MM

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1 STAT 207: The MM Algorithm

The MM algorithm

- relies on convexity arguments and is useful in high-dimensional problems such as image reconstruction.
- The first "M": majorize/minorize; the second "M": minimize/maximize depending on the problem.
- It substitutes a difficult optimization problem with a simpler one.
  - a) avoiding large matrix inversions,
  - b) linearizing the problem,
  - c) separating the variables,
  - d) dealing with equality and inequality constraints,
  - e) turning a nondifferentiable problem into a smooth problem.
- The price of simplifying the problem is iteration or iteration with slower convergence.
- The EM algorithm is a special case of the MM algorithm developed by statisticians that deals with missing data.
- Compared to other algorithms, the MM algorithm is
  - greater generality,
  - more obvious connection to convexity,
  - weaker reliance on difficult statistical principles.

1.1 Definition

A function $g(x|x_n)$ is said to **majorize** a function $f(x)$ at x_n provided

$$f(x_n) = g(x_n|x_n)f(x) \leq g(x|x_n), x \neq x_n.$$

Here x_n represents the current iterate in a search of the surface $f(x)$.

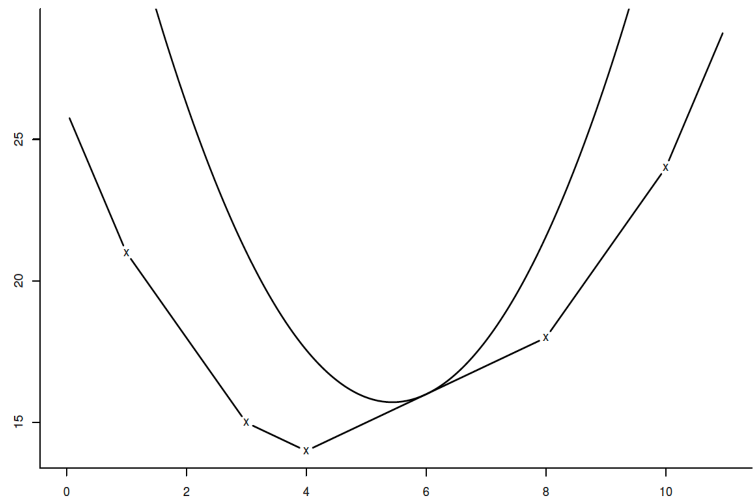


FIGURE 12.1. A Quadratic Majorizing Function for the Piecewise Linear Function $f(x) = |x-1| + |x-3| + |x-4| + |x-8| + |x-10|$ at the Point $x_n = 6$

In the **minimization** version of the MM algorithm, we minimize the surrogate majorizing function $g(x|x_n)$ rather than the actual function $f(x)$.

- If x_{n+1} denotes the minimum of the surrogate $g(x|x_n)$, then we can show that the MM procedure forces $f(x)$ downhill.

$$f(x_{n+1}) \leq g(x_{n+1}|x_n) \leq g(x_n|x_n) = f(x_n)$$

- The descent property lends the MM algorithm remarkable numerical stability.
- It depends only on decreasing the surrogate function $g(x|x_n)$, not on minimizing it.
- In practice, when the minimum of $g(x|x_n)$ cannot be found exactly.
- When $f(x)$ is strictly convex, one can show with a few additional mild hypotheses that the iterates x_n converge to the global minimum of $f(x)$ regardless of the initial point x_0.
- x_n is a stationary point of $g(x|x_n) - f(x)$, with

