Practice set 2

ANSWERS

Problem 2.1

Use the error formula for bisection method to get:

$$\begin{aligned}|\alpha - c_n| \leq \frac{3}{2^n} &\leq \varepsilon = 10^{-9}\\ \Leftrightarrow n &\geq \frac{\ln\left(\frac{3}{10^{-9}}\right)}{\ln 2} \approx 31.48.\end{aligned}$$

Therefore $n \geq 32$.

Problem 2.2

The smallest error tolerance that makes sense (since the root is between 1 and 2, no subnormal numbers are used) is the machine epsilon, in this case $\varepsilon = 2^{-m}$. Since we should have

$$\begin{aligned}n &\geq \frac{\ln\left(\frac{1}{2^{-m}}\right)}{\ln 2}\\ &= \frac{\ln\left(2^m\right)}{\ln 2}\\ &= m,\end{aligned}$$

clearly the number of halvings will be m.

Problem 2.3

Solution, obviously is $\alpha = 0$. Newton's method is

$$\begin{aligned}x_{n+1} &= x_n - \frac{x_n^2}{2x_n}\\ &= \frac{1}{2}x_n = \frac{1}{2}\left(\frac{1}{2}x_{n-1}\right) = \frac{1}{2^2}\left(\frac{1}{2}x_{n-2}\right)\\ &= \dots\\ &= \frac{x_0}{2^{n+1}},\end{aligned}$$

where x_0 is the initial guess. It can be seen that $x_{n+1} \to 0$ for any value of x_0, in other words, the Newton's method will converge, no matter what initial guess is chosen. In order to see the speed of convergence, first we can observe a similar behavior to the behavior of the bisection method (look for the error formula for bisection method).

Therefore, a linear convergence will be expected. Actually, we can easily derive a formula for the error in this case:

$$\begin{aligned}x_{n+1} - \alpha &= x_{n+1}\\ &= \frac{1}{2}x_n\\ &= \frac{1}{2}(x_n - \alpha)\end{aligned}$$

Thus, we have a linear convergence with linear rate $\frac{1}{2}$.

Theoretically, the convergence for Newton's method should be quadratic, but in this case the convergence speed is lower. (Because of the multiplicity of the root!)

Problem 2.4

Let a be given and set $\alpha = \sqrt{a}$ and $f(x) = x^2 - a$. Then for any $n \geq 0$ we have

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n^2 - a}{2x_n} \\ &= \frac{1}{2}\left(x_n + \frac{a}{x_n}\right). \end{aligned}$$

In class we have shown the formula

$$\begin{aligned} \sqrt{a} - x_{n+1} &= -\frac{(\sqrt{a} - x_n)^2}{2} \cdot \frac{f"(c_n)}{f'(x_n)} \\ &= -\frac{(\sqrt{a} - x_n)^2}{2} \cdot \frac{2}{2x_n} \\ &= -\frac{\left(\sqrt{a} - x_n\right)^2}{2x_n}. \\ \operatorname{Rel}(x_{n+1}) &= \frac{\sqrt{a} - x_{n+1}}{\sqrt{a}} \\ &= \frac{-\frac{\left(\sqrt{a} - x_n\right)^2}{2x_n}}{\sqrt{a}} \\ &= -\frac{\left(\sqrt{a} - x_n\right)^2 \cdot \sqrt{a}}{2x_n\left(\sqrt{a}\right)^2} \\ &= -\frac{\sqrt{a}}{2x_n}\left(\frac{\sqrt{a} - x_n}{\sqrt{a}}\right)^2 \\ &= -\frac{\sqrt{a}}{2x_n}\left(\operatorname{Rel}(x_n)\right)^2. \end{aligned}$$

For initial guess x_0 near $\sqrt{a}$, the method was shown to converge and therefore $x_n \to \sqrt{a}$. Thus for n big enough, $x_n \approx \sqrt{a}$ and the last formula becomes

$$\operatorname{Rel}(x_{n+1}) \approx -\frac{1}{2}\left(\operatorname{Rel}(x_n)\right)^2$$

