Linear Algebra
Test Follow Up

Felice Li

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1. Suppose W is a subspace of a real vector space V. Suppose A is the matrix representation of a linear transformation that maps V to V. Define $AW = \{A\mathbf{w} | \mathbf{w} \in W\}$.

   (a) Show AW is closed under vector addition and scalar multiplication (AW is a subspace of V.)

   Let $\mathbf{u}, \mathbf{v} \in AW$. Therefore, $\mathbf{u}$ can be written as $A\mathbf{w}_1$, and $\mathbf{v}$ can be written as $A\mathbf{w}_2$, where $\mathbf{w}_1, \mathbf{w}_2 \in W$. Summing $\mathbf{u} + \mathbf{v}$, we can rewrite it as $A\mathbf{w}_1 + A\mathbf{w}_2$, which is equal to $A(\mathbf{w}_1 + \mathbf{w}_2)$ due to linearity. Note that $\mathbf{w}_1 + \mathbf{w}_2 \in W$ because W is a subspace, and subspaces are closed under vector addition. Therefore, $A\mathbf{w}_1 + A\mathbf{w}_2 = A(\mathbf{w}_1 + \mathbf{w}_2) \in AW$.

   To show that AW is closed under scalar multiplication, let $\mathbf{u} \in AW$. Again note that $\mathbf{u} = A\mathbf{w}$, where $\mathbf{w} \in W$. Multiplying $\mathbf{u}$ by a constant k, we get $k\mathbf{u} = kA\mathbf{w} = A(k\mathbf{w})$. $k\mathbf{w}$ is still in W because W is a subspace and thus closed under scalar multiplication. Therefore, $k\mathbf{u} = kA\mathbf{w} = A(k\mathbf{w})$ is still in AW, and thus AW is closed under scalar multiplication.

   For the remainder of the problem, assume $\dim W = 2$, and let A be a 2x2 matrix with real entries.

   (b) Let $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ be a basis for W. Show that $\{A\mathbf{b}_1, A\mathbf{b}_2\}$ spans AW.

   Note that the span of $\{A\mathbf{b}_1, A\mathbf{b}_2\}$ is $mA\mathbf{b}_1 + nA\mathbf{b}_2$, where $m, n \in \mathbb{R}$. This is equivalent to $A(m\mathbf{b}_1) + A(n\mathbf{b}_2) = A(m\mathbf{b}_1 + n\mathbf{b}_2)$. Since $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis for W, $m\mathbf{b}_1 + n\mathbf{b}_2 \in W$. Therefore, $\{A\mathbf{b}_1, A\mathbf{b}_2\}$ spans AW.

   (c) The set $\{A\mathbf{b}_1, A\mathbf{b}_2\}$ is not necessarily a basis for AW. Under what conditions on A must $\{A\mathbf{b}_1, A\mathbf{b}_2\}$ be a basis for AW?

   For $\{A\mathbf{b}_1, A\mathbf{b}_2\}$ to be a basis for AW, the set must be independent. Therefore, there should exist only the trivial solution to $mA\mathbf{b}_1 + nA\mathbf{b}_2 = \mathbf{0}$. Factoring out the A, we have $A(m\mathbf{b}_1 + n\mathbf{b}_2) = \mathbf{0}$. Note that $\mathbf{w} = m\mathbf{b}_1 + n\mathbf{b}_2$, so we can write $A\mathbf{w} = \mathbf{0}$. For this to be true only when $\mathbf{w} = \mathbf{0}$ (the trivial solution), A must be one-to-one, and thus A must have independent columns.

   (d) What are the possible values of $\dim(AW)$?

   AW maps $\mathbb{R}^2 \rightarrow \mathbb{R}^2$. The maximum for $\dim(AW)$ would be 2, as there can be at most 2 basis vectors for a subspace in $\mathbb{R}^2$. However, if A has dependent columns, $\dim(AW)$ would be 1. Note that $\dim(AW)$ cannot be 0, as that would be not be a vector space. Therefore, $\dim(AW)$ can be 1 or 2.

2. Given two real vector spaces U and V, define

$$U + V = \{\mathbf{u} + \mathbf{v} |\ \mathbf{u} \in U, \mathbf{v} \in V\}$$

The space $U + V$ is closed under vector addition and scalar multiplication ($U + V$ is a vector space).

