.*

Similarly, by expressing all vectors in the reference frame, i.e., $\mathbf{r_b} = [\mathbf{i} \;\; \mathbf{j} \;\; \mathbf{k}]\mathbf{r_b}_{xyz} = r_{bx}\mathbf{i} + r_{by}\mathbf{j} + r_{bz}\mathbf{k}$ and $\boldsymbol{\omega} = [\mathbf{i} \;\; \mathbf{j} \;\; \mathbf{k}]\boldsymbol{\omega}_{xyz} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$, one can obtain $\mathbf{H_b}$ in the reference-frame $oxyz$ as

$$\begin{aligned} \mathbf{H_b} &= \int_B \{(\mathbf{r_b}\cdot\mathbf{r_b})\boldsymbol{\omega} - (\mathbf{r_b}\cdot\boldsymbol{\omega})\mathbf{r_b}\}\, dm \\ &= [\mathbf{i} \;\; \mathbf{j} \;\; \mathbf{k}] \int_B \left\{ (r_{bx}^2 + r_{by}^2 + r_{bz}^2) \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} - (r_{bx}\omega_x + r_{by}\omega_y + r_{bz}\omega_z) \begin{bmatrix} r_{bx} \\ r_{by} \\ r_{bz} \end{bmatrix} \right\} dm \\ &= [\mathbf{i} \;\; \mathbf{j} \;\; \mathbf{k}] \underbrace{\int_B \begin{bmatrix} r_{by}^2 + r_{bz}^2 & -r_{bx}r_{by} & -r_{bx}r_{bz} \\ -r_{by}r_{bx} & r_{bx}^2 + r_{bz}^2 & -r_{by}r_{bz} \\ -r_{bz}r_{bx} & -r_{bz}r_{by} & r_{bx}^2 + r_{by}^2 \end{bmatrix} dm}_{\mathbf{I}_{xyz}} \underbrace{\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}}_{\boldsymbol{\omega}_{xyz}} \end{aligned} \tag{62}$$

where $\mathbf{I}_{xyz}$ is the moment of inertia matrix of the rigid body with respect to the reference frame. However, different from $\mathbf{I}_{\tilde{x}\tilde{y}\tilde{z}}$, $\mathbf{I}_{xyz}$ is not a constant matrix anymore as $\mathbf{r_b}_{xyz} = \mathbf{R}\mathbf{r_b}_{\tilde{x}\tilde{y}\tilde{z}}$ depends on the rotational motion of the rigid body and is time-varying. It can be verified that

$$\mathbf{I}_{xyz} = \mathbf{R}\mathbf{I}_{\tilde{x}\tilde{y}\tilde{z}}\mathbf{R}^T \tag{63}$$

6.3 Kinetic energy

Kinetic energy of a rigid body can be obtained by integrating the kinetic energy of the infinitesimal point mass dm over the entire space of the rigid body as

$$\begin{aligned} T &= \tfrac{1}{2}\int \mathbf{v}\cdot\mathbf{v}dm = \tfrac{1}{2}\int(\dot{\mathbf{r}}_\mathbf{o}+\boldsymbol{\omega}\times\mathbf{r}_\mathbf{b})\cdot(\dot{\mathbf{r}}_\mathbf{o}+\boldsymbol{\omega}\times\mathbf{r}_\mathbf{b})dm \\ &= \tfrac{1}{2}[m\dot{\mathbf{r}}_\mathbf{o}\cdot\dot{\mathbf{r}}_\mathbf{o}+2\dot{\mathbf{r}}_\mathbf{o}\cdot(\boldsymbol{\omega}\times\textstyle\int_B \mathbf{r}_\mathbf{b}dm)+\int_B(\boldsymbol{\omega}\times\mathbf{r}_\mathbf{b})\cdot(\boldsymbol{\omega}\times\mathbf{r}_\mathbf{b})dm] \\ &= \tfrac{1}{2}[m\dot{\mathbf{r}}_\mathbf{o}\cdot\dot{\mathbf{r}}_\mathbf{o}+2\dot{\mathbf{r}}_\mathbf{o}\cdot(\boldsymbol{\omega}\times m\mathbf{r}_\mathbf{cb})+\textstyle\int_B(\boldsymbol{\omega}\times\mathbf{r}_\mathbf{b})\cdot(\boldsymbol{\omega}\times\mathbf{r}_\mathbf{b})dm] \end{aligned} \tag{64}$$

$$(\boldsymbol{\omega}\times\mathbf{r}_\mathbf{b})_{\tilde{x}\tilde{y}\tilde{z}} = -(\mathbf{r}_\mathbf{b}\times\boldsymbol{\omega})_{\tilde{x}\tilde{y}\tilde{z}} = -S_{r_{b\tilde{x}\tilde{y}\tilde{z}}}\boldsymbol{\omega}_{\tilde{\mathbf{x}}\tilde{\mathbf{y}}\tilde{\mathbf{z}}} \tag{65}$$

$$\begin{aligned} \|\boldsymbol{\omega}\times\mathbf{r}_\mathbf{b}\|^2_{\tilde{x}\tilde{y}\tilde{z}} &= (S_{r_{b\tilde{x}\tilde{y}\tilde{z}}}\boldsymbol{\omega}_{\tilde{\mathbf{x}}\tilde{\mathbf{y}}\tilde{\mathbf{z}}})^T(S_{r_{b\tilde{x}\tilde{y}\tilde{z}}}\boldsymbol{\omega}_{\tilde{\mathbf{x}}\tilde{\mathbf{y}}\tilde{\mathbf{z}}}) \\ &= \boldsymbol{\omega}_{\tilde{\mathbf{x}}\tilde{\mathbf{y}}\tilde{\mathbf{z}}}{}^T S^T_{r_{b\tilde{x}\tilde{y}\tilde{z}}}S_{r_{b\tilde{x}\tilde{y}\tilde{z}}}\boldsymbol{\omega}_{\tilde{\mathbf{x}}\tilde{\mathbf{y}}\tilde{\mathbf{z}}} \end{aligned} \tag{66}$$

