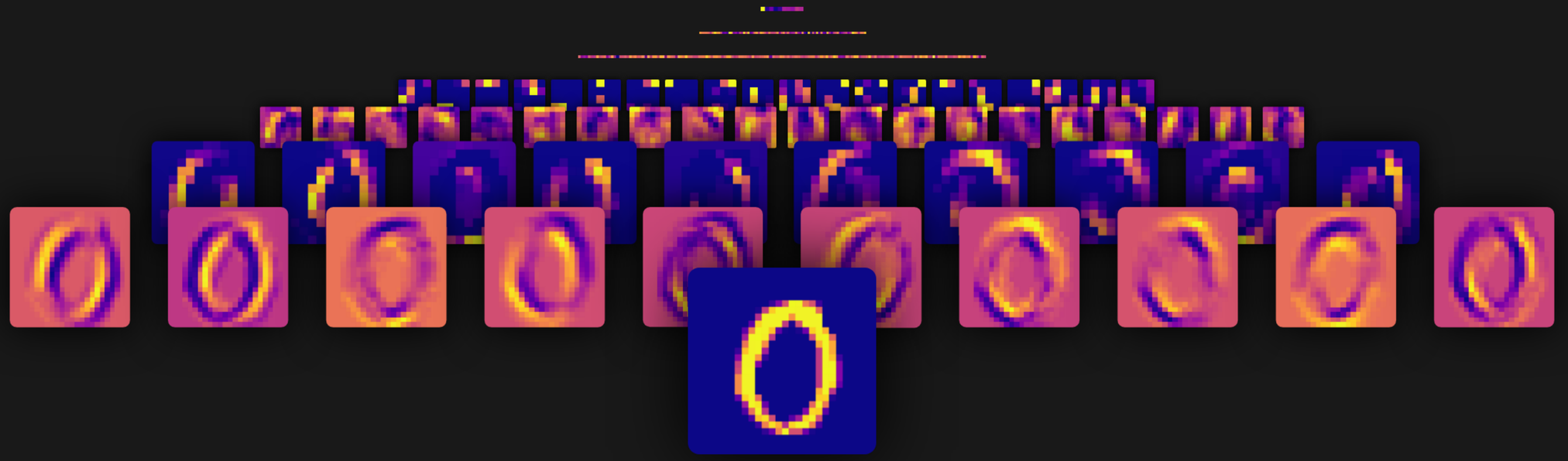
Fourier Analysis of Activations in Neural Networks

Bachelor-Thesis

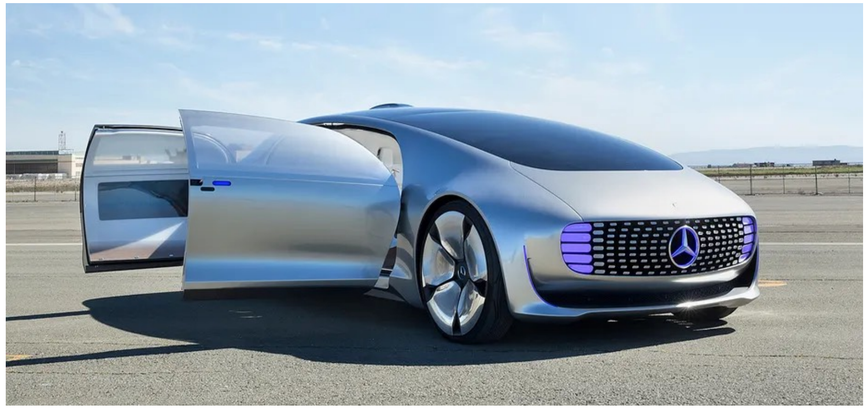


Felipa Schwarz, 2021

Analysis on Neural Networks

Motivation

Self-driving cars



https://www.wired.com/2017/04/mercedes-promises-self-driving-taxis-just-three-years/

Healthcare

https://www.medgadget.com/2019/08/healthcare-analytics-market-2019-industry-analysis-size-share-upcoming-trends-segmentation-forecast-to-2025-cagr-of-26-48.html

Cyber-security

https://www.news18.com/news/tech/hacker-group-darkside-suspected-of-carrying-cyber-attack-on-top-us-pipeline-operator-colonial-3723686.html

Ethics

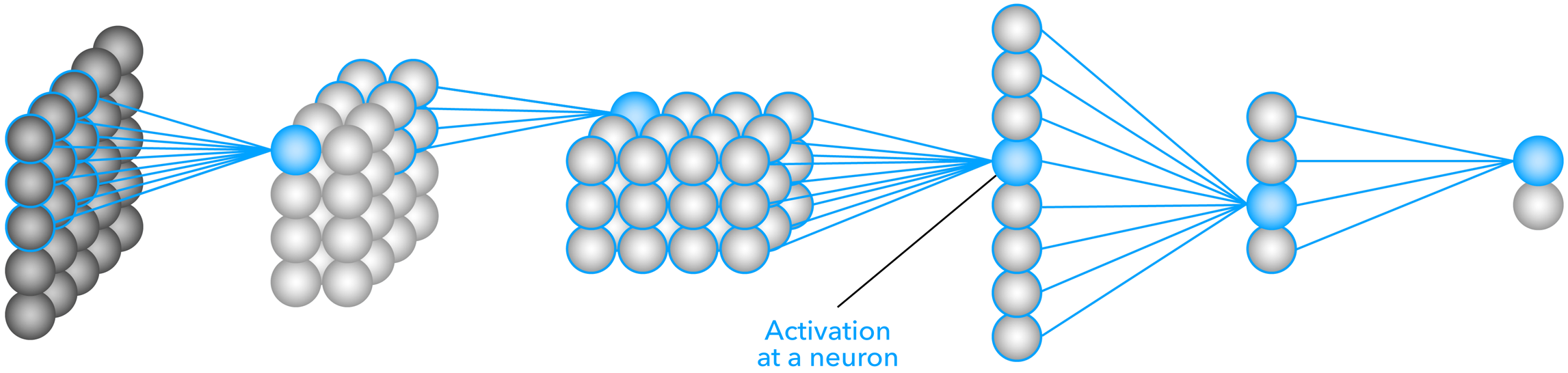
https://todaysveterinarybusiness.com/practice-without-prejudice/

Neural Networks (NN)

Activation pattern $\in \mathbb{R}^{\alpha+\beta}$

Input $\in \mathbb{R}^{\alpha}$ | Activations $\in \mathbb{R}^{\beta}$ | Output

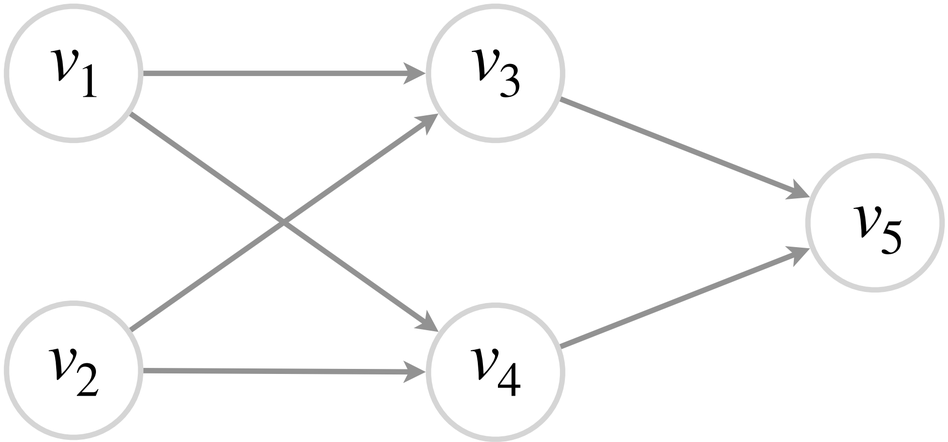
Layer 0 → Layer 1 → Layer 2 → Layer 3 → Layer 4 → Layer 5



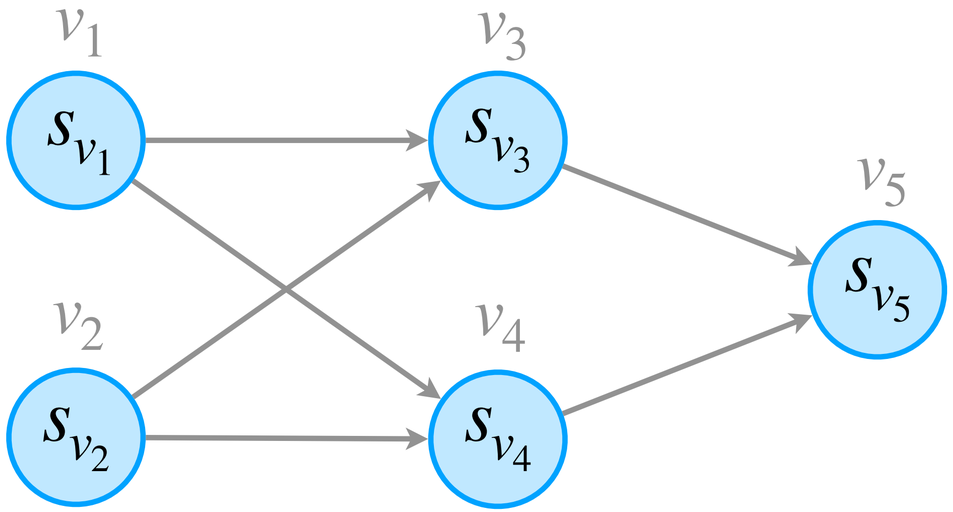
Causal Signal Processing (CSP)

Data-domain

Directed acyclic graph $G = (V, E)$



Signal $s \in \mathbb{R}^{|V|}$



Intuition: $s_x = \sum_{y \le x,\ y \in V} \hat{s}_y$

Causes

Unobserved weight

$$s_{v_3} = \hat{s}_{v_1} + \hat{s}_{v_2} + \hat{s}_{v_3}$$

Causal Signal Processing (CSP)

Data-domain

Signal s

$$s_x = \sum_{y \le x,\ y \in V} \hat{s}_y = \sum_{y \in V} \zeta(x,y)\ \hat{s}_y$$

Zeta

$$s = F^{-1}\ \hat{s}$$

Inverse Fourier transform F^{-1}

$$F^{-1}_{x,y} = \zeta(x,y) = \begin{cases} 1 & \text{if } y \le x\ , \\ 0 & \text{otherwise.} \end{cases}$$

Fourier-domain

Fourier coefficients $\hat{s}$

$$\hat{s}_x = \sum_{y \in V} \mu(x,y)\ s_y$$

Moebius

$$\hat{s} = F\ s$$

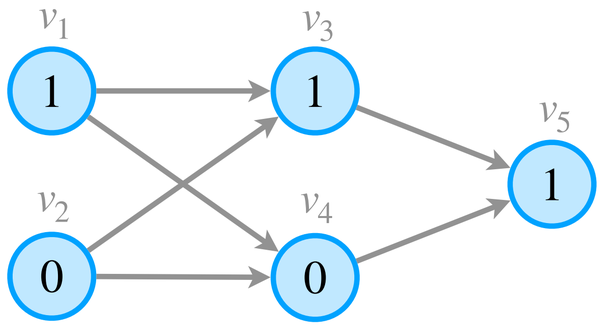
Fourier transform F

$$F_{x,y} = \mu(y,x) = \begin{cases} 1 & \text{if } x = y\ , \\ -\sum_{y \le z < x} \mu(y,z) & \text{otherwise.} \end{cases}$$

Causal Signal Processing (CSP)

Data-domain

Signal s

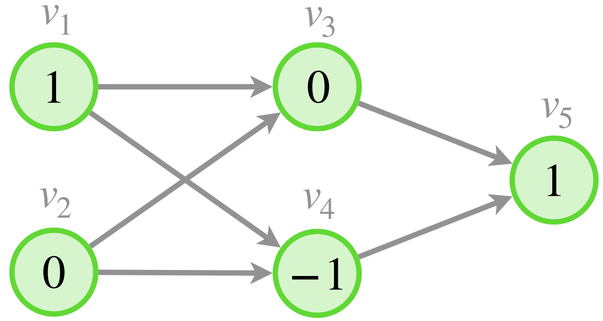


$$s = F^{-1} \hat{s}$$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{matrix}$$

Fourier-domain

Fourier coefficients $\hat{s}$



$$\hat{s} = F s$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 \\ 1 & 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{matrix}$$

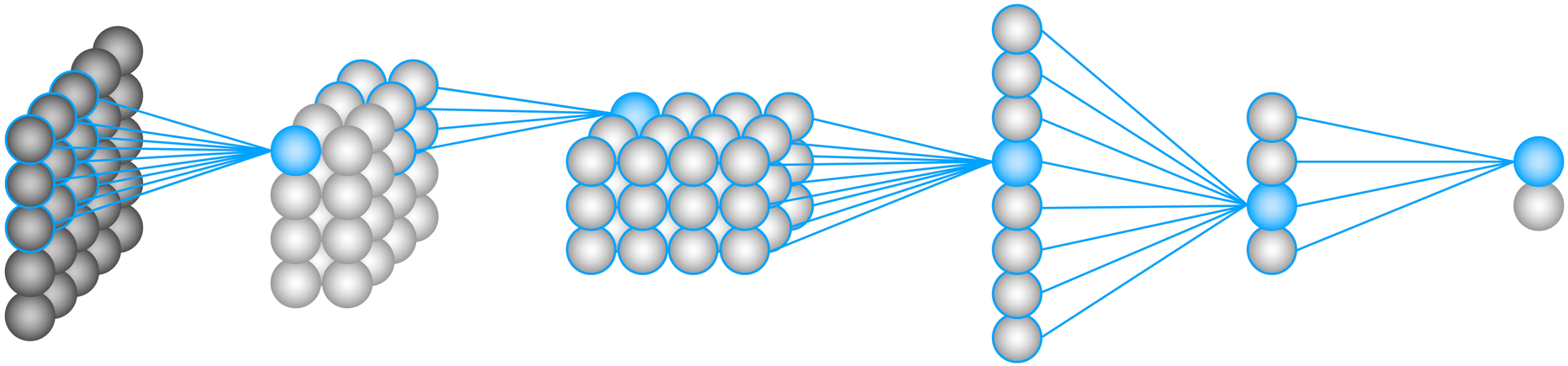
CSP on Neural Networks

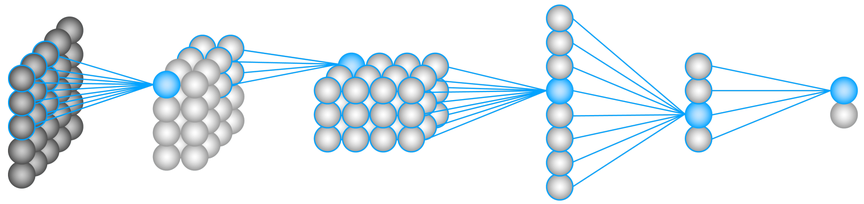
Thesis Project

Causal Signal Processing

Neural Network ⟹ DAG $G = (V, E)$ → Inverse Fourier Matrix F^{-1}

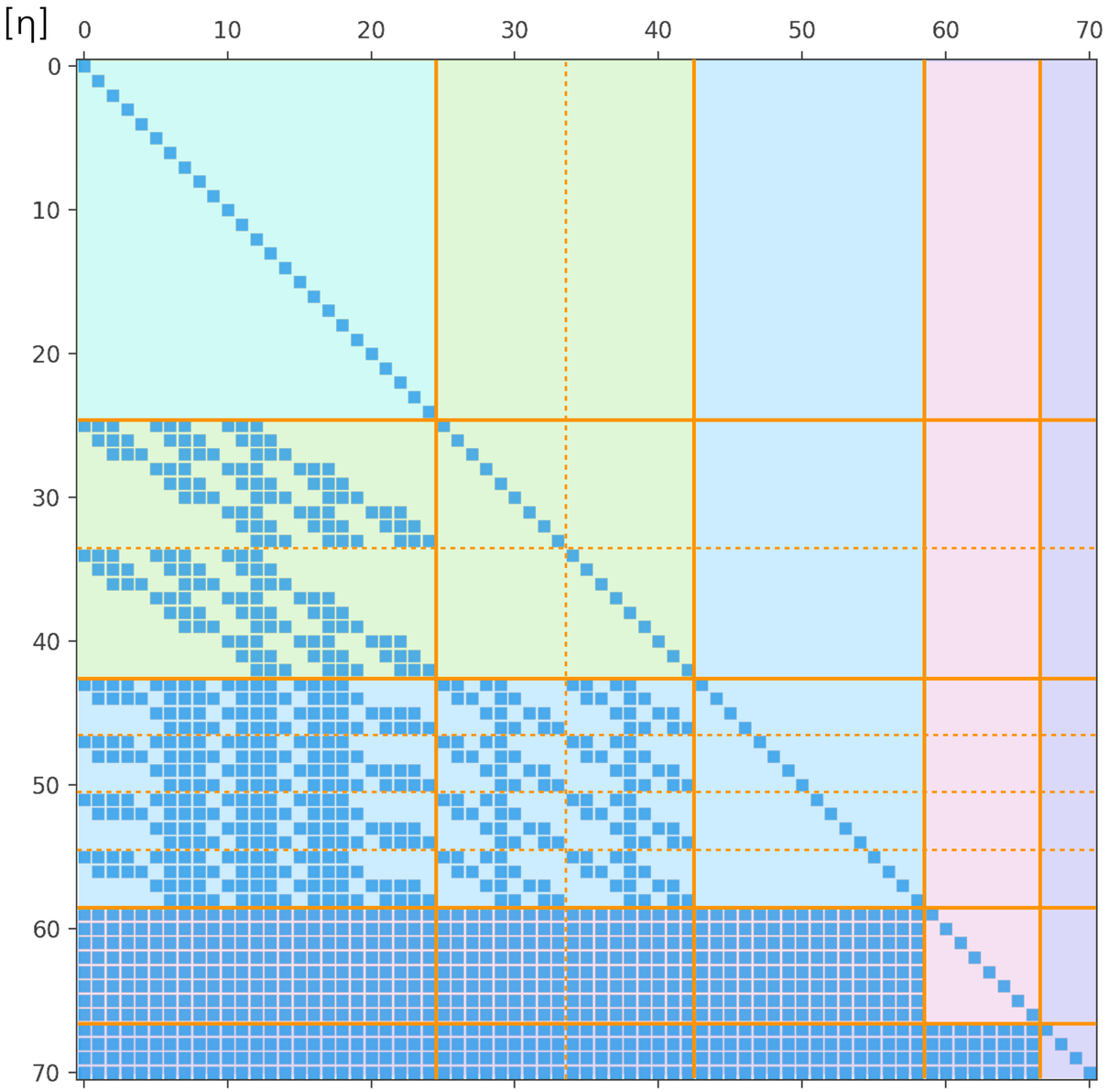
Activation pattern ⟹ Signal s → Solve: $F^{-1}\hat{s} = s$ → Fourier coefficients $\hat{s}$





Inverse Fourier Matrix F^{-1}

$$F^{-1}_{x,y} = \begin{cases} 1 & \text{if } y \leq x \\ 0 & \text{otherwise} \end{cases}$$



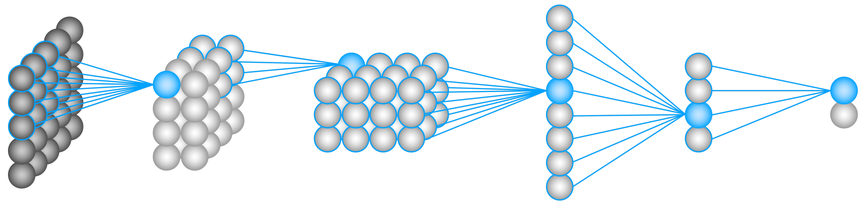
Layer 0: Input — 25 Neurons

Layer 1: Convolution — 18 Neurons

Layer 3: Convolution — 16 Neurons

Layer 4: Linear — 8 Neurons

Layer 5: Linear — 4 Neurons



Inverse Fourier Matrix F^{-1}

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Layer 0: Input

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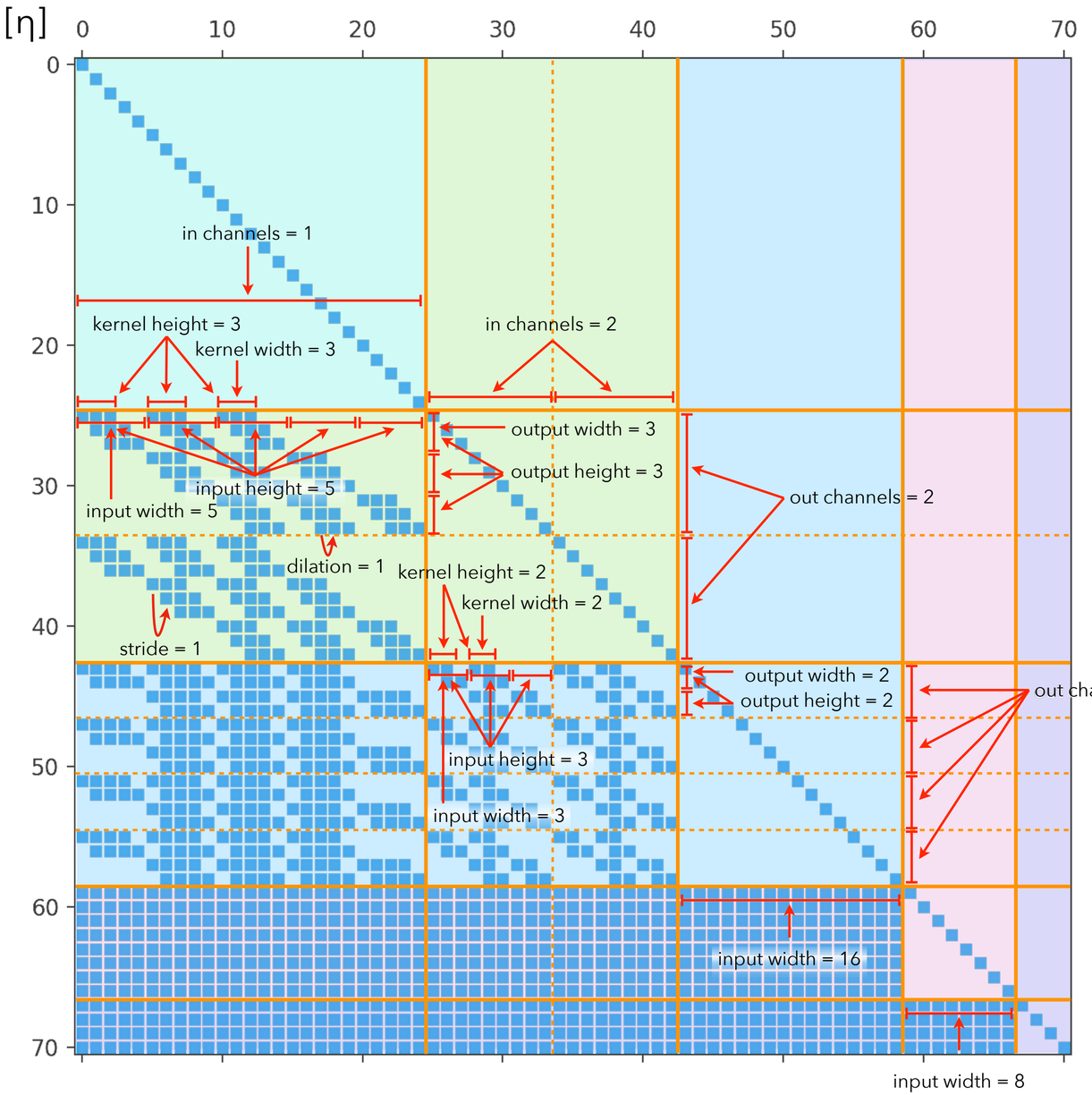
Layer 3: Convolution

Layer 4: Linear

Layer 5: Linear



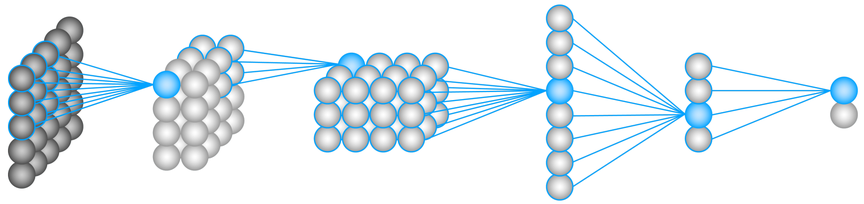
25 Neurons

18 Neurons

16 Neurons

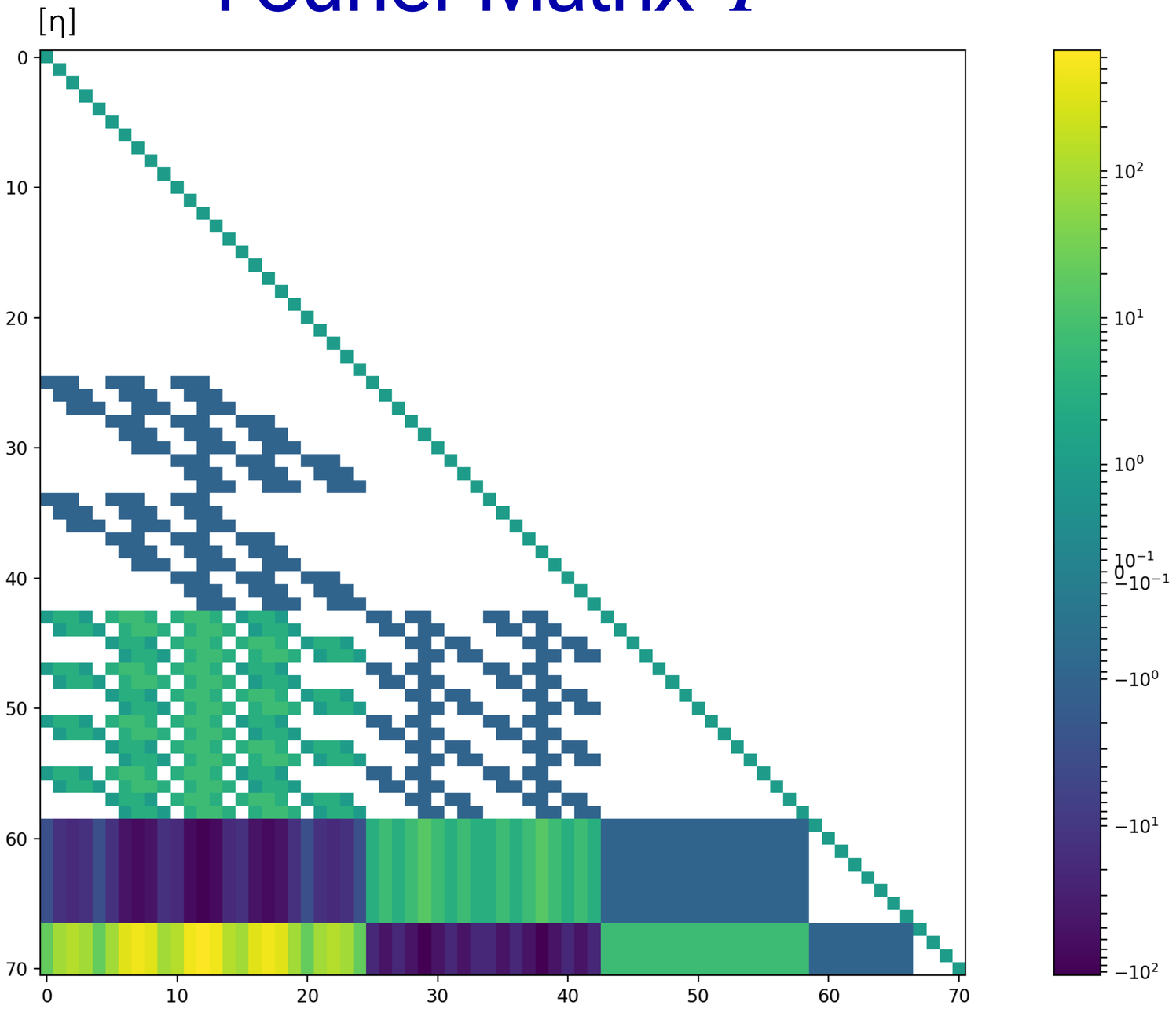
8 Neurons

4 Neurons

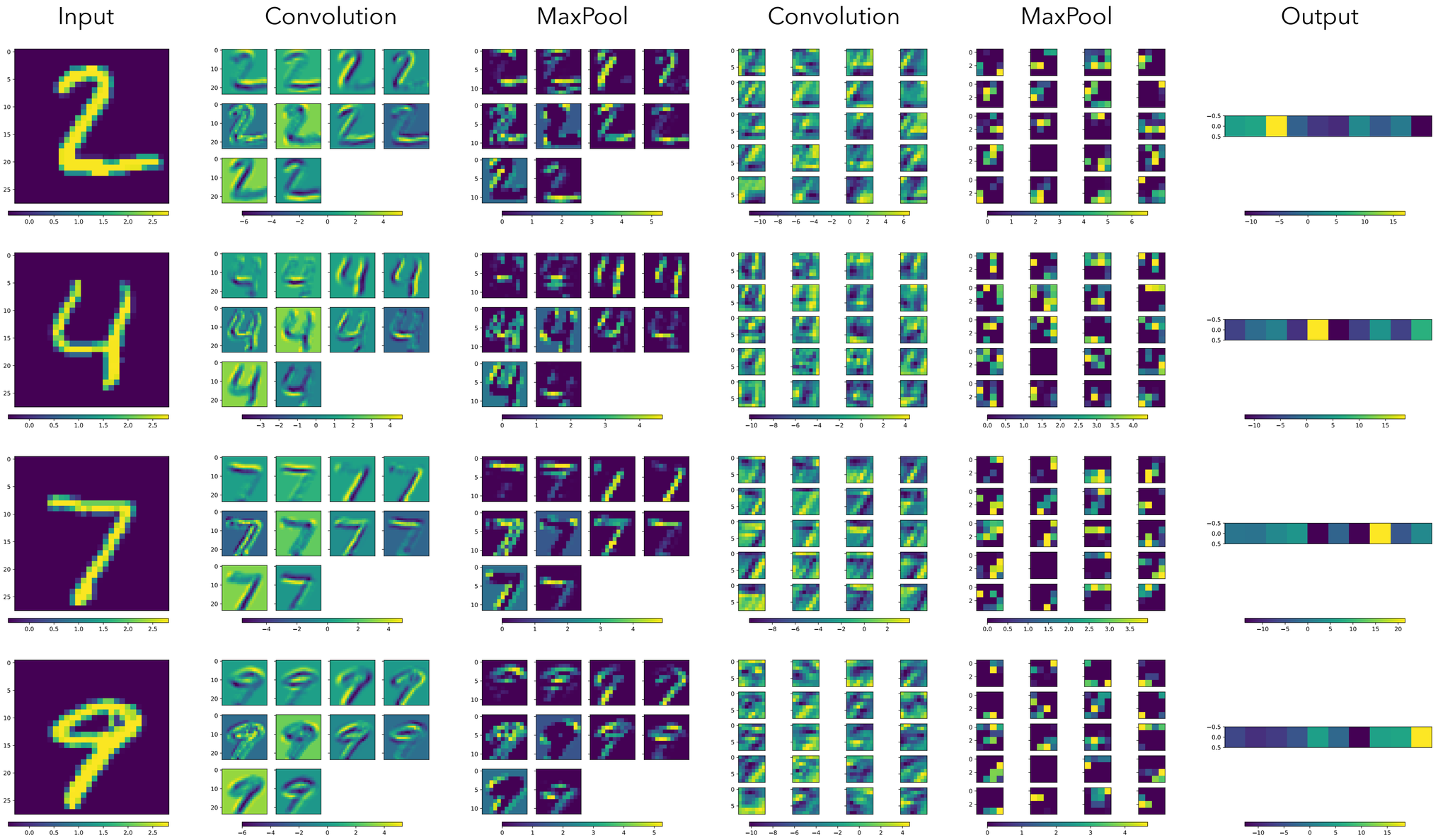


Fourier Matrix F

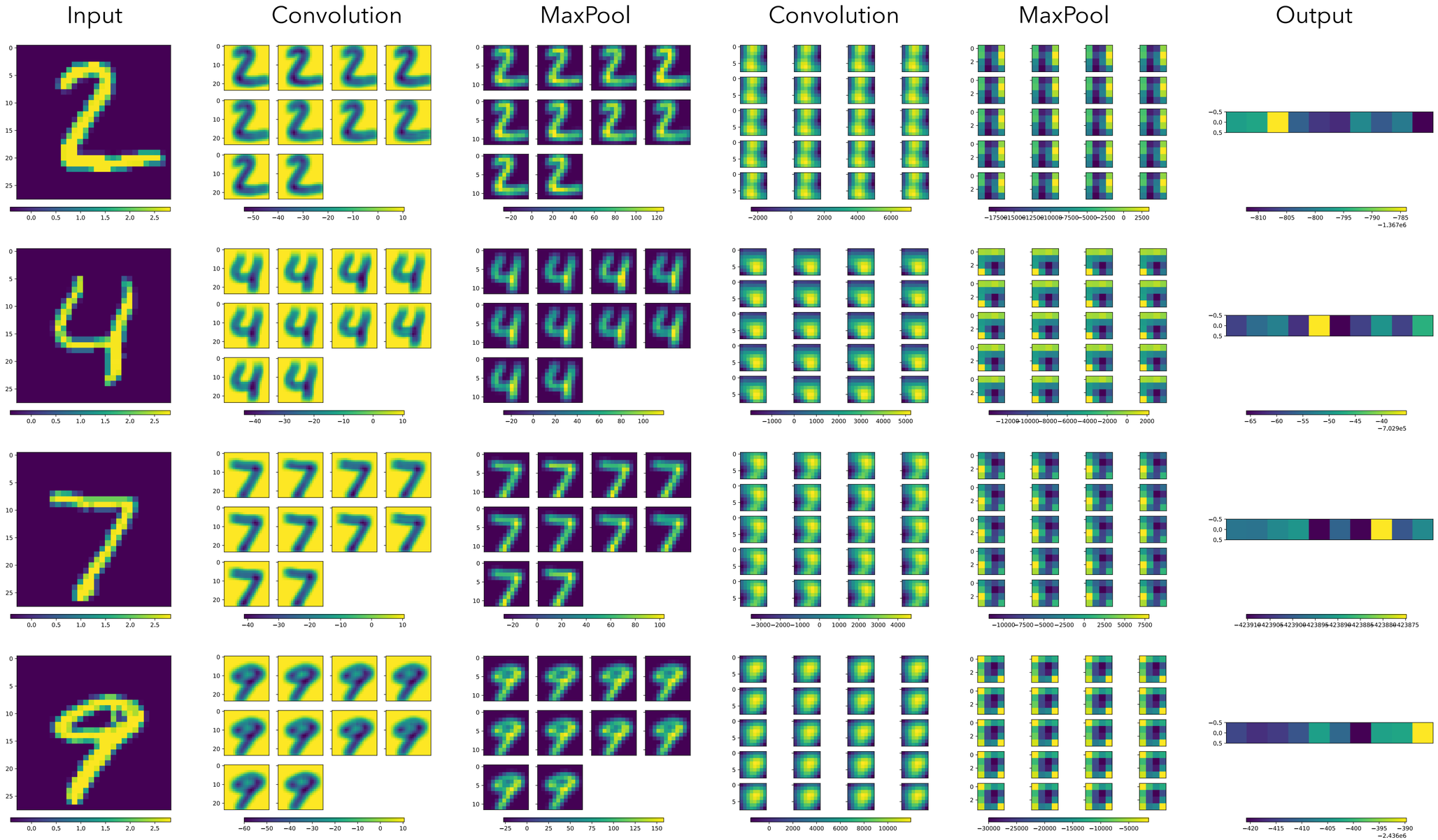
$$F_{x,y} = \begin{cases} 0 & \text{if } y \not\leq x \\ 1 & \text{if } x = y \\ -1 & \text{if } (y,x) \in E \\ -\sum_{\substack{y \leq w \leq z \\ (z,x) \in E}} \mu(y,w) & \text{otherwise} \end{cases}$$



