Instrumental Variables

Felipe Buchbinder

A refresher from our first class...

WHEN AN IMPORTANT X
IS MISSING IN OUR MODEL
BAD THINGS CAN HAPPEN


For example…?

Biased regression coefficients

Linear Regression refresher

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$
$$\boldsymbol{\epsilon} \sim \mathrm{N}(\mathbf{0}; \sigma^2 \mathbf{I})$$

$$\mathbf{b} = \left(\mathbf{X}^{\mathbf{T}}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathbf{T}}\mathbf{y}$$

b is unbiased, meaning $\mathbb{E}(\mathbf{b}) = \boldsymbol{\beta}$

Now suppose there's a variable, **U**, that affects **Y** but we fail to put it in our model. We simply calculate **b** without it!

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{U} + \boldsymbol{\epsilon}$$
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$$\boldsymbol{\epsilon} \sim \mathrm{N}(\mathbf{0}; \sigma^2\mathbf{I})$$

$$\mathbf{b} = \left(\mathbf{X}^{\mathbf{T}}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathbf{T}}\mathbf{y}$$

In this case, **b** is no longer an unbiased estimate of $\boldsymbol{\beta}$!

$$
\begin{aligned}
\mathbf{b} &= \left(\mathbf{X}^{\mathrm{T}}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\boldsymbol{y} \\
&= \left(\mathbf{X}^{\mathrm{T}}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}(\mathbf{X}\boldsymbol{\beta} + \mathbf{U} + \boldsymbol{\epsilon}) \\
&= \underbrace{\left(\mathbf{X}^{\mathrm{T}}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{X}}_{\mathbf{I}}\boldsymbol{\beta} + \left(\mathbf{X}^{\mathrm{T}}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{U} + \left(\mathbf{X}^{\mathrm{T}}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\boldsymbol{\epsilon} \\
&= \boldsymbol{\beta} + \left(\mathbf{X}^{\mathrm{T}}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{U} + \left(\mathbf{X}^{\mathrm{T}}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\boldsymbol{\epsilon}
\end{aligned}
$$

Taking the expected value...

$$
\mathbb{E}(\mathbf{b}) = \boldsymbol{\beta} + \left(\mathbf{X}^{\mathrm{T}}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{U} + \left(\mathbf{X}^{\mathrm{T}}\mathbf{X}\right)^{-1}\underbrace{\mathbb{E}\left(\mathbf{X}^{\mathrm{T}}\boldsymbol{\epsilon}\right)}_{0}
$$

$$\therefore$$

$$
\mathbb{E}(\mathbf{b}) = \boldsymbol{\beta} + \left(\mathbf{X}^{\mathrm{T}}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{U}
$$

Thus, in general, $\mathbb{E}(\mathbf{b}) \neq \boldsymbol{\beta}$:

$\mathbf{b}$ is a biased estimate of $\boldsymbol{\beta}$

Everything would be fine, if only we had $\mathbf{X}^{\mathrm{T}}\mathbf{U} = 0$ ...

$$\mathbf{b} = \boldsymbol{\beta} + \left(\mathbf{X}^{\mathrm{T}}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{U}$$

Everything would be fine, if only we had $\mathbf{X}^{\mathrm{T}}\mathbf{U} = 0$ …

$$\mathbf{b} = \boldsymbol{\beta} + \left(\mathbf{X}^{\mathrm{T}}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{U}$$

What if we had a variable $\mathbf{Z}$ that's correlated to $\mathbf{X}$ but not to $\mathbf{U}$? We could use <u>that</u> to get an unbiased estimate of $\boldsymbol{\beta}$

Instrumental Variable

An **instrumental variable** is a variable that affects Y *only through* X.

Mathematically, this means it is correlated to X but not to U.

Instrumental variable estimator

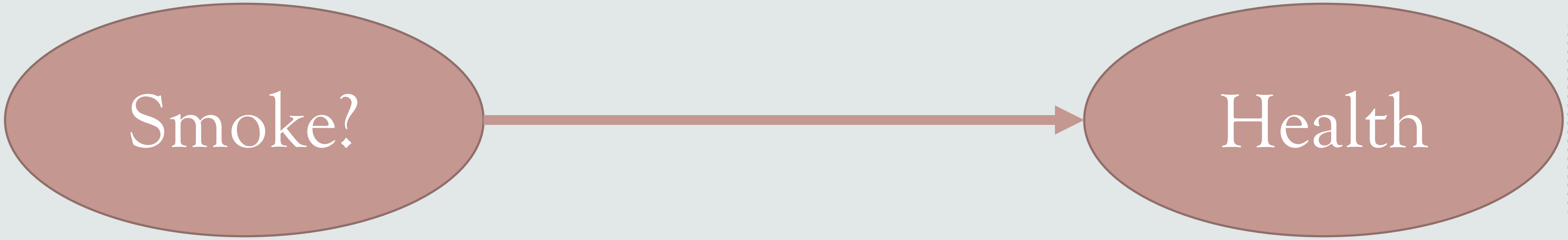
$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{U} + \boldsymbol{\epsilon}$$

