Digital Image Processing

Filtering in the Frequency Domain

Prof. Tiago Vieira

Universidade Federal de Alagoas

tvieira@ic.ufal.br

December 2, 2021

Contents

Frequency

The number of times that a periodic function repeats the same sequence of values during a unit variation of the independent variable.

Filter

A device or material for suppressing or minimizing waves or oscillations of certain frequencies.

- Periodic functions can be expressed as a sum of weighted sinusoids of different frequencies (Fourier series).
- Non-periodic functions can be expressed as an integral of weighted sinusoids (Fourier transform).
- Representations are interchangeable (spatial $\leftrightarrow$ frequency domains).

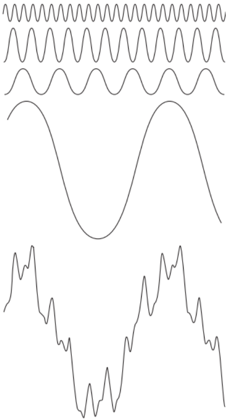


FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

Fourier Series

A function $f(t)$ periodic with period T, can be expressed as the sum of sines and cosines multiplied by appropriate coefficients.

$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t}} \tag{1}$$

where the coefficients are given by

$$\boxed{c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-j\frac{2\pi n}{T}t} dt}, \tag{2}$$

for $n = 0, \pm 1, \pm 2, \ldots$

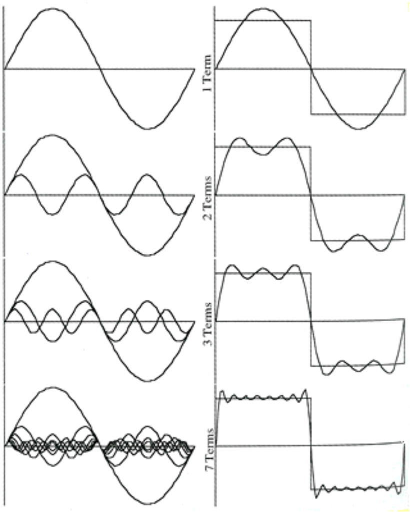
Using the Euler formula:

$$e^{j\theta} = \cos\theta + j\sin\theta \tag{3}$$

The representation of $f(t)$ in Fourier series can be expanded as:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \left[\cos\left(\frac{2\pi n}{T}t\right) + j\sin\left(\frac{2\pi n}{T}t\right)\right] \tag{4}$$

Example 1:



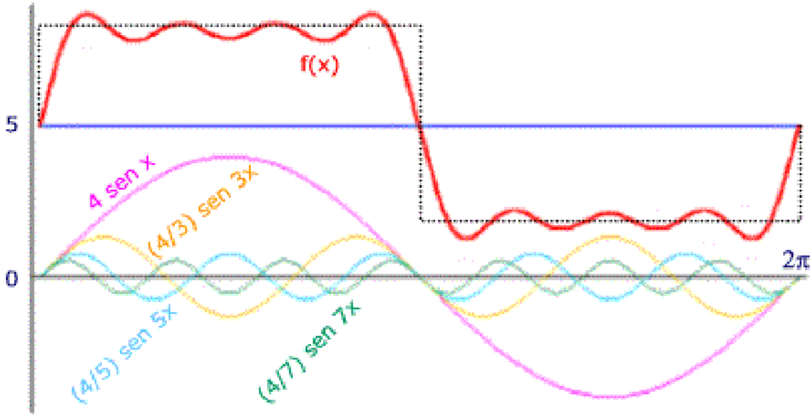
$$f_1(x) = \sin(2\pi ft).$$

$$f_2(x) = \sin(2\pi ft) + (1/3)\sin(6\pi ft).$$

$$\begin{aligned} f_3(x) = {} & \sin(2\pi ft) + \\ & (1/3)\sin(6\pi ft) + \\ & (1/5)\sin(10\pi ft). \end{aligned}$$

$$\begin{aligned} f_4(x) = {} & \sin(2\pi ft) + (1/3)\sin(6\pi ft) + \\ & (1/5)\sin(10\pi ft) + \\ & (1/7)\sin(14\pi ft) + \\ & (1/9)\sin(18\pi ft) + \\ & (1/11)\sin(22\pi ft) + \\ & (1/13)\sin(26\pi ft). \end{aligned}$$

Example 2:

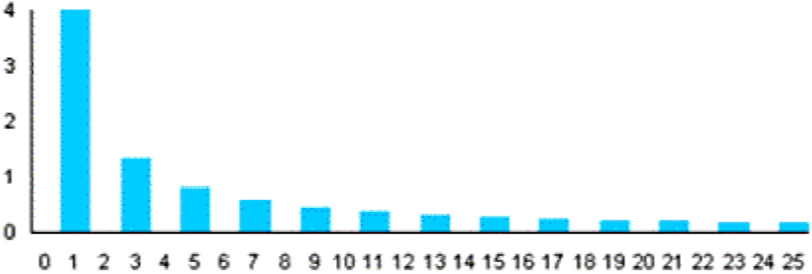


Note:

- First component $c_0 = 5$ is the signal's *dc*[1] component.
- Second component $4 \sin x$ is the *fundamental frequency*, since it has the same frequency as the signal.
- Next components $(\sin 3x,\ \sin 5x, \ldots)$ are the signal's *harmonics*.
- Every periodic signal is formed by
  - A *dc* component.
  - A harmonic signal.
  - Harmonic components.

