Strathmore UNIVERSITY

School of Computing & Engineering Sciences

Bachelor of Business Information & Technology

End of Semester Examinations

MAT 3301: Probability & Statistics II

Date: 6th September 2021 **Time: 2 Hours**

Instructions: Answer Question ONE and ANY other TWO questions in the booklet provided.

Question One [30 marks]

(a) Let $X_1, X_2, \cdots, X_n$ be a random sample from a distribution with *pdf* $f(x;\theta)$, where θ is an unknown parameter. $T(X_1, X_2, \cdots, X_n)$ is a function of $X_1, X_2, \cdots, X_n$ used to estimate θ. Explain the meaning of:
(i) $T(X_1, X_2, \cdots, X_n)$ is an unbiased estimator of θ. **[2 marks]**
(ii) $T(X_1, X_2, \cdots, X_n)$ is a consistent estimator of θ. **[2 marks]**

(b) The amount of time, Y, in minutes that a bank teller spends with a customer is known to have an exponential distribution with an average amount of 7 minutes. The density function is defined by

$$f(y) = \begin{cases} \frac{1}{7}e^{-\frac{1}{7}y}, & y > 0 \\ \\ 0, & \text{elsewhere} \end{cases}$$

Find the probability that the bank teller spends *seven* to *nine* minutes with a randomly selected customer. **[4 marks]**

(c) The time taken to assemble a car in a certain plant is a random variable having a normal distribution with a mean of 20 hours and a standard deviation of 2 hours. What is the probability that a car can be assembled at this plant in a period of
(i) less than 19.5 hours **[2 marks]**
(ii) between 20 hours and 22 hours **[3 marks]**

(d) A continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} \frac{1}{4}x, & 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Obtain:
(i) the CDF of X **[3 marks]**
(ii) $P(0.5 \le X \le 2)$ **[2 marks]**

(e) A laptop manufacturing company claims that its top quality laptops are good, on average, for atleast 200 months before expiry. A consumer protection agency tested 60 such laptops and found that on average they last 265 months with a standard deviation of 10 months. Construct:
(i) 90% confidence interval for the mean. **[3 marks]**
(ii) 99% confidence interval for the mean **[3 marks]**
(iii) Which confidence interval should the agency use for better precision? Why?**[2 marks]**

(f) A Mathematical Statistician determines that the revenue a City Restaurant makes in a week is a random variable, X, with moment generating function given by

$$M_X(t) = \frac{1}{(1-2500t)^4}$$

Calculate the variance the of the revenue the city restaurant makes in a week. **[4 marks]**

Question Two [20 marks]

(a) Suppose deposit amounts, X, by customers to a savings account are uniformly distributed over the interval (β, θ). Given also that the density of the distribution is defined by

$$f(x) = \begin{cases} \frac{1}{\theta-\beta}, & \beta \leq x \leq \theta \\ \\ 0, & \text{otherwise} \end{cases}$$

(i) Show that the function $f(x)$ is a *pdf*. **[3 marks]**
(ii) Find the mean of X. **[3 marks]**
(iii) Obtain the variance of X. **[6 marks]**
(iv) Find the moment generating function of X. **[4 marks]**

(b) Let $Y_1, Y_2, \cdots, Y_n$ be a Bernoulli random variable with parameter p, $0 < p < 1$ whose pmf is given by

$$f(y) = \begin{cases} p^y(1-p)^{1-y}, & y = 0, 1 \\ \\ 0, & \text{elsewhere} \end{cases}$$

Obtain the moment generating function of Y. **[4 marks]**

Question Three [20 marks]

(a) Let X be a continuous random variable whose probability density function is given by

$$f(x) = \begin{cases} e^{-2x} + \frac{1}{2}e^{-x}, & x > 0 \\ \\ 0, & \text{elsewhere} \end{cases}$$

(i) Write down the moment generating function (mgf) of X. **[3 marks]**
(ii) Use the mgf to compute $E(X)$ and $Var(X)$. **[6 marks]**

(b) Let $X_1, X_2, \cdots, X_n$ be independent and identically distributed ($i.i.d$) random sample drawn from a population with mean μ and variance σ^2. Given that

$$\hat{\mu} = \frac{X_1 + X_2}{2}$$

(i) Show that $\hat{\mu}$ is unbiased for μ **[2 marks]**
(ii) Find the variance of $\hat{\mu}$ **[2 marks]**

(c) The top 10% of applicants as measured by GRE scores will receive IT postgraduate scholarships in a certain University. If GRE scores are normally distributed with mean 600 and standard deviation 100, how high does your GRE score have to be to qualify for the scholarship? **[3 marks]**

(d) Family income is believed to be normally distributed with a mean of $Kshs$ 25,000 and a standard deviation of $Kshs$ 10,000. If poverty level is $Kshs$ 10,000;

(i) What percentage of population lives in poverty? **[2 marks]**

(ii) A new tax policy is expected to benefit "middle income" families, those with incomes between $Kshs$ 20,000 and $Kshs$ 30,000. What percentage of the population will benefit from the law? **[2 marks]**

Question Four [20 marks]

(a) Let X be a discrete random variable with probability mass function given by

X	0	1	2	3
$P(X = x)$	$\frac{1-\alpha}{3}$	$\frac{\alpha}{3}$	$\frac{2(\alpha-1)}{3}$	$\frac{2\alpha}{3}$

where $0 \leq \alpha \leq 1$ is a parameter. The following 12 independent observations were obtained from such a distribution.

$$1, \quad 2, \quad 1, \quad 0, \quad 3, \quad 1, \quad 2, \quad 1, \quad 0, \quad 3, \quad 0, \quad 1$$

Find the method of moments (MoM) estimator of α. **[6 marks]**

(b) A random sample of 5 employees $(x_1, x_2, x_3, x_4, x_5)$ who have taken an insurance policy in a certain company is drawn from a normal population with an unknown mean μ and variance σ^2. Consider the following estimators to estimate μ.

$$T_1 = \frac{x_1 + x_2 + x_3 + x_4 + x_5}{5}$$

$$T_2 = \frac{x_1 + x_2}{2} + x_3$$

$$T_3 = \frac{2x_1 + x_2 + \theta x_3}{3}$$

where θ is such that T_3 is unbiased estimator for μ.

(i) Find θ. **[4 marks]**

(ii) Are T_1 and T_2 unbiased? Show your workings. **[5 marks]**

(c) A survey by education loans board found that out of 200 students, 168 said they needed loans or scholarships to pay for their tuition and expenses. Find the 90% confidence interval for the population proportion of students needing loans or scholarships. **[5 marks]**

Question Five [20 marks]

(a) Let $Y_1, Y_2, \cdots, Y_n$ be independently and identically distributed random variables with mean μ and variance σ^2 i.e. $Y_i \sim N(\mu, \sigma^2)$. Obtain the maximum likelihood estimators ($MLEs$) of μ and σ^2. **[10 marks]**

(b) Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from a population with probability density function given by

$$f(x;\lambda) = \begin{cases} \lambda\, x^{\lambda-1}, & 0 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$$

where $0 < \lambda < \infty$ is an unknown parameter. Using the method of moments,

(i) Obtain an estimator of λ. **[6 marks]**

(ii) Suppose $x_1 = 0.2$, $x_2 = 0.6$, $x_3 = 0.5$ and $x_4 = 0.3$ is a random sample of size 4 obtained from the population, find the estimate of λ. **[4 marks]**