Linear Algebra
Inverses Take Home Exam

Felice Li

---

1. Let $M = \begin{bmatrix} -2 & 4 & -2 \\ 0 & 1 & -3 \\ 2 & -3 & 2 \end{bmatrix}$. (5 points)

   (a) Find the inverse of M using a technique developed in class. (Use computing technology to verify it, but find the inverse by hand.)
   To get M^{-1}, we perform the same operations on M to get it into reduced row echelon form on an identity matrix.

$$\left[\begin{array}{ccc|ccc} -2 & 4 & -2 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 2 & -3 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-\frac{1}{2}①} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 2 & -3 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-2①+③}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right] \xrightarrow{②\longleftrightarrow③} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -3 & 0 & 1 & 0 \end{array}\right] \xrightarrow{2②+①}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{3}{2} & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -3 & 0 & 1 & 0 \end{array}\right] \xrightarrow{-1②+③} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{3}{2} & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & -3 & -1 & 1 & -1 \end{array}\right] \xrightarrow{-\frac{1}{3}③}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{3}{2} & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{array}\right] \xrightarrow{-③+①} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{6} & \frac{1}{3} & \frac{5}{3} \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{array}\right].$$

   Thus, $M^{-1} = \begin{bmatrix} \frac{7}{6} & \frac{1}{3} & \frac{5}{3} \\ 1 & 0 & 1 \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$.

   (b) Use the result of part (a) to easily solve the equation $M\mathbf{x} = \mathbf{0}$. Explain why this makes sense.
   We can perform the following operations.

$$\begin{gathered} M\mathbf{x} = \mathbf{0} \\ M^{-1}M\mathbf{x} = M^{-1}\mathbf{0} \\ \mathbf{x} = \mathbf{0}. \end{gathered}$$

Additionally, we can note that since M is invertible, it must also be one-to-one. Thus, there should exist only one $\mathbf{x}$ such that $M\mathbf{x} = \mathbf{0}$, and that $\mathbf{x}$ must be $\mathbf{0}$. Otherwise, if there exists solutions other than $\mathbf{x} = \mathbf{0}$, there exists more than one solution.

(c) Use the result of part (a) to easily solve the equation $M\mathbf{x} = \begin{bmatrix} 12 \\ -3 \\ 3 \end{bmatrix}$

We can perform the following:

$$\begin{aligned} M\mathbf{x} &= \begin{bmatrix} 12 \\ -3 \\ 3 \end{bmatrix} \\ M^{-1}M\mathbf{x} &= M^{-1}\begin{bmatrix} 12 \\ -3 \\ 3 \end{bmatrix} \\ \mathbf{x} &= \begin{bmatrix} \frac{7}{6} & \frac{1}{3} & \frac{5}{3} \\ 1 & 0 & 1 \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 12 \\ -3 \\ 3 \end{bmatrix} \\ \mathbf{x} &= \begin{bmatrix} 18 \\ 15 \\ 6 \end{bmatrix}. \end{aligned}$$

$$\boxed{\text{Thus, } \mathbf{x} = \begin{bmatrix} 18 \\ 15 \\ 6 \end{bmatrix} \text{ is the solution for } M\mathbf{x} = \begin{bmatrix} 12 \\ -3 \\ 3 \end{bmatrix}.}$$

2. Consider the matrix $B = \begin{bmatrix} 1 & -3 & -1 \\ -2 & 5 & 1 \\ -3 & 5 & -1 \end{bmatrix}$. (5 points)

(a) Describe all vectors $\begin{bmatrix} x \\ y \\ x \end{bmatrix}$ such that $B\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}$.

