

# Instrumental Variables

Felipe Buchbinder



A refresher from  
our first class...



WHEN AN IMPORTANT X  
IS MISSING IN OUR MODEL  
BAD THINGS CAN HAPPEN





For example...?



Biased  
regression  
coefficients



# Linear Regression refresher

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$
$$\boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}; \sigma^2 \mathbf{I})$$

$$\mathbf{b} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y}$$

$\mathbf{b}$  is unbiased, meaning  $\mathbb{E}(\mathbf{b}) = \boldsymbol{\beta}$

Now suppose there's a variable,  $\mathbf{U}$ , that affects  $\mathbf{Y}$  but we fail to put it in our model. We simply calculate  $\mathbf{b}$  without it!

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$$\mathbf{b} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y}$$

In this case,  $\mathbf{b}$  is no longer an unbiased estimate of  $\boldsymbol{\beta}$ !



$$\begin{aligned}
\mathbf{b} &= (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y} \\
&= (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T (\mathbf{X} \boldsymbol{\beta} + \mathbf{U} + \boldsymbol{\epsilon}) \\
&= \underbrace{(\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{X}}_{\mathbf{I}} \boldsymbol{\beta} + (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{U} + (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \boldsymbol{\epsilon} \\
&= \boldsymbol{\beta} + (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{U} + (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \boldsymbol{\epsilon}
\end{aligned}$$

Taking the expected value...

$$\begin{aligned}
\mathbb{E}(\mathbf{b}) &= \boldsymbol{\beta} + (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{U} + (\mathbf{X}^T \mathbf{X})^{-1} \underbrace{\mathbb{E}(\mathbf{X}^T \boldsymbol{\epsilon})}_0 \\
&\quad \therefore \\
\mathbb{E}(\mathbf{b}) &= \boldsymbol{\beta} + (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{U}
\end{aligned}$$

Thus, in general,  $\mathbb{E}(\mathbf{b}) \neq \boldsymbol{\beta}$ :

**b** is a biased estimate of **β**

Everything would be fine, if only we had  $\mathbf{X}^T \mathbf{U} = 0 \dots$

$$\mathbf{b} = \boldsymbol{\beta} + (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{U}$$

Everything would be fine, if only we had  $\mathbf{X}^T \mathbf{U} = 0 \dots$

$$\mathbf{b} = \boldsymbol{\beta} + (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{U}$$

What if we had a variable  $\mathbf{Z}$  that's correlated to  $\mathbf{X}$  but not to  $\mathbf{U}$ ? We could use that to get an unbiased estimate of  $\boldsymbol{\beta}$

# Instrumental Variable

An **instrumental variable** is a variable that affects  $Y$  *only through*  $X$ .

Mathematically, this means it is correlated to  $X$  but not to  $U$ .



# Instrumental variable estimator

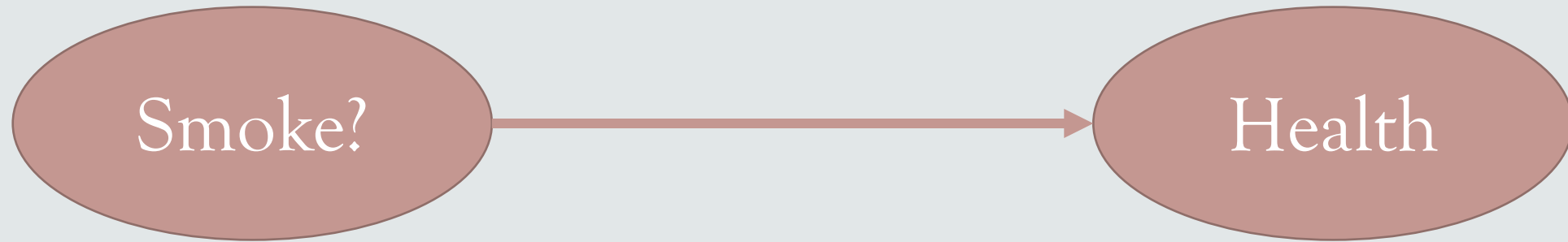
$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{U} + \boldsymbol{\epsilon}$$

$$\mathbf{Cov}(\mathbf{Z}; \mathbf{Y}) = \mathbf{Cov}(\mathbf{Z}; \mathbf{X}\boldsymbol{\beta} + \mathbf{U} + \boldsymbol{\epsilon})$$

$$\mathbf{Cov}(\mathbf{Z}; \mathbf{Y}) = \mathbf{Cov}(\mathbf{Z}; \mathbf{X})\boldsymbol{\beta} + \underbrace{\mathbf{Cov}(\mathbf{Z}; \mathbf{U})}_0 + \underbrace{\mathbf{Cov}(\mathbf{Z}; \boldsymbol{\epsilon})}_0$$

