

# VectorsMatrices

April 13, 2023

## 0.1 STAT 207: Vectors and matrices

**Zhe Fei (zhe.fei@ucr.edu)** Many scientific computational problems involve vectors and matrices.

The distinction between the set of real numbers,  $\mathbb{R}$ , and the set of floating-point numbers,  $\mathbb{F}$ , that we use in the computer has important implications for numerical computations.

We will consider the floating-point representation of elements of vectors and matrices and the computations in  $\mathbb{F}$ .

- In multidimensional calculus, vector and matrix norms quantify notions of topology and convergence.
- Norms are also devices for deriving explicit bounds, theoretical developments in numerical analysis rely heavily on norms.
- They are particularly useful in establishing convergence and in estimating rates of convergence of iterative methods for solving linear and nonlinear equations.
- Norms also arise in almost every other branch of theoretical numerical analysis. Functional analysis, which deals with infinite-dimensional vector spaces, uses norms on functions.

### 0.1.1 Elementary Properties of Vector Norms

Euclidean vector norm:

$$\|x\|_2 = \sqrt{\sum_{i=1}^m x_i^2}$$

A norm on  $R^m$  is formally defined by four properties:

- (a)  $\|x\| \geq 0$ ,
- (b)  $\|x\| = 0$  if and only if  $x = \mathbf{0}$ ,
- (c)  $\|cx\| = |c| \cdot \|x\|$  for all real number  $c$ ,
- (d)  $\|x + y\| \leq \|x\| + \|y\|$ .

$\ell$ -1 norm:

$$\|x\|_1 = \sum_{i=1}^m |x_i|,$$

$\ell$ - $\infty$  norm:

$$\|x\|_\infty = \max_i |x_i|.$$

For each of the norms  $\|x\|_p, p = 1, 2$ , and  $\infty$ , a sequence of vectors  $x_n$  converges to a vector  $y$  if and only if each component sequence  $x_{ni}$  converges to  $y_i$ . Thus, all three norms give the same topology on  $R^m$ .

**NAS Proposition 6.2.1** Let  $\|x\|$  be any norm on  $R^m$ . Then there exist positive constants  $k_l$  and  $k_u$  such that  $k_l\|x\|_1 \leq \|x\| \leq k_u\|x\|_1$  holds for all  $x \in R^m$ .

- $\sup_{x \neq 0} \|x\|/\|x\|^*$  is finite for any pair of norms  $\|x\|$  and  $\|x\|^*$ .
- For  $p < q$  from  $\{1, 2, \infty\}$ ,

$$\|x\|_q \leq \|x\|_p$$

$$\|x\|_p \leq m^{1/p-1/q} \|x\|_q$$

### 0.1.2 Elementary Properties of Matrix Norms

An  $m \times m$  matrix  $A = (a_{ij})$  is simply a vector in  $R^{m^2}$ .

It is preferred that a matrix norm is compatible with matrix multiplication.

In addition to (a) - (d), we require

$$(e) \|AB\| \leq \|A\| \cdot \|B\|$$

for any product of  $m \times m$  matrices  $A$  and  $B$ .

- Frobenius norm:  $\|A\|_F = \sqrt{\sum_{i=1}^m \sum_{j=1}^m a_{ij}^2} = \sqrt{\text{tr}(AA^T)} = \sqrt{\text{tr}(A^T A)}$
- Matrix norm induced from any vector norm  $\|x\|$  on  $R^m$ :

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\|.$$

- What is  $\|I\|$  and  $\|I\|_F$ ?

In infinite-dimensional settings, induced matrix norms are called operator norms.

**NAS Proposition 6.2.1** If  $A = (a_{ij})$  is an  $m \times m$  matrix, then

- (a)  $\|A\|_1 = \max_j \sum_i |a_{ij}|$ ,
- (b)  $\|A\|_2 = \sqrt{\rho(A^T A)}$ , which reduces to  $\rho(A)$  if  $A$  is symmetric,
- (c)  $\|A\|_2 = \max_{\|u\|_2=1, \|v\|_2=1} u^T A v$ ,
- (d)  $\|A\|_\infty = \max_i \sum_j |a_{ij}|$ .

