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In recent years, there has been an increasing interest in hyperspherical caps from machine learning domain. Concise formulas for the area of a hyperspherical cap are now available and being beneficial for researchers, but there is not one for the intersection of two hyperspherical caps in spite of its large potential in application. This paper provides concise formulas for the surface area of the intersection of two hyperspherical caps for every possible case.

Key Words: Hyperspace, Hypersphere, Hyperspherical cap, Surface area, Intersection

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In recent years, there has been an increasing interest in hyperspherical caps from machine learning domain. Concise formulas for the area of a hyperspherical cap are now available and being beneficial for researchers, but there is not one for the intersection of two hyperspherical caps in spite of its large potential in application. This paper provides concise formulas for the surface area of the intersection of two hyperspherical caps for every possible case.

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1. INTRODUCTION

Consider an $(n-1)$-sphere centered at the origin in n-dimensional Euclidean space

$$S^{n-1}(r) = \{x \in \mathbb{R}^n : \|x\| = r\}.$$

The surface area of the hypersphere is given by

$$A_n(r) = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} r^{n-1},$$

where, Γ is the gamma function.

Let us denote an n-dimensional hyperspherical cap as $C_n(r, v, \theta)$, where v is the axis, and θ is the colatitude angle, which is the largest angle formed by the axis and a vector on the hyperspherical cap. Recently, hyperspherical caps have found its extensive applications in machine learning domain: Durrant and Kaban (2013), Galego *et al.* (2013), Hughes (2008), and Piciarelli *et al.* (2008). In this regard, Li (2011) has provided a concise formula for the surface area of a hyperspherical cap instead of series or recursive forms derived by Cox *et al.* (2007), Cox *et al.* (2008), Ericson and Zinoviev (2001), and Hughes (2008). Li (2011)'s formula for the surface area of hyperspherical cap $C_n(r, v, \theta)$ for $\theta \in [0, \pi/2]$ is given by

$$A_n^{\theta}(r) = \frac{1}{2} A_n(r) I_{\sin^2\theta}\left(\frac{n-1}{2}, \frac{1}{2}\right),$$

where, $I_x(a, b)$ is the regularized incomplete beta function. However, there is no simple way to calculate the surface area of the intersection when we are given with two hyperspherical caps. For the ease of further applications, simple formulas for the intersection are definitely needed.

2. Surface Area of the Intersection of Two Hyperspherical Caps

Consider two hyperspherical caps $C_n(r, v_1, \theta_1)$ and $C_n(r, v_2, \theta_2)$ of $S^{n-1}(r)$ and let θ_v be the angle between two axes v_1 and v_2, that is $\theta_v = \arccos(v_1^T v_2 / (\|v_1\| \|v_2\|))$. Note that the intersection area of these two hyperspherical caps depends only on three parameters θ_1, θ_2, and θ_v. However, depending on these three parameters, there exist a number of different cases.

Let the surface area of the intersection of two hyperspherical caps be $A_n^{\theta_1,\theta_2,\theta_v}(r)$. In addition, let $\bar{\theta} = \pi - \theta$. We first cover ten representative cases. Afterwards, the remaining cases will be straightforward. Entire formulas for the intersection areas for every case are listed in Table 1. Additionally, we provide MATLAB codes for actual calculation in the Appendix.

2.1 Case 1

The first case is when the two hyperspherical caps do not intersect, i.e., $\theta_v \geq \theta_1 + \theta_2$. Thus the remaining cases should always have intersection. Figure 1 illustrates Case 1 in 2-dimensional Euclidean space. Obviously, the surface area of the intersection

