# Linear Algebra
# Basis Take Home Exam

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1. Suppose the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. Is the set $\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_2 + \mathbf{v}_3, \mathbf{v}_1 + \mathbf{v}_3\}$ independent or dependent? Justify your response.

   To check whether $\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_2 + \mathbf{v}_3, \mathbf{v}_1 + \mathbf{v}_3\}$ is linearly independent, we can check to see if there exists a non-trivial solution to

$$b_1(\mathbf{v}_1 + \mathbf{v}_2) + b_2(\mathbf{v}_2 + \mathbf{v}_3) + b_3(\mathbf{v}_1 + \mathbf{v}_3) = \mathbf{0}.$$

   Distributing the constants and rearranging the equation, we get:

$$\begin{gathered} b_1\mathbf{v}_1 + b_1\mathbf{v}_2 + b_2\mathbf{v}_2 + b_2\mathbf{v}_3 + b_3\mathbf{v}_1 + b_3\mathbf{v}_3 = \mathbf{0} \\ (b_1 + b_3)\mathbf{v}_1 + (b_1 + b_2)\mathbf{v}_2 + (b_2 + b_3)\mathbf{v}_3 = \mathbf{0}. \end{gathered}$$

   Note that since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, that means that only the trivial solution exists for:

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{0}.$$

   Thus, $a_1 = a_2 = a_3 = 0$. Therefore, we have:

$$\begin{gathered} b_1 + b_3 = a_1 = 0 \\ b_1 + b_2 = a_2 = 0 \\ b_2 + b_3 = a_3 = 0. \end{gathered}$$

   Solving this system of equation, we first linearly combine the first two equations to eliminate $b_1$:

$$\begin{gathered} b_2 - b_3 = 0 \\ b_2 + b_3 = 0. \end{gathered}$$

   Summing these two equations, we get:

$$\begin{gathered} 2b_2 = 0 \\ b_2 = 0. \end{gathered}$$

From this, it follows that $b_1$ and $b_3$ must also be 0. Therefore, the only solution that exists is the trivial solution, $b_1 = b_2 = b_3 = 0$, and thus $\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_2 + \mathbf{v}_3, \mathbf{v}_1 + \mathbf{v}_3\}$ is linearly independent.

2. Let $V$ be a vector space and let $W_1 \subseteq V$ and $W_2 \subseteq V$ be vector subspaces of $V$. The intersection of $W_1$ and $W_2$, written as $W_1 \cap W_2$, is defined to be the set of all vectors in $W_1$ and in $W_2$. Prove that $W_1 \cap W_2$ is a vector subspace of $V$ by proving that the set of vectors $W_1 \cap W_2$ is (1) closed under vector addition, and (2) closed under scalar multiplication.

   (a) Show that $W_1 \cap W_2$ is closed under vector addition.
   Let $\mathbf{u}, \mathbf{v} \in W_1 \cap W_2$, and the basis of $W_1$ be $\{\mathbf{a}, \mathbf{b}\}$ and the basis of $W_2$ be $\{\mathbf{c}, \mathbf{d}\}$. Since $\mathbf{u}$ is in both $W_1$ and $W_2$, $\mathbf{u}$ can be written as $A_1\mathbf{a} + B_1\mathbf{b}$ and $C_1\mathbf{c} + D_1\mathbf{d}$.

   In other words, $\mathbf{u} = A_1\mathbf{a} + B_1\mathbf{b} = C_1\mathbf{c} + D_1\mathbf{d} \;\; \forall \, \{A_1, B_1, C_1, D_1\} \in \mathbb{R}$

   The same thing applies to $\mathbf{v}$. Therefore, $\mathbf{v} = A_2\mathbf{a} + B_2\mathbf{b} = C_2\mathbf{c} + D_2\mathbf{d} \;\; \forall \, \{A_2, B_2, C_2, D_2\} \in \mathbb{R}$.

