Practice problems 2

Problem 2.1

Let the interval used in the bisection method have the length $b - a = 3$. Find the number of midpoints c_n that must be calculated with the bisection method to obtain an approximate root within an error tolerance of 10^{-9}.

Problem 2.2

Imagine you are finding a root α satisfying $1 < \alpha < 2$. If you are using a binary computer with m digits in its significand, what is the smallest error tolerance that makes sense in finding an approximation to α? If the original interval is $[1, 2]$ how many halving are needed to find an approximation to α with the maximum accuracy possible for this computer?

Problem 2.3

Work out what the Newton iteration is for $f(x) = x^2$. What is the solution to $f(x) = 0$? Will the sequence generated by Newton method converge to solution? How quickly? Relate this to the theory of Newton method.

Problem 2.4

On most computers, the computation of $\sqrt{a}$ is based on Newton's method. Set up the Newton's iteration for solving $x^2 - a = 0$, and show that it can be written in the form

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right), \qquad n \geq 0.$$

Derive the error and relative error formulas:

$$\begin{aligned}
\sqrt{a} - x_{n+1} &= -\frac{1}{2x_n}\left(\sqrt{a} - x_n\right)^2, \\
\mathrm{Rel}(x_{n+1}) &= -\frac{\sqrt{a}}{2x_n}\left(\mathrm{Rel}(x_n)\right)^2.
\end{aligned}$$

For initial guess x_0 near $\sqrt{a}$, the last formula becomes

$$\mathrm{Rel}(x_{n+1}) \approx -\frac{1}{2}\left(\mathrm{Rel}(x_n)\right)^2$$

Assuming $\mathrm{Rel}(x_0) = 0.1$, use this formula to estimate the relative error in $x_i, i = 1, 2, 3, 4$.

Problem 2.5

Derive formula

$$\mathrm{Rel}(x_{n+1}) = \left(\mathrm{Rel}(x_n)\right)^2$$

for the Newton's iterations used in computing $\frac{1}{b}$ for given b (formula was discussed in class without proof).

Problem 2.6

How many solutions are there to the equation $x = e^{-x}$? Will the iteration $x_{n+1} = e^{-x_n}$ converge for a suitable choice of x_0? Use Aitken extrapolation formula to estimate the error $\alpha - x_3$ for $x_0 = 0.57$.

Problem 2.7

The iteration

$$x_{n+1} = 2 - (1 + c)x_n + cx_n^3$$

will converge to $\alpha = 1$ for some values of c (provided that initial guess x_0 is chosen sufficiently close to α). Find the values of c for which convergence occurs. For what values of c, if any, convergence will be quadratic?

Problem 2.8

Consider the equation

$$x^7 - 28x^6 + 322x^5 - 1960x^4 + 6769x^3 - 13132x^2 - 5040 = 0$$

Change the coefficient of x^4 from -1960 to -1960.14. What is relative perturbation error in the coefficient of x^4? Calculate $\alpha(\varepsilon)$ for $\alpha(0) = 3$ and $\alpha(0) = 5$.

Problem 2.9

What is the order of convergence of the iteration

$$x_{n+1} = \frac{x_n(x_n^2 + 15)}{3x_n^2 + 5}$$

as it converges to the fixed point $\alpha = \sqrt{5}$?

Problem 2.10

Newton's method is used to find the root of $f(x) = 0$. The first few iterates are shown in the following table, giving a very slow speed of convergence. What can be said about the root α to explain the convergence? Knowing $f(x)$, how would you find an accurate value for α?

n	x_n	$x_{n-1} - x_n$
0	0.75	
1	0.752710	0.00271
2	0.754795	0.00208
3	0.756368	0.00157
4	0.757552	0.00118
5	0.758441	0.000889

Problem 2.11

Consider the following table of iterates from an iteration method which is convergent to a fixed point α of the function $g(x)$:

n	x_n	$x_n - x_{n-1}$
0	1.30499998	
1	1.25340617	$-5.159E-2$
2	1.21676284	$-3.664E-2$
3	1.19087998	$-2.588E-2$
4	1.17257320	$-1.831E-2$
5	1.15962919	$-1.294E-2$

(a) Does this appear to be a linearly convergent iteration method? If so, then estimate the rate of linear convergence. (b) Estimate the error in x_5 . (c) Give an improved estimate of α .