New extended formulation and column generation-based solution method for the scheduling problem with reversible and non-reversible energy sources

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1 New extended formulation and Dantzig-Wolfe decomposition

The problem definition requires a discrete time horizon $\mathcal{T} = \{1, \ldots, |\mathcal{T}|\}$, a set of preemptive activities $\mathcal{A} = \{1, \ldots, n\}$, each activity i having a duration p_i, a constant energy demand of b_i per time unit and a time window that starts at r_i and ends at d_i.

One reversible energy source and one non-reversible energy source are available to satisfy the power demand at each instant. The non-reversible source has a non decreasing piecewise linear efficiency function ρ, i.e. a cost or energy consumption of x produces an amount of usable energy of $\rho(x)$. In other words, $(\rho)^{-1}(x)$ is required from the non reversible source to satisfy a demand of x. The pwl function has a set of linear pieces $\mathcal{P}$ and a set of breakpoints $\mathcal{K}$. Each breakpoint of the piecewise linear function has a value o_k on the x-axis (demand) and c_k on the y-axis (cost). We assume that the pwl function is discontinuous at each breakpoint, i.e that ρ is discontinuous and $|\mathcal{K}| = 2|\mathcal{P}|$. In the special case where the pwl function is continuous on a given breakpoint, we duplicate this breakpoint so as to obtain $|\mathcal{K}| = 2|\mathcal{P}|$, as illustrated on figure 1. Let $\kappa_1(p)$ and $\kappa_2(p)$ denote the two breakpoints that are extremities of the line-segment $p \in \mathcal{P}$, and let $\mathfrak{p}(k)$ be the line-segment to which breakpoint k belongs. The reversible energy source has an ideal efficiency function (no energy losses), and the amount of energy stored inside at any instant is called the state of charge. The initial state of charge is denoted q_{init} and at each instant the state of charge must remain between two limits $q_{\min}$ and $q_{\max}$. The objective is to schedule the tasks so as to minimize the total energy consumption on the non-reversible energy source while ensuring that the final state of charge of the reversible source is greater of equal to its initial state of charge.

The presence of a piecewise linear function and the resulting additional variables in the mathematical model can result into a weak linear relaxation. To counter that, we propose a set partitioning-based reformulation with a purely linear objective-function. This formulation is based on the identification of sets of feasible tasks subsets per breakpoint of the piecewise linear function. A *feasible subset* is a set of activities that can be in progress simultaneously without exceeding any resource availability. Let l be a feasible subset and k a breakpoint. The set of activities belonging to l is denoted $\mathcal{A}_l (\subseteq \mathcal{A})$. Let $\mathcal{L}$ be the set of all l. Each pair $l \in \mathcal{L}, k \in \mathcal{K}$ is assigned a release date $r_l = \max_{i \in \mathcal{A}_l} r_i$, a due date $d_l = \min_{i \in \mathcal{A}_l} d_i$, an energy demand from tasks $b_l = \sum_{i \in \mathcal{A}_l} b_i$, an energy amount g^{in}_{lk} sent to the reversible source, an energy amount g^{out}_{lk} produced by the reversible source, and an energy cost $c_k = \rho^{-1}(o_k) = \rho^{-1}(b_l + g^{\text{in}}_{lk} - g^{\text{out}}_{lk})$ on the non-reversible source. The reversible source can not produce and receive energy simultaneously, i.e $g^{\text{in}}_{lk} * g^{\text{out}}_{lk} = 0$. The set of activity sets executable at instant t is denoted $\mathcal{L}_t$, and is composed of subsets l that verify $r_l \leq t < d_l$. The set of breakpoints compatible with the activity set l is denoted $k \in \mathcal{K}_t$, and is composed of breakpoints that verify: $o_{\kappa_2(p)} - b_l \geq -q_{\max}$ if $k = \kappa_1(p)$, or $o_{\kappa_1(p)} - b_l \leq q_{\max}$ if $k = \kappa_2(p)$. Finally, constant term a_{il} with $i \in \mathcal{A}$ and $l \in \mathcal{L}$ is equal to 1 if task i belongs to set $\mathcal{A}$ and 0 otherwise.