If the origin o of the body-frame is chosen to be the center of the gravity of the rigid body, $\mathbf{r}_\mathbf{cb}=0$, we can rewrite the kinetic energy as

$$\begin{aligned} T &= \tfrac{1}{2}[m\dot{\mathbf{r}}_\mathbf{o}\cdot\dot{\mathbf{r}}_\mathbf{o}+\textstyle\int_B(\boldsymbol{\omega}\times\mathbf{r}_\mathbf{b})\cdot(\boldsymbol{\omega}\times\mathbf{r}_\mathbf{b})dm] \\ &= \tfrac{1}{2}[m\mathbf{v}_\mathbf{o}\cdot\mathbf{v}_\mathbf{o}+\textstyle\int\|\boldsymbol{\omega}\times\mathbf{r}_\mathbf{b}\|^2dm] \\ &= \tfrac{1}{2}\{m\mathbf{v}_\mathbf{o}\cdot\mathbf{v}_\mathbf{o}+\boldsymbol{\omega}_{\tilde{\mathbf{x}}\tilde{\mathbf{y}}\tilde{\mathbf{z}}}{}^T\underbrace{[\int S^T_{r_{b\tilde{x}\tilde{y}\tilde{z}}}S_{r_{b\tilde{x}\tilde{y}\tilde{z}}}dm]}_{I_{\tilde{x}\tilde{y}\tilde{z}}}\boldsymbol{\omega}_{\tilde{\mathbf{x}}\tilde{\mathbf{y}}\tilde{\mathbf{z}}}\} \end{aligned} \tag{67}$$

6.4 Potential energy

Let $\mathbf{g}$ be the gravitational vector. Then the potential energy of a rigid body with the origin of the inertia frame being zero potential is

$$P = -\int_B \mathbf{g}\cdot\mathbf{r}dm = -\mathbf{g}\cdot\int_B \mathbf{r}dm = -m\mathbf{g}\cdot\mathbf{r}_\mathbf{o} \tag{68}$$

in which it is assumed that the origin o of the body-frame has been chosen to be the center of the gravity of the rigid body.

6.5 Dynamics of a Rigid Body via Newton-Euler Equations of Motion

The Newton's second law states that the rate of change of the linear momentum of a system is equal to the sum of all external forces acting on the system. For a rigid body shown in Fig.16, *assuming that the origin of the body frame is chosen as the center of the gravity of the rigid body, i.e.,* $\mathbf{r}_\mathbf{cb}=0$, noting (53) and applying the Newton's second law,

$$m\mathbf{g}+\sum_{i=1}^N \mathbf{F}_\mathbf{i} = \frac{d\mathbf{L}}{dt} = \frac{d\,(m\dot{\mathbf{r}}_\mathbf{o})}{dt} = m\ddot{\mathbf{r}}_\mathbf{o} \tag{69}$$

where N represents the number of external forces acting on the rigid body excluding the gravitational force. Eq. (65) reveals that the summation effect of all external forces acting on a rigid body is only related to the translational motion of the center of gravity of the rigid body.

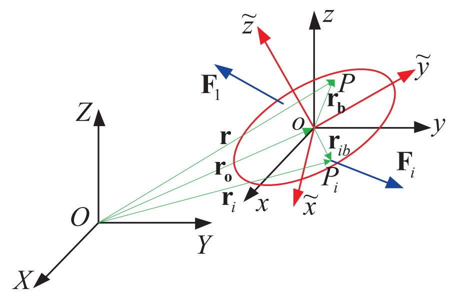

Fig. 16. Dynamics of a rigid body

Another fundamental equation of Newtonian mechanics is that the rate of change of the angular momentum of a system is equal to the sum of all moments generated by the external forces acting on the system. Before applying this law to the rigid body, *it is very important to keep in mind that all the fundamental equations in Newtonian mechanics are derived under the inertia frame assumption, i.e., the velocity, the angular velocity, and the angular momentum are all calculated with respect to the inertia frame*. Specifically, the calculation of linear momentum (53) uses the absolute velocity $\mathbf{v} = \dot{\mathbf{r}}$ rather than relative velocities to a non-inertia frame such as $\mathbf{v}_r$ in (51), and the angular velocity $\boldsymbol{\omega}$ used in the calculation of angular momentum by (54) to (62) should be the absolute angular velocity of the rigid body to the inertia frame $OXYZ$. With these pre-cautious, noting (55), one can apply the angular momentum law (with respect to the fixed origin O of the inertia frame $OXYZ$) to obtain

$$\mathbf{r_o} \times m\mathbf{g} + \sum_{i=1}^{N} (\mathbf{r_i} \times \mathbf{F_i}) = \tfrac{d\mathbf{H}}{dt} \tag{70}$$

where $\mathbf{r_i}$ represents the position vector of the point where the external force $\mathbf{F_i}$ is applied in the inertia frame $OXYZ$ as shown in Fig.16 and $\mathbf{H}$ is given by (54). Noting that $\mathbf{r_i}$ is related to the position vector of the same point in the body-frame $\mathbf{r_{ib}}$ by $\mathbf{r_i} = \mathbf{r_o} + \mathbf{r_{ib}}$, *under the assumption that the origin of the body frame is chosen as the center of the gravity of the rigid body*, from (55) and (70),

$$\begin{aligned} &\tfrac{d\mathbf{H}}{dt} = \tfrac{d\mathbf{H_o}}{dt} + \tfrac{d\mathbf{H_b}}{dt} = \mathbf{r_o} \times m\ddot{\mathbf{r}}_\mathbf{o} + \tfrac{d\mathbf{H_b}}{dt} \\ &\sum_{i=1}^{N} (\mathbf{r_i} \times \mathbf{F_i}) = \sum_{i=1}^{N} [(\mathbf{r_o} + \mathbf{r_{ib}}) \times \mathbf{F_i}] = \mathbf{r_o} \times \underbrace{\sum \mathbf{F_i}}_{m\ddot{\mathbf{r}}_\mathbf{o} - m\mathbf{g}} + \sum_{i=1}^{N} (\mathbf{r_{ib}} \times \mathbf{F_i}) \end{aligned} \tag{71}$$

Thus, Eq.(71) is simplified to

$$\frac{d\mathbf{H_b}}{dt} = \sum_{i=1}^{N}(\mathbf{r_{ib}} \times \mathbf{F_i}) \tag{72}$$

which shows that, as long as the moving point o is chosen to be the center of the gravity of the rigid body, the angular momentum law applies to the angular moment and the moments calculated with respect to this moving point as well – $\mathbf{H_b}$ is the angular moment of the rigid body with respect to the point o and the right hand side of (69) represents the moments by the external forces with respect to the point o.