$$\mathbf{Cov}(\mathbf{Z};\mathbf{Y}) = \mathbf{Cov}(\mathbf{Z};\mathbf{X}\boldsymbol{\beta} + \mathbf{U} + \boldsymbol{\epsilon})$$

$$\mathbf{Cov}(\mathbf{Z};\mathbf{Y}) = \mathbf{Cov}(\mathbf{Z};\mathbf{X})\boldsymbol{\beta} + \underbrace{\mathbf{Cov}(\mathbf{Z};\mathbf{U})}_{\mathbf{0}} + \underbrace{\mathbf{Cov}(\mathbf{Z};\boldsymbol{\epsilon})}_{\mathbf{0}}$$

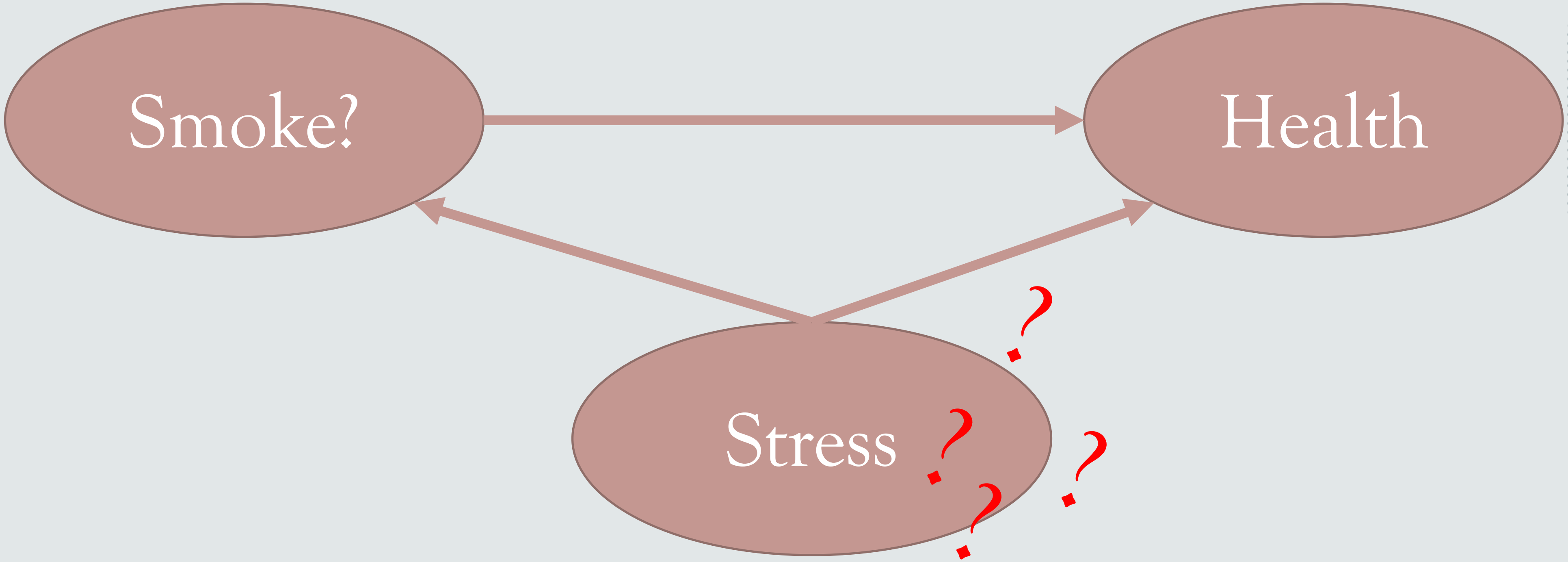
$$\boldsymbol{\beta} = \frac{\mathbf{Cov}(\mathbf{Z};\mathbf{Y})}{\mathbf{Cov}(\mathbf{Z};\mathbf{X})} = \left(\mathbf{Z}^{\mathbf{T}}\mathbf{X}\right)^{-1}\mathbf{Z}^{\mathbf{T}}\mathbf{Y} = \boldsymbol{\beta}_{\mathbf{IV}}$$