The following decision variables are needed for the extended mathematical formulation:

- x_{it}: binary, equal to 1 if task $i \in \mathcal{A}$ is executed at time period $t \in \mathcal{T}$

- y_{lkt}: continuous, positive, will be equal to the weight assigned to the column corresponding to the set of tasks $l \in \mathcal{L}$, the breakpoint $k \in \mathcal{K}$ and the time period $t \in \mathcal{T}$
- z_{pt}: binary, equal to 1 if the p^{th} linear piece of the pwl function is used at time period $t \in \mathcal{T}$ to satisfy the energy demand from tasks
- q_t: continuous, positive, equal to the state-of-charge of the reversible source at time period $t \in \mathcal{T}$
- $y_{\emptyset kt}$: continuous, positive. Corresponds to the case where no task is being executed at time period $t \in \mathcal{T}$, but the non-reversible energy source produces an amount of energy o_k that is stored into the reversible source, with $k \in \mathcal{K}$
- $z^{\emptyset}_{pt}$: binary, equal to 1 if the p^{th} linear piece of the piecewise linear function is used at time period $t \in \mathcal{T}$ but no task is being executed at that time period

An extended formulation of (P) is:

$$(\text{P}) \quad \min \sum_{t\in\mathcal{T}}\sum_{l\in\mathcal{L}_t}\sum_{k\in\mathcal{K}_l} c_k y_{lkt} + \sum_{t\in\mathcal{T}}\sum_{k\in\mathcal{K}_l} c_k y_{\emptyset kt} \tag{1}$$

s.t.

$$x_{it} - \sum_{l\in\mathcal{L}_t}\sum_{k\in\mathcal{K}_l} a_{il} y_{lkt} = 0, \qquad \forall i \in \mathcal{A}, \forall t \in \mathcal{T} \tag{2}$$

$$\sum_{t\in\mathcal{T}}\sum_{l\in\mathcal{L}_t}\sum_{k\in\mathcal{K}_l} a_{il} y_{lkt} \geq p_i, \qquad \forall i \in \mathcal{A} \tag{3}$$

$$z_{pt} - \sum_{l\in\mathcal{L}_t} y_{l,\kappa_1(p),t} - \sum_{l\in\mathcal{L}_t} y_{l,\kappa_2(p),t} = 0, \qquad \forall p \in \mathcal{P}, \forall t \in \mathcal{T} \tag{4}$$

$$z^{\emptyset}_{pt} - y_{\emptyset,\kappa_1(p),t} - y_{\emptyset,\kappa_2(p),t} = 0, \qquad \forall p \in \mathcal{P}, \forall t \in \mathcal{T} \tag{5}$$

$$\sum_{p\in\mathcal{P}} z_{pt} + \sum_{p\in\mathcal{P}} z^{\emptyset}_{pt} \leq 1, \qquad \forall t \in \mathcal{T} \tag{6}$$

$$q_0 = q_{\text{init}} \tag{7}$$

$$q_{|\mathcal{T}|} - q_0 \geq 0 \tag{8}$$

$$q_t \geq q_{\min}, \qquad \forall t \in \mathcal{T} \tag{9}$$

$$q_t \leq q_{\max}, \qquad \forall t \in \mathcal{T} \tag{10}$$

$$q_{t+1} - q_t + \sum_{l\in\mathcal{L}_t}\sum_{k\in\mathcal{K}_l} (g^{\text{out}}_{lk} - g^{\text{in}}_{lk}) y_{lkt} - g^{\text{in}}_{lk} y_{\emptyset,kt} = 0, \qquad \forall t \in \mathcal{T} \tag{11}$$

$$\sum_{l\in\mathcal{L}_t}\sum_{k\in\mathcal{K}_l} (o_k - g^{\text{in}}_{lk} + g^{\text{out}}_{lk}) y_{lkt} - \sum_{i\in\mathcal{A}} b_i x_{it} = 0, \qquad \forall t \in \mathcal{T} \tag{12}$$