Eqs. (69) and (72) are all the equations that one needs for the dynamics of a rigid body. These equations are written in vector forms which can be projected to any coordinate frames to obtain the specific component equations needed to solve the motion of the rigid body. This way of modeling is sometimes referred to as the vector mechanics. In the following, these equations will be projected to both the inertia frame and the body-frame as each of them has certain benefits and shortcomings.

Dynamic Equations in Inertia Frame

When projected to the inertia frame $OXYZ$, (69) becomes

$$m\underbrace{\begin{bmatrix}\ddot{r}_{oX}\\ \ddot{r}_{oY}\\ \ddot{r}_{oZ}\end{bmatrix}}_{\ddot{\mathbf{r}}_{oXYZ}} = \sum_{i=1}^{N}\underbrace{\begin{bmatrix}F_{iX}\\ F_{iY}\\ F_{iZ}\end{bmatrix}}_{\mathbf{F_i}_{XYZ}} + m\underbrace{\begin{bmatrix}g_X\\ g_Y\\ g_Z\end{bmatrix}}_{\mathbf{g}_{XYZ}} \tag{73}$$

where $\ddot{\mathbf{r}}_{\mathbf{o}} = \ddot{r}_{oX}\mathbf{I} + \ddot{r}_{oY}\mathbf{J} + \ddot{r}_{oZ}\mathbf{K}$, $\mathbf{F_i} = F_{iX}\mathbf{I} + F_{iY}\mathbf{J} + F_{iZ}\mathbf{K}$ and $\mathbf{g} = g_X\mathbf{I} + g_Y\mathbf{J} + g_Z\mathbf{K}$. With respect to the inertia frame, noting (62) and (43), (72) becomes

$$\begin{array}{l}\frac{d}{dt}\left\{[\mathbf{i}\;\;\mathbf{j}\;\;\mathbf{k}]\mathbf{I}_{xyz}\boldsymbol{\omega}_{xyz}\right\} = \sum_{i=1}^{N}\left\{[\mathbf{i}\;\;\mathbf{j}\;\;\mathbf{k}]\mathbf{S}_{\mathbf{r_{bi}}_{xyz}}\mathbf{F_i}_{xyz}\right\}\\ \Rightarrow [\mathbf{i}\;\;\mathbf{j}\;\;\mathbf{k}]\left\{\dot{\mathbf{I}}_{xyz}\boldsymbol{\omega}_{xyz} + \mathbf{I}_{xyz}\dot{\boldsymbol{\omega}}_{xyz}\right\} = [\mathbf{i}\;\;\mathbf{j}\;\;\mathbf{k}]\sum_{i=1}^{N}\left\{\mathbf{S}_{\mathbf{r_{bi}}_{xyz}}\mathbf{F_i}_{xyz}\right\}\end{array} \tag{74}$$

which leads to the following angular moment equation in component form

$$\mathbf{I}_{xyz}\dot{\boldsymbol{\omega}}_{xyz} + \dot{\mathbf{I}}_{xyz}\boldsymbol{\omega}_{xyz} = \sum_{i=1}^{N}\left\{\mathbf{S}_{\mathbf{r_{bi}}_{xyz}}\mathbf{F_i}_{xyz}\right\} \tag{75}$$

where, from (63) and (49),

$$\begin{aligned}\dot{\mathbf{I}}_{xyz} &= \dot{\mathbf{R}}\mathbf{I}_{\tilde{x}\tilde{y}\tilde{z}}\mathbf{R}^T + \mathbf{R}\mathbf{I}_{\tilde{x}\tilde{y}\tilde{z}}\dot{\mathbf{R}}^T\\ &= \mathbf{S}_{\boldsymbol{\omega}_{xyz}}\mathbf{R}\mathbf{I}_{\tilde{x}\tilde{y}\tilde{z}}\mathbf{R}^T + \mathbf{R}\mathbf{I}_{\tilde{x}\tilde{y}\tilde{z}}\mathbf{R}^T\mathbf{S}^T_{\boldsymbol{\omega}_{xyz}}\end{aligned} \tag{76}$$

Dynamic Equations in Body Frame

Similarly, when projected to the body frame $o\tilde{x}\tilde{y}\tilde{z}$, (69) becomes

$$m\underbrace{\begin{bmatrix}\ddot{r}_{o\tilde{x}}\\ \ddot{r}_{o\tilde{y}}\\ \ddot{r}_{o\tilde{z}}\end{bmatrix}}_{\ddot{\mathbf{r}}_{o\tilde{x}\tilde{y}\tilde{z}}} = \sum_{i=1}^{N}\underbrace{\begin{bmatrix}F_{i\tilde{x}}\\ F_{i\tilde{y}}\\ F_{i\tilde{z}}\end{bmatrix}}_{\mathbf{F_i}_{\tilde{x}\tilde{y}\tilde{z}}} + m\underbrace{\begin{bmatrix}g_{\tilde{x}}\\ g_{\tilde{y}}\\ g_{\tilde{z}}\end{bmatrix}}_{\mathbf{g}_{\tilde{x}\tilde{y}\tilde{z}}} \tag{77}$$

where $\ddot{\mathbf{r}}_{\mathbf{o}} = \ddot{r}_{o\tilde{x}}\tilde{\mathbf{i}} + \ddot{r}_{o\tilde{y}}\tilde{\mathbf{j}} + \ddot{r}_{o\tilde{z}}\tilde{\mathbf{k}}$, $\mathbf{F_i} = F_{i\tilde{x}}\tilde{\mathbf{i}} + F_{i\tilde{y}}\tilde{\mathbf{j}} + F_{i\tilde{z}}\tilde{\mathbf{k}}$ and $\mathbf{g} = g_{\tilde{x}}\tilde{\mathbf{i}} + g_{\tilde{y}}\tilde{\mathbf{j}} + g_{\tilde{z}}\tilde{\mathbf{k}}$. With respect to the body frame, noting (58) and (43) to (45), (72) becomes

