HW1

April 10, 2023

1 STAT 207 HW1

1.1 Your Name

All homeworks should be completed independently; make your answers and codes as concise as possible; avoid excessive outputs; submit BOTH your source code and output file to Canvas.

1.2 1. NAS Problem 1.1

Let f_n be the number of subsets of $\{1, ..., n\}$ that do not contain two consecutive integers. Show that $f_1 = 2$, $f_2 = 3$, and $f_n = f_{n-1} + f_{n-2}$ for $n > 2$.

1.3 2. NAS Problem 1.4

Prove that the characteristic polynomial $p_n(x) = \det(M - xI)$ of the $n \times n$ tridiagonal matrix

$$M = \begin{bmatrix} b_1 & c_2 & 0 & \cdots & 0 & 0 \\ a_2 & b_2 & c_3 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & a_{n-2} & b_{n-2} & c_{n-1} & 0 \\ 0 & \cdots & 0 & a_{n-1} & b_{n-1} & c_n \\ 0 & \cdots & 0 & 0 & a_n & b_n \end{bmatrix}$$

can be computed recursively by defining

$$\begin{aligned} p_0(x) &= 1 \\ p_1(x) &= (b_1 - x) \\ p_m(x) &= (b_m - x)p_{m-1}(x) - a_m c_m p_{m-2}(x), \quad m = 2, 3, \ldots, n. \end{aligned}$$

Why are the roots of $p_n(x)$ real in the symmetric case $a_m = c_m$ for all m? Devise a related recurrence to calculate the derivative $p_n'(x)$.

1.4 3. NAS Problem 1.10

Demonstrate that the example in class with integral y_n defined below

$$y_n = \int_0^1 \frac{x^n}{x+a} dx$$

can be expanded in the infinite series

$$y_n = \sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k+1)a^{k+1}}.$$

when $a > 1$. This does provide a reasonably stable method of computing y_n for large a. Program this solution and find suitable sequence k for n.

1.5 4. NAS Problem 5.8

For $0 \le a \le 4$, the function $f(x) = ax(1-x)$ maps the unit interval $[0,1]$ onto itself. Show that:

(a) The point 0 is a fixed point that is globally attractive when $a \le 1$ and locally repelling when $a > 1$. Note that the rate of convergence to 0 is less than geometric when $a = 1$.

(b) The point $1 - a^{-1}$ is a fixed point for $a > 1$. It is locally attractive when $1 < a < 3$ and locally repelling when $3 < a \le 4$.

(c) For $1 < a \le 2$, the fixed point $r = 1 - a^{-1}$ is globally attractive on $(0,1)$. (Hint: Write $f(x) - r = (x-r)(1-ax)$.)

1.6 5. Functional Iteration

1.6.1 5.1 Lotka data

Program the functional iteration for an extinction probability of surnames among white males in the United States, with the following progeny generating function from the 1920 census data,

$P(s) = .4982 + .2103s + .1270s^2 + .0730s^3 + .0418s^4 + .0241s^5 + .0132s^6 + .0069s^7 + .0035s^8 + .0015s^9 + .0005s^{10}$.

What is the order of the number of operations at each iteration in your program?

1.6.2 5.2

Functional iteration can often be accelerated. In searching for a fixed point of $x = f(x)$, consider the iteration scheme $x_n = f_\alpha(x_{n-1})$, where α is some constant and $f_\alpha(x) = (1-\alpha)x + \alpha f(x)$. Prove that any fixed point x_1 of $f(x)$ is also a fixed point of $f_\alpha(x)$ and vice versa. Since $|f'_\alpha(x_1)|$ determines the rate of convergence of $x_n = f_\alpha(x_{n-1})$ to x_∞, find the α that minimizes $|f'_\alpha(x_\infty)|$ when $|f'(x_\infty)| < 1$. Unfortunately, neither x_∞ nor $f'(x_\infty)$ is typically known in advance.

1.6.3 5.3

Use this idea to investigate numerically the following iteration

$$x_n = (1-\alpha)x_{n-1} + \alpha P(X_{n-1})$$

with $\alpha = 1/[1 - P'(0)]$ in the Lotka data. Is the convergence faster?

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