Recent developments in robust portfolios with a worst-case approach

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Robust models have a major role in portfolio optimization for resolving the sensitivity issue of the classical mean-variance model. In this paper, we survey developments of worst-case optimization while focusing on approaches for constructing robust portfolios. In addition to the robust formulations for the Markowitz model, we review work on deriving robust counterparts for value-at-risk and conditional value-at-risk problems as well as methods for combining uncertainty in factor models. Recent findings on properties of robust portfolios are introduced and we conclude by presenting our thoughts on future research directions.

Keywords: mean-variance model, robust portfolio, worst-case optimization, uncertainty sets

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1. Introduction

The mean-variance model introduced by Markowitz [1, 2] is one of the most important models in portfolio optimization and also the basis for asset allocation. Although it is one of the earliest optimization models for portfolio selection, improvements and variations of the model are continually being researched. Furthermore, it is still accepted as the basic theoretical framework for practitioners mainly because of its intuitive approach based on the risk-return tradeoff. However, due to its simplicity, there are several well-known documented shortcomings of the model. Even though conceptually optimal, portfolios constructed from the mean-variance model often have non-intuitive or extreme weights that cannot be carried out in active trading such as investment positions with large negative weights. As a resolution, although additional constraints such as no-shorting positions are imposed, it likely results in the construction of optimal portfolios very close to the boundary of the restrictions and therefore becomes dependent on the conditions imposed on allocation weights [3]. In addition to the inherent concerns caused from solving an optimizing problem, mean-variance optimization has problems due to the random nature of asset returns. In fact, the most documented issue with the Markowitz model is its high sensitivity to the inputs. Michaud [4] argues that risk and return estimates are subject to estimation error and even refers to the model as *estimation-error maximizers* because securities with large estimated returns, negative correlations, and small variances are over-weighted in the model even though these are in fact the securities with large estimation errors. He summarizes the main issues as the use of sample mean, the ignorance of market factors, and the assumption of a single optimal portfolio. Broadie [5] analyzes the effect of estimation errors by comparing the estimated frontier with the actual frontier, which is computed from the true parameter values for the same portfolio weights as the estimated efficient frontier, and the true frontier, which is an unrealistic case that uses the true values for finding the optimal portfolio. He points out that a portfolio on the estimated efficient frontier is likely to have better performance on the mean-variance scale compared to the corresponding portfolios on the actual frontier and the true frontier showing how the mean-variance model over-estimates the optimal portfolio. More importantly, Broadie surprisingly finds support for the inefficiency of mean-variance portfolios by noticing that the actual frontier lies below the true frontier. Moreover, Chopra and Ziemba [6] examine the effect of estimation errors in means, variances, and covariances of security returns on portfolio performance individually and conclude that errors in estimating expected returns of securities is at least 10 times more important than the error in estimated variances or covariances. For a more detailed analysis on estimation errors in mean security returns, Best and Grauer [7, 8] investigate the sensitivity of mean-variance optimal portfolios to changes in the individual return of securities. They find that even a small change in the estimated mean of a single security results in a relatively large adjustment in the corresponding portfolio weights. The unified concern from these reviews of the mean-variance model is that it should be used with extreme caution.

As an approach to resolve the sensitivity issue of the Markowitz model, robust portfolio optimization has been proposed by many researchers. In this paper, we survey developments in robust portfolios with emphasis on recent progress. Among several methods to increase robustness in portfolio performance, we focus on the worst-case optimization approach based on the mean-variance model. There has been considerable research comparing the return and risk of the two approaches because although optimizing a portfolio at its worst scenario may result in increased robustness, it may be overly conservative to show matching performance with its corresponding mean-variance portfolio [9-11]. However, since there has not been a conclusive finding regarding performance, in this survey we mainly review work on formulating robust problems along with a few recent developments on the properties of robust portfolios.

The organization of the paper follows a chronological order beginning from the introduction of robust optimization even before its application to portfolio allocation, which is briefly summarized in Section 2. In Section 3, earlier work on robust formulations for portfolio optimization is presented, and Section 4 defines uncertainty sets developed on the inputs of the mean-variance model. Section 5 introduces recent robust formulations that incorporate additional risk measures. Robust methods that combine factor models, which are suggested to improve earlier and later robust models, are listed in Section 6. Section 7 summarizes recent attempts to identify properties of robust portfolios. We conclude the paper in Section 8 by proposing future research directions.

2. Robust optimization

Prior to the establishment of robust optimization, randomness in parameters was often framed using stochastic programming [12]. Further robust methods such as robust estimators, resampling techniques, and Bayesian approaches were also developed to improve stability [13]. In this paper, we focus on robust models with worst-case optimization approaches that optimize the objective within a predefined uncertainty set.

The notion of uncertainty set for parameters was first mentioned by Soyster [14] as he defines an inexact linear programming problem where the true value is no longer certain but known to lie within a convex set. Later, the use of uncertainty sets was formally extended to introducing a robust formulation when El Ghaoui and Lebret [15] discuss the least-squares problem where the parameters are unknown-but-bounded matrices. In their work, a problem for finding the worst-case residual is described, which they refer to as the robust least-squares solution, and show how this solution can be computed using second-order cone or semidefinite programming. The idea of integrating unknown-but-bounded parameters into semidefinite programming is further studied by El Ghaoui, Oustry, and Lebret [16]. Their work includes formulations for robust semidefinite programming and demonstrates sufficient conditions that guarantee the existence of robust solutions. Similarly, Ben-Tal and Nemirovski [17] look for applicable uncertainty sets that provide computationally tractable robust formulations for linear programming problems and point out that ellipsoidal uncertainty sets formulate the original uncertain linear programming problems to robust conic quadratic programs that can be efficiently solved by interior-point methods. In a separate study, they expand the domain to generic convex optimization problems and show how ellipsoidal uncertainty sets result in tractable robust convex problems [18]. In a comparable work, Goldfarb and Iyengar [19] search for uncertainty sets with a focus on finding robust counterparts as second-order cone programs.

