Practice Set 3

ANSWERS

Problem 3.1

Consider a general cubic polynomial

$$P(x) = a + bx + cx^2 + dx^3.$$

Then

$$P'(x) = b + 2cx + 3dx^2.$$

From conditions

$$y_1 = P(0), \quad y_2 = P(1), \quad y_1' = P'(0), \quad y_2' = P'(1),$$

we get

$$\begin{aligned}
y_1 &= P(0) = a, \\
y_2 &= P(1) = a + b + c + d, \\
y_1' &= P'(0) = b, \\
y_2' &= P'(1) = b + 2c + 3d,
\end{aligned}$$

Solve this system with unknowns a, b, c and d

$$\begin{aligned}
y_1 &= a, \\
y_2 &= a + b + c + d, \\
y_1' &= b, \\
y_2' &= b + 2c + 3d,
\end{aligned}$$

and obtain solution

$$\begin{aligned}
a &= y_1, \\
b &= y_1', \\
c &= -3y_1 + 3y_2 - 2y_1' - y_2', \\
d &= 2y_1 - 2y_2 + y_1' + y_2',
\end{aligned}$$

Thus we have

$$\begin{aligned}
P(x) &= y_1 + y_1' x + \big(-3y_1 + 3y_2 - 2y_1' - y_2'\big)x^2 + \big(2y_1 - 2y_2 + y_1' + y_2'\big)x^3 \\
&= \big(2x^3 - 3x^2 + 1\big)y_1 + \big(-2x^3 + 3x^2\big)y_2 + \big(x^3 - 2x^2 + x\big)y_1' + \big(x^3 - x^2\big)y_2' \\
&= \big(1 + 2x\big)\big(1 - x\big)^2 y_1 + x^2\big(3 - 2x\big)y_2 + x\big(1 - x\big)^2 y_1' + x^2\big(x - 1\big)y_2' \\
&= H_1(x)y_1 + H_2(x)y_2 + H_3(x)y_1' + H_4(x)y_2'
\end{aligned}$$

with

$$\begin{aligned}
H_1(x) &= \big(1 + 2x\big)\big(1 - x\big)^2, \\
H_2(x) &= x^2\big(3 - 2x\big), \\
H_3(x) &= x\big(1 - x\big)^2, \\
H_4(x) &= x^2\big(x - 1\big),
\end{aligned}$$

Observe that

$$\begin{array}{llll}
H_1(0) = 1, & H_1(1) = 0, & H_1'(0) = 0, & H_1'(1) = 0, \\
H_2(0) = 0, & H_2(1) = 1, & H_2'(0) = 0, & H_2'(1) = 0, \\
H_3(0) = 0, & H_3(1) = 0, & H_3'(0) = 1, & H_3'(1) = 0, \\
H_4(0) = 0, & H_4(1) = 0, & H_4'(0) = 0, & H_4'(1) = 1,
\end{array}$$

Problem 3.2

Look for function $Q_1(x) = a + b\cos(\pi x) + c\sin(\pi x)$ such that

$$\begin{cases} a + b\cos(\pi \cdot 0) + c\sin(\pi \cdot 0) = Q_1(0) = 2 \\ a + b\cos(\pi \cdot \frac{1}{2}) + c\sin(\pi \cdot \frac{1}{2}) = Q(\frac{1}{2}) = 5 \\ a + b\cos(\pi \cdot 1) + c\sin(\pi \cdot 1) = Q_1(1) = 4 \end{cases}$$

This leads to the following system

$$\begin{cases} a + b = 2 \\ a + c = 5 \\ a - b = 4 \end{cases}$$

that has solution $a = 3$, $b = -1$, $c = 2$. Therefore the function is $Q_1(x) = 3 - \cos(\pi x) + 2\sin(\pi x)$.

