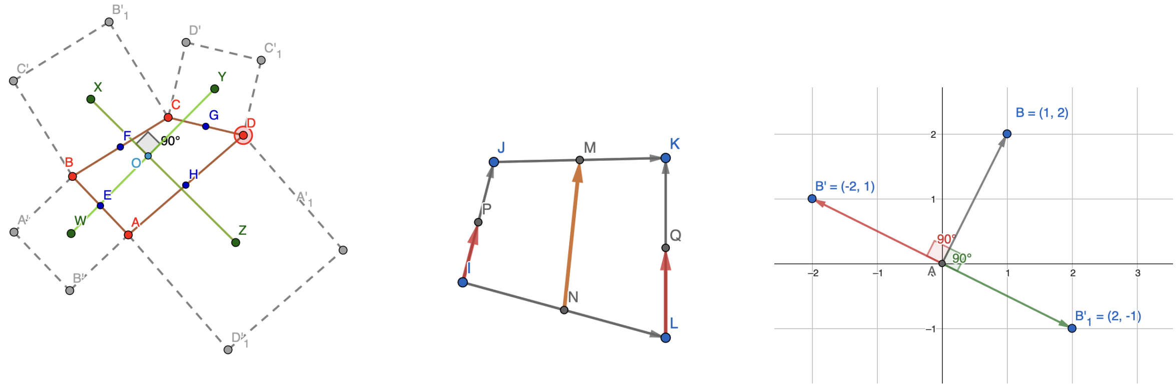
VECTOR PROOF OF VAN AUBEL'S THEOREM

1 INTRODUCTION

Given an arbitrary planar quadrilateral, place a square outwardly on each side, and connect the centers of opposite squares. Van Aubel's theorem states that the two lines have equal magnitudes and intersect at a right angle. In essence, $|\overrightarrow{ZX}| = |\overrightarrow{WY}|$ and $\overrightarrow{ZX} \cdot \overrightarrow{WY} = 0$. See here for an interactive demonstration. To start, we prove two lemmas.



(a) Van Aubel's Theorem (b) Lemma 1 (c) Lemma 2

1.1 Lemma 1: Midline of Quadrilaterals as Average of Sides

Lemma 1: Given a quadrilateral $IJKL$ and midpoints P, M, Q, N on IJ, JK, KL, and LI respectively, $\overrightarrow{NM}$ is the average of the vectors $\overrightarrow{IJ}$ and $\overrightarrow{LK}$. In other words, $\overrightarrow{NM} = \frac{1}{2}(\overrightarrow{IJ} + \overrightarrow{LK})$

By independence of the origin, let I be the origin. Let us assign variables to denote the other points.

$$I = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad J = \begin{bmatrix} a \\ b \end{bmatrix} \quad K = \begin{bmatrix} c \\ d \end{bmatrix} \quad L = \begin{bmatrix} e \\ f \end{bmatrix}$$

Since M and N are the midpoints of $\overrightarrow{JK}$ and $\overrightarrow{IL}$, we have $M = \begin{bmatrix} \frac{1}{2} \times (a+c) \\ \frac{1}{2} \times (b+d) \end{bmatrix} = \begin{bmatrix} \frac{a}{2} + \frac{c}{2} \\ \frac{b}{2} + \frac{d}{2} \end{bmatrix}$ and $N = \begin{bmatrix} \frac{1}{2} \times (0+e) \\ \frac{1}{2} \times (0+f) \end{bmatrix} = \begin{bmatrix} \frac{e}{2} \\ \frac{f}{2} \end{bmatrix}$. Now, we can write $\overrightarrow{NM} = \begin{bmatrix} \frac{a}{2} + \frac{c}{2} - \frac{e}{2} \\ \frac{b}{2} + \frac{d}{2} - \frac{f}{2} \end{bmatrix}$.