As briefly listed, the major developments in robust optimization in its earlier phase were led by research to derive robust formulations and to structure uncertainty sets for finding robust counterparts.[1] With the establishment of the basic robust framework, this technique has been applied to many areas with portfolio optimization being just one example.

3. Robust models for portfolio optimization

Adopting the mean and covariance matrix of returns as measures of portfolio return and risk, respectively, Markowitz [1, 2] finds the optimal portfolio for an investor's risk appetite represented by $\lambda \in \mathbb{R}$ by solving the following optimization problem for n risky assets,

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[1] For more complete references on robust optimization, see Bertimas, Brown, and Caramanis [20] and Ben-Tal, El Ghaoui, and Nemirovski [21].

$$\min_{x} \mu' x - \lambda x' \Sigma x$$

$$\text{s.t. } x' \iota = 1$$

where $\mu \in \mathbb{R}^n$ is the expected return, $\Sigma \in \mathbb{R}^n \times \mathbb{R}^n$ is the covariance matrix, $\iota \in \mathbb{R}^n$ is a vector of ones, and $x \in \mathbb{R}^n$ is the portfolio weights (which sum to one due to the equality constraint). Note that a higher value of λ indicates investors with greater risk-aversion. In addition, since the covariance matrix of asset returns is always positive semidefinite, the above is a convex quadratic programming problem. Under this framework, robust portfolios are portfolios that are relatively stable with respect to changes in μ and Σ. In other words, it is the portfolio that performs well even under unfavorable scenarios and is computed by finding the portfolio with the best performance in the worst possible case. Formulations introduced from now on assume that the distributions of returns are uncertain but are known to have partial information for structuring the uncertainty set.

Among numerous studies on robust optimization, as summarized in Section 2, Lobo and Boyd [22] provide an introduction to robust portfolio formulations, listing many uncertainty sets applicable to asset returns. Furthermore, Halldórsson and Tütüncü [23] consider an example of robust portfolio optimization in their investigation of saddle point problems. They introduce a robust formulation of the mean-variance model that focuses on optimizing the portfolio in the worst-case,

$$\max_{x \in \mathcal{X}} \left\{ \min_{(\mu, \Sigma) \in \mathcal{U}} \mu' x - \lambda x' \Sigma x \right\} \qquad (1)$$

where $\mathcal{X}$ is the set of feasible portfolios and $\mathcal{U}$ is the uncertainty set for the expected return and covariance matrix of asset returns. The set $\mathcal{X}$ usually contains portfolios with weights that sum to one with constraints on portfolio weights often imposed. An extensive definition of robust portfolio selection problems is provided by Goldfarb and Iyengar [24]. They specify

- the robust minimum variance optimization problem for a lower limit on portfolio return μ_p,[2]

$$\min_{x \in \mathcal{X}} \max_{\Sigma \in \mathcal{U}_\Sigma} x' \Sigma x \qquad (2)$$

[2] Here, the uncertainty of the covariance matrix is represented as a set $\mathcal{U}_\Sigma$, but a factor model approach is used in the original work for defining the uncertainty set. This is further described in Section 6.

$$\text{s.t.} \ \min_{\mu \in \mathcal{U}_\mu} \mu' x \geq \mu_p$$

- the robust maximum return optimization problem for an upper limit on portfolio variance σ_p^2,

$$\max_{x \in \mathcal{X}} \ \min_{\mu \in \mathcal{U}_\mu} \ \mu' x$$

$$\text{s.t.} \ \max_{\Sigma \in \mathcal{U}_\Sigma} x' \Sigma x \leq \sigma_p^2$$

- the robust maximum Sharpe ratio problem with risk-free rate r_f,

$$\max_{x \in \mathcal{X}} \ \min_{\{\mu \in \mathcal{U}_\mu, \Sigma \in \mathcal{U}_\Sigma\}} \frac{\mu' x - r_f}{\sqrt{x' \Sigma x}}$$

and show their reduction to second-order cone problems with the assumption of a factor model. Moreover, Tütüncü and Koenig [25] show the equivalence between the robust mean-variance problem (1) and the robust minimum variance problem (2). They also prove that the robust maximum Sharpe ratio problem is tractable even when the no-shorting constraint assumed by Goldfarb and Iyengar is removed. Costa and Paiva [26] apply polytope uncertainty sets to the expected return and covariance matrix of returns to consider robust tracking-error optimization problems and show that they can be solved as linear-matrix inequalities optimization problems. Revisiting the inefficiency of the actual frontiers relative to the estimated frontiers first noted by Broadie [5], Ceria and Stubbs [9] derive a robust model for reducing the discrepancy caused by the mean-variance model. In addition to robust versions of the basic minimum variance and maximum return optimization, they introduce models with more practical use in actual portfolio management.

Most of the formulations presented here are combined with box or ellipsoidal uncertainty sets on either the expected return or the covariance matrix of returns for finding robust portfolios. We review the approaches in the next section. Fabozzi *et al*. [13, 27] provide a thorough review on robust portfolio optimization.

4. Defining uncertainty sets

One of the most important aspects of robust optimization is defining the proper structure of uncertainty sets that will provide tractable solutions. Even though there has been intensive study on the geometry of uncertainty sets that are suitable for various optimization problems, we only cover a few examples that are widely used in portfolio optimization,[3] and leave interested readers to review Bertsimas, Brown, and Caramanis [20] along with Ben-Tal, El Ghaoui, and Nemirovski [21] and references therein.

4.1 Uncertainty in expected return

Since the effect of estimation error in portfolio performance is known be greater for the expected return of assets, there have been more studies focusing on the uncertainty for the expected returns. Even though we know that the future asset returns are unlikely to be the estimate of the expected returns $\hat{\mu}$ from past data, we can still predict that they will not be too far from the estimated value. Therefore, uncertainty sets for the expected asset returns describe a geometric structure around the estimate $\hat{\mu}$.