In order to find the quadratic interpolation polynomial use Newton's divided difference formula. First, need to compute Newton's divided differences:

x	y	D_1	D_2
0	2	6	−8
0.5	5	−2	
1	4		

And interpolating polynomial is $P_2(x) = 2 + 6x - 8x(x - \frac{1}{2}) = -8x^2 + 10x + 2$.

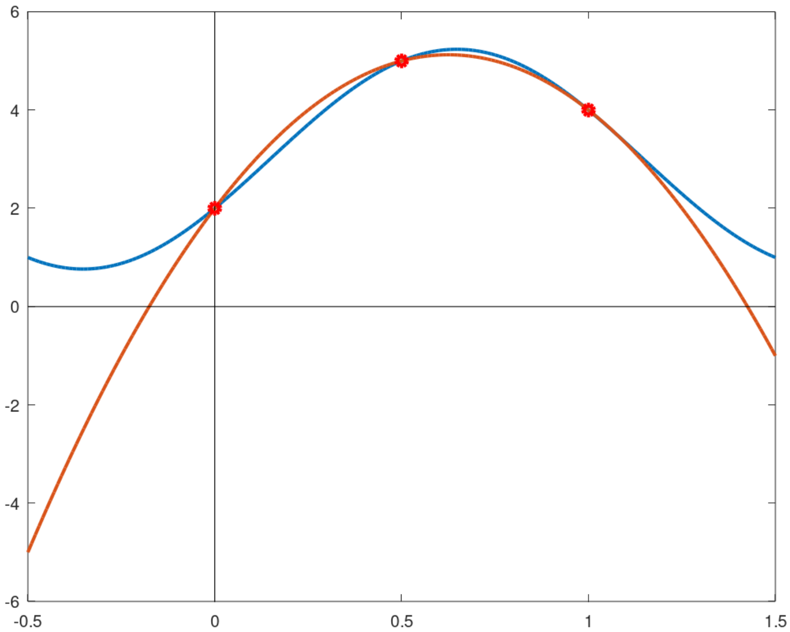

Figure 1: Graphs of quadratic interpolant $P_2(x) = -8x^2 + 10x + 2$ (red) and trigonometric interpolant $Q_1(x) = 3 - \cos(\pi x) + 2\sin(\pi x)$ (blue).

Problem 3.3

Use Newton's divided difference formula

x	y	D_1	D_2	D_3
0	−1	5	−7/2	13/15
1	4	−2	5/6	
2	2	4/3		
5	6			

Interpolating cubic polynomial is $P_3(x) = -1 + 5x - \frac{7}{2}x(x-1) + \frac{13}{15}x(x-1)(x-2)$.

Problem 3.4

Observe that data are satisfying relation $y = x^2$, i.e. interpolating points are located on parabola $y = x^2$. Since interpolation polynomial is unique, the answer is $P(x) = x^2$.

Problem 3.5

Let $N+1$ interpolation points be

$$x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N$$

and let $L_i(x)$, $i = 0, 1, \ldots, N$ be $N+1$ associated Lagrange basis functions. Need to prove that

$$\sum_{i=0}^{N} L_i(x) = 1. \tag{1}$$

Proof.
Rewrite the left side of the identity (1) in the form

$$1 \cdot L_0(x) + 1 \cdot L_1(x) + 1 \cdot L_2(x) + \cdots + 1 \cdot L_N(x) \tag{2}$$

and compare it with the formula for interpolation polynomial for the data (x_i, y_i), $i = 0, 1, \ldots, N$:

$$y_0 \cdot L_0(x) + y_1 \cdot L_1(x) + y_2 \cdot L_2(x) + \cdots + y_N \cdot L_N(x).$$

