

Interpolation under latent factor regression models

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Abstract

This work studies finite-sample properties of the risk of the minimum-norm interpolating predictor in high-dimensional regression models. If the effective rank of the covariance matrix Σ of the p regression features is much larger than the sample size n , we show that the min-norm interpolating predictor is not desirable, as its risk approaches the risk of trivially predicting the response by 0. However, our detailed finite sample analysis reveals, surprisingly, that this behavior is not present when the regression response and the features are *jointly* low-dimensional, following a widely used factor regression model. Within this popular model class, and when the effective rank of Σ is smaller than n , while still allowing for $p \gg n$, both the bias and the variance terms of the excess risk can be controlled, and the risk of the minimum-norm interpolating predictor approaches optimal benchmarks. Moreover, through a detailed analysis of the bias term, we exhibit model classes under which our upper bound on the excess risk approaches zero, while the corresponding upper bound in the recent work [3] diverges. Furthermore, we show that the minimum-norm interpolating predictor analyzed under the factor regression model, despite being model-agnostic, can have similar risk to model-assisted predictors based on principal components regression, in the high-dimensional regime.

Keywords: Interpolation, minimum-norm predictor, finite sample risk bounds, prediction, factor models, high-dimensional regression.

1 Introduction

Motivated by the widely observed phenomenon that interpolating deep neural networks generalize well despite having zero training error, there has been a recent wave of literature showing that this is a general behaviour that can occur for a variety of models and prediction methods [3–8, 18, 20, 27, 31–35, 42]. One of the simplest settings is the prediction of a real-valued response $y \in \mathbb{R}$ from vector-valued features $X \in \mathbb{R}^p$ via a linear predictor $\hat{y}_x := X^\top \hat{\alpha}$ with $\hat{\alpha}$ defined as the vector with the smallest Euclidean norm among all weight vectors that perfectly fit the training data (\mathbf{X}, \mathbf{y}) . The data consists of the $n \times p$ data matrix \mathbf{X} and response vector $\mathbf{y} \in \mathbb{R}^n$, obtained from n i.i.d. copies (X_i, y_i) , $i \in [n]$, of (X, y) , with $p > n$. The interpolation property of $\hat{\alpha}$ means that $\mathbf{X}\hat{\alpha} = \mathbf{y}$. We refer to the corresponding predictor as the minimum-norm interpolating predictor.

This paper is devoted to the finite sample statistical analysis of prediction via the minimum-norm interpolator $\hat{\alpha}$. We first note that ideally, the prediction risk $R(\hat{\alpha}) := \mathbb{E}_{X,y} [(X^\top \hat{\alpha} - y)^2]$

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of $\hat{\alpha}$ approaches the optimal risk $\inf_{\alpha \in \mathbb{R}^p} \mathbb{E}_{X,y} [(X^\top \alpha - y)^2]$. Unfortunately, this is often not the case. Theorem 1 of Section 2 below shows that $\|\hat{\alpha}\|$ decreases to 0 as n increases and $p \gg n$ under appropriate conditions on the covariance matrix Σ_X of X and variance σ_y^2 of y . Furthermore, if $\|\hat{\alpha}\|$ is very close to 0, one might expect $R(\hat{\alpha})$ to be approximately equal to $R(\mathbf{0})$, with $\mathbf{0} \in \mathbb{R}^p$. This would be undesirable as $R(\mathbf{0}) = \mathbb{E}[y^2] := \sigma_y^2$ is the non-optimal null risk of trivially predicting via the zero weight vector, ignoring the data. Theorem 3, stated in Section 2, confirms that indeed the ratio $R(\hat{\alpha})/R(\mathbf{0})$ approaches 1 in the regime $r_e(\Sigma_X) \gg n$. The *effective rank* $r_e(\Sigma_X)$ of the covariance matrix Σ_X of X is defined as the ratio between the trace of Σ_X and its operator norm, and is at most equal to its rank, $r_e(\Sigma_X) \leq p$. In particular, if Σ_X is well-conditioned, with $r_e(\Sigma_X) \asymp p$, then the prediction risk $R(\hat{\alpha})$ of the minimum norm interpolator approaches the trivial risk $R(\mathbf{0})$, whenever $p \gg n$.

This opens the question as to whether, in the high-dimensional $p > n$ setting, there exist underlying distributions of the data that allow $R(\hat{\alpha})$ to be close to an optimal risk benchmark. The recent work [3] provides a positive answer to this question, primarily focusing on sufficient conditions on the spectrum of Σ_X that can lead to consistent prediction. In this paper we show that the *joint* structure of (X, y) , not just the marginal structure of X as considered in [3], is key to understanding the conditions under which consistent prediction is possible with $\hat{\alpha}$. In particular, we provide a detailed and novel finite sample analysis of the prediction risk $R(\hat{\alpha})$ when the pair (X, y) follows a linear factor regression model, $y = Z^\top \beta + \varepsilon$, $X = AZ + E$, in the regime $r_e(\Sigma_X) < c \cdot n$ for an absolute constant $c > 0$. Here $(X, y) \in \mathbb{R}^p \times \mathbb{R}$ are observable random features and response, $Z \in \mathbb{R}^K$ is a vector of unobservable sub-Gaussian random latent factors with $K < p$, $A \in \mathbb{R}^{p \times K}$ is a loading matrix relating Z to X , and E and ε are mean zero sub-Gaussian noise terms independent of Z and each other. Under this model, the observation made in inequality (9) below shows that $r_e(\Sigma_X)$ is less than $c \cdot n$ as long as $K < c_1 \cdot n$ and the signal-to-noise ratio $\xi := \lambda_k(A \Sigma_Z A^\top) / \|\Sigma_E\| \gtrsim p/n \geq c_2 \cdot r_e(\Sigma_E)/n$ for suitable absolute constants $c_1, c_2 > 0$, where Σ_Z and Σ_E denote the covariance matrices of Z and E respectively.

Our primary contribution is the study of $R(\hat{\alpha})$ under the factor regression model, and in this regime. In Section 3 we present a detailed finite sample study of the risk $R(\hat{\alpha})$ of the model-agnostic interpolating predictor $\hat{y}_x = X^\top \hat{\alpha}$ in factor regression models with $p > n$ and $K < n$, but with K allowed to grow with n . Our main result is Theorem 7 in Section 3.3. It provides a finite sample bound on the *excess risk* $R(\hat{\alpha}) - \sigma_\varepsilon^2$ of $\hat{\alpha}$ in the high-dimensional setting $p > n$, relative to the natural risk benchmark $\mathbb{E}[\varepsilon^2] := \sigma_\varepsilon^2$ in the factor regression model; the excess risk relative to the benchmark $\inf_{\alpha \in \mathbb{R}^p} \mathbb{E}_{X,y} [(X^\top \alpha - y)^2]$ is also derived in this theorem. As a consequence, we obtain sufficient conditions under which the prediction risk $R(\hat{\alpha})$ approaches the optimal risk, by adapting to the embedded dimension K . The excess risk not only decreases beyond the interpolation boundary to a non-zero value as observed in [20], but does indeed decrease to zero, as desired. We remark that at least for Gaussian (X, y) , the recent work [3] provides an alternative bound to Theorem 7. However, Theorem 7 provides an improved rate for typical factor regression models, and in particular provides examples when the upper bound on the excess risk in [3] diverges, yet prediction is consistent; see Section 3.5 for a detailed comparison.

Table 1 below offers a snap-shot of our main results. The first row is a reminder that all results are established for $p > n$, while the second row separates the regimes of $r_e(\Sigma_X)$ larger or smaller than n , where $C > 1$ and $c > 0$ are absolute constants. The third row specifies the assumptions on (X, y) , namely general sub-Gaussian random variables or, in addition, the factor regression model. The last row gives finite sample bounds. The risk bounds in the bottom right panel are stated

under the assumptions that the operator norms $\|\Sigma_Z\|$ and $\|\Sigma_E\|$ are constant and $\text{r}_e(\Sigma_E) \asymp p$. These simplifying assumptions are made here for transparency of presentation and are not made in the body of the paper. The bottom right panel shows that the variance term V decreases if

$p > n$	
$\text{r}_e(\Sigma_X) > C \cdot n$	$\text{r}_e(\Sigma_X) < c \cdot n, \quad K < n$
(X, y) sub-Gaussian	(X, y) sub-Gaussian $y = \beta^\top Z + \varepsilon$ $X = AZ + E$
$\left \frac{R(\hat{\alpha})}{R(\mathbf{0})} - 1 \right \lesssim \sqrt{n/\text{r}_e(\Sigma_X)}$	$R(\hat{\alpha}) - \sigma_\varepsilon^2 \lesssim B_Z + V$ $B_Z = \ \beta\ ^2 \cdot p/(n \cdot \xi)$ $V = \{(n/p) + (K/n)\} \log n$

Table 1: Behavior of risk $R(\hat{\alpha})$. (i) $R(\hat{\alpha})$ approaches null risk $R(\mathbf{0})$ for well-conditioned matrices Σ_X when $p \gg n$ (left panel); (ii) Variance term vanishes when $p \gg n \log n$ and $K \log n \ll n$; Bias term vanishes for $\xi := \lambda_K(A\Sigma_Z A^\top)/\|\Sigma_E\| \gg \|\beta\|^2 p/n$ (right panel).

$p \gg n \log n$ and $K \log n \ll n$ and that the bias term B_Z decreases provided that the signal-to-noise ratio $\xi := \lambda_K(A\Sigma_Z A^\top)/\|\Sigma_E\|$ is large enough. Specifically, we need that $\xi \gg \|\beta\|^2 p/n$, which for $\|\beta\|^2 \lesssim K$ amounts to $\xi \gg p \cdot K/n$. For instance, as explained in Section 3.3, a common, natural situation is $\xi \asymp p$ and the bias is small for $K \ll n$. In clustering problems where the p coordinates of X can be clustered in K groups of approximately equal size $m \approx p/K$ as discussed in Section 3.3, we find $\xi \asymp p/K$. In that case, B_Z vanishes if $n \gg K^2$.

We emphasize that a condition on the effective rank of Σ_X alone is not enough to guarantee that $R(\hat{\alpha})$ is close to the optimal risk σ_ε^2 . As argued in Section 3.3, if we assume the model $X = AZ + E$, but instead of assuming that y is also a function of Z , as in this work, we only assume a standard linear model $y = X^\top \alpha + \varepsilon$, with $\alpha \in \mathbb{R}^p$, then the bias term *cannot* be ignored, unless $\|\alpha\| \rightarrow 0$, which is typically not the case in high dimensions. From this perspective, this work illustrates the critical role played in the risk analysis by a modeling assumption in which (X, y) are jointly low-dimensional.

Finally, we remark that prediction under factor regression models has been well studied, starting with classical factor analysis that can be traced back to the 1940s [23–26, 28–30], including the pertinent work [2]. A number of works ranging from purely Bayesian [1, 9, 13, 19] to variational Bayes [11] to frequentist [10, 14–17, 21, 22, 38–40] show that this class of models can be a useful framework for constructing and analyzing predictors of y from high-dimensional and correlated data. The literature on finite sample prediction bounds under factor regression models is relatively limited, with instances provided by [10, 14–17], and most existing results established for K fixed. Relevant for the work presented here, the (non-Bayesian) prediction schemes that have been studied in generic factor regression models are often variations of principal component regression in $K < n$ fixed dimensions, and therefore typically do not interpolate the data. From this perspective, the results of this paper complement this existing literature, by studying the behavior of interpolating predictors in factor regression. Furthermore, in Section 3.4 we derive an upper bound on the excess risk of prediction based on principal components, under the factor regression model, and find that it

is comparable to the excess risk bound of the interpolating predictor, in the regime $p \gg n$, provided that the covariance matrix Σ_E of the noise is well conditioned. This provides further motivation for the use of $\hat{\alpha}$ in the setting discussed here.

The rest of the paper is organized as follows.

Section 2 derives sufficient conditions on Σ_X and $\sigma_y^2 := \mathbb{E}[y^2]$ under which $\|\hat{\alpha}\|$ approaches zero, and related conditions that imply $R(\hat{\alpha})$ approaches $R(\mathbf{0})$.

Section 3 introduces the factor regression model and risk benchmarks.

Section 3.3 contains our main result regarding the approximate adaptation of the minimum-norm interpolating predictor in this model under conditions that prevent $\|\hat{\alpha}\|$ from approaching zero.

Section 3.4 is devoted to a comparison with prediction via principal component regression, under the factor regression model.

Section 3.5 presents a detailed comparison with [3], which provides risk bounds for $\hat{y}_x = X^\top \hat{\alpha}$, for sub-Gaussian data (X, y) , and offers sufficient conditions on Σ_X for optimal risk behavior, with emphasis on the optimality of the variance component of the risk. In Appendix A.3, we derive simplified versions of the generic bias and variance bounds obtained in [3] under the factor regression model.

Table 2 of Section 3.5 summarizes our findings that the bound on the excess risk in [3] is often larger in order of magnitude than the bound given in Theorem 7 of Section 3.3. In particular, we exhibit instances of the factor regression model class under which the excess risk upper bound in [3] diverges, yet our upper bound approaches zero.

All proofs and ancillary results are deferred to the Appendix. In particular, as a supplement to Section 2, a quasi-heuristic geometric explanation of why $\|\hat{\alpha}\|$ approaches 0, under suitable conditions on Σ_X and σ_y^2 is given in Appendix B. Furthermore, Theorem 19 in the Appendix complements Theorem 7 by showing the risk behavior of \hat{y}_x for $n > c \cdot p$ for an absolute constant $c > 0$, and is included for completeness.

1.1 Notation

Throughout the paper, for a vector $v \in \mathbb{R}^d$, $\|v\|$ denotes the Euclidean norm of v .

For any matrix $A \in \mathbb{R}^{n \times m}$, $\|A\|$ denotes the operator norm and A^+ the Moore-Penrose pseudo-inverse. See section D in the appendix for a definition of the pseudo-inverse and a summary its properties used in this paper.

For a positive semi-definite matrix $Q \in \mathbb{R}^{p \times p}$, and vector $v \in \mathbb{R}^p$, we define $\|v\|_Q^2 := v^\top Q v$, let $\lambda_1(Q) \geq \lambda_2(Q) \geq \dots \geq \lambda_p(Q)$ be its ordered eigenvalues, $\kappa(Q) := \lambda_1(Q)/\lambda_p(Q)$ its condition number, and $r_e(Q) := \text{tr}(Q)/\|Q\|$ its effective rank.

The identity matrix in dimension m is denoted I_m .

The set $\{1, 2, \dots, m\}$ is denoted $[m]$.

Letters c, c', c_1, C , etc., are used to denote absolute constants, and may change from line to line.

2 Interpolation and the null risk

Given i.i.d. observations $(X_1, y_1), \dots, (X_n, y_n)$, distributed as $(X, y) \in \mathbb{R}^p \times \mathbb{R}$, let $\mathbf{X} \in \mathbb{R}^{n \times p}$ be the corresponding data matrix with rows X_1, \dots, X_n , and let $\mathbf{y} := (y_1, \dots, y_n)^\top \in \mathbb{R}^n$. For the rest of the

paper, unless specified otherwise, we make the blanket assumptions that $p > n$ and $\sigma_y^2 := \mathbb{E}[y^2] > 0$.

We are interested in studying the prediction risk associated with the minimum ℓ_2 -norm estimator $\hat{\alpha}$ defined as

$$\hat{\alpha} := \arg \min \left\{ \|\alpha\| : \|\mathbf{X}\alpha - \mathbf{y}\| = \min_u \|\mathbf{X}u - \mathbf{y}\| \right\}. \quad (1)$$

We define the prediction risk for any $\alpha \in \mathbb{R}^p$ as

$$R(\alpha) := \mathbb{E}_{X,y}(X^\top \alpha - y)^2. \quad (2)$$

The expectation is over the new data point (X, y) , independent of the observed data (\mathbf{X}, \mathbf{y}) . In particular, since $\hat{\alpha}$ is independent of (X, y) , we have $R(\hat{\alpha}) = \mathbb{E}_{X,y} [(X^\top \hat{\alpha} - y)^2 | \mathbf{X}, \mathbf{y}] = \mathbb{E}_{X,y} [(X^\top \hat{\alpha} - y)^2]$. If the data matrix \mathbf{X} has full rank, then $\min_{u \in \mathbb{R}^p} \|\mathbf{X}u - \mathbf{y}\| = 0$ and

$$\hat{\alpha} := \arg \min_{\alpha: \mathbf{X}\alpha = \mathbf{y}} \|\alpha\|. \quad (3)$$

Regardless of the rank of \mathbf{X} , Equation (1) always has the closed form solution $\hat{\alpha} = \mathbf{X}^+ \mathbf{y}$, where \mathbf{X}^+ is the Moore-Penrose pseudo-inverse of \mathbf{X} ; we prove this fact in section C.1 for completeness. We begin our consideration of the minimum-norm estimator $\hat{\alpha} = \mathbf{X}^+ \mathbf{y}$ by showing that when the effective rank $r_e(\Sigma_X)$ is large enough and $\text{tr}(\Sigma_X)$ grows at a rate faster than $n \cdot \sigma_y^2$, the norm $\|\hat{\alpha}\|$ approaches zero as n grows. Proofs for this section are contained in Appendix A.1. We make the following distributional assumption.

Assumption 1. $X = \Sigma_X^{1/2} \tilde{X}$ and $y = \sigma_y \tilde{y}$, where $\tilde{X} \in \mathbb{R}^p$ has independent entries, and both \tilde{X} and \tilde{y} have zero mean, unit variance, and sub-Gaussian constants bounded by an absolute constant.

Theorem 1. Suppose Assumption 1 holds and $r_e(\Sigma_X) > C \cdot n$ for some large enough absolute constant $C > 0$. Then, with probability at least $1 - c \cdot e^{-c'n}$, for some absolute constants $c, c' > 0$,

$$\|\hat{\alpha}\|^2 \lesssim \frac{n \cdot \sigma_y^2}{\text{tr}(\Sigma_X)}. \quad (4)$$

Appendix B offers a quasi-heuristic geometric explanation of Theorem 1; it relies on Lemma 15, which shows that when $r_e(\Sigma_X)$ is large compared to n , the features X_1, \dots, X_n are close to being mutually orthogonal with high probability.

Next we connect the fact that $\|\hat{\alpha}\|$ decreases to zero in high dimensions to the behavior of the risk $R(\hat{\alpha})$. To this end, we first show that for any $\theta \in \mathbb{R}^p$, if $\|\theta\|_{\Sigma_X}^2$ is small relative to the null risk $R(\mathbf{0})$, then $R(\theta)$ will be close to $R(\mathbf{0})$.

Lemma 2. For any vector $\theta \in \mathbb{R}^p$,

$$\left| \frac{R(\theta)}{R(\mathbf{0})} - 1 \right| \leq \frac{\|\theta\|_{\Sigma_X}^2}{R(\mathbf{0})} + 2\sqrt{\frac{\|\theta\|_{\Sigma_X}^2}{R(\mathbf{0})}}. \quad (5)$$

Using $\|\hat{\alpha}\|_{\Sigma_X}^2 \leq \|\Sigma_X\| \|\hat{\alpha}\|^2$ and recalling $\sigma_y^2 = R(\mathbf{0})$, Theorem 1 implies the bound

$$\frac{\|\hat{\alpha}\|_{\Sigma_X}^2}{R(\mathbf{0})} \lesssim \frac{\|\Sigma_X\| \cdot n}{\text{tr}(\Sigma_X)} = \frac{n}{r_e(\Sigma_X)}, \quad (6)$$

invoking $r_e(\Sigma_X) = \text{tr}(\Sigma_X) / \|\Sigma_X\|$ in the second step. Combining this bound with Lemma 2, we find that the risk of the minimum-norm interpolator approaches the null risk whenever $n/r_e(\Sigma_X) \rightarrow 0$.

Theorem 3. Suppose Assumption 1 holds and $r_e(\Sigma_X) > C \cdot n$ for some absolute constant $C > 1$ large enough. Then, with probability at least $1 - ce^{-c'n}$ for absolute constants $c, c' > 0$,

$$\left| \frac{R(\hat{\alpha})}{R(\mathbf{0})} - 1 \right| \lesssim \sqrt{\frac{n}{r_e(\Sigma_X)}}. \quad (7)$$

As a consequence, $\hat{\alpha}$ is not a useful estimator in the regime $r_e(\Sigma_X) \gg n$, as trivially predicting with the null vector $\mathbf{0} \in \mathbb{R}^p$ will give asymptotically equivalent results. This occurs, for instance, when Σ_X is well conditioned and $p/n \rightarrow \infty$. Figure 2 in [20] depicts an example of this behavior: it plots $\mathbb{E}[\|\hat{\alpha} - \alpha\|^2 | \mathbf{X}]$ as a function of the ratio $\gamma = p/n$, where (X, y) follows the linear model $y = \alpha^\top X + \varepsilon$ with $\Sigma_X = I_p$.

This motivates our study, in Section 3, of a class of factor regression models that have covariance matrix Σ_X of low effective rank ($r_e(\Sigma_X)/n \not\rightarrow \infty$ as $n \rightarrow \infty$), while still allowing for the high-dimensional setting $p > n$. We prove that for this class the risk $R(\hat{\alpha})$ can approach the optimal value $\inf_{\alpha \in \mathbb{R}^p} R(\alpha)$, strictly less than $R(\mathbf{0})$.