$$\begin{aligned}&\tfrac{d}{dt}\left\{\begin{bmatrix}\tilde{\mathbf{i}} & \tilde{\mathbf{j}} & \tilde{\mathbf{k}}\end{bmatrix}\mathbf{I}_{\tilde{x}\tilde{y}\tilde{z}}\boldsymbol{\omega}_{\tilde{x}\tilde{y}\tilde{z}}\right\} = \sum_{i=1}^{N}\left\{\begin{bmatrix}\tilde{\mathbf{i}} & \tilde{\mathbf{j}} & \tilde{\mathbf{k}}\end{bmatrix}\mathbf{S}_{\mathbf{r_{bi}}\tilde{x}\tilde{y}\tilde{z}}\mathbf{F_i}_{\tilde{x}\tilde{y}\tilde{z}}\right\}\\ \Rightarrow &\begin{bmatrix}\tilde{\mathbf{i}} & \tilde{\mathbf{j}} & \tilde{\mathbf{k}}\end{bmatrix}\mathbf{I}_{\tilde{x}\tilde{y}\tilde{z}}\tfrac{d}{dt}\boldsymbol{\omega}_{\tilde{x}\tilde{y}\tilde{z}} + \begin{bmatrix}\dot{\tilde{\mathbf{i}}} & \dot{\tilde{\mathbf{j}}} & \dot{\tilde{\mathbf{k}}}\end{bmatrix}\mathbf{I}_{\tilde{x}\tilde{y}\tilde{z}}\boldsymbol{\omega}_{\tilde{x}\tilde{y}\tilde{z}} = \begin{bmatrix}\tilde{\mathbf{i}} & \tilde{\mathbf{j}} & \tilde{\mathbf{k}}\end{bmatrix}\sum_{i=1}^{N}\left\{\mathbf{S}_{\mathbf{r_{bi}}\tilde{x}\tilde{y}\tilde{z}}\mathbf{F_i}_{\tilde{x}\tilde{y}\tilde{z}}\right\}\end{aligned} \tag{78}$$

From (44) or (46), when projected to the body frame, the following angular moment equations in component form are obtained

$$\mathbf{I}_{\tilde{x}\tilde{y}\tilde{z}}\tfrac{d}{dt}\boldsymbol{\omega}_{\tilde{x}\tilde{y}\tilde{z}} + [\mathbf{S}_{\boldsymbol{\omega}_{\tilde{x}\tilde{y}\tilde{z}}}\tilde{\mathbf{i}}_{\tilde{x}\tilde{y}\tilde{z}} \;\; \mathbf{S}_{\boldsymbol{\omega}_{\tilde{x}\tilde{y}\tilde{z}}}\tilde{\mathbf{j}}_{\tilde{x}\tilde{y}\tilde{z}} \;\; \mathbf{S}_{\boldsymbol{\omega}_{\tilde{x}\tilde{y}\tilde{z}}}\tilde{\mathbf{k}}_{\tilde{x}\tilde{y}\tilde{z}}]\mathbf{I}_{\tilde{x}\tilde{y}\tilde{z}}\boldsymbol{\omega}_{\tilde{x}\tilde{y}\tilde{z}} = \sum_{i=1}^{N}\left\{\mathbf{S}_{\mathbf{r_{bi}}\tilde{x}\tilde{y}\tilde{z}}\mathbf{F_i}_{\tilde{x}\tilde{y}\tilde{z}}\right\} \tag{79}$$

Using the notation (59) and noting (47) and the fact that $\tilde{\mathbf{i}}_{\tilde{x}\tilde{y}\tilde{z}} = [1 \;\; 0 \;\; 0]^T$, $\tilde{\mathbf{j}}_{\tilde{x}\tilde{y}\tilde{z}} = [0 \;\; 1 \;\; 0]^T$, and $\tilde{\mathbf{k}}_{\tilde{x}\tilde{y}\tilde{z}} = [0 \;\; 0 \;\; 1]^T$, (79) is written in the traditional form as

$$\begin{aligned}&\begin{bmatrix}I_{\tilde{x}\tilde{x}} & I_{\tilde{x}\tilde{y}} & I_{\tilde{x}\tilde{z}}\\ I_{\tilde{y}\tilde{x}} & I_{\tilde{y}\tilde{y}} & I_{\tilde{y}\tilde{z}}\\ I_{\tilde{z}\tilde{x}} & I_{\tilde{z}\tilde{y}} & I_{\tilde{z}\tilde{z}}\end{bmatrix}\frac{d}{dt}\begin{bmatrix}\omega_{\tilde{x}}\\ \omega_{\tilde{y}}\\ \omega_{\tilde{z}}\end{bmatrix} + \begin{bmatrix}0 & -\omega_{\tilde{z}} & \omega_{\tilde{y}}\\ \omega_{\tilde{z}} & 0 & -\omega_{\tilde{x}}\\ -\omega_{\tilde{y}} & \omega_{\tilde{x}} & 0\end{bmatrix}\begin{bmatrix}I_{\tilde{x}\tilde{x}} & I_{\tilde{x}\tilde{y}} & I_{\tilde{x}\tilde{z}}\\ I_{\tilde{y}\tilde{x}} & I_{\tilde{y}\tilde{y}} & I_{\tilde{y}\tilde{z}}\\ I_{\tilde{z}\tilde{x}} & I_{\tilde{z}\tilde{y}} & I_{\tilde{z}\tilde{z}}\end{bmatrix}\begin{bmatrix}\omega_{\tilde{x}}\\ \omega_{\tilde{y}}\\ \omega_{\tilde{z}}\end{bmatrix}\\ &= \sum_{i=1}^{N}\begin{bmatrix}0 & -r_{bi\tilde{z}} & r_{bi\tilde{y}}\\ r_{bi\tilde{z}} & 0 & -r_{bi\tilde{x}}\\ -r_{bi\tilde{y}} & r_{bi\tilde{x}} & 0\end{bmatrix}\begin{bmatrix}F_{i\tilde{x}}\\ F_{i\tilde{y}}\\ F_{i\tilde{z}}\end{bmatrix}\end{aligned} \tag{80}$$

where $\boldsymbol{\omega} = \omega_{\tilde{x}}\tilde{\mathbf{i}} + \omega_{\tilde{y}}\tilde{\mathbf{j}} + \omega_{\tilde{z}}\tilde{\mathbf{k}}$ and $\mathbf{r_{bi}} = r_{bi\tilde{x}}\tilde{\mathbf{i}} + r_{bi\tilde{y}}\tilde{\mathbf{j}} + r_{bi\tilde{z}}\tilde{\mathbf{k}}$. When the axes of the body frame are chosen to be the principle rotational axes of the rigid body, $I_{\tilde{x}\tilde{x}} = I_1$, $I_{\tilde{y}\tilde{y}} = I_2$, $I_{\tilde{z}\tilde{z}} = I_3$, and $I_{\tilde{x}\tilde{y}} = I_{\tilde{x}\tilde{z}} = I_{\tilde{y}\tilde{z}} = 0$. Thus (80) is further simplified to

