Chapter 08
Preliminary of Projective Geometry

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Outline

- Vector Operations
- Fundamentals of Projective Geometry
  - Projective plane
  - Homogeneous coordinate
  - Lines and points on the projective plane

Vector operations

Vector representation

$$\vec{a} = x\vec{i} + y\vec{j} + z\vec{k} = \{x, y, z\}$$

Length (or norm) of a vector

$$\left|\vec{a}\right| = \sqrt{x^2 + y^2 + z^2}$$

Normalized vector (unit vector)

$$\frac{\vec{a}}{\left|\vec{a}\right|} = \{\frac{x}{\left|\vec{a}\right|}, \frac{y}{\left|\vec{a}\right|}, \frac{z}{\left|\vec{a}\right|}\}$$

We say $\vec{a} = \mathbf{0}$, if and only if $x = 0, y = 0, z = 0$

if $\vec{a} = (x_1, y_1, z_1)$, $\vec{b} = (x_2, y_2, z_2)$,

then $\vec{a} \pm \vec{b} = (x_1 \pm x_2, y_1 \pm y_2, z_1 \pm z_2)$,

Dot product (inner product)

$$\vec{a} \cdot \vec{b} = \left|\vec{a}\right|\left|\vec{b}\right| \cos\theta = x_1 x_2 + y_1 y_2 + z_1 z_2$$

Laws of dot product:

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}, \vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

Theorem

$$\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b} \quad \text{(why?)}$$
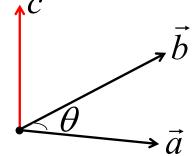
Vector operations

Cross product

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} i + \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix} j + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} k$$

Vector operations

Cross product

$\vec{c} = \vec{a} \times \vec{b}$ is also a vector, whose direction is determined by the right-hand law and

$\vec{c} \perp \vec{a}, \vec{c} \perp \vec{b}$

$|\vec{c}| = |\vec{a}||\vec{b}|\sin\theta$

$\boxed{\vec{c} \text{ represents the oriented area of the parallelogram taking } \vec{a} \text{ and } \vec{b} \text{ as two sides} \quad \text{(easy to prove)}}$

$\vec{r_1} \times \vec{r_2} = -\vec{r_2} \times \vec{r_1}$ (why?)

Vector operations

Cross product

Theorem
$$\vec{a} \parallel \vec{b} \Leftrightarrow \vec{a} \times \vec{b} = \mathbf{0} \quad \text{(why?)}$$

Theorem
$$\vec{a} \parallel \vec{b} \Leftrightarrow \exists \lambda, \mu, \text{ they are not equal to zero at the}$$
same time, and $\lambda\vec{a} + \mu\vec{b} = \mathbf{0}$ (easy to understand)

Property

$$\vec{r_1} \times (\vec{r_2} + \vec{r_3}) = \vec{r_1} \times \vec{r_2} + \vec{r_1} \times \vec{r_3}$$

Vector operations

Cross product

Definition

Suppose $\vec{a} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ $\qquad \vec{a}^{\wedge} \triangleq \begin{bmatrix} 0 & -z_1 & y_1 \\ z_1 & 0 & -x_1 \\ -y_1 & x_1 & 0 \end{bmatrix}$

Then,

$$\vec{a} \times \vec{b} = \vec{a}^{\wedge} \vec{b}$$

Vector operations

Mixed product (scalar triple product or box product)

$$(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Geometric Interpretation: it is the (signed) volume of the parallelepiped defined by the three vectors given
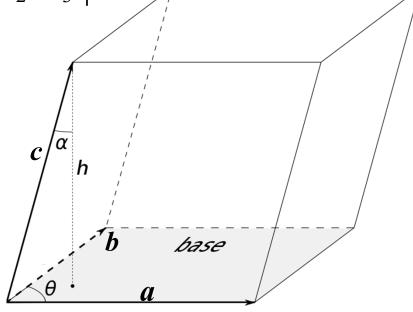
Vector operations