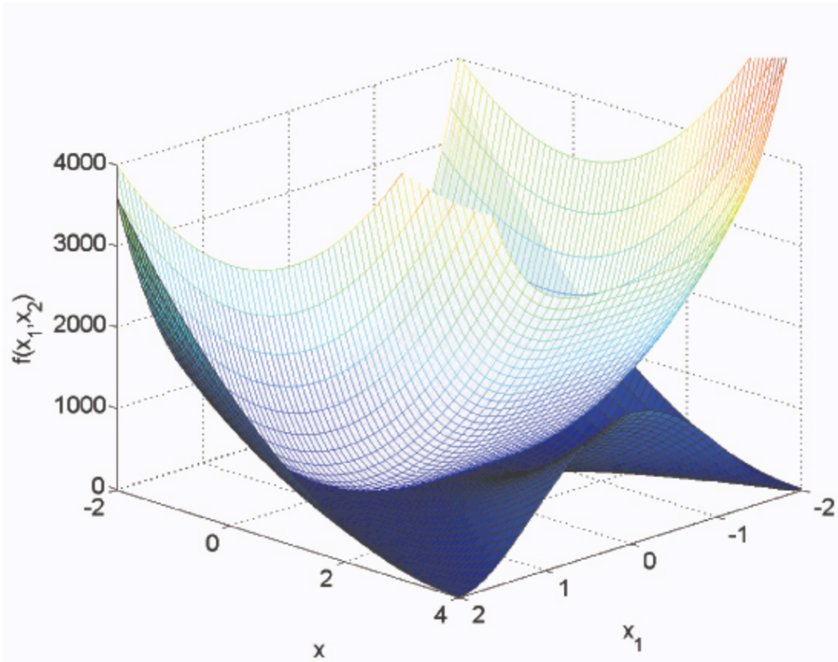
$$\nabla g(x_n|x_n) = \nabla f(x_n).$$

- The second differential $d^2 g(x_n|x_n) - d^2 f(x_n)$ is PSD.

Remarks

- The MM algorithm and the EM algorithm can be viewed as a vague philosophy for deriving an algorithm
- Examples of the value of a unifying principle and a framework for attacking concrete problems
- The strong connection of the MM algorithm to convexity and inequalities can strengthen skills in these areas

1.2 Majorizing Functions



Recall Jensen's inequality

$$f\left(\sum_{i=1}^{n}\alpha_i t_i\right) \le \sum_{i=1}^{n}\alpha_i f(t_i)$$

for any convex function $f(t)$. Apply it to f of a linear function $c^T x$,

$$f(c^T x) \le \sum_i \frac{c_i y_i}{c^T y} f\left(\frac{c^T y}{y_i} x_i\right) = g(x|y),$$

provided all components of the vectors c, x and y are positive.

- It reduces optimization over x to a sequence of one-dimensional optimizations over each component x_i.
- An alternative to relax the positivity restrictions:

  $$f(c^T x) \le \sum_i \alpha_i f\left(\frac{c_i}{\alpha_i}(x_i - y_i) + c^T y\right) = g(x|y),$$

  with $\alpha_i \ge 0, \sum_i \alpha_i = 1$ and $\alpha_i > 0$ whenever $c_i \ne 0$. For example,

  $$\alpha_i = \frac{|c_i|^p}{\sum_j |c_j|^p},$$

  for $p \ge 0$.

Linear majorization

$$f(x) \le f(y) + df(y)(x - y) = g(x|y)$$

for any concave function $f(x)$, for example, $\ln(x)$.

Assuming that $f(x)$ is twice differentiable, we look for a matrix B satisfying $B \succeq d^2 f(x)$ and $B \succ 0$. By second-order Taylor exapansion of $f(x)$ at y:

$$
\begin{aligned}
f(x) =& f(y) + df(y)(x-y) + \frac{1}{2}(x-y)^T d^2 f(z)(x-y) \\
\leq& f(y) + df(y)(x-y) + \frac{1}{2}(x-y)^T B(x-y) \\
=& g(x|y).
\end{aligned}
$$