$$\begin{aligned}I_1\tfrac{d}{dt}\omega_{\tilde{x}} + (I_3 - I_2)\,\omega_{\tilde{y}}\omega_{\tilde{z}} &= \sum_{i=1}^{N}(r_{bi\tilde{y}}F_{i\tilde{z}} - r_{bi\tilde{z}}F_{i\tilde{y}})\\ I_2\tfrac{d}{dt}\omega_{\tilde{y}} + (I_1 - I_3)\,\omega_{\tilde{z}}\omega_{\tilde{x}} &= \sum_{i=1}^{N}(r_{bi\tilde{z}}F_{i\tilde{x}} - r_{bi\tilde{x}}F_{i\tilde{z}})\\ I_3\tfrac{d}{dt}\omega_{\tilde{z}} + (I_2 - I_1)\,\omega_{\tilde{x}}\omega_{\tilde{y}} &= \sum_{i=1}^{N}(r_{bi\tilde{x}}F_{i\tilde{y}} - r_{bi\tilde{y}}F_{i\tilde{x}})\end{aligned} \tag{81}$$

With the roll, pitch, and yaw angles of ϕ, θ, ψ in describing the rotation of the body-frame with respect to the frame $oxyz$, from (37), the rotational dynamic equations in terms of the three Euler angles are obtained from (81) as

$$\underbrace{\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}}_{\mathbf{I}_{\tilde{x}\tilde{y}\tilde{z}}} \mathbf{\Omega}_{\phi\theta\psi} \begin{bmatrix} \ddot{\phi} \\ \ddot{\theta} \\ \ddot{\psi} \end{bmatrix} + \mathbf{I}_{\tilde{x}\tilde{y}\tilde{z}} \underbrace{\begin{bmatrix} 0 & 0 & -\dot{\theta}C_\theta \\ 0 & -\dot{\phi}S_\phi & \dot{\phi}C_\phi C_\theta - \dot{\theta}S_\phi S_\theta \\ 0 & -\dot{\phi}C_\phi & -\dot{\phi}S_\phi C_\theta - \dot{\theta}C_\phi S_\theta \end{bmatrix}}_{\dot{\mathbf{\Omega}}_{\phi\theta\psi}} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$
$$+ \underbrace{\begin{bmatrix} 0 & \dot{\theta}S_\phi - \dot{\psi}C_\phi C_\theta & \dot{\theta}C_\phi + \dot{\psi}S_\phi C_\theta \\ -\dot{\theta}S_\phi + \dot{\psi}C_\phi C_\theta & 0 & -\dot{\phi} + \dot{\psi}S_\theta \\ -\dot{\theta}C_\phi - \dot{\psi}S_\phi C_\theta & \dot{\phi} - \dot{\psi}S_\theta & 0 \end{bmatrix}}_{\mathbf{S}_{\boldsymbol{\omega}_{\tilde{x}\tilde{y}\tilde{z}}}} \mathbf{I}_{\tilde{x}\tilde{y}\tilde{z}} \mathbf{\Omega}_{\phi\theta\psi} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$
$$= \begin{bmatrix} \sum_{i=1}^{N} (r_{bi\tilde{y}} F_{i\tilde{z}} - r_{bi\tilde{z}} F_{i\tilde{y}}) \\ \sum_{i=1}^{N} (r_{bi\tilde{z}} F_{i\tilde{x}} - r_{bi\tilde{x}} F_{i\tilde{z}}) \\ \sum_{i=1}^{N} (r_{bi\tilde{x}} F_{i\tilde{y}} - r_{bi\tilde{y}} F_{i\tilde{x}}) \end{bmatrix} = \sum_{i=1}^{N} \left\{ \mathbf{S}_{\mathbf{r_{bi}}_{\tilde{x}\tilde{y}\tilde{z}}} \mathbf{F_i}_{\tilde{x}\tilde{y}\tilde{z}} \right\} \tag{82}$$

6.6 Dynamics of a Rigid Body via Lagrangian Equations

In the previous subsection, dynamics of a rigid body is obtained via Newtonian or vector mechanics, which involve the explicit calculation of all forces and moments acting on the rigid body. For a system consisting of several different rigid bodies, those forces and moments in the dynamics of a particular rigid body via Newtonian mechanics include all the interaction forces among the different rigid bodies as well, which may complicate the development of dynamic equations for the overall system significantly. To by-pass this problem, the Lagrangian equations of motions method can be used instead. To illustrate the benefits of such a method, it will be used to obtain the dynamics of a rigid body in this subsection and will be compared to the Newtonian method presented in the previous subsection.

The Lagrangian of a system is defined to be

$$L(\mathbf{q}, \dot{q}) = T(\mathbf{q}, \dot{q}) - P(\mathbf{q}) \tag{83}$$

where $\mathbf{q} \in \mathcal{R}^n$ represents the generalized coordinates of the system and T and P are the kinetic energy and potential energy of the system. With this scalar Lagrangian, dynamics of a system with holonomic constraints only can then be obtained by the Euler-Lagrange equations given by

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} - \frac{\partial L}{\partial \mathbf{q}} = \boldsymbol{\tau} \tag{84}$$

where $\boldsymbol{\tau}$ represents the vector of generalized forces acting on the system. For a rigid body, as shown in previous sections, its configuration is completely determined by the position of the origin of a body frame (i.e., $\mathbf{r_o}$) and the rotation of the body frame with respect to this origin (e.g., any set of three Euler angles). Thus the generalized coordinates of a rigid body can be chosen as $\mathbf{q} = [\mathbf{q_1}^T, \mathbf{q_2}^T]^T = [r_{oX}, r_{oY}, r_{oZ}, \phi, \theta, \psi]^T$, in which the first three coordinates $\mathbf{q_1} = [r_{oX}, r_{oY}, r_{oZ}]^T$ specify the position of the origin of the body-frame in the inertia frame and the last three coordinates $\mathbf{q_2} = [\phi, \theta, \psi]^T$ are the yaw, pitch, and roll angles of the body-frame with respect to the frame $oxyz$. With this set of generalized coordinates, noting (37) and, for simplicity, assuming that the axes of the body frame have been chosen to be the principle rotational axes of the rigid body, the kinetic energy of a rigid body (67) becomes