Effect of smoking on health

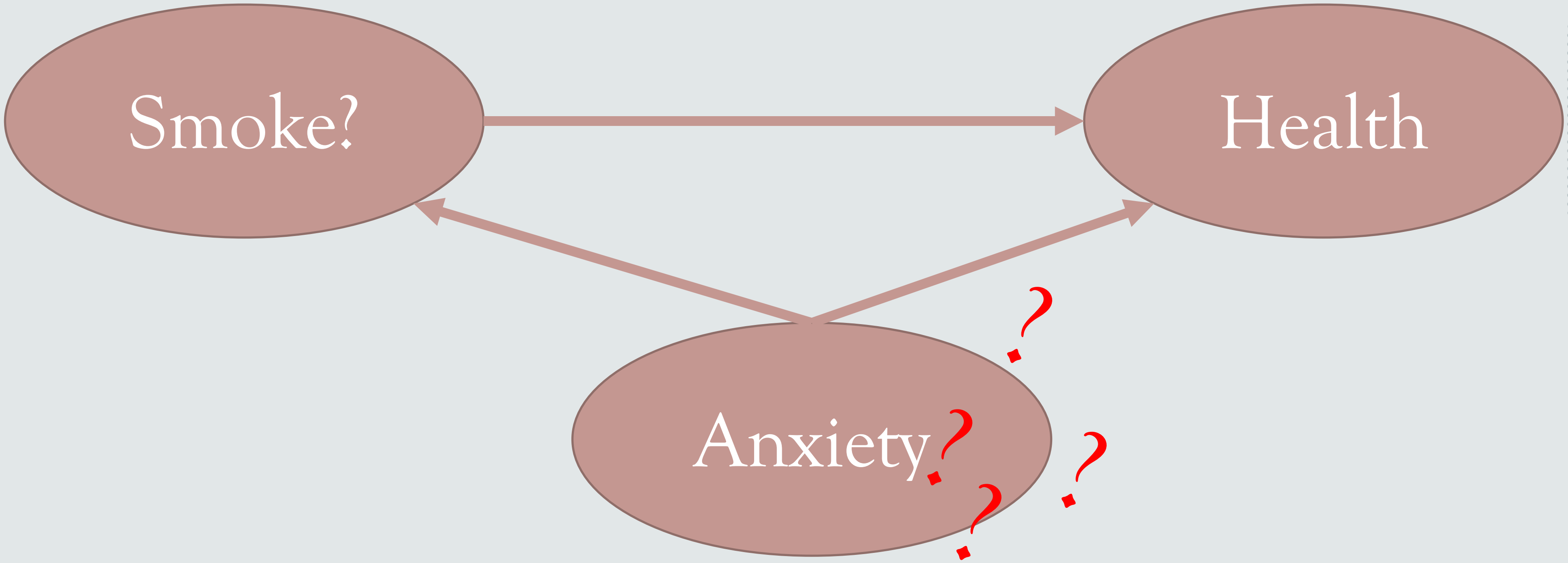


What variables may I be omitting here?