$$x_{it} \in \{0,1\}, \qquad \forall i \in \mathcal{A}, t \in \mathcal{T} \tag{13}$$

$$z_{pt}, z^{\emptyset}_{pt} \in \{0,1\}, \qquad \forall p \in \mathcal{P}, t \in \mathcal{T} \tag{14}$$

$$y_{lkt} \geq 0, \qquad t \in \mathcal{T}, \forall l \in \mathcal{L} \cup \{\emptyset\}, \forall k \in \mathcal{K} \tag{15}$$

$$y_{\emptyset,k,t} \geq 0, \qquad t \in \mathcal{T}, \forall k \in \mathcal{K} \tag{16}$$

$$q_t \geq 0, \qquad t \in \mathcal{T} \tag{17}$$

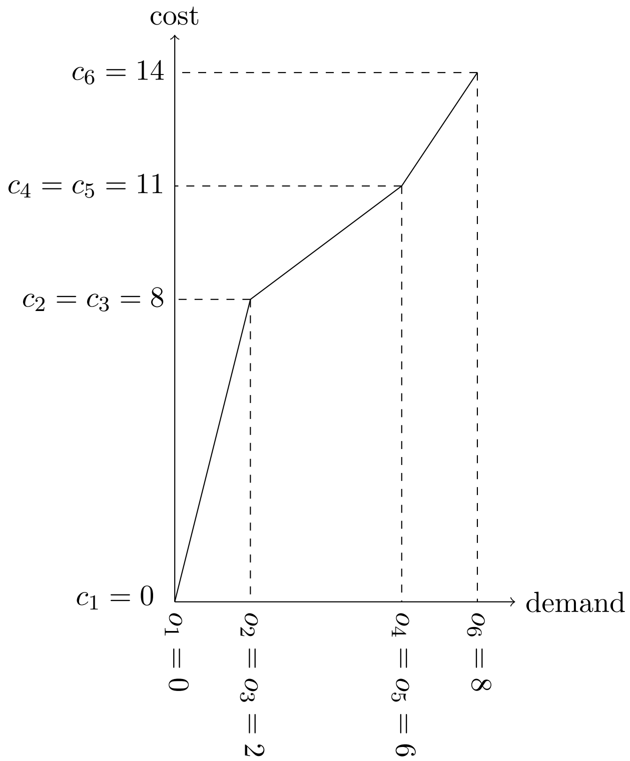
The objective function (1) aims at minimizing the total energy cost on the non reversible source. Constraints (2) link variables x and y. Constraints (3) ensure that the duration of each activity is satisfied. Constraints (4) link variables y and z. Constraints (5) link variables $y_{\emptyset}$ and $z^{\emptyset}$. Constraints (6) ensure that only one linear piece of the piecewise linear function is chosen at each instant. Constraint (7) sets the initial state of charge of the reversible source. Constraint (8) impose a final state of charge higher or equal to the initial state of charge. Constraints (11) computes the state of charge of time period $t+1$ in function of the state of charge of time period t and the amount of energy in and out of the reversible source. Constraints (12) ensures that the total power demand of tasks being executed is being satisfied with the columns selected. Finally, constraints (13), (14), (15), (16) and (17) specify the domain of each variable.

This formulation has a polynomial of constraints, a polynomial number of binary variables x and z, but an exponential number of continuous variables y, therefore its linear relaxation is suitable for a column generation solution method and the problem can be solved with branch-and-price.