$$A_n^{\theta_1,\theta_2,\theta_v}(r) = 0.$$

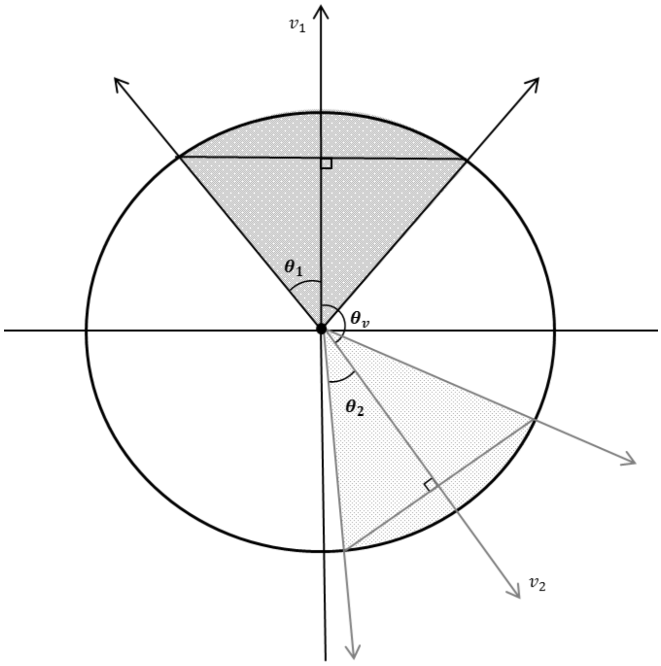


Figure 1: Case 1 in Table 1

Table 1: Surface area of the intersection of two hyperspherical caps

Case	Conditions					$A_n^{\theta_1,\theta_2,\theta_v}(r)$	θ_{min}
1	$\theta_v \geq \theta_1 + \theta_2$					0	-
2	$\theta_v < \theta_1 + \theta_2$	$\theta_1 \geq \min(\theta_2 + \theta_v, \pi)$				$A_n^{\theta_2}(r)$	-
3		$\theta_2 \geq \min(\theta_1 + \theta_v, \pi)$				$A_n^{\theta_1}(r)$	-
4		$\theta_1 + \theta_2 > 2\pi - \theta_v$	$\theta_1 \in [0, \pi/2]$			$A_n^{\theta_1}(r) - A_n^{\bar{\theta}_2}(r)$	-
5			$\theta_1 \in (\pi/2, \pi]$			$A_n^{\theta_2}(r) - A_n^{\bar{\theta}_1}(r)$	-
6		$\theta_1 + \theta_2 \leq 2\pi - \theta_v$	$\theta_1 \in [0, \pi/2)$	$\theta_2 \in [0, \pi/2)$	$\theta_2 > \theta_v,\ \cos(\theta_1)\cos(\theta_v) \geq \cos(\theta_2)$	$A_n^{\theta_1}(r) - J_n^{\theta_{min},\theta_1}(r) + J_n^{\theta_v+\theta_{min},\theta_2}(r)$	$\arctan(1/\tan(\theta_v) - \cos(\theta_2)/(\cos(\theta_1)\sin(\theta_v)))$
7					$\theta_1 > \theta_v,\ \cos(\theta_2)\cos(\theta_v) \geq \cos(\theta_1)$	$A_n^{\theta_2}(r) - J_n^{\theta_{min},\theta_2}(r) + J_n^{\theta_v+\theta_{min},\theta_1}(r)$	$\arctan(1/\tan(\theta_v) - \cos(\theta_1)/(\cos(\theta_2)\sin(\theta_v)))$
8					Otherwise	$J_n^{\theta_v-\theta_{min},\theta_1}(r) + J_n^{\theta_{min},\theta_2}(r)$	$\arctan(\cos(\theta_1)/(\cos(\theta_2)\sin(\theta_v)) - 1/\tan(\theta_v))$
9				$\theta_2 = \pi/2$	$\theta_v \in [0, \pi/2]$	$A_n^{\theta_1}(r) - J_n^{\theta_2-\theta_v,\theta_1}(r)$	-
10					$\theta_v \in (\pi/2, \pi]$	$J_n^{\theta_v-\theta_2,\theta_1}(r)$	-
11				$\theta_2 \in (\pi/2, \pi]$	$\bar{\theta}_2 > \bar{\theta}_v,\ \cos(\theta_1)\cos(\bar{\theta}_v) \geq \cos(\bar{\theta}_2)$	$J_n^{\theta_{min},\theta_1}(r) - J_n^{\bar{\theta}_v+\theta_{min},\bar{\theta}_2}(r)$	$\arctan(1/\tan(\bar{\theta}_v) - \cos(\bar{\theta}_2)/(\cos(\theta_1)\sin(\bar{\theta}_v)))$
12					$\theta_1 > \bar{\theta}_v,\ \cos(\bar{\theta}_2)\cos(\bar{\theta}_v) \geq \cos(\theta_1)$	$A_n^{\theta_1}(r) - (A_n^{\bar{\theta}_2}(r) + J_n^{\bar{\theta}_v+\theta_{min},\theta_1}(r) - J_n^{\theta_{min},\bar{\theta}_2}(r))$	$\arctan(1/\tan(\bar{\theta}_v) - \cos(\theta_1)/(\cos(\bar{\theta}_2)\sin(\bar{\theta}_v)))$
13					Otherwise	$A_n^{\theta_1}(r) - (J_n^{\theta_{min},\bar{\theta}_2}(r) - J_n^{\bar{\theta}_v-\theta_{min},\theta_1}(r))$	$\arctan(\cos(\theta_1)/(\cos(\bar{\theta}_2)\sin(\bar{\theta}_v)) - 1/\tan(\bar{\theta}_v))$
14			$\theta_1 = \pi/2$	$\theta_2 \in [0, \pi/2]$	$\theta_v \in [0, \pi/2]$	$A_n^{\theta_2}(r) - J_n^{\theta_1-\theta_v,\theta_2}(r)$	-
15					$\theta_v \in (\pi/2, \pi]$	$J_n^{\theta_v-\theta_1,\theta_2}(r)$	-
16				$\theta_2 \in (\pi/2, \pi]$	$\bar{\theta}_v \in [0, \pi/2]$	$A_n^{\theta_1}(r) - (A_n^{\bar{\theta}_2}(r) - J_n^{\theta_1-\bar{\theta}_v,\bar{\theta}_2}(r))$	-
17					$\bar{\theta}_v \in (\pi/2, \pi]$	$A_n^{\theta_1}(r) - J_n^{\bar{\theta}_v-\theta_1,\bar{\theta}_2}(r)$	-
18			$\theta_1 \in (\pi/2, \pi]$	$\theta_2 \in [0, \pi/2)$	$\theta_2 > \bar{\theta}_v,\ \cos(\bar{\theta}_1)\cos(\bar{\theta}_v) \geq \cos(\theta_2)$	$A_n^{\theta_2}(r) - (A_n^{\bar{\theta}_1}(r) - J_n^{\theta_{\min},\bar{\theta}_1}(r) + J_n^{\bar{\theta}_v+\theta_{\min},\theta_2}(r))$	$\arctan(1/\tan(\bar{\theta}_v) - \cos(\theta_2)/(\cos(\bar{\theta}_1)\sin(\bar{\theta}_v)))$
19					$\bar{\theta}_1 > \bar{\theta}_v,\ \cos(\theta_2)\cos(\bar{\theta}_v) \geq \cos(\bar{\theta}_1)$	$J_n^{\theta_{\min},\theta_2}(r) - J_n^{\bar{\theta}_v+\theta_{\min},\bar{\theta}_1}(r)$	$\arctan(1/\tan(\bar{\theta}_v) - \cos(\bar{\theta}_1)/(\cos(\theta_2)\sin(\bar{\theta}_v)))$
20					Otherwise	$A_n^{\theta_2}(r) - J_n^{\bar{\theta}_v-\theta_{\min},\bar{\theta}_1}(r) - J_n^{\theta_{\min},\theta_2}(r)$	$\arctan(\cos(\bar{\theta}_1)/(\cos(\theta_2)\sin(\bar{\theta}_v)) - 1/\tan(\bar{\theta}_v))$
21				$\theta_2 = \pi/2$	$\bar{\theta}_v \in [0, \pi/2]$	$A_n^{\theta_2}(r) - (A_n^{\bar{\theta}_1}(r) - J_n^{\theta_2-\bar{\theta}_v,\bar{\theta}_1}(r))$	-
22					$\bar{\theta}_v \in (\pi/2, \pi]$	$A_n^{\theta_2}(r) - J_n^{\bar{\theta}_v-\theta_2,\bar{\theta}_1}(r)$	-
23				$\theta_2 \in (\pi/2, \pi]$	$\bar{\theta}_2 > \theta_v.\ \cos(\bar{\theta}_1)\cos(\theta_v) \geq \cos(\bar{\theta}_2)$	$A_n(r) - (A_n^{\bar{\theta}_2}(r) - J_n^{\theta_v+\theta_{\min},\bar{\theta}_2}(r) + J_n^{\theta_{\min},\bar{\theta}_1}(r))$	$\arctan(1/\tan(\theta_v) - \cos(\bar{\theta}_2)/(\cos(\bar{\theta}_1)\sin(\theta_v)))$
24					$\bar{\theta}_1 > \theta_v,\ \cos(\bar{\theta}_2)\cos(\theta_v) \geq \cos(\bar{\theta}_1)$	$A_n(r) - (A_n^{\bar{\theta}_1}(r) - J_n^{\theta_v+\theta_{min},\bar{\theta}_1}(r) + J_n^{\theta_{min},\bar{\theta}_2}(r))$	$\arctan(1/\tan(\theta_v) - \cos(\bar{\theta}_1)/(\cos(\bar{\theta}_2)\sin(\theta_v)))$
25					Otherwise	$A_n(r) - (A_n^{\bar{\theta}_1}(r) + A_n^{\bar{\theta}_2}(r) - (J_n^{\theta_v-\theta_{\min},\bar{\theta}_1}(r) + J_n^{\theta_{\min},\bar{\theta}_2}(r)))$	$\arctan(\cos(\bar{\theta}_1)/(\cos(\bar{\theta}_2)\sin(\theta_v)) - 1/\tan(\theta_v))$

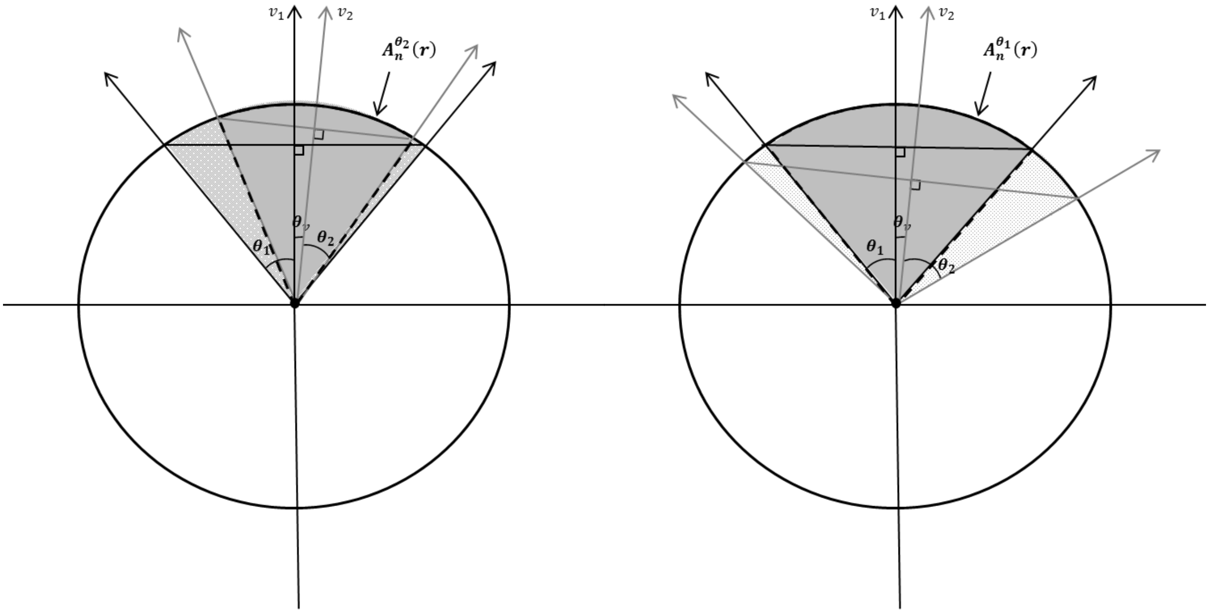
2.2 Case 2 and 3

Case 2 and 3 are when one of the two hyperspherical caps includes another. In Figure 2 and 3, the intersections of two hyperspherical caps are highlighted in dark gray and dotted line. It is trivial that for Case 2, the intersection area

