

## Practice problems 2

### Problem 2.1

Let the interval used in the bisection method have the length  $b - a = 3$ . Find the number of midpoints  $c_n$  that must be calculated with the bisection method to obtain an approximate root within an error tolerance of  $10^{-9}$ .

### Problem 2.2

Imagine you are finding a root  $\alpha$  satisfying  $1 < \alpha < 2$ . If you are using a binary computer with  $m$  digits in its significand, what is the smallest error tolerance that makes sense in finding an approximation to  $\alpha$ ? If the original interval is  $[1, 2]$  how many halving are needed to find an approximation to  $\alpha$  with the maximum accuracy possible for this computer?

### Problem 2.3

Work out what the Newton iteration is for  $f(x) = x^2$ . What is the solution to  $f(x) = 0$ ? Will the sequence generated by Newton method converge to solution? How quickly? Relate this to the theory of Newton method.

### Problem 2.4

On most computers, the computation of  $\sqrt{a}$  is based on Newton's method. Set up the Newton's iteration for solving  $x^2 - a = 0$ , and show that it can be written in the form

$$x_{n+1} = \frac{1}{2} \left( x_n + \frac{a}{x_n} \right), \quad n \geq 0.$$

Derive the error and relative error formulas:

$$\begin{aligned} \sqrt{a} - x_{n+1} &= -\frac{1}{2x_n} (\sqrt{a} - x_n)^2, \\ \text{Rel}(x_{n+1}) &= -\frac{\sqrt{a}}{2x_n} (\text{Rel}(x_n))^2. \end{aligned}$$

For initial guess  $x_0$  near  $\sqrt{a}$ , the last formula becomes

$$\text{Rel}(x_{n+1}) \approx -\frac{1}{2} (\text{Rel}(x_n))^2$$

Assuming  $\text{Rel}(x_0) = 0.1$ , use this formula to estimate the relative error in  $x_i, i = 1, 2, 3, 4$ .

### Problem 2.5

Derive formula

$$\text{Rel}(x_{n+1}) = (\text{Rel}(x_n))^2$$

for the Newton's iterations used in computing  $\frac{1}{b}$  for given  $b$  (formula was discussed in class without proof).

### Problem 2.6

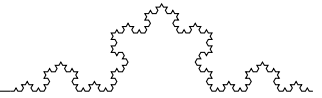
How many solutions are there to the equation  $x = e^{-x}$ ? Will the iteration  $x_{n+1} = e^{-x_n}$  converge for a suitable choice of  $x_0$ ? Use Aitken extrapolation formula to estimate the error  $\alpha - x_3$  for  $x_0 = 0.57$ .

### Problem 2.7

The iteration

$$x_{n+1} = 2 - (1 + c)x_n + cx_n^3$$

will converge to  $\alpha = 1$  for some values of  $c$  (provided that initial guess  $x_0$  is chosen sufficiently close to  $\alpha$ ). Find the values of  $c$  for which convergence occurs. For what values of  $c$ , if any, convergence will be quadratic?



### Problem 2.8

Consider the equation

$$x^7 - 28x^6 + 322x^5 - 1960x^4 + 6769x^3 - 13132x^2 - 5040 = 0$$

Change the coefficient of  $x^4$  from  $-1960$  to  $-1960.14$ . What is relative perturbation error in the coefficient of  $x^4$ ? Calculate  $\alpha(\varepsilon)$  for  $\alpha(0) = 3$  and  $\alpha(0) = 5$ .

### Problem 2.9

What is the order of convergence of the iteration

$$x_{n+1} = \frac{x_n(x_n^2 + 15)}{3x_n^2 + 5}$$

as it converges to the fixed point  $\alpha = \sqrt{5}$ ?

### Problem 2.10

Newton's method is used to find the root of  $f(x) = 0$ . The first few iterates are shown in the following table, giving a very slow speed of convergence. What can be said about the root  $\alpha$  to explain the convergence? Knowing  $f(x)$ , how would you find an accurate value for  $\alpha$ ?

$n$	$x_n$	$x_{n-1} - x_n$
0	0.75	
1	0.752710	0.00271
2	0.754795	0.00208
3	0.756368	0.00157
4	0.757552	0.00118
5	0.758441	0.000889

### Problem 2.11

Consider the following table of iterates from an iteration method which is convergent to a fixed point  $\alpha$  of the function  $g(x)$ :

$n$	$x_n$	$x_n - x_{n-1}$
0	1.30499998	
1	1.25340617	$-5.159E-2$
2	1.21676284	$-3.664E-2$
3	1.19087998	$-2.588E-2$
4	1.17257320	$-1.831E-2$
5	1.15962919	$-1.294E-2$

(a) Does this appear to be a linearly convergent iteration method? If so, then estimate the rate of linear convergence. (b) Estimate the error in  $x_5$ . (c) Give an improved estimate of  $\alpha$ .