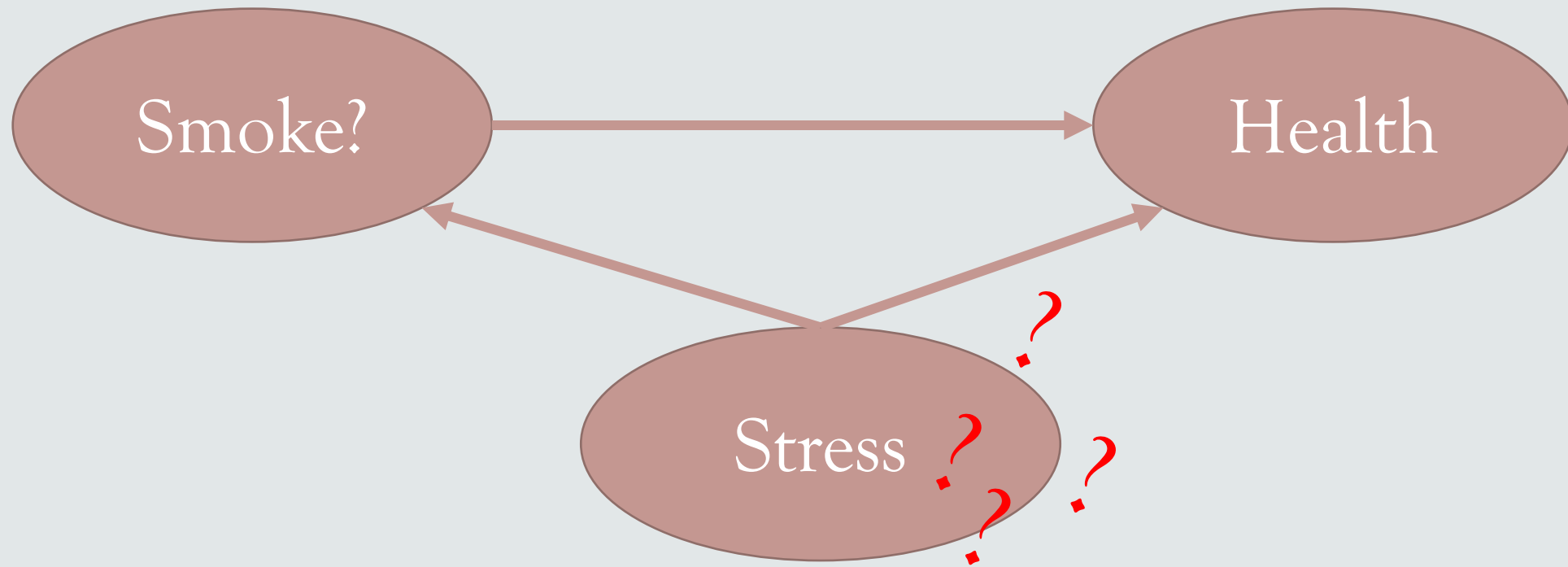
$$\boldsymbol{\beta} = \frac{\mathbf{Cov}(\mathbf{Z}; \mathbf{Y})}{\mathbf{Cov}(\mathbf{Z}; \mathbf{X})} = (\mathbf{Z}^T \mathbf{X})^{-1} \mathbf{Z}^T \mathbf{Y} = \boldsymbol{\beta}_{IV}$$