$$A_n^{\theta_1,\theta_2,\theta_v}(r) = A_n^{\theta_2}(r)$$

, and for Case 3,

$$A_n^{\theta_1,\theta_2,\theta_v}(r) = A_n^{\theta_1}(r).$$



(a) Case 2 (b) Case 3

Figure 2: Case 2 and 3 in Table 1

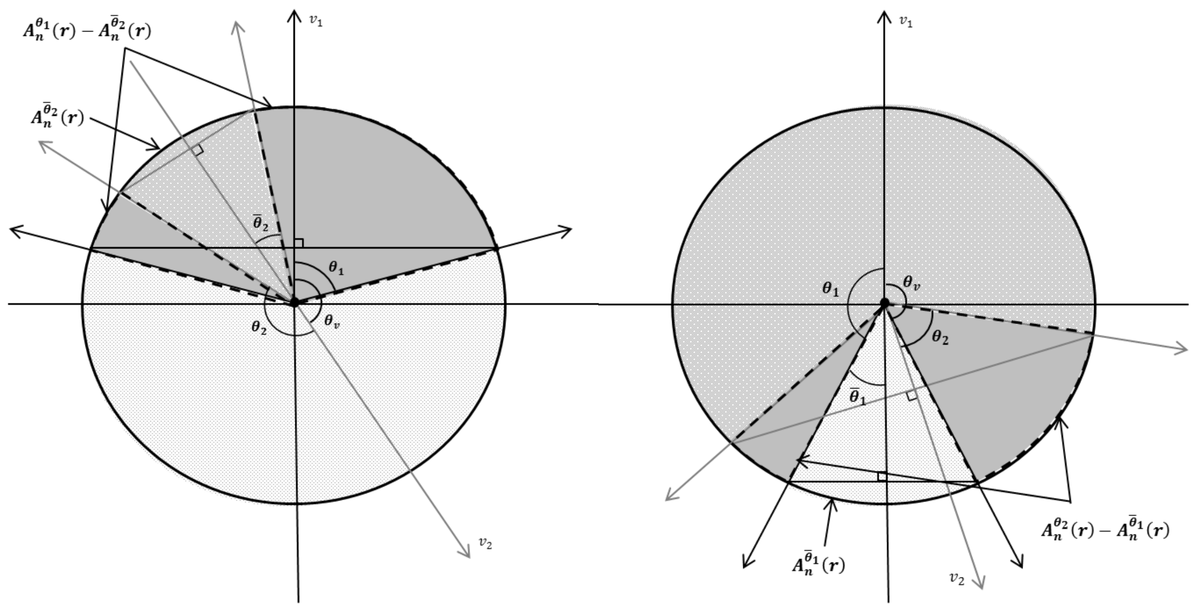
2.3 Case 4 and 5

Case 4 and 5 are when the two hyperspherical caps cover the whole $(n-1)$-sphere as illustrated in Figure 3. In Case 4, $\theta_1 \in [0, \pi/2]$, and in Case 5, $\theta_1 \in (\pi/2, \pi]$. In these cases, the intersection area includes non-intersecting area as can be seen in the figures. Thus, the surface area of the intersection for Case 4 is

$$A_n^{\theta_1,\theta_2,\theta_v}(r) = A_n^{\theta_1}(r) - A_n^{\bar{\theta}_2}(r)$$

, and for Case 5 it is

$$A_n^{\theta_1,\theta_2,\theta_v}(r) = A_n^{\theta_2}(r) - A_n^{\bar{\theta}_1}(r).$$



(a) Case 4 (b) Case 5

Figure 3: Case 4 and 5 in Table 1

2.4 Case 6, 7, and 8

For these three cases, the colatitude angles of both hyperspherical caps are less than $\pi/2$. These cases would be the most common cases in applications to various domains. Unlike the cases before, however, the calculation of intersection area is not quite simple. Figure 4 and 5 show Case 8 in different views. As can be seen in the figures, the intersection area can be divided into two parts by a hyperplane which contains the center of the hypersphere. The boundary line is represented in black real line. Each part can be regarded as a hyperspherical cap cut by a hyperplane passing through the center of the hypersphere. So we first calculate the surface area of each part separately and sum them up to obtain the whole intersection area.

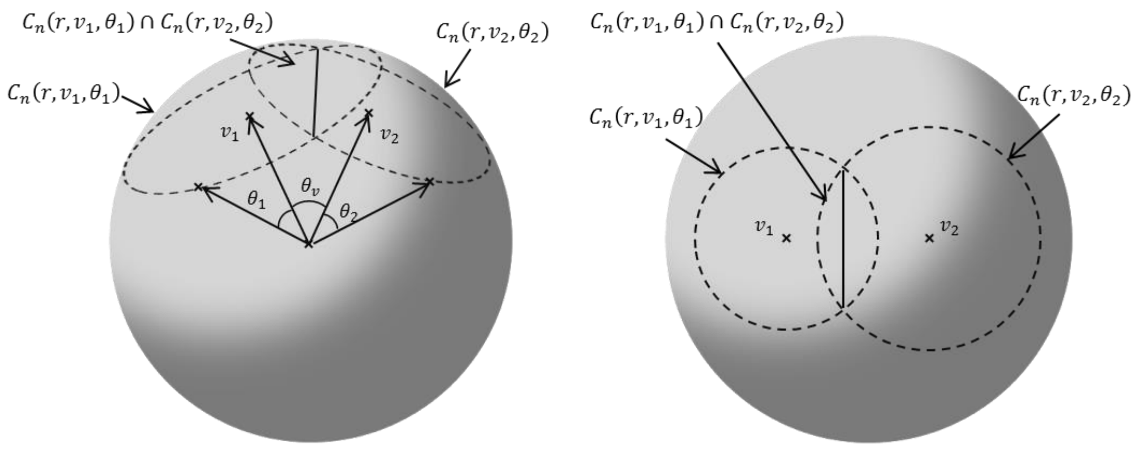


Figure 4: Case 8 in Table 1 (in $\mathbb{R}^3$)

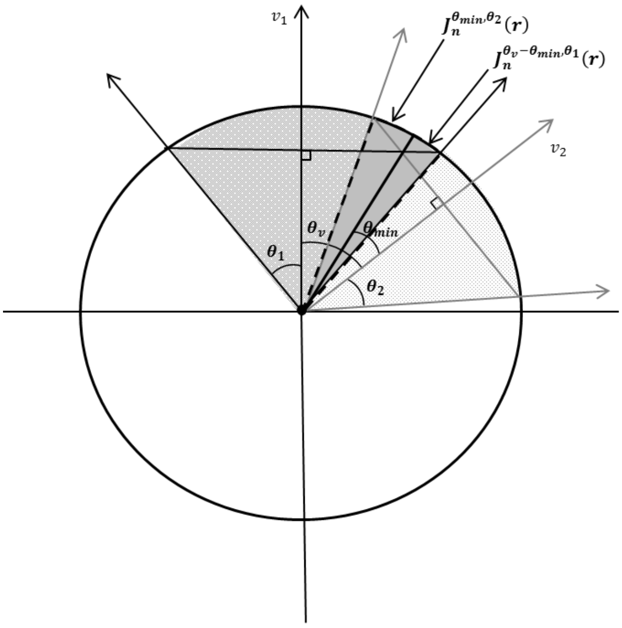


Figure 5: Case 8 in Table 1 (in $\mathbb{R}^2$)

Let us first calculate the left part, which can be considered as the second hyperspherical cap $C_n(r, v_2, \theta_2)$ cut by a hyperplane passing through the origin. The surface area can be obtained by the $(n-1)$-dimensional hyperspherical cap of radius $r\sin(\phi)$ with the colatitude angle $\arccos(\tan(\theta_{min})/\tan(\phi))$ with respect to $r\mathrm{d}\phi$ for $\phi \in [\theta_{min}, \theta_2]$ as in Figure 6. Here, through simple trigonometric function calculations, θ_{min} can be determined as $\arctan(\cos(\theta_1)\,/(\cos(\theta_2)\sin(\theta_v)) - 1/\tan(\theta_v))$.

