Linear Algebra
Inverses Take Home Exam

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1. Let $M = \begin{bmatrix} -2 & 4 & -2 \\ 0 & 1 & -3 \\ 2 & -3 & 2 \end{bmatrix}$. (5 points)

   (a) Find the inverse of M using a technique developed in class. (Use computing technology to verify it, but find the inverse by hand.)
   To get M^{-1}, we perform the same operations on M to get it into reduced row echelon form on an identity matrix.

$$\left[\begin{array}{ccc|ccc} -2 & 4 & -2 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 2 & -3 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-\frac{1}{2}①} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 2 & -3 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-2①+③}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right] \xrightarrow{②\longleftrightarrow③} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -3 & 0 & 1 & 0 \end{array}\right] \xrightarrow{2②+①}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{3}{2} & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -3 & 0 & 1 & 0 \end{array}\right] \xrightarrow{-1②+③} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{3}{2} & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & -3 & -1 & 1 & -1 \end{array}\right] \xrightarrow{-\frac{1}{3}③}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{3}{2} & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{array}\right] \xrightarrow{-③+①} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{6} & \frac{1}{3} & \frac{5}{3} \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{array}\right].$$

$$\boxed{\text{Thus, } M^{-1} = \begin{bmatrix} \frac{7}{6} & \frac{1}{3} & \frac{5}{3} \\ 1 & 0 & 1 \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}.}$$

   (b) Use the result of part (a) to easily solve the equation $M\mathbf{x} = \mathbf{0}$. Explain why this makes sense.
   We can perform the following operations.

$$M\mathbf{x} = \mathbf{0}$$
$$M^{-1}M\mathbf{x} = M^{-1}\mathbf{0}$$
$$\mathbf{x} = \mathbf{0}.$$

Additionally, we can note that since M is invertible, it must also be one-to-one. Thus, there should exist only one $\mathbf{x}$ such that $M\mathbf{x} = \mathbf{0}$, and that $\mathbf{x}$ must be $\mathbf{0}$. Otherwise, if there exists solutions other than $\mathbf{x} = \mathbf{0}$, there exists more than one solution.

(c) Use the result of part (a) to easily solve the equation $M\mathbf{x} = \begin{bmatrix} 12 \\ -3 \\ 3 \end{bmatrix}$

We can perform the following:

$$M\mathbf{x} = \begin{bmatrix} 12 \\ -3 \\ 3 \end{bmatrix}$$
$$M^{-1}M\mathbf{x} = M^{-1}\begin{bmatrix} 12 \\ -3 \\ 3 \end{bmatrix}$$
$$\mathbf{x} = \begin{bmatrix} \frac{7}{6} & \frac{1}{3} & \frac{5}{3} \\ 1 & 0 & 1 \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 12 \\ -3 \\ 3 \end{bmatrix}$$
$$\mathbf{x} = \begin{bmatrix} 18 \\ 15 \\ 6 \end{bmatrix}.$$

2. Consider the matrix $B = \begin{bmatrix} 1 & -3 & -1 \\ -2 & 5 & 1 \\ -3 & 5 & -1 \end{bmatrix}$. (5 points)

(a) Describe all vectors $\begin{bmatrix} x \\ y \\ x \end{bmatrix}$ such that $B\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}$.

We can obtain a system of equations from matrix multiplication.

$$B\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}$$
$$\begin{bmatrix} 1 & -3 & -1 \\ -2 & 5 & 1 \\ -3 & 5 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}$$
$$\begin{bmatrix} x - 3y - z \\ -2x + 5y + z \\ -3x + 5y - z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Setting the components of the matrices equal to each other, we obtain the following three equations:

$$x - 3y - z = 0$$

$$-2x + 5y + z = 0$$
$$-3x + 5y - z = 0$$

To solve this system of equations, first add the first and second equation together, then add the second and third equation to obtain the following equations:

$$-x + 2y = 0$$
$$-5x + 10y = 0$$

Note that the two equations are dependent, and thus there are multiple solutions. To generalize, set $y = t$ and solve for x in terms of t:

$$-x + 2y = 0$$
$$-x + 2t = 0$$
$$x = 2t$$

Now, to solve for z in terms of t as well, we substitute $x = 2t$ and $y = t$ into any of the original equations.

$$x - 3y - z = 0$$
$$2t - 3t - z = 0$$
$$z = -t$$

Thus, $\begin{bmatrix} x \\ y \\ x \end{bmatrix} = \begin{bmatrix} 2t \\ t \\ -t \end{bmatrix} \forall\, t \in \mathbb{R}$ produces $B \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}$.

(b) Explain how the result of (a) shows that the matrix B is not invertible.

Since there exists multiple solutions for $B \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}$, there exists multiple inputs to the same output, and thus it is evident that the transformation the matrix B represents is not one-to-one. As B is not one-to-one, it is not invertible.

(c) Suppose $B\mathbf{u} = \mathbf{0}$ and $B\mathbf{v} = \mathbf{0}$. Must it be the case that $B(\mathbf{u} + \mathbf{v}) = \mathbf{0}$? Explain.

Yes. Matrix multiplication distributes over addition due to linearity. Thus, the following operations can be performed

$$B(\mathbf{u} + \mathbf{v})$$
$$B\mathbf{u} + B\mathbf{v}$$
$$\mathbf{0} + \mathbf{0} = \mathbf{0}.$$

Thus, $B(\mathbf{u} + \mathbf{v}) = \mathbf{0}$.

