Homework 1

Problem 1.1

Let d_1d_2/m_1m_2 be your birthday. Find the binary single precision IEEE floating-point expression for the number $(d_1m_2)_{10}$. Also, find the binary double precision IEEE floating-point expression for the number $(d_2m_1)_{10}$. In both cases, specify significand, exponent and σ. Convert $(111\ldots1)_2$ to decimal form with the parentheses enclosing $(21 - m_1m_2)$ ones.

Problem 1.2

Some microcomputers in the past used a binary floating-point format with 8 bits for the exponent and 1 bit for the sign σ. The significand contained 31 bits, with no hiding of the leading bit 1. The arithmetic used rounding. To determine the accuracy of the representation, find the machine epsilon, integer M, and the largest number that can be represented exactly in this floating-point format. Also, find the accuracy of the rounding operation.

Problem 1.3

Consider a binary floating-point representation with significand containing 3 digits without hiding the leading 1 and $-1_{10} \leq e \leq 3_{10}$. List all numbers that can be stored exactly together with their decimal value. Plot these numbers on real axis. For this arithmetic, specify what are the corresponding floating-point representation of $\pi/4$ and $14/5$ if a) rounding is used; b) chopping is used?

Problem 1.4

Calculate the error, relative error and the number of significant digits in the following approximations $x_A \approx x_T$.

$$
\begin{aligned}
&a) \quad x_A = 28.271, \quad x_T = 28.254;\\
&b) \quad x_A = 0.028271, \quad x_T = 0.028254;\\
&c) \quad x_A = 19/7, \quad x_T = e;\\
&d) \quad x_A = 1.414, \quad x_T = \sqrt{2}.
\end{aligned}
$$

Problem 1.5

Avoid loss-of-significance errors in the following formulas

$$
\begin{aligned}
&a) \quad \log(x+1) - \log(x) \quad \text{for large values of } x;\\
&b) \quad \frac{e^x - 1}{x} \quad \text{for small values of } x;\\
&c) \quad \sin(x+a) - \sin(a) \quad \text{for small values of } x;\\
&d) \quad \sqrt[3]{x+1} - \sqrt[3]{x} \quad \text{for large values of } x;\\
&e) \quad \sqrt{1 + \frac{1}{x} - 1} \quad \text{for large values of } x.
\end{aligned}
$$

Problem 1.6

In the following function evaluations $f(x_A)$, assume the numbers x_A are correctly rounded to the number of digits shown. Bound the error $f(x_T) - f(x_A)$ and the relative error $Rel(x_A)$:

$$
\begin{aligned}
&a) \quad cos(1.473);\\
&b) \quad e^{2.231};\\
&c) \quad \sqrt{0.0275};\\
&d) \quad \arctan(4.7869).
\end{aligned}
$$

Problem 1.7

Extra Bonus point Bound

$$\int_0^{\pi} \frac{t^2}{1+t^4}\,dt - \int_0^{22/7} \frac{t^2}{1+t^4}\,dt.$$

Hint: Define function

$$f(x) = \int_0^{x} \frac{t^2}{1+t^4}\,dt.$$

and then bound $f(\pi) - f(22/7)$.

Practice problems 2

Problem 2.1

Let the interval used in the bisection method have the length $b - a = 3$. Find the number of midpoints c_n that must be calculated with the bisection method to obtain an approximate root within an error tolerance of 10^{-9}.

Problem 2.2

Imagine you are finding a root α satisfying $1 < \alpha < 2$. If you are using a binary computer with m digits in its significand, what is the smallest error tolerance that makes sense in finding an approximation to α? If the original interval is $[1, 2]$ how many halving are needed to find an approximation to α with the maximum accuracy possible for this computer?

Problem 2.3

Work out what the Newton iteration is for $f(x) = x^2$. What is the solution to $f(x) = 0$? Will the sequence generated by Newton method converge to solution? How quickly? Relate this to the theory of Newton method.

Problem 2.4

On most computers, the computation of $\sqrt{a}$ is based on Newton's method. Set up the Newton's iteration for solving $x^2 - a = 0$, and show that it can be written in the form

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right), \qquad n \geq 0.$$

Derive the error and relative error formulas:

$$\begin{aligned} \sqrt{a} - x_{n+1} &= -\frac{1}{2x_n}\left(\sqrt{a} - x_n\right)^2, \\ \mathrm{Rel}(x_{n+1}) &= -\frac{\sqrt{a}}{2x_n}\left(\mathrm{Rel}(x_n)\right)^2. \end{aligned}$$

For initial guess x_0 near $\sqrt{a}$, the last formula becomes

$$\mathrm{Rel}(x_{n+1}) \approx -\frac{1}{2}\left(\mathrm{Rel}(x_n)\right)^2$$

Assuming $\mathrm{Rel}(x_0) = 0.1$, use this formula to estimate the relative error in $x_i, i = 1, 2, 3, 4$.

Problem 2.5

Derive formula

$$\mathrm{Rel}(x_{n+1}) = \left(\mathrm{Rel}(x_n)\right)^2$$

for the Newton's iterations used in computing $\frac{1}{b}$ for given b (formula was discussed in class without proof).

Problem 2.6

How many solutions are there to the equation $x = e^{-x}$? Will the iteration $x_{n+1} = e^{-x_n}$ converge for a suitable choice of x_0? Use Aitken extrapolation formula to estimate the error $\alpha - x_3$ for $x_0 = 0.57$.

