Chapter 03
Linear Geometric Transformations

Lin ZHANG
School of Computer Science and Technology
Tongji University

Outline

- Linear Geometric Transformations on 2D Plane
- Group and Geometry
- Linear Geometric Transformations in 3D Space

Linear geometric transformation

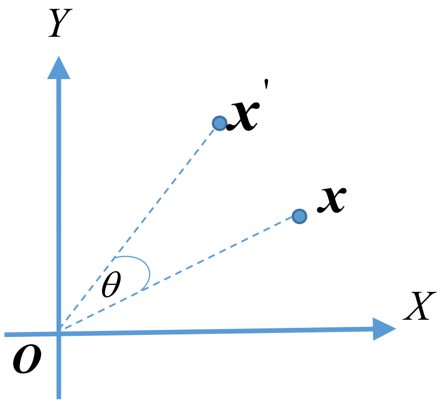
In $\mathbb{R}^n$, by a geometric transformation, a point $\boldsymbol{x} \in \mathbb{R}^n$ is mapped to $\boldsymbol{x}' \in \mathbb{R}^n$. If such a mapping can be represented by **an invertible matrix** $\boldsymbol{H} \in \mathbb{R}^{n \times n}$, i.e.,

$$\boldsymbol{x}' = \boldsymbol{H}\boldsymbol{x}$$

We say such a geometric transformation is a linear geometric transformation

Linear geometric transformations on 2D plane

- Rotation transformation (special orthogonal transformation)



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow$$

Homogeneous form

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\downarrow$$

$$\boldsymbol{x}' = \begin{bmatrix} \boldsymbol{R}_{2\times2} & \mathbf{0}_{2\times1} \\ \mathbf{0}_{1\times2} & 1 \end{bmatrix} \boldsymbol{x}$$

The DoF of the 2D rotation transformation is 1

A matrix that can represent a 2D rotation should be of the form,

$$\boldsymbol{H}_{3\times3} = \begin{bmatrix} \boldsymbol{R}_{2\times2} & \mathbf{0}_{2\times1} \\ \mathbf{0}_{1\times2} & 1 \end{bmatrix}, \boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{I}, \det(\boldsymbol{R}) = 1 \longrightarrow$$

Linear geometric transformations on 2D plane

- Rotation transformation (special orthogonal transformation)

For an $\boldsymbol{R}_{2\times 2}$. If $\boldsymbol{R}\boldsymbol{R}^T=\boldsymbol{I}$, $\det(\boldsymbol{R})=-1$, $\boldsymbol{R}$ actually represents an in-plane rotation with a reflection

An example

$$\boldsymbol{R}=\begin{bmatrix}\cos\frac{\pi}{6} & -\sin\frac{\pi}{6}\\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6}\end{bmatrix}\quad(\det(\boldsymbol{R})=1)$$

$$\boldsymbol{R}=\begin{bmatrix}-\cos\frac{\pi}{6} & -\sin\frac{\pi}{6}\\ -\sin\frac{\pi}{6} & \cos\frac{\pi}{6}\end{bmatrix}\quad(\det(\boldsymbol{R})=-1)$$

Linear geometric transformations on 2D plane

- Some notes about homogeneous coordinate

For a 2D point $\boldsymbol{x}$, its homogeneous coordinate is a 3D vector $(x_1, x_2, x_3)^T$

If $x_3 = 0$, $\boldsymbol{x}$ is an **infinity point**; otherwise, it is a **normal point**

$\boldsymbol{x}$'s homogeneous coordinate is not unique:

if $(x_1, x_2, x_3)^T$ is $\boldsymbol{x}$'s homogeneous coordinate, then, $\forall k \neq 0$, $k(x_1, x_2, x_3)^T$ is also $\boldsymbol{x}$'s homogeneous coordinate