The simplest uncertainty set that defines possible intervals for each asset, also known as the box uncertainty set, is written as

$$\mathcal{U}_\delta(\hat{\mu}) = \{\mu : |\mu_i - \hat{\mu}_i| \leq \delta_i, i = 1, \dots, n\}$$

where the estimation error for asset i is less than δ_i and therefore δ_i is the value that sets the size of the confidence region for each asset. Applying this uncertainty to the robust equation (1) with only the constraint that the weights sum to unity, the max-min problem

$$\max_x \left\{ \min_{\mu \in \mathcal{U}_\delta(\hat{\mu})} \mu' x - \lambda x' \Sigma x \right\}$$

$$\text{s.t. } x'\iota = 1$$

can be expressed as maximization problem,

$$\max_x \hat{\mu}' x - \delta' |x| - \lambda x' \Sigma x$$

$$\text{s.t. } x'\iota = 1 \; .$$

The most often used structure for tractability is an ellipsoidal set that measures the combined deviation of all assets,

$$\mathcal{U}_\delta(\hat{\mu}) = \left\{ \mu : (\mu - \hat{\mu})' \Sigma_\mu^{-1} (\mu - \hat{\mu}) \leq \delta^2 \right\}$$

[3] The structure of uncertainty sets mentioned in this section are described in further detail in Lobo and Boyd [22] and Fabozzi *et al*. [13, 27].

where Σ_μ is the covariance matrix of estimation errors for the expected asset returns.[4] Again, solving the inner problem of (1) with unity constraint and the ellipsoidal uncertainty set gives,

$$\max_x \hat{\mu}' x - \lambda x' \Sigma x - \delta \sqrt{x' \Sigma_\mu x}$$

$$\text{s.t. } x' \iota = 1 \ .$$

For empirical analysis, asset returns are often assumed to follow a normal distribution for finding the desirable confidence interval for the box uncertainty while the χ^2 distribution with the number of assets as the degree of freedom is used for setting the confidence region for the ellipsoidal model.

4.2 Uncertainty in covariance matrix

Similar to the box uncertainty set for the expected returns, specifying confidence intervals with lower and upper bounds are also often used for the covariance matrix of asset returns,

$$\mathcal{U}_\Sigma = \left\{ \Sigma : \ \underline{\Sigma} \leq \Sigma \leq \overline{\Sigma} \right\}$$

For example, Tütüncü and Koenig [25] show that when the uncertainty set is defined as below,

$$\mathcal{U} = \left\{ (\mu, \Sigma) : \ \mu \in \mathcal{U}_\mu, \Sigma \in \mathcal{U}_\Sigma \right\}$$

where $\mathcal{U}_\mu = \left\{ \mu : \underline{\mu} \leq \mu \leq \overline{\mu} \right\}$, then solving equation (1) with non-negativity constraints on portfolio weights is equivalent to solving the following maximization problem with boundary values,

$$\max_{x \in \mathcal{X}} \ \underline{\mu}' x - \lambda x' \overline{\Sigma} x \ .$$

Factor models are used to define the ambiguity in the covariance matrix of returns and we explain this method in Section 6.

4.3 Joint uncertainty set

As shown above, the vector of expected returns and covariance matrix of returns both can be modeled to be uncertain. But recently, Lu [29] refers to these approaches as using *separable* uncertainty sets and emphasizes that it might result in conservative or non-diversified portfolios

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[4] Stubbs and Vance [28] cover ways to empirically approximate the matrix Σ_μ.

because individual uncertainty sets do not fully contain information on joint behavior. Hence, he proposes a joint ellipsoidal uncertainty set where the desired confidence level can also be computed.[5]

5. Incorporating additional risk measures

While we have already covered the sensitivity issue of the mean-variance model, another criticism of the model that is crucial for investors is the use of variance for portfolio risk because not all investors perceive the upside and downside of fluctuations the same; downside fluctuations may lead to significant loss whereas upside movements can enhance the expected performance of portfolios. Naturally, portfolio optimization problems incorporating other risk measures have been studied and robust strategies have also been developed to resolve the two main issues of the mean-variance model. Here, we introduce robust models that optimize the worst-case value-at-risk (VaR) and conditional value-at-risk (CVaR) of portfolios.[6]

5.1 Worst-case VaR

Goldfarb and Iyengar [24] briefly discuss robust VaR selection problems as they find a robust alternative as a second-order cone programming problem. In addition, El Ghaoui, Oks, and Oustry [33] focus on worst-case VaR and argue that an equivalent tractable semidefinite program exists. In their work, VaR is defined as the smallest value γ such that the probability of loss exceeding γ is at most $1-\varepsilon$,[7]

$$
\begin{aligned}
VaR_{\varepsilon}(x) &= \min\ \{\gamma :\ \text{Prob}(\gamma \leq -r'x) \leq 1-\varepsilon\} \\
&= \min\ \{\gamma :\ \text{Prob}(\gamma \geq -r'x) \geq \varepsilon\} \\
&= \min \left\{\gamma :\ \int_{\gamma \geq -r'x} p(r)dr \geq \varepsilon \right\}
\end{aligned}
$$

where $\varepsilon \in (0,1]$, r is the random return vector that is known to have probability density $p(r)$,

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[5] We suggest the series of papers by Lu [29-31] for more detail.

[6] Fabozzi, Huang, and Zhou [32] also provide a comprehensive review on methods for minimizing worst-case VaR and CVaR.