We can obtain a system of equations from matrix multiplication.

$$\begin{aligned} B\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \mathbf{0} \\ \begin{bmatrix} 1 & -3 & -1 \\ -2 & 5 & 1 \\ -3 & 5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \mathbf{0} \\ \begin{bmatrix} x - 3y - z \\ -2x + 5y + z \\ -3x + 5y - z \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

Setting the components of the matrices equal to each other, we obtain the following three equations:

$$\begin{aligned} x - 3y - z &= 0 \\ -2x + 5y + z &= 0 \\ -3x + 5y - z &= 0 \end{aligned}$$

To solve this system of equations, first add the first and second equation together, then add the second and third equation to obtain the following equations:

$$\begin{aligned} -x + 2y &= 0 \\ -5x + 10y &= 0 \end{aligned}$$

Note that the two equations are dependent, and thus there are multiple solutions. To generalize, set $y = t$ and solve for x in terms of t:

$$\begin{aligned} -x + 2y &= 0 \\ -x + 2t &= 0 \\ x &= 2t \end{aligned}$$

Now, to solve for z in terms of t as well, we substitute $x = 2t$ and $y = t$ into any of the original equations.

$$\begin{aligned} x - 3y - z &= 0 \\ 2t - 3t - z &= 0 \\ z &= -t \end{aligned}$$

$$\boxed{\text{Thus, } \begin{bmatrix} x \\ y \\ x \end{bmatrix} = \begin{bmatrix} 2t \\ t \\ -t \end{bmatrix} \ \forall\, t \in \mathbb{R} \text{ produces } B \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}.}$$

(b) Explain how the result of (a) shows that the matrix B is not invertible.

Since there exists multiple solutions for $B \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}$, there exists multiple inputs to the same output, and thus it is evident that the transformation the matrix B represents is not one-to-one. As B is not one-to-one, it is not invertible.

(c) Suppose $B\mathbf{u} = \mathbf{0}$ and $B\mathbf{v} = \mathbf{0}$. Must it be the case that $B(\mathbf{u}+\mathbf{v}) = \mathbf{0}$? Explain.

Yes. Matrix multiplication distributes over addition due to linearity. Thus, the following operations can be performed

$$\begin{gathered} B(\mathbf{u} + \mathbf{v}) \\ B\mathbf{u} + B\mathbf{v} \\ \mathbf{0} + \mathbf{0} = \mathbf{0}. \end{gathered}$$

Thus, $B(\mathbf{u}+\mathbf{v}) = \mathbf{0}$.

In a different approach, we can set up the following, as we know $\mathbf{u}$ and $\mathbf{v}$ must right-multiply with B to result in $\mathbf{0}$:

$$\mathbf{u} = \begin{bmatrix} 2k \\ k \\ -k \end{bmatrix} \ \forall\, k \in \mathbb{R} \text{ and } \mathbf{v} = \begin{bmatrix} 2d \\ d \\ -d \end{bmatrix} \ \forall\, d \in \mathbb{R}.$$

Substituting the definitions of $\mathbf{u}$ and $\mathbf{v}$ into $B(\mathbf{u}+\mathbf{v})$, we have

$$B\left(\begin{bmatrix} 2k \\ k \\ -k \end{bmatrix} + \begin{bmatrix} 2d \\ d \\ -d \end{bmatrix}\right) = B\left(\begin{bmatrix} 2k+2d \\ k+d \\ -k-d \end{bmatrix}\right)$$

Since k and d are both constants, we can represent their sum as a different constant: $g = k + d$.

$$B\left(\begin{bmatrix} 2(k+d) \\ k+d \\ -(k+d) \end{bmatrix}\right) = B\left(\begin{bmatrix} 2g \\ g \\ -g \end{bmatrix}\right).$$

Since $\begin{bmatrix} 2g \\ g \\ -g \end{bmatrix}$ is still in the form $\begin{bmatrix} 2t \\ t \\ -t \end{bmatrix}$, and we know from part (a) that $B\begin{bmatrix} 2t \\ t \\ -t \end{bmatrix} = \mathbf{0}$, we can conclude that that

$$B\begin{bmatrix} 2g \\ g \\ -g \end{bmatrix} = B(\mathbf{u}+\mathbf{v}) = \mathbf{0}.$$

(d) Show the column vectors of B are linearly dependent by finding a non-trivial linear combination of the column vectors that is equal to the zero vector.