3 Minimum ℓ_2 -norm prediction in factor regression

The results and discussion of the previous section imply that in order for the generalized least squares estimator $\hat{\alpha}$ to have asymptotically better prediction performance than the trivial estimator $\mathbf{0} \in \mathbb{R}^p$, the ratio $r_e(\Sigma_X)/n$ must remain bounded as n and p grow, as a first requirement. We now introduce a class of models, and associated conditions, under which this happens. The model class is that of factor regression models, which is a latent factor model in which we single out one variable, $y \in \mathbb{R}$, to emphasize its role as the response relative to input covariates $X \in \mathbb{R}^p$, while both X and y are directly connected to a lower dimensional, unobserved, random vector $Z \in \mathbb{R}^K$, with mean zero and $K < n$. Specifically, the factor regression model postulates that

$$X = AZ + E, \quad y = Z^\top \beta + \varepsilon, \quad (8)$$

where $\beta \in \mathbb{R}^K$ is the latent variable regression vector, $A \in \mathbb{R}^{p \times K}$ is a unknown loading matrix, and $\varepsilon \in \mathbb{R}$ and $E \in \mathbb{R}^p$ are mean zero additive noise terms independent of one another and of Z . We let $\Sigma_E := \text{Cov}(E)$, $\Sigma_Z := \text{Cov}(Z)$ and $\sigma_\varepsilon^2 := \text{Var}(\varepsilon)$. Using that $\Sigma_X = A\Sigma_Z A^\top + \Sigma_E$ under (8), we find

$$\begin{aligned} r_e(\Sigma_X) &= \frac{\text{tr}(\Sigma_X)}{\|\Sigma_X\|} \\ &\leq \frac{\text{tr}(A\Sigma_Z A^\top) + \text{tr}(\Sigma_E)}{\|A\Sigma_Z A^\top\|} && (\text{since } \|\Sigma_X\| \geq \|A\Sigma_Z A^\top\|) \\ &\leq K + \frac{\text{tr}(\Sigma_E)}{\|A\Sigma_Z A^\top\|} && (\text{since } \text{tr}(A\Sigma_Z A^\top) \leq K\|A\Sigma_Z A^\top\|) \\ &\leq K + \frac{\|\Sigma_E\|}{\lambda_K(A\Sigma_Z A^\top)} \cdot \frac{\text{tr}(\Sigma_E)}{\|\Sigma_E\|}. && (\text{since } \|A\Sigma_Z A^\top\| \geq \lambda_K(A\Sigma_Z A^\top)) \end{aligned}$$

We thus have

$$\frac{r_e(\Sigma_X)}{n} \leq \frac{K}{n} + \frac{r_e(\Sigma_E)}{\xi}, \quad (9)$$

where

$$\xi := \lambda_K(A\Sigma_Z A^\top) / \|\Sigma_E\| \quad (10)$$

can be viewed as a signal-to-noise ratio since $\Sigma_X = A\Sigma_Z A^\top + \Sigma_E$. In standard factor regression models [2], $\Sigma_E = I_p$, in which case $\text{r}_e(\Sigma_E) = p$, but in our analysis we allow for a general Σ_E , with possibly smaller $\text{r}_e(\Sigma_E)$. From (9) it is clear that as long as

$$\frac{K}{n} \leq c_1 \quad \text{and} \quad \xi \geq c_2 \frac{\text{r}_e(\Sigma_E)}{n}, \quad (11)$$

for some absolute positive constants c_1, c_2 , then

$$\text{r}_e(\Sigma_X)/n \leq c_3, \quad (12)$$

for some positive constant c_3 , as K, p and n grow. The positive repercussion of this observation is that Theorem 3 no longer applies. This in turn opens up the possibility of showing that, under the data generating model (8) with restrictions (11), the risk $R(\hat{\alpha})$ will approach optimal risk benchmarks. We make the benchmark risks precise in Section 3.1, and show that the answer to this question is affirmative in Sections 3.2 and 3.3. We also clarify in these sections the importance of the factor regression model, in which (X, y) jointly have a low-dimensional structure, in contrast to the classical linear model $y = X^\top \alpha + \varepsilon$ with low-dimensional structure on X alone.

For the remainder of the paper we will assume that the data consist of n i.i.d. pairs (X_i, y_i) satisfying (8), in that

$$X_i = AZ_i + E_i, \quad y_i = Z_i^\top \beta + \varepsilon_i \quad \forall i \in [n], \quad (13)$$

where the latent factors $Z_1, \dots, Z_n \in \mathbb{R}^K$ are i.i.d. copies of Z , and the error terms $E_i \in \mathbb{R}^p$ and $\varepsilon_i \in \mathbb{R}$ for $i = 1, \dots, n$ are i.i.d. copies of E and ε , respectively. We recall that $\mathbf{X} \in \mathbb{R}^{n \times p}$ is the matrix with rows X_1, \dots, X_n and $\mathbf{y} \in \mathbb{R}^n$ is the vector with entries y_1, \dots, y_n . We similarly let $\mathbf{Z} \in \mathbb{R}^{n \times K}$ be the matrix with rows Z_1, \dots, Z_n .

3.1 Risk benchmarks

We will compare $R(\hat{\alpha})$ to two natural benchmarks. Under model (8), if $Z \in \mathbb{R}^K$ were observed, the optimal risk of a linear oracle with access to Z is

$$\min_{v \in \mathbb{R}^K} \mathbb{E} \left[(Z^\top v - y)^2 \right] = \mathbb{E}[\varepsilon^2] = \sigma_\varepsilon^2, \quad (14)$$

which we henceforth refer to as the oracle risk. Another natural benchmark to compare the risk $R(\hat{\alpha})$ to is the minimum risk possible for any linear predictor $\alpha^\top X$, namely $R(\alpha^*)$, where

$$\alpha^* \in \arg \min_{\alpha \in \mathbb{R}^p} R(\alpha). \quad (15)$$

Lemma 17 in Appendix C shows that for arbitrary zero-mean (X, y) with finite second moments, $\alpha^* = \Sigma_X^+ \Sigma_{Xy}$ is a minimizer of $R(\alpha)$. We can characterize the difference between these two benchmarks, σ_ε^2 and $R(\alpha^*)$, as follows. See Appendix A.2.1 for its proof.

Lemma 4 (Comparison of risk benchmarks). *Suppose model (8) holds and let ξ be the signal-to-noise ratio defined in (10). The following then holds.*

1. $R(\alpha^*) - \sigma_\varepsilon^2 \geq 0$.
2. $R(\alpha^*) - \sigma_\varepsilon^2 \leq \|\beta\|_{\Sigma_Z}^2 / \xi$, provided the matrices Σ_Z and Σ_E are invertible.

Although the optimal risk $R(\alpha^*)$ is always greater than the oracle risk σ_ε^2 , the bound $\|\beta\|_{\Sigma_Z}^2 / \xi$ on the difference $R(\alpha^*) - \sigma_\varepsilon^2$ is not a leading term in the excess risk bound given in Theorem 7. From this perspective, we can view these benchmarks as asymptotically equivalent, but with different interpretations.

3.2 Exact adaptation in factor regression models with noiseless features

We begin by considering an extreme case of model (8), in which $E = 0$ almost surely, and thus Σ_X is degenerate, with $\text{r}_e(\Sigma_X) \leq \text{rank}(\Sigma_X) = K$. When $K < n$, the factor regression model (8) closely resembles a low-dimensional classical regression model, with the caveat that A is not known and therefore Z cannot be treated as observed via $X = AZ$. Nevertheless, the first part of Theorem 5 below shows that $R(\hat{\alpha})$ does indeed mimic prediction in K observed dimensions. Let $\hat{y}_z := Z^\top \hat{\beta}$ based on the least-squares regression coefficients $\hat{\beta} := \mathbf{Z}^+ \mathbf{y}$ of \mathbf{y} onto \mathbf{Z} . Since \hat{y}_z is the classical least-squares prediction of y under model (8) that an oracle would use if it had access to the unobserved data matrix \mathbf{Z} , and the new, but unobservable, data point Z , we therefore call \hat{y}_z the *oracle* predictor. In contrast, a linear predictor of y from X based on (\mathbf{X}, \mathbf{y}) only is $\hat{y}_x = X^\top \hat{\alpha}$. Theorem 5.1 below shows that the realizable prediction equals the oracle prediction. The second part of the theorem gives lower and upper bounds on the risk that hold with high probability over the training data. Proofs for this section are contained in Appendix A.2.2. We make the following assumptions.

Assumption 2. The $p \times K$ matrix A and $K \times K$ matrix Σ_Z both have full rank equal to K .

Assumption 3. $E = \Sigma_E^{1/2} \tilde{E}$, where $\tilde{E} \in \mathbb{R}^p$ has independent entries with zero mean, unit variance, and sub-Gaussian constants bounded by an absolute constant.

Furthermore, $Z = \Sigma_Z^{1/2} \tilde{Z}$ and $\varepsilon = \sigma_\varepsilon \tilde{\varepsilon}$, where $\tilde{Z} \in \mathbb{R}^K$ and $\tilde{\varepsilon} \in \mathbb{R}$ have zero mean and sub-Gaussian constants bounded by an absolute constant.

Theorem 5 (Factor regression with noiseless features). *Under model (8) with $\Sigma_E = 0$, suppose that Assumption 2 holds.*

1. *Then, on the event that the matrix \mathbf{Z} has full rank K , we have $\hat{y}_x = \hat{y}_z$ and $R(\hat{\alpha}) = \mathbb{E}_{(X,y)}[(X^\top \hat{\alpha} - y)^2] = \mathbb{E}_{(Z,y)}[(Z^\top \hat{\beta} - y)^2]$.*
2. *Suppose that Assumption 3 also holds and that $n > C \cdot K$ for some large enough absolute constant $C > 0$. Then, with probability at least $1 - c/n$ for some absolute constant $c > 0$,*

$$R(\hat{\alpha}) - \sigma_\varepsilon^2 \lesssim \sigma_\varepsilon^2 \frac{K \log n}{n} \quad \text{and} \quad \mathbb{E}_\varepsilon[R(\hat{\alpha})] - \sigma_\varepsilon^2 \gtrsim \sigma_\varepsilon^2 \frac{K}{n}. \quad (16)$$

The risk bounds (16) are the same as the standard risk bounds for prediction in linear regression in K dimensions with observable design, despite A not being known under model (8). We note that, since $\text{rank}(\mathbf{X}) = K < n$, \mathbf{y} may not lie in the range of \mathbf{X} and so $\hat{\alpha}$ may not interpolate. Nonetheless, under model (8), with $E \neq 0$, and in the interpolating regime, we expect that the prediction performance of \hat{y}_x will still approximately mimic that of \hat{y}_z , as long as the signal, as measured by $\lambda_K(A^\top \Sigma_Z A)$, is strong relative to the noise, as measured by $\|\Sigma_E\|$. The next section is devoted to the detailed study of this fact.

3.3 Approximate adaptation of interpolating predictors in factor regression

In this section we present our main results on the excess risk, relative to the two benchmarks above, under the factor regression model (8) with $E \neq 0$. Our main result, Theorem 7 below, shows that although $\hat{\alpha}$ interpolates, in that $\mathbf{X}\hat{\alpha} = \mathbf{y}$, the excess risks can vanish as a result of approximate adaptation to the embedded low-dimensional structure of (8). The estimator $\hat{\alpha}$ is guaranteed to interpolate the data whenever $\text{rank}(\mathbf{X}) = n$, or equivalently, the smallest singular value $\sigma_n(\mathbf{X}) > 0$. The next proposition shows that the following set of conditions in terms of n , K and $\text{r}_e(\Sigma_E)$ guarantee this. Proofs for this section are contained in Appendix A.2.3.

Proposition 6. *Under model (8), suppose that Assumptions 2 and 3 hold, and that $\text{r}_e(\Sigma_E) > C \cdot n$ for some $C > 0$ large enough. Then, with probability at least $1 - c/n$, for some $c > 0$,*

$$\sigma_n^2(\mathbf{X}) \gtrsim \text{tr}(\Sigma_E) > 0,$$

and thus, in particular, $\hat{\alpha}$ interpolates: $\mathbf{X}\hat{\alpha} = \mathbf{y}$.

General existing bounds of the type $\sigma_n(\mathbf{X}) \gtrsim (\sqrt{p} - \sqrt{n})$ are by now well established in random matrix theory [37]. When $p > C \cdot n$ for some $C > 1$ and the entries of \mathbf{X} are i.i.d. sub-Gaussian with zero mean and unit variance, Theorem 1.1 in [37] implies that $\sigma_n^2(\mathbf{X}) \gtrsim p$ with high probability. By comparison, Proposition 6 holds for \mathbf{X} with i.i.d. sub-Gaussian rows with covariance matrix $\Sigma_X = A\Sigma_Z A^\top + \Sigma_E$.

Recall that, at the beginning of Section 3, we argued that whenever

$$n > C \cdot K \quad \text{and} \quad \xi := \lambda_K(A\Sigma_Z A^\top) / \|\Sigma_E\| > C \cdot \text{r}_e(\Sigma_E) / n,$$

for some $C > 0$, $\text{r}_e(\Sigma_X)/n$ remains bounded. In this case, Theorem 3 does not imply $R(\hat{\alpha})/R(\mathbf{0}) \rightarrow 1$, opening up the possibility that $\hat{\alpha}$ has asymptotically lower risk than $\mathbf{0}$. Theorem 5 above showed that $R(\hat{\alpha}) - \sigma_\varepsilon^2$ can in fact approach 0 under certain conditions when $E = 0$. The following result demonstrates that this can continue to hold even when $E \neq 0$.

Theorem 7 (Main result: Risk bound for factor regression). *Under model (8), suppose that Assumptions 2 and 3 hold and $n > C \cdot K$ and $\text{r}_e(\Sigma_E) > C \cdot n$ hold, for some $C > 0$. Then, with probability exceeding $1 - c/n$, for some $c > 0$,*

$$R(\hat{\alpha}) - R(\alpha^*) \leq R(\hat{\alpha}) - \sigma_\varepsilon^2 \lesssim \frac{\|\beta\|_{\Sigma_Z}^2}{\xi} \cdot \frac{\text{r}_e(\Sigma_E)}{n} + \sigma_\varepsilon^2 \frac{n \log n}{\text{r}_e(\Sigma_E)} + \sigma_\varepsilon^2 \frac{K \log n}{n}. \quad (17)$$

Recall $\xi := \lambda_K(A\Sigma_Z A^\top) / \|\Sigma_E\|$ is the signal-to-noise ratio.

The first inequality in (17) is an immediate consequence of the second part of Lemma 4 above.

We now discuss the three terms appearing in the upper bound (17) of Theorem 7. A comparison with the risk bound in Theorem 5 above, where the feature noise E is equal to zero, reveals that the term $\sigma_\varepsilon^2 K \log(n)/n$ in (17) is equal to the risk of the oracle predictor \hat{y}_z up to the multiplicative $\log n$ factor, and is small when $K \ll n$. The first two terms can be viewed as bias and variance components, respectively, that capture the impact of non-zero Σ_E . The first term (bias) is proportional to the effective rank $\text{r}_e(\Sigma_E)$, while the second term (variance) is inversely proportional to $\text{r}_e(\Sigma_E)$. As such, the variance term is implicitly regularized by the feature noise E , while for the bias to be small, we need the signal-to-noise ratio ξ to be sufficiently large. For example, suppose

that $\lambda_K(\Sigma_Z) > c_1$ and $c_2 < \lambda_p(\Sigma_E) \leq \|\Sigma_E\| < c_3$, both standard assumptions in factor models. Then,

$$\text{r}_e(\Sigma_E) \asymp p, \quad \text{and} \quad \xi = \frac{\lambda_K(A\Sigma_Z A^\top)}{\|\Sigma_E\|} \gtrsim \lambda_K(A^\top A). \quad (18)$$

Provided β has uniformly bounded entries and $\|\Sigma_Z\| \leq C$ so that $\|\beta\|_{\Sigma_Z}^2 \lesssim K$, the bias term in (17) can be bounded as

$$B_Z := \frac{\|\beta\|_{\Sigma_Z}^2}{\xi} \cdot \frac{\text{r}_e(\Sigma_E)}{n} \lesssim \frac{Kp}{n \cdot \lambda_K(A^\top A)}; \quad (19)$$

it thus approaches zero whenever

$$\lambda_K(A^\top A) \gg \frac{Kp}{n}. \quad (20)$$

We give a few examples of A that imply (20):

1. For a well-conditioned matrix $A \in \mathbb{R}^{p \times K}$ with entries taking values in a bounded interval, $\lambda_K(A^\top A) \asymp p$, and (20) holds when $K \ll n$, as already assumed.
2. Treating A as a realization of a random matrix with i.i.d. entries and $p \gg K$, then, by analogy with Proposition 6, we once again have $\lambda_K(A^\top A) \gtrsim p$, with high probability, and (20) holds for $K \ll n$.
3. In other situations, (20) is an assumption. It is a very natural, and mild, requirement in factor regression models, and if A is structured and sparse, (20) can be given further interpretation. For instance, the model $X = AZ + E$ has been used and analyzed in [12] for clustering the p components of X around the latent Z -coordinates, via an assignment matrix $A \in \{0, 1\}^{p \times K}$, and when Σ_E is an approximately diagonal matrix. Denoting the size of the smallest of the K non-overlapping clusters by m , for some integer $2 \leq m \leq p$, it is immediate to see (Lemma 20 in Appendix C.3) that $\lambda_K(A^\top A) \geq m$. Furthermore, when these K clusters are approximately balanced, then $m \approx p/K$ and (20) holds, provided $K^2 \ll n$.

We summarize this discussion in Corollary 8 below.

Corollary 8. *Under the same conditions as in Theorem 7, suppose, in particular, that $\lambda_K(\Sigma_Z)$ and $\|\Sigma_E\|$ are constant, $\text{r}_e(\Sigma_E) \asymp p$, and $\|\beta\|_{\Sigma_Z}^2 \lesssim K$. Then, with probability at least $1 - c/n$, for some absolute constant $c > 0$,*

$$R(\hat{\alpha}) - R(\alpha^*) \leq R(\hat{\alpha}) - \sigma_\varepsilon^2 \lesssim \frac{K}{\lambda_K(A^\top A)} \times \frac{p}{n} + \sigma_\varepsilon^2 \left(\frac{n}{p} + \frac{K}{n} \right) \log n. \quad (21)$$

In particular, if $\lambda_K(A^\top A) \gtrsim p/K$, and with probability at least $1 - c/n$, for some absolute constant $c > 0$,

$$R(\hat{\alpha}) - R(\alpha^*) \leq R(\hat{\alpha}) - \sigma_\varepsilon^2 \lesssim \frac{K^2}{n} + \sigma_\varepsilon^2 \left(\frac{n}{p} + \frac{K}{n} \right) \log n. \quad (22)$$

While Theorem 7 and Corollary 8 demonstrate that the bias term cannot be ignored, it also stresses the importance of the modeling framework we put forward in this work: (X, y) is jointly low-dimensional via the the factor regression model [3]

In contrast, if we consider instead the stand-alone factor model on $X = AZ + E$, and the classical linear model $y = X^\top \alpha + \varepsilon$, in which case $\alpha = \alpha^*$ given by (15), then the bias can be much

higher. This can be seen, for instance, via Theorem 2 in [3], which provides a bound on the excess risk $R(\hat{\alpha}) - R(\alpha^*)$, for generic Gaussian (X, y) , and α^* given by (15). The bias term in the bound of [3] is given by

$$\|\alpha^*\|^2 \|\Sigma_X\| \sqrt{\frac{r_e(\Sigma_X)}{n}} \quad (23)$$

and since $\|\Sigma_X\| \sqrt{r_e(\Sigma_X)/n} \not\rightarrow 0$, when $p \gg n$, we would need $\|\alpha^*\| \rightarrow 0$, at an appropriate rate, for the bias to be asymptotically negligible. For a general $\alpha^* \in \mathbb{R}^p$, this is a strong assumption, even if α^* is sparse. However, if the model (8) holds, the expression of the best linear predictor coefficient α^* simplifies and adapts to the low-dimensional structure of (X, y) . In particular, we find from Lemma 14 that at least when $\lambda_p(\Sigma_E) > c > 0$, we have $\|\alpha^*\|^2 \lesssim \|\beta\|_{\Sigma_Z}^2 / \xi$, which converges to zero as long as ξ grows fast enough. Furthermore, under our modelling assumptions, the generic bias term provided by Theorem 2 in [3] can be refined, as in Theorem 7. We provide a detailed comparison of these results in Section 3.5 and Appendix A.3 below.

Figure 1 illustrates the risk behavior proved in Theorem 7. Note the descent to zero in the regime $\gamma := p/n > 1$. For completeness, we also provide a bound on the risk $R(\hat{\alpha})$ for the low-dimensional case $p \ll n$, under model (8), in Appendix C.2.

3.4 Comparison to Principal Component Regression

Under the assumption that the covariance matrix Σ_X has an approximately low rank, such as in the factor regression model, a natural and practical prediction method is Principal Component Regression (PCR). Here the response \mathbf{y} is regressed onto the first \hat{K} principal components of the data matrix \mathbf{X} to estimate a vector of coefficients $(\mathbf{X}\hat{U}_{\hat{K}})^+ \mathbf{y}$. Typically \hat{K} is estimated from the data and $\hat{U}_{\hat{K}} \in \mathbb{R}^{p \times \hat{K}}$ has columns equal to the first \hat{K} eigenvectors of the sample covariance matrix $\mathbf{X}^\top \mathbf{X} / n$. We now show that in the high-dimensional regime $p \gg n$, under the factor regression model setting (8), the minimum-norm estimator $\hat{\alpha}$ is competitive even with the stylized version $\hat{\beta} := (\mathbf{X}U_K)^+ \mathbf{y}$ of PCR. This is a toy estimator as it uses the unknown dimension K and U_K , composed of the first K eigenvectors of the population covariance matrix Σ_X , in place of estimates \hat{K} and \hat{U}_K , respectively. We have the following risk bound for

$$R_{\text{PCR}}(\hat{\beta}) := \mathbb{E} \left[(X^\top U_K \hat{\beta} - y)^2 \right]. \quad (24)$$

See Appendix A.2.4 for its proof.

Theorem 9. *Under model (8), suppose that (X, y) are jointly Gaussian and that $\lambda_p(\Sigma_E) > 0$. Then if $n > C \cdot K \log n$ for some $C > 0$ large enough, with probability at least $1 - c/n$,*

$$R_{\text{PCR}}(\hat{\beta}) - \sigma_\varepsilon^2 \lesssim \kappa(\Sigma_E) \frac{\|\beta\|_{\Sigma_Z}^2}{\xi} \frac{p}{n} + \sigma_\varepsilon^2 \frac{K \log n}{n}, \quad (25)$$

where $\kappa(\Sigma_E) := \lambda_1(\Sigma_E) / \lambda_p(\Sigma_E)$ is the condition number of the matrix Σ_E .