Obviuosly (2) represents the interpolation polynomial that interpolates points $\{(x_0, 1), (x_1, 1), (x_2, 1), \ldots, (x_N, 1)\}$. These points lie on the line $y = 1$, therfore, since interpolation polynomial is unique, it follows that interpolation polynomial given by (2) is identical function 1. So

$$1 \cdot L_0(x) + 1 \cdot L_1(x) + 1 \cdot L_2(x) + \cdots + 1 \cdot L_N(x) \equiv 1.$$

and identity (1) is proved. ■

Problem 3.6

Let $P_2(x)$ be the quadratic interpolation polynomial of function e^{-x^2} at the points $x_0 = 0$, $x_1 = \frac{1}{2}$ and $x_2 = 1$. The interpolation error is given by

$$e^{-x^2} - P_2(x) = \frac{x(x - \frac{1}{2})(x-1)}{6} \left(e^{-x^2} \right)'''(\theta) \tag{3}$$

for some $\theta \in [0, 1]$. Let $h = \frac{1}{2}$. Using error formula (3) we get

$$\begin{aligned} \left| e^{-x^2} - P_2(x) \right| &= \frac{\left| x(x-h)(x-2h) \right|}{6} \left| \left(e^{-x^2} \right)'''(\theta) \right| \\ &\leq \frac{1}{6} \cdot \max_{x \in [0,1]} \left| x(x-h)(x-2h) \right| \cdot \max_{x \in [0,1]} \left| \left(e^{-x^2} \right)'''(x) \right| \end{aligned} \tag{4}$$

From Lecture 9 (pages $8 - 10$) we know that

$$\max_{x_0 \leq x \leq x_2} \left| (x - x_0)(x - x_1)(x - x_2) \right| = \frac{2h^3}{3\sqrt{3}}, \text{ with } h = x_1 - x_0 = x_2 - x_1.$$

Thus,

$$\max_{x \in [0,1]} \left| x(x-h)(x-2h) \right| = \frac{2 \cdot \frac{1}{2^3}}{3\sqrt{3}} = \frac{1}{12\sqrt{3}} \approx 0.048113 \tag{5}$$

Next compute the 3rd derivative of e^{-x^2}:

$$\begin{aligned} \left(e^{-x^2} \right)' &= -2x e^{-x^2} \\ \left(e^{-x^2} \right)'' &= (4x^2 - 2) e^{-x^2} \\ \left(e^{-x^2} \right)''' &= (-8x^3 + 12x) e^{-x^2} \end{aligned}$$

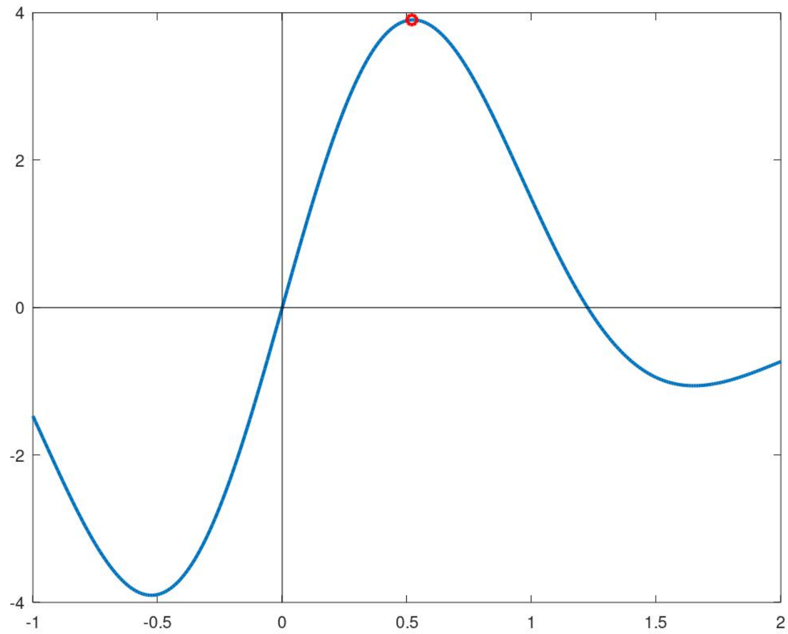

Figure 2: Graph of $(-8x^3 + 12x)e^{-x^2}$

In order to find the maximum of $(-8x^3 + 12x)e^{-x^2}$ on $[0, 1]$, plot its graph.
It follows from the graph that

$$\max_{x\in[0,1]} \left| \left(e^{-x^2} \right)^{'''}(x) \right| \approx 3.9032 \quad \text{at } x \approx 0.52 \tag{6}$$