Let a, b, and c be the coefficients for the linear combinations of the first, second, and third column vector. We set

$$a\begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix} + b\begin{bmatrix} -3 \\ 5 \\ 5 \end{bmatrix} + c\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \mathbf{0}.$$

Simplifying, we get

$$\begin{bmatrix} a-3b-c \\ -2a+5b+c \\ -3a+5b-c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Note that this system of equation is the same as the system in part (a), with $(x, y, z) = (a, b, c)$. Thus, the solution still applies: $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2t \\ t \\ -t \end{bmatrix} \ \forall\, t \in \mathbb{R}$.

Reviewing the solution in the context of linear combination of columns, we see that

$$2t \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix} + t \begin{bmatrix} -3 \\ 5 \\ 5 \end{bmatrix} + -t \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \mathbf{0} \; \forall \; t \in \mathbb{R}.$$

As this solution includes non-trivial linear combinations of the column vectors of B, the column vectors of B are linearly dependent.

3. Given a square matrix A, we wish to prove the following biconditional theorem: (3 points)

$$A \text{ is invertible if and only if } A\mathbf{x} = \mathbf{0} \text{ implies } \mathbf{x} = \mathbf{0}$$

(a) First, we prove that if A is invertible then $A\mathbf{x} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$. Assume that A is invertible. Use this to show that if $A\mathbf{x} = \mathbf{0}$ then $\mathbf{x} = \mathbf{0}$.

If A is invertible, A must also be one-to-one. Thus, there is only one $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$. For this to be true, $\mathbf{x}$ must be equal to $\mathbf{0}$. If there exists other non-zero vectors such that $A\mathbf{x} = \mathbf{0}$, there are multiple solutions for $A\mathbf{x} = \mathbf{0}$, which means A is not one-to-one and thus not invertible. Since we know A to indeed be invertible, $\mathbf{x} = \mathbf{0}$ must be true.

(b) Second, we prove that if $A\mathbf{x} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$, then A is invertible. Assume $A\mathbf{x} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$. Use this to prove that A is one-to-one, and so, is invertible.

To show this, first take:

$$\begin{gathered} A\mathbf{u} = A\mathbf{v} \\ A\mathbf{u} - A\mathbf{v} = \mathbf{0} \\ A(\mathbf{u} - \mathbf{v}) = \mathbf{0}. \end{gathered}$$

Letting $\mathbf{x} = \mathbf{u} - \mathbf{v}$, we see that

$$A\mathbf{x} = \mathbf{0}.$$

Since we assume that $A\mathbf{x} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$, we can see that

$$\begin{gathered} \mathbf{x} = \mathbf{0} = \mathbf{u} - \mathbf{v} \\ \mathbf{v} = \mathbf{u}. \end{gathered}$$

From this, we see that the same output ($A\mathbf{u} = A\mathbf{v}$) implies that the inputs are the same as well ($\mathbf{u} = \mathbf{v}$). Thus, A is one-to-one, and it is invertible.

4. Suppose the matrix A has independent columns. Explain why it must be the case that if $A\mathbf{x} = \mathbf{0}$ then $\mathbf{x} = \mathbf{0}$. What does this tell you about A if A is a square matrix? (2 points)

Matrix multiplication $A\mathbf{x}$ can be interpreted as taking a linear combination of the columns. For instance, taking a case where $A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} m \\ n \end{bmatrix}$,

$$A\mathbf{x} = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} = m \begin{bmatrix} a \\ c \\ e \end{bmatrix} + n \begin{bmatrix} b \\ d \\ f \end{bmatrix}.$$

Since the columns of A are independent, there exists only the trivial linear combination that results in the 0 vector. In other words, the coefficients of the column vectors must be 0, and that occurs when $\mathbf{x} = \mathbf{0}$.
Knowing that $A\mathbf{x} = 0$ necessitates that $\mathbf{x} = \mathbf{0}$, if A is a square matrix, it tells us that A is invertible (biconditional theorem from question 3).

5. Bonus: Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ and let A $[\mathbf{u}|\mathbf{v}]$. Under what conditions would A^TA be invertible? Under what conditions would AA^T be invertible?
Simplifying A^TA, we have

$$A^TA = \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{u} \cdot \mathbf{u} & \mathbf{u} \cdot \mathbf{v} \\ \mathbf{u} \cdot \mathbf{v} & \mathbf{v} \cdot \mathbf{v} \end{bmatrix}.$$

As we explored, if the columns of a matrix are independent, then the matrix is one-to-one and invertible. Thus, we want $\begin{bmatrix} \mathbf{u} \cdot \mathbf{u} \\ \mathbf{u} \cdot \mathbf{v} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{u} \cdot \mathbf{v} \\ \mathbf{v} \cdot \mathbf{v} \end{bmatrix}$ to be independent of each other.
After pondering these two column vectors, we see that if $\mathbf{u} \cdot \mathbf{v} = \mathbf{0}$, the two columns become $\begin{bmatrix} \mathbf{u} \cdot \mathbf{u} \\ \mathbf{0} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{0} \\ \mathbf{v} \cdot \mathbf{v} \end{bmatrix}$. These two column vectors are evidently independent.

$$\boxed{A^TA \text{ is invertible when } \mathbf{u} \cdot \mathbf{v} = \mathbf{0}.}$$

For AA^T, we can represent A as

$$A = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{bmatrix}.$$

With this, we can calculate AA^T

$$AA^T = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{bmatrix} \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} u_1u_1 + v_1v_1 & u_1u_2 + v_1v_2 & u_1u_3 + v_1v_3 \\ u_1u_2 + v_1v_2 & u_2u_2 + v_2v_2 & u_2u_3 + v_2v_3 \\ u_1u_3 + v_1v_3 & u_2u_3 + v_2v_3 & u_3u_3 + v_3v_3 \end{bmatrix}.$$

Note that AA^T is actually a diagonal matrix, as we can see some repetition of entries.

$$AA^T = \begin{bmatrix} u_1u_1 + v_1v_1 & \color{red}{u_1u_2 + v_1v_2} & \color{blue}{u_1u_3 + v_1v_3} \\ \color{red}{u_1u_2 + v_1v_2} & u_2u_2 + v_2v_2 & \color{orange}{u_2u_3 + v_2v_3} \\ \color{blue}{u_1u_3 + v_1v_3} & \color{orange}{u_2u_3 + v_2v_3} & u_3u_3 + v_3v_3 \end{bmatrix}.$$

We want the column vectors of AA^T to be independent so that it is invertible. In other words, there should exist only the trivial solution for a linear combination of the column vectors that result in $\mathbf{0}$.

$$a \begin{bmatrix} u_1u_1 + v_1v_1 \\ u_1u_2 + v_1v_2 \\ u_1u_3 + v_1v_3 \end{bmatrix} + b \begin{bmatrix} u_1u_2 + v_1v_2 \\ u_2u_2 + v_2v_2 \\ u_2u_3 + v_2v_3 \end{bmatrix} + c \begin{bmatrix} u_1u_3 + v_1v_3 \\ u_2u_3 + v_2v_3 \\ u_3u_3 + v_3v_3 \end{bmatrix} = \mathbf{0}.$$

Note that this can be rewritten as:

$$a \left(u_1 \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + v_1 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \right) + b \left(u_2 \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + v_2 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \right) + c \left(u_3 \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + v_3 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \right) = \mathbf{0}$$

We can see that the expressions a, b, and c are being multiplied with are simply linear combinations of $\mathbf{u}$ and $\mathbf{v}$. If $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent, non-trivial solutions would exist.

$\boxed{\text{Thus, } \mathbf{u} \text{ and } \mathbf{v} \text{ must be independent.}}$

I am still confused on this question but I know my answer isn't right... I tried $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, but A^TA was invertible (even though $\mathbf{u} \cdot \mathbf{v} \neq 0$) and AA^T wasn't invertible even though $\mathbf{u}$ and $\mathbf{v}$ are independent... LOL. I don't have enough time to fix it, but here's what I think went wrong:

- For A^TA, $\mathbf{u} \cdot \mathbf{v} = 0$ is one of the conditions where A^TA will be invertible, but it's too limiting. The two column vectors can be independent, even if they don't have zeros.
- For AA^T, I'm a little more lost. SOS.