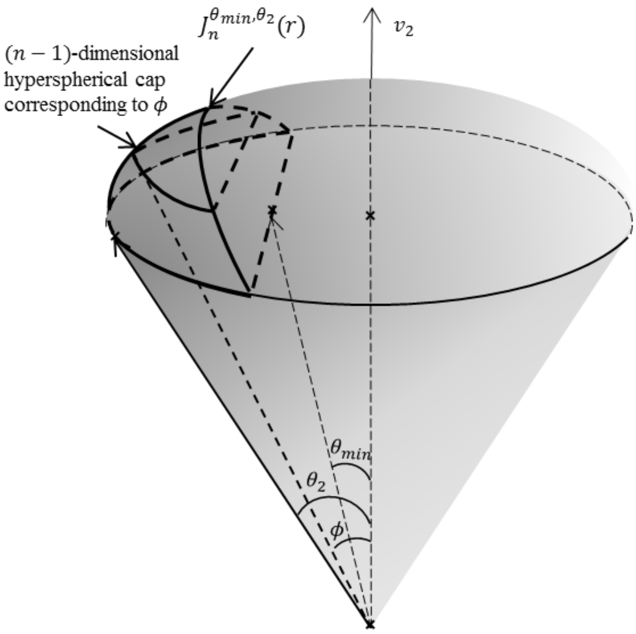


Figure 6: Left part of the intersection in Figure 6 as the integration of $(n-1)$-dimensional hyperspherical cap

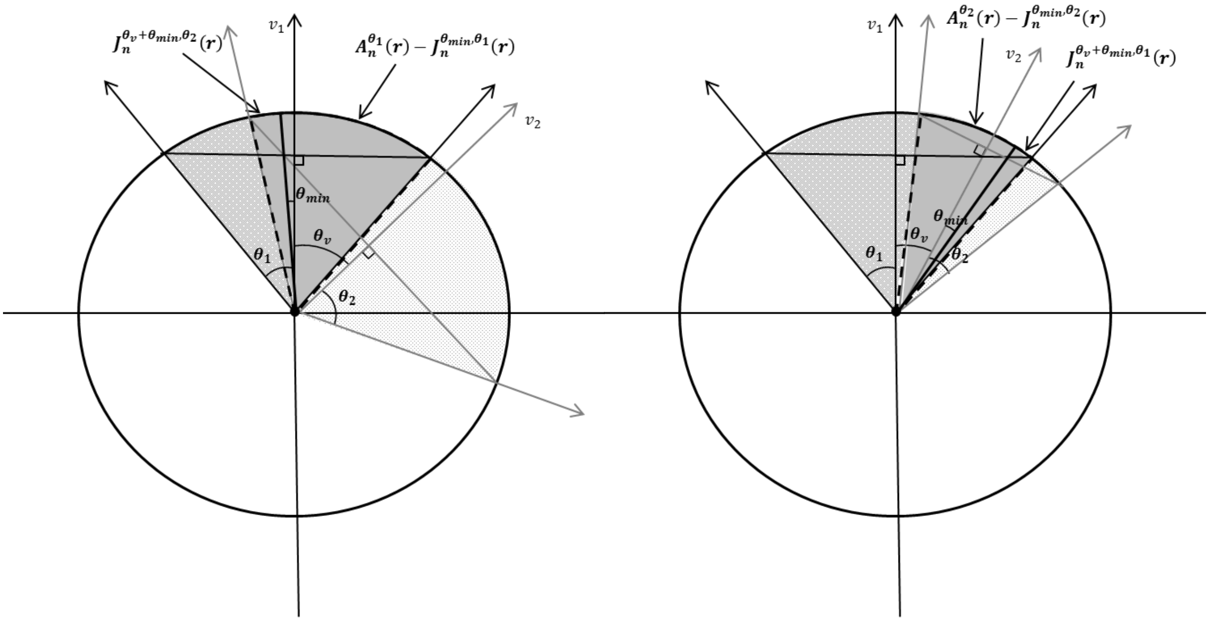
Therefore,

$$
\begin{aligned}
Left\ part &= \int_{\theta_{min}}^{\theta_2} A_{n-1}^{arccos\left(\frac{tan(\theta_{min})}{tan(\phi)}\right)}\big(r\sin(\phi)\big)r\mathrm{d}\phi \\
&= \int_{\theta_{min}}^{\theta_2} \frac{1}{2} A_{n-1}\big(r\sin(\phi)\big) I_{\sin\left(arccos\left(\frac{tan(\theta_{min})}{tan(\phi)}\right)\right)^2}\left(\frac{n-2}{2},\frac{1}{2}\right) r\mathrm{d}\phi \\
&= \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} r^{n-1} \int_{\theta_{min}}^{\theta_2} \sin(\phi)^{n-2}\, I_{1-\cos\left(arccos\left(\frac{tan(\theta_{min})}{tan(\phi)}\right)\right)^2}\left(\frac{n-2}{2},\frac{1}{2}\right)\mathrm{d}\phi \\
&= \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} r^{n-1} \int_{\theta_{min}}^{\theta_2} \sin(\phi)^{n-2}\, I_{1-\left(\frac{tan(\theta_{min})}{tan(\phi)}\right)^2}\left(\frac{n-2}{2},\frac{1}{2}\right)\mathrm{d}\phi \\
&=: J_n^{\theta_{min},\theta_2}(r)
\end{aligned}
$$

As above, we denote the area of the left part as $J_n^{\theta_{min},\theta_2}(r)$. Similarly, the surface area of the right part becomes $J_n^{\theta_v-\theta_{min},\theta_1}(r)$. Hence, for Case 8, the surface area of the intersection

$$A_n^{\theta_1,\theta_2,\theta_v}(r) = J_n^{\theta_{min},\theta_2}(r) + J_n^{\theta_v-\theta_{min},\theta_1}(r).$$

In Case 8, the dividing hyperplane locates between the two axes of hyperspherical caps v_1 and v_2. For Case 6 and 7, however, the dividing hyperplane is located at one side of the two axes. As shown in Figure 7, Case 6 is when the boundary is on the left side of v_1 and v_2 and the other side for Case 7.



(a) Case 6 (b) Case 7

Figure 7: Case 6 and 7 in Table 1

The calculations of intersection area for these two cases are straight-forward by looking at the figures. The surface area of the intersection for Case 6 is

$$A_n^{\theta_1,\theta_2,\theta_v}(r) = A_n^{\theta_1}(r) - J_n^{\theta_{min},\theta_1}(r) + J_n^{\theta_v+\theta_{min},\theta_2}(r)$$

, and for Case 7 it is

$$A_n^{\theta_1,\theta_2,\theta_v}(r) = A_n^{\theta_2}(r) - J_n^{\theta_{min},\theta_2}(r) + J_n^{\theta_v+\theta_{min},\theta_1}(r)$$

,where for Case 6,

$$\theta_{min} = \arctan(1/\tan(\theta_v) - \cos(\theta_2)/(\cos(\theta_1)\sin(\theta_v)))$$

,and for Case 7,

$$\theta_{min} = \arctan(1/\tan(\theta_v) - \cos(\theta_1)/(\cos(\theta_2)\sin(\theta_v))).$$

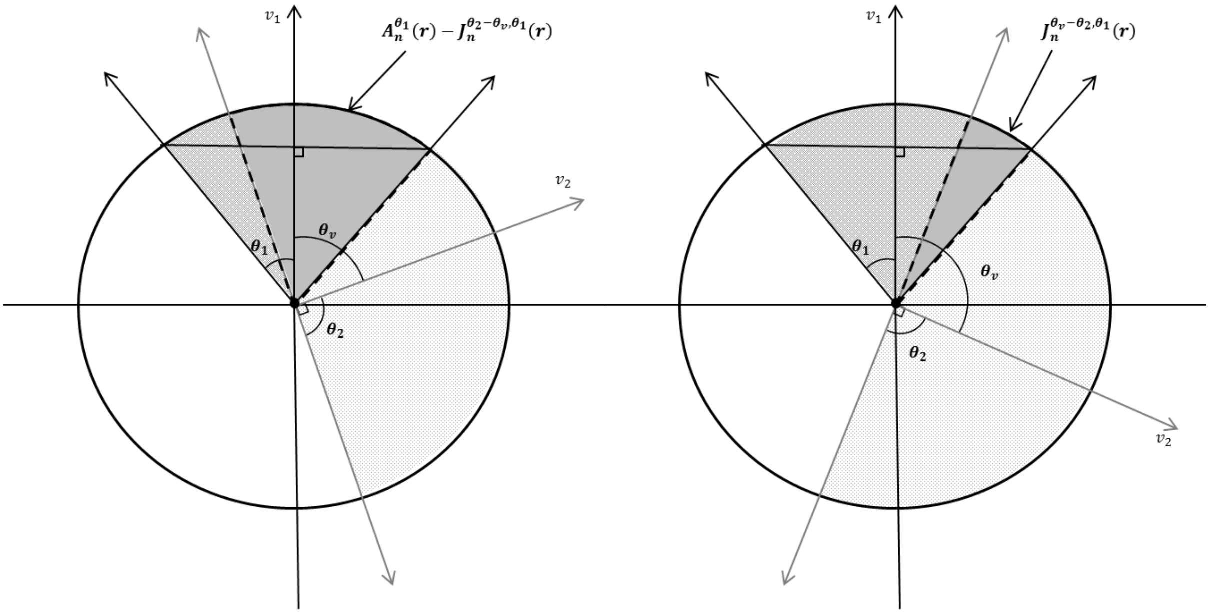
2.5 Case 9 and 10

Case 9 and 10 are when $\theta_2 = \pi/2$, i.e., one of the two hyperspherical caps is actually a hemisphere. In Case 9, $\theta_v \in [0, \pi/2]$, and in Case 10, $\theta_v \in (\pi/2, \pi]$. These cases also can be regarded as a hyperspherical cap cut by a hyperplane which passes through the origin of hypersphere. Thus, the calculations are straight-forward. For Case 9,

$$A_n^{\theta_1,\theta_2,\theta_v}(r) = A_n^{\theta_1}(r) - J_n^{\theta_2-\theta_v,\theta_1}(r)$$