Provided $\kappa(\Sigma_E)$ is bounded above by an absolute constant, the upper bounds for the minimum-norm and PCR predictors are comparable. Indeed, Theorem 7 implies the bound

$$R(\hat{\alpha}) - \sigma_\varepsilon^2 \lesssim \frac{\|\beta\|_{\Sigma_Z}^2}{\xi} \frac{p}{n} + \sigma_\varepsilon^2 \log n \left(\frac{K}{n} + \frac{n}{p} \right) \quad (26)$$

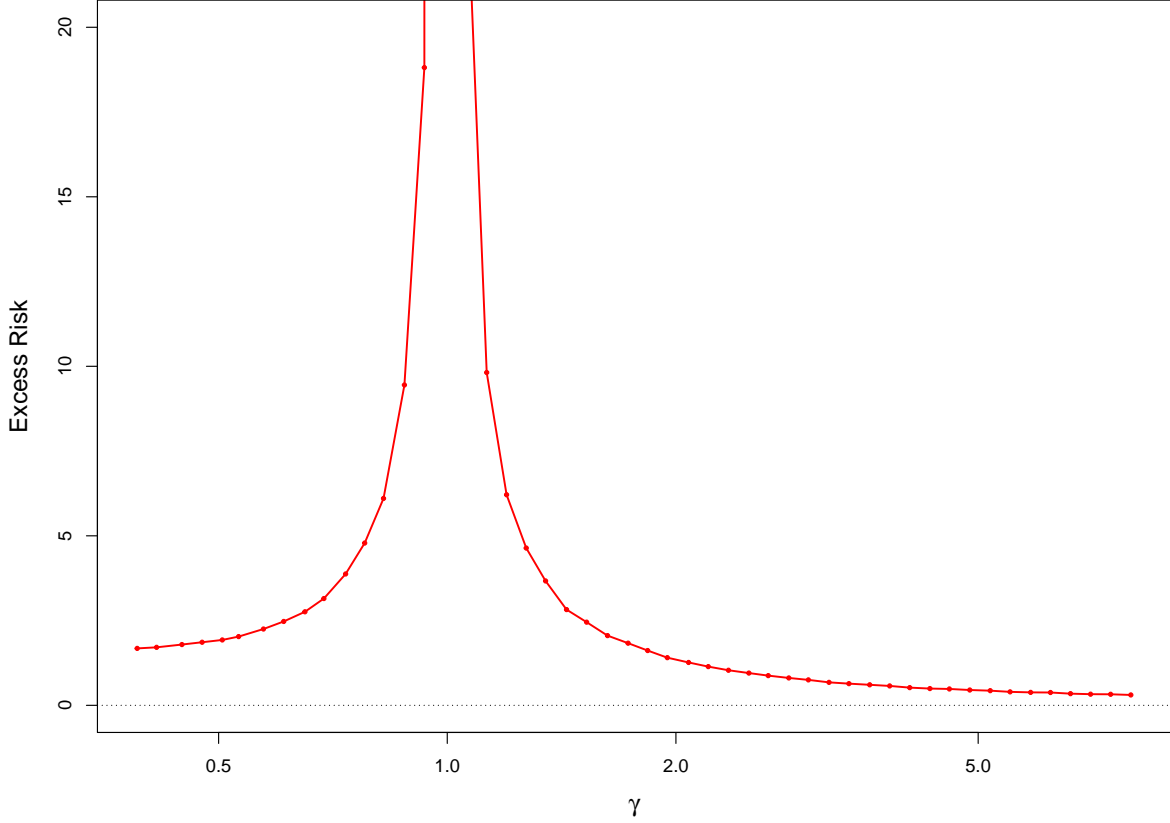


Figure 1: Excess prediction risk $R(\hat{\alpha}) - \sigma_\varepsilon^2$ of the minimum-norm predictor under the factor regression model as a function of $\gamma = p/n$. Here K increases linearly from 16 to 64, $n = \lceil K^{1.5} \rceil$ and thus increases from 64 to 512, and p increases from 33 to 4066. Further, $\Sigma_E = I_p$, $\Sigma_Z = I_K$, $\beta = (1, \dots, 1)^\top$, and $A = \sqrt{p} \cdot V_K$, where V_K is generated by taking the first K rows of a randomly generated $p \times p$ orthogonal matrix V .

for the excess risk of the minimum-norm predictor. The additional term $\sigma_\varepsilon^2 n \log n/p$ in this bound is absent in the PCR prediction bound (25) above, but in the regime $p \gg n$ it can become negligible. It is perhaps surprising that under the factor regression model, the interpolator $\hat{\alpha}$ can not only provide consistent prediction, but can in fact have excess risk comparable to a genuine K -dimensional predictor widely used in practice and tailored to the problem setting.

3.5 Comparison to existing finite sample bounds

The recent paper [3] gives a bias-variance type bound on the excess prediction risk $R(\hat{\alpha}) - R(\alpha^*)$ of the minimum-norm predictor $\hat{y}_x = X^\top \hat{\alpha}$ considered in this work. In contrast to our study, [3] does not consider model (8), and in fact assumes $\mathbb{E}[y|X] = X^\top \alpha^*$, which is typically not satisfied under (8) for general sub-Gaussian (X, y) . When the data are jointly Gaussian this assumption is however satisfied under model (8), and for this common case Table 2 compares the respective bounds on the bias and variance terms corresponding to our Theorem 7 and Theorem 2 of [3], respectively. The

	Bias	Variance
Theorem 7	$\ \beta\ _{\Sigma_Z}^2 \cdot p/(n \cdot \xi)$	$\sigma_\varepsilon^2 \log n \{(n/p) + (K/n)\}$
Theorem 2 in [3]	$\ \beta\ _{\Sigma_Z}^2 \cdot \max \left\{ \sqrt{p/(n \cdot \xi)}, p/(n \cdot \xi) \right\}$	$\sigma_\varepsilon^2 \log n \{(n/p) + (K/n)\}$

Table 2: Comparison of risk bounds

entries in the second row of Table 2 correspond to the results in [3] under model (8), simplified in this table for ease of comparison¹. Further details and discussion on the comparison of these two results are deferred to Appendix A.3.

We first note that the variance terms in Table 2 match and that when $p/(n \cdot \xi) \geq 1$, the bias terms match as well. However, $p \ll n \cdot \xi$ is a necessary condition for either bound to converge to zero (assuming $\|\beta\|_{\Sigma_Z}^2$ is bounded below). In this case, the bound in [3] becomes $\|\beta\|_{\Sigma_Z}^2 \sqrt{p/(n \cdot \xi)}$, which is larger than our bias bound in Theorem 7 by a factor of $\sqrt{p/(n \cdot \xi)}$. In fact, the upper bound on the excess risk in [3] can diverge while our bound in Theorem 7 vanishes. For instance, if β is a non-sparse vector in \mathbb{R}^K with $\|\beta\|_{\Sigma_Z}^2 \asymp K$, this phenomenon occurs if the signal-to-noise ratio ξ lies in the range $Kp/n \lesssim \xi \lesssim K^2p/n$. This illustrates that the general bound provided in [3] is not always tight.

We make one further remark by way of comparison with [3]. They define a class of *benign* covariance matrices that can lead to consistent prediction with $\hat{\alpha}$, but this class is only defined under the assumption $\|\Sigma_X\| = 1$. However, under the natural assumption $\|\Sigma_E\| > c > 0$ in the factor model,

$$\|\Sigma_X\| \geq \|A\Sigma_Z A^\top\| \gtrsim \frac{\lambda_K(A\Sigma_Z A^\top)}{\|\Sigma_E\|} = \xi,$$

so $\xi \rightarrow \infty$ implies $\|\Sigma_X\| \rightarrow \infty$. In this sense, the class of factor regression models studied here are not considered in the *benign* definition of [3], yet are natural and can lead to consistent prediction.

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¹ For simplicity, we assume for this comparison that the matrices Σ_X and Σ_E are invertible and that the condition numbers $\kappa(\Sigma_E)$ and $\kappa(A\Sigma_Z A^\top)$ are bounded above by an absolute constant. Consequently, the effective rank $r_e(\Sigma_E)$ satisfies $c \cdot p \leq r_e(\Sigma_E) \leq p$, for some $c \in (0, 1)$.

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A Proofs for the main text

A.1 Proofs for Section 2

Proof of Theorem 1

We work on the event

$$\mathcal{K} := \{\sigma_n^2(\mathbf{X}) \gtrsim \text{tr}(\Sigma_X), \|\mathbf{y}\|^2 \lesssim n\sigma_y^2\}. \quad (27)$$

On this event, recalling $\hat{\alpha} = \mathbf{X}^+ \mathbf{y}$ and invoking identity (116) in Appendix D,

$$\|\hat{\alpha}\|^2 \leq \|\mathbf{X}^+\|^2 \|\mathbf{y}\|^2 = \frac{\|\mathbf{y}\|^2}{\sigma_n^2(\mathbf{X})} \lesssim \sigma_y^2 \frac{n}{\text{tr}(\Sigma_X)},$$

which proves the claim of the theorem. It remains to show that the event \mathcal{K} occurs with high probability. To this end, note that since we suppose Assumption 1 holds, we have $\mathbf{X} = \tilde{\mathbf{X}} \Sigma_X^{1/2}$, and thus

$$\sigma_n^2(\mathbf{X}) = \lambda_n(\mathbf{X} \mathbf{X}^\top) = \lambda_n(\tilde{\mathbf{X}} \Sigma_X \tilde{\mathbf{X}}),$$

where $\tilde{\mathbf{X}}$ has i.i.d. entries that have zero mean, unit variance, and sub-Gaussian constants bounded by an absolute constant. Theorem 10 in Appendix A.2.3 below thus implies that if $r_e(\Sigma_X) > C \cdot n$ for $C > 0$ large enough, then with probability at least $1 - e^{-cn}$,

$$\sigma_n^2(\mathbf{X}) \geq \text{tr}(\Sigma_X)/2 - c_0 \|\Sigma_X\| n = \text{tr}(\Sigma_X) \cdot [1/2 - c_0 n / r_e(\Sigma_X)].$$

Using that $n/r_e(\Sigma_X) < 1/C$ and choosing C large enough,

$$\mathbb{P}(\sigma_n^2(\mathbf{X}) \gtrsim \text{tr}(\Sigma_X)) \geq 1 - e^{-cn}. \quad (28)$$

By Assumption 1, $\mathbf{y} = \sigma_\varepsilon \tilde{\mathbf{y}}$. Since $\tilde{y}_1, \dots, \tilde{y}_n$ have zero mean and sub-Gaussian constants bounded by an absolute constant, Bernstein's inequality (Corollary 2.8.3 of [41]) implies that

$$\mathbb{P}(\|\tilde{\mathbf{y}}\|^2 \gtrsim n) = \mathbb{P}\left(\left|\sum_{i=1}^n \tilde{y}_i^2\right| \gtrsim n\right) \leq 2e^{-2cn}.$$

Thus,

$$\mathbb{P}(\|\mathbf{y}\|^2 \gtrsim \sigma_y^2 n) = \mathbb{P}(\sigma_y^2 \|\tilde{\mathbf{y}}\|^2 \gtrsim \sigma_y^2 n) = \mathbb{P}(\|\tilde{\mathbf{y}}\|^2 \gtrsim n) \leq 2e^{-2cn}.$$

Combining this with (28) completes the proof that $\mathbb{P}(\mathcal{K}) \geq 1 - ce^{-c'n}$. ■

Theorem 10 and its proof

The proof of Theorem 1 above made crucial use of the following result.

Theorem 10. *Suppose \mathbf{W} is an $n \times r$ random matrix with independent columns and subgaussian entries that have zero mean and unit variance. Then for any positive semi-definite matrix $\Sigma \in \mathbb{R}^{r \times r}$ and some $c' > 0$ large enough, with probability at least $1 - 2e^{-cn}$,*

$$M^2 \text{tr}(\Sigma)/2 - c' M^2 \|\Sigma\| n \leq \lambda_n(\mathbf{W} \Sigma \mathbf{W}^\top) \leq \lambda_1(\mathbf{W} \Sigma \mathbf{W}^\top) \leq 3M^2 \text{tr}(\Sigma)/2 + c' M^2 \|\Sigma\| n,$$

where $M := \max_{i,j} \|\mathbf{W}_{ij}\|_{\psi_2}$.

A similar result for diagonal Σ has been derived in Lemma 9 of [3]. We make use of the Hanson-Wright inequality in our proof to deal with non-diagonal Σ . Theorem 4.6.1 in [41] provides similar two-sided bounds for the smallest and largest eigenvalue of $\mathbf{W}\Sigma\mathbf{W}^\top$, when $\Sigma = I_r$.

Proof. First we note that we can prove the result for $M = 1$ without loss of generality. Indeed, suppose we have proven it for $M = 1$. Then for any $M > 0$,

$$\mathbf{W}\Sigma\mathbf{W}^\top = \left(\mathbf{W}\frac{1}{M}\right)(M^2\Sigma)\left(\mathbf{W}^\top\frac{1}{M}\right).$$

Then since $\max_{ij} \|\mathbf{W}_{ij}/M\|_{\psi_2} = \max_{ij} \|\mathbf{W}_{ij}\|_{\psi_2}/M = 1$, we can apply the theorem to find that with probability at least $1 - 2e^{-cn}$,

$$\text{tr}(M^2\Sigma)/2 - c'\|M^2\Sigma\|n \leq \lambda_n(\mathbf{W}\Sigma\mathbf{W}^\top) \leq \lambda_1(\mathbf{W}\Sigma\mathbf{W}^\top) \leq 3\text{tr}(M^2\Sigma)/2 + c'\|M^2\Sigma\|n,$$

which implies the claim of the theorem. We therefore assume henceforth that $M = 1$. We will prove that for some $c' \geq 1$,

$$\|\mathbf{W}\Sigma\mathbf{W}^\top - \text{tr}(\Sigma)I_n\| \leq c'\|\Sigma\|n + \text{tr}(\Sigma)/2 \quad (29)$$

with probability at least $1 - 2e^{-cn}$. Equation (29) implies that for any $v \in \mathbb{R}^n$ with $\|v\| = 1$,

$$|v^\top \mathbf{W}\Sigma\mathbf{W}^\top v - \text{tr}(\Sigma)| \leq c'\|\Sigma\|n + \text{tr}(\Sigma)/2,$$

and so

$$\text{tr}(\Sigma)/2 - c'\|\Sigma\|n \leq v^\top \mathbf{W}\Sigma\mathbf{W}^\top v \leq 3\text{tr}(\Sigma)/2 + c'\|\Sigma\|n.$$

Taking the minimum and maximum over $v \in S^{n-1}$ then gives the desired result.

We now prove (29). Let \mathcal{N} be a $1/4$ -net of S^{n-1} with $|\mathcal{N}| \leq 9^n$, which exists by Corollary 4.2.13 of [41]. Then by Exercise 4.4.3 of [41],

$$\|\mathbf{W}\Sigma\mathbf{W}^\top - \text{tr}(\Sigma)I_n\| = \sup_{v \in S^{n-1}} |v^\top \mathbf{W}\Sigma\mathbf{W}^\top v - \text{tr}(\Sigma)| \leq 2 \sup_{v \in \mathcal{N}} |v^\top \mathbf{W}\Sigma\mathbf{W}^\top v - \text{tr}(\Sigma)|, \quad (30)$$

where we use that $\mathbf{W}\Sigma\mathbf{W}^\top - \text{tr}(\Sigma)I_n$ is symmetric in the first step.

Now fix $v \in S^{n-1}$ and define $B = \mathbf{W}^\top v \in \mathbb{R}^r$. Observe that B has mean zero entries that are independent because the columns of \mathbf{W} are independent. Furthermore,

$$\|B_i\|_{\psi_2} = \left\| \sum_j \mathbf{W}_{ji} v_j \right\|_{\psi_2} \leq \sum_j \|\mathbf{W}_{ji}\|_{\psi_2} v_j \leq \max_{\ell i} \|\mathbf{W}_{\ell i}\|_{\psi_2} \sum_j v_j = M = 1,$$

where we used $\|v\| = 1$ in the last step. Thus, by the Hanson-Wright inequality (Theorem 6.2.1 in [41]),

$$\mathbb{P}\left(|B^\top \Sigma B - \mathbb{E}B^\top \Sigma B| \geq c_1 t\right) \leq 2 \exp\left\{-c_2 \min\left(t/\|\Sigma\|, t^2/\|\Sigma\|_F^2\right)\right\}, \quad (31)$$

where we can choose $c_1 > 0$ large enough such that $c_2 \geq 12$.

Note that

$$\mathbb{E}B^\top \Sigma B = \sum_{i,j,k,l} \mathbb{E}v_i \mathbf{W}_{ij} \Sigma_{jl} \mathbf{W}_{kl} v_k = \sum_{ij} v_i^2 \Sigma_{jj} \mathbb{E}\mathbf{W}_{ij}^2 = \|v\|^2 \text{tr}(\Sigma) = \text{tr}(\Sigma), \quad (32)$$

where in the second step we use that \mathbf{W} has independent mean zero entries, in the third step we use that $\mathbb{E}\mathbf{W}_{ij}^2 = 1$ for all i, j , and in the final step we use that $\|v\| = 1$.

Choosing $t = \|\Sigma\|n/2 + \sqrt{n\|\Sigma\|_F^2/2}$ in (31) and using that $c_2 \geq 12$, we observe that

$$c_2 t / \|\Sigma\| = c_2 n / 2 + c_2 \sqrt{n\|\Sigma\|_F^2 / (2\|\Sigma\|)} \geq c_2 n / 2 \geq 3n,$$

and

$$c_2 t^2 / \|\Sigma\|_F^2 = c_2 [n\|\Sigma\| / (2\|\Sigma\|_F) + \sqrt{n/2}]^2 \geq c_2 n / 4 \geq 3n.$$

Thus,

$$\mathbb{P}\left(|B^\top \Sigma B - \text{tr}(\Sigma)| \geq c_1 \|\Sigma\|n/2 + c_1 \sqrt{n\|\Sigma\|_F^2/2}\right) \leq 2e^{-3n}, \quad (33)$$

where we used (32). Finally, using

$$\|\Sigma\|_F^2 = \text{tr}(\Sigma^2) \leq \|\Sigma\| \text{tr}(\Sigma),$$

and the inequality $2ab \leq a^2 + b^2$,

$$c_1 \sqrt{n\|\Sigma\|_F^2/2} \leq c_1 \sqrt{(c_1 n \|\Sigma\|)(\text{tr}(\Sigma)/c_1)/2} \leq c_1^2 n \|\Sigma\| / 4 + \text{tr}(\Sigma) / 4.$$

Thus, by (33), and for $c' = c_1 + c_1^2/2$ large enough,

$$\mathbb{P}\left(|B^\top \Sigma B - \text{tr}(\Sigma)| \geq c' \|\Sigma\|n/2 + \text{tr}(\Sigma)/4\right) \leq 2e^{-3n}. \quad (34)$$

Denoting $c' \|\Sigma\|n/2 + \text{tr}(\Sigma)/4$ by L , we thus have

$$\begin{aligned} \mathbb{P}\left(\|\mathbf{W}\Sigma\mathbf{W}^\top - \text{tr}(\Sigma)I_n\| \geq c' \|\Sigma\|n + \text{tr}(\Sigma)/2\right) &= \mathbb{P}\left(\|\mathbf{W}\Sigma\mathbf{W}^\top - \text{tr}(\Sigma)I_n\| \geq 2L\right) \\ &\leq \mathbb{P}\left(2 \sup_{v \in \mathcal{N}} |v^\top \mathbf{W}\Sigma\mathbf{W}^\top v - \text{tr}(\Sigma)| \geq 2L\right) \quad (\text{by (30)}) \\ &\leq \sum_{v \in \mathcal{N}} \mathbb{P}\left(|v^\top \mathbf{W}\Sigma\mathbf{W}^\top v - \text{tr}(\Sigma)| \geq L\right) \quad (\text{union bound}) \\ &\leq 2 \times 9^n e^{-3n} \quad (\text{by (34)}) \\ &= 2e^{n \log(9) - 3n} \leq 2e^{-cn}, \end{aligned}$$

where we define $c = 3 - \log(9) > 0$ in the last step. This shows (29) and completes the proof. \blacksquare

Proof of Theorem 3

By Theorem 1, if $r_e(\Sigma_X) > C \cdot n$ for $C > 0$ large enough, then with probability at least $1 - ce^{-c'n}$,

$$\frac{\|\hat{\alpha}\|_{\Sigma_X}^2}{R(\mathbf{0})} \lesssim \frac{n}{r_e(\Sigma_X)}. \quad (35)$$

Thus, by Theorem 2, with at least the same probability,

$$\left|\frac{R(\hat{\alpha})}{R(\mathbf{0})} - 1\right| \leq \frac{\|\hat{\alpha}\|_{\Sigma_X}^2}{R(\mathbf{0})} + 2\sqrt{\frac{\|\hat{\alpha}\|_{\Sigma_X}^2}{R(\mathbf{0})}} \lesssim \frac{n}{r_e(\Sigma_X)} + \sqrt{\frac{n}{r_e(\Sigma_X)}}.$$

Setting $C' = \max(C, 1)$, when $r_e(\Sigma_X) > C'n \geq n$, so $n/r_e(\Sigma_X) > 1$, we find

$$\frac{n}{r_e(\Sigma_X)} + \sqrt{\frac{n}{r_e(\Sigma_X)}} \leq 2\sqrt{\frac{n}{r_e(\Sigma_X)}}.$$

Thus with probability at least $1 - ce^{-c'n}$,

$$\left| \frac{R(\hat{\alpha})}{R(\mathbf{0})} - 1 \right| \lesssim \sqrt{\frac{n}{r_e(\Sigma_X)}},$$

and the proof is complete. ■

Proof of Lemma 2

We first show that $\Sigma_X \alpha^* = \Sigma_{Xy}$, where $\alpha^* = \Sigma_X^+ \Sigma_{Xy}$. To this end, observe that

$$\begin{aligned} \text{Cov}((I - \Sigma_X \Sigma_X^+)X) &= (I_p - \Sigma_X \Sigma_X^+) \mathbb{E}[XX^\top] (I_p - \Sigma_X \Sigma_X^+) \\ &= (I_p - \Sigma_X \Sigma_X^+) \Sigma_X (I_p - \Sigma_X^+ \Sigma_X) \\ &= 0, \end{aligned}$$

where we use that $\Sigma_X \Sigma_X^+ \Sigma_X = \Sigma_X$ (see Appendix D). Thus $(I_p - \Sigma_X \Sigma_X^+)X = 0$ a.s., so

$$\Sigma_X \alpha^* = \Sigma_X \Sigma_X^+ \Sigma_{Xy} = \mathbb{E}[\Sigma_X \Sigma_X^+ Xy] = \mathbb{E}[Xy] = \Sigma_{Xy}. \quad (36)$$