$$\begin{aligned} T &= \tfrac{1}{2}\Bigg\{m\left(\dot{r}_{oX}^2 + \dot{r}_{oY}^2 + \dot{r}_{oZ}^2\right) + \\ &\qquad \left[\dot{\phi}\ \dot{\theta}\ \dot{\psi}\right] \underbrace{\begin{bmatrix} 1 & 0 & -S_\theta \\ 0 & C_\phi & S_\phi C_\theta \\ 0 & -S_\phi & C_\phi C_\theta \end{bmatrix}^T}_{\boldsymbol{\Omega}^T_{\phi\theta\psi}} \underbrace{\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}}_{\mathbf{I}_{\tilde{x}\tilde{y}\tilde{z}}} \underbrace{\begin{bmatrix} 1 & 0 & -S_\theta \\ 0 & C_\phi & S_\phi C_\theta \\ 0 & -S_\phi & C_\phi C_\theta \end{bmatrix}}_{\boldsymbol{\Omega}_{\phi\theta\psi}} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}\Bigg\} \\ &= \tfrac{1}{2}\left\{m\dot{\mathbf{q}}_\mathbf{1}^T\dot{\mathbf{q}}_\mathbf{1} + \dot{\mathbf{q}}_\mathbf{2}^T \underbrace{\boldsymbol{\Omega}^T_{\phi\theta\psi}\mathbf{I}_{\tilde{x}\tilde{y}\tilde{z}}\boldsymbol{\Omega}_{\phi\theta\psi}}_{\mathbf{M_2}(\mathbf{q_2})} \dot{\mathbf{q}}_\mathbf{2}\right\} \end{aligned} \tag{85}$$

where the rotational inertia matrix $\mathbf{M_2}(\mathbf{q_2})$ is symmetric and positive definite and is uniquely determined by $\mathbf{q_2}$ as

$$\mathbf{M_2}(\mathbf{q_2}) = \boldsymbol{\Omega}^\mathbf{T}_{\phi\theta\psi}\mathbf{I}_{\tilde{\mathbf{x}}\tilde{\mathbf{y}}\tilde{\mathbf{z}}}\boldsymbol{\Omega}_{\phi\theta\psi} \tag{86}$$

Thus,

$$\frac{\partial L}{\partial \dot{\mathbf{q}}_\mathbf{1}} = \frac{\partial T}{\partial \dot{\mathbf{q}}_\mathbf{1}} = m\dot{\mathbf{q}}_\mathbf{1} \quad \Rightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}_\mathbf{1}} = m\ddot{\mathbf{q}}_\mathbf{1} \tag{87}$$

$$\frac{\partial L}{\partial \mathbf{q_1}} = -\frac{\partial P(\mathbf{q})}{\partial \mathbf{q_1}} = m\mathbf{g}_{\tilde{\mathbf{x}}\tilde{\mathbf{y}}\tilde{\mathbf{z}}} \tag{88}$$

$$\begin{gathered} \tfrac{\partial L}{\partial \dot{\mathbf{q}}_\mathbf{2}} = \tfrac{\partial T}{\partial \dot{\mathbf{q}}_\mathbf{2}} = \mathbf{M_2}(\mathbf{q_2})\dot{\mathbf{q}}_\mathbf{2} \quad \Rightarrow \quad \tfrac{\mathbf{d}}{\mathbf{dt}}\tfrac{\partial \mathbf{L}}{\partial \dot{\mathbf{q}}_\mathbf{2}} = \mathbf{M_2}(\mathbf{q_2})\ddot{\mathbf{q}}_\mathbf{2} + \dot{\mathbf{M}}_\mathbf{2}\dot{\mathbf{q}}_\mathbf{2}, \\ \dot{\mathbf{M}}_\mathbf{2} = \dot{\boldsymbol{\Omega}}_{\boldsymbol{\phi\theta\psi}}{}^\mathbf{T}\mathbf{I}_{\tilde{\mathbf{x}}\tilde{\mathbf{y}}\tilde{\mathbf{z}}}\boldsymbol{\Omega}_{\phi\theta\psi} + \boldsymbol{\Omega}^\mathbf{T}_{\phi\theta\psi}\mathbf{I}_{\tilde{\mathbf{x}}\tilde{\mathbf{y}}\tilde{\mathbf{z}}}\dot{\boldsymbol{\Omega}}_{\boldsymbol{\phi\theta\psi}} \end{gathered} \tag{89}$$

$$\frac{\partial L}{\partial \mathbf{q_2}} = \frac{\partial T}{\partial \mathbf{q_2}} = \frac{1}{2}\begin{bmatrix} \dot{\mathbf{q}}_\mathbf{2}^T \frac{\mathbf{M_2}(\mathbf{q_2})}{\partial \phi}\dot{\mathbf{q}}_\mathbf{2} \\ \dot{\mathbf{q}}_\mathbf{2}^T \frac{\mathbf{M_2}(\mathbf{q_2})}{\partial \theta}\dot{\mathbf{q}}_\mathbf{2} \\ \dot{\mathbf{q}}_\mathbf{2}^T \frac{\mathbf{M_2}(\mathbf{q_2})}{\partial \psi}\dot{\mathbf{q}}_\mathbf{2} \end{bmatrix} \tag{90}$$

where $\dot{\boldsymbol{\Omega}}_{\boldsymbol{\phi\theta\psi}}$ is as shown in (82) and