Mixed product (scalar triple product or box product)

$$(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} = |\boldsymbol{a} \times \boldsymbol{b}||\boldsymbol{c}|\cos\alpha$$

$$= \underbrace{|\boldsymbol{a}||\boldsymbol{b}|\sin\theta}_{\text{Base}} \cdot \underbrace{|\boldsymbol{c}|\cos\alpha}_{h}$$

Vector operations

Mixed product (scalar triple product or box product)

$$(a, b, c) = (a \times b) \cdot c = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Property:

$$(a, b, c) = (b, c, a) = (c, a, b)$$

$$(a, b, c) = -(b, a, c) = -(a, c, b)$$

why?

Vector operations

Mixed product (scalar triple product or box product)

Theorem

$$\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \text{ are coplanar} \iff (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) = 0$$

why?
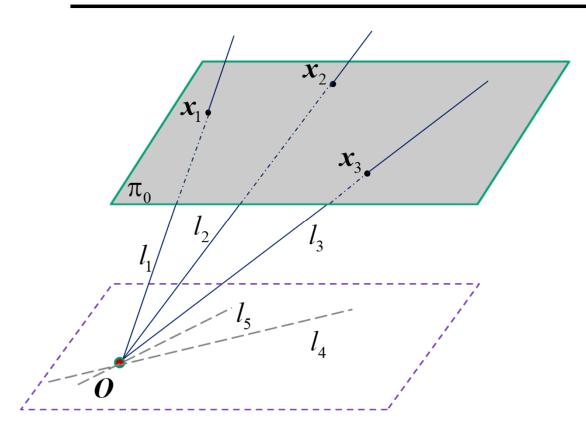
$\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are coplanar $\iff \exists \lambda, \mu, \nu,$ they are not equal to zero at the same time, and $\lambda \boldsymbol{a} + \mu \boldsymbol{b} + \nu \boldsymbol{c} = \boldsymbol{0}$

Outline

- Vector Operations

- Fundamentals of Projective Geometry
  - Projective plane
  - Homogeneous coordinate
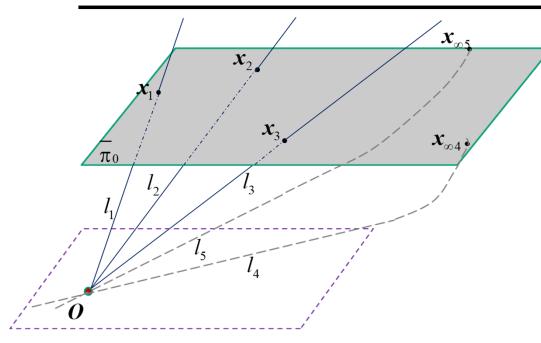 - Lines and points on the projective plane

Projective plane

- Consider a spatial point $\boldsymbol{O}$ and an Euclidean plane π_0
- In general, the line l passing through $\boldsymbol{O}$ intersects with π_0 at one point $\boldsymbol{x}$ and we say l and $\boldsymbol{x}$ are **associated**
- For every point on π_0, it can be uniquely associated to one line passing through $\boldsymbol{O}$; However, the converse of this is not true: the lines parallel to π_0 do not intersect with π_0 and thus they cannot be associated to any points

How to modify π_0 to make its points associated to lines passing through $\boldsymbol{O}$ in a one-to-one manner?
Add infinity points to π_0!