Fixing $\theta \in \mathbb{R}^p$, we have

$$\begin{aligned} R(\theta) - R(\mathbf{0}) &= \mathbb{E}[(X^\top \theta - y)^2] - \mathbb{E}[y^2] \\ &= \theta^\top \mathbb{E}[XX^\top] \theta - 2\theta^\top \mathbb{E}[Xy] \\ &= \|\theta\|_{\Sigma_X}^2 - 2\theta^\top \Sigma_{Xy} \\ &= \|\theta\|_{\Sigma_X}^2 - 2\theta^\top \Sigma_X \alpha^* \quad (\text{by (36)}), \end{aligned}$$

so by the Cauchy-Schwarz inequality,

$$|R(\theta) - R(\mathbf{0})| \leq \|\theta\|_{\Sigma_X}^2 + 2\|\theta\|_{\Sigma_X} \|\alpha^*\|_{\Sigma_X}. \quad (37)$$

Next observe that

$$R(\mathbf{0}) = \mathbb{E}[y^2] = \mathbb{E}(y - X^\top \alpha^* + X^\top \alpha^*)^2 = R(\alpha^*) + \|\alpha^*\|_{\Sigma_X}^2 \geq \|\alpha^*\|_{\Sigma_X}^2,$$

where we use that by (36),

$$\mathbb{E}(X^\top \alpha^*)(X^\top \alpha^* - y) = \alpha^{*\top} \Sigma_X \alpha^* - \alpha^{*\top} \Sigma_{Xy} = 0.$$

Thus, $\|\alpha^*\|_{\Sigma_X}^2 \leq R(\mathbf{0})$, so by (37),

$$|R(\theta) - R(\mathbf{0})| \leq \|\theta\|_{\Sigma_X}^2 + 2\|\theta\|_{\Sigma_X} \sqrt{R(\mathbf{0})}. \quad (38)$$

Dividing both sides by $R(\mathbf{0})$ gives the final result. ■

A.2 Proofs for Section 3

A.2.1 Proof of Lemma 4 from Section 3.1

Using $y = Z^\top \beta + \varepsilon$ and the fact that ε is independent of X and Z ,

$$R(\alpha^*) = \mathbb{E}[(\alpha^{*\top} X - y)]^2 = \mathbb{E}[(\alpha^{*\top} X - Z^\top \beta)]^2 + \sigma_\varepsilon^2 \geq \sigma_\varepsilon^2,$$

which proves the first claim. Using $X = AZ + E$, we further find

$$R(\alpha^*) - \sigma_\varepsilon^2 = \mathbb{E}[(\alpha^{*\top} X - Z^\top \beta)]^2 = \alpha^{*\top} \Sigma_X \alpha^* + \beta^\top \Sigma_Z \beta - 2\alpha^{*\top} A \Sigma_Z \beta. \quad (39)$$

Now suppose Σ_E and Σ_Z are invertible as in the second claim. Then in particular,

$$\lambda_p(\Sigma_X) \geq \lambda_p(\Sigma_E) > 0,$$

so Σ_X is invertible and thus $\Sigma_X^+ = \Sigma_X^{-1}$. Also, $\Sigma_X y = \mathbb{E}[Xy] = A \Sigma_Z \beta$, so

$$\alpha^* = \Sigma_X^+ \Sigma_X y = \Sigma_X^{-1} A \Sigma_Z \beta.$$

Plugging this into (39) and simplifying, we find

$$R(\alpha^*) - \sigma_\varepsilon^2 = \beta^\top \left[\Sigma_Z - \Sigma_Z A^\top \Sigma_X^{-1} A \Sigma_Z \right] \beta. \quad (40)$$

By the Woodbury matrix identity,

$$\Sigma_X^{-1} = (A \Sigma_Z A^\top + \Sigma_E)^{-1} = \Sigma_E^{-1} - \Sigma_E^{-1} A (\Sigma_Z^{-1} + A^\top \Sigma_E^{-1} A)^{-1} A^\top \Sigma_E^{-1},$$

so letting $G := \Sigma_Z^{-1} + A^\top \Sigma_E^{-1} A$,

$$\Sigma_Z A^\top \Sigma_X^{-1} A \Sigma_Z = \Sigma_Z A^\top \Sigma_E^{-1} A \Sigma_Z - \Sigma_Z A^\top \Sigma_E^{-1} A G^{-1} A^\top \Sigma_E^{-1} A \Sigma_Z.$$

Now using $A^\top \Sigma_E^{-1} A = G - \Sigma_Z^{-1}$, we find

$$\begin{aligned} \Sigma_Z A^\top \Sigma_X^{-1} A \Sigma_Z &= \Sigma_Z (G - \Sigma_Z^{-1}) \Sigma_Z - \Sigma_Z (G - \Sigma_Z^{-1}) G^{-1} (G - \Sigma_Z^{-1}) \Sigma_Z \\ &= \Sigma_Z G \Sigma_Z - \Sigma_Z - \Sigma_Z (I_K - \Sigma_Z^{-1} G^{-1}) (G - \Sigma_Z^{-1}) \Sigma_Z \\ &= \Sigma_Z G \Sigma_Z - \Sigma_Z - [\Sigma_Z G \Sigma_Z - \Sigma_Z - \Sigma_Z + G^{-1}] \\ &= \Sigma_Z - G^{-1}. \end{aligned}$$

Using this to simplify (40), we find

$$R(\alpha^*) - \sigma_\varepsilon^2 = \beta^\top G^{-1} \beta,$$

proving the equality in the second claim of the lemma. To show the upper bound $\beta^\top G^{-1} \beta \leq \|\beta\|_{\Sigma_Z}^2 / \xi$, we use

$$\beta^\top G^{-1} \beta = \beta^\top \Sigma_Z^{1/2} (\Sigma_Z^{1/2} G \Sigma_Z^{1/2})^{-1} \Sigma_Z^{1/2} \beta \leq \|\beta\|_{\Sigma_Z}^2 \|(\Sigma_Z^{1/2} G \Sigma_Z^{1/2})^{-1}\| = \|\beta\|^2 / \lambda_K(\Sigma_Z^{1/2} G \Sigma_Z^{1/2}),$$

and that since $\Sigma_Z^{1/2} G \Sigma_Z^{1/2} = I_K + \Sigma_Z^{1/2} A^\top \Sigma_E^{-1} A \Sigma_Z^{1/2}$,

$$\lambda_K(\Sigma_Z^{1/2} G \Sigma_Z^{1/2}) \geq \lambda_K(\Sigma_Z A^\top \Sigma_E^{-1} A \Sigma_Z) \geq \lambda_K(A \Sigma_Z A^\top) \lambda_p(\Sigma_E^{-1}) = \lambda_K(A \Sigma_Z A^\top) / \|\Sigma_E\| = \xi,$$

which together give the desired bound. ■

A.2.2 Proofs for Section 3.2

Proof of Theorem 5

Part 1: By (112) of Lemma 21 in Appendix D below,

$$\mathbf{X}^+ = (\mathbf{Z}A^\top)^+ = (\mathbf{Z}^+\mathbf{Z}A^\top)^+(\mathbf{Z}A^\top A^{+\top})^+. \quad (41)$$

Since $K < n$, $\mathbf{Z}^+\mathbf{Z} = I_K$ on the event where \mathbf{Z} is of full rank K by Lemma 21. Similarly, since A is of dimensions $p \times K$ and $\text{rank}(A) = K$ by Assumption 2,

$$A^\top A^{+\top} = (A^+A)^\top = A^+A = I_K,$$

where we also use that A^+A is symmetric by (111) of Appendix D. Using these two results in (41), we find

$$A^\top \mathbf{X}^+ = A^\top [A^{+\top} \mathbf{Z}^+] = (A^\top A^{+\top}) \mathbf{Z}^+ = \mathbf{Z}^+.$$

Thus, recalling $\hat{\alpha} = \mathbf{X}^+ \mathbf{y}$, we find that on the event where \mathbf{Z} is full rank,

$$\hat{y}_x = X^\top \hat{\alpha} = Z^\top A^\top \mathbf{X}^+ \mathbf{y} = Z^\top \mathbf{Z}^+ \mathbf{y} = Z^\top \hat{\beta} = \hat{y}_z. \quad (42)$$

Part 2: We will work on the event

$$\mathcal{A} := \left\{ \|\tilde{\mathbf{Z}}^+ \tilde{\varepsilon}\|^2 \lesssim \log(n) \text{tr}(\tilde{\mathbf{Z}}^{+\top} \tilde{\mathbf{Z}}^+), \ c_1 n \leq \sigma_K^2(\tilde{\mathbf{Z}}) \leq \|\tilde{\mathbf{Z}}\|^2 \leq c_2 n \right\},$$

which occurs with probability at least $1 - c/n$, as shown below, where $\mathbf{Z} = \tilde{\mathbf{Z}} \Sigma_Z^{1/2}$ and $\varepsilon = \sigma_\varepsilon \tilde{\varepsilon}$ by Assumption 3. Using the independence of ε and Z together with (42), the excess risk can be written as

$$R(\hat{\alpha}) - \sigma_\varepsilon^2 = \mathbb{E}[(X^\top \hat{\alpha} - Z^\top \beta)^2] = \mathbb{E}[(Z^\top \hat{\beta} - Z^\top \beta)^2] = \|\hat{\beta} - \beta\|_{\Sigma_Z}^2. \quad (43)$$

On the event \mathcal{A} , and using $\lambda_K(\Sigma_Z) > 0$ by Assumption 2,

$$\sigma_K^2(\mathbf{Z}) = \lambda_K(\mathbf{Z}\mathbf{Z}^\top) = \lambda_K(\tilde{\mathbf{Z}}\Sigma_Z\tilde{\mathbf{Z}}^\top) \geq \lambda_K(\Sigma_Z) \cdot \sigma_n^2(\tilde{\mathbf{Z}}) \gtrsim \lambda_K(\Sigma_Z) \cdot n > 0, \quad (44)$$

so $\text{rank}(\mathbf{Z}) = K$ and thus $\mathbf{Z}^+\mathbf{Z} = I_K$ by Lemma 21 below. We thus have

$$\hat{\beta} = \mathbf{Z}^+ \mathbf{y} = \mathbf{Z}^+ \mathbf{Z} \beta + \mathbf{Z}^+ \varepsilon = \beta + \mathbf{Z}^+ \varepsilon,$$

so by (43),

$$R(\hat{\alpha}) - \sigma_\varepsilon^2 = \|\mathbf{Z}^+ \varepsilon\|_{\Sigma_Z}^2 = \|\Sigma_Z^{1/2} \mathbf{Z}^+ \varepsilon\|^2. \quad (45)$$

By (112) of Lemma 21,

$$\Sigma_Z^{1/2} \mathbf{Z}^+ = \Sigma_Z^{1/2} (\tilde{\mathbf{Z}} \Sigma_Z^{1/2})^+ = \Sigma_Z^{1/2} (\tilde{\mathbf{Z}}^+ \tilde{\mathbf{Z}} \Sigma_Z^{1/2})^+ (\tilde{\mathbf{Z}} \Sigma_Z^{1/2} \Sigma_Z^{-1/2})^+ = \Sigma_Z^{1/2} \Sigma_Z^{-1/2} \tilde{\mathbf{Z}}^+ = \tilde{\mathbf{Z}}^+, \quad (46)$$

where we used that $\tilde{\mathbf{Z}}^+ \tilde{\mathbf{Z}} = I_K$ since $\text{rank}(\tilde{\mathbf{Z}}) = \text{rank}(\tilde{\mathbf{Z}}^+) = K$ on \mathcal{A} . Thus by (45),

$$R(\hat{\alpha}) - \sigma_\varepsilon^2 = \|\tilde{\mathbf{Z}}^+ \varepsilon\|^2 = \sigma_\varepsilon^2 \|\tilde{\mathbf{Z}}^+ \tilde{\varepsilon}\|^2 \lesssim \sigma_\varepsilon^2 \log(n) \text{tr}(\tilde{\mathbf{Z}}^{+\top} \tilde{\mathbf{Z}}^+). \quad (47)$$

We then use that $\text{rank}(\tilde{\mathbf{Z}}^+) = K$ and that $\|\tilde{\mathbf{Z}}^+\| = 1/\sigma_K(\tilde{\mathbf{Z}})$ from Lemma 21 in Appendix D below to find that on \mathcal{A} ,

$$\text{tr}(\tilde{\mathbf{Z}}^{+\top} \tilde{\mathbf{Z}}^+) \leq K \|\tilde{\mathbf{Z}}^{+\top} \tilde{\mathbf{Z}}^+\| = K \|\tilde{\mathbf{Z}}^+\|^2 = \frac{K}{\sigma_K^2(\tilde{\mathbf{Z}})} \lesssim \frac{K}{n}.$$

Plugging this into (47) completes the proof of the upper bound.

For the lower bound, first observe that on \mathcal{A} ,

$$\mathbb{E}_{\boldsymbol{\varepsilon}} R(\hat{\alpha}) - \sigma_{\boldsymbol{\varepsilon}}^2 = \mathbb{E}_{\boldsymbol{\varepsilon}} \|\tilde{\mathbf{Z}}^+ \boldsymbol{\varepsilon}\|^2 = \sigma_{\boldsymbol{\varepsilon}}^2 \text{tr}(\tilde{\mathbf{Z}}^{+\top} \tilde{\mathbf{Z}}^+) \geq \sigma_{\boldsymbol{\varepsilon}}^2 K \lambda_K(\tilde{\mathbf{Z}}^{+\top} \tilde{\mathbf{Z}}^+) = \sigma_{\boldsymbol{\varepsilon}}^2 K \sigma_K^2(\tilde{\mathbf{Z}}^+),$$

so using $\sigma_K(\tilde{\mathbf{Z}}^+) = 1/\|\tilde{\mathbf{Z}}\|$ by Lemma 21 again,

$$\mathbb{E}_{\boldsymbol{\varepsilon}} R(\hat{\alpha}) - \sigma_{\boldsymbol{\varepsilon}}^2 \geq \sigma_{\boldsymbol{\varepsilon}}^2 \frac{K}{\|\tilde{\mathbf{Z}}\|^2} \gtrsim \sigma_{\boldsymbol{\varepsilon}}^2 \frac{K}{n}.$$

Bounding $\mathbb{P}(\mathcal{A})$

Since $\tilde{\mathbf{Z}}$ has independent rows with entries that are zero mean, unit variance, and have sub-Gaussian constants bounded by an absolute constant, Theorem 4.6.1 of [41] gives that with probability at least $1 - 2/n$,

$$\sqrt{n} - c''(\sqrt{K} + \sqrt{\log n}) \leq \sigma_n(\tilde{\mathbf{Z}}) \leq \|\tilde{\mathbf{Z}}\| \leq \sqrt{n} + c''(\sqrt{K} + \sqrt{\log n}).$$

and thus

$$\sqrt{n} \cdot [1 - c''(\sqrt{K/n} + \sqrt{\log(n)/n})] \leq \sigma_n(\tilde{\mathbf{Z}}) \leq \|\tilde{\mathbf{Z}}\| \leq \sqrt{n} \cdot [1 + c''(\sqrt{K/n} + \sqrt{\log(n)/n})].$$

Using that $n > CK$ we can choose C large enough such that

$$c''(\sqrt{K/n} + \sqrt{\log(n)/n}) < c_0 < 1,$$

and thus

$$\mathbb{P}\left(c_3 n \leq \sigma_K^2(\tilde{\mathbf{Z}}) \leq \|\tilde{\mathbf{Z}}\|^2 \leq c_4 n\right) \geq 1 - 2/n. \quad (48)$$

The bound

$$\mathbb{P}\left(\|\tilde{\mathbf{Z}}^+ \tilde{\boldsymbol{\varepsilon}}\|^2 \lesssim \log(n) \text{tr}[\tilde{\mathbf{Z}}^{+\top} \tilde{\mathbf{Z}}^+]\right) \geq 1 - e^{-cn}$$

follows from Lemma 11, which we state below. Combining this with (48) proves that \mathcal{A} occurs with probability at least $1 - c/n$. \blacksquare

The following result is a slightly adapted version of Lemma 19 from [3] and the discussion that follows.

Lemma 11. *Suppose $\tilde{\boldsymbol{\varepsilon}} \in \mathbb{R}^n$ has independent entries with sub-Gaussian constants bounded by an absolute constant, and suppose $M \in \mathbb{R}^{n \times n}$ is a positive semidefinite matrix independent of $\tilde{\boldsymbol{\varepsilon}}$. Then, with probability at least $1 - e^{-cn}$,*

$$\tilde{\boldsymbol{\varepsilon}}^\top M \tilde{\boldsymbol{\varepsilon}} \lesssim \log(n) \cdot \text{tr}(M).$$

A.2.3 Proofs for Section 3.3

In this section we begin with the proof of our main result, Theorem 7, which relies on Proposition 6, proved subsequently.

Proof of Theorem 7

Step 1: Decomposing the risk

Using that Z , E and ε are independent of one another and of $\hat{\alpha}$, we have

$$\begin{aligned} R(\hat{\alpha}) &= \mathbb{E}[(X^\top \hat{\alpha} - y)^2] \\ &= \mathbb{E}[(Z^\top A^\top \hat{\alpha} - Z^\top \beta - \varepsilon + E^\top \hat{\alpha})^2] \\ &= \sigma_\varepsilon^2 + \|\Sigma_E^{1/2} \hat{\alpha}\|^2 + \|\Sigma_Z^{1/2} (A^\top \hat{\alpha} - \beta)\|^2. \end{aligned}$$

Since $\hat{\alpha} = \mathbf{X}^+ \mathbf{y} = \mathbf{X}^+ \mathbf{Z} \beta + \mathbf{X}^+ \varepsilon$,

$$\|\Sigma_E^{1/2} \hat{\alpha}\|^2 \leq 2\|\Sigma_E^{1/2} \mathbf{X}^+ \mathbf{Z} \beta\|^2 + 2\|\Sigma_E^{1/2} \mathbf{X}^+ \varepsilon\|^2 := 2B_1 + 2V_1.$$

Similarly,

$$\|\Sigma_Z^{1/2} (A^\top \hat{\alpha} - \beta)\|^2 \leq 2\|\Sigma_Z^{1/2} (A^\top \mathbf{X}^+ \mathbf{Z} - I_K) \beta\|^2 + 2\|\Sigma_Z^{1/2} A^\top \mathbf{X}^+ \varepsilon\|^2 := 2B_2 + 2V_2.$$

We thus have $R(\hat{\alpha}) - \sigma_\varepsilon^2 \lesssim B + V$, where we view $B := B_1 + B_2$ as a bound on the bias component of the risk and $V := V_1 + V_2$ as a bound on the variance component. In what follows, we bound the four terms

$$\begin{aligned} B_1 &= \|\Sigma_E^{1/2} \mathbf{X}^+ \mathbf{Z} \beta\|^2 \\ B_2 &= \|\Sigma_Z^{1/2} (A^\top \mathbf{X}^+ \mathbf{Z} - I_K) \beta\|^2 \\ V_1 &= \|\Sigma_E^{1/2} \mathbf{X}^+ \varepsilon\|^2 \\ V_2 &= \|\Sigma_Z^{1/2} A^\top \mathbf{X}^+ \varepsilon\|^2. \end{aligned}$$

Step 2: Bounding the risk

Recall that $\mathbf{Z} = \tilde{\mathbf{Z}} \Sigma_Z^{1/2}$ and $\varepsilon = \sigma_\varepsilon \varepsilon_w$ from Assumption 3. We will bound the bias and variance on the event $\mathcal{E} := \mathcal{E}_1 \cap \mathcal{E}_2$, where

$$\mathcal{E}_1 := \left\{ \sigma_n^2(\mathbf{X}) \geq c_1 \text{tr}(\Sigma_E), \|\mathbf{E}\|^2 \leq c_2 \text{tr}(\Sigma_E), c_3 n \leq \sigma_K^2(\tilde{\mathbf{Z}}) \leq \|\tilde{\mathbf{Z}}\|^2 \leq c_4 n \right\},$$

$$\mathcal{E}_2 := \left\{ \tilde{\varepsilon}^\top \mathbf{X}^{+\top} \Sigma_X \mathbf{X}^+ \tilde{\varepsilon} \lesssim \log(n) \text{tr}(\mathbf{X}^{+\top} \Sigma_X \mathbf{X}^+) \right\},$$

where c_1 to c_4 are absolute constants such that, as shown in the last step of the proof, \mathcal{E} occurs with probability at least $1 - c/n$ for some $c > 0$.