In a different approach, we can set up the following:

$$\mathbf{u} = \begin{bmatrix} 2k \\ k \\ -k \end{bmatrix} \forall\, k \in \mathbb{R} \text{ and } \mathbf{v} = \begin{bmatrix} 2d \\ d \\ -d \end{bmatrix} \forall\, d \in \mathbb{R}.$$

Substituting the definitions of $\mathbf{u}$ and $\mathbf{v}$ into $B(\mathbf{u}+\mathbf{v})$, we have

$$B\left(\begin{bmatrix} 2k \\ k \\ -k \end{bmatrix} + \begin{bmatrix} 2d \\ d \\ -d \end{bmatrix}\right) = B\left(\begin{bmatrix} 2k+2d \\ k+d \\ -k-d \end{bmatrix}\right)$$

Since k and d are both constants, we can represent their sum as a different constant: $g = k + d$.

$$B\left(\begin{bmatrix} 2(k+d) \\ k+d \\ -(k+d) \end{bmatrix}\right) = B\left(\begin{bmatrix} 2g \\ g \\ -g \end{bmatrix}\right).$$

Since $\begin{bmatrix} 2g \\ g \\ -g \end{bmatrix}$ is still in the form $\begin{bmatrix} 2t \\ t \\ -t \end{bmatrix}$, and we know from part (a) that $B\begin{bmatrix} 2t \\ t \\ -t \end{bmatrix} = \mathbf{0}$, we can conclude that that

$$B\begin{bmatrix} 2g \\ g \\ -g \end{bmatrix} = B(\mathbf{u}+\mathbf{v}) = \mathbf{0}.$$

(d) Show the column vectors of B are linearly dependent by finding a non-trivial linear combination of the column vectors that is equal to the zero vector.

Let a, b, and c be the coefficients for the linear combinations of the first, second, and third column vector. We set

$$a\begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix} + b\begin{bmatrix} -3 \\ 5 \\ 5 \end{bmatrix} + c\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \mathbf{0}.$$

Simplifying, we get

$$\begin{bmatrix} a-3b-c \\ -2a+5b+c \\ -3a+5b-c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Note that this system of equation is the same as the system in part (a), with $(x, y, z) = (a, b, c)$. Thus, the solution still applies: $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2t \\ t \\ -t \end{bmatrix} \forall\, t \in \mathbb{R}$.

Reviewing the solution in the context of linear combination of columns, we see that

$$2t\begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix} + t\begin{bmatrix} -3 \\ 5 \\ 5 \end{bmatrix} + -t\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \mathbf{0}\ \forall\, t \in \mathbb{R}.$$

As this solution includes non-trivial linear combinations of the column vectors of B, the column vectors of B are linearly dependent.

3. Given a square matrix A, we wish to prove the following biconditional theorem: (3 points)

$$A \text{ is invertible if and only if } A\mathbf{x} = \mathbf{0} \text{ implies } \mathbf{x} = \mathbf{0}$$

   (a) First, we prove that if A is invertible then $A\mathbf{x} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$. Assume that A is invertible. Use this to show that if $A\mathbf{x} = \mathbf{0}$ then $\mathbf{x} = \mathbf{0}$.
   If A is invertible, A must also be one-to-one. Thus, there is only one $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$. For this to be true, $\mathbf{x}$ must be equal to $\mathbf{0}$. If there exists other non-zero vectors such that $A\mathbf{x} = \mathbf{0}$, there are multiple solutions for $A\mathbf{x} = \mathbf{0}$, which means A is not one-to-one and thus not invertible. Since we know A to indeed be invertible, $\mathbf{x} = \mathbf{0}$ must be true.

   (b) Second, we prove that if $A\mathbf{x} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$, then A is invertible. Assume $A\mathbf{x} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$. Use this to prove that A is one-to-one, and so, is invertible.
   To show this, first take:

$$\begin{gathered} A\mathbf{u} = A\mathbf{v} \\ A\mathbf{u} - A\mathbf{v} = \mathbf{0} \\ A(\mathbf{u} - \mathbf{v}) = \mathbf{0}. \end{gathered}$$

   Letting $\mathbf{x} = \mathbf{u} - \mathbf{v}$, we see that

$$A\mathbf{x} = \mathbf{0}.$$

   Since we assume that $A\mathbf{x} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$, we can see that

$$\begin{gathered} \mathbf{x} = \mathbf{0} = \mathbf{u} - \mathbf{v} \\ \mathbf{v} = \mathbf{u}. \end{gathered}$$

   From this, we see that the same output ($A\mathbf{u} = A\mathbf{v}$) implies that the inputs are the same as well ($\mathbf{u} = \mathbf{v}$). Thus, A is one-to-one, and it is invertible.

4. Suppose the matrix A has independent columns. Explain why it must be the case that if $A\mathbf{x} = \mathbf{0}$ then $\mathbf{x} = \mathbf{0}$. What does this tell you about A if A is a square matrix? (2 points)

   Matrix multiplication $A\mathbf{x}$ can be interpreted as taking a linear combination of the columns. For instance, taking a case where $A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} m \\ n \end{bmatrix}$,

$$A\mathbf{x} = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} = m \begin{bmatrix} a \\ c \\ e \end{bmatrix} + n \begin{bmatrix} b \\ d \\ f \end{bmatrix}.$$

   Since the columns of A are independent, there exists only the trivial linear combination that results in the 0 vector. In other words, the coefficients of the column

vectors must be 0, and that occurs when $\mathbf{x} = \mathbf{0}$.
Knowing that $A\mathbf{x} = 0$ necessitates that $\mathbf{x} = \mathbf{0}$, if A is a square matrix, it tells us that A is invertible (biconditional theorem from question 3).

5. Bonus: Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ and let A $[\mathbf{u}|\mathbf{v}]$. Under what conditions would $A^T A$ be invertible? Under what conditions would AA^T be invertible?
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