, and for Case 10,

$$A_n^{\theta_1,\theta_2,\theta_v}(r) = J_n^{\theta_v-\theta_2,\theta_1}(r).$$



(a) Case 9 (b) Case 10

Figure 8: Case 9 and 10 in Table 1

2.6 Case 11 -25

Remaining 15 cases can now be proved by simply looking at the figures similar to the figures presented before. See Figures 9 to 23 in the Appendix.

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A. APPENDIX: MATLAB CODES

```
function area = cal_intersection(n,r,t1,t2,tv)
% Surface Area of the Intersection of Two Hyperspherical Caps
% n: dimension
% r: radius
% t1: theta_1
% t2: theta_2
% tv: theta_v

% Check t1, t2 and tv
if t1 < 0 || t1 > pi || t2 < 0 || t2 > pi || tv < 0 || tv > pi
    disp('error');
    return
end

% Calculate t1_bar, t2_bar and tv_bar
t1_bar = pi - t1;
t2_bar = pi - t2;
tv_bar = pi - tv;

% Dividing Cases
if tv >= t1 + t2
    % Case 1 : Do Not intersect
    area = 0;
elseif tv < t1 + t2
    % Case 2 ~ 25 : Intersect
    if t1 >= min(tv+t2,pi)
        % Case 2
        area = A(n,r,t2);
    elseif t2 >= min(tv+t1,pi)
        % case 3
        area = A(n,r,t1);
    else
        if t1 + t2 > 2*pi - tv
            if t1 <= pi/2
                % Case 4
                area = A(n,r,t1) - A(n,r,t2_bar);
            else
                % case 5
                area = (A(n,r,t2) - A(n,r,t1_bar));
            end
        else
            if t1 < pi/2
                if t2 < pi/2
                    if  t2 > tv && cos(t1)*cos(tv) >= cos(t2)
                        % Case 6
                        tmin = atan(1/tan(tv) -
cos(t2)/(cos(t1)*sin(tv)));
                        area = A(n,r,t1) - J(n,r,tmin,t1) +
J(n,r,tv+tmin,t2);
                    elseif t1 > tv && cos(t2)*cos(tv) >= cos(t1)
                        % Case 7
```

```
                    tmin = atan(1/tan(tv) -
cos(t1)/(cos(t2)*sin(tv)));
                    area = A(n,r,t2) - J(n,r,tmin,t2) +
J(n,r,tv+tmin,t1);
                else
                    % Case 8
                    tmin = atan(cos(t1)/(cos(t2)*sin(tv)) -
1/tan(tv));
                    area = J(n,r,tv-tmin,t1) + J(n,r,tmin,t2);
                end
            elseif t2 == pi/2
                if tv <= pi/2
                    % Case 9
                    area = A(n,r,t1) - J(n,r,t2-tv,t1);
                else
                    % Case 10
                    area = J(n,r,tv-t2,t1);
                end
            elseif t2 > pi/2
                % Case 11, 12, 13 (calculate recursively)
                area = A(n,r,t1) -
cal_intersection(n,r,t1,t2_bar,tv_bar);
            end
        elseif t1 == pi/2
            if t2 <= pi/2
                if tv <= pi/2
                    % Case 14
                    area = A(n,r,t2) - J(n,r,t1-tv,t2);
                else
                    % Case 15
                    area = J(n,r,tv-t1,t2);
                end
            elseif t2 > pi/2
               if tv_bar <= pi/2
                   % Case 16
                   area = A(n,r,t1) - (A(n,r,t2_bar) - J(n,r,t1-
tv_bar,t2_bar));
               else
                   % Case 17
                   area = A(n,r,t1) - J(n,r,tv_bar-t1,t2_bar);
               end
            end
        elseif t1 > pi/2
            if t2 <= pi/2
                % Case 18, 19, 20 (calculate recursively)
                area = A(n,r,t2) -
cal_intersection(n,r,t1_bar,t2,tv_bar);
            elseif t2 == pi/2
                if tv_bar < pi/2
                    % Case 21
                    area = A(n,r,t2) - (A(n,r,t1_bar) - J(n,r,t2-
tv_bar,t1_bar));
                else
                    % Case 22
```

```
                        area = A(n,r,t2) - J(n,r,tv_bar-t2,t1_bar);
                    end
                elseif t2 > pi/2
                    % Case 23, 24, 25 (calculate recursively)
                    area = A(n,r,pi) - A(n,r,t1_bar) - A(n,r,t2_bar) +
cal_intersection(n,r,t1_bar,t2_bar,tv);
                end
            else
                disp('error');
                return
            end
        end
    end
end
end
```

```
function area = J(n,r,t1,t2)
% Suface Area of a Hyperspherical Cap Cut by a Hyperplane
% n: dimension
% r: radius
% t1: theta_1 (minimum angle from the axis)
% t2: theta_2 (maximum angle from the axis)

% Check t1 and t2
if t1 < 0 || t1 > pi/2 || t2 < 0 || t2 > pi/2
    disp('error');
    return
end
% Number of slices for numerical integration
k = 10000;
% Integration from t1 to t2
dt = (t2-t1)/k;
t = t1+dt:dt:t2-dt;
% Regularized Incomplete Beta Function within the integration
I = betainc(1-(tan(t1)./tan(t)).^2,(n-2)/2,1/2);
% Take integral by sum
J = sin(t).^(n-2).*I;
J = sum(J*dt);
area = J*pi^((n-1)/2)/gamma((n-1)/2)*r^(n-1);
end
```

```
function area = A(n,r,t)
% Surface Area of a Hyperspherical Cap
% n: dimension
% r: radius
% t: theta (colatitude angle)

if t <= pi/2
    area = pi^(n/2) / gamma(n/2) * r^(n-1) * betainc(sin(t)^2,(n-
1)/2,1/2);
else
    area = 2 * pi^(n/2) / gamma(n/2) * r^(n-1) * (1 - 1/2 * be-
tainc(sin(t)^2,(n-1)/2,1/2));
end
end
```

B. APPENDIX: FIGURES FOR CASE 11 – 25

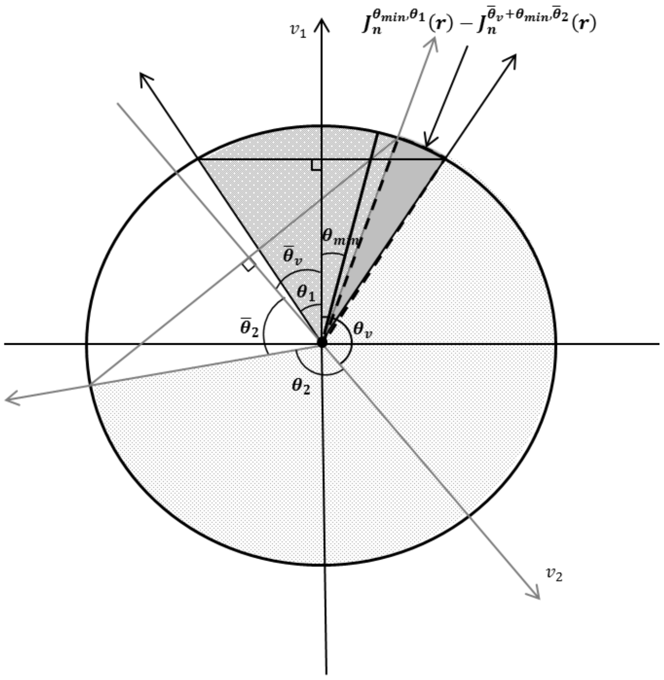

Figure 9: Case 11 in Table 1

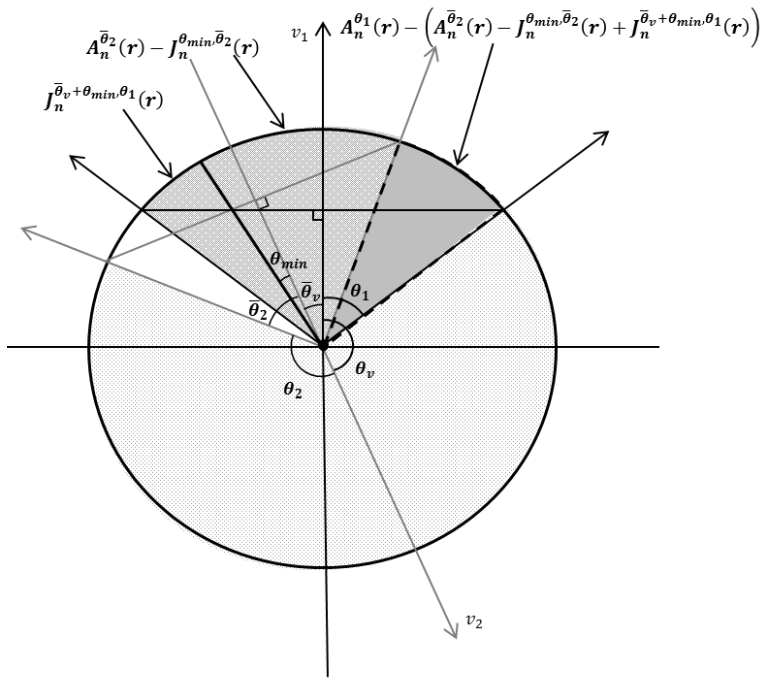

Figure 10: Case 12 in Table 1

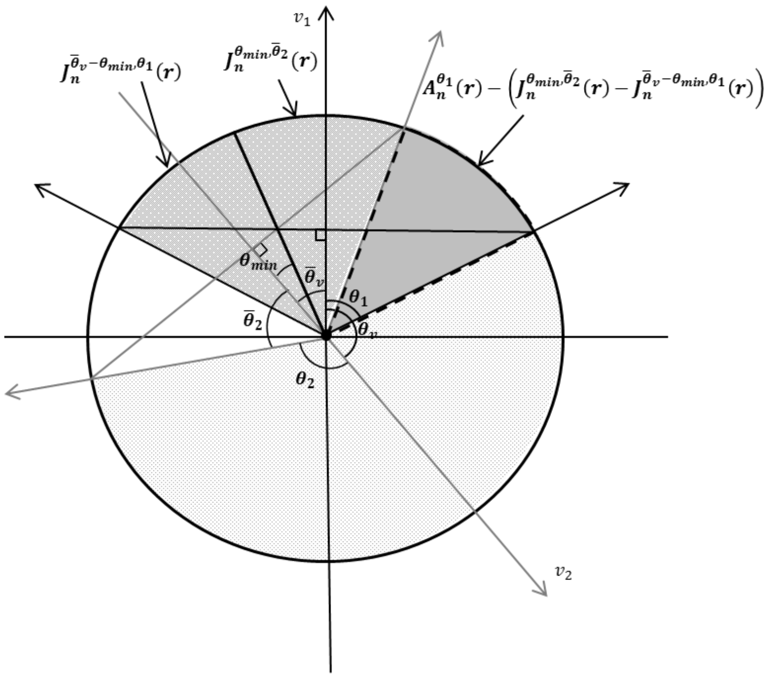

Figure 11: Case 13 in Table 1

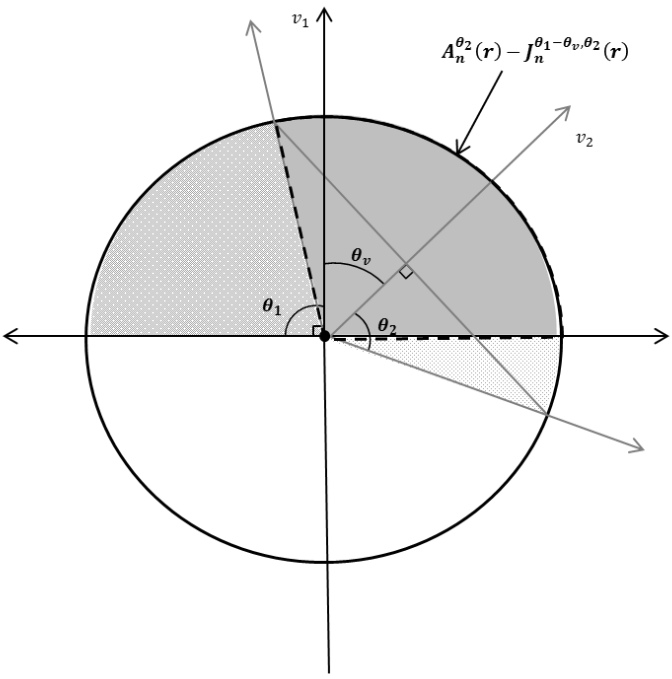

Figure 12: Case 14 in Table 1

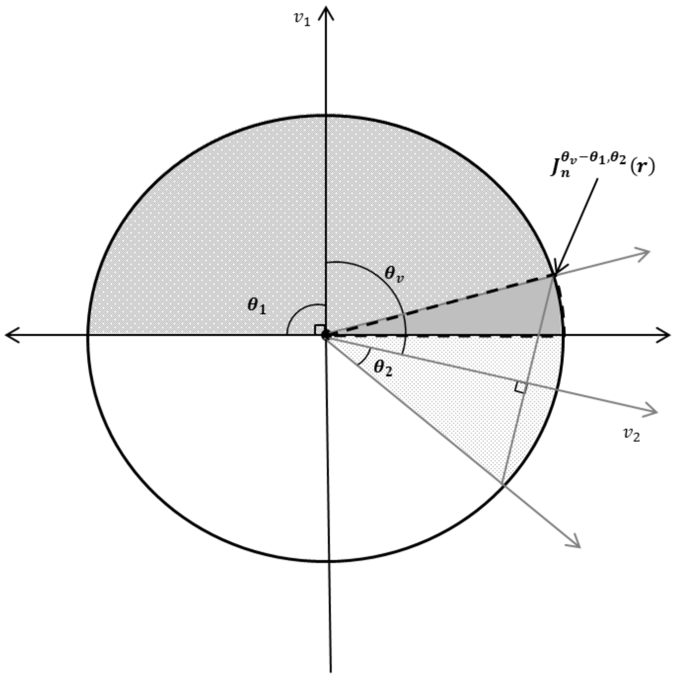

Figure 13: Case 15 in Table 1

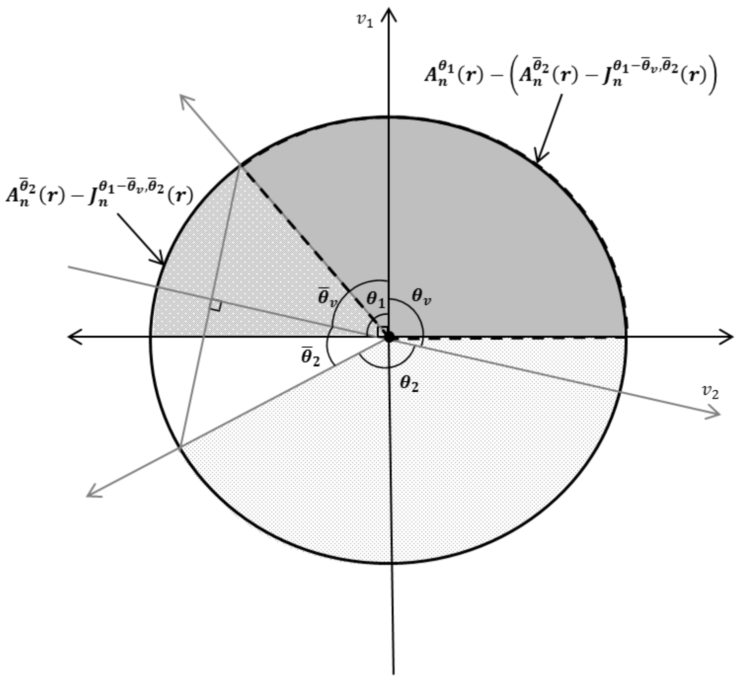

Figure 14: Case 16 in Table 1

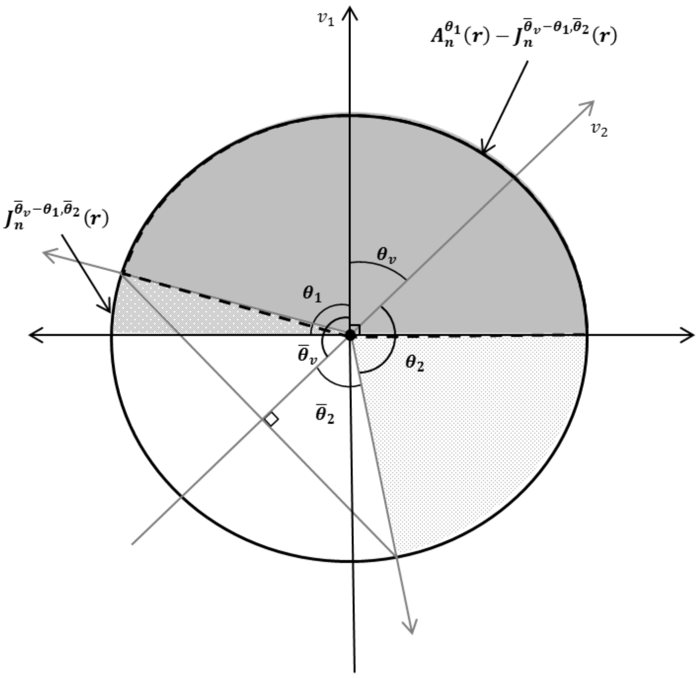

Figure 15: Case 17 in Table 1

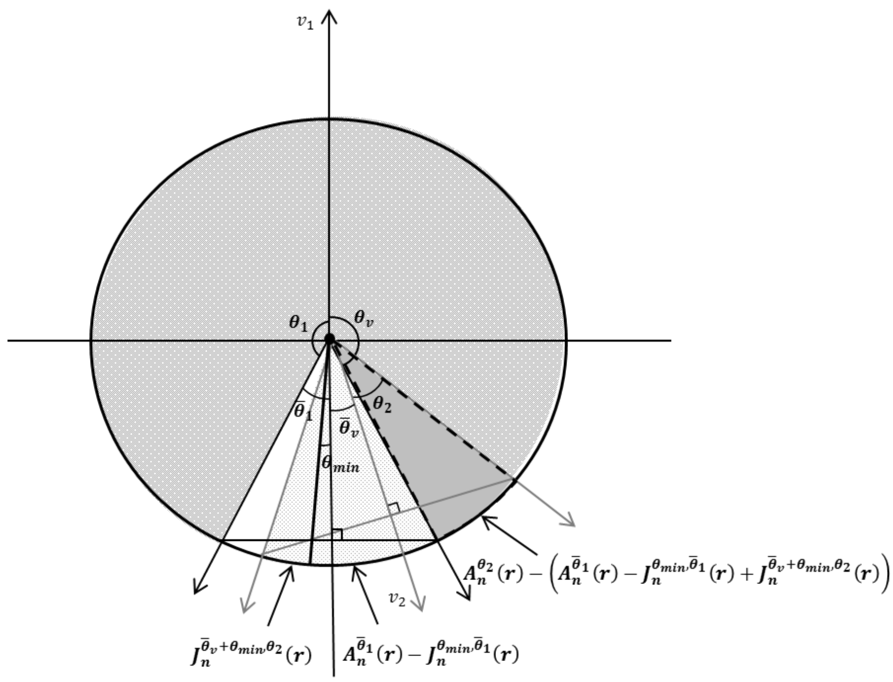

Figure 16: Case 18 in Table 1

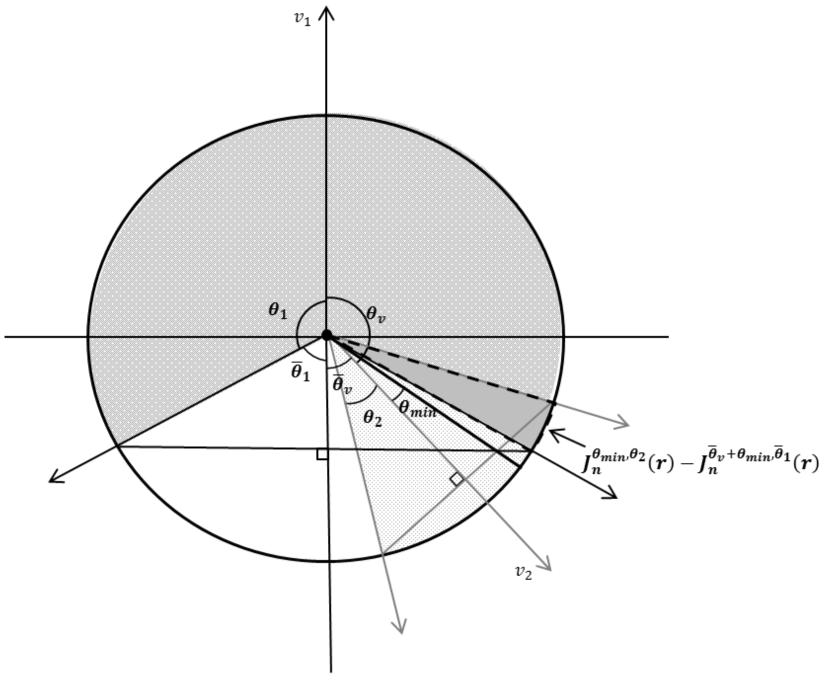

Figure 17: Case 19 in Table 1

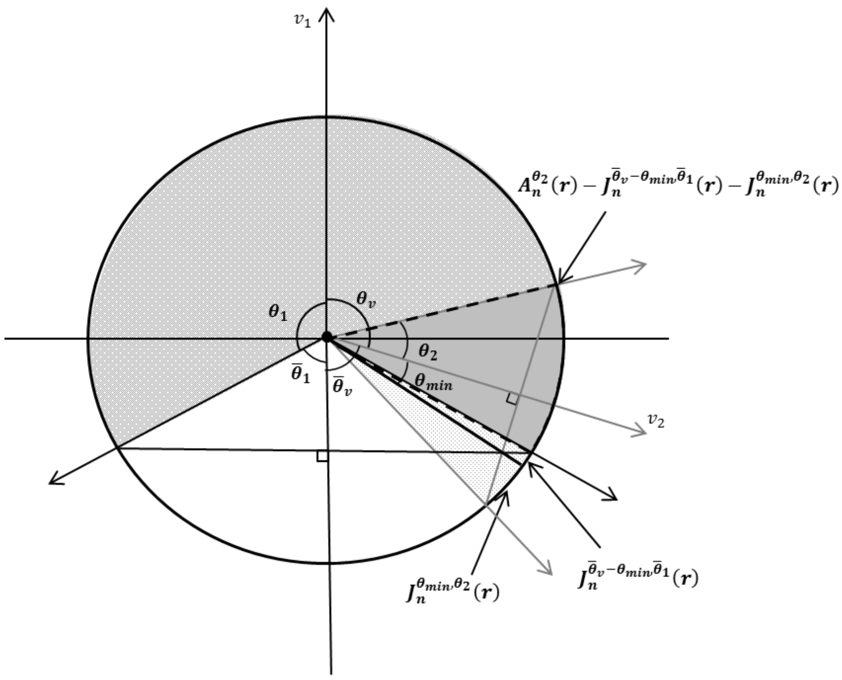

Figure 18: Case 20 in Table 1

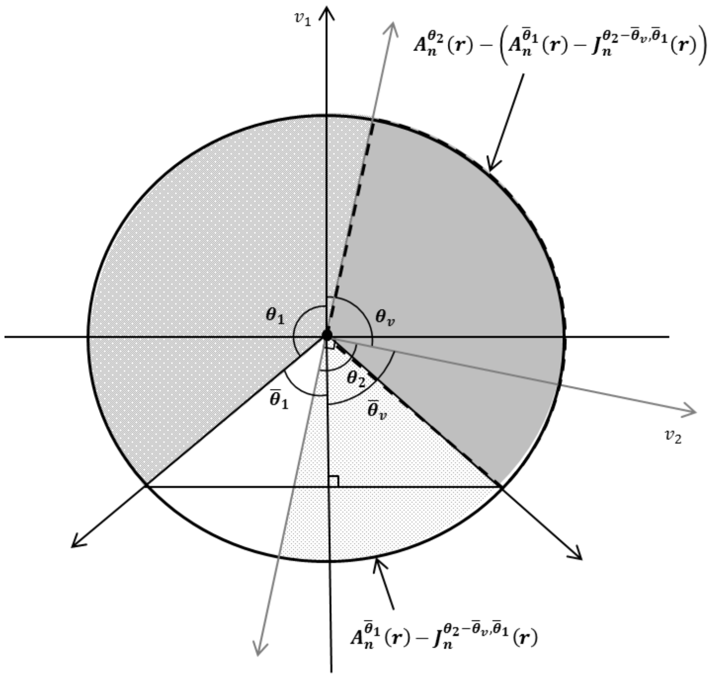

Figure 19: Case 21 in Table 1

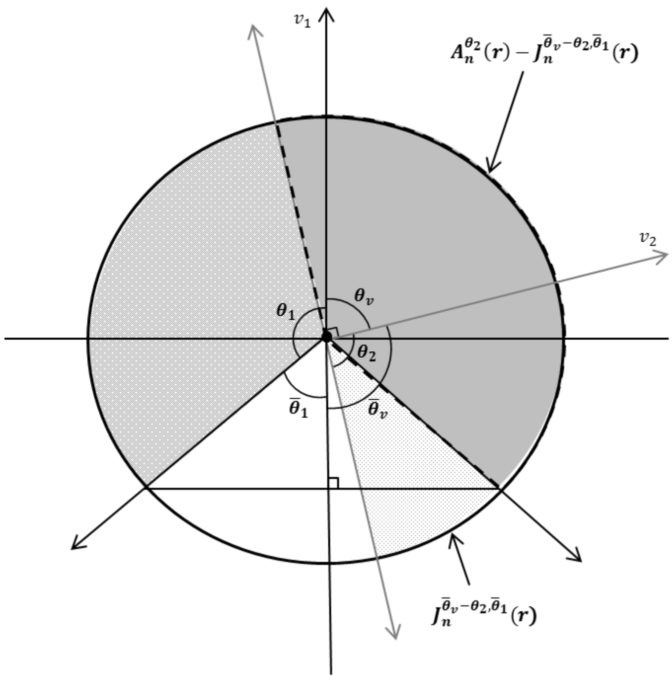

Figure 20: Case 22 in Table 1

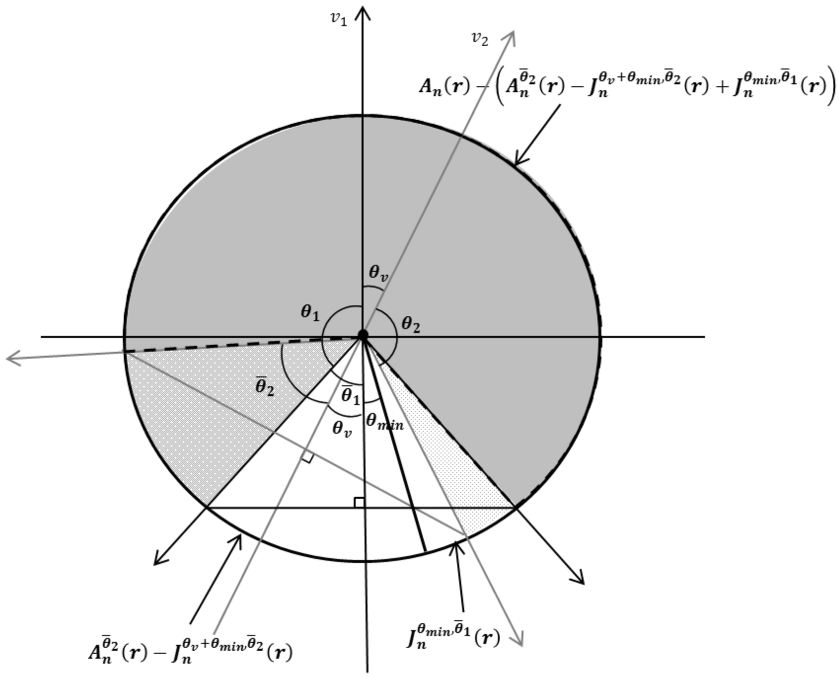

Figure 21: Case 23 in Table 1

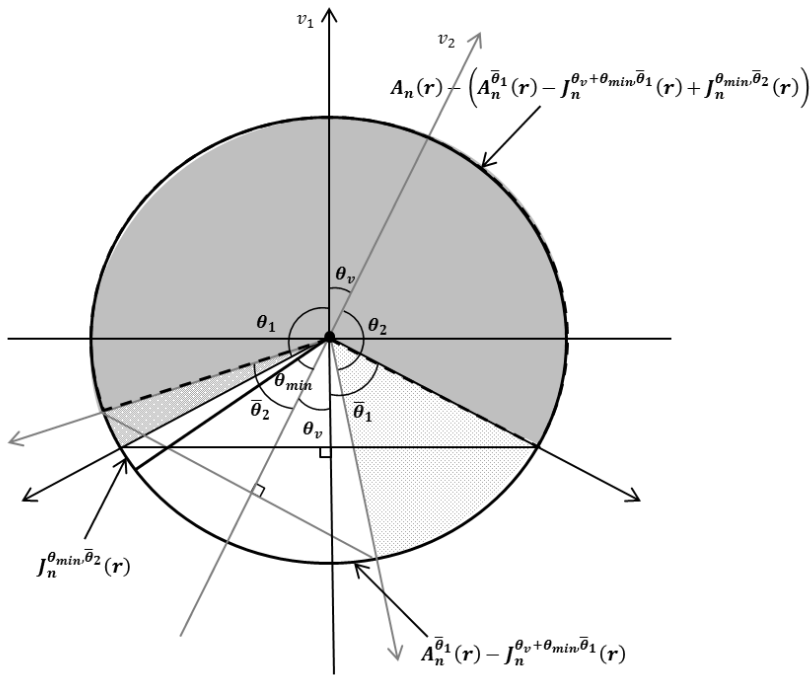

Figure 22: Case 24 in Table 1

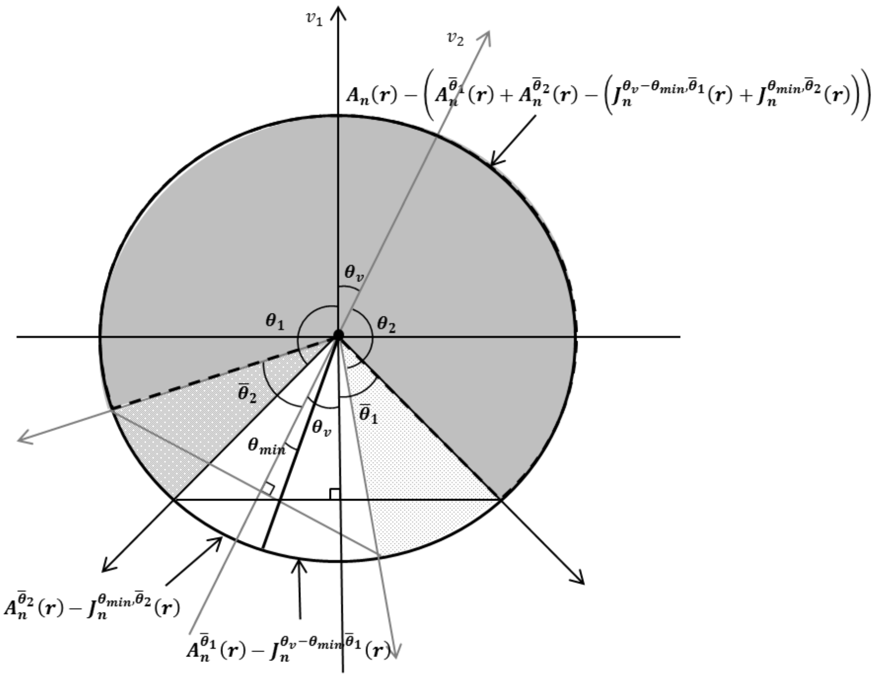

Figure 23: Case 25 in Table 1