Bounding the bias component

On the event \mathcal{E} , $\sigma_n(\mathbf{X}) > 0$ and by Assumption 2 and (44) above, $\sigma_n^2(\mathbf{Z}) \gtrsim \lambda_K(\Sigma_Z)n > 0$. Thus \mathbf{X} and \mathbf{Z} are of rank n and K respectively, so by Lemma 21 of Appendix D, $\mathbf{X}\mathbf{X}^+ = I_n$ and $\mathbf{Z}^+\mathbf{Z} = I_K$. It follows that

$$\begin{aligned} \mathbf{Z}^+ - A^\top \mathbf{X}^+ &= \mathbf{Z}^+ \mathbf{X} \mathbf{X}^+ - A^\top \mathbf{X}^+ && (\text{since } \mathbf{X} \mathbf{X}^+ = I_n) \\ &= (\mathbf{Z}^+ \mathbf{X} - A^\top) \mathbf{X}^+ \\ &= (\mathbf{Z}^+ [\mathbf{Z} A^\top + \mathbf{E}] - A^\top) \mathbf{X}^+ && (\text{since } \mathbf{X} = \mathbf{Z} A^\top + \mathbf{E}) \\ &= \mathbf{Z}^+ \mathbf{E} \mathbf{X}^+, && (\text{since } \mathbf{Z}^+ \mathbf{Z} = I_K) \end{aligned} \quad (49)$$

and thus again using $\mathbf{Z}^+ \mathbf{Z} = I_K$

$$B_2 = \|\Sigma_Z^{1/2} (A^\top \mathbf{X}^+ \mathbf{Z} - I_K) \beta\|^2 = \|\Sigma_Z^{1/2} (A^\top \mathbf{X}^+ - \mathbf{Z}^+) \mathbf{Z} \beta\|^2 = \|\Sigma_Z^{1/2} \mathbf{Z}^+ \mathbf{E} \mathbf{X}^+ \mathbf{Z} \beta\|^2.$$

By (46) above and the fact that \mathbf{Z} is full rank on \mathcal{E} , $\Sigma_Z^{1/2} \mathbf{Z}^+ = \tilde{\mathbf{Z}}^+$, so on \mathcal{E} ,

$$B_2 = \|\tilde{\mathbf{Z}}^+ \mathbf{E} \mathbf{X}^+ \mathbf{Z} \beta\|^2 \leq \frac{\|\mathbf{E}\|^2}{\sigma_K^2(\tilde{\mathbf{Z}})} \|\mathbf{X}^+ \mathbf{Z} \beta\|^2 \lesssim \frac{\text{tr}(\Sigma_E) \|\mathbf{X}^+ \mathbf{Z} \beta\|^2}{n},$$

where we also used that $\|\tilde{\mathbf{Z}}^+\|^2 = 1/\sigma_K^2(\tilde{\mathbf{Z}})$. Since $B_1 = \|\Sigma_E^{1/2} \mathbf{X}^+ \mathbf{Z} \beta\|^2 \leq \|\Sigma_E\| \|\mathbf{X}^+ \mathbf{Z} \beta\|^2$, and

$$\|\Sigma_E\| = \text{tr}(\Sigma_E) \frac{\|\Sigma_E\|}{\text{tr}(\Sigma_E)} = \frac{\text{tr}(\Sigma_E)}{n} \cdot \frac{n}{\text{r}_e(\Sigma_E)} \lesssim \frac{\text{tr}(\Sigma_E)}{n},$$

where we used the assumption $\text{r}_e(\Sigma_E) > c_1 n$ in the last step, we also have that on \mathcal{E} ,

$$B = B_1 + B_2 \lesssim \frac{\text{tr}(\Sigma_E) \|\mathbf{X}^+ \mathbf{Z} \beta\|^2}{n}. \quad (50)$$

To bound $\|\mathbf{X}^+ \mathbf{Z} \beta\|^2$, we first use $A^\top A^+ = I_K$ and $\mathbf{Z} A^\top = \mathbf{X} - \mathbf{E}$ to find

$$\|\mathbf{X}^+ \mathbf{Z} \beta\|^2 = \|\mathbf{X}^+ \mathbf{Z} A^\top A^+ \beta\|^2 \leq 2 \|\mathbf{X}^+ \mathbf{X} A^+ \beta\|^2 + 2 \|\mathbf{X}^+ \mathbf{E} A^+ \beta\|^2.$$

The second term can be bounded, on the event \mathcal{E} , by

$$\frac{\|\mathbf{E}\|^2 \|A^+ \beta\|^2}{\sigma_n^2(\mathbf{X})} \lesssim \|A^+ \beta\|^2.$$

On the other hand, the first term can be bounded as $\|\mathbf{X}^+ \mathbf{X} A^+ \beta\|^2 \leq \|A^+ \beta\|^2$ using the fact that $\mathbf{X}^+ \mathbf{X}$ is a projection matrix, so we find that on \mathcal{E} ,

$$\|\mathbf{X}^+ \mathbf{Z} \beta\|^2 \lesssim \|A^+ \beta\|^2. \quad (51)$$

Finally, we have

$$\|A^+ \beta\|^2 = \beta^\top (A^\top A)^{-1} \beta = \beta^\top \Sigma_Z^{1/2} (\Sigma_Z^{1/2} A^\top A \Sigma_Z^{1/2})^{-1} \Sigma_Z^{1/2} \beta \leq \frac{\|\beta\|_{\Sigma_Z}^2}{\lambda_K(A \Sigma_Z A^\top)}. \quad (52)$$

Combining this with (51) and plugging into (50), we find that on the event \mathcal{E} ,

$$B \lesssim \frac{\|\beta\|_{\Sigma_Z}^2}{\lambda_K(A \Sigma_Z A^\top)} \frac{\text{tr}(\Sigma_E)}{n} = \frac{\|\beta\|_{\Sigma_Z}^2 \|\Sigma_E\|}{\lambda_K(A \Sigma_Z A^\top)} \cdot \frac{\text{tr}(\Sigma_E)}{\|\Sigma_E\| n} = \frac{\|\beta\|_{\Sigma_Z}^2}{\xi} \frac{\text{r}_e(\Sigma_E)}{n}. \quad (53)$$

Bounding the variance component

First note that

$$V = V_1 + V_2 = \|\Sigma_E^{1/2} \mathbf{X}^+ \boldsymbol{\varepsilon}\|^2 + \|\Sigma_Z^{1/2} A^\top \mathbf{X}^+ \boldsymbol{\varepsilon}\|^2 = \boldsymbol{\varepsilon}^\top \mathbf{X}^{+\top} \Sigma_X \mathbf{X}^+ \boldsymbol{\varepsilon} = \sigma_\varepsilon^2 \tilde{\boldsymbol{\varepsilon}} \mathbf{X}^{+\top} \Sigma_X \mathbf{X}^+ \tilde{\boldsymbol{\varepsilon}},$$

so on the event \mathcal{E} ,

$$V \lesssim \sigma_\varepsilon^2 \log(n) \text{tr}(\mathbf{X}^{+\top} \Sigma_X \mathbf{X}^+) = \sigma_\varepsilon^2 \log(n) \left\{ \text{tr}(\mathbf{X}^{+\top} \Sigma_E \mathbf{X}^+) + \text{tr}(\mathbf{X}^{+\top} A \Sigma_Z A^\top \mathbf{X}^+) \right\}, \quad (54)$$

where we use $\Sigma_X = A \Sigma_Z A^\top + \Sigma_E$ in the second step. The first term in (54) can be bounded as

$$\text{tr}(\mathbf{X}^{+\top} \Sigma_E \mathbf{X}^+) \leq \|\Sigma_E\| \cdot n \|\mathbf{X}^{+\top} \mathbf{X}^+\| = \|\Sigma_E\| \frac{n}{\sigma_n^2(\mathbf{X})} \lesssim \frac{n}{r_e(\Sigma_E)}, \quad (55)$$

where in the first step we used that $\text{rank}(\mathbf{X}^+) = \text{rank}(\mathbf{X}) = n$ and in the last step that $\sigma_n^2(\mathbf{X}) \gtrsim \text{tr}(\Sigma_E)$ on \mathcal{E} .

For the second term in (54),

$$\begin{aligned} \text{tr}(\mathbf{X}^{+\top} A \Sigma_Z A^\top \mathbf{X}^+) &\leq K \|\Sigma_Z^{1/2} A^\top \mathbf{X}^+\|^2 && \text{(since } \text{rank}(A \Sigma_Z A^\top) = K) \\ &= K \|\Sigma_Z^{1/2} (\mathbf{Z}^+ - \mathbf{Z}^+ \mathbf{E} \mathbf{X}^+)\|^2 && \text{(by (49) above)} \\ &\leq 2K \|\tilde{\mathbf{Z}}^+\|^2 + 2K \|\tilde{\mathbf{Z}}^+\|^2 \|\mathbf{E}\|^2 \|\mathbf{X}^+\|^2, \end{aligned}$$

where we use that $\Sigma_Z^{1/2} \mathbf{Z}^+ = \tilde{\mathbf{Z}}^+$ from (46) in the final step. Continuing, we find

$$\text{tr}(\mathbf{X}^{+\top} A \Sigma_Z A^\top \mathbf{X}^+) \lesssim \frac{K}{\sigma_K^2(\tilde{\mathbf{Z}})} \left(1 + \frac{\|\mathbf{E}\|^2}{\sigma_n^2(\mathbf{X})} \right) \lesssim \frac{K}{n}, \quad (56)$$

where we use the bounds defining \mathcal{E}_1 in the last inequality. Combining (56) and (55) with (54), we conclude that on \mathcal{E} ,

$$V \lesssim \sigma_\varepsilon^2 \frac{n \log n}{r_e(\Sigma_E)} + \sigma_\varepsilon^2 \frac{K \log n}{n}.$$

Combining this with the bias bound (53) gives the bound in the statement of the theorem. It remains to control $\mathbb{P}(\mathcal{E})$.

Step 3: Bounding $\mathbb{P}(\mathcal{E})$

We have $\mathbb{P}(\mathcal{E}^c) \leq \mathbb{P}(\mathcal{E}_1^c) + \mathbb{P}(\mathcal{E}_2^c)$. The bound $\mathbb{P}(\mathcal{E}_2^c) \leq e^{-cn}$ follows immediately from Lemma 11 in Appendix A.2.2 above, using the fact that $\tilde{\boldsymbol{\varepsilon}}$ has independent entries with sub-Gaussian constants bounded by an absolute constant. Considering $\mathbb{P}(\mathcal{E}_1^c)$, we have

$$\mathbb{P}(\mathcal{E}_1^c) \leq \mathbb{P}\{\sigma_n^2(\mathbf{X}) \leq c_1 \text{tr}(\Sigma_E)\} + \mathbb{P}\{\|\mathbf{E}\|^2 \geq c_2 \text{tr}(\Sigma_E)\} + \mathbb{P}\{c_3 n \leq \sigma_K^2(\tilde{\mathbf{Z}}) \leq \|\tilde{\mathbf{Z}}\|^2 \leq c_4 n\}$$

The three terms above can be bounded as follows. Recall that we assume $n > CK$ and $r_e(\Sigma_E) > Cn$ for some $C > 1$ large enough.

1. Since $r_e(\Sigma_E) > Cn$, Proposition 6 can be applied to conclude

$$\mathbb{P}\{\sigma_n^2(\mathbf{X}) \leq c_1 \text{tr}(\Sigma_E)\} \leq 2e^{-cn}.$$

2. By Assumption 3, $\mathbf{E} = \tilde{\mathbf{E}}\Sigma_E^{1/2}$, where $\tilde{\mathbf{E}}$ has independent entries with zero mean, unit variance, and sub-Gaussian constants bounded by an absolute constant. Thus,

$$\|\mathbf{E}\|^2 = \|\mathbf{E}\mathbf{E}^\top\| = \|\tilde{\mathbf{E}}\Sigma_E\tilde{\mathbf{E}}^\top\|,$$

and by applying Theorem 10 with $\tilde{\mathbf{E}}$ and Σ_E we find that with probability at least $1 - 2e^{-cn}$,

$$\|\mathbf{E}\|^2 \leq \text{tr}(\Sigma_E) + c'\|\Sigma_E\|n = \text{tr}(\Sigma_E) \cdot (1 + c'n/\text{r}_e(\Sigma_E)) \lesssim \text{tr}(\Sigma_E),$$

where the last inequality holds since $n/\text{r}_e(\Sigma_E) < 1/C$. Thus for $c_2 > 0$,

$$\mathbb{P}\{\|\mathbf{E}\|^2 \geq c_2\text{tr}(\Sigma_E)\} \leq 2e^{-cn}.$$

3. By (48) we have that with probability at least $1 - 2/n$,

$$c_3n \leq \sigma_K^2(\tilde{\mathbf{Z}}) \leq \|\tilde{\mathbf{Z}}\|^2 \leq c_4n.$$

Combining the previous three steps shows that $\mathbb{P}(\mathcal{E}_1^c) \leq c/n$. ■

Proof of Proposition 6

We will work on the event

$$\mathcal{F} := \{\sigma_n^2(\mathbf{E}U_{(K+1):p}) \geq c_4\text{tr}(\Sigma_E), \|\tilde{\mathbf{Z}}\|^2 \leq c_5n\},$$

where $U_{(K+1):p} \in \mathbb{R}^{p \times (p-K)}$ has columns equal to the orthonormal eigenvectors of Σ_X corresponding to the smallest $p - K$ eigenvalues.

Bounding $\mathbb{P}(\mathcal{F})$

By Assumption 3, $\mathbf{E} = \tilde{\mathbf{E}}\Sigma_E^{1/2}$, where $\tilde{\mathbf{E}}$ has independent sub-Gaussian entries with zero mean, unit variance, sub-Gaussian constants bounded by an absolute constant. Thus, letting

$$Q = U_{(K+1):p}U'_{(K+1):p},$$

we have

$$\sigma_n^2(\mathbf{E}U_{(K+1):p}) = \lambda_n(\mathbf{E}Q\mathbf{E}^\top) = \lambda_n(\tilde{\mathbf{E}}\Sigma_E^{1/2}Q\Sigma_E^{1/2}\tilde{\mathbf{E}}^\top).$$

We can now apply Theorem 10, stated and proved above in Section A.1, with $\tilde{\mathbf{E}}$ and $\Sigma_E^{1/2}Q\Sigma_E^{1/2}$. Noting that $M = \max_{ij} \|\tilde{\mathbf{E}}\|_{\psi_2}$ is bounded by an absolute constant by Assumption 3, this implies that with probability at least $1 - 2e^{-cn}$,

$$\sigma_n^2(\mathbf{E}U_{(K+1):p}) \geq \text{tr}(\Sigma_E^{1/2}Q\Sigma_E^{1/2})/2 - c'\|\Sigma_E^{1/2}Q\Sigma_E^{1/2}\|n. \quad (57)$$

Since Q is a projection matrix, $\|\Sigma_E^{1/2}Q\Sigma_E^{1/2}\| \leq \|\Sigma_E\|\|Q\| = \|\Sigma_E\|$. Furthermore,

$$\begin{aligned} \text{tr}(\Sigma_E^{1/2}Q\Sigma_E^{1/2}) &= \text{tr}(\Sigma_E Q) \\ &= \text{tr}(\Sigma_E) - \text{tr}(\Sigma_E(I - Q)) \\ &\geq \text{tr}(\Sigma_E) - K\|\Sigma_E(I - Q)\| \end{aligned} \quad (\text{since } \text{rank}(I - Q) = K)$$

$$\begin{aligned}
&\geq \text{tr}(\Sigma_E) - K\|\Sigma_E\|\|I - Q\| \\
&= \text{tr}(\Sigma_E) - K\|\Sigma_E\| && (\text{since } \|I - Q\| = 1) \\
&\geq \text{tr}(\Sigma_E) - n\|\Sigma_E\|. && (\text{since } n \geq K)
\end{aligned}$$

Plugging these two results into (57), we find that with probability at least $1 - 2e^{-cn}$,

$$\sigma_n^2(\mathbf{E}U_{(K+1):p}) \geq \text{tr}(\Sigma_E)/2 - (1/2 + c')n\|\Sigma_E\| = \text{tr}(\Sigma_E) \cdot [1/2 - (1/2 + c')n/r_e(\Sigma_E)] \gtrsim \text{tr}(\Sigma_E), \quad (58)$$

where in the last inequality we use that $n/r_e(\Sigma_E) < 1/C$ and choose C large enough.

Also, since $\tilde{\mathbf{Z}}$ has independent rows with entries that have zero mean, unit variance, and sub-Gaussian constants bounded by an absolute constant, we have that by Theorem 4.6.1 of [41],

$$\|\tilde{\mathbf{Z}}\|^2 \leq c_2 n,$$

with probability at least $1 - e^{-c'n}$. Combining this with 58 we conclude that

$$\mathbb{P}(\mathcal{F}) \geq 1 - ce^{-c'n}.$$

Bounding $\sigma_n(\mathbf{X})$ on \mathcal{F}

We now show that $\sigma_n^2(\mathbf{X}) \gtrsim \text{tr}(\Sigma_E)$ holds on the event \mathcal{F} . Let $\Sigma_X = UDU^\top$ with $U \in \mathbb{R}^{p \times p}$ orthogonal and $D = \text{diag}(\lambda_1(\Sigma_X), \dots, \lambda_p(\Sigma_X))$. Define $U_K \in \mathbb{R}^{p \times K}$ to be the sub-matrix of U containing the first K columns, and define $U_{(K+1):p}$ to be composed of the last $p - K$ columns of U . Then

$$I_p = UU^\top = U_K U_K^\top + U_{(K+1):p} U_{(K+1):p}^\top,$$

so

$$\lambda_n(\mathbf{X}\mathbf{X}^\top) = \lambda_n(\mathbf{X}U_K U_K^\top \mathbf{X}^\top + \mathbf{X}U_{(K+1):p} U_{(K+1):p}^\top \mathbf{X}^\top) \geq \lambda_n(\mathbf{X}U_{(K+1):p} U_{(K+1):p}^\top \mathbf{X}^\top),$$

where we use the min-max formula for eigenvalues in the last step. This implies

$$\sigma_n(\mathbf{X}) \geq \sigma_n(\mathbf{X}U_{(K+1):p}). \quad (59)$$

By Weyl's inequality for singular values, and using $\mathbf{X} = \mathbf{Z}A^\top + \mathbf{E}$,

$$|\sigma_n(\mathbf{X}U_{(K+1):p}) - \sigma_n(\mathbf{E}U_{(K+1):p})| \leq \|\mathbf{Z}A^\top U_{(K+1):p}\|,$$

so by (59),

$$\sigma_n(\mathbf{X}) \geq \sigma_n(\mathbf{X}U_{(K+1):p}) \geq \sigma_n(\mathbf{E}U_{(K+1):p}) - \|\mathbf{Z}A^\top U_{(K+1):p}\| \gtrsim \sqrt{\text{tr}(\Sigma_E)} - \|\mathbf{Z}A^\top U_{(K+1):p}\|, \quad (60)$$

where the last inequality holds on the event \mathcal{F} . We show below that $\|\mathbf{Z}A^\top U_{(K+1):p}\| \lesssim \sqrt{n\|\Sigma_E\|}$ on \mathcal{F} , which implies that

$$\sigma_n(\mathbf{X}) \gtrsim \sqrt{\text{tr}(\Sigma_E)} - c\sqrt{n\|\Sigma_E\|} = \sqrt{\text{tr}(\Sigma_E)} \cdot (1 - c\sqrt{n/r_e(\Sigma_E)}) \gtrsim \sqrt{\text{tr}(\Sigma_E)},$$

where in the last inequality we use that $n/r_e(\Sigma_E) < 1/C$ and choose C large enough.

Upper bound of $\|\mathbf{Z}A^\top U_{(K+1):p}\|$

On the event \mathcal{F} ,

$$\|\mathbf{Z}A^\top U_{(K+1):p}\|^2 = \|\tilde{\mathbf{Z}}\Sigma_Z^{1/2}A^\top U_{(K+1):p}\| \leq \|\tilde{\mathbf{Z}}\|^2 \|\Sigma_Z^{1/2}A^\top U_{(K+1):p}\|^2 \lesssim n \|\Sigma_Z^{1/2}A^\top U_{(K+1):p}\|^2. \quad (61)$$

Furthermore, using $\Sigma_X = A\Sigma_ZA^\top + \Sigma_E$, and that $U_{(K+1):p}^\top \Sigma_X U_{(K+1):p} = D_{(K+1):p}$ where we define $D_{(K+1):p} := \text{diag}(\lambda_{K+1}(\Sigma_X), \dots, \lambda_p(\Sigma_X))$,

$$\begin{aligned} \|\Sigma_Z^{1/2}A^\top U_{(K+1):p}\|^2 &= \|U_{(K+1):p}^\top A\Sigma_ZA^\top U_{(K+1):p}\| \\ &= \|U_{(K+1):p}^\top \Sigma_X U_{(K+1):p} - U_{(K+1):p}^\top \Sigma_E U_{(K+1):p}\| \\ &= \|D_{(K+1):p} - U_{(K+1):p}^\top \Sigma_E U_{(K+1):p}\| \\ &\leq \lambda_{K+1}(\Sigma_X) + \|U_{(K+1):p}^\top \Sigma_E U_{(K+1):p}\| \\ &\leq \lambda_{K+1}(\Sigma_X) + \|\Sigma_E\| \|U_{(K+1):p}^\top U_{(K+1):p}\| \\ &= \lambda_{K+1}(\Sigma_X) + \|\Sigma_E\|, \end{aligned}$$

where we use $U_{(K+1):p}^\top U_{(K+1):p} = I_{p-K}$ in the last step. Thus, using that

$$\lambda_{K+1}(\Sigma_X) = \lambda_{K+1}(\Sigma_X) - \lambda_{K+1}(A\Sigma_ZA^\top) \leq \|\Sigma_E\|$$

by Weyl's inequality and the fact that $\lambda_{K+1}(A\Sigma_ZA^\top) = 0$, we find

$$\|\Sigma_Z^{1/2}A^\top U_{(K+1):p}\|^2 \leq 2\|\Sigma_E\|.$$

Combining this with (61), we find that on \mathcal{F} ,

$$\|\mathbf{Z}A^\top U_{(K+1):p}\| \lesssim \sqrt{n\|\Sigma_E\|}.$$

■

A.2.4 Proof of Theorem 9 from Section 3.4

Define $D_K = U_K^\top \Sigma_X U_K$ and note that since U_K and Σ_X are full rank, so is D_K , and thus D_K is invertible. Furthermore, define $\eta = y - X^\top \alpha^*$ with variance $\sigma_\eta^2 = \mathbb{E}[\eta^2]$, and the sample version $\boldsymbol{\eta} = \mathbf{y} - \mathbf{X}\alpha^*$. We work on the event $\mathcal{D} := \mathcal{D}_1 \cap \mathcal{D}_2$, where

$$\mathcal{D}_1 := \left\{ \sigma_K^2(\mathbf{X}U_K D_K^{-1/2}) \gtrsim n, \|\mathbf{X}\Sigma_X^{-1/2}\|^2 \lesssim p \right\},$$

and

$$\mathcal{D}_2 := \left\{ \|(\mathbf{X}U_K D_K^{-1/2})^+ \boldsymbol{\eta}\|^2 \lesssim \log(n) \cdot \sigma_\eta^2 \cdot \text{tr}[(\mathbf{X}U_K D_K^{-1/2})^{+\top} (\mathbf{X}U_K D_K^{-1/2})^+] \right\}.$$

As the last step of this proof, we will show that $\mathbb{P}(\mathcal{D}) \geq 1 - c'/n$.