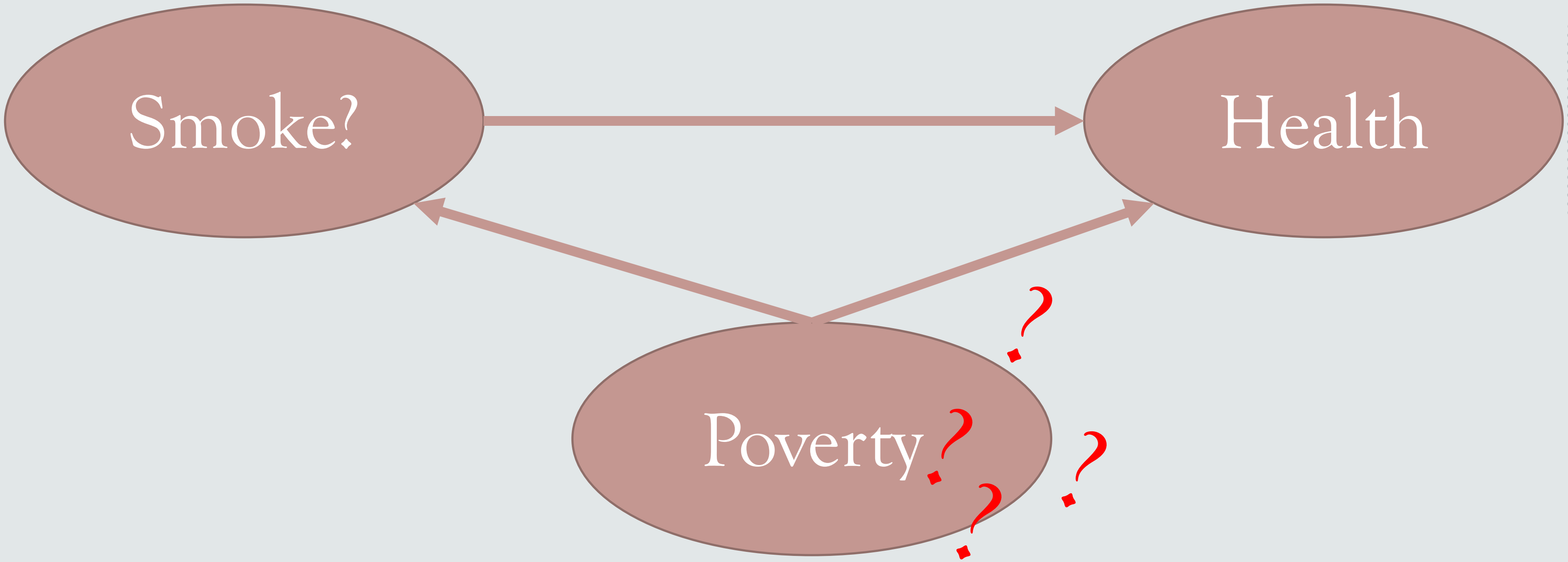
Effect of smoking on health



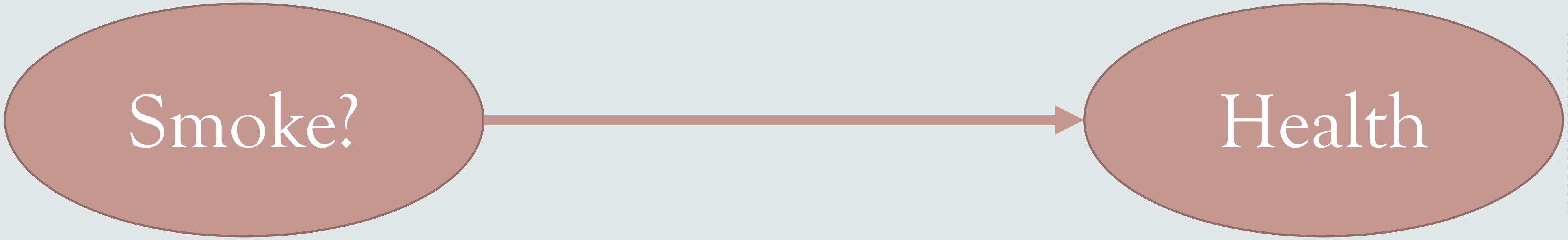
Effect of smoking on health



Effect of smoking on health

