# STAT 207: Advanced Optimization Topics

In general,

- **unconstrained** optimization problems are easier to solve than **constrained** optimization problems,
- **equality constrained** problems are easier to solve than **inequality constrained** problems

## Inequality constraint - interior point method

- We consider the constrained optimization

$$\min f_0(x)$$

subject to

$$\begin{aligned} f_j(x) &\leq 0, \quad 1 \leq j \leq m; \\ Ax &= b, \end{aligned}$$

where $f_0, \ldots, f_m : R^n \to R$ are convex and twice continuously differentiable, and $A$ as full row rank.

- Assume the problem is solvable with optimal point $\mathbf{x}^*$ and optimal value $f_0(\mathbf{x}^*) = p^*$.
- KKT conditions:

$$\begin{aligned} \mathbf{A}\mathbf{x}^* = \mathbf{b},\ f_i(\mathbf{x}^*) \leq 0, i = 1, \ldots, m \quad &\text{(primal feasibility)} \\ \lambda_i^* \geq 0, i = 1, \ldots, m \\ \nabla f_0(\mathbf{x}^*) + \sum_{i=1}^m \lambda_i^* \nabla f_i(\mathbf{x}^*) + \mathbf{A}^T \boldsymbol{\nu}^* = 0 \quad &\text{(dual feasibility)} \\ \lambda_i^* f_i(\mathbf{x}^*) = 0, \quad i = 1, \ldots, m. \end{aligned}$$

### Barrier method¶

- Convert the problem to implicitly include the inequality constraints in the objective and minimize

$$f_0(\mathbf{x}) + \sum_{i=1}^m I_-(f_i(\mathbf{x}))$$

subject to $\mathbf{A}\mathbf{x} = \mathbf{b}$, where

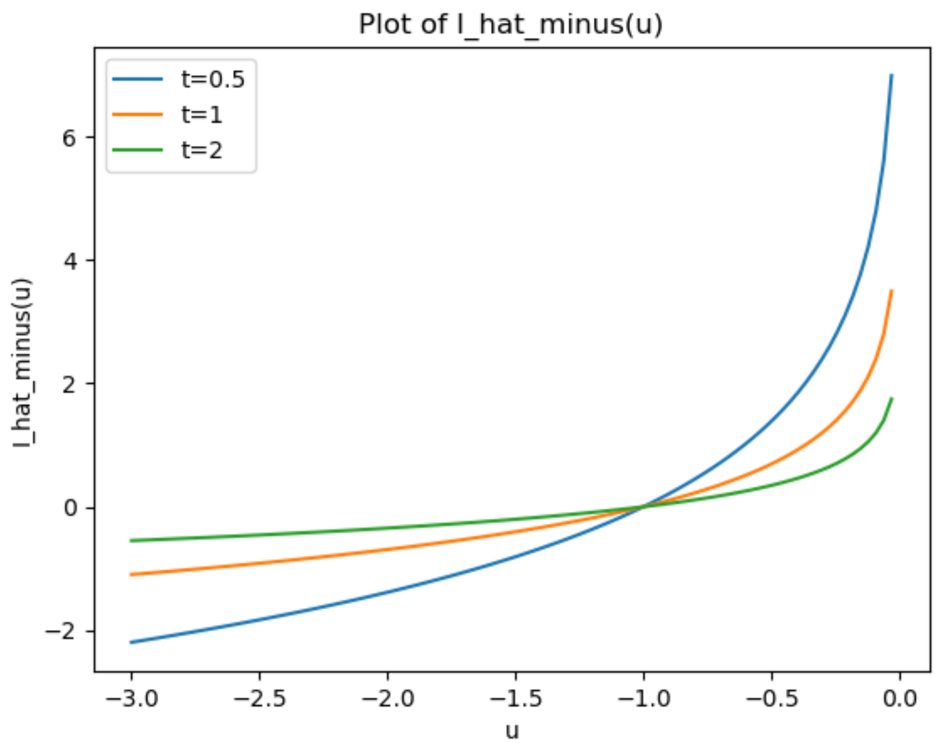
$$I_-(u) = \begin{cases} 0 & \text{if } u \leq 0 \\ \infty & \text{if } u > 0 \end{cases}.$$

- And to approximate $I_-$ by a differentiable function

$$\hat{I}_-(u) = -(1/t)\log(-u), \quad u < 0,$$

where $t > 0$ is a parameter tuning the approximation accuracy. As $t$ increases, the approximation becomes more accurate.

In [3]:
```python
import numpy as np
import matplotlib.pyplot as plt
import warnings

# Ignore the specific warning
warnings.filterwarnings("ignore", category=RuntimeWarning)

def I_hat_minus(u, t):
    return -(1/t) * np.log(-u)

u = np.linspace(-3, 0, 100)
t_values = [0.5, 1, 2]

for t in t_values:
    plt.plot(u, I_hat_minus(u, t), label=f't={t}')

plt.xlabel('u')
plt.ylabel('I_hat_minus(u)')
plt.legend()
plt.title('Plot of I_hat_minus(u)')
plt.grid(False)
plt.show()
```




- The barrier method solves a sequence of equality-constraint problems

$$\begin{aligned} \min \quad & t f_0(\mathbf{x}) - \sum_{i=1}^m \log(-f_i(\mathbf{x})) \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \end{aligned}$$

with increased parameter $t$ at each step and starting each Newton minimization at the solution for the previous value of $t$.