Letting $\eta := y - X^\top \alpha^*$, we have

$$\mathbb{E}[X\eta] = \mathbb{E}[Xy] - \mathbb{E}[XX^\top]\alpha^* = \Sigma_{Xy} - \Sigma_X \Sigma_X^+ \Sigma_{Xy} = 0, \quad (62)$$

where we used (36) in the last step. Thus,

$$\begin{aligned}
R_{\text{PCR}}(\hat{\beta}) &:= \mathbb{E}[(X^\top U_K \hat{\beta} - y)^2] \\
&= \mathbb{E}[(X^\top U_K \hat{\beta} - X^\top \alpha^* - \eta)^2] \\
&= \mathbb{E}[(X^\top U_K \hat{\beta} - X^\top \alpha^*)^2] + \mathbb{E}[\eta^2] \quad (\text{by 62}) \\
&= \|U_K \hat{\beta} - \alpha^*\|_{\Sigma_X}^2 + R(\alpha^*).
\end{aligned} \tag{63}$$

We then define the projection matrix $P = U_K U_K^\top$, and write

$$\mathbf{y} = \mathbf{X}\alpha^* + \boldsymbol{\eta} = \mathbf{X}P\alpha^* + \mathbf{X}(I_p - P)\alpha^* + \boldsymbol{\eta}.$$

Using $\hat{\beta} = (\mathbf{X}U_K)^+ \mathbf{y}$ we thus find

$$\begin{aligned}
U_K \hat{\beta} &= U_K (\mathbf{X}U_K)^+ \mathbf{y} \\
&= U_K (\mathbf{X}U_K)^+ \mathbf{X}P\alpha^* + U_K (\mathbf{X}U_K)^+ \mathbf{X}(I_p - P)\alpha^* + U_K (\mathbf{X}U_K)^+ \boldsymbol{\eta}.
\end{aligned}$$

From the fact that $\mathbf{X}U_K$ is an $n \times K$ matrix with $K < n$ and $\text{rank}(\mathbf{X}U_K) = K$ on the event \mathcal{D}_1 , we have $(\mathbf{X}U_K)^+ \mathbf{X}U_K = I_K$ by Lemma 21 below. Thus, using $P = U_K U_K^\top$ we have $(\mathbf{X}U_K)^+ \mathbf{X}P = U_K^\top$. Applying this in the previous display, we find

$$U_K \hat{\beta} = P\alpha^* + U_K (\mathbf{X}U_K)^+ \mathbf{X}(I_p - P)\alpha^* + U_K (\mathbf{X}U_K)^+ \boldsymbol{\eta}.$$

It thus follows from the decomposition (63) that

$$\begin{aligned}
R_{\text{PCR}}(\hat{\beta}) - R(\alpha^*) &= \|U_K \hat{\beta} - \alpha^*\|_{\Sigma_X}^2 \\
&\lesssim \|(I_p - P)\alpha^*\|_{\Sigma_X}^2 + \|U_K (\mathbf{X}U_K)^+ \mathbf{X}(I_p - P)\alpha^*\|_{\Sigma_X}^2 + \|U_K (\mathbf{X}U_K)^+ \boldsymbol{\eta}\|_{\Sigma_X}^2 \\
&=: B_1 + B_2 + V.
\end{aligned} \tag{64}$$

Bounding B_1

We find

$$B_1 = \|\Sigma_X^{1/2}(I_p - P)\alpha^*\|^2 \leq \|\Sigma_X^{1/2}(I_p - P)\|^2 \|\alpha^*\|^2 = \|(I - P)\Sigma_X(I - P)\| \|\alpha^*\|^2. \tag{65}$$

Since $I - P$ is a projection onto the span of the last $p - K$ eigenvectors of Σ_X with eigenvalues $\lambda_{K+1}(\Sigma_X), \dots, \lambda_p(\Sigma_X)$, we have $\|(I - P)\Sigma_X(I - P)\| = \lambda_{K+1}(\Sigma_X)$. By Weyl's inequality,

$$\lambda_{K+1}(\Sigma_X) = \lambda_{K+1}(\Sigma_X) - \lambda_{K+1}(A\Sigma_Z A^\top) \leq \|\Sigma_E\|,$$

where we used that $\lambda_{K+1}(A\Sigma_Z A^\top) = 0$ in the first step since $A\Sigma_Z A^\top$ is rank K . Thus

$$\|\Sigma_X^{1/2}(I_p - P)\|^2 \leq \|\Sigma_E\|,$$

and combining this with (65) we find

$$B_1 \leq \|\Sigma_E\| \|\alpha^*\|^2. \tag{66}$$

Using that $\Sigma_X = A\Sigma_Z A^\top + \Sigma_E$, Lemma 13 of Appendix A.3 can be applied to find

$$\|\alpha^*\|^2 \leq \kappa(\Sigma_E) \|\beta\|_{\Sigma_Z}^2 / \lambda_K(A\Sigma_Z A^\top),$$

so

$$B_1 \leq \kappa(\Sigma_E) \frac{\|\Sigma_E\|}{\lambda_K(A\Sigma_Z A^\top)} \cdot \|\beta\|_{\Sigma_Z}^2 = \kappa(\Sigma_E) \frac{\|\beta\|_{\Sigma_Z}^2}{\xi}. \tag{67}$$

Bounding B_2

Recalling $D_K = U_K^\top \Sigma_X U_K$,

$$\begin{aligned} B_2 &= \alpha^{*\top} (I_p - P) \mathbf{X}^\top (\mathbf{X} U_K)^{+\top} U_K^\top \Sigma_X U_K (\mathbf{X} U_K)^+ \mathbf{X} (I - P) \alpha^* \\ &= \|D_K^{1/2} (\mathbf{X} U_K)^+ \mathbf{X} (I_p - P) \alpha^*\|^2. \end{aligned} \quad (68)$$

Observe that by Lemma 21 of Appendix D,

$$(\mathbf{X} U_K D_K^{-1/2})^+ = [(\mathbf{X} U_K)^+ (\mathbf{X} U_K) D_K^{-1/2}]^+ \cdot [\mathbf{X} U_K D_K^{-1/2} D_K^{1/2}]^+ = D_K^{1/2} (\mathbf{X} U_K)^+, \quad (69)$$

where we used that $\mathbf{X} U_K$ is a full rank $n \times K$ matrix with $K < n$ so $(\mathbf{X} U_K)^+ (\mathbf{X} U_K) = I_K$. Using this in (68) yields

$$\begin{aligned} B_2 &= \|(\mathbf{X} U_K D_K^{-1/2})^+ \mathbf{X} (I_p - P) \alpha^*\|^2 \\ &\leq \frac{\|\mathbf{X} (I_p - P) \alpha^*\|^2}{\sigma_K^2 (\mathbf{X} U_K D_K^{-1/2})} \\ &\leq \frac{\|\mathbf{X} \Sigma_X^{-1/2}\|^2}{\sigma_K^2 (\mathbf{X} U_K D_K^{-1/2})} \cdot \|\Sigma_X^{1/2} (I_p - P) \alpha^*\|^2 \\ &\lesssim \frac{p}{n} \|\Sigma_X^{1/2} (I_p - P) \alpha^*\|^2, \end{aligned}$$

where the last step holds on \mathcal{D} . Recalling that $\|\Sigma_X^{1/2} (I_p - P) \alpha^*\|^2 = B_1$ and using (67), we find that

$$B_2 \lesssim \kappa(\Sigma_E) \frac{\|\beta\|_{\Sigma_Z}^2}{\xi} \cdot \frac{p}{n}. \quad (70)$$

Bounding V

We have on \mathcal{D} ,

$$\begin{aligned} V &= \boldsymbol{\eta}^\top (\mathbf{X} U_K)^{+\top} U_K^\top \Sigma_X U_K (\mathbf{X} U_K)^+ \boldsymbol{\eta} \\ &= \boldsymbol{\eta}^\top (\mathbf{X} U_K)^{+\top} D_K (\mathbf{X} U_K)^+ \boldsymbol{\eta} \\ &= \|D_K^{1/2} (\mathbf{X} U_K)^+ \boldsymbol{\eta}\|^2 \\ &= \|(\mathbf{X} U_K D_K^{-1/2})^+ \boldsymbol{\eta}\|^2 && \text{(by (69))} \\ &\lesssim \sigma_\eta^2 \cdot \log(n) \cdot \text{tr}[(\mathbf{X} U_K D^{-1/2})^{+\top} (\mathbf{X} U_K D^{-1/2})^+] && \text{(on } \mathcal{D}_2) \\ &\leq \sigma_\eta^2 \cdot \log(n) \cdot K \cdot \|(\mathbf{X} U_K D^{-1/2})^+\|^2 && \text{(since } \text{rank}(\mathbf{X} U_K D^{-1/2}) = K) \\ &= \sigma_\eta^2 \cdot \frac{K \log n}{\sigma_K^2 (\mathbf{X} U_K D^{-1/2})} \\ &\lesssim \sigma_\eta^2 \cdot \frac{K \log n}{n}. && \text{(on } \mathcal{D}_1). \end{aligned}$$

Recalling $\eta = y - X^\top \alpha^*$ so $\sigma_\eta^2 = R(\alpha^*)$, we use from Lemma 4 that

$$\sigma_\eta^2 \leq \sigma_\varepsilon^2 + \frac{\|\beta\|_{\Sigma_Z}^2}{\xi},$$

so

$$V \lesssim \frac{\|\beta\|_{\Sigma_Z}^2}{\xi} \frac{K \log n}{n} + \sigma_\varepsilon^2 \frac{K \log n}{n}.$$

Combining this with (67) and (70) gives

$$\begin{aligned} R_{\text{PCR}}(\hat{\beta}) - R(\alpha^*) &\lesssim \kappa(\Sigma_E) \frac{\|\beta\|_{\Sigma_Z}^2}{\xi} \cdot \frac{p}{n} + \frac{\|\beta\|_{\Sigma_Z}^2}{\xi} \frac{K \log n}{n} + \sigma_\varepsilon^2 \frac{K \log n}{n} \\ &\lesssim \kappa(\Sigma_E) \frac{\|\beta\|_{\Sigma_Z}^2}{\xi} \cdot \frac{p}{n} + \sigma_\varepsilon^2 \frac{K \log n}{n}, \end{aligned}$$

where we use that

$$K \log n < c \cdot n \lesssim p \leq \kappa(\Sigma_E)p$$

in the last step. This gives the bound as in the theorem statement.

Bounding $\mathbb{P}(\mathcal{D})$

We first bound the probability $\mathbb{P}(\mathcal{D}_1)$. Note that the matrix $\mathbf{X}U_K D_K^{-1/2}$ has independent Gaussian rows $D_K^{-1/2} U_K^\top X_i$, with covariance

$$\mathbb{E}[D_K^{-1/2} U_K^\top X_i X_i^\top U_K D_K^{-1/2}] = D_K^{-1/2} U_K^\top U_K D_K^{-1/2} = D_K^{-1/2} D_K D_K^{-1/2} = I_K,$$

and so $\mathbf{X}U_K D_K^{-1/2}$ i.i.d. $N(0, 1)$ entries. Thus, by Theorem 4.6.1 of [41], with probability at least $1 - 2/n$,

$$\sigma_K(\mathbf{X}U_K D_K^{-1/2}) \geq \sqrt{n} - c(\sqrt{K} + \sqrt{\log n}) = \sqrt{n} \cdot [1 - c\sqrt{K/n} - c\sqrt{\log(n)/n}] \gtrsim \sqrt{n}, \quad (71)$$

where in the last step we use the assumption that $n > CK > C$ and choose C large enough.

Similarly, $\mathbf{X}\Sigma_X^{-1/2}$ is a $n \times p$ matrix with i.i.d. $N(0, 1)$ entries, so again by Theorem 4.6.1 of [41], with probability at least $1 - 2e^{-n}$,

$$\|\mathbf{X}\Sigma_X^{-1/2}\| \leq \sqrt{n} + c(\sqrt{p} + \sqrt{n}) \lesssim \sqrt{p}. \quad (72)$$

Using a union bound to combine this with (71), we find

$$\mathbb{P}(\mathcal{D}_1) \geq 1 - c'/n,$$

for some $c' > 0$.

To bound $\mathbb{P}(\mathcal{D}_2)$, first note that by (62) and the assumption that (X, y) are Gaussian, \mathbf{X} and $\boldsymbol{\eta}$ are independent. Furthermore, $\tilde{\boldsymbol{\eta}} = \boldsymbol{\eta}/\sigma_\eta$ has independent $N(0, 1)$ entries. We can thus apply Lemma 11 from Appendix A.2.2 above with

$$M = (\mathbf{X}U_K D_K^{-1/2})^{+\top} (\mathbf{X}U_K D_K^{-1/2})^+$$

to conclude that with probability at least $1 - e^{-cn}$,

$$\|(\mathbf{X}U_K D_K^{-1/2})^+ \boldsymbol{\eta}\|^2 = \boldsymbol{\eta}^\top M \boldsymbol{\eta} = \sigma_\eta^2 \tilde{\boldsymbol{\eta}}^\top M \tilde{\boldsymbol{\eta}} \lesssim \sigma_\eta^2 \cdot \log(n) \cdot \text{tr}(M),$$

and so $\mathbb{P}(\mathcal{D}_2^c) \leq e^{-cn}$. ■

A.3 Detailed comparison of the bias and variance terms in Section 3.5

In this sections we give a detailed comparison between our Theorem 7 and Theorem 2 in [3]. We assume throughout this section that the matrices Σ_X and Σ_E are invertible and the condition number $\kappa(\Sigma_E)$ of the matrix Σ_E is bounded above by an absolute constant c_1 .

First define the effective ranks

$$r_k(\Sigma_X) := \frac{\sum_{i>k} \lambda_i(\Sigma_X)}{\lambda_{i+1}}, \quad R_k(\Sigma_X) := \frac{(\sum_{i>k} \lambda_i(\Sigma_X))^2}{\sum_{i>k} \lambda_i^2(\Sigma_X)}.$$

The bound of Bartlett et. al. is stated to hold for probability at least $1 - \delta$ for a general $\delta < 1$ such that $\log(1/\delta) > n/c$ for an absolute constant $c > 1$. Taking $\delta = e^{-c'n}$ (for an appropriate c') to ease comparison with our results, the bound then states that with when model 8 holds, (X, y) are jointly Gaussian, $\text{rank}(\Sigma_X) \geq n$, and n is large enough, with probability at least $1 - e^{-c'n}$,

$$R(\hat{\alpha}) - R(\hat{\alpha}^*) \lesssim B + V,$$

where

$$B := \|\alpha^*\|^2 \|\Sigma_X\| \max \left\{ \sqrt{\frac{r_0(\Sigma_X)}{n}}, \frac{r_0(\Sigma_X)}{n}, 1 \right\}, \quad (73)$$

and

$$V := \sigma_\varepsilon^2 \log(n) \left(\frac{n}{R_{K^*}(\Sigma_X)} + \frac{K^*}{n} \right) \quad (74)$$

are bounds on the bias and variance respectively, and

$$K^* = \min\{k \geq 0 : r_k(\Sigma_X)/n \geq b\}, \quad (75)$$

where $b > 1$ is an absolute constant.

We now compare these two terms to the corresponding terms in our bound in Theorem 7.

A.3.1 Comparison of variance terms

We first compare the variance term V to corresponding variance term in our Theorem 7, display (17). Note that as long as the SNR

$$\xi := \lambda_K(A\Sigma_Z A^\top) / \|\Sigma_E\|$$

grows fast enough, $K^* = K$ for large enough n , where K is the dimension of the latent variables $Z \in \mathbb{R}^K$ in the factor regression model.

Lemma 12. *If $K/n = o(1)$, $r_e(\Sigma_E)/n \rightarrow \infty$, and $\xi \rightarrow \infty$, such that $\xi^{-1}r_e(\Sigma_E)/n = o(1)$, then $K^* = K$ for all n large enough.*

Thus, under the conditions stated in Lemma 12 and for n large enough,

$$V := \sigma_\varepsilon^2 \log(n) \left(\frac{n}{R_K(\Sigma_X)} + \frac{K}{n} \right).$$

We can bound $R_K(\Sigma_X)$ above via

$$R_K(\Sigma_X) = \frac{(\sum_{i=K+1}^p \lambda_i(\Sigma_X))^2}{\sum_{i=K+1}^p \lambda_i^2(\Sigma_X)} \leq \frac{(p-K)^2 \|\Sigma_E\|^2}{(p-K) \lambda_p^2(\Sigma_E)} \leq p \kappa(\Sigma_E)^2 \lesssim p,$$

where in the final step we use the assumption that $\kappa(\Sigma_E) < c_1$. Thus,

$$V \geq \sigma_\varepsilon^2 \log(n) \left(\frac{n}{p} + \frac{K}{n} \right) \quad (76)$$

When $\kappa(\Sigma_E) < c_1$, $p \lesssim r_e(\Sigma_E) \leq p$, and so the variance term in the bound of our Theorem 7 is can be written as

$$\sigma_\varepsilon^2 \log(n) \left(\frac{n}{r_0(\Sigma_E)} + \frac{K}{n} \right) \lesssim \sigma_\varepsilon^2 \log(n) \left(\frac{n}{p} + \frac{K}{n} \right).$$

Thus, comparing with (76), we see that under the stated conditions our variance bound is the same as that of Bartlett et. al., up to absolute constants.

Proof of Lemma 12. We will prove that

$$\frac{r_\ell(\Sigma_X)}{n} \leq \frac{K}{n} (1 + \xi^{-1}) + \frac{1}{\xi} \frac{r_e(\Sigma_E)}{n}, \quad \text{for } 0 \leq \ell \leq K-1 \quad (77)$$

and that

$$\frac{r_K(\Sigma_X)}{n} \geq \frac{r_e(\Sigma_E)}{n} - \frac{K}{n}. \quad (78)$$

Together with the definition of K^* in (75), these two bounds imply Lemma 12.

First note that for $0 \leq \ell \leq K$,

$$\begin{aligned} \sum_{i=\ell+1}^p \lambda_i(\Sigma_X) &= \text{tr}(\Sigma_X) - \sum_{i=1}^{\ell} \lambda_i(\Sigma_X) \\ &= \text{tr}(\Sigma_E) + \text{tr}(A \Sigma_Z A^\top) - \sum_{i=1}^{\ell} \lambda_i(\Sigma_X) \\ &= \text{tr}(\Sigma_E) + \sum_{i=\ell+1}^K \lambda_i(A \Sigma_Z A^\top) + \sum_{i=1}^{\ell} (\lambda_i(A \Sigma_Z A^\top) - \lambda_i(\Sigma_X)), \end{aligned} \quad (79)$$

where the sums from $\ell+1$ to K and from 1 to ℓ are defined to be zero when $\ell = K$ and $\ell = 0$, respectively.

Proof of (77). By Weyl's inequality,

$$|\lambda_i(A \Sigma_Z A^\top) - \lambda_i(\Sigma_X)| \leq \|\Sigma_E\|, \quad (80)$$

so by (79),

$$\sum_{i=\ell+1}^p \lambda_i(\Sigma_X) \leq \text{tr}(\Sigma_E) + (K - \ell) \lambda_{\ell+1}(A \Sigma_Z A^\top) + \ell \|\Sigma_E\|$$

$$\leq \text{tr}(\Sigma_E) + K\lambda_{\ell+1}(A\Sigma_Z A^\top) + K\|\Sigma_E\|. \quad (81)$$

From the min-max formula for eigenvalues we also have

$$\lambda_{\ell+1}(\Sigma_X) = \lambda_{\ell+1}(A\Sigma_Z A^\top + \Sigma_E) \geq \lambda_{\ell+1}(A\Sigma_Z A^\top). \quad (82)$$

Combining (81) and (82), we find

$$\begin{aligned} r_\ell(\Sigma_X) &= \frac{\sum_{i=\ell+1}^p \lambda_i(\Sigma_X)}{\lambda_{\ell+1}(\Sigma_X)} \\ &\leq K \left(1 + \frac{\|\Sigma_E\|}{\lambda_{\ell+1}(A\Sigma_Z A^\top)} \right) + \frac{\text{tr}(\Sigma_E)}{\lambda_{\ell+1}(A\Sigma_Z A^\top)} \\ &\leq K \left(1 + \frac{\|\Sigma_E\|}{\lambda_K(A\Sigma_Z A^\top)} \right) + \frac{\text{tr}(\Sigma_E)}{\lambda_K(A\Sigma_Z A^\top)} \\ &= K(1 + \xi^{-1}) + \xi^{-1} \text{r}_e(\Sigma_E), \end{aligned}$$

which completes the proof of (77).

Proof of (78). Equation (79) for $\ell = K$ is

$$\sum_{i=K+1}^p \lambda_i(\Sigma_X) = \text{tr}(\Sigma_E) + \sum_{i=1}^K (\lambda_i(A\Sigma_Z A^\top) - \lambda_i(\Sigma_X)).$$

Again using (80),

$$\sum_{i=K+1}^p \lambda_i(\Sigma_X) \geq \text{tr}(\Sigma_E) - K\|\Sigma_E\|. \quad (83)$$

Since

$$\begin{aligned} \lambda_{K+1}(\Sigma_X) &= \lambda_{K+1}(\Sigma_X) - \lambda_{K+1}(A\Sigma_Z A^\top) \quad (\text{since } \lambda_{K+1}(A\Sigma_Z A^\top) = 0) \\ &\leq \|\Sigma_E\| \quad (\text{Weyl's inequality}). \end{aligned} \quad (84)$$

Combining (83) and (84), we find

$$r_K(\Sigma_X) = \frac{\sum_{i=K+1}^p \lambda_i(\Sigma_X)}{\lambda_{K+1}(\Sigma_X)} \geq \text{r}_e(\Sigma_E) - K,$$

which proves (78). ■

A.3.2 Comparison of bias terms

A more interesting comparison arises between the bias term B and the corresponding bias term in Theorem 7, display (17). Here we will see how the approach we take in this paper, explicitly taking advantage of the structure of the factor regression model, leads to a stronger bound under certain conditions

Lemma 13. Suppose $\xi := \lambda_K(A\Sigma_Z A^\top)/\|\Sigma_E\| > 1$. Then

$$B \geq \frac{\|\beta\|_{\Sigma_Z}^2}{2\kappa(\Sigma_E)}(1 - \xi^{-1}) \max\left(\sqrt{\frac{r_0(\Sigma_X)}{n}}, \frac{r_0(\Sigma_X)}{n}\right), \quad (85)$$

where

$$\frac{r_0(\Sigma_X)}{n} \geq \frac{1}{2} \frac{r_0(A\Sigma_Z A^\top)}{n} + \frac{1}{2\kappa(A\Sigma_Z A^\top)} \frac{1}{\xi} \frac{r_e(\Sigma_E)}{n}. \quad (86)$$

In particular, if $\xi > c_1 > 1$ and $\kappa(\Sigma_E) < c_2$, $\kappa(A\Sigma_Z A^\top) < c_2$ for absolute constants c_1, c_2 ,

$$B \gtrsim \|\beta\|_{\Sigma_Z}^2 \max\left(\sqrt{\frac{1}{\xi} \frac{p}{n}}, \frac{1}{\xi} \frac{p}{n}\right). \quad (87)$$

Compared to our bias bound $\|\beta\|_{\Sigma_Z}^2 p/(n \cdot \xi)$ in Theorem 7, there is an additional quantity $r_0(A\Sigma_Z A^\top)/n$ of order $O(K/n)$. Ignoring this quantity, provided both $\kappa(\Sigma_E)$ and $\kappa(A\Sigma_Z A^\top)$ are uniformly bounded, we obtain the lower bound (87). When $p/(n \cdot \xi) < 1$, this rate is worse by a factor $\sqrt{p/(n \cdot \xi)}$, compared to the bias term $\|\beta\|_{\Sigma_Z}^2 p/(n \cdot \xi)$ in Theorem 7.