### 0.1.3 Norm Preserving Linear Transformations

- Orthogonal matrix:  $OO^T = I$ ;
- Orthonormal set of vectors  $S$ : every vector in  $S$  has magnitude 1 and the set of vectors are mutually orthogonal.

Following  $(\det O)^2 = 1$ , we can divide the orthogonal matrices into *rotations* with  $\det O = 1$  and *reflections* with  $\det O = -1$ . Take  $2 \times 2$  matrices for example,

- Rotation of angle  $\theta$  can be presented as:

$$O = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

- Reflection of a point across the line at angle  $\theta/2$  with the  $x$  axis:

$$O = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}$$

#### Remarks

- The set of orthogonal matrices forms a group under matrix multiplication.
- The identity matrix is the unit of the group.
- The rotations constitute a subgroup of the orthogonal group, but the reflections do not since the product of two reflections is a rotation.

Orthogonal transformations preserve inner products and Euclidean norms:  $(Ou)^T Ov = u^T O^T Ov = u^T v$ .

As a result, all eigenvalues of  $O$  lie on the boundary of the unit circle.

Norm invariance for vectors also leads to norm invariance for matrices.

- $\|OA\|_2^2 = \|A\|_2^2$
- $\|AO\|_2^2 = \|A\|_2^2$
- $\|OA\|_F^2 = \|A\|_F^2$
- $\|AO\|_F^2 = \|A\|_F^2$

**Householder matrices** Given a unit vector  $u$ , the Householder matrix  $H = I - 2uu^T$  represents a reflection across the plane perpendicular to  $u$ .

$H$  is orthogonal and symmetric,

$$HH^T = I - 4uu^T + 4u\|u\|_2^2 u^T = I.$$

- $Hv = v$  whenever  $v$  lies in the plane perpendicular to  $u$ .
- $Hu = -u$ .
- $H$  has one eigenvalue equal to  $-1$  and all others equal to 1.

### 0.1.4 Iterative Solution of Linear Equations

Many numerical problems involve iterative schemes of the form

$$x_n = Bx_{n-1} + w \quad (1)$$

for solving the vector-matrix equation  $(I - B)x = w$ . The map  $f(x) = Bx + w$  satisfies

$$\|f(y) - f(x)\| = \|B(y - x)\| \leq \|B\| \cdot \|y - x\|$$

and is contractive for a vector norm  $\|x\|$  if  $\|B\| < 1$ .

**NAS Proposition 6.5.1** Let  $B$  be an arbitrary matrix with spectral radius  $\rho(B)$ . Then  $\rho(B) < 1$  if and only if  $\|B\| < 1$  for some induced matrix norm. The inequality  $\|B\| < 1$  implies:

- (a)  $\lim_{n \rightarrow \infty} \|B^n\| = 0$ ,
- (b)  $(I - B)^{-1} = \sum_{n=0}^{\infty} B^n$ ,
- (c)  $\frac{1}{1 - \|B\|} \leq \|(I - B)^{-1}\| \leq \frac{1}{1 - \|B\|}$ .

Linear iteration is especially useful in solving the equation  $Ax = b$  for  $x$  when an approximation  $C$  to  $A^{-1}$  is known, let  $B = I - CA$  and  $w = Cb$  and iterate equation (1).

**Jacobi's Method** Suppose  $A = (a_{ij})$  is strictly diagonally dominant, let  $D = \text{diag}(a_{ii})$  be the diagonal matrix. Then  $C = D^{-1}$  is an approximate inverse of  $A$ , and  $B = I - CA$  satisfies  $\|B\|_{\infty} < 1$ .

```
[1]: import numpy as np

def jacobi(A, b, x0, tol=1e-6, max_iter=1000):
    """
    Solves the system of linear equations Ax = b using Jacobi's method.
    """
    n = len(A)
    C = np.diag(1/np.diag(A))
    B = np.identity(n) - C@A
    w = C@b

    x = B@x0 + w
    k = 1
    while max(abs(x-x0)) > tol:
        x0 = x
        x = B@x0 + w
        k += 1

    if k < max_iter:
        print(f'Converged in {k} iterations.')
        return x
    else:
        print(f'Maximum iterations reached.')
        return x
```

```
[2]: # Example usage
A = np.array([[4, 1, 1], [2, 5, 1], [1, 2, 4]])
b = np.array([4, 1, 2])
x0 = np.array([0, 0, 0])

x = jacobi(A, b, x0)
print(f'Solution: {x}')

print(A@x)
```

Converged in 28 iterations.