For a normal point $\boldsymbol{x}=(x_1, x_2, x_3)^T$, it has a unique **normalized homogeneous coordinate** $(x_1/x_3, x_2/x_3, 1)^T$; For an infinity point $\boldsymbol{x}=(x_1, x_2, 0)^T$, it does not have a normalized homogeneous coordinate

Linear geometric transformations on 2D plane

- Some notes about homogeneous coordinate

For a normal point $\boldsymbol{x}$, its homogeneous coordinate and non-homogeneous coordinate can convert to each other:

From non-homogeneous coordinate to homogeneous coordinate

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightarrow \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} \quad k\begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix}, \forall k \neq 0$$

non-homogeneous coordinate → normalized homogeneous coordinate, homogeneous coordinate

From homogeneous coordinate to non-homogeneous coordinate

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, x_3 \neq 0 \rightarrow \begin{pmatrix} x_1 / x_3 \\ x_2 / x_3 \end{pmatrix}$$

homogeneous coordinate → non-homogeneous coordinate

How about an infinity point?

Can homogeneous coordinate bring us new information?

Linear geometric transformations on 2D plane

- Euclidean transformation (special Euclidean transformation)
  - On the basis of a rotation transformation, composing it with a translation transformation results in an Euclidean transformation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \Longrightarrow$$

Homogeneous form

$$\boldsymbol{x}' = \begin{bmatrix} \boldsymbol{R}_{2\times2} & \boldsymbol{t}_{2\times1} \\ \boldsymbol{0}_{1\times2} & 1 \end{bmatrix} \boldsymbol{x}$$

The DoF of the 2D Euclidean transformation is 3

A matrix that can represent a 2D Euclidean transformation should be of the form,

$$\boldsymbol{H}_{3\times3} = \begin{bmatrix} \boldsymbol{R}_{2\times2} & \boldsymbol{t}_{2\times1} \\ \boldsymbol{0}_{1\times2} & 1 \end{bmatrix}, \boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{I}, \det(\boldsymbol{R}) = 1$$

Linear geometric transformations on 2D plane

- Similarity transformation
  - On the basis of an Euclidean transformation, composing it with an isotropic scaling results in a similarity transformation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}, s \neq 0 \Rightarrow \boldsymbol{x}' = \begin{bmatrix} s\boldsymbol{R}_{2\times2} & \boldsymbol{t}_{2\times1} \\ \boldsymbol{0}_{1\times2} & 1 \end{bmatrix} \boldsymbol{x}, s \neq 0$$

Homogeneous form

The DoF of the 2D similarity transformation is 4

A matrix that can represent a 2D similarity transformation should be of the form,

$$\boldsymbol{H}_{3\times3} = \begin{bmatrix} s\boldsymbol{R}_{2\times2} & \boldsymbol{t}_{2\times1} \\ \boldsymbol{0}_{1\times2} & 1 \end{bmatrix}, \boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{I}, \det(\boldsymbol{R}) = 1, s \neq 0$$

Linear geometric transformations on 2D plane

- Affine transformation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

where, $\boldsymbol{A} \triangleq \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $\det(\boldsymbol{A})>0$

Homogeneous form

$$\boldsymbol{x}' = \begin{bmatrix} \boldsymbol{A}_{2\times2} & \boldsymbol{t}_{2\times1} \\ \boldsymbol{0}_{1\times2} & 1 \end{bmatrix} \boldsymbol{x} \;, \det(\boldsymbol{A})>0$$

The DoF of the 2D affine transformation is 6

A matrix that can represent a 2D affine transformation should be of the form,

$$\boldsymbol{H}_{3\times3} = \begin{bmatrix} \boldsymbol{A}_{2\times2} & \boldsymbol{t}_{2\times1} \\ \boldsymbol{0}_{1\times2} & 1 \end{bmatrix}, \det(\boldsymbol{A}) > 0$$

