Practice Problems Set 3

Problem 3.1

Find a polynomial $P(x)$ of degree ≤ 3 for which

$$\begin{array}{ll} P(0) = y_1 & P(1) = y_2 \\ P'(0) = y_1' & P'(1) = y_2' \end{array}$$

with y_1, y_2, y_1', y_2' given constants.
The resulting polynomial is called **cubic Hermite interpolating polynomial**.
HINT: Write $P(x) = y_1 H_1(x) + y_2 H_2(x) + y_1' H_3(x) + y_2' H_4(x)$ with H_i cubic polynomials satisfying appropriate properties, in analogy with Lagrange interpolating polynomials.

Problem 3.2

Find the function $P(x) = a + b\cos(\pi x) + c\sin(\pi x)$, which interpolates the data

x	0	0.5	1
y	2	5	4

This is so-called **trigonometric interpolation**. Also, find the quadratic polynomial interpolating this data. In each instance, draw the graph of the interpolating function.

Problem 3.3

Find cubic polynomial interpolating the data

x	0	1	2	5
y	-1	4	2	6

Problem 3.4

Find the polynomial interpolating the data

x	-3	1	2	4	5
y	9	1	4	16	25

Problem 3.5

Prove that

$$\sum_{i=0}^{N} L_i(x) = 1,$$

where $L_i(x)$ are Lagrange basis functions associated to $N+1$ interpolation points.

Problem 3.6

Consider the polynomial interpolation of the function $f(x) = e^{-x^2}$ on $[0,1]$ at the points $x_0 = 0$, $x_1 = 0.5$ and $x_2 = 1$. Estimate the maximum of the polynomial interpolation error for $x \in [0,1]$, i.e. give an upper bound for this error.

Problem 3.7

Is the following a cubic spline on the interval $0 \leq x \leq 2$?

$$s(x) = \begin{cases} (x-1)^3, & 0 \leq x \leq 1 \\ 2(x-1)^3, & 1 \leq x \leq 2 \end{cases}$$

Problem 3.8

Consider the data

x	0	1/2	1	2	3
y	0	1/4	1	-1	-1

(a) Find the piecewise linear interpolating function for the data; (b) Find the piecewise quadratic interpolating function for the data. (c) Find the natural cubic spline that interpolates the data. Graph all three graphs for $0 \leq x \leq 3$.

Problem 3.9

Compute the error bound for the minimax approximation of the function $f(x) = e^{3x-1}$ on the $[-1, 2]$ and $n = 5$.

Problem 3.10

How many multiplications and additions are needed to compute Chebyshev polynomials $T_0(x), T_1(x), T_2(x), \ldots, T_n(x)$ for a particular value of x?

Problem 3.11

Let $q(x)$ be a polynomial of degree $\leq n - 1$, and consider

$$\max_{-1 \leq x \leq 1} |x^n - q(x)|$$

What is the smallest possible value for this quantity? Solve for the $q(x)$ for which this value is attained.

Problem 3.12

For $n, m \geq 0$ and $n \neq m$ show

$$\int_{-1}^{1} \frac{T_n(x) T_m(x)}{\sqrt{1 - x^2}} dx = 0$$

This is called the orthogonality property for the Chebyshev polynomials.