

Homework 1

Problem 1.1

Let d_1d_2/m_1m_2 be your birthday. Find the binary single precision IEEE floating-point expression for the number $(d_1m_2)_{10}$. Also, find the binary double precision IEEE floating-point expression for the number $(d_2m_1)_{10}$. In both cases, specify significand, exponent and σ . Convert $(111\dots1)_2$ to decimal form with the parentheses enclosing $(21 - m_1m_2)$ ones.

Problem 1.2

Some microcomputers in the past used a binary floating-point format with 8 bits for the exponent and 1 bit for the sign σ . The significand contained 31 bits, with no hiding of the leading bit 1. The arithmetic used rounding. To determine the accuracy of the representation, find the machine epsilon, integer M , and the largest number that can be represented exactly in this floating-point format. Also, find the accuracy of the rounding operation.

Problem 1.3

Consider a binary floating-point representation with significand containing 3 digits without hiding the leading 1 and $-1_{10} \leq e \leq 3_{10}$. List all numbers that can be stored exactly together with their decimal value. Plot these numbers on real axis. For this arithmetic, specify what are the corresponding floating-point representation of $\pi/4$ and $14/5$ if a) rounding is used; b) chopping is used?

Problem 1.4

Calculate the error, relative error and the number of significant digits in the following approximations $x_A \approx x_T$.

- a) $x_A = 28.271, \quad x_T = 28.254;$
- b) $x_A = 0.028271, \quad x_T = 0.028254;$
- c) $x_A = 19/7, \quad x_T = e;$
- d) $x_A = 1.414, \quad x_T = \sqrt{2}.$

Problem 1.5

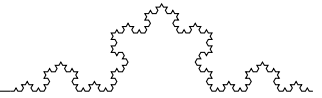
Avoid loss-of-significance errors in the following formulas

- a) $\log(x+1) - \log(x)$ for large values of x ;
- b) $\frac{e^x - 1}{x}$ for small values of x ;
- c) $\sin(x+a) - \sin(a)$ for small values of x ;
- d) $\sqrt[3]{x+1} - \sqrt[3]{x}$ for large values of x ;
- e) $\sqrt{1 + \frac{1}{x}} - 1$ for large values of x .

Problem 1.6

In the following function evaluations $f(x_A)$, assume the numbers x_A are correctly rounded to the number of digits shown. Bound the error $f(x_T) - f(x_A)$ and the relative error $Rel(x_A)$:

- a) $\cos(1.473);$
- b) $e^{2.231};$
- c) $\sqrt{0.0275};$
- d) $\arctan(4.7869).$

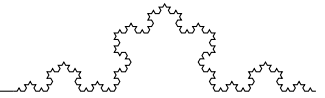
**Problem 1.7****Extra Bonus point** Bound

$$\int_0^\pi \frac{t^2}{1+t^4} dt - \int_0^{22/7} \frac{t^2}{1+t^4} dt.$$

Hint: Define function

$$f(x) = \int_0^x \frac{t^2}{1+t^4} dt.$$

and then bound $f(\pi) - f(22/7)$.



Practice problems 2

Problem 2.1

Let the interval used in the bisection method have the length $b - a = 3$. Find the number of midpoints c_n that must be calculated with the bisection method to obtain an approximate root within an error tolerance of 10^{-9} .

Problem 2.2

Imagine you are finding a root α satisfying $1 < \alpha < 2$. If you are using a binary computer with m digits in its significand, what is the smallest error tolerance that makes sense in finding an approximation to α ? If the original interval is $[1, 2]$ how many halving are needed to find an approximation to α with the maximum accuracy possible for this computer?

Problem 2.3

Work out what the Newton iteration is for $f(x) = x^2$. What is the solution to $f(x) = 0$? Will the sequence generated by Newton method converge to solution? How quickly? Relate this to the theory of Newton method.

Problem 2.4

On most computers, the computation of \sqrt{a} is based on Newton's method. Set up the Newton's iteration for solving $x^2 - a = 0$, and show that it can be written in the form

$$x_{n+1} = \frac{1}{2} \left(x_n + \frac{a}{x_n} \right), \quad n \geq 0.$$

Derive the error and relative error formulas:

$$\begin{aligned} \sqrt{a} - x_{n+1} &= -\frac{1}{2x_n} (\sqrt{a} - x_n)^2, \\ \text{Rel}(x_{n+1}) &= -\frac{\sqrt{a}}{2x_n} (\text{Rel}(x_n))^2. \end{aligned}$$

For initial guess x_0 near \sqrt{a} , the last formula becomes

$$\text{Rel}(x_{n+1}) \approx -\frac{1}{2} (\text{Rel}(x_n))^2$$

Assuming $\text{Rel}(x_0) = 0.1$, use this formula to estimate the relative error in $x_i, i = 1, 2, 3, 4$.

Problem 2.5

Derive formula

$$\text{Rel}(x_{n+1}) = (\text{Rel}(x_n))^2$$

for the Newton's iterations used in computing $\frac{1}{b}$ for given b (formula was discussed in class without proof).

Problem 2.6

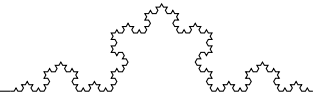
How many solutions are there to the equation $x = e^{-x}$? Will the iteration $x_{n+1} = e^{-x_n}$ converge for a suitable choice of x_0 ? Use Aitken extrapolation formula to estimate the error $\alpha - x_3$ for $x_0 = 0.57$.