[7] In our definition, we solve for probability at most $1-\varepsilon$, instead of ε to be consistent with formulations discussed in the following sections.

and general loss is computed by the negative portfolio return $-r'x$.[8] Although the entire distribution must be known to compute VaR, it is a more accurate risk measure for distributions that are heavy-tailed, which is often the case for the returns of financial assets. They extend the above definition to find the worst-case VaR with respect to a set of probability distributions $\mathcal{P}$,

$$WVaR_{\varepsilon}(x) = \min\left\{\gamma : \sup_{p(r)\in\mathcal{P}} \text{Prob}(\gamma \leq -r'x) \leq 1 - \varepsilon\right\}.$$

Consequently, the corresponding minimization problem is solved for a robust solution,

$$\min_{x\in\mathcal{X}} WVaR_{\varepsilon}(x).$$

Furthermore, El Ghaoui, Oks, and Oustry prove that the worst-case VaR with level ε being less than γ is equivalent to the proposition that for every $\mu \in \mathbb{R}^n$ such that

$$\begin{bmatrix} \Sigma & (\mu - \hat{\mu}) \\ (\mu - \hat{\mu})' & \kappa(\varepsilon)^2 \end{bmatrix} \succcurlyeq 0\,,$$

then $-\mu'x \leq \gamma$ is true where $\kappa(\varepsilon) = \sqrt{\frac{\varepsilon}{1-\varepsilon}}$. In addition, the proposition is also analogous to saying that there exists an $n \times n$ symmetric matrix Δ and $v \in \mathbb{R}$ such that

$$\langle \Delta, \Sigma \rangle + \kappa(\varepsilon)^2 v - \hat{\mu}'x \leq \gamma\,, \quad \begin{bmatrix} \Delta & x/2 \\ x'/2 & v \end{bmatrix} \succcurlyeq 0\,.$$

where $\langle \Delta, \Sigma \rangle = Tr(\Delta\Sigma)$. Therefore, with the assumption that the mean and covariance matrix of asset returns is known to belong to a given set without exact knowledge of the probability distributions, they derive the following robust formulation,

$$\min_{x\in\mathcal{X}} \max_{(\mu,\hat{\mu},\Sigma)} -\mu'x$$

$$\text{s.t. } (\hat{\mu}, \Sigma) \in \mathcal{U}, \qquad \begin{bmatrix} \Sigma & (\mu - \hat{\mu}) \\ (\mu - \hat{\mu})' & \kappa(\varepsilon)^2 \end{bmatrix} \succcurlyeq 0$$

where $\mathcal{U}$ is the set of possible values for $\hat{\mu}$ and Σ. Alternatively, robustness can be achieved by the min-max problem,

$$\min_{(x,v,\Delta)} \max_{(\hat{\mu},\Sigma)} \langle \Delta, \Sigma \rangle + \kappa(\varepsilon)^2 v - \hat{\mu}'x$$

$$\text{s.t. } (\hat{\mu}, \Sigma) \in \mathcal{U}, \qquad \Sigma \succcurlyeq 0, \qquad x \in \mathcal{X}$$

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[8] Portfolio loss is often written in the literature as a function of x and r, $f(x,r)$, but here we denote it by $-r'x$ since it is most likely computed as the negative portfolio return in most cases.

$$\begin{bmatrix} \Delta & x/2 \\ x'/2 & v \end{bmatrix} \succcurlyeq 0 \ .$$

They conclude that the structure of $\mathcal{X}$ and $\mathcal{U}$ determine the tractability of the robust formulation, and specifically state that it becomes a semidefinite programing problem if the two sets are expressed by linear inequalities in x, $\hat{\mu}$, and Σ. As examples, they discuss cases when the pair $(\hat{\mu}, \Sigma)$ belongs to a polytope as well as when $\hat{\mu}$ and Σ are each bounded in an interval.

Introducing an extra layer of uncertainty, Huang, Fabozzi, and Fukushima [34] disregard the assumption on deterministic exit time of investments. In reality, there may be unexpected events in the stock market or exogenous personal reason to withdraw from or rebalance existing asset holdings. Hence, they develop a model for both the uncertainty in the distribution of exit time as well as the uncertainty in the return distribution conditional on exit time. They look into the discrete case of exit time τ partly because possible exiting moments are not continuous in reality. Thus, for discrete points of time $\{t_1, t_2, \dots, t_m\}$ with probability of exit,

$$\text{Prob}(\tau = t_i) = \lambda_i \quad \text{where} \quad \textstyle\sum_{i=1}^{m} \lambda_i = 1,\ \lambda_i \geq 0 \quad \text{for} \quad i = 1, \dots, m$$

the unconditional mean and covariance of returns is,

$$\hat{\mu} = \sum_{i=1}^{m} \lambda_i \hat{\mu}_i \ , \qquad \Sigma = \sum_{i=1}^{m} \lambda_i \Sigma_i$$

where $\hat{\mu}_i$ and Σ_i are the conditional mean and covariance of returns at time t_i, respectively. Now let Λ contain all possible values of λ_i with some information on the distribution of exit time and $\mathcal{U}_i$ contain all possible mean and covariance matrix of conditional distribution on exit time $\tau = t_i$. Then the robust problem with uncertain exit time becomes,

$$\min_{x,v,\Delta} \ \beta$$

$$\text{s.t.} \ \max_{\lambda \in \Lambda} \max_{(\hat{\mu}_i, \Sigma_i) \in \mathcal{U}_i} \sum_{i=1}^{m} \lambda_i [\langle \Delta, \Sigma_i \rangle - \hat{\mu}_i' x] + \kappa(\varepsilon)^2 v \leq \beta$$

$$x \in \mathcal{X}, \quad \begin{bmatrix} \Delta & x/2 \\ x'/2 & v \end{bmatrix} \succcurlyeq 0 \ .$$

Similar to El Ghaoui, Oks, and Oustry [33], Huang, Fabozzi, and Fukushima [34] illustrate when the pair $(\hat{\mu}, \Sigma)$ belongs to a polytope for each exit time with the distribution of exit time either in an interval or a semi-ellipsoid to show its reduction to semidefinite programming problems.