# Effect of smoking on health

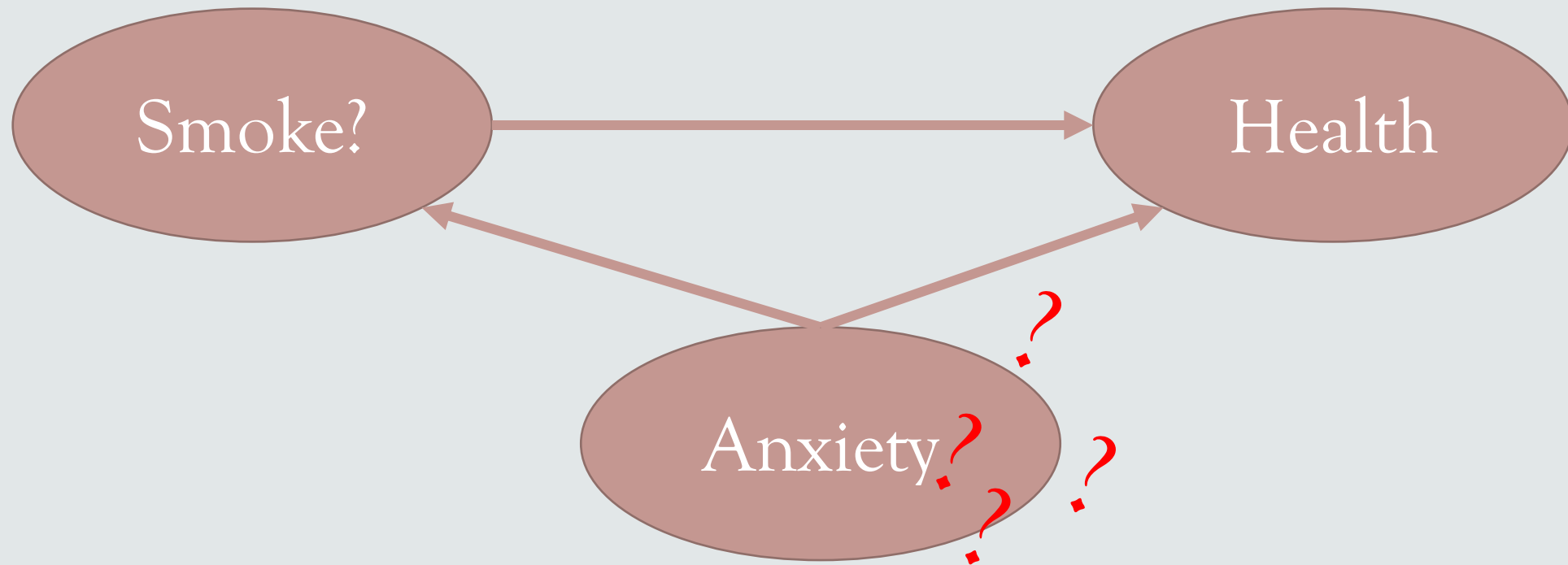


What variables may I be omitting here?