Let $\operatorname{Rel}(x_0) = 0.1$, Then

$$\begin{aligned} \operatorname{Rel}(x_1) &\approx -\frac{1}{2}\left(\operatorname{Rel}(x_0)\right)^2 = -0.005; \\ \operatorname{Rel}(x_2) &\approx -\frac{1}{2}\left(\operatorname{Rel}(x_1)\right)^2 \approx -1.25 \cdot 10^{-5}; \\ \operatorname{Rel}(x_3) &\approx -\frac{1}{2}\left(\operatorname{Rel}(x_2)\right)^2 \approx -7.8125 \cdot 10^{-11}; \\ \operatorname{Rel}(x_4) &\approx -\frac{1}{2}\left(\operatorname{Rel}(x_3)\right)^2 \approx -3.0518 \cdot 10^{-21}. \end{aligned}$$

Problem 2.5

Let $f(x) = b - \frac{1}{x}$. Root is $\alpha = \frac{1}{b}$. The relative error according to its definition is

$$\begin{aligned} \operatorname{Rel}(x_{n+1}) &= \frac{\alpha - x_{n+1}}{\alpha} \\ &= \frac{\frac{1}{b} - x_{n+1}}{\frac{1}{b}} \\ &= 1 - bx_{n+1}. \end{aligned}$$

In class we showed that Newton's method for the equation $b - \frac{1}{x} = 0$ becomes

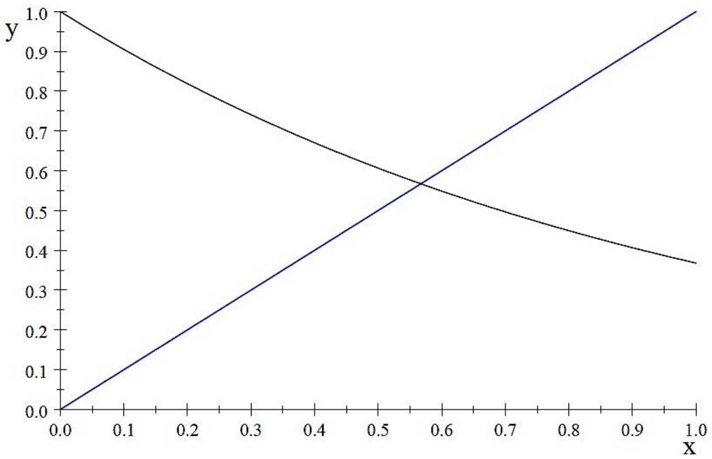
$$x_{n+1} = x_n(2 - bx_n).$$

Thus, combining the last two formulas we have

$$\begin{aligned}
\mathrm{Rel}(x_{n+1}) &= 1 - bx_{n+1} \\
&= 1 - bx_n(2 - bx_n) \\
&= 1 - 2bx_n + b^2x_n^2 \\
&= (1 - bx_n)^2 \\
&= (\mathrm{Rel}(x_n))^2
\end{aligned}$$

Problem 2.6

From the graph it can be seen that we have only one root α between 0.5 and 0.7.



Let $g(x) = e^{-x} \Rightarrow g'(x) = -e^{-x}$. Since

$$\max_{x\in[0.5,0.7]} |g'(x)| = e^{-0.5} \approx 0.6065 < 1,$$

the fixed point iterates $x_{n+1} = e^{-x_n}$ will converge for $x_0 \in [0.5, 0.7]$ (Actually, any $x_0 > 0$ will do fine.). The Aitken extrapolation formula for the error $\alpha - x_3$ is

$$\alpha - x_3 \approx \frac{\lambda_3}{1 - \lambda_3}(x_3 - x_2),$$

where

$$\lambda_3 = \frac{x_3 - x_2}{x_2 - x_1}.$$

We have

i	x_i	λ_i
0	0.57	
1	$5.6553E-1$	
2	$5.6806E-1$	
3	$5.6662E-1$	$-5.6734E-1$

Thus

$$\alpha - x_3 \approx 5.2084E-4.$$

Problem 2.7

Convergence will happen if

$$|g'(\alpha)| < 1$$

where $g(x) = 2 - (1+c)x + cx^3$ and $\alpha = 1$. Therefore, we have the following condition

$$\begin{aligned} |-(1+c)+3c| &< 1 \Leftrightarrow \\ |-1+2c| &< 1 \Leftrightarrow \\ -1 &< -1+2c < 1 \Leftrightarrow \\ 0 &< c < 1 \end{aligned}$$

Convergence will be at least quadratic, if $g'(1) = 0 \Leftrightarrow -1+2c = 0 \Leftrightarrow c = \frac{1}{2}$. It should be remarked that $g''(1) = 3x \Rightarrow g''(1) \neq 0$. Therefore, convergence will be exactly quadratic.