5.2 Worst-case conditional value-at-risk

Even though VaR became popular to deal with the deficiency of the variance as a risk measure due to symmetric penalization on both the negative and the positive direction, it has its own shortcomings as a risk measure. As Rockafeller and Uryasev [35, 36], among others, point out, VaR is not a coherent risk measure [37] and it does not reflect any information on the extreme points beyond the threshold value. The latter is just as important as the former because stock markets often experience extreme left-tail events. In contrast, conditional value-at-risk (CVaR) is more sensitive to left-tail outliers since it takes the expected value of loss that exceeds the value of $VaR_\varepsilon(x)$. Moreover, CVaR is a coherent risk measure, and it is defined as,

$$CVaR_\varepsilon(x) = \frac{1}{1-\varepsilon} \int_{-r'x \geq VaR_\varepsilon(x)} (-r'x)p(r)dr \,.$$

Furthermore, Rockafeller and Uryasev prove that $CVaR_\varepsilon(x)$ can be rewritten as

$$CVaR_\varepsilon(x) = \min_\gamma F_\varepsilon(x, \gamma)$$

which becomes a trivial problem since $F_\varepsilon(x, \gamma)$ is convex and continuously differentiable,

$$F_\varepsilon(x, \gamma) = \gamma + \frac{1}{1-\varepsilon} \int_{r \in \mathbb{R}^n} [-r'x - \gamma]^+ \, p(r)dr$$

where $[a]^+ = \max\{a, 0\}$. Consequently, they formulate the minimization of CVaR as,

$$\min_{x \in \mathcal{X}} CVaR_\varepsilon(x) = \min_{(x,\gamma)} F_\varepsilon(x, \gamma) \,.$$

The above minimization formula allows CVaR to be applied to various optimization approaches and eventually led to worst-case CVaR problems for portfolio management. Zhu and Fukushima [38] first prove that the worst-case CVaR is still a coherent risk measure and define the problem as

$$WCVaR_\varepsilon(x) = \sup_{p(r) \in \mathcal{P}} CVaR_\varepsilon(x) \,.$$

where the density function for asset returns belongs to a set $\mathcal{P}$ of probability distributions. For simplicity, if we first assume that the returns r follow a discrete distribution, the sample space can be written as

$$\{r_1, r_2, \ldots, r_S\} \quad \text{with} \quad \mathrm{Prob}(r_i) = \pi_i$$

where S is the size of the sample space and

$$\sum_{i=1}^{S} \pi_i = 1,\ \pi_i \geq 0 \quad \text{for} \quad i = 1, \dots, S\ .$$

Then the CVaR for a fixed x and π can be defined by,

$$CVaR_\varepsilon(x,\pi) = \min_\gamma F_\varepsilon(x,\gamma,\pi)$$

where $\pi = (\pi_1, \dots, \pi_S)'$ and

$$F_\varepsilon(x,\gamma,\pi) = \gamma + \frac{1}{1-\varepsilon}\sum_{i=1}^{S} \pi_i[-r_i'x-\gamma]^+\ .$$

Thus, assuming that π belongs to a set of all possible discrete distributions $\mathcal{P}_\pi$, the worst-case CVaR for a portfolio x becomes,

$$WCVaR_\varepsilon(x) = \sup_{\pi\in\mathcal{P}_\pi} CVaR_\varepsilon(x,\pi) = \sup_{\pi\in\mathcal{P}_\pi} \min_\gamma F_\varepsilon(x,\gamma,\pi)\ .$$

Furthermore, Zhu and Fukushima show that if $\mathcal{P}_\pi$ is a compact convex set, then the worst-case CVaR problem can be represented as a min-max problem for every x,

$$WCVaR_\varepsilon(x) = \min_\gamma \max_{\pi\in\mathcal{P}_\pi} F_\varepsilon(x,\gamma,\pi)\ .$$

Finally, the problem of minimizing the worst-case CVaR over a feasible set of portfolios is formulated as,

$$\begin{aligned}
&\min_{(x,u,\gamma,\theta)} \theta \\
&\text{s.t. } x \in \mathcal{X} \\
&\quad \max_{\pi\in\mathcal{P}_\pi} \gamma + \frac{1}{1-\varepsilon}\pi'u \leq \theta \\
&\quad u_i \geq -r_i'x - \gamma\ , \qquad i = 1, \dots, S \\
&\quad u_i \geq 0\ , \qquad i = 1, \dots, S\ .
\end{aligned}$$

In particular, Zhu and Fukushima derive formulations using box and ellipsoidal uncertainty in discrete distributions that result in linear programming and second-order cone programming problems, respectively.

In parallel to their work on VaR [34], Huang *et al*. [39] apply the uncertainty in exit time to the robust CVaR model. They follow the definition of worst-case CVaR from an earlier manuscript of the work by Zhu and Fukushima [38] and also assume a discrete distribution of time as previously described in Section 5.1,

$$\mathrm{Prob}(\tau = t_i) = \lambda_i \quad \text{where} \quad \textstyle\sum_{i=1}^{m} \lambda_i = 1,\ \lambda_i \geq 0 \quad \text{for} \quad i = 1, \ldots, m.$$

Accordingly, the distribution of asset returns $p(\cdot)$ when exit time is uncertain can be expressed as,

$$p(\cdot) = \sum_{i=1}^{m} \lambda_i p_i(\cdot)$$

where $p_i(\cdot)$ is the density function of asset returns conditional on exit time t_i. Based on the formula for minimizing CVaR by Rockafeller and Uryasev [35, 36], Huang *et al*. develop the robust problem for minimizing worst-case CVaR assuming that the distribution of exit time belongs to a certain set,

$$\min_{x \in \mathcal{X}} \sup_{p(\cdot) \in \mathcal{P}_{\mathrm{M}}} \min_{\gamma} \gamma + \frac{1}{1-\varepsilon} \int_{r \in \mathbb{R}^n} [-r'x - \gamma]^+ \, p(r) dr$$

where $\mathcal{P}_{\mathrm{M}}$ contains the possible probability distributions of returns

$$\mathcal{P}_{\mathrm{M}} = \left\{ \sum_{i=1}^{m} \lambda_i p_i(\cdot) : (\lambda_1, \ldots, \lambda_m)' \in \Omega \right\}$$

and the densities of exit time, which we are assumed to have knowledge of, are represented as

$$\Omega \subseteq \left\{ (\lambda_i, \ldots, \lambda_i)' : \sum_{i=1}^{m} \lambda_i = 1, \lambda_i \geq 0, i = 1, \ldots, m \right\}.$$