[1] *Direct current*

- Coefficients c_n are the amplitudes of the harmonics.
- Frequency spectrum containing the first 25 harmonics:



Impulses

A unit impulse of a continuous variable located at $t = 0$ denoted is *defined* as

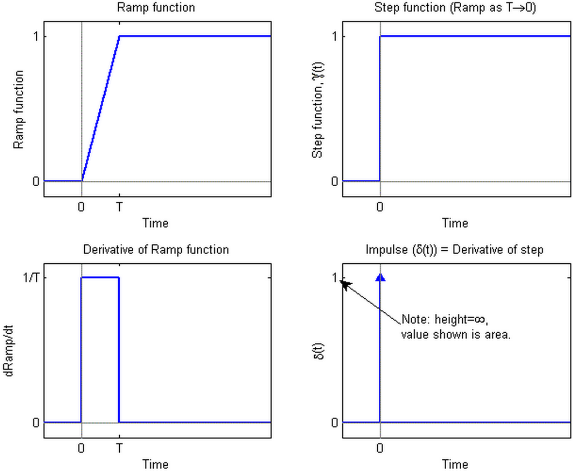
$$\delta(t) = \begin{cases} \infty, & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases} \tag{5}$$

and satisfies

$$\int_{-\infty}^{\infty} \delta(t) dt = 1. \tag{6}$$

An impulse can be viewed as a spike occurring at $t = 0$ of:

- Infinite amplitude.
- Zero duration.
- Unit area.



Sifting property:

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0) \tag{7}$$

Sifting simply yields the value of the function (t) at the location of the impulse.
More generally:

$$\boxed{\int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt = f(t_0)}. \tag{8}$$

The unit discrete impulse $\delta(x)$:

$$\delta(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases} \tag{9}$$

Also:

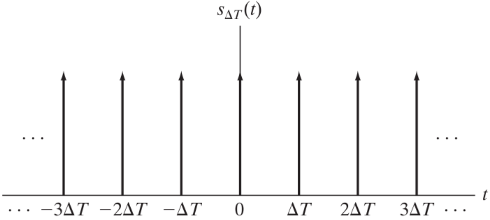
$$\sum_{x=-\infty}^{\infty} \delta(x) = 1. \tag{10}$$

And the sifting property:

$$\sum_{x=-\infty}^{\infty} f(x)\delta(x - x_0) = f(x_0). \tag{11}$$

The *impulse train* is defined as the sum of infinitely many periodic impulses ΔT apart:

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T). \tag{12}$$



The impulse train is important to understand the *sampling* process and its properties.

Fourier Transform

The Fourier transform of a continuous function $f(t)$ of a continuous variable t, is defined by the equation

$$\boxed{\mathcal{F}\left\{f(t)\right\} = F(\mu) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi\mu t}dt}, \tag{13}$$

Conversely, given $F(\mu)$;

$$\boxed{f(t) = \mathcal{F}^{-1}\{F(\mu)\} = \int_{-\infty}^{\infty} F(\mu)e^{j2\pi\mu t}d\mu}. \tag{14}$$

These two equations comprise the *Fourier transform pair*.

Using Euler's formula:

$$\begin{aligned} F(\mu) &= \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt \\ &= \int_{-\infty}^{\infty} f(t)[\cos(2\pi\mu t) + j\sin(2\pi\mu t)] \end{aligned} \tag{15}$$

- The Fourier transform is an expansion of $f(t)$ multiplied by sinusoidal terms whose frequencies are determined by the values of μ.
- After integration, only the frequency remains, so the domain of the Fourier transform is called the *frequency* domain (cycles per second).

The Fourier transform of a real function is, in general, complex:

- Fourier spectrum:

$$|F(\mu)| = \left[R^2(\mu) + I^2(\mu)\right]^{1/2}. \tag{16}$$

- Phase:

$$\phi(\mu) = \tan^{-1}\left[\frac{I(\mu)}{R(\mu)}\right]. \tag{17}$$

- Power spectrum:

$$P(\mu) = |F(\mu)|^2 = R^2(\mu) + I^2(\mu). \tag{18}$$

Example 1: The Fourier transform of a simple function

$$f(t) = \begin{cases} A, & |t| < W/2 \\ 0, & \text{otherwise} \end{cases}$$

Example 1: The Fourier transform of a simple function

$$
\begin{aligned}
F(\mu) &= \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi\mu t}\, dt = \int_{-W/2}^{W/2} A e^{-j2\pi\mu t}\, dt \\
&= \frac{-A}{j2\pi\mu}\left[e^{-j2\pi\mu t}\right]_{-W/2}^{W/2} = \frac{-A}{j2\pi\mu}\left[e^{-j\pi\mu W} - e^{j\pi\mu W}\right] \\
&= \frac{A}{j2\pi\mu}\left[e^{j\pi\mu W} - e^{-j\pi\mu W}\right] \\
&= AW \frac{\sin(\pi\mu W)}{(\pi\mu W)}
\end{aligned}
$$