# Effect of smoking on health

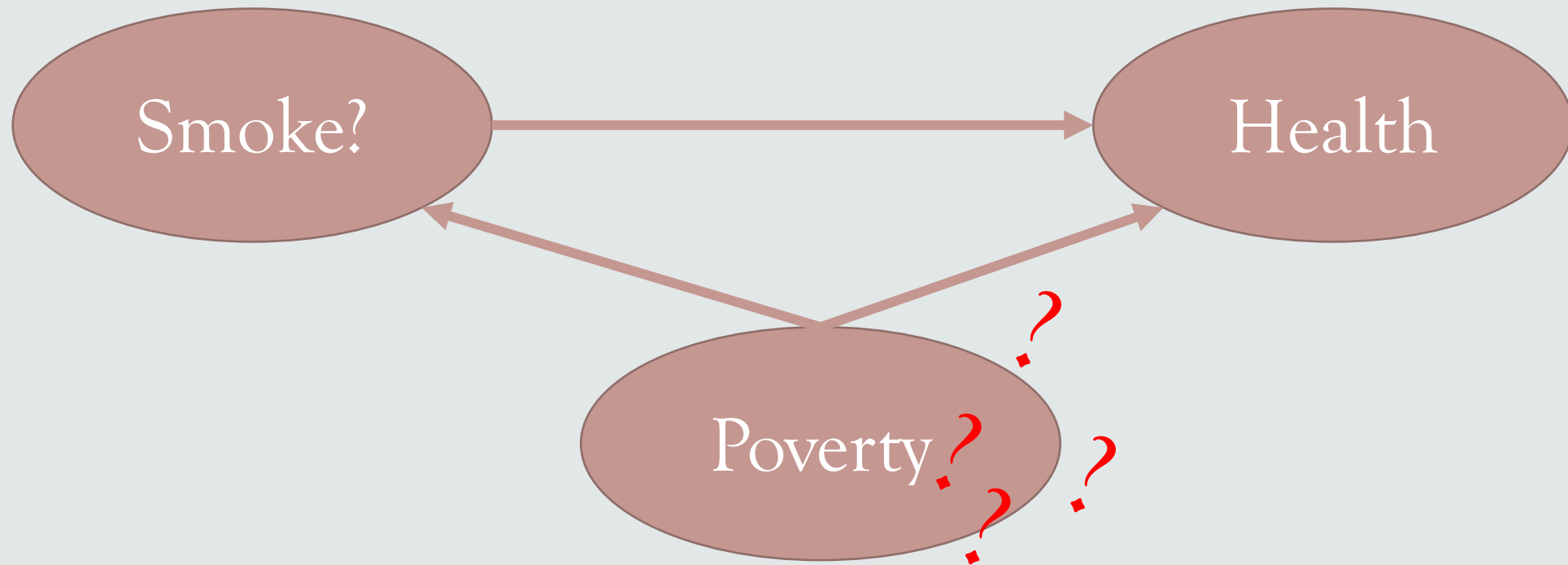


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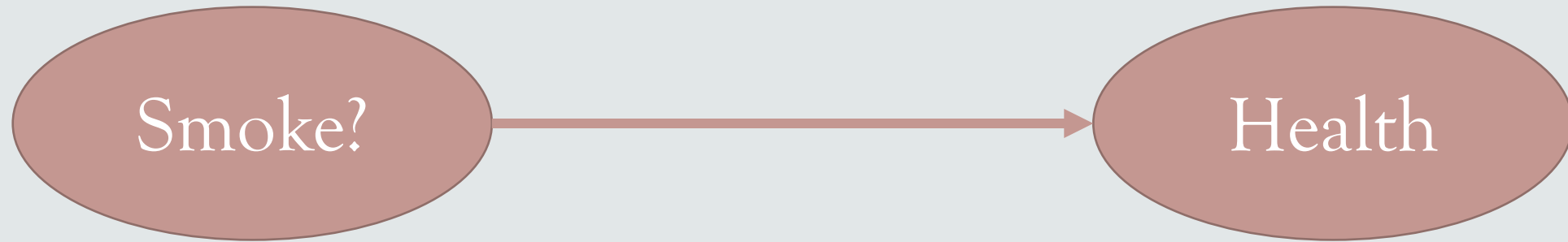




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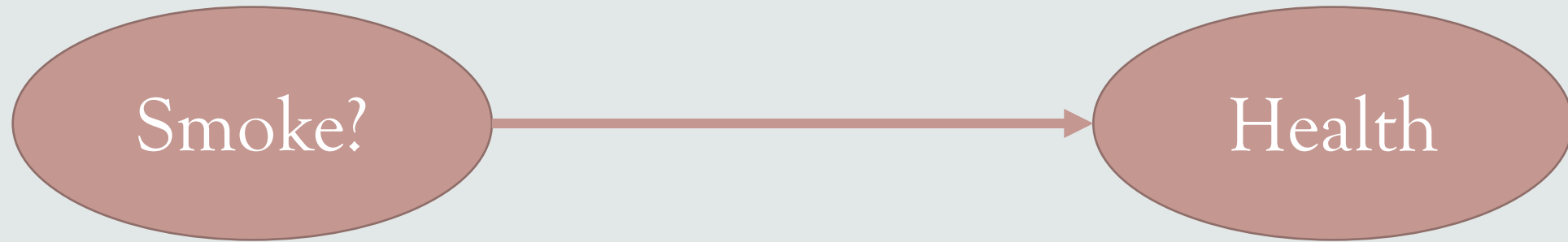