(a) Suppose U and V are finite dimensional, and let B_1 and B_2 be bases for U and V respectively. Explain why $B_1 \cup B_2$ spans $U + V$.
$U + V$ is the span of vectors in U and V, as per the definition. $B_1 \cup B_2$ includes the basis vectors for both U and V. Therefore, the span of $B_1 \cup B_2$ spans all vectors in U and V, and thus spans $U + V$.
Alternatively, using a slightly more algebraic process to illustrate the same point, every vector in $U + V$ can be written in terms of vectors in $B_1 \cup B_2$. Let $\mathbf{x} \in U + V$, and thus $\mathbf{x} = \mathbf{u} + \mathbf{v}$. Note that $\mathbf{u}$ can be written with basis vectors in B_1, and $\mathbf{v}$ can be written with basis vectors in B_2. Therefore, $\mathbf{x} =$ [linear combinations of elements in B_1]+[linear combinations of elements in B_2]. In other words, for any arbitrary $\mathbf{x} \in U + V$, $\mathbf{x}$ is in the span of $B_1 \cup B_2$. So, $B_1 \cup B_2$ spans $U + V$.

(b) Is $B_1 \cup B_2$ a basis for $U + V$?
Not necessarily. Some elements of B_1 may be dependent upon elements of B_2, and thus $B_1 \cup B_2$ wouldn't be a basis as it is not an independent set.

(c) Suppose $U \subseteq V$. What is $U + V$?
Rewriting the initial definition with the fact that all vectors in U are also in V, we have

$$\begin{aligned} U + V &= \{\mathbf{u} + \mathbf{v} |\ \mathbf{u} \in U, \mathbf{v} \in V\} \\ U + V &= \{\mathbf{u} + \mathbf{v} |\ \mathbf{u} \in V, \mathbf{v} \in V\} \\ U + V &= \{k\mathbf{u} |\ \mathbf{u} \in V\}. \end{aligned}$$

Therefore, $U + V$ is the span of vectors in V, or the subspace V.

(d) Write an equation relating $\dim U$, $\dim V$, $\dim(U + V)$, $\dim(U \cap V)$
Based on the result from part (a), where $B_1 \cup B_2$ spans $U + V$, we know that $B_1 \cup B_2$ must have at least as many vectors in the basis of $U + V$. Therefore, we know

$$\dim U + \dim V \geq \dim(U + V).$$

To make this into an equation, we need to figure out how much greater $\dim U + \dim V$ is than $\dim(U + V)$. When B_1 and B_2 have dependent vectors, those dependent vectors are accounted for twice. Therefore, we can subtract the overlap, which leaves us with the following equation:

$$\dim U + \dim V - \dim(U \cap V) = \dim(U + V).$$

This tracks with the insight in part (c): if $U \subseteq V$, $U \cap V$ would be all the elements in U, and thus $\dim(U \cap V) = \dim U$. We see that this works with our equation, and we find that $\dim V = \dim(U + V)$. Based on part (c), $U + V = V$, so we ultimately have $\dim V = \dim V$, which works.

3. Given two vector spaces U and V, we say that U and V are orthogonal subspaces if $\forall \mathbf{u} \in U$ and $\forall \mathbf{v} \in V$ we have $\mathbf{u} \cdot \mathbf{v} = 0$. (Note: when U and V are orthogonal subspaces we can write $U \perp V$).
   Prove that if U and V are orthogonal subspaces, then $U \cap V = \{\mathbf{0}\}$
   Let $\mathbf{w} \in U \cap V$. Since it's in both U and V, and we are given that U and V are orthogonal subspaces, we can write

$$\mathbf{w} \cdot \mathbf{w} = 0$$
$$||\mathbf{w}||||\mathbf{w}||cos\theta = 0.$$

   The two possibilities are if $\mathbf{w} = \mathbf{0}$ or if $cos\theta = 0$. However, we know that $cos\theta = 0$ is impossible, as that would require $\mathbf{w}$ to be perpendicular to itself. Therefore, $\forall \mathbf{w} \in U \cap V, \mathbf{w} = \mathbf{0}$. In other words, $U \cap V = \{\mathbf{0}\}$.