Problem 2.7

The iteration

$$x_{n+1} = 2 - (1 + c)x_n + cx_n^3$$

will converge to $\alpha = 1$ for some values of c (provided that initial guess x_0 is chosen sufficiently close to α). Find the values of c for which convergence occurs. For what values of c , if any, convergence will be quadratic?



Problem 2.8

Consider the equation

$$x^7 - 28x^6 + 322x^5 - 1960x^4 + 6769x^3 - 13132x^2 - 5040 = 0$$

Change the coefficient of x^4 from -1960 to -1960.14 . What is relative perturbation error in the coefficient of x^4 ? Calculate $\alpha(\varepsilon)$ for $\alpha(0) = 3$ and $\alpha(0) = 5$.

Problem 2.9

What is the order of convergence of the iteration

$$x_{n+1} = \frac{x_n(x_n^2 + 15)}{3x_n^2 + 5}$$

as it converges to the fixed point $\alpha = \sqrt{5}$?

Problem 2.10

Newton's method is used to find the root of $f(x) = 0$. The first few iterates are shown in the following table, giving a very slow speed of convergence. What can be said about the root α to explain the convergence? Knowing $f(x)$, how would you find an accurate value for α ?

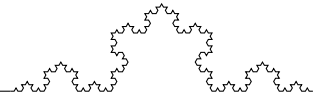
n	x_n	$x_{n-1} - x_n$
0	0.75	
1	0.752710	0.00271
2	0.754795	0.00208
3	0.756368	0.00157
4	0.757552	0.00118
5	0.758441	0.000889

Problem 2.11

Consider the following table of iterates from an iteration method which is convergent to a fixed point α of the function $g(x)$:

n	x_n	$x_n - x_{n-1}$
0	1.30499998	
1	1.25340617	$-5.159E-2$
2	1.21676284	$-3.664E-2$
3	1.19087998	$-2.588E-2$
4	1.17257320	$-1.831E-2$
5	1.15962919	$-1.294E-2$

(a) Does this appear to be a linearly convergent iteration method? If so, then estimate the rate of linear convergence. (b) Estimate the error in x_5 . (c) Give an improved estimate of α .



Practice Problems Set 3

Problem 3.1

Find a polynomial $P(x)$ of degree ≤ 3 for which

$$\begin{aligned} P(0) &= y_1 & P(1) &= y_2 \\ P'(0) &= y'_1 & P'(1) &= y'_2 \end{aligned}$$

with y_1, y_2, y'_1, y'_2 given constants.

The resulting polynomial is called **cubic Hermite interpolating polynomial**.

HINT: Write $P(x) = y_1 H_1(x) + y_2 H_2(x) + y'_1 H_3(x) + y'_2 H_4(x)$ with H_i cubic polynomials satisfying appropriate properties, in analogy with Lagrange interpolating polynomials.

Problem 3.2

Find the function $P(x) = a + b \cos(\pi x) + c \sin(\pi x)$, which interpolates the data

x	0	0.5	1
y	2	5	4

This is so-called **trigonometric interpolation**. Also, find the quadratic polynomial interpolating this data. In each instance, draw the graph of the interpolating function.

Problem 3.3

Find cubic polynomial interpolating the data

x	0	1	2	5
y	-1	4	2	6

Problem 3.4

Find the polynomial interpolating the data

x	-3	1	2	4	5
y	9	1	4	16	25

Problem 3.5

Prove that

$$\sum_{i=0}^N L_i(x) = 1,$$

where $L_i(x)$ are Lagrange basis functions associated to $N + 1$ interpolation points.

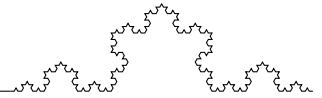
Problem 3.6

Consider the polynomial interpolation of the function $f(x) = e^{-x^2}$ on $[0, 1]$ at the points $x_0 = 0$, $x_1 = 0.5$ and $x_2 = 1$. Estimate the maximum of the polynomial interpolation error for $x \in [0, 1]$, i.e. give an upper bound for this error.

Problem 3.7

Is the following a cubic spline on the interval $0 \leq x \leq 2$?

$$s(x) = \begin{cases} (x-1)^3, & 0 \leq x \leq 1 \\ 2(x-1)^3, & 1 \leq x \leq 2 \end{cases}$$

**Problem 3.8**

Consider the data

x	0	1/2	1	2	3
y	0	1/4	1	-1	-1

(a) Find the piecewise linear interpolating function for the data; (b) Find the piecewise quadratic interpolating function for the data. (c) Find the natural cubic spline that interpolates the data. Graph all three graphs for $0 \leq x \leq 3$.

Problem 3.9

Compute the error bound for the minimax approximation of the function $f(x) = e^{3x-1}$ on the $[-1, 2]$ and $n = 5$.

Problem 3.10

How many multiplications and additions are needed to compute Chebyshev polynomials $T_0(x), T_1(x), T_2(x), \dots, T_n(x)$ for a particular value of x ?

Problem 3.11

Let $q(x)$ be a polynomial of degree $\leq n - 1$, and consider

$$\max_{-1 \leq x \leq 1} |x^n - q(x)|$$

What is the smallest possible value for this quantity? Solve for the $q(x)$ for which this value is attained.

Problem 3.12

For $n, m \geq 0$ and $n \neq m$ show

$$\int_{-1}^1 \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}} dx = 0$$

This is called the orthogonality property for the Chebyshev polynomials.