Then by defining

$$F_{\varepsilon}^{i}(x, \gamma) = \gamma + \frac{1}{1-\varepsilon} \int_{r \in \mathbb{R}^n} [-r'x - \gamma]^+ \, p_i(r) dr$$

the robust optimization problem for selecting portfolio under uncertain exit time becomes,

$$\min_{x \in \mathcal{X}} WCVaR_{\varepsilon}(x) = \min_{(x,\gamma)} \max_{\lambda \in \Lambda} \sum_{i=1}^{m} \lambda_i F_{\varepsilon}^{i}(x, \gamma) \, .$$

Huang *et al*. illustrate when either no information or partial information on exit time is given and show that both cases can be efficiently solved as linear programs.

The main developments on robust approaches for portfolio selection, summarized so far in this paper, are referred to as *absolute* robust optimization in Huang *et al*. [40] because the models assume a uniform distribution in the uncertainty set and therefore focus on optimizing the worst-case within the set. However, they argue that the worst scenario might happen only with extremely low probability, in which case the resulting portfolio will be very conservative, and propose a *relative* robust conditional value-at-risk framework, based on Zhu and Fukushima [38], that also considers the best solution for each case. The relative robust CVaR for a fixed portfolio x with respect to a set of possible discrete probability distributions $\mathcal{P}_{\pi}$ is,

$$RCVaR_{\varepsilon}(x) = \sup_{\pi\in\mathcal{P}_{\pi}} \{CVaR_{\varepsilon}(x,\pi) - CVaR_{\varepsilon}(z^*(\pi),\pi)\}$$

where $z^*(\pi) = \mathrm{argmin}_{z\in\mathcal{X}}\, CVaR_{\varepsilon}(z,\pi)$. For each distribution π, $CVaR_{\varepsilon}(z^*(\pi),\pi)$ is the smallest possible CVaR and $z^*(\pi)$ can be considered as the benchmark portfolio. Therefore, $CVaR_{\varepsilon}(x,\pi) - CVaR_{\varepsilon}(z^*(\pi),\pi)$ computes the relative risk of portfolio x to its benchmark portfolio measured by CVaR. Furthermore, if there are finitely many possibilities of π, $RCVaR_{\varepsilon}(x)$ can be expressed using convex functions $F_{\varepsilon}(x,\gamma,\pi)$ for portfolio x as,

$$\begin{aligned} RCVaR_{\varepsilon}(x) &= \max_{\pi\in\mathcal{P}_{\mathrm{M}}} \{CVaR_{\varepsilon}(x,\pi) - CVaR_{\varepsilon}(z^*(\pi),\pi)\} \\ &= \max_{\pi\in\mathcal{P}_{\mathrm{M}}} \left\{\min_{\gamma} F_{\varepsilon}(x,\gamma,\pi) - \min_{(z,\eta)} F_{\varepsilon}(z,\eta,\pi)\right\} \\ &= \max_{i\in\{1,\dots,l\}} \left\{\min_{\gamma} F_{\varepsilon}(x,\gamma,\pi^i) - \min_{(z,\eta)} F_{\varepsilon}(z,\eta,\pi^i)\right\} \end{aligned}$$

where there are l possible distributions, $\mathcal{P}_{\mathrm{M}} = \{\pi^i\colon\ i = 1,\dots,l\}$. Since $RCVaR_{\varepsilon}(x)$ finds the greatest relative risk with respect to all possible distributions, the problem for selecting the robust optimal portfolio is written as a min-max problem,

$$\min_{x\in\mathcal{X}} \max_{i\in\{1,\dots,l\}} \left\{\min_{\gamma} F_{\varepsilon}(x,\gamma,\pi^i) - \min_{(z,\eta)} F_{\varepsilon}(z,\eta,\pi^i)\right\}$$

and they conclude with a proof that the following problem,

$$\begin{aligned} &\min_{(x,\gamma,\theta)} \theta \\ &\text{s.t. } F_{\varepsilon}(x,\gamma_i,\pi^i) - \min_{(z,\eta)} F_{\varepsilon}(z,\eta,\pi^i) \le \theta, \qquad i = 1,\dots,l \end{aligned}$$

is equivalent to solving the relative robust CVaR problem.

6. Combining with uncertainty in factor models

Even though the existence of a factor model is not explicitly assumed in the Markowitz model, there have been attempts to search for ambiguity that originate from fundamental factors in the market. These methods contain practical implication since factor models are often used as return-generating models. In this section, we introduce several approaches that combine the ambiguity in asset returns arising from the returns of factors as well as the unreliability of factor models used for generating those individual returns.