   Summing $\mathbf{u}$ and $\mathbf{v}$, we have:
   $$\mathbf{u} + \mathbf{v} = A_1\mathbf{a} + B_1\mathbf{b} + A_2\mathbf{a} + B_2\mathbf{b} = C_1\mathbf{c} + D_1\mathbf{d} + C_2\mathbf{c} + D_2\mathbf{d}$$
   $$\mathbf{u} + \mathbf{v} = (A_1 + A_2)\mathbf{a} + (B_1 + B_2)\mathbf{b} = (C_1 + C_2)\mathbf{c} + (D_1 + D_2)\mathbf{d}.$$
   As we can see, $\mathbf{u} + \mathbf{v}$ can be written in two ways, first as a linear combination of the vectors in the basis of $W_1$ and again as a linear combination of the vectors in the basis of $W_2$. Thus, $\mathbf{u} + \mathbf{v}$ is in both $W_1$ and $W_2$, and so $\mathbf{u} + \mathbf{v} \in W_1 \cap W_2$.

   Note that this is a specific case, where $W_1$ and $W_2$ are dimension 2. However, this argument expands to other dimensions. Essentially, the sum of vectors $\mathbf{u}$ and $\mathbf{v}$ can be written as linear combinations of only the vectors in the basis of $W_1$ and of only the vectors in the basis of $W_2$. Thus, $\mathbf{u} + \mathbf{v}$ is in both $W_1$ and $W_2$, or $W_1 \cap W_2$, and we can finally conclude that $W_1 \cap W_2$ is closed under vector addition.

   (b) Show that $W_1 \cap W_2$ is closed under scalar multiplication.
   Let $\mathbf{u} \in W_1 \cap W_2$. Again, let the basis of $W_1$ be $\{\mathbf{a}, \mathbf{b}\}$ and the basis of $W_2$ be $\{\mathbf{c}, \mathbf{d}\}$. We can express $\mathbf{u}$ as $A_1\mathbf{a} + B_1\mathbf{b}$ and $C_1\mathbf{c} + D_1\mathbf{d}$. Multiplying $\mathbf{u}$ by a scalar constant, $k$, we have:
   $$k\mathbf{u} = (kA_1)\mathbf{a} + (kB_1)\mathbf{b} = (kC_1)\mathbf{c} + (kD_1)\mathbf{d}.$$
   Note that $k\mathbf{u}$ is still in $W_1$ and $W_2$, as it can be expressed as a vector in both subspaces.
   Again, this expands to $W_1$ and $W_2$ with higher dimensions – a scalar multiple of any linear combinations of basis vectors are still linear combinations of basis vectors. Therefore, $W_1 \cap W_2$ is closed under scalar multiplication.

3. Let $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

   (a) Explain why span $S \subseteq \mathbb{R}^3$.

   Span $S$ can be written as $\left\{ a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \forall\, a, b \in \mathbb{R} \right\}$. Simplifying this, we get

$$\begin{bmatrix} a \\ a \\ 0 \end{bmatrix} + \begin{bmatrix} -b \\ b \\ b \end{bmatrix} = \begin{bmatrix} a-b \\ a+b \\ b \end{bmatrix}.$$

   A vector of the form $\begin{bmatrix} a-b \\ a+b \\ b \end{bmatrix} \in \mathbb{R}^3$. However, it does not account for all vectors in $\mathbb{R}^3$. Therefore, span $S \subseteq \mathbb{R}^3$.
   Another interpretation of this is that span $S$ is the linear combination of two 3-vectors, and that creates a plane. A plane is a subset of $\mathbb{R}^3$.

   (b) Show that $\exists \mathbf{u} \in \mathbb{R}^3$ such that $\mathbf{u} \notin$ span $S$.

   The vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in \mathbb{R}^3$ would not be in span $S$. In this case, $b = 1$, which would result in two different values for $a$. For the first entry, we have

$$\begin{aligned} a - b &= 1 \\ a &= 2. \end{aligned}$$

   For the second entry, we have

$$\begin{aligned} a + b &= 1 \\ a &= 0. \end{aligned}$$

   Therefore, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in \mathbb{R}^3$ but $\notin$ span $S$, as it cannot be written in the form of $\begin{bmatrix} a-b \\ a+b \\ b \end{bmatrix}$.

   (c) Give an example of a vector $\mathbf{v} \in \mathbb{R}^3$ where $S \cup \{\mathbf{v}\}$ does not span $\mathbb{R}^3$. Justify your response.

   Taking $\mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$, $S \cup \{\mathbf{v}\}$ does not span $\mathbb{R}^3$. $S \cup \{\mathbf{v}\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \right\}$.

Since $\mathbf{v}$ is a scalar multiple of a vector that already exists in $S$, $\begin{bmatrix}1\\1\\0\end{bmatrix}$, no new information is being added and thus $S \cup \{\mathbf{v}\}$ does not span $\mathbb{R}^3$.

In another view, the minimum number of vectors required to span $\mathbb{R}^3$ is the dimension of the basis, which is 3 in this case. For a set of 3 vectors to span $\mathbb{R}^3$, they need to be independent. However, since $S \cup \{\mathbf{v}\}$ has only 3 vectors and they are linearly dependent, we can deduce that $S \cup \{\mathbf{v}\}$ does not span $\mathbb{R}^3$.

(d) Find a vector $\mathbf{w}$ so that the set of vectors $T = S \cup \{\mathbf{w}\}$ is pairwise orthogonal. Explain why $T$ must be a basis for $\mathbb{R}^3$.

To find $\mathbf{w}$ such that $S \cup \{\mathbf{w}\}$ is pairwise orthogonal, we can set up the following equations:

$$\mathbf{w} \cdot \begin{bmatrix}1\\1\\0\end{bmatrix} = 0$$
$$\mathbf{w} \cdot \begin{bmatrix}-1\\1\\1\end{bmatrix} = 0.$$

Let $\mathbf{w} = \begin{bmatrix}x\\y\\z\end{bmatrix}$. Thus, we have

$$\begin{bmatrix}x\\y\\z\end{bmatrix} \cdot \begin{bmatrix}1\\1\\0\end{bmatrix} = 0$$
$$\begin{bmatrix}x\\y\\z\end{bmatrix} \cdot \begin{bmatrix}-1\\1\\1\end{bmatrix} = 0.$$

This gives us the following system:

$$x + y = 0$$
$$-x + y + z = 0.$$

Summing the two equations together, we get

$$2y + z = 0$$

Let $z = -2$, which gives us $y = 1$. Using that to find $x$, we find that $x = -1$. Thus, $w = \begin{bmatrix}-1\\1\\-2\end{bmatrix}$ results in $T = S \cup \{\mathbf{w}\} = \left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\-2\end{bmatrix} \right\}$ being pairwise orthogonal.

$T$ must be a basis for $\mathbb{R}^3$ because it has 3 elements, which is suitable for a

3-dimensional space. It also spans $\mathbb{R}^3$ and is linearly independent. Using a geometric approach, since span $S$ spans a plane, adding the normal of that plane to the set would allow the set to span $\mathbb{R}^3$.

(e) Let $M$ be the matrix whose columns are the vectors of $S$. Compute $MM^T$. Given a vector $\mathbf{v} \in \mathbb{R}^3$, every vector $(MM^T)\mathbf{v}$ lies in a plane. Find the equation of that plane.

$M = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$. To find $MM^T$:

$$MM^T = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & 1 \end{bmatrix}.$$

To find the plane of $(MM^T)\mathbf{v}$, first let $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. We see that

$$(MM^T)\mathbf{v} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Seeing this as a linear combination of the column vectors of $MM^T$, we have

$$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + z \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

However, note that this is also equal to span$\left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$. In other words, $(MM^T)\mathbf{v} = \text{span}\left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

To find the plane represented by span$\left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$, we can find the normal to those vectors. Let the normal $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Thus,

$$\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \cdot \mathbf{n} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \cdot \mathbf{n} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$\begin{bmatrix}-1\\1\\1\end{bmatrix}\cdot\mathbf{n}=\begin{bmatrix}-1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}a\\b\\c\end{bmatrix}=0$$

Simplifying this, we have

$$\begin{aligned}2a-c&=0\\2b+c&=0\\-a+b+c&=0.\end{aligned}$$

Summing the first two systems of equations and simplifying, we get $a+b=0$. Thus, we now have

$$\begin{aligned}a+b&=0\\-a+b+c&=0.\end{aligned}$$

Note that this is the same system as the system we solved in 3d (which is kinda interesting...), with the exception that the variables $(x,y,z)=(a,b,c)$. Therefore, our solution for 3d stands, and $\mathbf{n}=\begin{bmatrix}-1\\1\\-2\end{bmatrix}$.

In conclusion, since our normal to the plane is $\begin{bmatrix}-1\\1\\-2\end{bmatrix}$, the equation of our plane is $-x+y-2z=0$.

4. Let $W$ be the set of vectors $v=\begin{bmatrix}x\\y\\z\end{bmatrix}$ in $\mathbb{R}^3$ that satisfy the equation $x+y+z=0$.

   (a) Show that $W$ is a subspace of $\mathbb{R}^3$ by showing $W$ is closed under vector addition and scalar multiplication.

   Any vector in $\mathbb{R}^3$ that satisfy the equation $x+y+z=0$ can be written in the form $v=\begin{bmatrix}x\\y\\-x-y\end{bmatrix}$. For $\mathbf{u},\mathbf{v}\in W$, let $\mathbf{u}=\begin{bmatrix}x_u\\y_u\\-x_u-y_u\end{bmatrix}$ and $\mathbf{v}=\begin{bmatrix}x_v\\y_v\\-x_v-y_v\end{bmatrix}$.

   Summing $\mathbf{u}+\mathbf{v}$, we have:

$$\mathbf{u}+\mathbf{v}=\begin{bmatrix}x_u\\y_u\\-x_u-y_u\end{bmatrix}+\begin{bmatrix}x_v\\y_v\\-x_v-y_v\end{bmatrix}=\begin{bmatrix}x_u+x_v\\y_u+y_v\\-x_u-y_u-x_v-y_v\end{bmatrix}.$$

   Note that this is still in the form $v=\begin{bmatrix}x\\y\\-x-y\end{bmatrix}$, and thus $\mathbf{u}+\mathbf{v}\in W$. Therefore, $W$ is closed under vector addition.

   For scalar multiplication, let $\mathbf{u}\in W$ and let $\mathbf{u}=\begin{bmatrix}x_u\\y_u\\-x_u-y_u\end{bmatrix}$. Multiplying by a constant $k$ gives us

$$k\mathbf{u} = \begin{bmatrix} kx_u \\ ky_u \\ k(-x_u - y_u) \end{bmatrix} = \begin{bmatrix} kx_u \\ ky_u \\ -kx_u - ky_u \end{bmatrix}.$$

This is again still in the form $v = \begin{bmatrix} x \\ y \\ -x-y \end{bmatrix}$, and thus, $k\mathbf{u} \in W$ and $W$ is closed under scalar multiplication.

(b) Find a basis for $W$. Call this $S$.

$S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$. Both vectors in the set span $x + y + z = 0$ and are linearly independent, and thus the set is a basis.

(c) Give an example of a vector $\mathbf{u}$ such that $S \cup \{\mathbf{u}\}$ is a basis for $\mathbb{R}^3$. Justify your response in 1-2 sentences.

The vector $\mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ results in $S \cup \{\mathbf{u}\}$ being a basis for $\mathbb{R}^3$. Since $\mathbb{R}^3$ has a dimension of 3, and there must be three independent vectors that span $\mathbb{R}^3$ in its basis, and $S \cup \{\mathbf{u}\}$ satisfies that.

(d) The subspace $U$ of vectors of $\mathbb{R}^3$ that satisfy the equation $x + y - z = 0$ has a basis $T = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$. Could $T \cup S$ be a basis for $\mathbb{R}^3$? Explain.

No. $T \cup S$ would have 4 vectors. There is no overlap between the vectors in $T$ and any potential vectors in $S$, as none of the vectors in $T$ satisfy $x + y + z = 0$. Furthermore, since there are 4 vectors in the set and we are working with a 3-dimensional vector space, $T \cup S$ must be linearly dependent. As it is linearly dependent, it cannot be a basis.

5. Find an orthonormal basis $B = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ for $\mathbb{R}^3$ such that $\{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis for $W$.

Using the Gram-Schmidt process, we can turn our basis from part (b), $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$, into an orthogonal basis. For ease of reference, let $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. First, we project $\mathbf{u}$ onto $\mathbf{v}$:

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{v}||^2}\mathbf{v} = \frac{-1}{2} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix}.$$

To find the orthogonal component, we can subtract $\text{proj}_{\mathbf{v}}\mathbf{u}$ from $\mathbf{u}$:

$$\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -1 \\ \frac{1}{2} \end{bmatrix}.$$

Thus, $\left\{ \begin{bmatrix} \frac{1}{2} \\ -1 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is an orthogonal basis for $W$.

To find $\mathbf{b}_3$ such that $\left\{ \begin{bmatrix} \frac{1}{2} \\ -1 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{b}_3 \right\}$ is a basis for $\mathbb{R}^3$, we can let $\mathbf{b}_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.
We know that $\mathbf{b}_3$ needs to me orthogonal to $\mathbf{b}_1$ and $\mathbf{b}_2$, and thus we can set up the following equations:

$$\mathbf{b}_3 \cdot \begin{bmatrix} \frac{1}{2} \\ -1 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} \\ -1 \\ \frac{1}{2} \end{bmatrix} = 0$$

$$\mathbf{b}_3 \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = 0.$$

From this, we get

$$\frac{x}{2} - y + \frac{z}{2} = 0$$
$$-x + z = 0.$$

As there are 3 variables and 2 equations, we have an extra degree of freedom. Thus, we can let $x = 1$, and based on the second equation, $z = 1$. Substituting those values into the first equation, we get that $y = 1$ as well. Therefore, the vector $\mathbf{b}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is orthogonal to the subspace $W$.
Currently, we have an orthogonal basis for $\mathbb{R}^3$ wherein the first two vectors also form the basis for $W$, $\left\{ \begin{bmatrix} \frac{1}{2} \\ -1 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$. To make it an orthonormal basis, we can scale each vector such that the magnitude is 1.

$$\left\| a \begin{bmatrix} \frac{1}{2} \\ -1 \\ \frac{1}{2} \end{bmatrix} \right\| = 1.$$

$$\left|\left|\begin{bmatrix}\frac{a}{2}\\-a\\\frac{a}{2}\end{bmatrix}\right|\right| = 1.$$
$$\sqrt{(\tfrac{a}{2})^2+(-a)^2+(\tfrac{a}{2})^2} = 1.$$
$$\sqrt{\tfrac{3a^2}{2}} = 1.$$
$$a = \sqrt{\tfrac{2}{3}}.$$

Therefore, the vector $\begin{bmatrix}\frac{1}{2}\\-1\\\frac{1}{2}\end{bmatrix}$ is scaled by $\sqrt{\frac{2}{3}}$, resulting in $\begin{bmatrix}\sqrt{\frac{1}{6}}\\-\sqrt{\frac{2}{3}}\\\sqrt{\frac{1}{6}}\end{bmatrix}$.

Performing the same thing for $\begin{bmatrix}-1\\0\\1\end{bmatrix}$, we get that it must be scaled by $\frac{\sqrt{2}}{2}$, resulting in $\begin{bmatrix}-\frac{\sqrt{2}}{2}\\0\\\frac{\sqrt{2}}{2}\end{bmatrix}$. Lastly, $\begin{bmatrix}1\\1\\1\end{bmatrix}$ needs to be scaled by a factor of $\frac{\sqrt{3}}{3}$, resulting in $\begin{bmatrix}\frac{\sqrt{3}}{3}\\\frac{\sqrt{3}}{3}\\\frac{\sqrt{3}}{3}\end{bmatrix}$.

In conclusion, the orthonormal basis $B = \left\{\begin{bmatrix}\sqrt{\frac{1}{6}}\\-\sqrt{\frac{2}{3}}\\\sqrt{\frac{1}{6}}\end{bmatrix}, \begin{bmatrix}-\frac{\sqrt{2}}{2}\\0\\\frac{\sqrt{2}}{2}\end{bmatrix}, \begin{bmatrix}\frac{\sqrt{3}}{3}\\\frac{\sqrt{3}}{3}\\\frac{\sqrt{3}}{3}\end{bmatrix}\right\}$ exists for $\mathbb{R}^3$ such that $\left\{\begin{bmatrix}\sqrt{\frac{1}{6}}\\-\sqrt{\frac{2}{3}}\\\sqrt{\frac{1}{6}}\end{bmatrix}, \begin{bmatrix}-\frac{\sqrt{2}}{2}\\0\\\frac{\sqrt{2}}{2}\end{bmatrix}\right\}$ is a basis for $W$.