Proof of Lemma 13. By Lemma 14, which we isolate below for reference elsewhere,

$$\|\Sigma_X\| \|\alpha^*\|^2 \geq \frac{\|\beta\|_{\Sigma_Z}^2}{2\kappa(\Sigma_E)}(1 - \xi^{-1}),$$

which implies (85). To prove (86), we first recall that $r_0(\Sigma_X) = \text{tr}(\Sigma_X)/\|\Sigma_X\|$ and $\Sigma_X = A\Sigma_Z A^\top + \Sigma_E$, which implies that

$$\frac{r_0(\Sigma_X)}{n} = \frac{\text{tr}(A\Sigma_Z A^\top)}{n\|\Sigma_X\|} + \frac{\text{tr}(\Sigma_E)}{n\|\Sigma_X\|}.$$

Observing that $\|\Sigma_X\| \leq \|A\Sigma_Z A^\top\| + \|\Sigma_E\| \leq 2\|A\Sigma_Z A^\top\|$, where we use that $\|\Sigma_E\| \leq \|A\Sigma_Z A^\top\|$ by the assumption $\xi > 1$, we find

$$\begin{aligned} \frac{r_0(\Sigma_X)}{n} &\geq \frac{1}{2} \frac{r_0(A\Sigma_Z A^\top)}{n} + \frac{1}{2} \frac{\text{tr}(\Sigma_E)}{n\|A\Sigma_Z A^\top\|} \\ &= \frac{1}{2} \frac{r_0(A\Sigma_Z A^\top)}{n} + \frac{1}{2} \frac{\lambda_K(A\Sigma_Z A^\top)}{\|A\Sigma_Z A^\top\|} \frac{\|\Sigma_E\|}{\lambda_K(A\Sigma_Z A^\top)} \frac{\text{tr}(\Sigma_E)}{n\|\Sigma_E\|} \\ &= \frac{1}{2} \frac{r_0(A\Sigma_Z A^\top)}{n} + \frac{1}{2\kappa(A\Sigma_Z A^\top)} \frac{1}{\xi} \frac{r_e(\Sigma_E)}{n}, \end{aligned}$$

which proves (86). ■

Lemma 14. Let $\Sigma_X = A\Sigma_Z A^\top + \Sigma_E$ with Σ_Z and Σ_E invertible, and let $\alpha^* = \Sigma_X^{-1} A\Sigma_Z \beta$. Then if $\xi = \lambda_K(A\Sigma_Z A^\top)/\|\Sigma_E\| > 1$,

$$\frac{\|\beta\|_{\Sigma_Z}^2}{2\|\Sigma_X\|\kappa(\Sigma_E)}(1 - \xi^{-1}) \leq \|\alpha^*\|^2 \leq \kappa(\Sigma_E) \frac{\|\beta\|_{\Sigma_Z}^2}{\lambda_K(A\Sigma_Z A^\top)}$$

Proof. Defining $\bar{A} = A\Sigma_Z^{1/2}$ and $\bar{\beta} = \Sigma_Z^{1/2}\beta$, we have $\alpha^* = \Sigma_X^{-1}\bar{A}\bar{\beta}$. Now recall that since A and Σ_Z are full rank, so is \bar{A} and thus $\bar{A}^+\bar{A} = \bar{A}^\top\bar{A}^{+\top} = I_K$ (see Appendix D). Thus,

$$\begin{aligned}\alpha^* &= \Sigma_X^{-1}\bar{A}\bar{\beta} \\ &= \Sigma_X^{-1}\bar{A}\bar{A}^\top\bar{A}^{+\top}\bar{\beta} \\ &= \Sigma_X^{-1}(\Sigma_X - \Sigma_E)\bar{A}^{+\top}\bar{\beta} && (\text{since } \Sigma_X = \bar{A}\bar{A}^\top + \Sigma_E) \\ &= (I_p - \Sigma_X^{-1}\Sigma_E)\bar{A}^{+\top}\bar{\beta}.\end{aligned}$$

By the Woodbury matrix identity applied to $\Sigma_X^{-1} = (\bar{A}\bar{A}^\top + \Sigma_E)^{-1}$,

$$I_p - \Sigma_X^{-1}\Sigma_E = \Sigma_E^{-1}\bar{A}\bar{G}^{-1}\bar{A}^\top,$$

where $\bar{G} := I_K + \bar{A}^\top\Sigma_E^{-1}\bar{A}$. Using this in the previous display,

$$\alpha^* = \Sigma_E^{-1}\bar{A}\bar{G}^{-1}\bar{A}^\top\bar{A}^{+\top}\bar{\beta} = \Sigma_E^{-1}\bar{A}\bar{G}^{-1}\bar{\beta}, \quad (88)$$

where we again use $\bar{A}^+\bar{A} = \bar{A}^\top\bar{A}^{+\top} = I_K$ in the second step.

Lower bound

Using (88), we find

$$\begin{aligned}\|\alpha^*\|^2 &= \bar{\beta}^\top\bar{G}^{-1}\bar{A}^\top\Sigma_E^{-2}\bar{A}\bar{G}^{-1}\bar{\beta} \\ &\geq \lambda_p(\Sigma_E^{-1}) \cdot \bar{\beta}^\top\bar{G}^{-1}\bar{A}^\top\Sigma_E^{-1}\bar{A}\bar{G}^{-1}\bar{\beta} \\ &= \frac{1}{\|\Sigma_E\|}\bar{\beta}^\top\bar{G}^{-1}(\bar{G} - I_K)\bar{G}^{-1}\bar{\beta} && (\text{since } \bar{A}^\top\Sigma_E^{-1}\bar{A} = \bar{G} - I_K) \\ &= \frac{1}{\|\Sigma_E\|}(\bar{\beta}^\top\bar{G}^{-1}\bar{\beta} - \bar{\beta}^\top\bar{G}^{-2}\bar{\beta}) \\ &\geq \frac{1}{\|\Sigma_E\|}\bar{\beta}^\top\bar{G}^{-1}\bar{\beta} \left(1 - \frac{1}{\lambda_K(\bar{G})}\right) && (\text{since } \bar{\beta}^\top\bar{G}^{-2}\bar{\beta} \leq \bar{\beta}^\top\bar{G}^{-1}\bar{\beta} \cdot \|\bar{G}^{-1}\|) \\ &\geq \frac{1}{\|\Sigma_E\|} \frac{\|\bar{\beta}\|^2}{\|\bar{G}\|} \left(1 - \frac{1}{\lambda_K(\bar{G})}\right).\end{aligned} \quad (89)$$

Recalling $\bar{G} = I_K + \bar{A}^\top\Sigma_E^{-1}\bar{A}$,

$$\lambda_K(\bar{G}) \geq \lambda_K(\bar{A}^\top\Sigma_E^{-1}\bar{A}) \geq \frac{\lambda_K(\bar{A}^\top\bar{A})}{\|\Sigma_E\|} = \xi, \quad (90)$$

where we recall $\bar{A} = A\Sigma_Z^{1/2}$ in the final step. We also have

$$\|\bar{G}\| = 1 + \|\bar{A}^\top\Sigma_E^{-1}\bar{A}\| \leq 1 + \frac{\|\bar{A}^\top\bar{A}\|}{\lambda_p(\Sigma_E)} \leq 2 \frac{\|\bar{A}^\top\bar{A}\|}{\lambda_p(\Sigma_E)}, \quad (91)$$

where in the last step we used the assumption that $\xi > 1$ and thus

$$\frac{\|\bar{A}^\top\bar{A}\|}{\lambda_p(\Sigma_E)} \geq \frac{\lambda_K(\bar{A}^\top\bar{A})}{\|\Sigma_E\|} = \frac{\lambda_K(A\Sigma_Z A^\top)}{\|\Sigma_E\|} = \xi > 1.$$

Using bounds (90) and (91) in (89) we find

$$\|\alpha^*\|^2 \geq \frac{\|\bar{\beta}\|^2}{2\kappa(\Sigma_E)\|\bar{A}^\top \bar{A}\|} (1 - \xi^{-1}) = \frac{\|\beta\|_{\Sigma_Z}^2}{2\kappa(\Sigma_E)\|A\Sigma_Z A^\top\|} (1 - \xi^{-1}),$$

where in the last step we recall $\bar{\beta} = \Sigma_Z^{1/2}\beta$ and $\bar{A} = A\Sigma_Z^{1/2}$. We then use $\Sigma_X = A\Sigma_Z A^\top + \Sigma_E$ to get the lower bound $\|\Sigma_X\| \geq \|A\Sigma_Z A^\top\|$ which gives the final lower bound,

$$\|\alpha^*\|^2 \geq \frac{\|\beta\|_{\Sigma_Z}^2}{2\kappa(\Sigma_E)\|\Sigma_X\|} (1 - \xi^{-1}).$$

Upper bound

Again beginning with (88), we find

$$\begin{aligned} \|\alpha^*\|^2 &= \bar{\beta}^\top \bar{G}^{-1} \bar{A}^\top \Sigma_E^{-2} \bar{A} \bar{G}^{-1} \bar{\beta} \\ &\leq \frac{1}{\lambda_p(\Sigma_E)} \cdot \bar{\beta}^\top \bar{G}^{-1} \bar{A}^\top \Sigma_E^{-1} \bar{A} \bar{G}^{-1} \bar{\beta} \\ &= \frac{1}{\lambda_p(\Sigma_E)} \cdot \bar{\beta}^\top \bar{G}^{-1} (\bar{G} - I_K) \bar{G}^{-1} \bar{\beta} && (\text{since } \bar{G} = \bar{A}^\top \Sigma_E^{-1} \bar{A} + I_K) \\ &= \frac{1}{\lambda_p(\Sigma_E)} \cdot \left[\bar{\beta}^\top \bar{G}^{-1} \bar{\beta} - \bar{\beta}^\top \bar{G}^{-2} \bar{\beta} \right] \\ &\leq \frac{1}{\lambda_p(\Sigma_E)} \cdot \bar{\beta}^\top \bar{G}^{-1} \bar{\beta} \\ &\leq \frac{1}{\lambda_p(\Sigma_E)} \cdot \frac{\|\bar{\beta}\|^2}{\lambda_K(\bar{G})}. \end{aligned}$$

Combining this with

$$\lambda_K(\bar{G}) = 1 + \lambda_K(\bar{A}^\top \Sigma_E^{-1} \bar{A}) \geq \lambda_K(\bar{A}^\top \bar{A}) / \|\Sigma_E\|,$$

and using $\bar{A} = A\Sigma_Z^{1/2}$ and $\bar{\beta} = \Sigma_Z^{1/2}\beta$, we find

$$\|\alpha^*\|^2 \leq \frac{\|\Sigma_E\|}{\lambda_p(\Sigma_E)} \cdot \frac{\|\beta\|_{\Sigma_Z}^2}{\lambda_K(A\Sigma_Z A^\top)} = \kappa(\Sigma_E) \frac{\|\beta\|_{\Sigma_Z}^2}{\lambda_K(A\Sigma_Z A^\top)},$$

as claimed. ■

B Geometric explanation of the vanishing norm of $\hat{\alpha}$

In this section we offer a quasi-heuristic geometric explanation of the conclusion of Theorem 1. First recall the well-known fact that in high dimensions, independent random vectors with identity covariance matrix are almost orthogonal with high probability (see, for example, Remark 3.2.5 in [41]). In particular, if $\Sigma_X = I_p$, then the independent, p -dimensional, observations X_1, \dots, X_n will be approximately orthogonal with high probability when $p \rightarrow \infty$. We can formalize this and extend it to the case of observations with generic Σ_X as follows.

Lemma 15. Define the unit vectors $\hat{X}_i := X_i/\|X_i\|$ for $1 \leq i \leq p$ and suppose Assumption 1 holds. Then if $\text{r}_e(\Sigma_X) > C \cdot n$ for $C > 1$ large enough, with probability at least $1 - ce^{-c'n}$,

$$\max_{1 \leq i < j \leq n} |\hat{X}_i^\top \hat{X}_j| \lesssim \sqrt{\frac{n}{\text{r}_e(\Sigma_X)}}.$$

In particular, for n fixed, the vectors $\hat{X}_1, \dots, \hat{X}_n$ will be arbitrarily close to being mutually orthogonal with probability tending to 1 as $\text{r}_e(\Sigma_X) \rightarrow \infty$, where we recall that for well conditioned matrices $\text{r}_e(\Sigma_X) \approx p$, and otherwise $\text{r}_e(\Sigma_X) < p$.

For the purpose of our heuristic argument, suppose these vectors were *exactly* orthogonal. The constraint $\mathbf{X}\hat{\alpha} = \mathbf{y}$ is equivalent to

$$\hat{X}_i^\top \hat{\alpha} = \frac{y_i}{\|X_i\|}, \quad (92)$$

which simply states that the component of $\hat{\alpha}$ along the direction of the unit vector \hat{X}_i is equal to $y_i/\|X_i\|$. Since we assume $\{\hat{X}_1, \dots, \hat{X}_n\}$ is an orthonormal set, this implies that

$$\hat{\alpha} = v + \sum_{i=1}^n \frac{y_i}{\|X_i\|} \hat{X}_i, \quad (93)$$

where v is some vector orthogonal to the span of $\hat{X}_1, \dots, \hat{X}_n$, so $\mathbf{X}v = 0$. Again since we assume $\hat{X}_1, \dots, \hat{X}_n$ are orthonormal, this implies

$$\|\hat{\alpha}\|^2 = \|v\|^2 + \sum_{i=1}^n \frac{y_i^2}{\|X_i\|^2}. \quad (94)$$

Since $\hat{\alpha}$ is the minimum norm solution to $\mathbf{X}\alpha = \mathbf{y}$ and $\mathbf{X}v = 0$, it must be that $v = 0$, so

$$\|\hat{\alpha}\|^2 = \sum_{i=1}^n \frac{y_i^2}{\|X_i\|^2} \lesssim \frac{\|\mathbf{y}\|^2}{\text{tr}(\Sigma_X)} \lesssim \frac{n\sigma_y^2}{\text{tr}(\Sigma_X)}, \quad (95)$$

where the last two statements hold in high probability and we use that $\|X_i\|^2$ concentrates around its mean $\text{tr}(\Sigma_X)$, and $\|\mathbf{y}\|^2/n$ concentrates around σ_y^2 . The bound (95) above is exactly what is proved rigorously in Theorem 1.

Supposing for simplicity that $\text{tr}(\Sigma_X) \approx p$, the above argument can be summarized as follows. The requirement $\mathbf{X}\hat{\alpha} = \mathbf{y}$ forces each of the n components of $\hat{\alpha}$ along the (nearly) orthogonal directions $\hat{X}_1, \dots, \hat{X}_n$ to be of order σ_y/\sqrt{p} , and the requirement that $\hat{\alpha}$ have minimum norm among the interpolating vectors forces the remaining $p - n$ components of $\hat{\alpha}$ to be zero. This implies $\|\hat{\alpha}\|^2 \lesssim \sigma_y^2 n/p$, which tends to zero in the high-dimensional regime $p \gg n \cdot \sigma_y^2$.

Proof of Lemma 15. Define the event

$$\mathcal{G} := \left\{ \max_{i \neq j} |X_i^\top X_j| \lesssim \sqrt{n \text{tr}(\Sigma_X) \|\Sigma\|}, \min_i \|X_i\|^2 \gtrsim \text{tr}(\Sigma_X) \right\}.$$

Observe that on this event

$$\max_{i \neq j} |\hat{X}_i^\top \hat{X}_j| \leq \frac{\max_{i \neq j} |X_i^\top X_j|}{\min_i \|X_i\|^2} \lesssim \frac{\sqrt{n \|\Sigma_X\| \text{tr}(\Sigma_X)}}{\text{tr}(\Sigma_X)} = \sqrt{\frac{n}{\text{r}_e(\Sigma_X)}}.$$

Thus all that remains to prove is that $\mathbb{P}(\mathcal{G}^c) \leq ce^{-c'n}$, which we now show.

First note that by Assumption 1, $\|X_i\|^2 = \tilde{X}_i^\top \Sigma_X \tilde{X}_i$, where \tilde{X}_i has independent, mean zero, unit variance entries with sub-Gaussian constants bounded by an absolute constant. Thus by an application of the Hanson-Wright inequality, as in (34) from the proof of Theorem 10,

$$\mathbb{P}(|\|X_i\|^2 - \text{tr}(\Sigma_X)| \geq c_0\|\Sigma_X\|n + \text{tr}(\Sigma_X)/4) \leq 2e^{-3n},$$

so

$$\mathbb{P}(\|X_i\|^2 \geq c_0\|\Sigma_X\|n + 5\text{tr}(\Sigma_X)/4) \leq 2e^{-3n}.$$

Thus, using the assumption that $r_e(\Sigma_X) > C \cdot n$ and so $\text{tr}(\Sigma_X) \gtrsim \|\Sigma_X\|n$,

$$\mathbb{P}(\|X_i\|^2 \gtrsim \text{tr}(\Sigma_X)) \leq 2e^{-c_3n}.$$

Using a union bound, we thus find

$$\mathbb{P}\left(\min_i \|X_i\|^2 \gtrsim \text{tr}(\Sigma_X)\right) \leq c_4e^{-c_3n}. \quad (96)$$

Next, using that for $i \neq j$, $X_i^\top X_j = \tilde{X}_i^\top \Sigma_X \tilde{X}_j$ with \tilde{X}_i and \tilde{X}_j independent, where both vectors have independent entries with zero mean and unit variance, we can apply Lemma 16 below to find, for $i \neq j$,

$$\mathbb{P}\left(|X_i^\top X_j| \geq \|\Sigma_X\|n + \sqrt{n\|\Sigma_X\|_F^2}\right) \leq 2e^{-c'n}. \quad (97)$$

Then using

$$\|\Sigma_X\|_F^2 = \text{tr}(\Sigma_X^2) \leq \|\Sigma_X\|\text{tr}(\Sigma_X),$$

we have

$$\begin{aligned} \|\Sigma_X\|n + \sqrt{n\|\Sigma_X\|_F^2} &\leq \|\Sigma_X\|n + \sqrt{n\text{tr}(\Sigma_X)\|\Sigma_X\|} \\ &= \sqrt{n\text{tr}(\Sigma_X)\|\Sigma_X\|} \left(1 + \sqrt{\frac{n}{r_e(\Sigma_X)}}\right) \\ &\lesssim \sqrt{n\text{tr}(\Sigma_X)\|\Sigma_X\|}, \end{aligned}$$

where we use $r_e(\Sigma_X) > C \cdot n$ in the last step. Combining this with (97), we find that

$$\mathbb{P}\left(|X_i^\top X_j| \gtrsim \sqrt{n\text{tr}(\Sigma_X)\|\Sigma_X\|}\right) \leq 2e^{-c'n}.$$

By a union bound we thus have

$$\mathbb{P}\left(\max_{i \neq j} |X_i^\top X_j| \gtrsim \sqrt{n\text{tr}(\Sigma_X)\|\Sigma_X\|}\right) \leq ce^{-c'n}.$$

Combining this with the bound (96) completes the proof that $\mathbb{P}(\mathcal{G}^c) \leq ce^{c't}$ and thus the proof as a whole. \blacksquare

Lemma 16. Suppose $W_1, W_2 \in \mathbb{R}^p$ are independent random vectors which both have zero mean, identity covariance matrix, and independent entries. Then for any positive semi-definite Σ

$$\mathbb{P} \left\{ |W_1^\top \Sigma W_2| \geq M^2 \|\Sigma\| t + M^2 \sqrt{t \|\Sigma\|_F^2} \right\} \leq 2e^{-c't},$$

where $M := \max(\|W_1\|_{\psi_2}, \|W_2\|_{\psi_2})$.

We give a proof of this simple lemma below for completeness. It uses elements of the proof of the Hanson-Wright inequality.

Proof. We prove concentration of the one-sided tail:

$$\mathbb{P} \left\{ W_1^\top \Sigma W_2 \geq \|M^2 \Sigma\| t + \sqrt{t \|M^2 \Sigma\|_F^2} \right\} \leq 2^{-c't}.$$

The final result follows from applying this result to $-\Sigma$ and using a union bound to combine the two tails. Furthermore, we can prove the result for $M = 1$ without loss of generality. Indeed, for any $M > 0$,

$$W_1^\top \Sigma W_2 = (W_1^\top / M)(M^2 \Sigma) W_2 / M,$$

and $\max(\|W_1/M\|_{\psi_2}, \|W_2/M\|_{\psi_2}) = 1$, and applying the theorem for $M = 1$ implies

$$\mathbb{P} \left\{ |W_1^\top \Sigma W_2| \geq \|M^2 \Sigma\| t + \sqrt{t \|M^2 \Sigma\|_F^2} \right\} \leq 2e^{-c't},$$

which implies the result for arbitrary M .