Earlier in this paper, we discussed the work by Goldfarb and Iyengar [24] where they extend the mean-variance model to develop robust strategies by engaging uncertainty sets for the input parameters. Specifically, they assume a factor model for random asset returns $r \in \mathbb{R}^n$,

$$r = \mu + V'f + \varepsilon \sim \mathcal{N}(\mu, V'FV + D)$$

where $f \sim \mathcal{N}(0, F) \in \mathbb{R}^m$ is the random return of $m(< n)$ factors, $V \in \mathbb{R}^{m\times n}$ is the matrix of factor loadings, and $\varepsilon \sim \mathcal{N}(0, D) \in \mathbb{R}^n$ is the vector of residual returns. Then the return of a portfolio x can be expressed by the factor model as,

$$r_x = r'x = \mu'x + f'Vx + \varepsilon'x \sim \mathcal{N}(\mu'x, x'(V'FV + D)x)\,.$$

While the covariance matrix of factor returns is assumed to be known, the uncertainty in asset returns comes from the ambiguity in mean returns, factor loadings, and residual covariance matrix, which are specified respectively as

$$\mathcal{U}_\mu = \{\mu : \mu = \mu_0 + \xi, |\xi_i| \le \gamma_1, i = 1, \dots, n\}$$

$$\mathcal{U}_V = \left\{V : V = V_0 + W, \|W_i\|_g \le \rho_i, i = 1, \dots, n\right\}$$

$$\mathcal{U}_D = \left\{D : D = diag(d), d_i \in \left[\underline{d}_i, \overline{d}_i\right], i = 1, \dots, n\right\}$$

where W_i is the ith column of W and $\|w\|_g = \sqrt{w'Gw}$ is the elliptic norm of w with respect to a symmetric positive definite matrix G. The factor model along with these sets are used to find robust counterparts as second-order cone programming of the optimization problems with objectives for minimum variance, maximum return, maximum Sharpe ratio, and optimization under the VaR constraint.

Similarly, Ma, Zhao, and Qu [41] assume a factor model where the factor returns and their covariance are known but the uncertainty coexists in expected return of assets, factor loadings, and residual errors of the factor model. In contrast to the optimization problems tackled by Goldfarb and Iyengar, Ma *et al*. focus on maximizing the worst expected utility over the uncertainty set of distributions,

$$\max_{x \in \mathcal{X}} \min_{(\mu,V,D)\in\Pi} \min_{r\sim(\mu,\Sigma)} E[\mathcal{U}_\theta(r_x)]$$

where $\theta = (\theta_1, \theta_2, \theta_3)'$ and

$$\Pi = \left\{(\mu, V, D) : \mu \in \mathcal{U}_\mu, V \in \mathcal{U}_V, D \in \mathcal{U}_D\right\}$$

$$\mathcal{U}_\mu = \{\mu : \mu = \mu_0 + diag(\theta_1)\gamma, \gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)', \theta_1 \in (0,1)^n\}$$

$$\mathcal{U}_V = \left\{V : V_j = V_{0j} + \theta_{2j}\rho_j, \rho_j = \left(\rho_{1j}, \rho_{2j}, \dots, \rho_{nj}\right)', j = 1,2, \dots, m, \theta_2 \in (0,1)^n\right\}$$

$$\mathcal{U}_D = \{D : D = D_0 + diag(\theta_3)diag(\eta), \eta = (\eta_1, \eta_2, \dots, \eta_n)', \theta_3 \in (0,1)^n\}.$$

Specifically, they focus on investors with concave utility for gains and convex utility for losses about the uncertainty for the model and show that an explicit formula for the solution as well as the efficient frontier can be achieved by solving a parametric quadratic programming problem.

Instead of exploring the inexact estimations of individual components of the factor model such as factor loadings and residual returns, Garlappi, Uppal, and Wang [42] put emphasis on situations where investors are not only uncertain about the factor returns but are also doubtful that the factor model they use is not an accurate representation of the true asset returns. The expected return vector and covariance matrix can be expressed using individual assets and factors together as,

$$\mu = \begin{bmatrix}\mu_a \\ \mu_f\end{bmatrix}, \qquad \Sigma = \begin{bmatrix}\Sigma_{aa} & \Sigma_{af} \\ \Sigma_{af} & \Sigma_{ff}\end{bmatrix}$$

where r_{at} is the excess returns of individual assets and r_{ft} is the excess return of the factors at period *t*, and a regression model of excess returns is assumed,

$$r_{at} = \alpha + \beta r_{ft} + \varepsilon_t .$$

In this case, investors with doubts about the factor returns as well as the factor model can solve the following min-max problem,

$$\max_{x} \min_{\mu_a,\mu_f} x'\mu - \frac{\lambda}{2}x'\Sigma x$$

$$\text{s.t. } (\hat{\mu}_a - \mu_a)'\Sigma_{aa}^{-1}(\hat{\mu}_a - \mu_a) \le \varepsilon_a$$

$$(\hat{\mu}_f - \mu_f)' \Sigma_{ff}^{-1} (\hat{\mu}_f - \mu_f) \leq \varepsilon_f$$

The first constraint represents aversion to model uncertainty since $\hat{\mu}_a = \mathrm{E}[\alpha + \beta r_f] = \beta \mu_f$ if the asset-pricing model holds. The second constraint simply measures the uncertainty in factor returns. For example, setting $\varepsilon_f = 0$ and $\varepsilon_a > 0$ represents investors that are more uncertain about the assets than the factor returns and therefore will hold factor portfolios. By the two additional constraints, investors can find a balanced portfolio between the two uncertainties.

Ruan and Fukushima [43] revisit the work on worst-case CVaR by Zhu and Fukushima [38] and make an effort to combine the earlier method with factor models. Computing the CVaR of a portfolio assumes knowledge of the vector of random returns r with probability density $p(r)$. Nonetheless, in reality, it is very unlikely that the distribution of random returns is known with absolute confidence. As a consequence, in robust formulations, the randomness in the distribution of returns is integrated with a set of probability distributions $\mathcal{P}$ and the worst-case CVaR is compared among all possible density functions in this set. Since the factor model provides a formula for expected returns of securities, Ruan and Fukushima use the three-factor model by Fama and French [44] to find more informative probability density functions. Under the assumption of a multivariate normal distribution and a fixed covariance matrix, they pay close attention to the expected returns during the *falling period*, which is recognized as the period when the market index return shows a declining pattern, in order to make the model more robust to future declines. By solving the minimization problem of the worst-case CVaR with historical data from the Tokyo stock exchange, they show that as predicted their model was more robust during the financial crisis of 2008 than the classical mean-variance model.