$$
\begin{aligned}
\frac{\mathbf{M_2(q_2)}}{\partial\phi} &= \frac{\partial\boldsymbol{\Omega}_{\phi\theta\psi}}{\partial\phi}^T \mathbf{I}_{\tilde{x}\tilde{y}\tilde{z}}\boldsymbol{\Omega}_{\phi\theta\psi} + \boldsymbol{\Omega}^T_{\phi\theta\psi}\mathbf{I}_{\tilde{x}\tilde{y}\tilde{z}}\frac{\partial\boldsymbol{\Omega}_{\phi\theta\psi}}{\partial\phi}, \\
\frac{\partial\boldsymbol{\Omega}_{\phi\theta\psi}}{\partial\phi} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -S_\phi & C_\phi C_\theta \\ 0 & -C_\phi & -S_\phi C_\theta \end{bmatrix} \\
\frac{\mathbf{M_2(q_2)}}{\partial\theta} &= \frac{\partial\boldsymbol{\Omega}_{\phi\theta\psi}}{\partial\theta}^T \mathbf{I}_{\tilde{x}\tilde{y}\tilde{z}}\boldsymbol{\Omega}_{\phi\theta\psi} + \boldsymbol{\Omega}^T_{\phi\theta\psi}\mathbf{I}_{\tilde{x}\tilde{y}\tilde{z}}\frac{\partial\boldsymbol{\Omega}_{\phi\theta\psi}}{\partial\theta}, \\
\frac{\partial\boldsymbol{\Omega}_{\phi\theta\psi}}{\partial\theta} &= \begin{bmatrix} 0 & 0 & -C_\theta \\ 0 & 0 & -S_\phi S_\theta \\ 0 & 0 & -C_\phi S_\theta \end{bmatrix} \\
\frac{\mathbf{M_2(q_2)}}{\partial\psi} &= \frac{\partial\boldsymbol{\Omega}_{\phi\theta\psi}}{\partial\psi}^T \mathbf{I}_{\tilde{x}\tilde{y}\tilde{z}}\boldsymbol{\Omega}_{\phi\theta\psi} + \boldsymbol{\Omega}^T_{\phi\theta\psi}\mathbf{I}_{\tilde{x}\tilde{y}\tilde{z}}\frac{\partial\boldsymbol{\Omega}_{\phi\theta\psi}}{\partial\psi}, \quad \frac{\partial\boldsymbol{\Omega}_{\phi\theta\psi}}{\partial\psi} = 0
\end{aligned}
\tag{91}
$$

The Euler-Lagrange equations (84) for $\mathbf{q_1}$ thus become

$$
m\ddot{\mathbf{q}}_\mathbf{1} - m\mathbf{g}_{\tilde{\mathbf{x}}\tilde{\mathbf{y}}\tilde{\mathbf{z}}} = \boldsymbol{\tau}_\mathbf{1} \tag{92}
$$

and for $\mathbf{q_2}$

$$
\mathbf{M_2(q_2)}\ddot{\mathbf{q}}_\mathbf{2} + \dot{\mathbf{M}}_\mathbf{2}\dot{\mathbf{q}}_\mathbf{2} - \frac{\mathbf{1}}{\mathbf{2}}\begin{bmatrix} \dot{\mathbf{q}}_\mathbf{2}{}^T\frac{\mathbf{M_2(q_2)}}{\partial\phi}\dot{\mathbf{q}}_\mathbf{2} \\ \dot{\mathbf{q}}_\mathbf{2}{}^T\frac{\mathbf{M_2(q_2)}}{\partial\theta}\dot{\mathbf{q}}_\mathbf{2} \\ \dot{\mathbf{q}}_\mathbf{2}{}^T\frac{\mathbf{M_2(q_2)}}{\partial\psi}\dot{\mathbf{q}}_\mathbf{2} \end{bmatrix} = \boldsymbol{\tau}_\mathbf{2} \tag{93}
$$

To obtain the generalized force vectors $\boldsymbol{\tau}_\mathbf{1}$ and $\boldsymbol{\tau}_\mathbf{2}$, consider the virtual work δW done by all non-potential forces acting on the rigid body (i.e., $\mathbf{F_i}, i = 1, \ldots, N$) for an infinitesimal virtual displacement of $\delta\mathbf{q} = [\delta\mathbf{q_1}{}^T, \delta\mathbf{q_2}{}^T]^T = [\delta r_{oX}, \delta r_{oX}, \delta r_{oX}, \delta\phi, \delta\theta, \delta\psi]^T$. Since $\mathbf{r_{ib}}$ is a body-fixed vector, $\dot{\mathbf{r}}_\mathbf{ib} = \boldsymbol{\omega}\times\mathbf{r_{ib}}$ or $d\mathbf{r_{ib}} = (\boldsymbol{\omega}dt)\times\mathbf{r_{ib}} = -\mathbf{r_{ib}}\times(\boldsymbol{\omega}dt)$. Noting (37), $(\boldsymbol{\omega}dt)_{\tilde{x}\tilde{y}\tilde{z}} = \boldsymbol{\Omega}_{\phi\theta\psi}d\mathbf{q_2}$ and, viewing (44), the expression of the infinitesimal vector $d\mathbf{r_{ib}}$ in the body-frame is

$$
(d\mathbf{r_{ib}})_{\tilde{x}\tilde{y}\tilde{z}} = -\mathbf{S}_{\mathbf{r_{ib}}\tilde{x}\tilde{y}\tilde{z}}(\boldsymbol{\omega}dt)_{\tilde{x}\tilde{y}\tilde{z}} = -\mathbf{S}_{\mathbf{r_{ib}}\tilde{x}\tilde{y}\tilde{z}}\boldsymbol{\Omega}_{\phi\theta\psi}d\mathbf{q_2} \tag{94}
$$

It is thus clear that the virtual displacement vector $\delta\mathbf{r_{ib}}$ caused by the infinitesimal virtual displacement of $\delta\mathbf{q}$ has the following expression in the body-frame

$$
(\delta\mathbf{r_{ib}})_{\tilde{x}\tilde{y}\tilde{z}} = -\mathbf{S}_{\mathbf{r_{ib}}\tilde{x}\tilde{y}\tilde{z}}\boldsymbol{\Omega}_{\phi\theta\psi}\delta\mathbf{q_2} \tag{95}
$$

Noting that $(\delta\mathbf{r_o})_{\tilde{x}\tilde{y}\tilde{z}} = \delta\mathbf{q_1}$, the virtual work δW can now be calculated as

$$
\begin{aligned}
\delta W &= \sum_{i=1}^N \mathbf{F_i}\cdot\delta\mathbf{r_i} = \sum_{i=1}^N \mathbf{F_i}\cdot(\delta\mathbf{r_o} + \delta\mathbf{r_{ib}}) \\
&= \sum_{i=1}^N \mathbf{F_i}\cdot\delta\mathbf{r_o} + \sum_{i=1}^N \mathbf{F_i}\delta\mathbf{r_{ib}}) \\
&= \underbrace{\sum_{i=1}^N \mathbf{F_i}{}^T_{\tilde{x}\tilde{y}\tilde{z}}}_{\boldsymbol{\tau}_\mathbf{1}{}^T}\delta\mathbf{q_1} + \underbrace{\left[-\sum_{i=1}^N \mathbf{F_i}{}^T_{\tilde{x}\tilde{y}\tilde{z}} - \mathbf{S}_{\mathbf{r_{ib}}\tilde{x}\tilde{y}\tilde{z}}\boldsymbol{\Omega}_{\phi\theta\psi}\right]}_{\boldsymbol{\tau}_\mathbf{2}{}^T}\delta\mathbf{q_2}
\end{aligned}
\tag{96}
$$