Thus assume $M = 1$. We first use that for all $s, \tau > 0$,

$$\mathbb{P} \left\{ W_1^\top \Sigma W_2 \geq s \right\} \leq e^{-\tau s} \mathbb{E} e^{\tau W_1^\top \Sigma W_2}. \quad (98)$$

Then, by a replacement trick given by Lemma 6.2.3 of [41], and using $M = 1$, we find that

$$\mathbb{E} e^{\tau W_1^\top \Sigma W_2} \leq \mathbb{E} e^{c\tau g_1^\top \Sigma g_2},$$

where g_1 and g_2 are i.i.d. $N(0, I_p)$. Thus, by Lemma 6.2.2 of [41], which gives a bound on the moment generating function of $g_1^\top \Sigma g_2$,

$$\mathbb{E} e^{\tau W_1^\top \Sigma W_2} \leq \mathbb{E} e^{c\tau g_1^\top \Sigma g_2} \leq e^{c_1 \tau^2 \|\Sigma\|_F^2},$$

where the last inequality holds for all τ satisfying $|\tau| \leq c_2/\|\Sigma\|$. Plugging this into (98), for $0 \leq \tau \leq c_2/\|\Sigma\|$ we have

$$\mathbb{P} \left\{ W_1^\top \Sigma W_2 \geq s \right\} \leq \exp \left(-\tau s + c_1 \tau^2 \|\Sigma\|_F^2 \right).$$

Optimizing over $0 \leq \tau \leq c_2/\|\Sigma\|$ gives the optimal choice $\tau = \min \left(s/(2c_1 \|\Sigma\|_F^2), c_2/\|\Sigma\| \right)$ and we get

$$\mathbb{P} \left\{ W_1^\top \Sigma W_2 \geq s \right\} \leq \exp \left[-\min \left(\frac{s^2}{4c_1 \|\Sigma\|_F^2}, \frac{c_2 s}{2\|\Sigma\|} \right) \right]. \quad (99)$$

Choosing $s = \|\Sigma\|t + \sqrt{t\|\Sigma\|_F^2}$ in (99) and noting that for this s ,

$$s/\|\Sigma\| = t + \sqrt{t\|\Sigma\|_F^2/\|\Sigma\|} \geq t,$$

and

$$s^2/\|\Sigma\|_F^2 = \left(t\|\Sigma\|/\|\Sigma\|_F + \sqrt{t}\right)^2 \geq t,$$

we arrive at the final result,

$$\mathbb{P} \left\{ W_1^\top \Sigma W_2 \geq \|\Sigma\|t + \sqrt{t\|\Sigma\|_F^2} \right\} \leq e^{-c't}.$$

■

C Supplementary Results

C.1 Closed form solutions of min-norm estimator and minimizer of $R(\alpha)$

Lemma 17. *For zero mean random variables $X \in \mathbb{R}^p$ and $y \in \mathbb{R}$, suppose $\Sigma_X := \mathbb{E}[XX^\top]$ and $\sigma_y^2 := \mathbb{E}[y^2]$ are finite, and let $\Sigma_{Xy} = \mathbb{E}[Xy]$. Then $\alpha^* := \Sigma_X^+ \Sigma_{Xy}$ is a minimizer of $R(\alpha)$:*

$$R(\alpha^*) = \min_{\alpha \in \mathbb{R}^p} R(\alpha).$$

Proof. We have

$$R(\alpha) = \mathbb{E}[(X^\top \alpha - y)^2] = \alpha^\top \Sigma_X \alpha + \sigma_y^2 - 2\alpha^\top \Sigma_{Xy},$$

so since $R(\alpha)$ is convex, α is a minimizer if and only if

$$\nabla_\alpha R(\alpha) = 2\Sigma_X \alpha - 2\Sigma_{Xy} = 0.$$

By (36), $\Sigma_X \alpha^* = \Sigma_{Xy}$, so the claim is proved.

■

For $\mathbf{X} \in \mathbb{R}^{n \times p}$ and $\mathbf{y} \in \mathbb{R}^n$, let

$$\hat{\alpha} := \arg \min \left\{ \|\alpha\| : \|\mathbf{X}\alpha - \mathbf{y}\| = \min_u \|\mathbf{X}u - \mathbf{y}\| \right\}.$$

We then have the following result.

Lemma 18. $\hat{\alpha} = \mathbf{X}^+ \mathbf{y}$.

Proof. Step 1: Existence and uniqueness of $\hat{\alpha}$. Since

$$\nabla_u \|\mathbf{X}u - \mathbf{y}\|^2 = 2\mathbf{X}^\top \mathbf{X}u - 2\mathbf{X}^\top \mathbf{y},$$

and $\|\mathbf{X}u - \mathbf{y}\|^2$ is convex in u , u is a minimizer of $u \mapsto \|\mathbf{X}u - \mathbf{y}\|^2$ if and only if

$$\mathbf{X}^\top \mathbf{X}u = \mathbf{X}^\top \mathbf{y}. \tag{100}$$

By the properties of the pseudo-inverse, $\mathbf{X}^\top \mathbf{X} \mathbf{X}^+ = \mathbf{X}^\top$, so

$$\mathbf{X}^\top \mathbf{X} (\mathbf{X}^+ \mathbf{y}) = \mathbf{X}^\top \mathbf{y},$$

and thus $\mathbf{X}^+ \mathbf{y}$ is a minimizer of $\|\mathbf{X}u - \mathbf{y}\|$. The set of vectors u satisfying $\mathbf{X}^\top \mathbf{X}u = \mathbf{X}^\top \mathbf{y}$ is also convex, so $\hat{\alpha}$ is a minimizer of a strictly convex function $\|\cdot\|$ over a non-empty convex set. Such a minimizer exists and is unique, so $\hat{\alpha}$ exists and is unique.

Step 2: formula for $\hat{\alpha}$. Since $\hat{\alpha}$ is a minimizer of $\|\mathbf{X}u - \mathbf{y}\|$, it must satisfy 100, i.e.

$$\mathbf{X}^\top \mathbf{X} \hat{\alpha} = \mathbf{X}^\top \mathbf{y}. \quad (101)$$

We can write

$$\hat{\alpha} = \mathbf{X}^+ \mathbf{X} \hat{\alpha} + (I - \mathbf{X}^+ \mathbf{X}) \hat{\alpha},$$

and using $\mathbf{X} \mathbf{X}^+ \mathbf{X} = \mathbf{X}$ as well as the fact that $\mathbf{X}^+ \mathbf{X}$ is symmetric (see Appendix D), a quick calculation gives

$$\|\hat{\alpha}\|^2 = \|\mathbf{X}^+ \mathbf{X} \hat{\alpha}\|^2 + \|(I - \mathbf{X}^+ \mathbf{X}) \hat{\alpha}\|^2.$$

Thus $\|\mathbf{X}^+ \mathbf{X} \hat{\alpha}\| \leq \|\hat{\alpha}\|^2$, and also

$$\mathbf{X}^\top \mathbf{X} (\mathbf{X}^+ \mathbf{X} \hat{\alpha}) = \mathbf{X}^\top \mathbf{X} \hat{\alpha} = \mathbf{X}^\top \mathbf{y},$$

where we used $\mathbf{X} \mathbf{X}^+ \mathbf{X} = \mathbf{X}$ in the first step and 101 in the second step. Thus $\mathbf{X}^+ \mathbf{X} \hat{\alpha}$ is a minimizer of $\|\cdot\|$ among minimizers of $\|\mathbf{X}u - \mathbf{y}\|$. Since by Step 1 above $\hat{\alpha}$ is the unique such minimizer, $\mathbf{X}^+ \mathbf{X} \hat{\alpha} = \hat{\alpha}$. Thus,

$$\begin{aligned} \hat{\alpha} &= \mathbf{X}^+ \mathbf{X} \hat{\alpha} \\ &= (\mathbf{X}^\top \mathbf{X})^+ \mathbf{X}^\top \mathbf{X} \hat{\alpha} && \text{(since } \mathbf{X}^+ = (\mathbf{X}^\top \mathbf{X})^+ \mathbf{X}^\top \text{)} \\ &= (\mathbf{X}^\top \mathbf{X})^+ \mathbf{X}^\top \mathbf{y} && \text{(by 101)} \\ &= \mathbf{X}^+ \mathbf{y}. && \text{(since } \mathbf{X}^+ = (\mathbf{X}^\top \mathbf{X})^+ \mathbf{X}^\top \text{)} \end{aligned}$$

■

C.2 Risk of $\hat{\alpha}$ under the factor regression model for $p \ll n$

For completeness, we provide a risk bound for the minimum-norm estimator $\hat{\alpha}$ under the factor regression model in the low-dimensional regime $p \ll n$.

Theorem 19. *Under model 8, suppose that Assumptions 1, 2 & 3 hold. Then if $n > C \cdot p$ for some $C > 0$ large enough and $p \geq K$, with probability at least $1 - c/n$,*

$$R(\hat{\alpha}) - \sigma_\varepsilon^2 \lesssim \kappa(\Sigma_E) \frac{\|\beta\|_{\Sigma_Z}^2}{\xi} + \frac{p}{n} \sigma_\varepsilon^2 \log n,$$

where $\kappa(\Sigma_E) = \lambda_1(\Sigma_E)/\lambda_p(\Sigma_E)$ is the condition number of Σ_E .

Proof. As in the proof of Theorem 7 found in section A.2.3 above,

$$R(\hat{\alpha}) \leq 2(B_1 + B_2) + 2(V_1 + V_2),$$

where

$$\begin{aligned}
B_1 &= \|\Sigma_E^{1/2} \mathbf{X}^+ \mathbf{Z} \beta\|^2 \\
B_2 &= \|\Sigma_Z^{1/2} (A^\top \mathbf{X}^+ \mathbf{Z} - I_K) \beta\|^2 \\
V_1 &= \|\Sigma_E^{1/2} \mathbf{X}^+ \boldsymbol{\varepsilon}\|^2 \\
V_2 &= \|\Sigma_Z^{1/2} A^\top \mathbf{X}^+ \boldsymbol{\varepsilon}\|^2.
\end{aligned}$$

We will bound these four terms on the event $\mathcal{B} = \mathcal{B}_1 \cap \mathcal{B}_2$, where

$$\mathcal{B}_1 := \{\|\tilde{\mathbf{E}}\|^2 < c_1 n, \sigma_K^2(\tilde{\mathbf{Z}}) > c_2 n, \sigma_p^2(\tilde{\mathbf{X}}) \geq c_3 n\}$$

and

$$\mathcal{B}_2 := \left\{ \tilde{\boldsymbol{\varepsilon}}^\top \mathbf{X}^{+\top} \Sigma_X \mathbf{X}^+ \tilde{\boldsymbol{\varepsilon}} \leq c_5 \log(n) \cdot \text{tr}(\mathbf{X}^{+\top} \Sigma_X \mathbf{X}^+) \right\}.$$

As the last step of the proof, we will show that $\mathbb{P}(\mathcal{B}) \geq 1 - c/n$.

Bounding the bias component

First observe that since $K < n$, when \mathbf{Z} is full rank, $\mathbf{Z}^+ \mathbf{Z} = I_K$ and so

$$A^\top \mathbf{X}^+ = \mathbf{Z}^+ \mathbf{Z} A^\top \mathbf{X}^+ = \mathbf{Z}^+ (\mathbf{X} - \mathbf{E}) \mathbf{X}^+ = \mathbf{Z}^+ \mathbf{X} \mathbf{X}^+ - \mathbf{Z}^+ \mathbf{E} \mathbf{X}^+.$$

Thus,

$$\begin{aligned}
B_2 &= \|(A^\top \mathbf{X}^+ \mathbf{Z} - I_K) \beta\|^2 \\
&= \|(\mathbf{Z}^+ \mathbf{X} \mathbf{X}^+ \mathbf{Z} - I_K) \beta - \mathbf{Z}^+ \mathbf{E} \mathbf{X}^+ \mathbf{Z} \beta\|_{\Sigma_Z}^2 \\
&\leq 2\|(\mathbf{Z}^+ \mathbf{X} \mathbf{X}^+ \mathbf{Z} - I_K) \beta\|_{\Sigma_Z}^2 + 2\|\mathbf{Z}^+ \mathbf{E} \mathbf{X}^+ \mathbf{Z} \beta\|_{\Sigma_Z}^2.
\end{aligned} \tag{102}$$

Note that since $p \geq K$, by Assumption 2, $\text{rank}(A) = K$ so by Lemma 21 of Appendix D,

$$A^\top A^{+\top} = I_K. \tag{103}$$

We thus have

$$\begin{aligned}
\|(\mathbf{Z}^+ \mathbf{X} \mathbf{X}^+ \mathbf{Z} - I_K) \beta\|_{\Sigma_Z}^2 &= \|(\mathbf{Z}^+ \mathbf{X} \mathbf{X}^+ \mathbf{Z} - \mathbf{Z}^+ \mathbf{Z}) \beta\|_{\Sigma_Z}^2 \\
&= \|\tilde{\mathbf{Z}}^+ (\mathbf{X} \mathbf{X}^+ - I_p) \mathbf{Z} \beta\|^2 \\
&\leq \frac{\|(\mathbf{X} \mathbf{X}^+ - I_p) \mathbf{Z} \beta\|^2}{\sigma_K^2(\tilde{\mathbf{Z}})} \\
&\lesssim \frac{1}{n} \|(\mathbf{X} \mathbf{X}^+ - I_p) \mathbf{Z} \beta\|^2 && \text{(on } \mathcal{B}) \\
&= \frac{1}{n} \|(\mathbf{X} \mathbf{X}^+ - I_p) \mathbf{Z} A^\top A^{+\top} \beta\|^2 && \text{(by (103))} \\
&= \frac{1}{n} \|(\mathbf{X} \mathbf{X}^+ - I_p) (\mathbf{X} - \mathbf{E}) A^{+\top} \beta\|^2 && \text{(since } \mathbf{X} = \mathbf{Z} A^\top + \mathbf{E}) \\
&= \frac{1}{n} \|(\mathbf{X} \mathbf{X}^+ - I_p) \mathbf{E} A^{+\top} \beta\|^2 && \text{(since } \mathbf{X} \mathbf{X}^+ \mathbf{X} = \mathbf{X})
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{n} \|\mathbf{X}\mathbf{X}^+ - I_p\| \cdot \|\mathbf{E}A^{+\top}\beta\|^2 \\
&\leq \frac{1}{n} \|\mathbf{E}A^{+\top}\beta\|^2 \\
&\lesssim \frac{n\|\Sigma_E\|}{n} \frac{\|\beta\|_{\Sigma_Z}^2}{\lambda_K(A\Sigma_ZA^\top)} \quad (\text{on } \mathcal{B} \text{ and by (52)}) \\
&= \frac{\|\beta\|_{\Sigma_Z}^2}{\xi}, \tag{104}
\end{aligned}$$

where in the penultimate step we used

$$\|A^{+\top}\beta\|^2 \leq \frac{\|\beta\|_{\Sigma_Z}^2}{\lambda_K(A\Sigma_ZA^\top)} \tag{105}$$

from (52). We can bound the second term in 102 as follows:

$$\begin{aligned}
\|\mathbf{Z}^+\mathbf{E}\mathbf{X}^+\mathbf{Z}\beta\|_{\Sigma_Z}^2 &= \|\tilde{\mathbf{Z}}^+\mathbf{E}\mathbf{X}^+\mathbf{Z}\beta\|^2 \\
&\leq \frac{\|\mathbf{E}\|^2}{\sigma_K^2(\tilde{\mathbf{Z}})} \|\mathbf{X}^+\mathbf{Z}\beta\|^2 \\
&\lesssim \|\Sigma_E\| \cdot \|\mathbf{X}^+\mathbf{Z}\beta\|^2 \quad (\text{on } \mathcal{B}) \\
&= \|\Sigma_E\| \cdot \|\mathbf{X}^+\mathbf{Z}A^\top A^{+\top}\beta\|^2 \quad (\text{since } A^\top A^{+\top} = I_K) \\
&= \|\Sigma_E\| \cdot \|\mathbf{X}^+(\mathbf{X} - \mathbf{E})A^{+\top}\beta\|^2 \quad (\text{since } \mathbf{X} = \mathbf{Z}A^\top + \mathbf{E}) \\
&\leq 2\|\Sigma_E\| \cdot \|\mathbf{X}^+\mathbf{X}A^{+\top}\beta\|^2 + 2\|\Sigma_E\| \cdot \|\mathbf{X}^+\mathbf{E}A^{+\top}\beta\|^2 \\
&\lesssim \|\Sigma_E\| \|A^{+\top}\beta\|^2 + \|\Sigma_E\| \frac{\|\mathbf{E}\|}{\sigma_p^2(\mathbf{X})} \|A^{+\top}\beta\|^2 \quad (\text{since } \|\mathbf{X}^+\mathbf{X}\| \leq 1) \\
&\lesssim \|\Sigma_E\| \cdot \kappa(\Sigma_E) \|A^{+\top}\beta\|^2 \\
&\leq \kappa(\Sigma_E) \frac{\|\beta\|_{\Sigma_Z}^2}{\xi}. \quad (\text{by (105)})
\end{aligned}$$

Using this and (104) in (102), and using the fact that $\kappa(\Sigma_E) > 1$, we find that on the event \mathcal{B} ,

$$B_2 \lesssim \kappa(\Sigma_E) \frac{\|\beta\|_{\Sigma_Z}^2}{\xi}. \tag{106}$$

Bounding the variance component

We have

$$\begin{aligned}
V_1 + V_2 &= \varepsilon^\top \mathbf{X}^{+\top} \Sigma_X \mathbf{X}^+ \varepsilon \\
&= \sigma_\varepsilon^2 \tilde{\varepsilon}^\top \mathbf{X}^{+\top} \Sigma_X \mathbf{X}^+ \tilde{\varepsilon} \quad (\text{by Assumption 3}) \\
&\lesssim \sigma_\varepsilon^2 \log(n) \text{tr}(\mathbf{X}^{+\top} \Sigma_X \mathbf{X}^+) \quad (\text{on } \mathcal{B}_2) \\
&\leq \sigma_\varepsilon^2 \log(n) \cdot p \|\mathbf{X}^{+\top} \Sigma_X \mathbf{X}^+\| \quad (\text{since } \text{rank}(\mathbf{X}^+) = p) \\
&= \sigma_\varepsilon^2 \log(n) \cdot p \|\Sigma_X^{1/2} \mathbf{X}^+\|^2. \tag{107}
\end{aligned}$$

From Assumption 1, $\mathbf{X} = \tilde{\mathbf{X}}\Sigma_X^{1/2}$, and from Lemma 21 of Appendix D below,

$$(\tilde{\mathbf{X}}\Sigma_X^{1/2})^+ = (\tilde{\mathbf{X}}^+ \tilde{\mathbf{X}}\Sigma_X^{1/2})^+ (\tilde{\mathbf{X}}\Sigma_X^{1/2}\Sigma_X^{-1/2})^+ = \Sigma_X^{-1/2}\tilde{\mathbf{X}}^+.$$

Using this in (107), we find

$$V_1 + V_2 \lesssim \sigma_\varepsilon^2 \log(n) \cdot p \|\tilde{\mathbf{X}}^+\|^2 = \sigma_\varepsilon^2 \log(n) \frac{p}{\sigma_p^2(\tilde{\mathbf{X}})}$$

Proof that $\mathbb{P}(\mathcal{B}) \geq 1 - c/n$.

The bounds $\mathbb{P}(\mathcal{B}_1) \geq 1 - c/n$ and $\mathbb{P}(\mathcal{B}_2) \geq 1 - e^{-cn}$ follow respectively from Theorem 4.6.1 of [41] and Lemma 11 in Appendix A.2.2 above, by similar reasoning as in the proof of Theorem 7, for example. \blacksquare

C.3 Signal to noise ratio bound for clustered variables

We present here a lower bound on the signal-to-noise ratio $\xi = \lambda_K(A\Sigma_Z A^\top)/\|\Sigma_E\|$ in terms of the number $|I_a|$ of features related to cluster a only, for $1 \leq a \leq K$. We recall the definition

$$I_a := \{i \in [p] : |A_{ia}| = 1, A_{ib} = 0 \text{ for } b \neq a\}.$$

Lemma 20. $\xi \geq \min_a |I_a| \cdot \lambda_K(\Sigma_Z)/\|\Sigma_E\|$.

Proof. For any $v \in \mathbb{R}^K$ with $\|v\| = 1$,

$$\begin{aligned} v^\top A^\top A v &= \|Av\|^2 = \sum_{i=1}^p \left(\sum_{a=1}^K A_{ia} v_a \right)^2 \\ &\geq \sum_{i \in I} \left(\sum_{a=1}^K A_{ia} v_a \right)^2 \\ &= \sum_{b=1}^K \sum_{i \in I_b} A_{ib}^2 v_b^2 \\ &= \sum_{b=1}^K |I_b| v_b^2 \quad (|A_{ib}| = 1 \text{ for } i \in I_b) \\ &\geq \min_a |I_a| \cdot \sum_{b=1}^K v_b^2 = \min_a |I_a|. \quad (\text{since } \|v\| = 1). \end{aligned}$$

Thus, using $\lambda_K(A\Sigma_Z A^\top) \geq \lambda_K(\Sigma_Z)\lambda_K(A^\top A)$,

$$\xi = \lambda_K(A\Sigma_Z A^\top)/\|\Sigma_E\| \geq \lambda_K(A^\top A)\lambda_K(\Sigma_Z)/\|\Sigma_E\| \geq \min_a |I_a| \lambda_K(\Sigma_Z)/\|\Sigma_E\|,$$

which completes the proof. \blacksquare

D Properties of the Moore-Penrose pseudo-inverse

We state the definition and some properties of the pseudo-inverse in this section for completeness. The material here can be found in [36], along with proofs of some of the statements. For a matrix $B \in \mathbb{R}^{n \times m}$, there exists a unique matrix B^+ , which we define as the pseudo-inverse of B , satisfying the following four conditions:

$$BB^+B = B \quad (108)$$

$$B^+BB^+ = B^+ \quad (109)$$

$$BB^+ \text{ is symmetric} \quad (110)$$

$$B^+B \text{ is symmetric} \quad (111)$$

We will use the following properties of the pseudo-inverse in this paper.

Lemma 21. *For any $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{m \times d}$,*

$$(BC)^+ = (B^+BC)^+(BCC^+)^+. \quad (112)$$

Furthermore, for any matrix $B \in \mathbb{R}^{n \times m}$ of rank r , with smallest non-zero singular value $\sigma_r(B)$,

$$B^\top BB^+ = B^\top \quad (113)$$

$$B^+B = I_m \text{ if } r = m \quad (114)$$

$$BB^+ = I_n \text{ if } r = n \quad (115)$$

$$\|B^+\| = 1/\sigma_r(B) \quad (116)$$

$$\text{rank}(B^+) = \text{rank}(B) = r. \quad (117)$$