7. Properties of robust portfolios

Along with the introduction of many robust formulations developed to deal with the unreliable parameters of the portfolio allocation problem, empirical analyses comparing the robustness and performance of proposed models with the classical mean-variance portfolios have been conducted. On the other hand, there has been relatively little effort to find special properties of robust portfolios.

Tütüncü and Koenig [25] look for characteristics of robust portfolios by comparing optimal portfolios between the mean-variance model and the robust minimum variance portfolio model designed by Goldfarb and Iyengar [24] with uncertainty sets specified by the lower and upper bounds. One of their most interesting findings, which is somewhat counter-intuitive, is that robust portfolios tend to invest in fewer asset classes compared to mean-variance portfolios. Among large-cap growth, large-cap value, small-cap growth, and small-cap value stocks along with fixed income securities, robust portfolios show very high concentration in large-cap value stocks whereas mean-variance portfolios include a mixture of fixed income securities, large-cap, and small-cap stocks depending on the level of risk aversion. In addition, when analyzing the change in portfolio composition over long periods, they report evidence that robust portfolios have significantly less turnover, meaning it is more suitable for actual trading due to its low rebalancing cost.

With a similar objective, Pflug and Wozabal [45] analyze the level of return, risk, and composition of robust portfolios. They propose a model with a successive method for approximating the uncertainty set and investigate portfolios as the robustness is increased. Instead of comparing robust portfolios with classical mean-variance portfolios, they analyze robust portfolios with different confidence levels. Another distinction between their study and that of Tütüncü and Koenig is that they only use six securities that are mostly large-cap stocks. For their test setting, they find that increasing the size of confidence sets does not reduce performance much but increases the level of diversification.

One of the first robust approaches was by Soyster [14] who uses convex uncertainty sets for a linear optimization problem. However, his method is too conservative and to that end there have been many proposals to reduce the conservativeness of the resulting portfolios. Although the majority of effort focused on the structure of uncertainty sets, Bertsimas and Sim [46] take a different route by proposing a model that controls the level of conservativeness. They introduce a new parameter that manages the tradeoff between the robustness and conservativeness of the solution, by setting the maximum number of uncertain parameters in its worst-case, to measure the *price of robustness*. Applying the model to portfolio allocation, they confirm that the worst portfolio return increases as the robustness is increased.

Based on the above Bertsimas-Sim model, Gregory, Darby-Downman, and Mitra [47] investigate a measure of the *cost of robustness* that they propose — basically the difference in total portfolio return between the portfolio with the highest mean return, measured by a single asset with the highest return, and the robust solution. Even though most work on robust formulations apply ellipsoidal uncertainty sets, they stick to the simpler polyhedral set to provide preliminary results on the effect of the change in scale of the uncertainty set. They argue that the cost of robustness increases with higher scale of the uncertainty set, and in most cases the probability of underperformance decreases since the uncertainty set contains more scenarios.[9] Furthermore, they find that increasing the length of historical data used in the optimization decreases the cost of robustness.

Most recently, there has been a series of studies attempting to provide further understanding on the characteristics of robust portfolios. The unified observation of the research is that the worst-case formulation for robust optimization may also affect the factor exposure of portfolios. Kim, Kim, and Fabozzi [48] first mathematically show under mild assumptions that increased robustness in the portfolio optimization formulation with ellipsoidal uncertainty sets results in portfolios that are more dependent on fundamental factor movements. As an extensive empirical support, Kim *et al*. [49] use historical returns under various settings to provide evidence that portfolios with higher robustness show higher correlation with the Fama-French three factors [44]. However, Kim *et al*. [50] employ a method that puts more weight on securities that perform especially well during worst market conditions without explicitly solving a min-max optimization problem in order to confirm that focusing on worst scenarios is necessary for constructing robust portfolios. Finally, Kim *et al*. [51] derive several new formulations that add robustness to the optimal portfolios while controlling the factor exposure. Simulation and empirical results complete the argument that the new robust methods can be used by portfolio managers who have target levels for the factor exposure of their portfolios.

8. Future directions

In this paper, we survey studies in robust portfolio optimization that utilize worst-case approaches. For the past decade, many formulations and uncertainty sets have been applied to portfolio allocation to increase the robustness of mean-variance portfolios, which are known to be sensitive to even small changes in its input parameters. In addition to the classical Markowitz model, robust counterparts of portfolio optimization problems based on value-at-risk and conditional value-at-risk were also proposed. However, all developments of robust formulations introduced so far only

[9] To increase the scale of the uncertainty set, they actually test with several point estimates and deviation parameters in addition to a scaling factor.

cover a single-period investment. A natural next step is to extend the approaches to a multi-period setting, which will also help its practical use since real-world portfolio management is performed over extended periods of time and, as a result, single-period models may be suboptimal in this case.

Actually, there already has been some research in robust portfolio optimization for multi-periods. Ben-Tal, Margalit, and Nemirovski [52] review the multi-period asset allocation problem by Dantzig and Infanger [53] and introduce uncertainty to the model. They experimentally compare the robust model with a multi-stage stochastic programming model and note its high potential due to its computational efficiency while not showing any worse performance than stochastic programming. Bertsimas and Pachamanova [54] extend the model and derive formulations that are linear and therefore can be solved more efficiently. In addition, they compare their method with single-period models to provide support for the better performance of their model. Considering the promising performance of robust portfolios reported in these studies, we think extensions from the single-period case will bring considerable improvements on the use of robust models.

As mentioned in the Section 7, there have been recent advancements in finding interesting properties of robust portfolios. These results give further meaning to robust portfolios but we believe that more studies on their behavior and economic value are needed. Finally, more contribution in multi-stage settings along with the characteristic of multi-period robust portfolios will not only help theoretical understanding of the approach but will also assist its practical use.

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