Linear geometric transformations on 2D plane

- Projective transformation

$$c\begin{pmatrix} x_1^{'} \\ x_2^{'} \\ x_3^{'} \end{pmatrix} = \underbrace{\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}}_{\boldsymbol{H}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Homogeneous form

$$\Rightarrow c\boldsymbol{x}^{'} = \boldsymbol{H}_{3\times 3}\boldsymbol{x}, \ \boldsymbol{H} \text{ is non-singular}$$

where $\boldsymbol{H}$ is non-singular, and c is a scalar related to $\boldsymbol{x}^{'}$

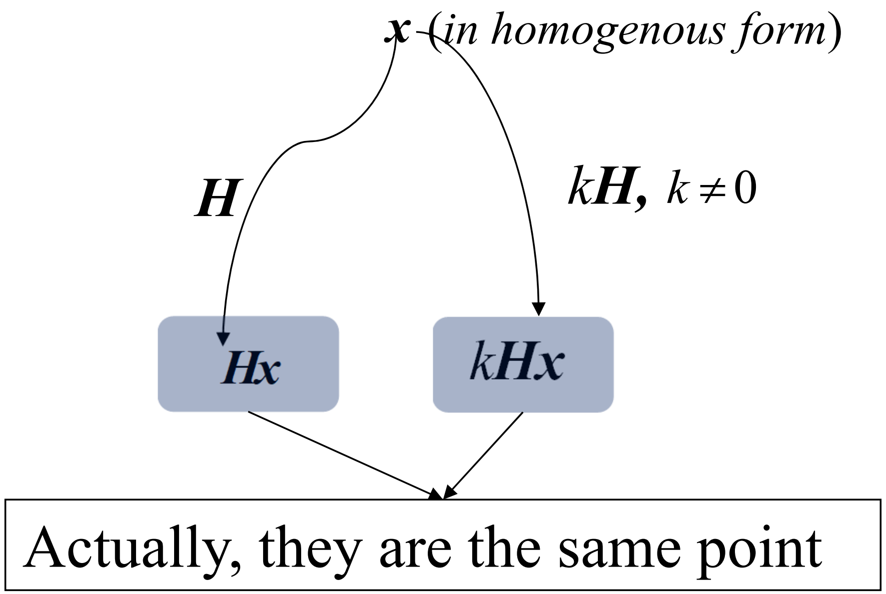
A matrix that can represent a projective transformation should be of the form,

$$\boldsymbol{H}_{3\times 3}, \ \boldsymbol{H} \text{ is non-singular}$$

Linear geometric transformations on 2D plane

- Projective transformation

It needs to be noted that though ***H*** has 9 numbers, the DoF of the 2D projective transformation it represents is 8!



H and $k\boldsymbol{H}$ $(k \neq 0)$ represents the same projective transformation

H's DoF is 8

Outline

- Linear Geometric Transformations on 2D Plane
- Group and Geometry
- Linear Geometric Transformations in 3D Space

Group and geometry

- Erlangen program
  - In mathematics, the Erlangen program is a method of characterizing geometries based on group theory and projective geometry, proposed by Felix Klein
  - It is named after the University Erlangen-Nürnberg (爱尔兰根-纽伦堡大学), where Klein worked
  - With every geometry, Klein associated an underlying group. The hierarchy of geometries is thus mathematically represented as a hierarchy of these groups, and hierarchy of their **invariants**. For example, lengths, angles and areas are preserved with respect to the Euclidean group, while only the incidence structure and the cross-ratio are preserved under the most general projective transformations. A concept of parallelism, which is preserved in affine geometry, is not meaningful in projective geometry. Then, by abstracting the underlying groups from the geometries, the relationships between them can be re-established at the group level

Group and geometry

Felix Christian Klein, the father of the Erlangen program

Felix Christian Klein (25 April 1849–22 June 1925) was a German mathematician and mathematics educator, known for his work in group theory, complex analysis, non-Euclidean geometry, and the associations between geometry and group theory. His 1872 Erlangen program classified geometries by their basic symmetry groups and was an influential synthesis of much of the mathematics of the time.