Solution: [ 0.96874976 -0.26562527 0.39062468]  
[3.99999845 0.99999787 1.99999795]

**Landweber's Iteration Scheme** In practice, the approximate inverse  $C$  can be rather crude.  $C = \epsilon A^T$  for  $\epsilon$  small.

- $A^T A$  is positive definite with eigenvalues  $0 < \lambda_1 \leq \dots \leq \lambda_m$ .
- $I - \epsilon A^T A$  has eigenvalues  $1 - \epsilon \lambda_1, \dots, 1 - \epsilon \lambda_m$ .
- Need  $1 - \epsilon \lambda_m > -1$ , so that  $\|I - \epsilon A^T A\|_2 < 1$ .

**Equilibrium Distribution of a Markov Chain** Movement among the  $m$  states of a Markov chain is governed by its  $m \times m$  transition matrix  $P = (p_{ij})$ , whose entries are nonnegative and satisfy  $\sum_j p_{ij} = 1$  for all  $i$ .

A column vector  $x$  with nonnegative entries and norm  $\|x\|_1 = \sum_i x_i = 1$  is said to be an equilibrium (stationary) distribution for  $P$  provided  $x^T P = x^T$ , or equivalently  $Qx = x$  for  $Q = P^T$ .

Let  $S = \{x : x_i \geq 0, i = 1, 2, \dots, m, \sum_i x_i = 1\}$ . Assume for some power  $k$ ,  $Q^k$  has all positive entries, then consider the matrix  $R = Q^k - c\mathbf{1}\mathbf{1}^T$ .

- We have the map  $x \rightarrow Q^k x$  is contractive on  $S$  with unique fixed point  $x_\infty$ .
- $Qx_\infty = x_\infty$ .
- $\lim_{n \rightarrow \infty} Q^n x = x_\infty$  for all  $x \in S$ .

This power method is also used in [PageRank](#):

- Eldén L (2007) *Matrix Methods in Data Mining and Pattern Recognition*. SIAM, Philadelphia
- Langville AN, Meyer CD (2006) *Google's PageRank and Beyond: The Science of Search Engine Rankings*. Princeton University Press, Princeton NJ

### 0.1.5 Condition Number of a Matrix

Consider the matrix

$$A = \begin{pmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{pmatrix}$$

that is symmetric and positive definite, and  $Ax = b$ .

- for  $b = (32, 23, 33, 31)^T$ , we find  $x = (1, 1, 1, 1)^T$ ,
- for  $b + \Delta b = (32.1, 22.9, 33.1, 30.9)^T$ , the solution is violently perturbed  $x + \Delta x = (9.2, -12.6, 4.5, -1.1)^T$ ,
- for  $b = (4, 3, 3, 1)^T$ ,  $x = (1, -1, 1, -1)^T$ ,
- if we perturb  $A$  to  $A + 0.01I$ , the solution is  $x + \Delta x = (.59, -.32, .82, -.89)^T$ .

How to explain these patterns? Define the condition number

$$\text{cond}(A) = \|A\| \cdot \|A^{-1}\|$$

of matrix  $A$  relative to the given norm. We can have

•

$$\frac{\|\Delta x\|}{\|x\|} \leq \text{cond}(A) \frac{\|\Delta b\|}{\|b\|},$$

if  $Ax = b$  and  $A(x + \Delta x) = b + \Delta b$ .

•

$$\frac{\|\Delta x\|}{\|x + \Delta x\|} \leq \text{cond}(A) \frac{\|\Delta A\|}{\|A\|},$$

if  $Ax = b$  and  $(A + \Delta A)(x + \Delta x) = b$ .

The condition number  $\text{cond}_2(A)$  relative to the matrix norm  $\|A\|_2$  is the ratio of the largest and smallest eigenvalues of  $A$ , which are  $\lambda_1 = 0.01015$ ,  $\lambda_4 = 30.2887$ . Therefore,  $\text{cond}_2(A) = 2984$ .

[ ]: