2.NonlinearFunctions

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1 STAT 207: Solution of Nonlinear Equations

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- NAS Chapter 5

Solving linear and nonlinear equations is a major preoccupation of applied mathematics and statistics.

Three simple techniques:

- bisection,
- functional iteration,
- Newton's method.

The latter two generalize to higher-dimensional problems.

1.1 The Balance Scale Challenge:

Among 8 apples, there is one bad apple that is heavier than all others. You can use a balance scale to compare groups of objects, what is the minimum number of weighings (comparisons) needed to identify the bad apple.

1.2 Bisection

- To find solutions to the equation $g(x) = 0$.
- Does not require derivatives.
- Under minimal assumptions on $g(x)$, bisection is guaranteed to converge to some root.

Suppose that $g(x)$ is continuous, and an interval $[a, b]$ has been identified such that $g(a)$ and $g(b)$ are of opposite sign.

- If we bisect $[a, b]$ a total of n times, then the final bracketing interval has length $2^{-n}(b - a)$.
- For n large enough, we can stop and approximate the bracketed root by the midpoint of the final bracketing interval.
- If we want to locate nearly all of the roots of $g(x)$ on $[a, b]$, then we can subdivide $[a, b]$ into many small adjacent intervals and apply bisection to each small interval in turn.

1.2.1 Computation of Quantiles by Bisection

Given a continuous distribution function $F(x)$, find the α-quantile of $F(x)$ is equivalent to solving the equation

$$g(x) = F(x) - \alpha = 0.$$

To find a bracketing interval to start the process:

- take an arbitrary initial point a and examine $g(a)$,
- if $g(a) < 0$, then look for the first positive integer k with $g(a+k) > 0$. When this integer is found, the interval $[a+k-1, a+k]$ brackets the α-quantile,
- if $g(a) > 0$, then look for the first negative integer k with $g(a+k) < 0$, and $[a+k, a+k+1]$ brackets the α-quantile,

Why this k is garenteed to be found?

- a can be the mean,
- the increment can be the standard deviation of $F(x)$.

[3]:
```
import scipy.stats as stats
import math
```

[4]:
```
def t_dist_prob(t, n):
    return 1 - stats.t.cdf(t, n)

n = 5
prob = 0.95
a = 0
k = math.sqrt(n/(n-2))
k

def find_b(f, a, k):
    fa = f(a)
    if fa > 0:
        k = -k
    b = a+k
    fb = f(b)
    while fa * fb >= 0:
        b = b+k
        fb = f(b)
    return b

def bisection(f, a, b, tol=1e-6):
    fa, fb = f(a), f(b)
    assert fa * fb < 0, "f(a) and f(b) must have opposite signs"
    while b - a > tol:
        c = (a + b) / 2
        fc = f(c)
```

```
        if fc == 0:
            return c
        elif fa * fc < 0:
            b, fb = c, fc
        else:
            a, fa = c, fc
    return (a + b) / 2
```

[6]:
```
f = lambda t: stats.t.cdf(t, n) - prob ## anonymous function
n = 5
prob = 0.95
a = 0
k = math.sqrt(n/(n-2))
print(k)
b = find_b(f,a,k)
b
```

```
1.2909944487358056
```

[6]:
```
2.581988897471611
```

[4]:
```
a = b-k

t_star = bisection(f, a, b)

print("The 0.95 quantile of the t-distribution with n=5 degrees of freedom is:
  ↪", t_star)
```

```
The 0.95 quantile of the t-distribution with n=5 degrees of freedom is:
2.015048562682933
```

1.3 Functional Iteration

We are interested in finding a root of the equation $g(x) = 0$, and let $f(x) = g(x) + x$, then this equation is trivially equivalent to the equation $x = f(x)$.

In many examples, the iterates $x_n = f(x_{n-1})$ converge to a root of $g(x)$ starting from any point x_0 nearby.

A root of $g(x)$ is said to be a **fixed point** of $f(x)$.

Precise sufficient conditions for the existence of a unique fixed point of $f(x)$ and convergence to it are offered by the following proposition.

(NAS) Proposition 5.3.1 Suppose the function $f(x)$ defined on a closed interval I satisfies the conditions:

(a) $f(x) \in I$ whenever $x \in I$,

(b) $|f(y) - f(x)| \leq \lambda |y - x|$ for any two points x and y in I.

Then, provided the Lipschitz constant λ is in $[0, 1)$, $f(x)$ has a unique fixed point $x_\infty \in I$, and the functional iterates $x_n = f(x_{n-1})$ converge to x_∞ regardless of their starting point $x_0 \in I$. Furthermore, we have the precise error estimate:

$|x_n - x_\infty| \leq \frac{\lambda^n}{1-\lambda}|x_1 - x_0|$.

A function $f(x)$ having a Lipschitz constant $\lambda < 1$ is said to be *contractive*. In practice, λ is taken to be any convenient upper bound of $|f'(x)|$ on the interval I. Such a choice is valid because of the mean value equality $f(x) - f(y) = f'(z)(x - y)$, where z is some number between x and y.

- A fixed point x_∞ with $|f'(x_\infty)| < 1$ is called attractive,
  - If $f'(x_\infty) \in (-1, 0)$,
  - If $f'(x_\infty) \in (0, 1)$,
- If $|f'(x_\infty)| > 1$, x_∞ is said to be repelling.
- The case $|f'(x_\infty)| = 1$ is indeterminate.

1.3.1 Fractional Linear Transformations

Finding a real root of $x = f(x)$ by functional iteration, where

$$f(x) = \frac{ax + b}{cx + d}.$$

1.3.2 Example: Extinction Probabilities by Functional Iteration

A branching process is when particles reproduce independently at the end of each generation based on the same probabilistic law.

The probability that a particle is replaced by k daughter particles at the next generation is denoted by p_k.

The goal is to find the probability s_∞ that the process eventually goes extinct, given that it starts with a single particle at generation 0.

Conditional on the number of daughter particles k born to the initial particle, if extinction is to occur, then each line of descent emanating from a daughter particle must die out.By independence of reproduction, all k lines of descent go extinct with probability s_∞^k.

It follows that s_∞^k satisfies the functional equation

$$s = \sum_{k=0}^{\infty} p_k s^k = P(s),$$

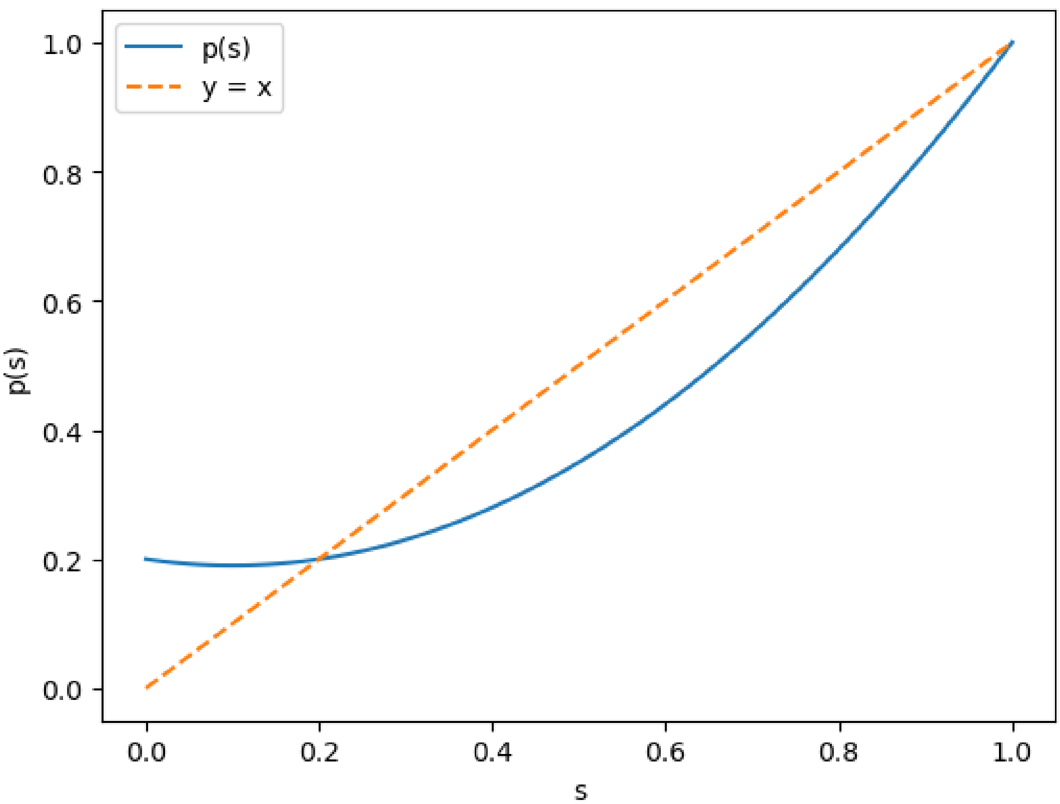
where $P(s)$ is the generating function of the progeny distribution.

Let s_n be the probability that extinction occurs in the branching process at or before generation n.

Then $s_0 = 0, s_1 = p_0 = P(s_0)$, and, in general, $s_{n+1} = P(s_n)$.

Since $P''(s) = \sum_{k=2}^{\infty} k(k-1)p_k s^{k-2} \geq 0$, the curve $P(s)$ is convex.

[5]:
```python
import numpy as np
import matplotlib.pyplot as plt

# Define p(s) as a convex function
def p(s):
    return s**2 + 0.2 - 0.2*s

# Create an array of s values between 0 and 1
s = np.linspace(0, 1, 100)

# Plot p(s) and the diagonal line y = x
plt.plot(s, p(s), label='p(s)')
plt.plot(s, s, label='y = x', linestyle='--')

# Add labels and legend
plt.xlabel('s')
plt.ylabel('p(s)')
plt.legend()

# Show the plot
plt.show()
```



There is a second intersection point to the left of $s = 1$ if and only if the slope of $P(s)$ at $s = 1$ is strictly greater than 1.

The slope $P'(1) = \sum_{k=0} kp_k$ equals the mean number of particles of the progeny distribution.

- $P'(1) \leq 1$
- $P'(1) > 1$

HW Program the functional iteration for an extinction probability of surnames among white males in the United States.

1.4 Newton's Method

Newton's method can be motivated by the mean value theorem.

Let x_{n-1} approximate the root x_∞ of the equation $g(x) = 0$.

$$g(x_{n-1}) = g(x_{n-1}) - g(x_\infty) = g'(z)(x_{n-1} - x_\infty),$$

for some z between x_{n-1} and x_∞.

We can define the update

$$x_n = x_{n-1} - \frac{g(x_{n-1})}{g'(x_{n-1})} = f(x_{n-1}),$$

by substituting x_{n-1} for z.

When does Newton's Method work?

The local convergence properties follow from

$$f'(x_\infty) =$$

Further let $e_n = x_n - x_\infty$, with a second-order Taylor expansion around x_∞,

$$e_n =$$

We have the **quadratic convergence**

$$\lim_{n\to\infty} \frac{e_n}{e_{n-1}^2} = \frac{1}{2} f''(x_\infty).$$

All else being equal, quadratic convergence is preferred to linear convergence.

The following two examples highlight favorable circumstances ensuring global convergence of Newton's method on a properly defined domain.

1.4.1 Division without Dividing

Forming the reciprocal of a number a is equivalent to solving for a root of the equation $g(x) = a - x^{-1}$. Newton's method iterates according to

$$\begin{aligned} x_n =& x_{n-1} - \frac{a - x_{n-1}^{-1}}{x_{n-1}^{-2}} \\ =& x_{n-1}(2 - a x_{n-1}), \end{aligned}$$

which involves multiplication and subtraction but no division.

- if $x_0 \in (0, 1/a)$
- if $x_0 \in [1/a, 2/a)$

1.4.2 Extinction Probabilities by Newton's Method

Newton's method starts with $x_0 = 0$ and iterates according to

$$x_n = x_{n-1} + \frac{P(x_{n-1}) - x_{n-1}}{1 - P'(x_{n-1})}$$

How it is guaranteed that no division by 0 in the Newton's iterates?

Compared to the sequence $s_n = P(s_{n-1})$ generated by functional iteration, both schemes start at 0. **Can you show by induction that (a) $x_n \leq s_\infty$, (b) $x_{n-1} \leq x_n$, and (c) $s_n \leq x_n$ hold for all $n > 0$?**

Given these properties, Newton's method typically converges much faster than functional iteration.

1.4.3 Golden Section Search

A simple numerical algorithm for minimization.

- Golden section search is reliable and applies to any continuous function $f(x)$.
- It cannot generalize to higher dimensions and its relatively slow rate of convergence.

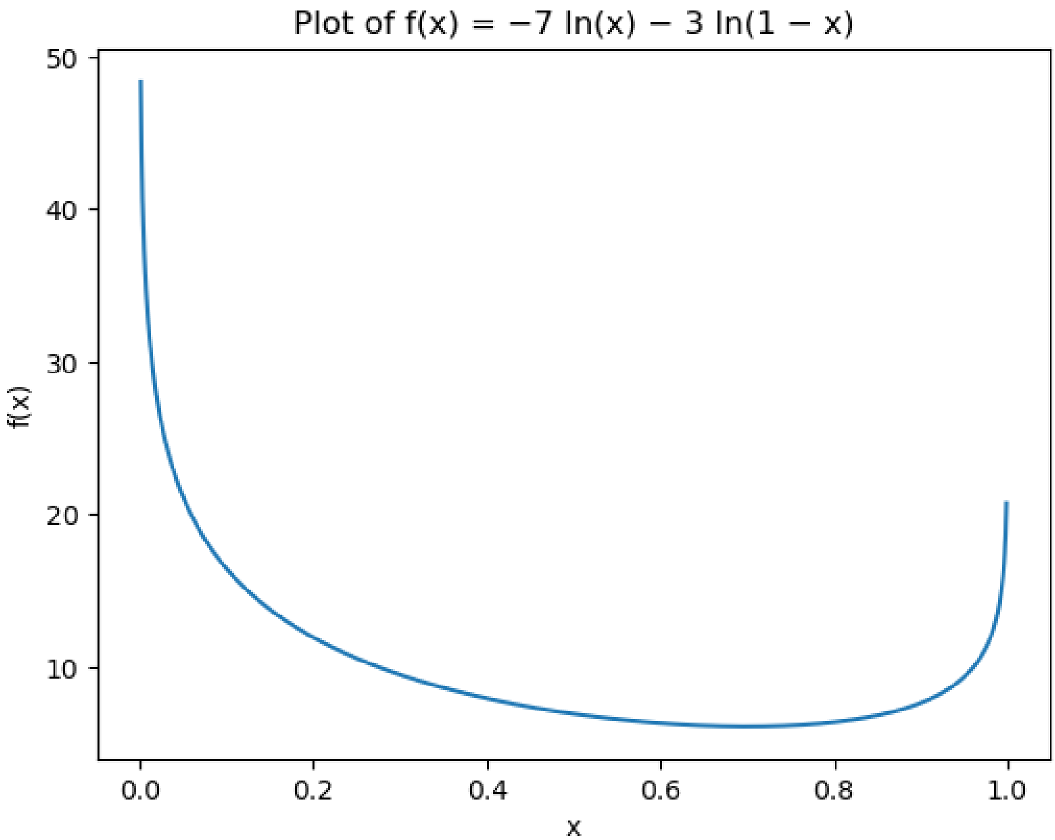
Golden section search:

- start with three points $a < b < c$ satisfying $f(b) < \min\{f(a), f(c)\}$,
- find the next bracketing interval that contains the minimum with $a < b < d$ or $b < d < c$.

Consider the function $f(x) = -7 \ln x - 3 \ln(1 - x)$.

[6]:
```
import numpy as np
import matplotlib.pyplot as plt

# Define the function to plot
def f(x):
    return -7 * np.log(x) - 3 * np.log(1 - x)

# Create an array of x values between 0 and 1
x = np.linspace(0.001, 0.999, 1000)

# Plot the function
plt.plot(x, f(x))

# Set the title and labels
plt.title('Plot of f(x) = -7 ln(x) - 3 ln(1 - x)')
plt.xlabel('x')
plt.ylabel('f(x)')
```

```
# Show the plot
plt.show()
```



[7]:
```
import math

def f(x):
    return -7 * math.log(x) - 3 * math.log(1 - x)

def golden_section_search(f, a, b, tol=1e-6):
    gr = (math.sqrt(5) + 1) / 2  # golden ratio
    c = b - (b - a) / gr
    d = a + (b - a) / gr
    while abs(c - d) > tol:
        if f(c) < f(d):
            b = d
        else:
            a = c
        c = b - (b - a) / gr
        d = a + (b - a) / gr
        print([a,c,b])
```

```
    return (b + a) / 2

# find the minimum of f(x) on [0, 1]
x_min = golden_section_search(f, 0.01, 0.99)
print("x_min =", x_min)
print("f(x_min) =", f(x_min))
```

```
[0.3843266910251031, 0.6156733089748969, 0.99]
[0.6156733089748969, 0.758653382050206, 0.99]
[0.6156733089748969, 0.7040398538493817, 0.8470199269246909]
[0.6156733089748969, 0.6702868371757214, 0.758653382050206]
[0.6702868371757214, 0.7040398538493817, 0.758653382050206]
[0.6702868371757214, 0.6911473487028853, 0.7249003653765457]
[0.6911473487028853, 0.7040398538493816, 0.7249003653765457]
[0.6911473487028853, 0.699115355083553, 0.7120078602300494]
[0.6911473487028853, 0.696071847468714, 0.7040398538493816]
[0.696071847468714, 0.6991153550835529, 0.7040398538493816]
[0.696071847468714, 0.6979528386197036, 0.7009963462345427]
[0.6979528386197036, 0.6991153550835529, 0.7009963462345427]
[0.6991153550835529, 0.6998338297706932, 0.7009963462345427]
[0.6991153550835529, 0.6995593968602621, 0.7002778715474024]
[0.6995593968602621, 0.6998338297706933, 0.7002778715474024]
[0.6998338297706933, 0.7000034386369713, 0.7002778715474024]
[0.6998338297706933, 0.6999386538148464, 0.7001082626811244]
[0.6999386538148464, 0.7000034386369712, 0.7001082626811244]
[0.6999386538148464, 0.6999786930368747, 0.7000434778589996]
[0.6999786930368747, 0.7000034386369712, 0.7000434778589996]
[0.6999786930368747, 0.6999939866588064, 0.700018732258903]
[0.6999939866588064, 0.7000034386369713, 0.700018732258903]
[0.6999939866588064, 0.6999998283025732, 0.700009280280738]
[0.6999939866588064, 0.6999975969932044, 0.7000034386369712]
[0.6999975969932044, 0.6999998283025731, 0.7000034386369712]
[0.6999975969932044, 0.6999989760182337, 0.7000012073276025]
x_min = 0.6999994021604035
f(x_min) = 6.1086430205574445
```

1.4.4 Minimization by Cubic Interpolation

Cubic interpolation offers a faster but less reliable method of minimization than golden section search.

The idea is to fit a cubic polynomial to three points in the function (or four values $f(x)$ and $f'(x)$ of two points), and then finding the minimum of that polynomial. It then replaces the worst point with the new point, and continues the process until convergence.

[8]:
```
# Define the cubic interpolation function
def cubic_interpolation(f, x0, x1, x2, tol):
    # Iterate until convergence
```

```python
    while abs(x2 - x0) > tol:
        f0, f1, f2 = f(x0), f(x1), f(x2)

        # Compute the cubic interpolation point
        a = f0
        b = (f1 - f0) / (x1 - x0)
        c = ((f2 - f0) / (x2 - x0) - b) / (x2 - x1)
        x = (x0 + x1) / 2 - b / (2 * c)

        # Evaluate the function at the new point
        fx = f(x)
        # Update the bracketing points
        if x < x1:
            if fx < f1:
                x2, x1, x0 = x1, x, x0
                f2, f1, f0 = f1, fx, f0
            else:
                x0 = x
                f0 = fx
        else:
            if fx < f1:
                x0, x1, x2 = x1, x, x2
                f0, f1, f2 = f1, fx, f2
            else:
                x2 = x
                f2 = fx
        print([x0,x1,x2])
    return x1
```

[9]:
```python
# Define initial bracketing points
x0 = 0.01
x1 = 0.5
x2 = 0.9

# Set tolerance for convergence
tol = 1e-6


x1 = cubic_interpolation(f, x0, x1, x2, tol)
f1 = f(x1)

# Print the minimum point and function value
print("Minimum point:", x1)
print("Minimum value:", f1)
```

```
[0.5, 0.685153578272752, 0.9]
[0.6690844407433396, 0.685153578272752, 0.9]
```

```
[0.685153578272752, 0.6918826232157623, 0.9]
[0.6918826232157623, 0.6957839546710956, 0.9]
[0.6957839546710956, 0.697698380103852, 0.9]
[0.697698380103852, 0.6987730421036226, 0.9]
[0.6987730421036226, 0.6993327307919798, 0.9]
[0.6993327307919798, 0.6996409149293873, 0.9]
[0.6996409149293873, 0.6998051816445571, 0.9]
[0.6998051816445571, 0.6998947858117166, 0.9]
[0.6998947858117166, 0.6999429869523967, 0.9]
[0.6999429869523967, 0.6999691667405551, 0.9]
[0.6999691667405551, 0.699983301876257, 0.9]
[0.699983301876257, 0.6999909645273977, 0.9]
[0.6999909645273977, 0.6999951080155089, 0.9]
[0.6999951080155089, 0.6999973523277518, 0.9]
[0.6999973523277518, 0.6999985666612467, 0.9]
[0.6999985666612467, 0.6999992241811399, 0.9]
[0.6999992241811399, 0.6999995800037265, 0.9]
[0.6999995800037265, 0.6999997726862445, 0.9]
[0.6999997726862445, 0.6999998769093836, 0.9]
[0.6999998769093836, 0.6999999333848749, 0.9]
[0.6999999333848749, 0.6999999638935326, 0.9]
[0.6999999638935326, 0.6999999804676557, 0.9]
[0.6999999804676557, 0.6999999889199975, 0.9]
[0.6999999889199975, 0.699999995635209, 0.9]
[0.699999995635209, 0.6999999957205442, 0.9]
[0.699999995635209, 0.6999999957205442, 0.7000001311439525]
Minimum point: 0.6999999957205442
Minimum value: 6.108643020548935
```

1.4.5 Stopping Criteria

In solving a nonlinear equation $g(x) = 0$, one can declare convergence when

- $|g(x_n)|$ is small or
- x_n does not change much from one iteration to the next

```
[ ]:
```