The linear relaxation of (EF) serves as the master problem (MP) of the DantzigWolfe decomposition. At each iteration α of a column generation we solve a restricted master problem (RMPα) obtained by restricting $\mathcal{L}, \mathcal{K}$ to a subset of activity sets and breakpoints $\mathcal{L}_\alpha, \mathcal{K}_\alpha \in \mathcal{L}, \mathcal{K}$, then try to generate one or several pairs "activity set,breakpoint" of negative reduced cost using the dual variables values of the solution found. If no such pair can be generated, then the current solution is optimal for the current linear relaxation (MP). Otherwise, the best pairs with negative reduced cost are added to $\mathcal{L}_\alpha$ to obtain $\mathcal{L}_{\alpha+1}$and the new master problem (RMP$\alpha + 1$) is solved.

Theorem 1 *The extended formulation (EF) dominates the compact formulation (CF), i.e. the linear relaxation of (EF) is stronger than the linear relaxation of (CF).*

Proof: Any solution of the linear relaxation of (EF) can be converted into a solution of the linear relaxation of (CF). Therefore the linear relaxation of (EF) is at least as strong as the linear relaxation of (CF). It remains to be proven that there exist solutions of (EF) that are not feasible for (CF), in order to prove that the two linear relaxations are not just equivalent. To do so, we exhibit hereafter such a case, with $|\mathcal{A}| = 1, |\mathcal{T}| = 1, b_i = 4, r_i = 1, d_i = 2, q_{\max} = 1.5, q_{\text{init}} = 0$ and the pwl function ρ illustrated the following figure:



In this case, for the extended formulation (EF), only one feasible set of tasks $l = \{1\}$ exists, therefore $\mathcal{L} = \{\{1\}\}$. Each column y_{lkt} of formulation (EF) corresponds to a pair "set of tasks l, breakpoint k" that verify $o_{\kappa_2(p)} - b_l \geq -q_{\max}$ if $k = \kappa_1(p)$, or $o_{\kappa_1(p)} - b_l \leq q_{\max}$ if $k = \kappa_2(p)$. Therefore only two columns exist in (EF): "$l = 1, k = 3$" and "$l = 1, k = 4$". The resulting linear relaxation is: $x_{1,1} = 1.0, y_{1,2,1} = y_{1,3,1} = 0.5, z_2 = 1.0$ and $y_{1,1,1} = y_{1,2,1} = y_{1,5,1} = y_{1,6,1} = z_1 = z_3 = 0$. Its costs is equal to 9.5. Meanwhile, the linear relaxation of (CF) produces the fractional solution of cost 7 ($x_{1,1} = 1.0, z_{1,1} = z_{3,1} = 0.5, w_{3,1} = 4, w_{1,1} = 0$ and $w_{3,1} = z_{2,1} = 0$)

In conclusion, the linear relaxation of (EF) is stronger than the linear relaxation of (CF). This also proves that the subproblem is NP-hard.

$\square$

Let $\alpha_{it}, \beta_i, \gamma_{\mathfrak{p}(k)t}, \eta_{pt}, \theta_t, \sigma, \rho, \mu_t, \nu_t, \delta_t, \phi_t, \lambda_{it}, \omega_{pt}$ and $\omega^{\emptyset}_{pt}$ be the dual variables associated to constraints (2)-(14). The dual of the linear relaxation can be formulated as:

$$\text{(DRP)} \quad \max \quad \sum_{i\in\mathcal{A}} p_i\beta_i + \sum_{t\in\mathcal{T}} \theta_t + q_{\text{init}}\sigma + \sum_{t\in\mathcal{T}} q_{\min}\mu_t + \sum_{t\in\mathcal{T}} q_{\max}\nu_t + \sum_{i\in\mathcal{A}}\sum_{i\in\mathcal{T}} \lambda_{it} \tag{18}$$

s.t.