Note that $\frac{1}{2}(\overrightarrow{IJ} + \overrightarrow{LK}) = \frac{1}{2}\overrightarrow{IJ} + \frac{1}{2}\overrightarrow{LK} = \frac{1}{2}\begin{bmatrix} a \\ b \end{bmatrix} + \frac{1}{2}\begin{bmatrix} c-e \\ d-f \end{bmatrix} = \begin{bmatrix} \frac{a}{2} + \frac{c}{2} - \frac{e}{2} \\ \frac{b}{2} + \frac{d}{2} - \frac{f}{2} \end{bmatrix}$. Thus, we have proven that $\overrightarrow{NM} = \frac{1}{2}(\overrightarrow{IJ} + \overrightarrow{LK})$

1.2 Lemma 2: Important Orthogonality Property

Given vectors a, b, c, and d such that $a \cdot b = 0$ and $c \cdot d = 0$, show that $(a+c) \cdot (b+d) = 0$. See here for an interactive demonstration.

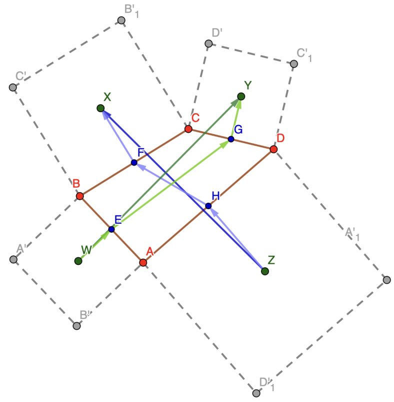
Note that rotating a vector $\begin{bmatrix} x \\ y \end{bmatrix}$ counterclockwise 90° would result in $\begin{bmatrix} -y \\ x \end{bmatrix}$, and clockwise 90° would result in $\begin{bmatrix} y \\ -x \end{bmatrix}$. For a quick proof, $\begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} -y \\ x \end{bmatrix} = 0$ and $\begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} y \\ -x \end{bmatrix} = 0$. See diagram (c) above for visual.

As a and b are orthogonal to each other and c and d are orthogonal to each other, we can let $a = \begin{bmatrix} x \\ y \end{bmatrix}$, $b = \begin{bmatrix} -y \\ x \end{bmatrix}$, $c = \begin{bmatrix} u \\ v \end{bmatrix}$, $d = \begin{bmatrix} -v \\ u \end{bmatrix}$.

$$(a+c)\cdot(b+d) = a\cdot b + a\cdot d + c\cdot b + c\cdot d = a\cdot d + c\cdot b = -xv + yu + -yu + xv = 0.$$

2 PROVING EQUAL LENGTHS

We can add midpoints for each side of the quadrilateral, namely points E, F, G, H. Note that we can write the lines of interest, $\overrightarrow{ZX}$ and $\overrightarrow{WY}$, in the following manner:



$$\overrightarrow{ZX} = \overrightarrow{ZH} + \overrightarrow{HF} + \overrightarrow{FX}$$
$$\overrightarrow{WY} = \overrightarrow{WE} + \overrightarrow{EG} + \overrightarrow{GY}$$

Using lemma 1, we can say that $\overrightarrow{HF} = \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{DC}$. Since the sides are squares, we also know that $|\overrightarrow{ZH}| = \frac{1}{2}|\overrightarrow{AD}|$ and $|\overrightarrow{FX}| = \frac{1}{2}|\overrightarrow{BC}|$. Thus,

$$|\overrightarrow{ZX}| = \frac{1}{2}|\overrightarrow{AD}| + \frac{1}{2}|\overrightarrow{AB}| + \frac{1}{2}|\overrightarrow{DC}| + \frac{1}{2}|\overrightarrow{BC}|$$

Following a similar logic,

$$|\overrightarrow{WY}| = \frac{1}{2}|\overrightarrow{AB}| + \frac{1}{2}|\overrightarrow{BC}| + \frac{1}{2}|\overrightarrow{AD}| + \frac{1}{2}|\overrightarrow{DC}|$$

As shown, $|\overrightarrow{ZX}| = |\overrightarrow{WY}|$.

3 PROVING ORTHOGONALITY

We want to show that $\overrightarrow{ZX} \perp \overrightarrow{WY}$, or $\overrightarrow{ZX}\cdot\overrightarrow{WY} = 0$.

By lemma 1, $\frac{1}{2}\overrightarrow{BC} + \frac{1}{2}\overrightarrow{AD} = \overrightarrow{EG}$.

Defining some points, let

$$\frac{1}{2}\overrightarrow{BC} = \begin{bmatrix} u \\ v \end{bmatrix} \text{ and } \frac{1}{2}\overrightarrow{AD} = \begin{bmatrix} m \\ n \end{bmatrix}, \text{ and thus } \overrightarrow{EG} = \begin{bmatrix} u+m \\ v+n \end{bmatrix}$$

Rotating $\frac{1}{2}\overrightarrow{BC}$ and $\frac{1}{2}\overrightarrow{AD}$ counterclockwise 90° would map them onto $\overrightarrow{FX}$ and $\overrightarrow{ZH}$ respectively. As explained in lemma 2 and seen in figure (c), we can write $\overrightarrow{FX} = \begin{bmatrix} -v \\ u \end{bmatrix}$ and $\overrightarrow{ZH} = \begin{bmatrix} -n \\ m \end{bmatrix}$. Note that

$$(\overrightarrow{FX} + \overrightarrow{ZH})\cdot\overrightarrow{EG} = \begin{bmatrix} -v-n \\ u+m \end{bmatrix}\cdot\begin{bmatrix} u+m \\ v+n \end{bmatrix} = 0.$$

A similar argument holds for $\frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{DC} = \overrightarrow{HF}$. Let $\frac{1}{2}\overrightarrow{AB} = \begin{bmatrix} w \\ x \end{bmatrix}$ and $\frac{1}{2}\overrightarrow{DC} = \begin{bmatrix} y \\ z \end{bmatrix}$. Here, we rotate them clockwise 90° instead, resulting in them mapping onto $\overrightarrow{WE}$ and $\overrightarrow{GY}$. Now, $\overrightarrow{WE} = \begin{bmatrix} x \\ -w \end{bmatrix}$ and $\overrightarrow{GY} = \begin{bmatrix} z \\ -y \end{bmatrix}$. Note that

$$(\overrightarrow{WE} + \overrightarrow{GY})\cdot\overrightarrow{HF} = \begin{bmatrix} x+z \\ -w-y \end{bmatrix}\cdot\begin{bmatrix} w+y \\ x+z \end{bmatrix} = 0.$$

Now, we have

$$(\overrightarrow{FX} + \overrightarrow{ZH})\cdot\overrightarrow{EG} = 0 \text{ and } (\overrightarrow{WE} + \overrightarrow{GY})\cdot\overrightarrow{HF} = 0.$$

Applying the commutative property of dot products to the second equation, we get:

$$(\overrightarrow{FX} + \overrightarrow{ZH})\cdot\overrightarrow{EG} = 0 \text{ and } \overrightarrow{HF}\cdot(\overrightarrow{WE} + \overrightarrow{GY}) = 0.$$

From this, we can use lemma 1 (given $a\cdot b = 0$ and $c\cdot d = 0$, $(a+c)\cdot(b+d) = 0$) we know that

$$((\overrightarrow{FX} + \overrightarrow{ZH}) + \overrightarrow{HF})\cdot(\overrightarrow{EG} + (\overrightarrow{WE} + \overrightarrow{GY})) = 0$$

Since $\overrightarrow{ZX} = \overrightarrow{ZH} + \overrightarrow{HF} + \overrightarrow{FX}$ and $\overrightarrow{WY} = \overrightarrow{WE} + \overrightarrow{EG} + \overrightarrow{GY}$, we conclude that $\overrightarrow{ZX}\cdot\overrightarrow{WY} = 0$.