Substituting (6) and (5) in inequaity (4) we obtain

$$\left| e^{-x^2} - P_2(x) \right| \leq \frac{1}{6} \cdot 0.048113 \cdot 3.9032 \approx 0.031299 = 3.13E - 2$$

Problem 3.7

In order to be a cubic spline, function $s(x)$ should have the properties (see Lecture 11):

1. $s(x)$ is a piecewise cubic polynomial;
2. $s(x)$, $s'(x)$ and $s''(x)$ should be continuous functions.

Let

$$s(x) = \begin{cases} (x-1)^3, & 0 \leq x \leq 1 \\ 2(x-1)^3, & 1 \leq x \leq 2 \end{cases}$$

Obviously condition 1 is satisfied. Then

$$s'(x) = \begin{cases} 3(x-1)^2, & 0 \leq x \leq 1 \\ 6(x-1)^2, & 1 \leq x \leq 2 \end{cases}$$

and

$$s''(x) = \begin{cases} 6(x-1), & 0 \leq x \leq 1 \\ 12(x-1), & 1 \leq x \leq 2 \end{cases}$$

It can be easily checked that $s_-(1) = 0 = s_+(1)$, $s'_-(1) = 0 = s'_+(1)$ and $s''_-(1) = 0 = s''_+(1)$. Thus, condition 2 is also satisfied. Therefore the function $s(x)$ is a cubic spline. Moreover, since $s''(0) = -6 \neq 0$ and $s''(2) = 12 \neq 0$, this is not a "natural" cubic spline.

Problem 3.8

(a) Piecewise linear interpolant $P_{1,p}(x)$ is

$$P_{1,p}(x) = \begin{cases} \frac{1}{2}x, & 0 \le x \le \frac{1}{2} \\ \frac{3}{2}x - \frac{1}{2}, & \frac{1}{2} \le x \le 1 \\ -2x+3, & 1 \le x \le 2 \\ -1, & 2 \le x \le 3 \end{cases}$$

(b) Use Newton's divided differences to find piecewise quadratic interpolant $P_{2,p}(x)$.

x	y	D_1	D_2
0	0	1/2	1
1/2	1/4	3/2	
1	1		

x	y	D_1	D_2
1	1	−2	1
2	−1	0	
3	−1		

On $\{0, 1/2, 1\}$ quadratic polynomial is $0 + 1/2(x-0) + 1(x-0)(x-1/2) = x^2$
and on $\{1, 2, 3\}$ quadratic polynomial is $1 + (-2)(x-1) + 1(x-1)(x-2) = x^2 - 5x + 5$.
Therefore, piecewise quadratic interpolant is:

$$P_{1,p}(x) = \begin{cases} x^2, & 0 \le x \le 1 \\ x^2 - 5x + 5, & 1 \le x \le 3 \end{cases}$$

(c) Since it was not specifically required to obtain the natural cubic spline analytically, we will use GNU Octave/MATLAB built-in functions to get it.

In GNU Octave define the interpolating data and apply *csape* function:

```
>>xx=[0, 0.5, 1, 2, 3];
>>yy=[0, 0.25, 1, -1, -1];
>>pp=csape(xx, yy, 'variational')
pp =
  scalar structure containing the fields:

    form = pp

    breaks =
       0.00000    0.50000    1.00000    2.00000    3.00000

    coefs =
       1.80952    0.00000    0.04762    0.00000
      -5.04762    2.71429    1.40476    0.25000
       2.52381   -4.85714    0.33333    1.00000
      -0.90476    2.71429   -1.80952   -1.00000

    pieces =  4
    order =  4
    dim =  1
```