$$\alpha_{it} - b_i\phi_t \leq 0, \qquad \forall i \in \mathcal{A}, \forall t \in \mathcal{T} \tag{19}$$

$$\sum_{i\in\mathcal{A}} a_{il}(-\alpha_{it} + \beta_i) - \gamma_{\mathfrak{p}(k),t} + (g^{\text{out}}_{lk} - g^{\text{in}}_{lk})(\delta_t + \phi_t) + o_k\phi_t \leq c_k, \qquad \forall l \in \mathcal{L}, \forall k \in \mathcal{K}, \forall t \in \mathcal{T} \tag{20}$$

$$\gamma_{pt} + \theta_t + \omega_{pt} \leq 0, \qquad \forall p \in \mathcal{P}, \forall t \in \mathcal{T} \tag{21}$$

$$\sigma - \rho \leq 0, \tag{22}$$

$$\mu_t + \nu_t \leq 0, \qquad \forall t \in \mathcal{T} \setminus \{|\mathcal{T}|\} \tag{23}$$

$$\mu_{|\mathcal{T}|} + \nu_{|\mathcal{T}|} + \sigma \leq 0, \tag{24}$$

$$\eta_{\mathfrak{p}(k)t} - \delta_t \leq 0, \qquad \forall k \in \mathcal{K}, \forall t \in \mathcal{T} \tag{25}$$

$$\eta_{pt} + \theta_t + \omega^{\emptyset}_{pt}, \qquad \forall p \in \mathcal{P}, \forall t \in \mathcal{T} \tag{26}$$

$$\alpha_{it} \in \mathbb{R}, \lambda_{it} \leq 0, \qquad \forall i \in \mathcal{A}, \forall t \in \mathcal{T} \tag{27}$$

$$\beta_i \geq 0, \qquad \forall i \in \mathcal{A} \tag{28}$$

$$\gamma_{pt} \in \mathbb{R}, \eta_{pt} \in \mathbb{R}, \omega_{pt} \leq 0, \omega^{\emptyset}_{pt} \leq 0, \qquad \forall p \in \mathcal{P}, \forall t \in \mathcal{T} \tag{29}$$

$$\theta_t \leq 0, \mu_t \geq 0, \nu_t \leq 0, \delta_t \in \mathbb{R}, \phi_t \in \mathbb{R}, \qquad \forall t \in \mathcal{T} \tag{30}$$

$$\sigma \in \mathbb{R}, \rho \geq 0, \tag{31}$$

where constraints (20) are the dual constraints associated to variables y_{lkt}.

Any column y_{lkt} missing from (EF) corresponds to a constraint (20) missing from its dual. Therefore, identifying columns of negative reduced cost is equivalent to identifying missing violated inequalities (20). As a consequence, the subproblem resolution (also called pricing procedure) consists in building an activity set $\widetilde{l}$, a breakpoint $\widetilde{p}$ and identifying an instant-time $\widetilde{t}$ that maximizes the difference between the left-hand-side and the right-hand-side of inequality (20). If the difference is strictly positive, then the corresponding column $y_{\widetilde{l}\widetilde{k}\widetilde{t}}$ is introduced into the model. Otherwise it can be disregarded and the current optimal solution of (EF) is the optimal solution of the current restricted master problem.

1.1 Subproblem SP1$_{\widetilde{t}}$: with fixed t

Generating the columns of negative reduced cost consists in building the triplet $(\widetilde{l}, \widetilde{k}, \widetilde{t})$ that maximizes the violation of constraints (20). Based on the idea that predefining $\widetilde{t}$ leads to a subproblem easier to solve, SP1$_{\widetilde{t}}$ inputs the current dual variables values, but also a predefined time $\widetilde{t}$, and then outputs the best activity set $\widetilde{l}$ and breakpoint $\widetilde{k}$ obtained by solving the following model based on binary decision variables u_i and v_k equal to one for each of the chosen task i or chosen breakpoint k:

$$(\text{SP1}_{\widetilde{t}}) \quad \max \quad \textstyle\sum_{i\in\mathcal{A}} (\beta_i - \alpha_{it} + \delta_t b_i + \phi_t b_i)\, u_i - \sum_{k\in\mathcal{K}} \left(c_k + \gamma_{\mathfrak{p}(k),t} + o_k\delta_t\right) v_k \tag{32}$$

s.t.