The magnitude of the transform is commonly used:

$$
|F(\mu)| = AT\left|\frac{\sin(\pi\mu W)}{\pi\mu W}\right| \quad (19)
$$

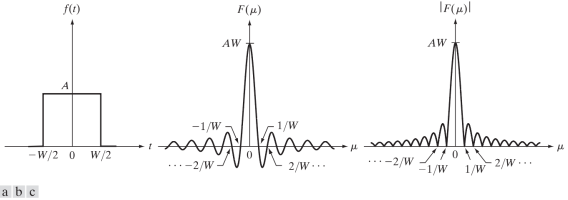


FIGURE 4.4 (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.

Example 1: The Fourier transform of a simple function

- The zeros in $|F(\mu)|$ are inversely proportional to the width W.

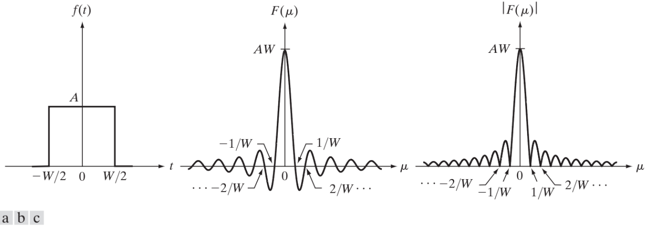


FIGURE 4.4 (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.

Example 2: The Fourier transform of the impulse

$$\delta(t) = \begin{cases} \infty, & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

Example 2: The Fourier transform of the impulse

$$\begin{aligned} F(\mu) &= \int_{-\infty}^{\infty} \delta(t)\, e^{-j2\pi\mu t} dt \\ &= \int_{-\infty}^{\infty} e^{-j2\pi\mu t}\, \delta(t)\, dt \\ &= e^{-j2\pi\mu 0} = e^{0} \\ &= 1 \end{aligned}$$

The Fourier transform of an impulse located at the origin of the spatial domain is a constant in the frequency domain.

Example 3: The Fourier transform of a shifted impulse

$$
\begin{aligned}
F(\mu) &= \int_{-\infty}^{\infty} \delta(t - t_0)\, e^{-j2\pi\mu t}\, dt \\
&= \int_{-\infty}^{\infty} e^{-j2\pi\mu t}\, \delta(t - t_0)\, dt \\
&= e^{-j2\pi\mu t_0} \\
&= \cos(2\pi\mu t_0) - j\sin(2\pi\mu t_0)
\end{aligned}
$$

The last two lines are equivalent representations of a unit circle centered on the origin of the complex plane.

Example 4: The Fourier transform of the impulse train

Step 1

Use Fourier series to represent the impulse train

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi(n/\Delta T)t}$$

$$c_n = \frac{1}{\Delta T}\int_{-\Delta T/2}^{\Delta T/2} s_{\Delta T}(t) e^{-j2\pi(n/\Delta T)t} dt = \frac{1}{\Delta T}$$

i. e.,

$$\boxed{s_{\Delta T}(t) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{j2\pi(n/\Delta T)t}}$$

Step 2

Consider the Fourier transform of the shifted impulse

$$\delta(t - t_0) \leftrightharpoons e^{-j2\pi\mu t_0} \tag{20}$$

Step 3

Use symmetry to find the Fourier transform of the complex exponential, based on the Fourier transform of the impulse

Symmetry property

If $f(t) \leftrightharpoons F(\mu)$, then $F(t) \leftrightharpoons f(-t)$

I. e., if $\delta(t - t_0) \leftrightharpoons e^{-j2\pi\mu t_0}$, we have

$$e^{-j2\pi t_0 t} \leftrightharpoons \delta(-\mu - t_0)$$

make $-t_0 = a$

$$\boxed{e^{j2\pi at} \leftrightharpoons \delta(-\mu + a) = \delta(\mu - a)}$$

Step 4

From the Fourier series representation of the impulse train, use the results of steps 2 and 3 to obtain the Fourier transform of the impulse train

$$
\begin{aligned}
s_{\Delta T}(t) =& \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} e^{j2\pi(n/\Delta T)t} \\
&\leftrightharpoons \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta(\mu - n/\Delta T).
\end{aligned} \tag{21}
$$

The Fourier transform ...

... of an impulse train with period ΔT is also an impulse train, whose period is $1/\Delta T$.

Convolution

- The convolution of two functions involves flipping (rotating by 180°) one function about its origin and sliding it past the other.
- In the convolution of two continuous functions $f(t)$ and $h(t)$, the sum of products reduces to the integral:

$$f(t) \star h(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau. \tag{22}$$

where:

  - t is the *displacement*.
  - τ is a dummy variable.
  - The minus sign does the rotation.

Given the following property:

Displacement property of the Fourier transform

If $h(t) \leftrightharpoons H(\mu)$, then

$$h(t-\tau) = e^{-j2\pi\mu\tau} H(\mu). \tag{23}$$

Let's compute the Fourier transform of the convolution integral:

The Fourier transform of the convolution integral is given by

$$
\begin{aligned}
\mathcal{F}\left\{f(t)\star h(t)\right\} =& \mathcal{F}\left\{\int_{-\infty}^{\infty} f(t)h(t-\tau)d\tau\right\} \\
=& \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau\right]e^{-j2\pi\mu t}dt \\
=& \int_{-\infty}^{\infty} f(\tau)\left[\int_{-\infty}^{\infty} h(t-\tau)e^{-2j\pi\mu t}dt\right]d\tau \\
=& \int_{-\infty}^{\infty} f(\tau)\left[e^{-j2\pi\mu\tau}H(\mu)\right]d\tau \\
=& H(\mu)\int_{-\infty}^{\infty} f(\tau)e^{-j2\pi\mu\tau}d\tau \\
=& H(\mu)F(\mu)
\end{aligned}
\tag{24}
$$

The convolution theorem

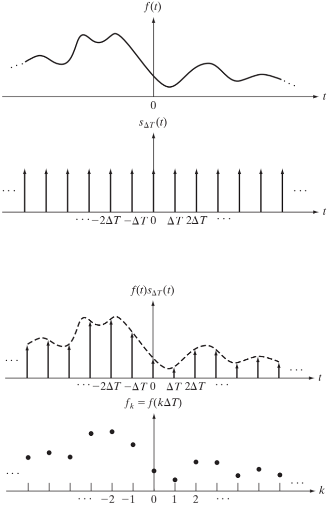
- The Fourier transform of the convolution of two functions in the spatial domain is equal to the product in the frequency domain of the Fourier transforms of the two functions.

$$\boxed{f(t) \star h(t) \leftrightharpoons F(\mu)H(\mu)} \tag{25}$$

- Equivalently; the convolution in the frequency domain is analogous to multiplication in the spatial domain:

$$\boxed{f(t)h(t) \leftrightharpoons F(\mu) \star H(\mu)} \tag{26}$$

Sampling:



$f(t)$
0
t
$s_{\Delta T}(t)$
$\cdots$
t
$\cdots -2\Delta T \; -\Delta T \; 0 \quad \Delta T \; 2\Delta T \quad \cdots$
$f(t)s_{\Delta T}(t)$
$\cdots$
t
$\cdots -2\Delta T \; -\Delta T \; 0 \quad \Delta T \; 2\Delta T \quad \cdots$
$f_k = f(k\Delta T)$
$\cdots$
k
$\cdots \; -2 \; -1 \quad 0 \quad 1 \quad 2 \quad \cdots$

The sampled function $\tilde{f}(t)$ is modeled by the product of the function $f(t)$ with the impulse train:

$$\tilde{f}(t) = f(t) \cdot s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T) \tag{27}$$

Each component of this summation is an impulse weighted by the value of $f(t)$ at the location of the impulse:

$$\begin{aligned} f_k =& \int_{-\infty}^{\infty} f(t)\delta(t - k\Delta T)dt \\ =& f(k\Delta T). \end{aligned} \tag{28}$$

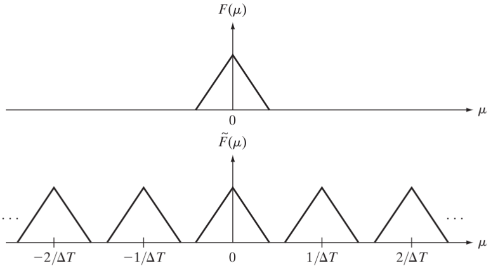
Since the Fourier transform of the product of two functions in the spatial domain is the convolution of the transforms of the two functions in the frequency domain, we have:

$$f(t)s_{\Delta T}(t) \leftrightharpoons F(\mu) \star S(\mu) \tag{29}$$

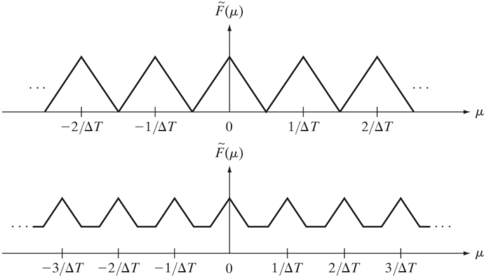
And since $s_{\Delta T}(t) \leftrightharpoons \dfrac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta(\mu - n/\Delta T)$ we have that

$$
\begin{aligned}
\tilde{F}(\mu) &= F(\mu) \star S(\mu) \\
&= \int_{-\infty}^{\infty} F(\tau)\, S(\mu - \tau)\, d\tau \\
&= \frac{1}{\Delta T} \int_{-\infty}^{\infty} F(\tau) \sum_{n=-\infty}^{\infty} \delta\left(\mu - \tau - \frac{n}{\Delta T}\right) d\tau \\
&= \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} F(\tau)\, \delta\left(\mu - \tau - \frac{n}{\Delta T}\right) d\tau \\
&= \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta T}\right)
\end{aligned}
$$

I. e., the Fourier transform of the sampled function is an infinite, periodic sequence of copies of the transform of the original, continuous function.

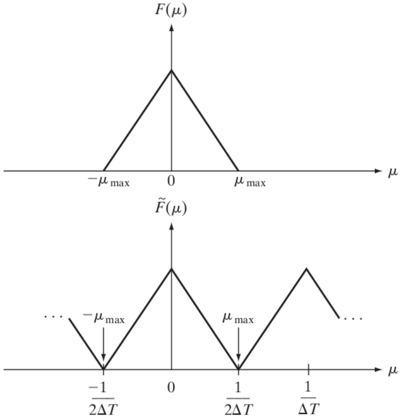


$F(\mu)$
μ
0
$\tilde{F}(\mu)$
. . .
. . .
μ
$-2/\Delta T$
$-1/\Delta T$
0
$1/\Delta T$
$2/\Delta T$



$\tilde{F}(\mu)$
. . .
. . .
μ
$-2/\Delta T$
$-1/\Delta T$
0
$1/\Delta T$
$2/\Delta T$
$\tilde{F}(\mu)$
. . .
. . .
μ
$-3/\Delta T$
$-2/\Delta T$
$-1/\Delta T$
0
$1/\Delta T$
$2/\Delta T$
$3/\Delta T$

- A function $f(t)$ is band limited if $F(\mu) = 0$ outside an interval $[-\mu_{min}, \mu_{max}]$.
- If one period of the Fourier transform of the sampled function $\tilde{F}(\mu)$ can be isolated, we can recover $f(t)$ computing its inverse Fourier transform!

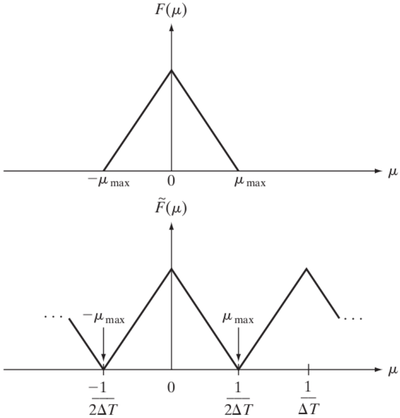


- Sufficient separation is guaranteed if

$$\frac{1}{2\Delta T} \geq \mu_{max}. \qquad (30)$$

Sampling theorem

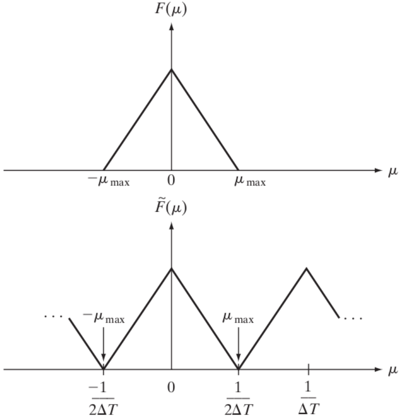
The sampling frequency must be higher than twice the signal's highest frequency.

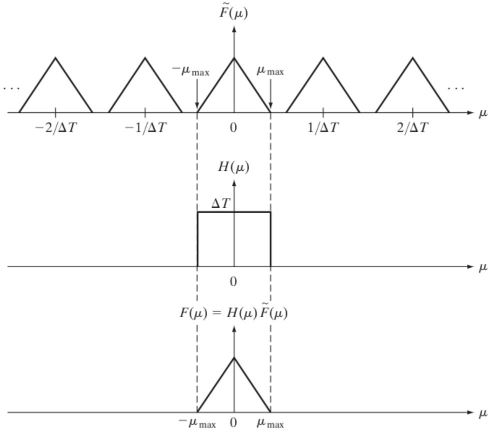


- Equivalently, sampling a signal with $1/\Delta T$ allow to "capture" a highest frequency equals $1/(2\Delta T)$.

The Nyquist rate

The Nyquist rate equals exactly twice the signal's highest frequency.





1. Sample $f(t)$ to obtain $\tilde{F}(\mu)$.
2. Filter $\tilde{F}(\mu)$ to obtain

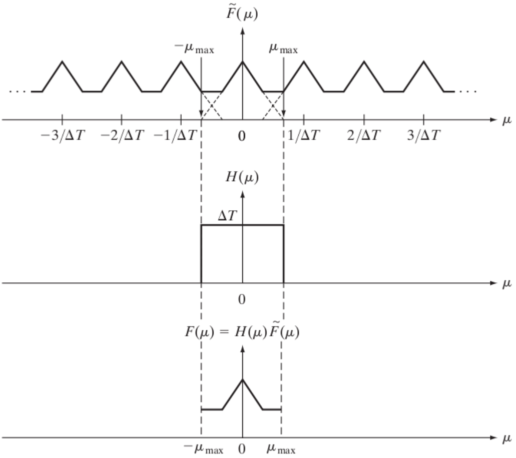
$$F(\mu) = H(\mu)F(\mu)$$

3. Recover $f(t)$ using

$$f(t) = \mathcal{F}^{-1}\{F(\mu)\}$$

Aliasing

Happens when a band-limited function is sampled at a rate smaller than twice its highest frequency.



Aliasing

Aliasing is a process in which high frequency components of a continuous function “masquerade” as lower frequencies in the sampled function.

Aliasing:

- **Is always present** because, in practice, the function must have limited duration.

Limiting duration implies:

- multiplying $f(t)$ by

$$h(t) = \begin{cases} 1, & 0 \leq t \leq T \\ 0, & \text{otherwise} \end{cases}$$

  which implies;

- making the convolution $F(\mu) \star H(\mu)$, where $H(\mu)$ is the infinite sinc function.

- Having to limit the duration of a function prevents its perfect recover using sampling.
- Aliasing is inevitable!

Anti-aliasing

Aliasing can be *reduced* by *smoothing* the input function to attenuate its higher frequencies **prior** to sampling.

Example

- Infinite band-limited $f(t) = \sin(\pi t)$.
- $\mu_{max} = 0.5$ Hz, i. e., $\Delta T < 1s$ is required.
- What happens if $\Delta T = 1s$?
- With $\Delta T \gtrsim 2$ The frequency of the dots is approximately one tenth of $f(t)$.

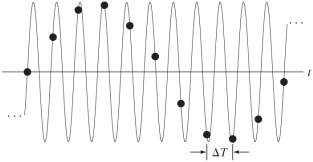


FIGURE 4.10 Illustration of aliasing. The under-sampled function (black dots) looks like a sine wave having a frequency much lower than the frequency of the continuous signal. The period of the sine wave is 2 s, so the zero crossings of the horizontal axis occur every second. ΔT is the separation between samples.

Sampling representation in the spatial domain

$$f(t) \leftrightharpoons F(\mu) = \tilde{F}(\mu)H(\mu) = h(t) \star \tilde{F}(\mu) \qquad (31)$$

Remembering that

$$\tilde{f}(t) = f(t) \cdot s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)$$

and

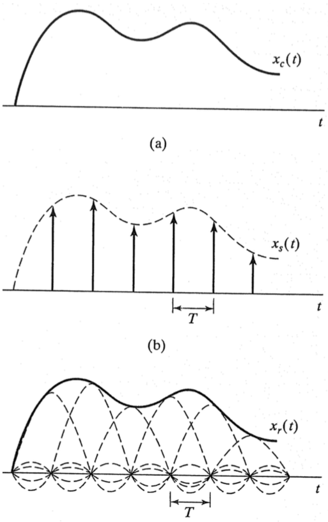
$$f(t) \star h(t) = \int_{-\infty}^{\infty} f(\tau)h(t - \tau)d\tau$$

we have

$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} f(n\Delta T)\text{sinc}\left[\frac{t - n\Delta T}{n\Delta T}\right]} \qquad (32)$$

Sampling in the spatial domain:

$$f(t) = \sum_{n=-\infty}^{\infty} f(n\Delta T)\mathrm{sinc}\left[\frac{t - n\Delta T}{n\Delta T}\right]$$



- The perfectly reconstructed function is an infinite sum of sinc functions weighted by the sample values, and has the important property that the reconstructed function is identically equal to the sample values at multiple integer increments of ΔT.
- Previous equation requires an infinite number of terms for the interpolations between samples. In practice, this implies that we have to look for approximations that are finite interpolations between samples.

Discrete Fourier Transform (DFT) of one variable

We've seen that:

- The Fourier transform of a sampled, band-limited function $\tilde{f}(t)$ extending from $-\infty$ to ∞ is a;
  - continuous;
  - periodic;

  function $\tilde{F}(\mu)$ extending from $-\infty$ to ∞.
- We know that

$$\tilde{F}(\mu) = \sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta T}\right)$$

  but we want a way to compute $\tilde{F}(\mu)$ directly from $\tilde{f}(t)$.

Obtaining the DFT from the continuous Fourier transform of a sampled function.

$$\begin{aligned}
\tilde{F}(\mu) &= \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T) \right] e^{-j2\pi\mu t} dt \\
&= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\delta(t - n\Delta T) e^{-j2\pi\mu t} dt \qquad (33) \\
&= \sum_{n=-\infty}^{\infty} f_n e^{-j2\pi\mu n\Delta T}
\end{aligned}$$

where

$$f_n = \int_{-\infty}^{\infty} f(t)\delta(t - n\Delta T) dt. \qquad (34)$$

I. e.,

$$\boxed{\tilde{F}(\mu) = \sum_{n=-\infty}^{\infty} f_n e^{-j2\pi\mu n\Delta T}} \tag{35}$$

Notice that, although f_n is a discrete function, its Fourier transform $\tilde{F}(\mu)$ is:

- Continuous, and;
- Infinitely periodic with period $1/\Delta T$.

Therefore, we need only one period of $\tilde{F}(\mu)$ to characterize it.

So:

- Obtain M equally spaced samples of $\tilde{F}(\mu)$ over $\mu = 0$ to $\mu = 1/\Delta T$.
- The samples occur at

$$\mu = \frac{m}{M\Delta T}, \qquad m = 0, 1, 2, \ldots, M-1. \tag{36}$$

Resulting:

$$\boxed{F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M}}, \qquad m = 0, 1, 2, \ldots, M-1. \tag{37}$$

This is the discrete Fourier transform pair:

$$\boxed{F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M}}, \qquad m = 0, 1, 2, \ldots, M-1.$$

Conversely:

$$\boxed{f_n = \frac{1}{M} \sum_{m=0}^{M-1} F_m e^{j2\pi nm/M}}, \qquad n = 0, 1, \ldots, M-1. \tag{38}$$

Notation change:

- So far, t and μ were used to denote **continuous** spatial and frequency domains.
- We'll use (x, y) and (u, v) to denote **discrete** spatial and frequency domains for two variables.
- We'll also use (t, z) and (μ, ν) to denote **continuous** 2-D variables in the spatial and frequency domain, respectively.

The DFT pair becomes:

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} \qquad u = 0, 1, \ldots, M-1. \tag{39}$$

and

$$f(x) = \frac{1}{M} \sum_{u=0}^{M-1} F(u) e^{j2\pi ux/M} \qquad x = 0, 1, \ldots, M-1. \tag{40}$$

The discrete equivalence of the convolution is

$$\boxed{f(x) \star h(x) = \sum_{m=0}^{M-1} f(m) h(x-m)}, \tag{41}$$

for $x = 0, 1, \ldots, M-1$.

Given that $f(x)$ consists of M samples of $f(t)$, ΔT units apart, the duration of the record comprising the set $\{f(x)\}, \quad x = 0, 1, \ldots, M-1$, is

$$T = M\Delta T. \tag{42}$$

The corresponding spacing Δu in the discrete frequency domain is

$$\Delta u = \frac{1}{M\Delta T} = \frac{1}{T}. \tag{43}$$

The entire frequency range spanned by the M components of the DFT is

$$\Omega = M\Delta u = \frac{1}{\Delta T}. \tag{44}$$

Thus:

- The resolution in frequency Δu depends inversely on the duration T over which the continuous function $f(t)$ is samples.
- The range of frequencies spanned by the DFT depends inversely on the sampling interval ΔT.

Example with 4 samples of $f(t)$:

$$F(0) = 11$$
$$F(1) = -3 + 2j$$
$$F(2) = -(1 + 0j)$$
$$F(3) = -3(+2j)$$

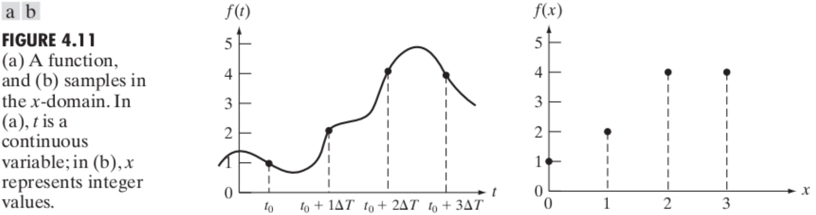


FIGURE 4.11 (a) A function, and (b) samples in the x-domain. In (a), t is a continuous variable; in (b), x represents integer values.

The 2-D impulse

The impulse $\delta(t, z)$ of continuous variables t and z is defined as

$$\delta(t, z) = \begin{cases} \infty & \text{if } t = z = 0 \\ 0 & \text{otherwise} \end{cases} \tag{45}$$

and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(t, z) dt dz = 1. \tag{46}$$

The sifting property of the 2-D impulse:

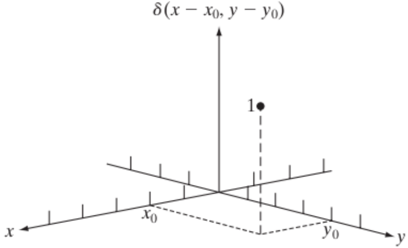
$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t, z) \delta(t - t_0, z - z_0) dt dz = f(t_0, z_0). \tag{47}$$

The 2-D impulse for discrete variables x and y is defined as

$$\delta(x, y) = \begin{cases} 1 & \text{if } x = y = 0 \\ 0 & \text{otherwise} \end{cases} \tag{48}$$

It sifting property:

$$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(x, y)\delta(x - x_0, y - y_0) = f(x_0, y_0). \tag{49}$$



The 2-D continuous Fourier Transform pair

Let $f(t, z)$ be a continuous function of two continuous variables t and z.
The two dimensional Fourier transform pair is given by

$$F(\mu, \nu) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t, z) e^{-j2\pi(\mu t + \nu z)} dt dz, \tag{50}$$

and

$$f(t, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\mu, \nu) e^{j2\pi(\mu t + \nu z)} d\mu d\nu. \tag{51}$$

Where μ and ν are the frequency variables.

Example

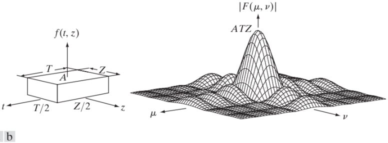


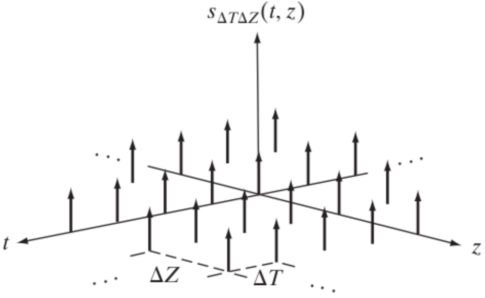
FIGURE 4.13 (a) A 2-D function, and (b) a section of its spectrum (not to scale). The block is longer along the t-axis, so the spectrum is more "contracted" along the μ-axis. Compare with Fig. 4.4.

$$
\begin{aligned}
F(\mu, \nu) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t, z)\, e^{-j2\pi(\mu t + \nu z)}\, dt\, dz \\
&= \int_{-T/2}^{T/2}\int_{-Z/2}^{Z/2} A\, e^{-j2\pi(\mu t + \nu z)}\, dt\, dz \\
&= ATZ\left[\frac{\sin(\pi\mu T)}{(\pi\mu T)}\right]\left[\frac{\sin(\pi\nu Z)}{(\pi\nu Z)}\right]
\end{aligned}
$$

Sampling function (2-D impulse train):

$$s_{\Delta T \Delta Z} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(t - m\Delta T, z - n\Delta z). \tag{52}$$

where ΔT and Δz are the separations between samples in t- and z- axis of function $f(t, z)$.



- $f(t, z)$ is band-limited if $F(\mu, \nu) = 0$ for $|\mu| \leq \mu_{max}$ and $|\nu| \leq \nu_{max}$.

Two-dimensional sampling theorem

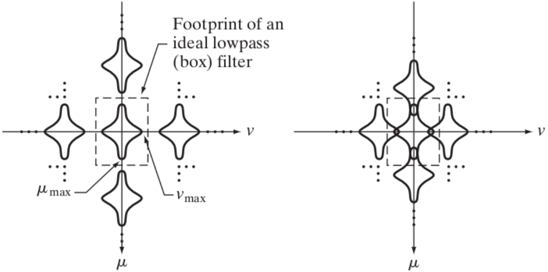
A continuous, band-limited function $f(t, z)$ can be recovered with no error from a set of its samples if the sampling intervals are

$$\Delta T < \frac{1}{2\mu_{max}}, \tag{53}$$

and

$$\Delta Z < \frac{1}{2\nu_{max}}. \tag{54}$$

No information is lost if a 2-D, band-limited continuous function is represented by samples acquired at rates greater than twice the highest frequency content of the function in both μ- and ν-directions.



a b

FIGURE 4.15 Two-dimensional Fourier transforms of (a) an over-sampled, and (b) under-sampled band-limited function.

Aliasing in images

We've seen that:

- Limiting the function's duration introduces corrupting frequency components extending to infinity in the frequency domain.

Image aliasing

Two principal manifestations of aliasing in images;

- spatial (under sampling), and;
- temporal (wagon wheel).

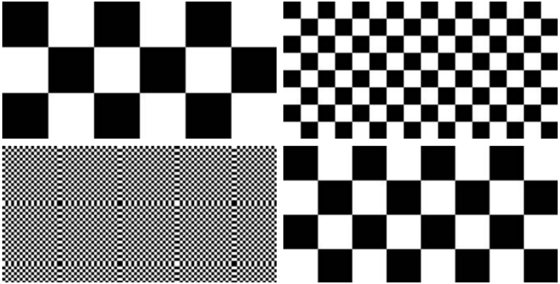
Spatial aliasing

- Jagged[2] lines.
- Spurious highlights.
- Appearance of frequency patterns.

[2] Denteado, chanfrado

Example

- Consider a 96×96 sensor.
- Use it to digitize checkerboard patterns.



a b
c d

FIGURE 4.16 Aliasing in images. In (a) and (b), the lengths of the sides of the squares are 16 and 6 pixels, respectively, and aliasing is visually negligible. In (c) and (d), the sides of the squares are 0.9174 and 0.4798 pixels, respectively, and the results show significant aliasing. Note that (d) masquerades as a “normal” image.

Remember:

- Anti-aliasing must be done at the *front-end*, i. e., before sampling occurs.

Thank you!

tvieira@ic.ufal.br