During his tenure at the University of Göttingen, Klein was able to turn it into a center for mathematical and scientific research through the establishment of new lectures, professorships, and institutes. His seminars covered most areas of mathematics then known as well as their applications. Klein also devoted considerable time to mathematical instruction, and promoted mathematics education reform at all grade levels in Germany and abroad.

Group and geometry

Definition: Group

For a set $\mathcal{G}$, $\circ$ is the operation defined on the elements of $\mathcal{G}$. If $\mathcal{G}$ satisfies the following 4 conditions, we say $\mathcal{G}$ with the operation $\circ$ is a group:

1) closure: $\forall g_1, g_2 \in \mathcal{G}, \exists g_3 \in \mathcal{G}, g_3 = g_1 \circ g_2$
2) Associativity: $\forall g_1, g_2, g_3 \in \mathcal{G}, g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3$
3) There exists an identity element $e \in \mathcal{G}$:
$$\forall g \in \mathcal{G}, g \circ e = e \circ g = g$$
4) For every element g in $\mathcal{G}$, there is an inverse element g^{-1} in $\mathcal{G}$, which satisfies,
$$g \circ g^{-1} = g^{-1} \circ g = e$$

Group and geometry

- With the common matrix multiplication operation as the set operation:
  - The matrices that express the 2D rotation transformation form a group (special orthogonal group)
  - The matrices that express the 2D Euclidean transformation form a group (special Euclidean group)
  - The matrices that express the 2D similarity transformation form a group
  - The matrices that express the 2D affine transformation form a group
  - The matrices that express the 2D projective transformation form a group

Group and geometry

An example: The matrices that express the 2D rotation transformation form a group

$$\mathcal{G} = \left\{ \boldsymbol{R} \in \mathbb{R}^{2\times 2} :| \boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{I}, \det(\boldsymbol{R}) = 1 \right\}$$

1) closure:

$$\left.\begin{array}{r} \text{For } g_1 = \boldsymbol{R}_1 \in \mathcal{G}, g_2 = \boldsymbol{R}_2 \in \mathcal{G},\ g_1 \circ g_2 = \boldsymbol{R}_1\boldsymbol{R}_2 \\ (g_1 \circ g_2)(g_1 \circ g_2)^T = (\boldsymbol{R}_1\boldsymbol{R}_2)(\boldsymbol{R}_1\boldsymbol{R}_2)^T = \boldsymbol{R}_1\boldsymbol{R}_2\boldsymbol{R}_2^T\boldsymbol{R}_1^T = \boldsymbol{I} \\ \det(g_1 \circ g_2) = \det(\boldsymbol{R}_1\boldsymbol{R}_2) = \det(\boldsymbol{R}_1)\det(\boldsymbol{R}_2) = 1 \end{array}\right\} \Longrightarrow g_1 \circ g_2 \in \mathcal{G}$$

2) associativity:

Since the operation $\circ$ is the common matrix multiplication, it naturally satisfies the associativity

Group and geometry

An example: The matrices that express the 2D rotation transformation form a group

$$\mathcal{G}=\left\{\boldsymbol{R}\in\mathbb{R}^{2\times 2}:|\ \boldsymbol{R}\boldsymbol{R}^T=\boldsymbol{I},\det(\boldsymbol{R})=1\right\}$$

3) There exists an identity:

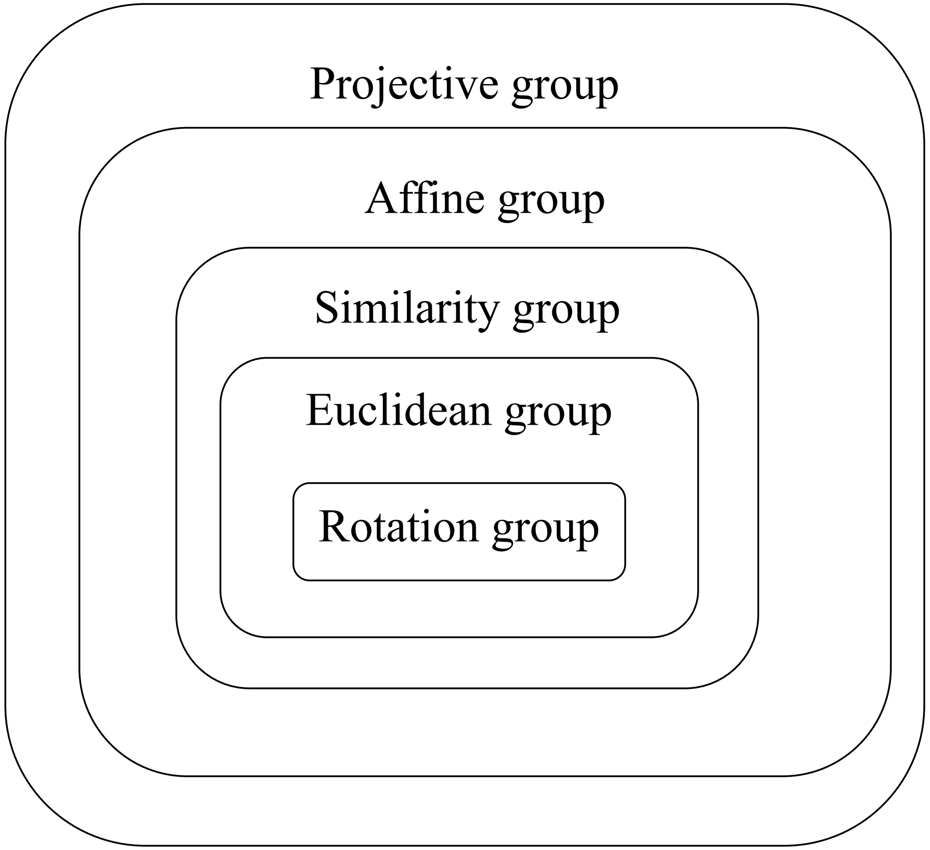
$$\left.\begin{array}{r}\boldsymbol{I}_{2\times 2}\in\mathcal{G}\\ \forall g\in\mathcal{G}, g\circ\boldsymbol{I}=g\boldsymbol{I}=g, \boldsymbol{I}\circ g=\boldsymbol{I}g=g\end{array}\right\}\ \boldsymbol{I}\text{ is the identity of }\mathcal{G}$$

4) For each element in $\mathcal{G}$, it has an inverse element in $\mathcal{G}$:

Suppose $g=\boldsymbol{R}\in\mathcal{G}$. Let's consider $\boldsymbol{R}^{-1}$.

$$\left.\begin{array}{r}\boldsymbol{R}^{-1}\left(\boldsymbol{R}^{-1}\right)^T=\boldsymbol{R}^T\boldsymbol{R}=\boldsymbol{I},\ \det\left(\boldsymbol{R}^{-1}\right)=\dfrac{1}{\det(\boldsymbol{R})}=1 \Longrightarrow \boldsymbol{R}^{-1}\in\mathcal{G}\\ g\circ\boldsymbol{R}^{-1}=\boldsymbol{R}\boldsymbol{R}^{-1}=\boldsymbol{I}\\ \boldsymbol{R}^{-1}\circ g=\boldsymbol{R}^{-1}\boldsymbol{R}=\boldsymbol{I}\end{array}\right\}\ g^{-1}=\boldsymbol{R}^{-1}$$

Group and geometry



For each group, there is a basic geometric invariant

For the rotation transformation or the Euclidean transformation group, the basic invariant is the **length** between any two points

For the similarity transformation group, the basic invariant is the **similarity ratio**