Matrix `coefs` contains the coefficients of piecewise cubic polynomial:
if row i of matrix `coefs` is $[a\,b\,c\,d]$ then $s(x) = a(x-x_i)^3 + b(x-x_i)^2 + c(x-x_i) + d$ if $x \in [x_i, x_{i+1}]$

Therefore, we have the following natural cubic spline function that interpolates our data.

$$sn(x) = \begin{cases} 1.80952x^3 + 0.04762x, & 0 \le x \le \frac{1}{2} \\ -5.04762(x-0.5)^3 + 2.71429(x-0.5)^2 + 1.40476(x-0.5) + 0.25, & \frac{1}{2} \le x \le 1 \\ 2.52381(x-1)^3 - 4.85714(x-1)^2 + 0.33333(x-1) + 1.0, & 1 \le x \le 2 \\ -0.90476(x-2)^3 + 2.71429(x-2)^2 - 1.80952(x-2) - 1.0, & 2 \le x \le 3 \end{cases}$$

Let's plot all three interpolants:

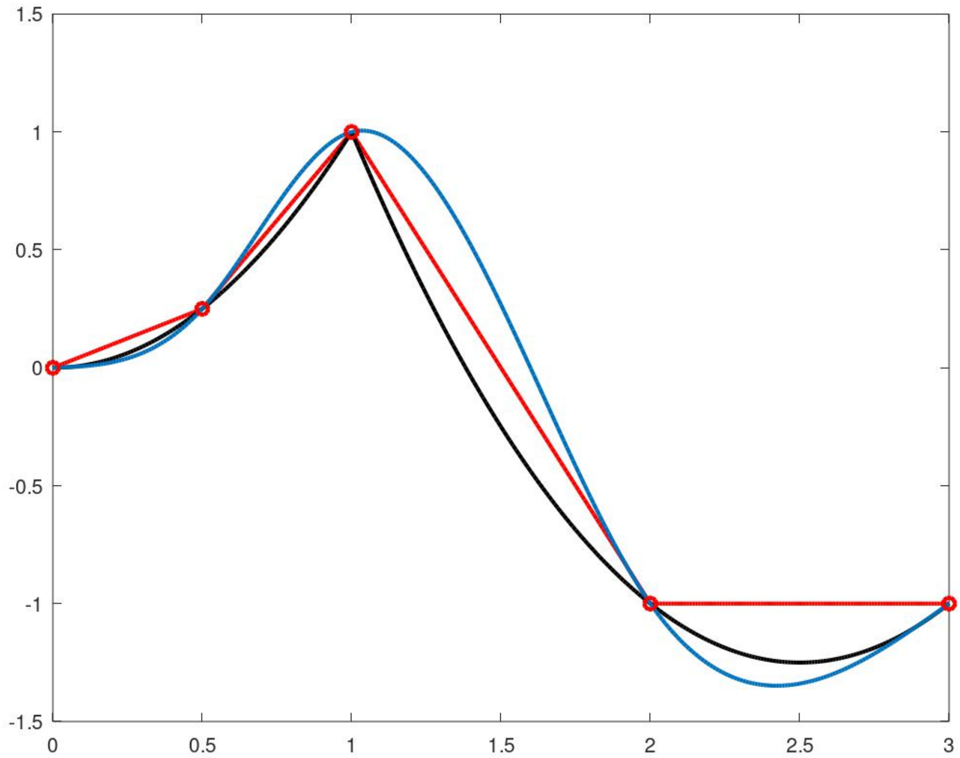

Figure 3: Graphs of piecewise linear interpolant (red), piecewise quadratic interpolant (black) and natural cubic spline interpolant (blue)

IMPORTANT REMARK:
Beware and read help/refference documentation initially, since, as it was mentioned in class lecture, there are variuous boundary conditions, and thus many types of spline functions exist. For example,