- The function $\phi(\mathbf{x}) = -\sum_{i=1}^m \log(-f_i(\mathbf{x}))$ is called the **logarithmic barrier** or **log barrier** function.
- Denote the solution at $t$ by $\mathbf{x}^*(t)$. Using the duality theory, we can show

$$f_0(\mathbf{x}^*(t)) - p^* \leq m/t.$$

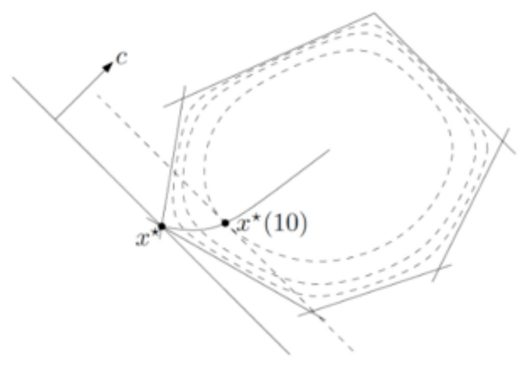



**Figure 11.2** Central path for an LP with $n = 2$ and $m = 6$. The dashed curves show three contour lines of the logarithmic barrier function $\phi$. The central path converges to the optimal point $x^*$ as $t \to \infty$. Also shown is the point on the central path with $t = 10$. The optimality condition (11.9) at this point can be verified geometrically: The line $c^T x = c^T x^*(10)$ is tangent to the contour line of $\phi$ through $x^*(10)$.

\begin{center}
\includegraphics[width=0.8\textwidth]{barrier.jpg}
\end{center}

- Barrier method has to start from a **strictly feasible point**. We can find such a point by solving

$$\min s$$

subject to

$$\begin{aligned} f_j(x) &\leq s, \quad 1 \leq j \leq m; \\ Ax &= b, \end{aligned}$$

by the barrier method.

## Penalty Method

- Unlike the barrier method that works from the interior of the feasible region, the penalty method works from the outside of the feasible region inward.
- Construct a continuous nonnegative penalty $p(x)$ that is 0 on the feasible region and positive outside it.
- Optimize

$$f_0(x) + \lambda_k p(x)$$

for an increasing sequence $\lambda_k$.

**Example** (Linear Regression with Linear Constraints) Consider the regression problem of minimizing $\|Y - X\beta\|_2^2$ subject to the linear constraints $V\beta = d$. If we take the penalty function $p(\beta) = \|V\beta - d\|_2^2$, then we must minimize at each stage the function

$$h_k(\beta) = \|Y - X\beta\|_2^2 + \lambda_k \|V\beta - d\|_2^2.$$

Setting the gradient

$$\nabla h_k(\beta) = -2X^T(Y - X\beta) + 2\lambda_k V^T(V\beta - d) = 0$$

yields the sequence of solutions

$$\beta_k = (X^T X + \lambda_k V^T V)^{-1}(X^T Y + \lambda_k V^T d).$$

- Ascent and descent properties of the penalty and barrier methods.

**Proposition 16.2.1**: Consider two real-valued functions $f(x)$ and $g(x)$ on a common domain and two positive constants $\alpha < \omega$. Suppose the linear combination $f(x) + \alpha g(x)$ attains its minimum value at $y$, and the linear combination $f(x) + \omega g(x)$ attains its minimum value at $z$. Then, $f(y) \leq f(z)$ and $g(y) \geq g(z)$.

- Global convergence for the penalty method.

**Proposition 16.2.2** Suppose that both the objective function $f(x)$ and the penalty function $p(x)$ are continuous on $\mathbb{R}^m$, and the penalized functions $h_k(x) = f(x) + \lambda_k p(x)$ are coercive on $\mathbb{R}^m$. Then, one can extract a corresponding sequence of minimum points $x_k$ such that $f(x_k) \leq f(x_{k+1})$. Furthermore, any cluster point of this sequence resides in the feasible region $C = \{x : p(x) = 0\}$ and attains the minimum value of $f(x)$ within $C$. Finally, if $f(x)$ is coercive and possesses a unique minimum point in $C$, then the sequence $x_k$ converges to that point.

- Global convergence for the barrier method.

**Proposition 16.2.3** Suppose the real-valued function $f(x)$ is continuous on the bounded open set $U$ and its closure $V$. Additionally, suppose the barrier function $b(x)$ is continuous and coercive on $U$. If the tuning constants $\mu_k$ decrease to 0, then the linear combinations $h_k(x) = f(x) + \mu_k b(x)$ attain their minima at a sequence of points $x_k$ in $U$ satisfying the descent property $f(x_{k+1}) \leq f(x_k)$. Furthermore, any cluster point of the sequence furnishes the minimum value of $f(x)$ on $V$. If the minimum point of $f(x)$ in $V$ is unique, then the sequence $x_k$ converges to this point.

- Possible defects of of the penalty and barrier methods:
  - iterations within iterations
  - choosing the tuning parameter sequence
  - numerical instability

In [4]:
```python
## jupyter nbconvert --TagRemovePreprocessor.remove_input_tags='{"hide_code"}' --to pdf Opt_top.ipynb
```