1.2.1 Example 12.4 Linear Regression

- How MM avoids matrix inversion

1.3 MM Example: t-distribution

- A typical example of 12.5 Elliptically Symmetric Densities

Given iid data $w_1, \dots, w_n$ from multivariate t-distribution $t_p(\mu, \Sigma, \nu)$, the log-likelihood is

$$
\begin{aligned}
L(\mu, \Sigma, \nu) = & -\frac{np}{2}\log(\pi\nu) + n\left[\log\Gamma\left(\frac{\nu+p}{2}\right) - \log\Gamma\left(\frac{\nu}{2}\right)\right] - \frac{n}{2}\log\det(\Sigma) \\
& + \frac{n}{2}(\nu+p)\log\nu - \frac{\nu+p}{2}\sum_{j=1}^{n}\log\left[\nu + (w_j-\mu)^\top\Sigma^{-1}(w_j-\mu)\right].
\end{aligned}
$$

- Since $t \to -\log t$ is a convex function, we can invoke the supporting hyperplane inequality to minorize the terms $-\log[\nu + \delta(w_j, \mu; \Sigma)]$:

$$
\begin{aligned}
-\log[\nu + \delta(w_j, \mu; \Sigma)] \geq & -\log[\nu^{(t)} + \delta(w_j, \mu^{(t)}; \Sigma^{(t)})] - \frac{\nu + \delta(w_j, \mu; \Sigma) - \nu^{(t)} - \delta(w_j, \mu^{(t)}; \Sigma^{(t)})}{\nu^{(t)} + \delta(w_j, \mu^{(t)}; \Sigma^{(t)})} \\
= & -\frac{\nu + \delta(w_j, \mu; \Sigma)}{\nu^{(t)} + \delta(w_j, \mu^{(t)}; \Sigma^{(t)})} + c^{(t)},
\end{aligned}
$$

where $c^{(t)}$ is a constant irrelevant to the optimization.

- Minorization function:

$$
\begin{aligned}
g(\mu, \Sigma, \nu) = & -\frac{np}{2}\log(\pi\nu) + n\left[\log\Gamma\left(\frac{\nu+p}{2}\right) - \log\Gamma\left(\frac{\nu}{2}\right)\right] - \frac{n}{2}\log\det(\Sigma) \\
& + \frac{n}{2}(\nu+p)\log\nu - \frac{\nu+p}{2}\sum_{j=1}^{n}\frac{\nu + \delta(w_j, \mu; \Sigma)}{\nu^{(t)} + \delta(w_j, \mu^{(t)}; \Sigma^{(t)})} + c^{(t)}.
\end{aligned}
$$

HW NAS Problem 12.15 and 12.16.