- **natural** (known in Matlab and GNU Octave as 'variational') cubic spline uses condition $s''(x_1) = s''(x_N) = 0$.
- **not-a-knot** cubic spline uses condition that $s'''(x)$ is continuous at x_2 and x_{N-1}.
- **complete** (also known as clamped) cubic spline uses condition $s'(x_1) = A$ and $s'(x_2) = B$ with values A and B provided beforehand.
- **periodic** cubic spline uses condition $s'(x_1) = s'(x_N)$.
- **second** cubic spline uses condition $s'(x_1) = C$ and $s'(x_2) = D$ with values A and B provided beforehand.

In order to obtain natural cubic spline we will use *csape* function provided by MATLAB library. For using it in GNU Octave you need to download and install additional package *splines*. Let Google be with you! I was able to do it, so will you. The available by default with GNU Octave basic distribution function *spline* can do only not-a-knot and clamped cubic splines.

As a bonus let's compare 3 different cubic spline interpolants.

- natural cubic spline obtained previously;
- clamped cubic spline (obtained using `spline` function from GNU Octave);
- not-a-knot cubic spline (aobtained using `spline` function).

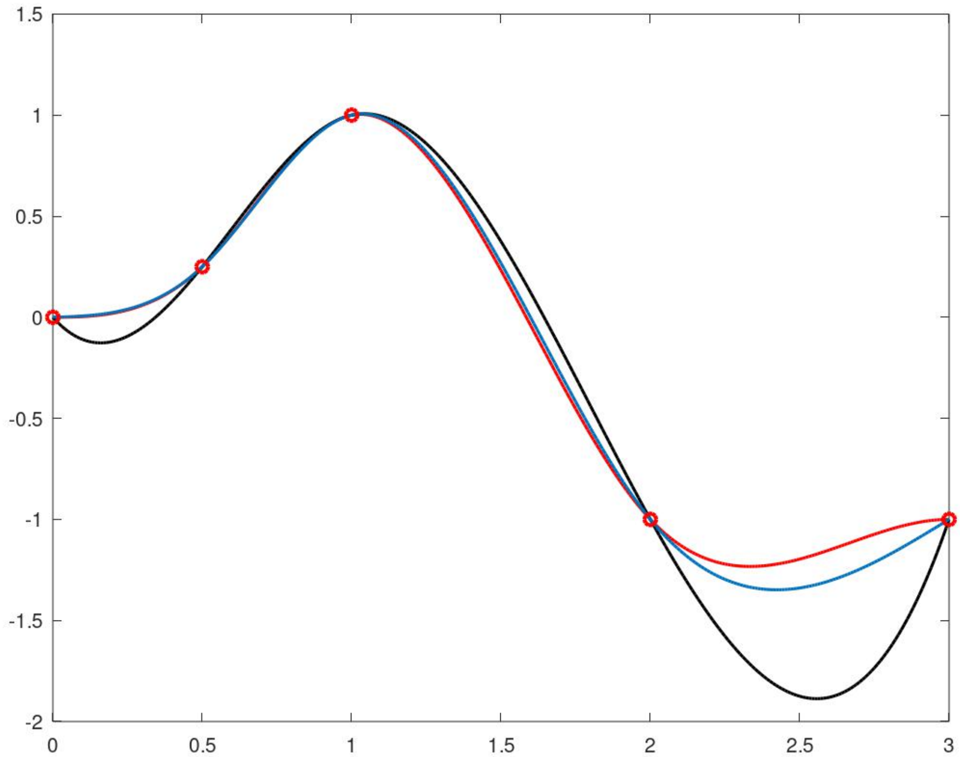

Figure 4: Natural cubic spline (blue), clamped cubic spline (red), not-a-knot cubic spline (black)

Problem 3.9

Consider the error formula for minimax approximation (see Lecture 10, page 5)

$$\rho_n(f) \le \frac{\left(\frac{b-a}{2}\right)^{n+1}}{(n+1)!\,2^n} \max_{x\in[a,b]} \left|f^{(n+1)}(x)\right|$$