Activations s



Fourier Coefficients $\hat{s}$



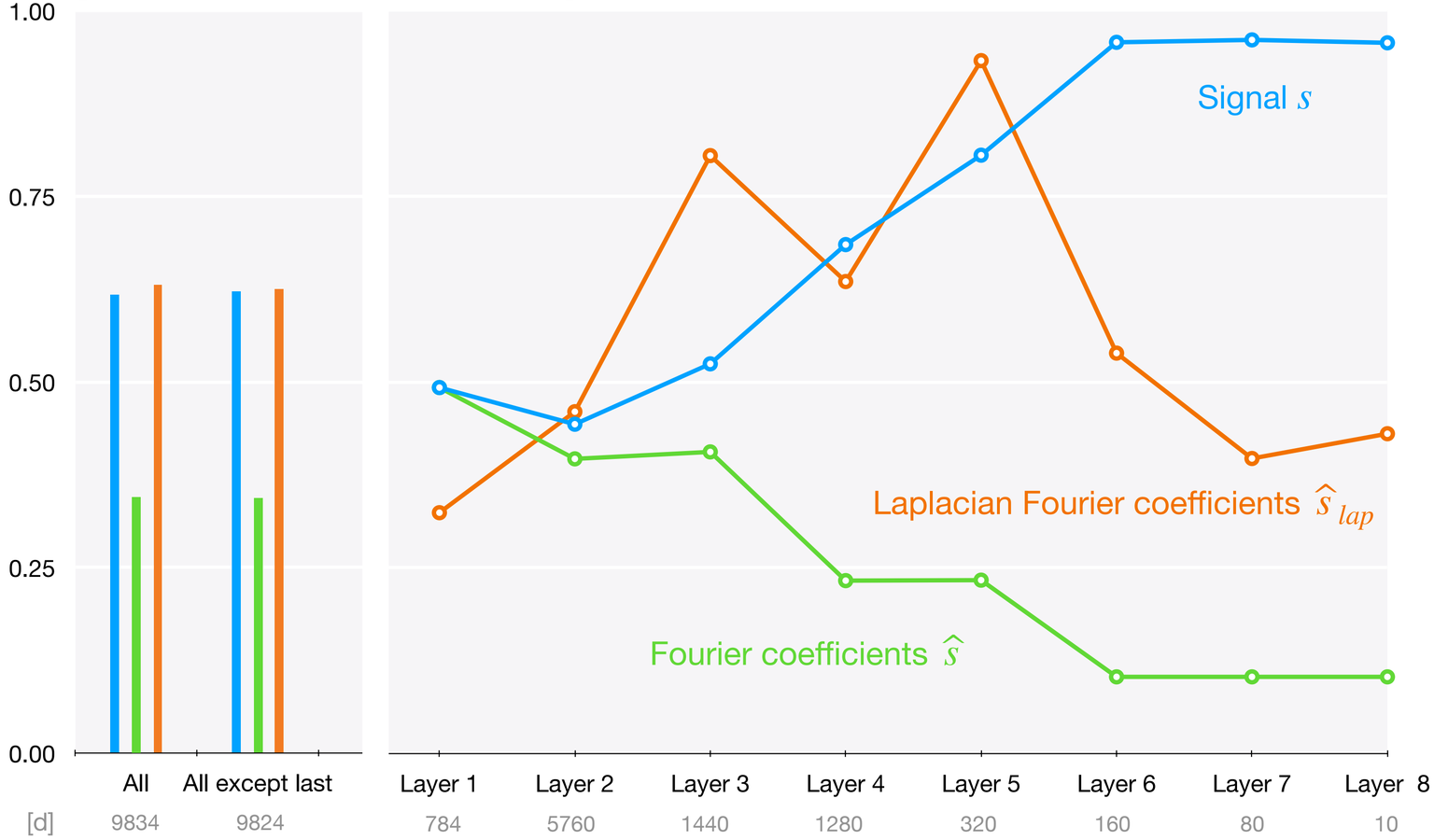
Clustering

$$\hat{s}_x = F_{x,:}\, s$$

$$\hat{s}_x = c_{s,i} + s_x$$

K-means clustering on the Euclidean distance: K = 10, N = 1000

$$\left[\mathbf{v} = \frac{2 \times homogeneity \times completeness}{homogeneity + completeness} \right]$$



Network trained on MNIST

pip install nsp