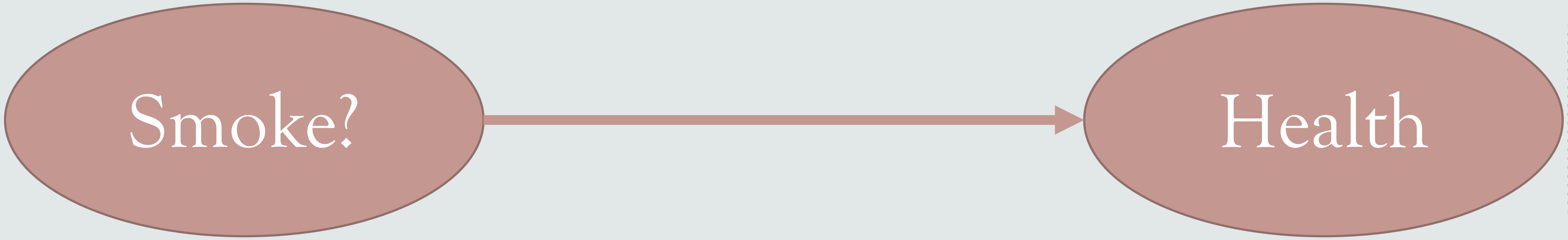


Effect of smoking on health



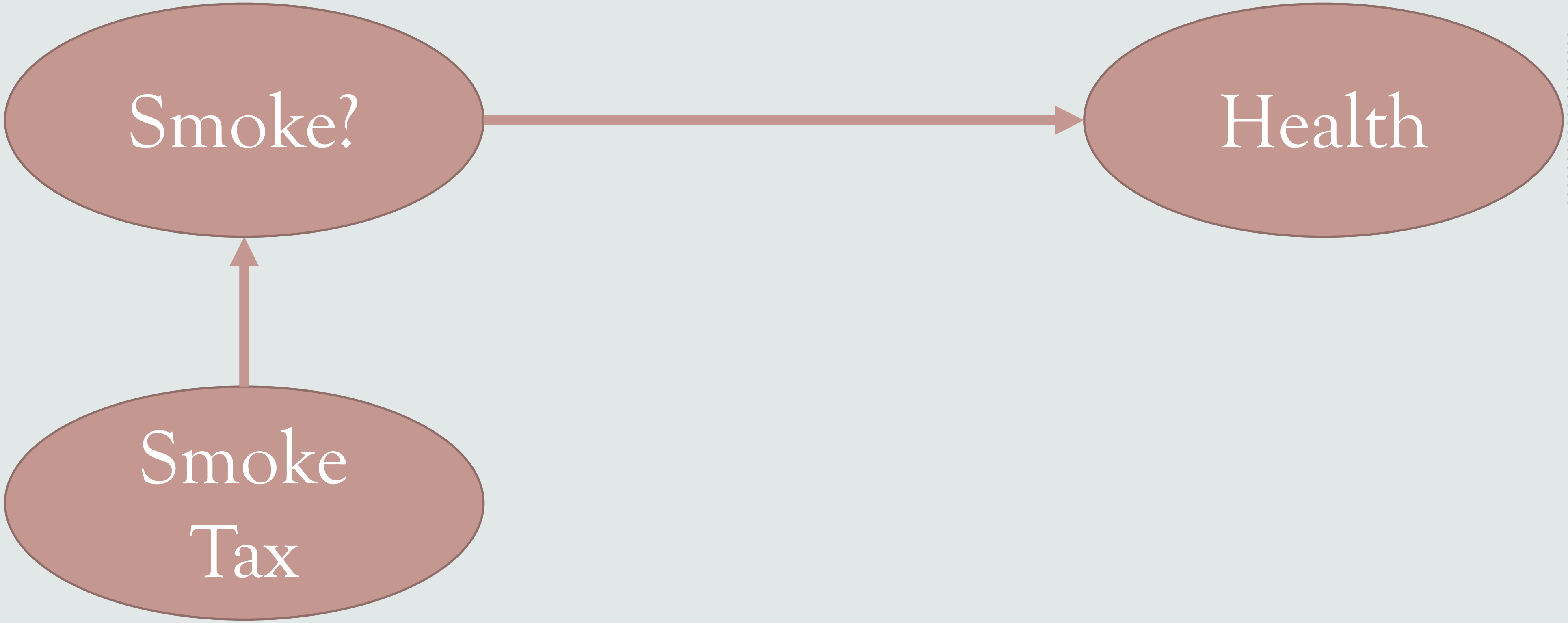
What characteristics must a variable have to work as an instrument in this case?