1.4 MM Example: non-negative matrix factorization (NNMF)

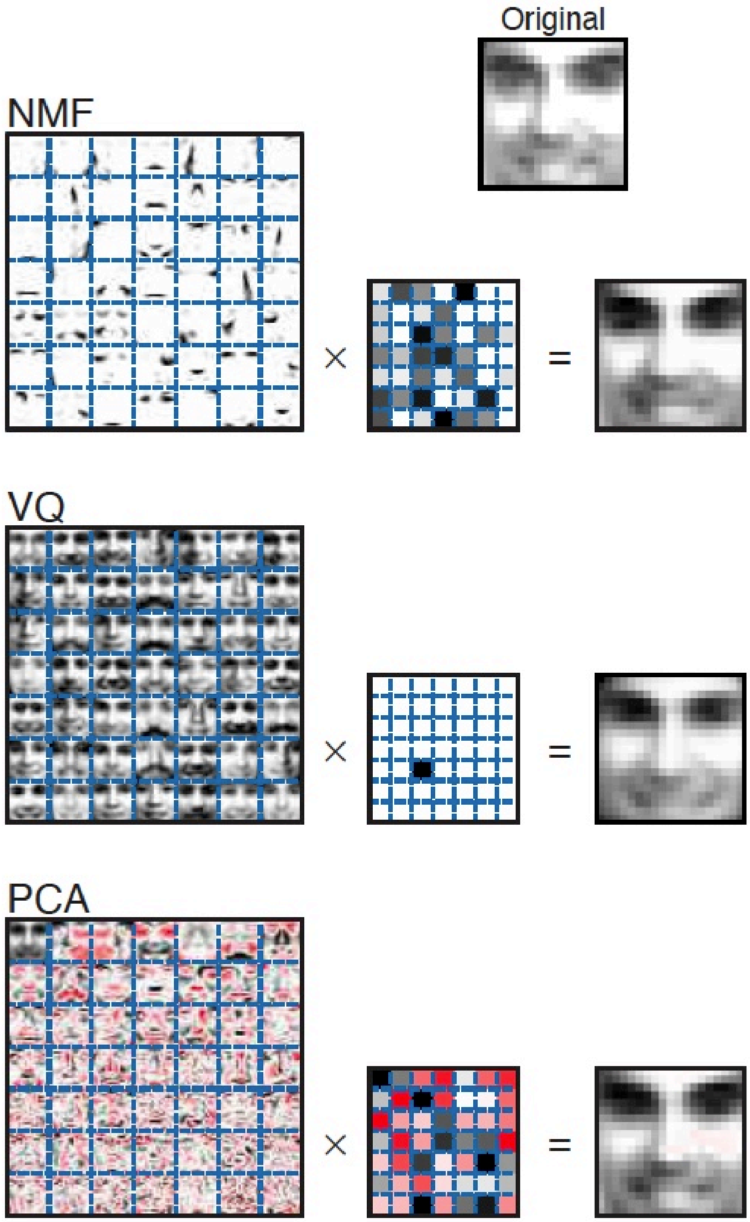


FIGURE 14.33. *Non-negative matrix factorization (NMF), vector quantization (VQ, equivalent to k-means clustering) and principal components analysis (PCA) applied to a database of facial images. Details are given in the text. Unlike VQ and PCA, NMF learns to represent faces with a set of basis images resembling parts of faces.*

- Nonnegative matrix factorization (NNMF) was introduced by Lee and Seung (1999, 2001) as an analog of principal components and vector quantization with applications in data compression and clustering.

- In mathematical terms, one approximates a data matrix $X \in \mathbb{R}^{m \times n}$ with nonnegative entries x_{ij} by a product of two low-rank matrices $V \in \mathbb{R}^{m \times r}$ and $W \in \mathbb{R}^{r \times n}$ with nonnegative entries v_{ik} and w_{kj}.

- Consider minimization of the squared Frobenius norm

$$L(V,W) = \|X - VW\|_F^2 = \sum_i \sum_j (x_{ij} - \sum_k v_{ik}w_{kj})^2, \quad v_{ik} \geq 0, w_{kj} \geq 0,$$

  which should lead to a good factorization.

- $L(V,W)$ is not convex, but bi-convex. The strategy is to alternately update V and W.
- The key is the majorization, via convexity of the function $(x_{ij} - x)^2$,

$$(x_{ij} - \sum_k v_{ik}w_{kj})^2 \leq \sum_k \frac{a_{ikj}^{(t)}}{b_{ij}^{(t)}}(x_{ij} - \frac{b_{ij}^{(t)}}{a_{ikj}^{(t)}} v_{ik}w_{kj})^2,$$

  where

$$a_{ikj}^{(t)} = v_{ik}^{(t)}w_{kj}^{(t)}, \quad b_{ij}^{(t)} = \sum_k v_{ik}^{(t)}w_{kj}^{(t)}.$$

- This suggests the alternating multiplicative updates

$$\begin{aligned} v_{i,k}^{(t+1)} &\leftarrow v_{i,k}^{(t)} \frac{\sum_j x_{i,j}w_{k,j}^{(t)}}{\sum_j b_{i,j}^{(t)}w_{k,j}^{(t)}} \\ b_{ij}^{(t+1/2)} &\leftarrow \sum_k v_{ik}^{(t+1)}w_{kj}^{(t)} \\ w_{k,j}^{(t+1)} &\leftarrow w_{k,j}^{(t)} \frac{\sum_i v_{i,k}^{(t+1)}x_{i,j}}{\sum_i v_{i,k}^{(t+1)}b_{i,j}^{(t+1/2)}} \end{aligned}$$