Substituting $a=-1$, $b=2$, $n=5$ we get error estimate formula

$$\rho_5(f) \le \frac{\left(\frac{3}{2}\right)^6}{6!\,2^5} \max_{x\in[-1,2]} \left|f^{(6)}(x)\right| = \frac{3^6}{6!\,2^{11}} \max_{x\in[-1,2]} \left|f^{(6)}(x)\right|,$$

where $f(x)=e^{3x-1}$. Compute derivatives

$$\left(e^{3x-1}\right)' = 3e^{3x-1}, \quad \left(e^{3x-1}\right)'' = 9e^{3x-1}, \quad \ldots, \quad \left(e^{3x-1}\right)^{(6)} = 3^6 e^{3x-1}$$

Since

$$\max_{x\in[-1,2]} \left|f^{(6)}(x)\right| = 3^6 \max_{x\in[-1,2]} \left|e^{3x-1}\right| = 3^6\cdot e^{3\cdot 2-1} = 3^6 e^5$$

error estiamte formula becomes

$$\rho_5(e^{3x-1}) \le \frac{3^{12}e^5}{6!\,2^{11}} \approx 53.489$$

It seems that this is a very crude estimate, but keep in mind that fucntion e^{3x-1} on interval $[-1,2]$ grows fast from 0.018 up to 148.41. So we can repeat the same computations for $n=10$ and $n=15$ to get

$$\rho_{10}(e^{3x-1}) \le \frac{3^{22}e^5}{11!\,2^{21}} \approx 5.564E-2, \quad \rho_{15}(e^{3x-1}) \le \frac{3^{32}e^5}{16!\,2^{31}} \approx 6.1207E-6$$

Problem 3.10

Recall the triple recurrsion formula for Chebyshev polynomials (see Lecture10, page 8)

$$\begin{aligned} T_0(x) &= 1, \\ T_1(x) &= x, \\ T_{n+1}(x) &= (2x) \cdot T_n(x) - T_{n-1}(x), \end{aligned}$$

Evaluation of T_0 and T_1 is straightforward, then T_2 will need 2 multiplications and 1 addition, and every next polynomial will need 1 more multiplication and 1 more addition. Therefore, evaluation at a particular x of Chebyshev polynomials $T_0(x), T_1(x), T_2(x), \ldots T_n(x)$ will need n multiplications and $n-1$ additions.

Problem 3.11

Let $q(x)$ be a polynomial of degree $n-1$ and consider the polynomial $x^n - q(x)$. It is a polynomial of degree n, moreover it is a monic polynomial (see Lecture 10). According to the **Theorem on minimum size property** from page 11, the degree n monic polynomial with the smallest maximum on $[-1,1]$ is the modified Chebyshev polynomial $\widetilde{T}_n(x)$, and its maximum value on $[-1,1]$ is $\frac{1}{2^{n-1}}$. Thus,

$$\max_{x\in[-1,1]} \left|x^n - q(x)\right| = \frac{1}{2^{n-1}}$$

and it is achieved for $q(x) = x^n - \widetilde{T}_n(x)$.

Problem 3.12

Proof.
Consider substitution

$$x = \cos\theta, \quad \theta = \arccos x, \quad \sqrt{1-x^2} = \sqrt{1-\cos^2\theta} = \sin\theta, \quad dx = d(\cos\theta) = -\sin\theta\, d\theta.$$

Then

$$\begin{aligned} \int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}\, dx &= \int_{\pi}^{0} \frac{\cos(n\theta)\cos(m\theta)(-\sin\theta)\, d\theta}{\sin\theta} \\ &= \int_{0}^{\pi} \cos(n\theta)\cos(m\theta)\, d\theta \\ &= 0, \text{ if } n \neq m. \end{aligned} \qquad \blacksquare$$