Effect of smoking on health

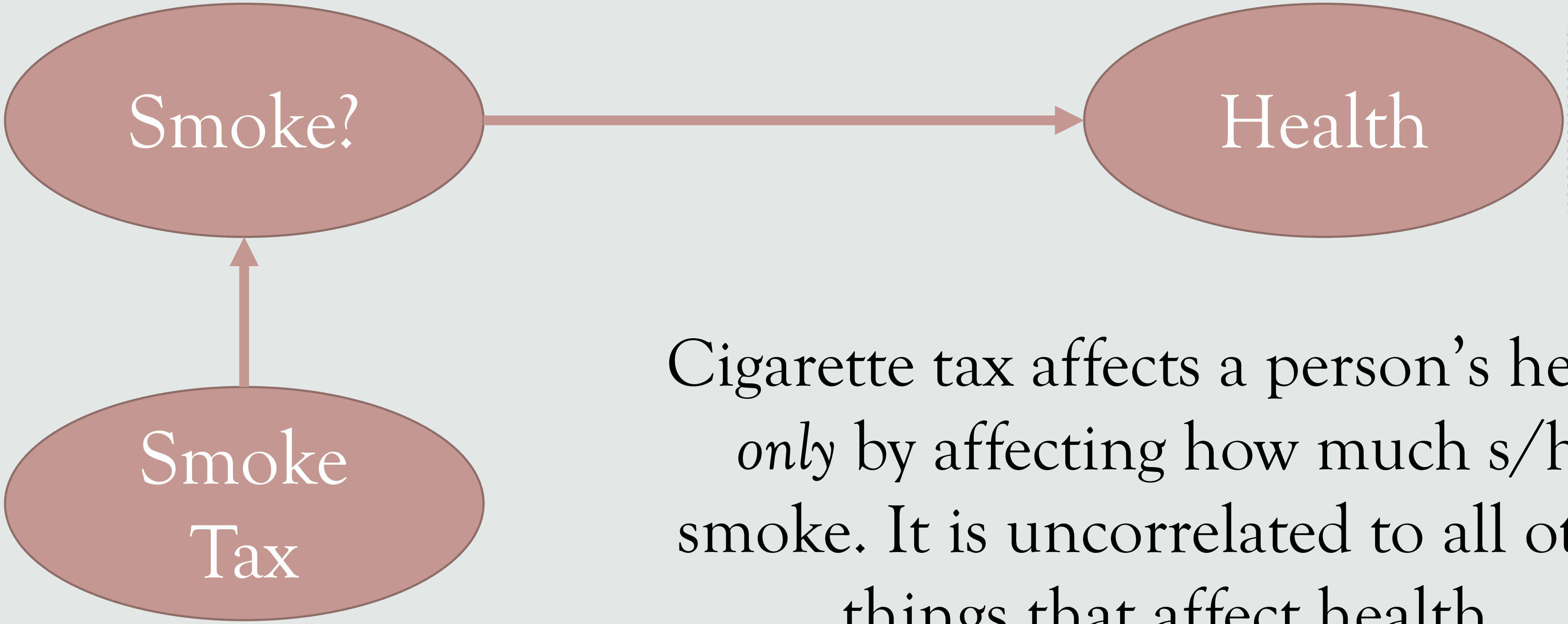


Can you think of any variable
that we could use as an instrument?

Effect of smoking on health



Effect of smoking on health



Cigarette tax affects a person's health *only* by affecting how much s/he smoke. It is uncorrelated to all other things that affect health.

Pearl's (2000) conditions for a good instrument

The equations of interest are "structural," not "regression".

The error term U stands for all exogenous factors that affect Y when X is held constant.

The instrument Z should be independent of U.

$(\mathbf{Z}^{\mathbf{T}}\mathbf{U} = 0)$

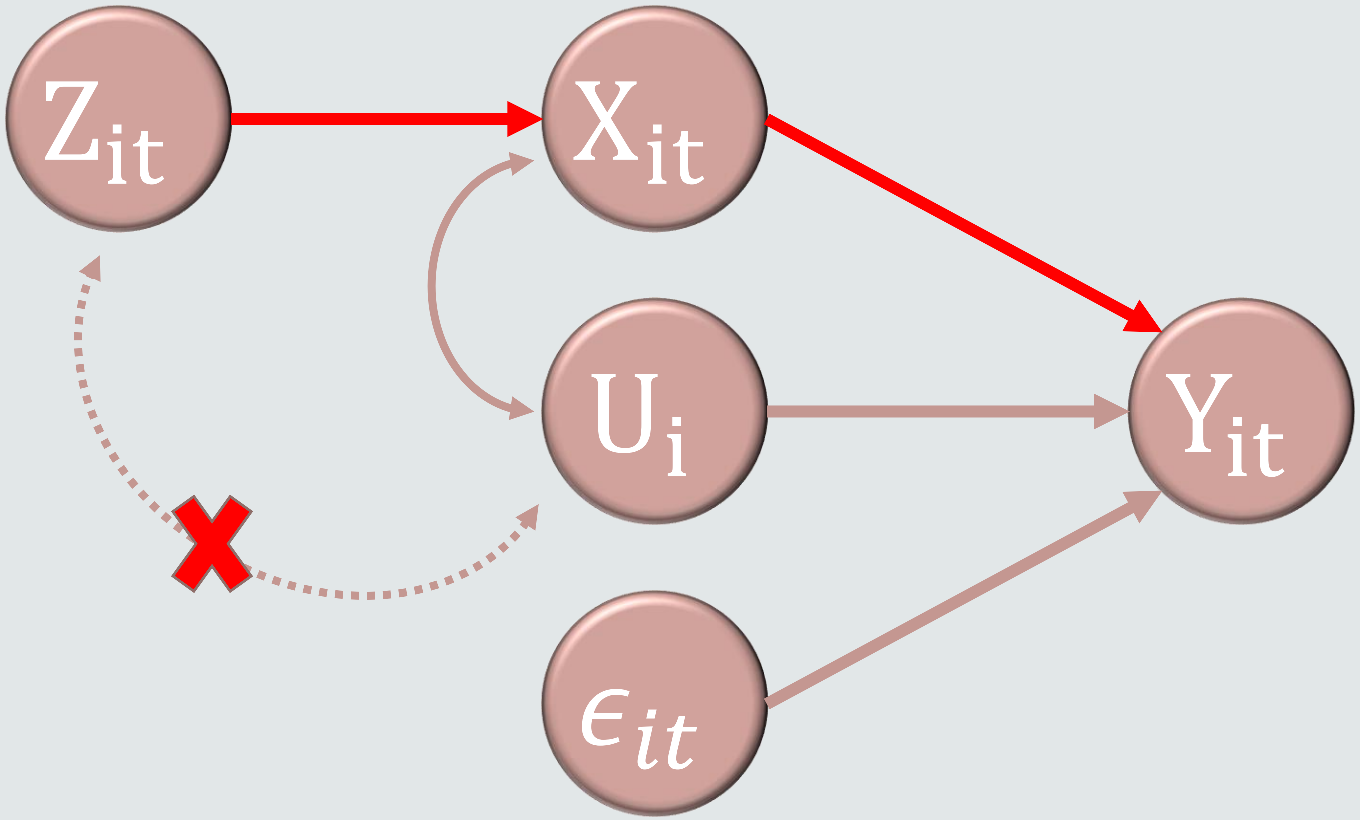
The instrument Z should not affect Y when X is held constant