$$\sum_{i\in\mathcal{A}} b_i u_i - \sum_{k\in\mathcal{K}} o_{\kappa_2(\mathfrak{p}(k)+1)} v_k \le q_{\max} \tag{33}$$

$$\sum_{i\in\mathcal{A}} b_i u_i - \sum_{k\in\mathcal{K}} o_{\kappa_2(\mathfrak{p}(k)-1)} v_k \ge q_{\max} \tag{34}$$

$$\sum_{k\in\mathcal{K}} v_k = 1 \tag{35}$$

$$u_i \le a_{i\widetilde{t}}, \qquad i \in \mathcal{A} \tag{36}$$

$$u_i \in \{0,1\}, \qquad i \in \mathcal{A} \tag{37}$$

$$v_k \in \{0,1\}, \qquad k \in \mathcal{K}. \tag{38}$$

SP1$_{\widetilde{t}}$ is variant of the knapsack problem where :

- $|\mathcal{A}|$ items are available, each with a weight b_i and a profit $\beta_i - \alpha_{it} + \delta_t b_i + \phi_t b_i$
- $|\mathcal{K}|$ knapsacks are available, each with a fixed cost $c_k + \gamma_{\mathfrak{p}(k),t} + o_k\delta_t$ and a capacity $q_{\max} + o_{\kappa_2(\mathfrak{p}(k)+1)}$.
- the objective is to choose one knapsack and a set of items to store into the knapsack, so as to maximize the total profit minus the knapsack cost whist respecting the capacity of the knapsack

To the best of our knowledge this problem has not been studied in the literature, but without constraint (35), it would be equivalent to the fixed-charge multiple knapsack problem (FCMK) studied by [Yamada and Takeoka, 2009].

1.2 Subproblem SP2$_{\widetilde{k},\widetilde{t}}$: with fixed t and k

SP2$_{\widetilde{k}\widetilde{t}}$ inputs the current dual variables values, a predefined time $\widetilde{t}$, and a predefined breakpoint $\widetilde{k}$, then outputs the best activity set $\widetilde{l}$ obtained by solving the following model

$$(\mathrm{SP2}_{\widetilde{\mathrm{k}}\widetilde{\mathrm{t}}}) \quad \max \quad \textstyle\sum_{i\in\mathcal{A}}(\beta_i - \alpha_{it})u_i + (\delta_t + \phi_t)\sum_{i\in\mathcal{A}} b_i u_i - \big(c_k + \gamma_{\mathfrak{p}(k),t} + o_k\delta_t\big) \tag{39}$$

s.t.

$$o_{\kappa_2(\mathfrak{p}(k)-1)} - q_{\max} \le \sum_{i\in\mathcal{A}} b_i u_i \le q_{\max} + o_{\kappa_2(\mathfrak{p}(k)+1)}, \tag{40}$$

$$u_i \le a_{i\widetilde{t}}, \qquad i \in \mathcal{A} \tag{41}$$

$$u_i \in \{0,1\}, \qquad i \in \mathcal{A}. \tag{42}$$

The objective function (39) maximizes the violation of constraint (20). Constraint (40) ensures that the total energy demand of the activity set is within the limits of the corresponding breakpoint. Constraints (41) forbid the selection of activities with a time window outside of time $\widetilde{t}$. Finally constraints (42) impose the domain of decision variables u_i. This formulation has a fully linear objective function, a polynomial number of variables and constraints. It is therefore suitable for a black box MILP solver. Note that the term $c_k+\gamma_{\mathfrak{p}(k),t}+o_k\delta_t$ is constant for each subproblem. Therefore the resulting problem is equivalent to a classical binary knapsack problem with an additional constraint imposing a minimum total weight $q_{\min}$. Each object has a profit $\beta_i - \alpha_{it} + \delta_t b_i + \phi_t b_i$. The capacity of the knapsack is $q_{\max}$.

References

[Yamada and Takeoka, 2009] Yamada, T. and Takeoka, T. (2009). An exact algorithm for the fixed-charge multiple knapsack problem. *European Journal of Operational Research*, 192(2):700 – 705.