Definition: Similarity ratio

Consider four points $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$ and $\boldsymbol{x}_4$, the similarity ratio is $\frac{\|\boldsymbol{x}_1\boldsymbol{x}_2\|}{\|\boldsymbol{x}_3\boldsymbol{x}_4\|}$

Before the transformation, there are four points $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$ and $\boldsymbol{x}_4$; after the transformation, they are mapped as $\boldsymbol{x}_1'$, $\boldsymbol{x}_2'$, $\boldsymbol{x}_3'$, and $\boldsymbol{x}_4'$

If the transformation is a similarity transformation, then the similarity ratio is preserved, i.e.,

$$\frac{\|\boldsymbol{x}_1\boldsymbol{x}_2\|}{\|\boldsymbol{x}_3\boldsymbol{x}_4\|} = \frac{\|\boldsymbol{x}_1'\boldsymbol{x}_2'\|}{\|\boldsymbol{x}_3'\boldsymbol{x}_4'\|}$$

Group and geometry

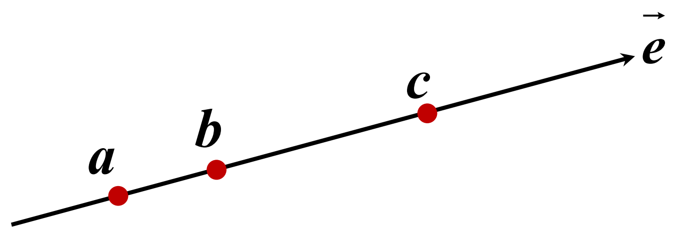
For the affine transformation group, the basic invariant is the **simple ratio**

Definition: Simple ratio

Given three collinear points ***a***, ***b***, and ***c***, the simple ratio (***a***, ***b***, ***c***) is defined as:

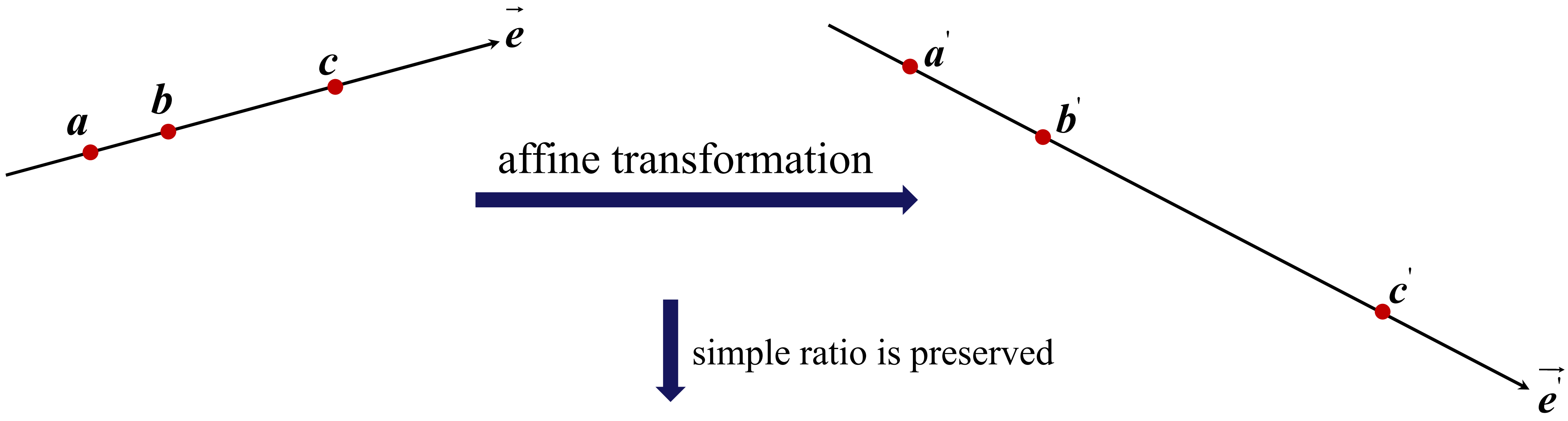
$$(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})=\frac{\boldsymbol{ac}}{\boldsymbol{cb}}$$

where ***ac*** is the signed distance from ***a*** to ***c***, and ***cb*** is the signed distance from ***c*** to ***b***



Group and geometry

For the affine transformation group, the basic invariant is the **simple ratio**



$$\left(a,b,c\right)=\left(a^{'},b^{'},c^{'}\right)$$

Group and geometry

For the projective transformation group, the basic invariant is the **cross ratio**

Definition: cross ratio

Given four **collinear** points ***a***, ***b***, ***c***, and ***d***, the cross ratio (***a***,***b***,***c***,***d***) is defined as:

$$(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})=\frac{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})}{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{d})}=\frac{\boldsymbol{ac}}{\boldsymbol{cb}}/\frac{\boldsymbol{ad}}{\boldsymbol{db}}=\frac{\boldsymbol{ac}}{\boldsymbol{cb}}\cdot\frac{\boldsymbol{db}}{\boldsymbol{ad}}$$

(The cross ratio is defined based on simple ratio)

Before the transformation, there are four **collinear** points $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$ and $\boldsymbol{x}_4$; after the transformation, they are mapped as $\boldsymbol{x}_1'$, $\boldsymbol{x}_2'$, $\boldsymbol{x}_3'$, and $\boldsymbol{x}_4'$

If the transformation is a projective transformation, then the cross ratio is preserved, i.e.,

$$(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3,\boldsymbol{x}_4)=(\boldsymbol{x}_1',\boldsymbol{x}_2',\boldsymbol{x}_3',\boldsymbol{x}_4')$$

Group and geometry

An open question:

What are the benefits to use group theory to study geometry?

Outline

- Linear Geometric Transformations on 2D Plane
- Group and Geometry
- Linear Geometric Transformations in 3D Space

Linear Geometric Transformations in 3D Space

二维空间与三维空间下的线性几何变换（n为空间维度，n=2, 3）

变换名称	矩阵表达式	二维情况下自由度个数	三维情况下自由度个数	不变量
旋转变换	$\begin{bmatrix} \boldsymbol{R}_{n\times n} & \boldsymbol{0}_{n\times 1} \\ \boldsymbol{0}_{1\times n} & 1 \end{bmatrix}$，$\boldsymbol{R}$ 为正交矩阵且 $\det(R)=1$	1	3	长度，角度，面积（体积）
欧氏变换	$\begin{bmatrix} \boldsymbol{R}_{n\times n} & \boldsymbol{t}_{n\times 1} \\ \boldsymbol{0}_{1\times n} & 1 \end{bmatrix}$，$\boldsymbol{R}$ 为正交矩阵且 $\det(\boldsymbol{R})=1$	3	6	长度，角度，面积（体积）
相似变换	$\begin{bmatrix} s\boldsymbol{R}_{n\times n} & \boldsymbol{t}_{n\times 1} \\ \boldsymbol{0}_{1\times n} & 1 \end{bmatrix}$，$\boldsymbol{R}$ 为正交矩阵且 $\det(\boldsymbol{R})=1$ ，且 $\det\left(s\boldsymbol{R}\right)=s^n\det\left(\boldsymbol{R}\right)>0$	4	7	相似比，角度，面积（体积）比
仿射变换	$\begin{bmatrix} \boldsymbol{A}_{n\times n} & \boldsymbol{t}_{n\times 1} \\ \boldsymbol{0}_{1\times n} & 1 \end{bmatrix}$，$\det(\boldsymbol{A})>0$	6	12	简单比，面积（体积）比，平行关系
射影变换	$\boldsymbol{H}_{(n+1)\times(n+1)}$，$\boldsymbol{H}$ 为非奇异矩阵	8	15	交比，共线关系


Q+A