(exclusion restriction)

The instrument Z should not be independent of X.

$(\mathbf{Z}^{\mathbf{T}}\mathbf{X} \neq 0)$

Representation as a Bayesian Causal Network



Weak instruments (Bound, Jaeger & Baker, 1995)

- $\mathrm{Corr}(\mathbf{Z}; \mathbf{X})$ is low
- Can lead to *even more biased coefficients than simply ignoring the problem of omitted variables* and running plain, old, OLS.
- Can also lead to inflated standard errors (impact on p-values?)

Weak instruments yield biased coefficients (1/2)

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{U} + \epsilon$$

$$\mathbf{Cov}(\mathbf{Z};\mathbf{Y}) = \mathbf{Cov}(\mathbf{Z};\mathbf{X}\boldsymbol{\beta} + \mathbf{U} + \epsilon)$$

$$\mathbf{Cov}(\mathbf{Z};\mathbf{Y}) = \underbrace{\mathbf{Cov}(\mathbf{Z};\mathbf{X})}_{\approx 0}\,\boldsymbol{\beta}_{\mathbf{WIV}} + \underbrace{\mathbf{Cov}(\mathbf{Z};\mathbf{U})}_{\text{not exactly 0}} + \underbrace{\mathbf{Cov}(\mathbf{Z};\epsilon)}_{0}$$

$$\boldsymbol{\beta}_{\mathbf{WIV}} = \frac{\mathbf{Cov}(\mathbf{Z};\mathbf{Y})}{\mathbf{Cov}(\mathbf{Z};\mathbf{X})} + \frac{\mathbf{Cov}(\mathbf{Z};\mathbf{U})}{\mathbf{Cov}(\mathbf{Z};\mathbf{X})} = \boldsymbol{\beta} + \frac{\mathbf{Cov}(\mathbf{Z};\mathbf{U})}{\mathbf{Cov}(\mathbf{Z};\mathbf{X})}$$

Weak instruments yield biased coefficients (1/2)

$$\boldsymbol{\beta}_{\mathbf{WIV}} = \boldsymbol{\beta} + \frac{\mathbf{Cov(Z; U)}}{\mathbf{Cov(Z; X)}}$$