from which the generalized forces are obtained as

$$
\begin{aligned}
\boldsymbol{\tau_1} &= \sum_{i=1}^{N} \mathbf{F_i}_{\tilde{x}\tilde{y}\tilde{z}} \\
\boldsymbol{\tau_2} &= -\boldsymbol{\Omega}_{\phi\theta\psi}^{T} \sum_{i=1}^{N} \mathbf{S}_{\mathbf{r_{ib}}\tilde{x}\tilde{y}\tilde{z}}^{T} \mathbf{F_i}_{\tilde{x}\tilde{y}\tilde{z}} = \boldsymbol{\Omega}_{\phi\theta\psi}^{T} \sum_{i=1}^{N} \mathbf{S}_{\mathbf{r_{ib}}\tilde{x}\tilde{y}\tilde{z}} \mathbf{F_i}_{\tilde{x}\tilde{y}\tilde{z}}
\end{aligned}
\tag{97}
$$

in which the fact that $\mathbf{S}_{\mathbf{r_{ib}}\tilde{x}\tilde{y}\tilde{z}}$ is a skew-symmetric matrix has been used. It is thus clear that (92) is the same as the one obtained using the Newton equations in (77) and (93) is the Newton-Euler equations (82) multiplied by the matrix $\boldsymbol{\Omega}_{\phi\theta\psi}^{T}$ from the left. However, different from the equations (82) where it is hard to see the essential structure of the system dynamics, the equations (93) obtained through the Lagrangian approach have the clear structural form of

$$
\mathbf{M_2}(\mathbf{q_2})\ddot{\mathbf{q}}_\mathbf{2} + \mathbf{C_2}(\mathbf{q_2}, \dot{\mathbf{q}}_\mathbf{2})\dot{\mathbf{q}}_\mathbf{2} = \boldsymbol{\tau_2} = \boldsymbol{\Omega}_{\phi\theta\psi}^{\mathbf{T}} \sum_{\mathbf{i=1}}^{\mathbf{N}} \mathbf{S}_{\mathbf{r_{ib}}\tilde{\mathbf{x}}\tilde{\mathbf{y}}\tilde{\mathbf{z}}} \mathbf{F}_{\mathbf{i}\tilde{\mathbf{x}}\tilde{\mathbf{y}}\tilde{\mathbf{z}}} \tag{98}
$$

where the matrix $\mathbf{C_2}(\mathbf{q_2}, \dot{\mathbf{q}}_\mathbf{2})$ is defined as

$$
\mathbf{C_2} = \frac{\mathbf{1}}{\mathbf{2}} \left\{ \dot{\mathbf{M}}_\mathbf{2} + \left[\frac{\mathbf{M_2}(\mathbf{q_2})}{\partial\phi}\dot{\mathbf{q}}_\mathbf{2} \quad \frac{\mathbf{M_2}(\mathbf{q_2})}{\partial\theta}\dot{\mathbf{q}}_\mathbf{2} \quad \frac{\mathbf{M_2}(\mathbf{q_2})}{\partial\psi}\dot{\mathbf{q}}_\mathbf{2} \right] - \begin{bmatrix} \dot{\mathbf{q}}_\mathbf{2}{}^{T} \frac{\mathbf{M_2}(\mathbf{q_2})}{\partial\phi} \\ \dot{\mathbf{q}}_\mathbf{2}{}^{T} \frac{\mathbf{M_2}(\mathbf{q_2})}{\partial\theta} \\ \dot{\mathbf{q}}_\mathbf{2}{}^{T} \frac{\mathbf{M_2}(\mathbf{q_2})}{\partial\psi} \end{bmatrix} \right\} \tag{99}
$$

In obtaining (98), the following fact has been used

$$
\begin{aligned}
\dot{\mathbf{M}}_\mathbf{2}\dot{\mathbf{q}}_\mathbf{2} &= \left(\frac{\mathbf{M_2}(\mathbf{q_2})}{\partial\phi}\dot{\phi} + \frac{\mathbf{M_2}(\mathbf{q_2})}{\partial\theta}\dot{\theta} + \frac{\mathbf{M_2}(\mathbf{q_2})}{\partial\psi}\dot{\psi} \right) \dot{\mathbf{q}}_\mathbf{2} \\
&= \left[\frac{\mathbf{M_2}(\mathbf{q_2})}{\partial\phi}\dot{\mathbf{q}}_\mathbf{2} \quad \frac{\mathbf{M_2}(\mathbf{q_2})}{\partial\theta}\dot{\mathbf{q}}_\mathbf{2} \quad \frac{\mathbf{M_2}(\mathbf{q_2})}{\partial\psi}\dot{\mathbf{q}}_\mathbf{2} \right] \dot{\mathbf{q}}_\mathbf{2}
\end{aligned}
\tag{100}
$$

Noting that

$$
\dot{\mathbf{M}}_\mathbf{2} - 2\mathbf{C_2} = -\left[\frac{\mathbf{M_2}(\mathbf{q_2})}{\partial\phi}\dot{\mathbf{q}}_\mathbf{2} \quad \frac{\mathbf{M_2}(\mathbf{q_2})}{\partial\theta}\dot{\mathbf{q}}_\mathbf{2} \quad \frac{\mathbf{M_2}(\mathbf{q_2})}{\partial\psi}\dot{\mathbf{q}}_\mathbf{2} \right] + \begin{bmatrix} \dot{\mathbf{q}}_\mathbf{2}{}^{T} \frac{\mathbf{M_2}(\mathbf{q_2})}{\partial\phi} \\ \dot{\mathbf{q}}_\mathbf{2}{}^{T} \frac{\mathbf{M_2}(\mathbf{q_2})}{\partial\theta} \\ \dot{\mathbf{q}}_\mathbf{2}{}^{T} \frac{\mathbf{M_2}(\mathbf{q_2})}{\partial\psi} \end{bmatrix} \tag{101}
$$

the matrix $\dot{\mathbf{M}}_\mathbf{2} - 2\mathbf{C_2}(\mathbf{q_2}, \dot{\mathbf{q}}_\mathbf{2})$ is skew-symmetric, which is one of the properties normally presented for the dynamics of a multi degrees-of-freedom (DOF) robot manipulator.