Problem 5

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a)
So we have
$$\sum_{x=0}^{N-1} \exp\left(-2\pi ikx/N\right) = \sum_{x=0}^{N-1} \left(\exp\left(-2\pi ik/N\right)\right)^x = \sum_{x=0}^{N-1} \alpha^x.$$
For any geometric sum, we know that
$$\sum_{k=0}^{n} ar^k = a\left(\frac{1-r^{n+1}}{1-r}\right).$$
Therefore, we have that
$$\sum_{x=0}^{N-1} e^{-2\pi ikx/N} = \frac{1-\alpha^N}{1-\alpha} = \frac{1-\exp\left(-2\pi ik\right)}{1-\exp\left(-2\pi ik/N\right)}.$$

b)
If we set $k=0$, we get the indeterminate form $0/0$, so we can use l'Hôpital's Rule:
$$\lim_{k\to 0}\frac{1-\exp\left(-2\pi ik\right)}{1-\exp\left(-2\pi ik/N\right)} = \lim_{k\to 0}\frac{2\pi i\exp\left(-2\pi ik\right)}{(2\pi i/N)\exp\left(-2\pi ik/N\right)} = N\frac{2\pi i}{2\pi i} = N$$

c)
Let's say we have some non-integer sine wave
$$y_m = \sin\left(2\pi qm\right)$$
for some non-integer q. Then, the DFT is given by
$$Y_k = \sum_{m=0}^{N-1} \sin\left(2\pi qm\right)\exp\left(-2\pi ikm/N\right) = \sum_{m=0}^{N-1}\left(\frac{\exp\left(2\pi iqm\right)-\exp\left(-2\pi iqm\right)}{2i}\right)\exp\left(-2\pi ikm/N\right)$$
$$Y_k = \frac{1}{2i}\sum_{m=0}^{N-1}\left(\exp\left(-2\pi i\left(k-Nq\right)m/N\right)-\exp\left(-2\pi i\left(k+Nq\right)m/N\right)\right)$$

$$Y_k = \frac{1}{2i}\left(\frac{1-\exp\left(-2\pi i\left(k-Nq\right)\right)}{1-\exp\left(-2\pi i\left(k-Nq\right)/N\right)} - \frac{1-\exp\left(-2\pi i\left(k+Nq\right)\right)}{1-\exp\left(-2\pi i\left(k+Nq\right)/N\right)}\right)$$

The difference between the NumPy DFT and this analytical solution are shown in the graph 'sine_dft.png' produced by the Python file 'problem5.py' They are both pretty close to a Dirac delta, but they do seem to curve a little bit at the bottom of the spike, meaning that it is not a perfect delta.

d)
As we can see from the plot 'sine_dft_window.png', the addition of the window does seem to tighten up the bottom of the delta a little bit. See problem5.py.

e)
See 'problem5.py' for the numerical proof.
So since we are doing a convolution in the Fourier space, it's useful to order the FT of the window properly, such that

$$\text{window FT} = \left[0, 0, ..., 0, -N/4, N/2, -N/4, 0, ..., 0, 0\right].$$

Therefore, when we do the convolution, the new Fourier transform components Y_k' are going to be related to the original FT components Y_k by

$$Y_k' = -\frac{N}{4}Y_{k-1} + \frac{N}{2}Y_k - \frac{N}{4}Y_{k+1}.$$