$$\mathbf{bias} = \frac{\mathbf{Cov(Z; U)}}{\mathbf{Cov(Z; X)}}$$

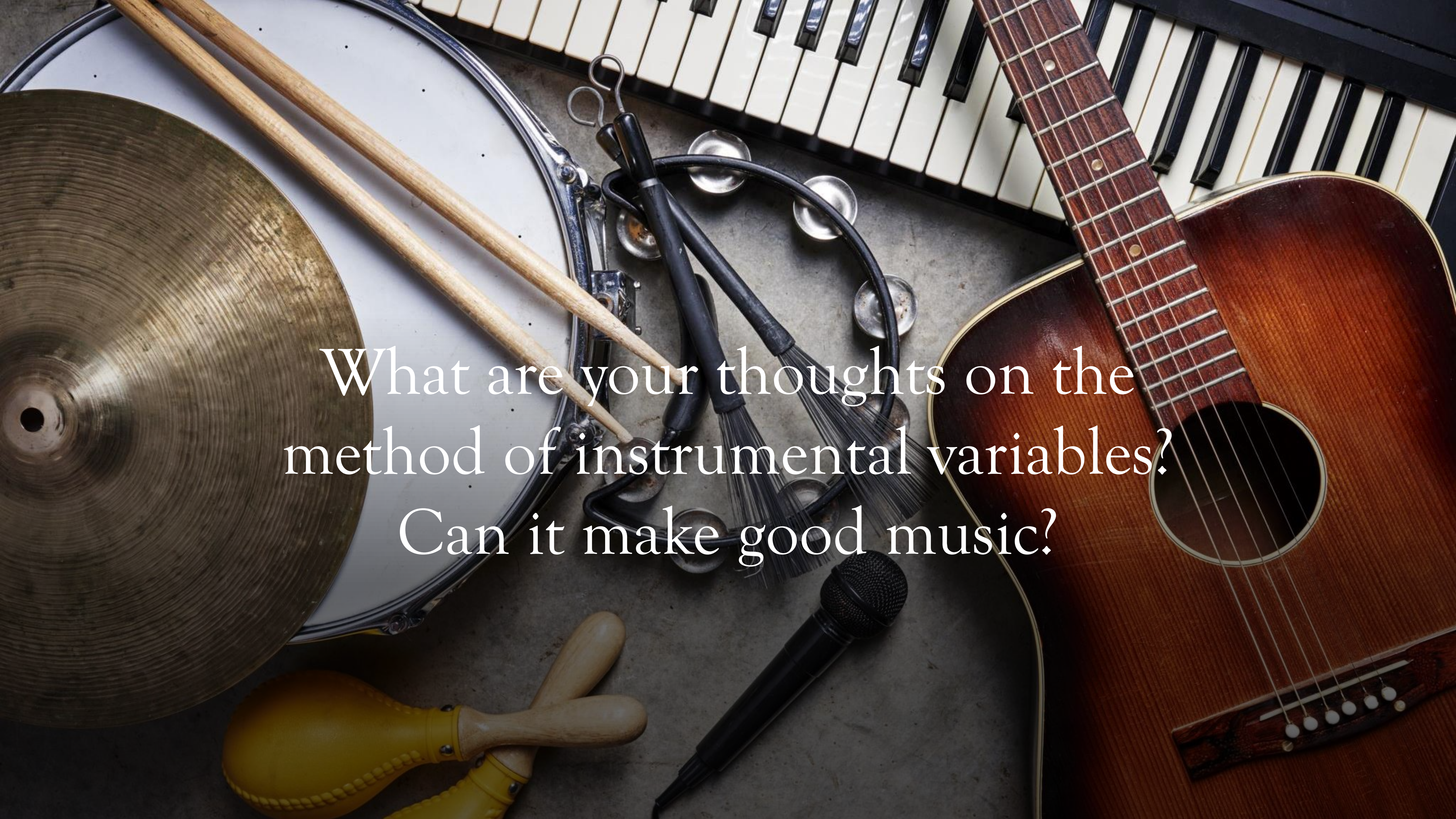
As $\mathbf{Cov(Z; X)} \rightarrow \mathbf{0}$, $\mathbf{bias} \rightarrow +\infty$ even if $\mathbf{Cov(Z; U)}$ is very small.

Weak instruments yield inflated standard errors

$$\boldsymbol{\beta}_{\mathbf{IV}} = \left(\mathbf{Z}^{\mathbf{T}}\mathbf{X}\right)^{-1}\mathbf{Z}^{\mathbf{T}}\mathbf{Y}$$

$$\mathrm{V}(\boldsymbol{\beta}_{\mathbf{IV}}) = \left(\mathbf{Z}^{\mathbf{T}}\mathbf{X}\right)^{-1}\sigma^2 \to +\infty$$

$$\text{as } \mathbf{Z}^{\mathbf{T}}\mathbf{X} = \mathbf{Cov}(\mathbf{Z};\mathbf{X}) \to 0$$



What are your thoughts on the method of instrumental variables? Can it make good music?