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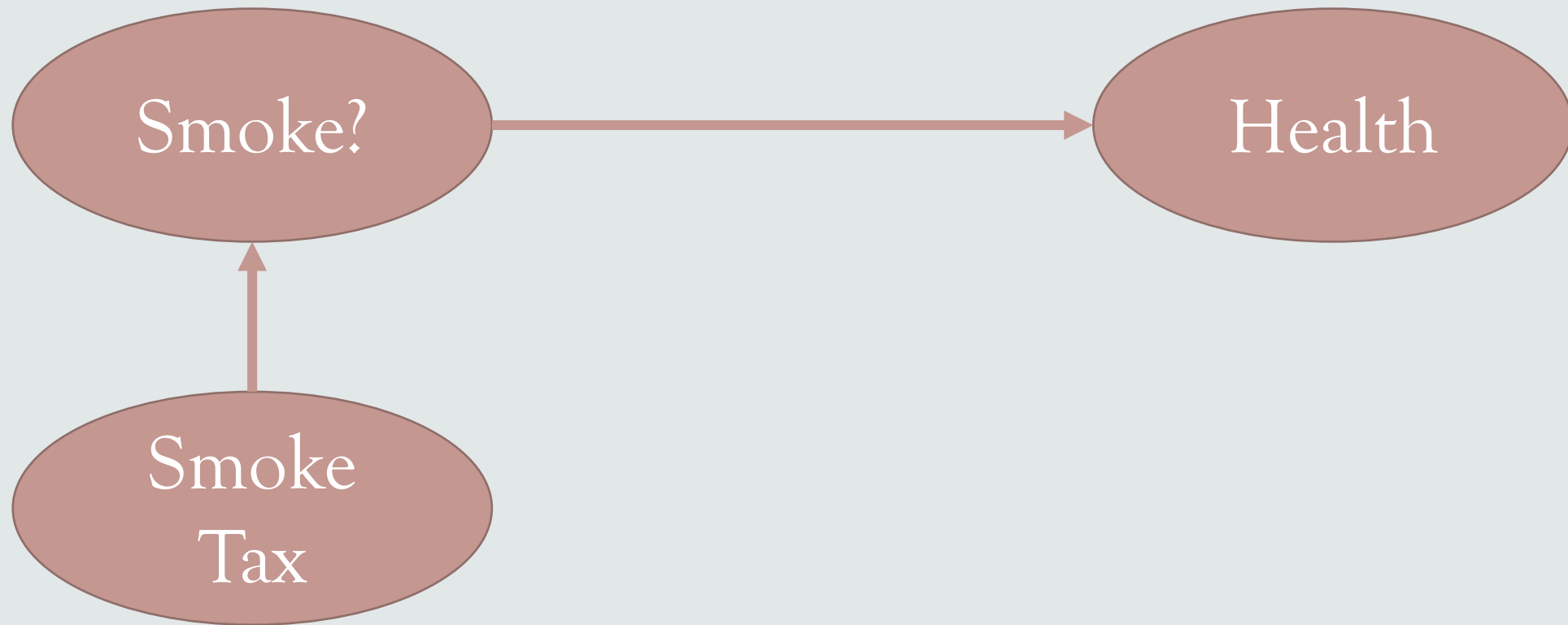
What characteristics must a variable have to work as an instrument in this case?

# Effect of smoking on health



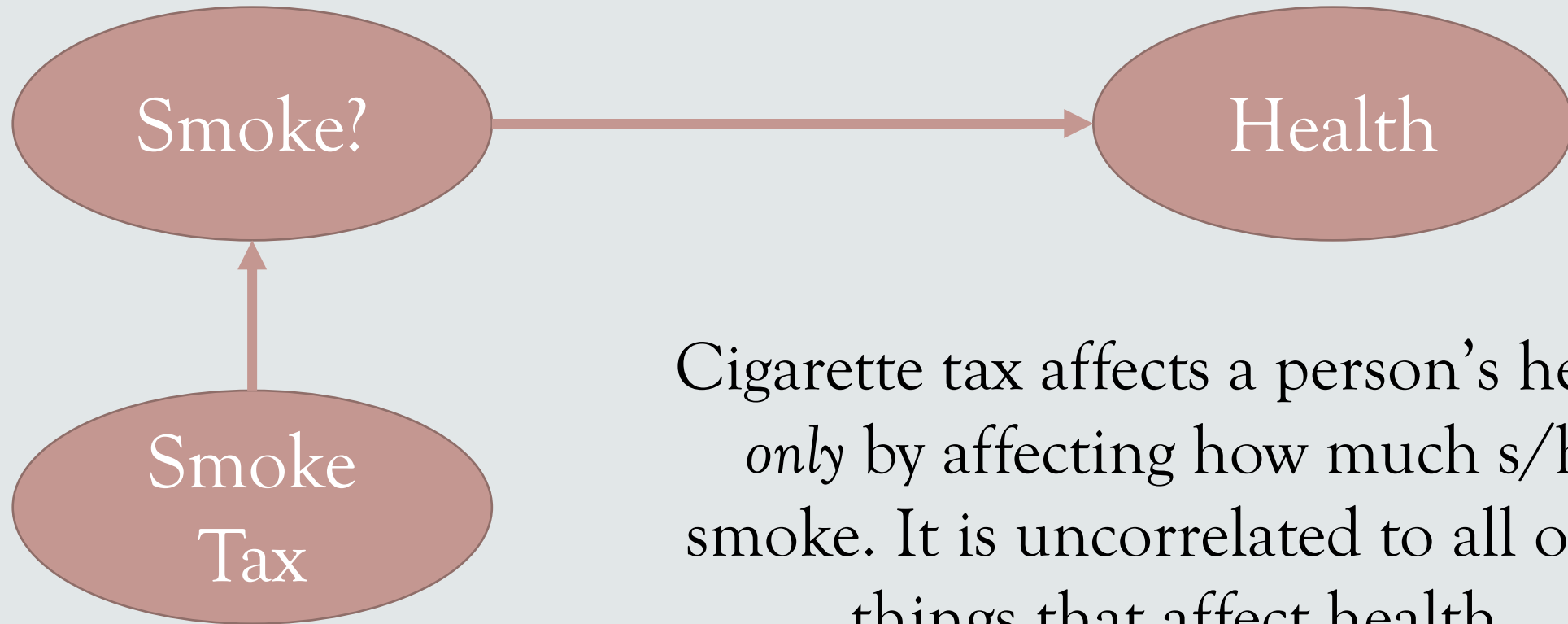
Can you think of any variable  
that we could use as an instrument?