- The update in matrix notation is extremely simple:

$$\begin{aligned} \mathbf{B}^{(t)} &= \mathbf{V}^{(t)}\mathbf{W}^{(t)} \\ \mathbf{V}^{(t+1)} &= \mathbf{V}^{(t)} \odot \left(\mathbf{X}\mathbf{W}^{(t)^T}\right) \oslash \left(\mathbf{B}^{(t)}\mathbf{W}^{(t)^T}\right) \\ \mathbf{B}(t+\frac{1}{2}) &= \mathbf{V}^{(t+1)}\mathbf{W}^{(t)} \\ \mathbf{W}^{(t+1)} &= \mathbf{W}^{(t)} \odot \left(\mathbf{X}^T\mathbf{V}^{(t+1)}\right) \oslash \left(\mathbf{B}^{(t+\frac{1}{2})^T}\mathbf{V}(t)^T\right) \end{aligned}$$

where $\odot$ denotes elementwise multiplication and $\oslash$ denotes elementwise division. If we start with $v_{ik}, w_{kj} > 0$, parameter iterates stay positive.

1.4.1 Python Implementation

[1]:
```
import numpy as np
from PIL import Image
from sklearn.decomposition import NMF
from skimage import data
```

[2]:
```
img = data.astronaut()
img.shape
```

[2]: (512, 512, 3)

[ ]:
```python
img_show = Image.fromarray(img)
img_show
```

[ ]:

[4]:
```python
img = img_show.convert('L')
img_array = np.array(img)
img_array.shape
```

[4]: (512, 512)

[5]:
```python
# Perform non-negative matrix factorization
model = NMF(n_components=50, init='random', random_state=0)
```

```
W = model.fit_transform(img_array)
H = model.components_
```

```
/Users/feiz/miniforge3/envs/tensorflow/lib/python3.9/site-
packages/sklearn/decomposition/_nmf.py:1665: ConvergenceWarning: Maximum number
of iterations 200 reached. Increase it to improve convergence.
  warnings.warn(
```

[6]:
```
print(W.shape)
print(H.shape)
```

```
(512, 50)
(50, 512)
```

[7]:
```
# Reconstruct image from compressed data
compressed_img = np.dot(W, H)
compressed_img = np.clip(compressed_img, 0, 255).astype(np.uint8)

# Convert image back to PIL format and save
compressed_img = Image.fromarray(compressed_img.reshape(img.size))
compressed_img
```

[7]:

[8]:
```
compressed_img.save('compressed_astronaut.jpg')
```

1.5 MM example: Netflix and matrix completion

Netflix database consists of about $m \approx 10^6$ users and about $n \approx 25,000$ movies. Users rate movies; the ratings are recorded into matrix $A \in R^{m \times n}$. Only 1% (over 100 million) of the ratings are observed.

- Netflix challenge: impute the unobserved ratings for personalized recommendation. (http://en.wikipedia.org/wiki/Netflix_Prize)
- $\Omega = \{(i,j) : \text{observed entries}\}$ index the set of observed entries and $P_\Omega(\mathbf{M})$ denote the projection of matrix $\mathbf{M}$ to Ω. The problem

$$\min_{\text{rank}(\mathbf{X}) \le r} \frac{1}{2} \|P_\Omega(\mathbf{Y}) - P_\Omega(\mathbf{X})\|_F^2 = \frac{1}{2} \sum_{(i,j)\in\Omega} (y_{ij} - x_{ij})^2$$

  is non-convex and hard.
- Convex relaxation of the rank minimization problem is:

$$\min_{\mathbf{X}} f(\mathbf{X}) = \frac{1}{2} \|\mathbf{P}_\Omega(\mathbf{Y}) - \mathbf{P}_\Omega(\mathbf{X})\|_F^2 + \lambda \|\mathbf{X}\|_*,$$