Projective plane

π_0 + infinity line = Projective plane or augmented Euclidean plane $\overline{\pi_0}$

With the following rules, we add infinity points to π_0:

- ✓ It is stipulated that lines passing through O and parallel to π_0 (such as l_4 and l_5) also intersect π_0, and the intersection points are called **infinity points**
- ✓ For two different lines passing through O and parallel to π_0, they intersect π_0 at two distinct infinity points
- ✓ It is stipulated that each line in π_0 has a corresponding *unique* infinity point; two lines with the same orientation (i.e., parallel lines) share the same infinity point, while two lines with different orientations have different infinity points
- ✓ It is stipulated that all infinity points on π_0 form the **infinity line**

The points on the projective plane $\overline{\pi_0}$ can be associated to lines passing through O in a one-to-one manner

Projective plane

Based on the stipulations we have made above, some obvious inferences can be drawn

✓ Two parallel lines in π_0 also intersect, meeting at their common infinity point

✓ A plane passing through O and parallel to π_0 also intersects π_0, and the resulting intersection line is the infinity line

✓ Since we have stipulated that each line in π_0 has only one infinity point, this means that for each line, when it extends infinitely in both directions, it will reach the same infinity point; In other words, on the projective plane, a line is conceptually "closed", similar to a circle

✓ Let l be a line passing through O and parallel to π_0, which intersects π_0 at the infinity point $\boldsymbol{p}_\infty$ on π_0. If l' is a line in π_0 parallel to l, then the infinity point of l' is $\boldsymbol{p}_\infty$, meaning that l and l' actually intersect at $\boldsymbol{p}_\infty$

The most significant difference between the projective plane and the ordinary Euclidean plane is that on the projective plane, the concept of parallelism no longer exists, and any two lines will intersect. On the projective plane, two lines may intersect at a normal point; if the two lines are parallel in the sense of Euclidean geometry, they will intersect at an infinity point; a normal line and the infinity line will intersect at an infinity point

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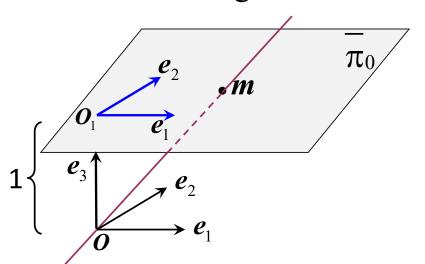
Homogeneous coordinate

- To further study the geometric relationships on the projective plane, it is necessary to introduce coordinates for points

- However, ordinary plane coordinate systems cannot express the coordinates of infinity points, which compels us to seek a new strategy for representing the coordinates of points on the projective plane

Homogeneous coordinate

Homogeneous coordinate

- What is homogeneous coordinate?

In plane $\overline{\pi}_0$, under the 2D frame $[\boldsymbol{O}_1; \boldsymbol{e}_1, \boldsymbol{e}_2]$, consider the point $\boldsymbol{m}(x_0, y_0)$

Coordinate of any point (except $\boldsymbol{O}$) on line $\boldsymbol{Om}$ under the frame $[\boldsymbol{O}; \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3]$ is the homogeneous coordinate of $\boldsymbol{m}$
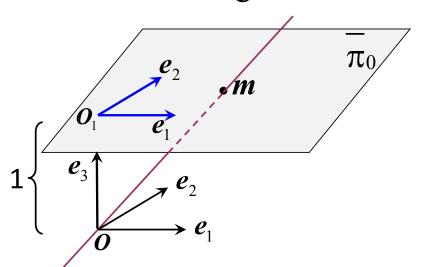
These points can be represented as $k(x_0, y_0, 1)^T, k \neq 0$

- ✓ If (x_1, x_2, x_3) is the homogeneous coordinate of $\boldsymbol{m}$, then $k(x_1, x_2, x_3)$ (k!=0) is also the homogeneous coordinate of $\boldsymbol{m}$ since (x_1, x_2, x_3) and $k(x_1, x_2, x_3)$ lie on the same line passing through $\boldsymbol{O}$; the homogeneous coordinates of a point are not unique but are proportional to one another
- ✓ If (x_1, x_2, x_3) and (y_1, y_2, y_3) are not proportional, they must lie on different lines passing through $\boldsymbol{O}$, and thus represent different points on π_0; the homogeneous coordinates of different points on π_0 are not proportional

Homogeneous coordinate

- What is homogeneous coordinate?

In plane $\bar{\pi}_0$, under the 2D frame $[\boldsymbol{O}_1; \boldsymbol{e}_1, \boldsymbol{e}_2]$, consider the point $\boldsymbol{m}(x_0, y_0)$

Coordinate of any point (except $\boldsymbol{O}$) on line $\boldsymbol{Om}$ under the frame $[\boldsymbol{O}; \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3]$ is the homogeneous coordinate of $\boldsymbol{m}$
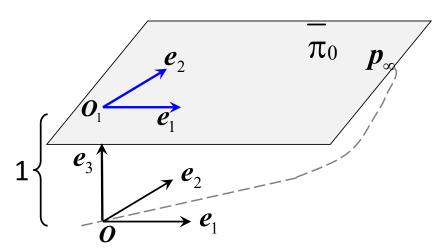
These points can be represented as $k\left(x_0, y_0, 1\right)^T, k \neq 0$

How about the infinity points?

Homogeneous coordinate

- What is homogeneous coordinate?

For one infinity point $\boldsymbol{p}_\infty$, it is associated to one line parallel to $\overline{\pi_0}$; under the frame $[\boldsymbol{O}; \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3]$, the points of this line should be of the form $(x_1, x_2, 0)$ (x_1 and x_2 cannot both be zero) , and thus, the coordinate of an infinity point should be of the form $k(x_1, x_2, 0)$ (k!=0)

✓ Similar as the normal point case, if $(x_1, x_2, 0)$ and $(y_1, y_2, 0)$ are proportional, actually they represent the same infinity point; for different infinity points, their coordinates are not proportional to one another
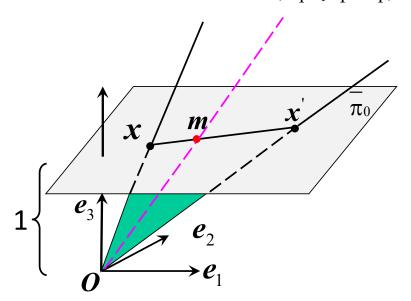
Outline

- Vector Operations

- Fundamentals of Projective Geometry

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  - Lines and points on the projective plane

Lines and Points on the Projective Plane

- The line determined by two points

On a projective plane, please determine the line passing two points $\boldsymbol{x} = (x_1, y_1, z_1)^T$, $\boldsymbol{x}' = (x_2, y_2, z_2)^T$
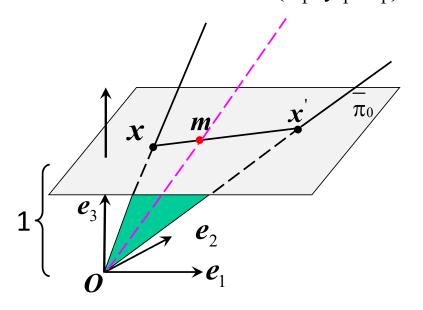
$\boldsymbol{Ox}$, $\boldsymbol{Ox}'$ determine two lines

The line determined by $\boldsymbol{xx}'$ actually is the intersection between $\boldsymbol{Oxx}'$ and $\overline{\pi_0}$

Lines and Points on the Projective Plane

- The line determined by two points

On a projective plane, please determine the line passing two points $\boldsymbol{x} = (x_1, y_1, z_1)^T$, $\boldsymbol{x}' = (x_2, y_2, z_2)^T$

Thus, $\boldsymbol{m}(x, y, z)$ locates on $\boldsymbol{xx}'$

$\Leftrightarrow \boldsymbol{Om}$ resides on the plane $\boldsymbol{Oxx}'$

$\Leftrightarrow \boldsymbol{Ox}, \boldsymbol{Om}, \boldsymbol{Ox}'$ are coplanar

$$\Leftrightarrow \begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0$$

Lines and Points on the Projective Plane

- The line determined by two points

On a projective plane, please determine the line passing two points $\boldsymbol{x} = (x_1, y_1, z_1)^T, \boldsymbol{x}' = (x_2, y_2, z_2)^T$

$$\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0 \iff \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} x + \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix} y + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} z = 0$$

$$\boldsymbol{l} \triangleq \left(\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix}, \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \right)^T$$

$\boldsymbol{l}$ is called the **homogeneous coordinate of the line** determined by $\boldsymbol{x}$ and $\boldsymbol{x}'$ and actually $\boldsymbol{l} = \boldsymbol{x} \times \boldsymbol{x}'$

Lines and Points on the Projective Plane

- The line determined by two points

> **Theorem**
>
> On the projective plane, the line passing two points x and x' is
> $$l = x \times x'$$

Example: Please give the coordinate of the infinity line on the projective plane

The infinity line can be determined by any two infinity points. Suppose that the selected two infinity points are $p_{\infty 1}(x_1, y_1, 0)$ and $p_{\infty 2}(x_2, y_2, 0)$ and they are not proportional to each other. The homogeneous coordinate of the line determined by them is,

$$l_\infty = p_{\infty 1} \times p_{\infty 2} = \left(x_1, y_1, 0\right) \times \left(x_2, y_2, 0\right) = \left(0, 0, \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}\right)$$

Usually, the coordinate of the infinity line can be represented as $k(0,0,1)$ ($k!=0$)

Lines and Points on the Projective Plane

- The line determined by two points

  On the projective plane, a line's homogeneous coordinate is $\boldsymbol{l}=(a, b, c)^T$. Then, the equation of this line is,

  $$\boldsymbol{l}^T \boldsymbol{x} = 0$$

  where $\boldsymbol{x}$ is the homogeneous coordinate of the point lying on this line

Lines and Points on the Projective Plane

- The intersection point determined by two lines

On the projective plane, there are two lines $\boldsymbol{l}=(a_1,b_1,c_1)$, $\boldsymbol{l'}=(a_2,b_2,c_2)$. The equations of the two lines represented by $\boldsymbol{l}$ and $\boldsymbol{l'}$ are,

$$\begin{cases} a_1x + b_1y + c_1z = 0 \\ a_2x + b_2y + c_2z = 0 \end{cases}$$

Please determine the intersection point of $\boldsymbol{l}$ and $\boldsymbol{l'}$

Lines and Points on the Projective Plane

- The intersection point determined by two lines

Case 1: l and l' are two normal lines and they intersect at a normal point

Denote the intersection point of l and l' by $x_0=(x_{10}, x_{20}, x_{30})$. Then we have

$$\begin{cases} a_1 x_{10} + b_1 x_{20} + c_1 x_{30} = 0 \\ a_2 x_{10} + b_2 x_{20} + c_2 x_{30} = 0 \end{cases}$$

Inhomogeneous form of x_0 is $\left(X_0 = \dfrac{x_{10}}{x_{30}}, Y_0 = \dfrac{x_{20}}{x_{30}} \right)$

$$\begin{cases} a_1 X_0 + b_1 Y_0 + c_1 = 0 \\ a_2 X_0 + b_2 Y_0 + c_2 = 0 \end{cases}$$

The intersection point is

$$X_0 = \frac{\begin{vmatrix} -c_1 & b_1 \\ -c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, Y_0 = \frac{\begin{vmatrix} a_1 & -c_1 \\ a_2 & -c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

x_0's homogeneous form is $k\left(\dfrac{\begin{vmatrix} -c_1 & b_1 \\ -c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \dfrac{\begin{vmatrix} a_1 & -c_1 \\ a_2 & -c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, 1 \right), k \neq 0$

let $k = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

$x_0 = \left(\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}, \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \right)$

$x_0 = l \times l'$

Lines and Points on the Projective Plane

- The intersection point determined by two lines

Case 2: l and l' are two normal lines and they intersect at an infinity point

   Please derive the result by yourself

Case 3: l is a normal line and l' is the infinity line, and they of course intersect at an infinity point

   Please derive the result by yourself

From the analysis of the above three cases, we can have the following conclusion:
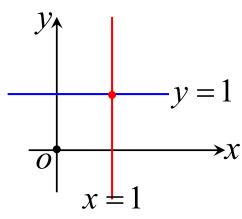
Theorem

On the projective plane, the intersection of two lines l and l' is the point

$$x = l \times l'$$

Lines and Points on the Projective Plane

- The intersection point determined by two lines

Example: find the cross point of the lines $x = 1, y = 1$

Homogeneous form

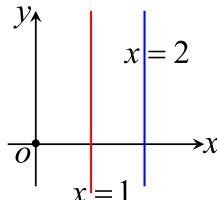$$\begin{cases} x_1 + 0x_2 + (-1)x_3 = 0 \\ 0x_1 + 1x_2 + (-1)x_3 = 0 \end{cases}$$

Homogeneous coordinates of the two lines are $(1, 0, -1)^T, (0, 1, -1)^T$

Cross point is

$$(1, 0, -1)^T \times (0, 1, -1)^T = (1, 1, 1)$$

Lines and Points on the Projective Plane

- The intersection point determined by two lines

Example: find the cross point of the lines $x = 1, x = 2$

$\downarrow$ Homogeneous form

$$\begin{cases} 1x_1 + 0x_2 + (-1)x_3 = 0 \\ 1x_1 + 0x_2 + (-2)x_3 = 0 \end{cases}$$

Homogeneous coordinates of the two lines are $(1, 0, -1)^T, (1, 0, -2)^T$

Cross point is

$$(1, 0, -1)^T \times (1, 0, -2)^T = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 1 & 0 & -2 \end{vmatrix} = (0, 1, 0)$$

Lines and Points on the Projective Plane

- Duality

  In projective geometry, lines and points can swap their positions

  $$x^T l = 0 \quad \textit{How to interpret?}$$

  If x is a variable, it represents the points lying on the line l;

  If l is a variable, it represents the lines passing a fixed point x

  The line passing two points x, x' is $l = x \times x'$
  The cross point of two lines l, l' is $x = l \times l'$

Duality Principle:

To any theorem of projective geometry, there corresponds a dual theorem, which may be derived by interchanging the roles of points and lines in the original theorem

More results you need to be familiar

- A set of parallel lines intersect at the same infinity point

- The homogeneous coordinate of the infinity line is $k(0,0,1)$

- The infinity point of a line can be identified as its intersection with the infinity line. E.g, on a projective plane, the infinity point of the X-axis is $k(1,0,0)$

- In 3D projective space, the infinity plane π_∞ are composed of points of the form $(x_1, x_2, x_3, x_4 = 0)$; You can also consider that the infinity plane comprises all the possible directions in 3D space

More results you need to be familiar

- Projective transformation

π_0, π_1 are two projective planes, $\mathbf{H} \in \mathbb{R}^{3\times3}$ is a matrix

$$\forall \boldsymbol{x}' \in \pi_1, \exists \text{ a unique } \boldsymbol{x} \in \pi_0, c_{\boldsymbol{x}'} \boldsymbol{x}' = \boldsymbol{H}\boldsymbol{x}$$

$$\forall \boldsymbol{x} \in \pi_0, \exists \text{ a unique } \boldsymbol{x}' \in \pi_1, c_{\boldsymbol{x}} \boldsymbol{x} = \boldsymbol{H}^{-1}\boldsymbol{x}'$$

We say π_0, π_1 can be projectively transformed to each other and $\boldsymbol{H}$ is the projective transformation matrix between them. For the 2-D case, $\boldsymbol{H}$ is also called as homography

Note 1: If π_0 can be projectively transformed to π_1, the projective transformation from π_0 to π_1 is unique up to a scale factor

Note 2: The above definition is for 2D case. It can be straightforwardly extended to other dimensions

More results you need to be familiar

- Projective transformation (typical examples in CV)