# Effect of smoking on health





# Effect of smoking on health



Cigarette tax affects a person's health *only* by affecting how much s/he smoke. It is uncorrelated to all other things that affect health.

# Pearl's (2000) conditions for a good instrument

The equations of interest are "structural," not "regression".

The error term  $U$  stands for all exogenous factors that affect  $Y$  when  $X$  is held constant.

The instrument  $Z$  should be independent of  $U$ .

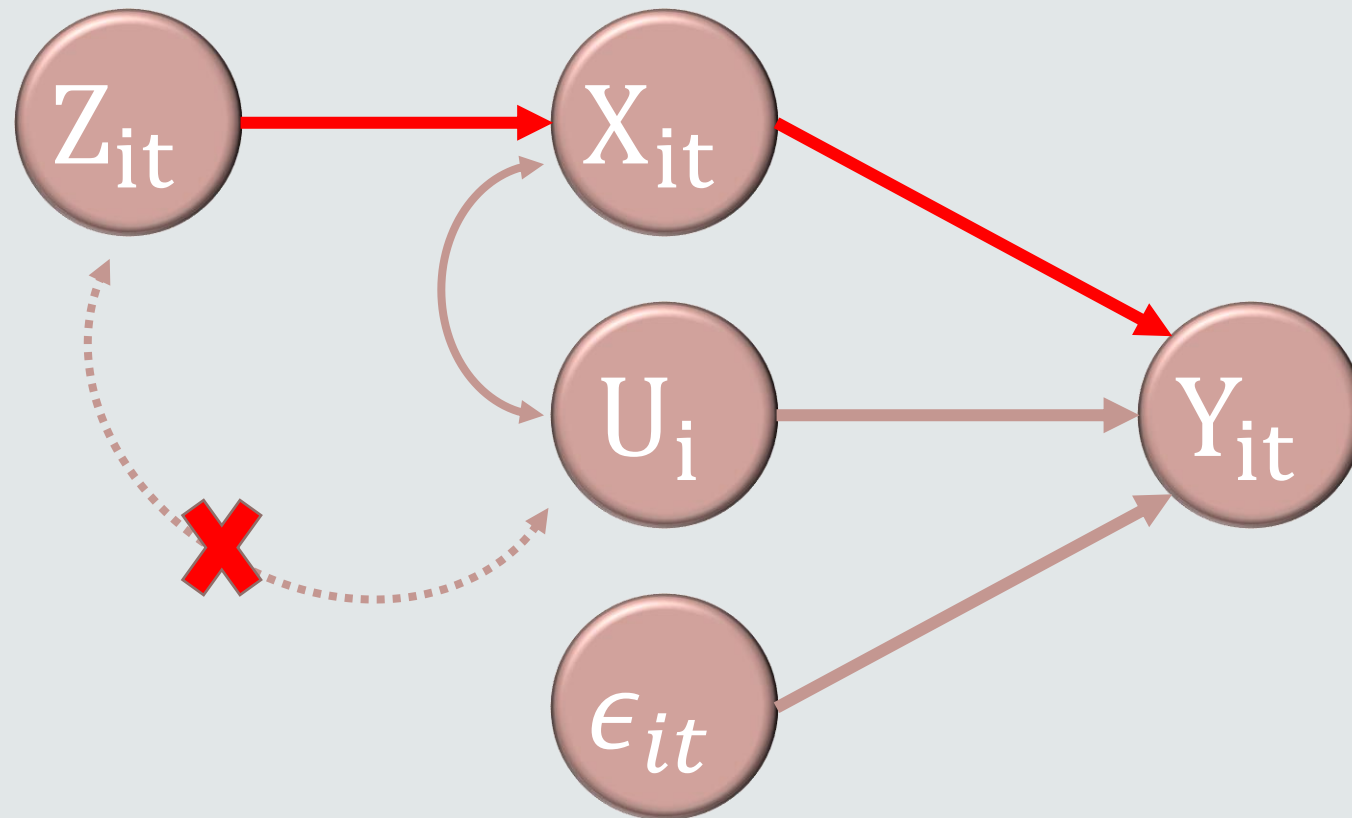
$$(\mathbf{Z}^T \mathbf{U} = 0)$$

The instrument  $Z$  should not affect  $Y$  when  $X$  is held constant  
(exclusion restriction)

The instrument  $Z$  should not be independent of  $X$ .

$$(\mathbf{Z}^T \mathbf{X} \neq 0)$$

## Representation as a Bayesian Causal Network



# Weak instruments (Bound, Jaeger & Baker, 1995)

- $\text{Corr}(\mathbf{Z}; \mathbf{X})$  is low
- Can lead to *even more biased coefficients than simply ignoring the problem of omitted variables and running plain, old, OLS.*
- Can also lead to inflated standard errors (impact on *p*-values?)





Weak instruments yield biased coefficients (1/2)

$$Y = X\beta + U + \epsilon$$

$$\text{Cov}(Z; Y) = \text{Cov}(Z; X\beta + U + \epsilon)$$

$$\text{Cov}(Z; Y) = \underbrace{\text{Cov}(Z; X)}_{\approx 0} \beta_{\text{WIV}} + \underbrace{\text{Cov}(Z; U)}_{\text{not exactly } 0} + \underbrace{\text{Cov}(Z; \epsilon)}_0$$

$$\beta_{\text{WIV}} = \frac{\text{Cov}(Z; Y)}{\text{Cov}(Z; X)} + \frac{\text{Cov}(Z; U)}{\text{Cov}(Z; X)} = \beta + \frac{\text{Cov}(Z; U)}{\text{Cov}(Z; X)}$$