Problem 2.7

The iteration

$$x_{n+1} = 2 - (1 + c)x_n + cx_n^3$$

will converge to $\alpha = 1$ for some values of c (provided that initial guess x_0 is chosen sufficiently close to α). Find the values of c for which convergence occurs. For what values of c, if any, convergence will be quadratic?

Problem 2.8

Consider the equation

$$x^7 - 28x^6 + 322x^5 - 1960x^4 + 6769x^3 - 13132x^2 - 5040 = 0$$

Change the coefficient of x^4 from -1960 to -1960.14. What is relative perturbation error in the coefficient of x^4? Calculate $\alpha(\varepsilon)$ for $\alpha(0) = 3$ and $\alpha(0) = 5$.

Problem 2.9

What is the order of convergence of the iteration

$$x_{n+1} = \frac{x_n(x_n^2 + 15)}{3x_n^2 + 5}$$

as it converges to the fixed point $\alpha = \sqrt{5}$?

Problem 2.10

Newton's method is used to find the root of $f(x) = 0$. The first few iterates are shown in the following table, giving a very slow speed of convergence. What can be said about the root α to explain the convergence? Knowing $f(x)$, how would you find an accurate value for α?

n	x_n	$x_{n-1} - x_n$
0	0.75	
1	0.752710	0.00271
2	0.754795	0.00208
3	0.756368	0.00157
4	0.757552	0.00118
5	0.758441	0.000889

Problem 2.11

Consider the following table of iterates from an iteration method which is convergent to a fixed point α of the function $g(x)$:

n	x_n	$x_n - x_{n-1}$
0	1.30499998	
1	1.25340617	$-5.159E-2$
2	1.21676284	$-3.664E-2$
3	1.19087998	$-2.588E-2$
4	1.17257320	$-1.831E-2$
5	1.15962919	$-1.294E-2$

(a) Does this appear to be a linearly convergent iteration method? If so, then estimate the rate of linear convergence. (b) Estimate the error in x_5 . (c) Give an improved estimate of α .

Practice Problems Set 3

Problem 3.1

Find a polynomial $P(x)$ of degree ≤ 3 for which

$$\begin{aligned} P(0) &= y_1 \qquad & P(1) &= y_2 \\ P'(0) &= y_1' \qquad & P'(1) &= y_2' \end{aligned}$$

with y_1, y_2, y_1', y_2' given constants.
The resulting polynomial is called **cubic Hermite interpolating polynomial**.
HINT: Write $P(x) = y_1 H_1(x) + y_2 H_2(x) + y_1' H_3(x) + y_2' H_4(x)$ with H_i cubic polynomials satisfying appropriate properties, in analogy with Lagrange interpolating polynomials.

Problem 3.2

Find the function $P(x) = a + b\cos(\pi x) + c\sin(\pi x)$, which interpolates the data

x	0	0.5	1
y	2	5	4

This is so-called **trigonometric interpolation**. Also, find the quadratic polynomial interpolating this data. In each instance, draw the graph of the interpolating function.

Problem 3.3

Find cubic polynomial interpolating the data

x	0	1	2	5
y	-1	4	2	6

Problem 3.4

Find the polynomial interpolating the data

x	-3	1	2	4	5
y	9	1	4	16	25

Problem 3.5

Prove that

$$\sum_{i=0}^{N} L_i(x) = 1,$$

where $L_i(x)$ are Lagrange basis functions associated to $N+1$ interpolation points.

Problem 3.6

Consider the polynomial interpolation of the function $f(x) = e^{-x^2}$ on $[0,1]$ at the points $x_0 = 0$, $x_1 = 0.5$ and $x_2 = 1$. Estimate the maximum of the polynomial interpolation error for $x \in [0,1]$, i.e. give an upper bound for this error.

Problem 3.7

Is the following a cubic spline on the interval $0 \leq x \leq 2$?

$$s(x) = \begin{cases} (x-1)^3, & 0 \leq x \leq 1 \\ 2(x-1)^3, & 1 \leq x \leq 2 \end{cases}$$

Problem 3.8

Consider the data

x	0	1/2	1	2	3
y	0	1/4	1	-1	-1

(a) Find the piecewise linear interpolating function for the data; (b) Find the piecewise quadratic interpolating function for the data. (c) Find the natural cubic spline that interpolates the data. Graph all three graphs for $0 \leq x \leq 3$.

Problem 3.9

Compute the error bound for the minimax approximation of the function $f(x) = e^{3x-1}$ on the $[-1, 2]$ and $n = 5$.

Problem 3.10

How many multiplications and additions are needed to compute Chebyshev polynomials $T_0(x), T_1(x), T_2(x), \ldots, T_n(x)$ for a particular value of x?

Problem 3.11

Let $q(x)$ be a polynomial of degree $\leq n - 1$, and consider

$$\max_{-1 \leq x \leq 1} |x^n - q(x)|$$

What is the smallest possible value for this quantity? Solve for the $q(x)$ for which this value is attained.

Problem 3.12

For $n, m \geq 0$ and $n \neq m$ show

$$\int_{-1}^{1} \frac{T_n(x) T_m(x)}{\sqrt{1 - x^2}} dx = 0$$

This is called the orthogonality property for the Chebyshev polynomials.