Problem 2.8

$$|\text{Rel}(-1960.14)| = \frac{-1960 - (-1960.14)}{-1960} = 7.1429E - 5$$

Consider the root $\alpha(0) = 3$. Let $g(x) = x^4$ and $f(x) = x^7 - 28x^6 + 322x^5 - 1960x^4 + 6769x^3 - 13132x^2 + 13680x - 5040$. Then $f'(x) = 7x^6 - 168x^5 + 1610x^4 - 7840x^3 + 20307x^2 - 26264x + 13680$ and $f'(3) = 660$. From the formula

$$\begin{aligned} \alpha(\varepsilon) &\approx \alpha(0) - \varepsilon\frac{g(\alpha(0))}{f'(\alpha(0))} \\ &= 3 - \varepsilon\frac{g(3)}{f'(3)} \\ &= 3 - 0.14 \cdot \frac{3^4}{660} \\ &\approx 2.9828. \end{aligned}$$

Consider the root $\alpha(0) = 5$. Then $f'(5) = 660$. Similarly,

$$\begin{aligned} \alpha(\varepsilon) &\approx 5 - \varepsilon\frac{g(5)}{f'(5)} \\ &= 5 - 0.14 \cdot \frac{5^4}{660} \\ &\approx 4.8674. \end{aligned}$$

Problem 2.9

Let

$$g(x) = \frac{x(x^2+15)}{3x^2+5}.$$

First, check that $\sqrt{5}$ is a fixed point for function g. Indeed $g(\sqrt{5}) = \sqrt{5}$. Then compute

$$\begin{aligned} g'(x) &= \frac{9x^4 - 30x^2 + 75}{(3x^2+5)^2} \\ g'(\sqrt{5}) &= 0. \end{aligned}$$

Thus, the order of convergence is at least quadratic.

$$\begin{aligned} g''(x) &= \frac{12x(-3x^4 + 31x^2 - 80)}{(3x^2+5)^4} \\ g''(\sqrt{5}) &= 0. \end{aligned}$$

Therefore, the order of convergence is at least cubic. It can be checked that

$$g^{(3)}(\sqrt{5}) \neq 0$$

So, the order of convergence is cubic.

Problem 2.10

n	x_n	$x_{n-1} - x_n$	λ_n
0	0.75		
1	0.752710	0.00271	
2	0.754795	0.00208	0.76753
3	0.756368	0.00157	0.75481
4	0.757552	0.00118	0.75159
5	0.758441	0.000889	0.75339

We can see that $\lambda_n \to 0.75 = \frac{3}{4}$. Therefore, we can say that our root have multiplicity 4. In order to find an accurate value of α, we compute the $f^{(3)}(x)$ and apply the Newton's method to this equation $f^{(3)}(x) = 0$. Since α is a simple root of $f^{(3)}(x)$,Newton′s method should converge much faster.

Problem 2.11

n	x_n	$x_n - x_{n-1}$	λ_n
0	1.30499998		
1	1.25340617	$-5.159E-2$	
2	1.21676284	$-3.664E-2$	$7.102E-1$
3	1.19087998	$-2.588E-2$	$7.063E-1$
4	1.17257320	$-1.831E-2$	$7.075E-1$
5	1.15962919	$-1.294E-2$	$7.06717E-1$

Since λ_n obviusly are converging to 0.707, we have $g'(\alpha) \approx 0.707$ and

$$\alpha - x_{n+1} \approx 0.707(\alpha - x_{n+1})$$

which means that convergence is linear with linear rate approximately 0.707. By Aitken error estimation formula we have

$$\begin{aligned} \alpha - x_5 &\approx \frac{\lambda_5}{1-\lambda_5}(x_5 - x_4) \\ &\approx \frac{0.706717}{1-0.76717} \cdot (-0.001294) \\ &\approx -3.92772E-3. \end{aligned}$$

By Aitken extrapolation formula

$$\begin{aligned} \alpha &\approx x_5 + \frac{\lambda_5}{1-\lambda_5}(x_5 - x_4) \\ &\approx 1.15962919 + (-3.92772E-3) \\ &= 1.15570147. \end{aligned}$$