Weak instruments yield biased coefficients (1/2)

$$\beta_{\text{WIV}} = \beta + \frac{\text{Cov}(\mathbf{Z}; \mathbf{U})}{\text{Cov}(\mathbf{Z}; \mathbf{X})}$$

$$\text{bias} = \frac{\text{Cov}(\mathbf{Z}; \mathbf{U})}{\text{Cov}(\mathbf{Z}; \mathbf{X})}$$

As  $\text{Cov}(\mathbf{Z}; \mathbf{X}) \rightarrow 0$ ,  $\text{bias} \rightarrow +\infty$  even if  $\text{Cov}(\mathbf{Z}; \mathbf{U})$  is very small.

# Weak instruments yield inflated standard errors

$$\begin{aligned}\beta_{IV} &= (\mathbf{Z}^T \mathbf{X})^{-1} \mathbf{Z}^T \mathbf{Y} \\ V(\beta_{IV}) &= (\mathbf{Z}^T \mathbf{X})^{-1} \sigma^2 \rightarrow +\infty \\ \text{as } \mathbf{Z}^T \mathbf{X} = \mathbf{Cov}(\mathbf{Z}; \mathbf{X}) &\rightarrow 0\end{aligned}$$



A top-down view of various musical instruments and accessories arranged on a dark, textured surface. On the left is a large, circular brass cymbal. Above it are two wooden drumsticks. To the right, a portion of a piano keyboard is visible. Below the keyboard, there's a black microphone with a silver grille. To the right of the microphone is an acoustic guitar with a dark wood finish. In the center, there are several small, round, metallic percussion instruments (possibly shakers or bells) and a black cord. At the bottom left, there's a yellow, bulbous percussion instrument with a wooden handle. The text is overlaid in the center, in a white, serif font.

What are your thoughts on the  
method of instrumental variables?  
Can it make good music?