  where $\|\mathbf{X}\|_* = \|\sigma(\mathbf{X})\|_1 = \sum_i \sigma_i(\mathbf{X})$ is the nuclear norm.
- **Majorization step:**

$$\begin{aligned}
f(\mathbf{X}) =& \frac{1}{2} \sum_{(i,j)\in\Omega} (y_{ij} - x_{ij})^2 + \frac{1}{2} \sum_{(i,j)\notin\Omega} 0 + \lambda \|\mathbf{X}\|_* \\
\le& \frac{1}{2} \sum_{(i,j)\in\Omega} (y_{ij} - x_{ij})^2 + \frac{1}{2} \sum_{(i,j)\notin\Omega} (x_{ij}^{(t)} - x_{ij})^2 + \lambda \|\mathbf{X}\|_* \\
=& \frac{1}{2} \|\mathbf{X} - \mathbf{Z}^{(t)}\|_F^2 + \lambda \|\mathbf{X}\|_* \\
=& g(\mathbf{X}|\mathbf{X}^{(t)}),
\end{aligned}$$

where $\mathbf{Z}^{(t)} = \mathbf{P}_\Omega(\mathbf{Y}) + \mathbf{P}_{\Omega^\perp}(\mathbf{X}^{(t)})$. (Fill in missing entries by the current imputation)

- **Minimization step:** Rewrite the surrogate function

$$g(\mathbf{X}|\mathbf{X}^{(t)}) = \frac{1}{2}\|\mathbf{X}\|_F^2 - \mathrm{tr}(\mathbf{X}^T\mathbf{Z}^{(t)}) + \frac{1}{2}\|\mathbf{Z}^{(t)}\|_F^2 + \lambda\|\mathbf{X}\|_*$$

Let σ_i be the (ordered) sigular values of $\mathbf{X}$ and ω_i be the (ordered) sigular values of $\mathbf{Z}^{(t)}$. Observe

$$\|\mathbf{X}\|_F^2 = \sum_i \sigma_i^2 \|\mathbf{Z}^{(t)}\|_F^2 = \sum_i \omega_i^2$$

and by von Neumann's inequality,

$$\mathrm{tr}(\mathbf{X}^T\mathbf{Z}^{(t)}) \leq \sum_i \sigma_i\omega_i$$

with equality achieved if and only if the left and right singular vectors of the two matrices coincide. Therefore we should choose $\mathbf{X}$ with the same singular vectors as $\mathbf{Z}^{(t)}$ and

$$\begin{aligned} g(\mathbf{X}|\mathbf{X}^{(t)}) =& \frac{1}{2}\sum_i \sigma_i^2 - \sum_i \sigma_i\omega_i + \frac{1}{2}\sum_i \omega_i^2 + \lambda\sum_i \sigma_i \\ =& \frac{1}{2}\sum_i (\sigma_i - \omega_i)^2 + \lambda\sum_i \sigma_i, \end{aligned}$$

with minimizer as

$$\sigma_i^{(t+1)} = \max(0, \omega_i - \lambda) = (\omega_i - \lambda)_+.$$

- Algorithm for matrix completion:
  - Initialize $\mathbf{X}^{(0)} \in \mathbb{R}^{m\times n}$
  - Repeat
    * $\mathbf{Z}^{(t)} \leftarrow \mathcal{P}_\Omega(\mathbf{Y}) + \mathcal{P}_{\Omega^\perp}(\mathbf{X}^{(t)})$
    * Compute SVD: $\mathbf{U}\mathrm{diag}(\mathbf{w})\mathbf{V}^\top \leftarrow \mathbf{Z}(t)$
    * $\mathbf{X}^{(t+1)} \leftarrow \mathbf{U}\mathrm{diag}[(\mathbf{w} - \lambda)_+]\mathbf{V}^\top$
  - objective value converges

[ ]: