# Füredi graphs and the projective norm graphs

Felipe Barquero Jiménez*  Tibor Szabó†

**Abstract**

The $K_{2,s+1}$-free graphs $F(q^t, s)$ defined by Füredi were investigated recently by Livinsky when $s = \frac{q^t-1}{q-1}$. In this note we determine the exact relationship of these graph to the $K_{t+1,t!+1}$-free projective normgraphs $NG(q,t)$ and find that they are the same up to a few exceptional vertices. We prove this via defining a $K_{t+1,t!+1}$-free generalization of the Erdős-Rényi polarity graph, which contains both as an induced subgraph.

## 1 Introduction

For a prime power $q$ and positive integer $s|q-1$, Füredi [3] defined the graphs $F(q,s)$ as follows. Let $H \subseteq \mathbb{F}_q^*$ denote the unique $s$-element subgroup of the multiplicative group of the $q$-element field $\mathbb{F}_q$ and define a relation $\sim$ on $\mathbb{F}_q^2$ by $(x,y) \sim (x',y')$ if there exists $h \in H$ such that $x' = hx$ and $y' = hy$. Note that $\sim$ is an equivalence relation since $H$ is a subgroup. The vertex set of the graph is

$$V(F(q,s)) = (\mathbb{F}_{q^t} \times \mathbb{F}_{q^t}) \setminus \{(0,0)\}/\sim$$

and two vertices $u = \langle u_0, u_1\rangle$ and $v = \langle v_0, v_1\rangle$ are adjacent in $F(q,s)$ if

$$u_0v_0 + u_1v_1 \in H.$$

It is not difficult to see that the definition of the edge relation respects the equivalence classes. Füredi [3] proved that $F(q,s)$ is $K_{2,s+1}$-free, and since $e(F(q,s) \geq \frac{1}{2}\frac{q^2-1}{s}(q-1) \geq \frac{\sqrt{s}}{2}n^{3/2}$, the Kővári-Sós-Turán upper bound on the Turán number of $K_{2,s+1}$ is asymptotically tight.

Given a prime power $q$ and integer $t \geq 1$, the projective norm graphs $NG(q,t+1)$ of Alon, Rónyai, and Szabó [1], have vertex set $\mathbb{F}_{q^t} \times \mathbb{F}_q^*$, and have two vertices $(X,\lambda)$ and $(Y,\mu)$ adjacent if

$$N(X+Y) = \lambda\mu,$$

where $N : \mathbb{F}_{q^t} \to \mathbb{F}_q$ is the corresponding norm function. With the crucial help of the Key Lemma of [5] it is proved that $NG(q,t+1)$ is $K_{t+1,t!+1}$-free, which

*Freie Universität Berlin, Institut für Mathematik, Berlin, Germany. Email:...
†Freie Universität Berlin, Institut für Mathematik, Berlin, Germany. Email: szabo@math.fu-berlin.de. Supported by GIF grant G-1347-304.6/2016.

establishes that the right order of magnitude of the Turán number $ex(n, K_{t,s})$ is $n^{2-\frac{1}{t+1}}$ for every $s > t!$.

In a recent preprint Livinsky [7] proved that for every $t \geq 2$, the Füredi graph $F(q^t, s)$ with $s = \frac{q^t-1}{q-1}$ does not contain $K_{t+1,t!+1}$. The proofs, both in the elementary case of $t = 2$ for $K_{3,3}$-free-ness as well as the general case using the Key Lemma of [5], resemble the proofs of $K_{t+1,t!+1}$-freeness of the projective normgraph $NG(q, t+1)$ from [1]

In this note we show that this similarity is no coincidence: $F\left(q^t, \frac{q^t-1}{q-1}\right)$ is the same as $N(q, t+1)$, up to a few vertices here and there.

When $q^t \equiv 1 \pmod 4$, i.e. when there exists an element $\beta \in \mathbb{F}_{q^t}$ with $\beta^2 = -1$, then the embedding

$$\langle x, y \rangle \mapsto \left( \frac{2x + 2\beta y}{x - \beta y}, N\left( \frac{2}{x - \beta y} \right) \right) \in \mathbb{F}_{q^t} \times \mathbb{F}_q^*$$

shows that $NG(q, t+1)$ is isomorphic to the subgraph of $F\left(q^t, \frac{q^t-1}{q-1}\right)$ induced by the vertices with non-zero second coordinate. This map respects the equivalence classes and is injective since $N(h) = 1$ if and only if $h \in H$ in the definition of $F\left(q^t, \frac{q^t-1}{q-1}\right)$. It is also straightforward to check that the map is a graph isomorphism.

To treat the general case we introduce a $K_{t+1,t!+1}$-free generalization of the Erdős-Rényi polarity graph, which turns out to contain both $F\left(q^t, \frac{q^t-1}{q-1}\right)$ and $N(q, t+1)$ as induced subgraphs, and has roughly the same number of vertices as those. Let, again, $H \leq \mathbb{F}_q^*$ be the subgroup of order $s$ and consider the relation $\sim$ over the projective plane $PG(2, q)$ where

$$[a, b, c] \sim [a', b', c'] \iff a = \lambda a', b = \lambda b', c = h\lambda c' \text{ for some } \lambda \in \mathbb{F}_q^* \text{ and } h \in H$$

The relation $\sim$ is an equivalence relation because $H$ is group; the equivalence class of a point $[a, b, c]$ will be denoted by $\langle a, b, c \rangle$. The equivalence classes $\langle a, b, 1 \rangle$ have exactly $s$ elements for $(a, b) \neq (0, 0)$, the classes $\langle a, b, 0 \rangle$ and $\langle 0, 0, 1 \rangle$ have exactly one element. The vertex set of $\mathrm{Pol}(q, s)$ is $PG(2, q)/\sim$ and the edge set consists of the pairs $u = \langle a, b, c \rangle, v = \langle x, y, z \rangle$ for which

$$\text{there exists } h \in H \text{ such that } ax + by + hcz = 0.$$

This definition respects the equivalence classes, because if $[\lambda a, \lambda b, h_1 \lambda c]$ and $[\mu x, \mu y, h_2 \mu z]$ are two other representatives then

$$(\lambda a)(\mu x) + (\lambda b)(\mu y) + (h h_1^{-1} h_2^{-1})(h_1 \lambda c)(h_2 \mu z) = \lambda\mu(ax + by + hcz) = 0,$$

where $h h_1^{-1} h_2^{-1} \in H$. Note that if $-1$ is a square then the non-singleton equivalence classes form a subgraph of $\mathrm{Pol}(q, s)$ is isomorphic to $F(q, s)$.

To state our theorem we need to introduce a version of the Füredi graph, where an arbitrary non-degenerate symmetric bilinear form $b : \mathbb{F}_q \times \mathbb{F}_q \to \mathbb{F}_q$ is used to define the edges. We define the graph $F(q,s,b)$ on the same vertex set $V(F(q,s)) = (\mathbb{F}_{q^t} \times \mathbb{F}_{q^t}) \setminus \{(0,0)\}/\sim$ as before, but now two vertices $u = \langle u_0, u_1 \rangle$ and $v = \langle v_0, v_1 \rangle$ are adjacent in $F(q,s,b)$ if

$$b((u_0,u_1),(v_0,v_1)) \in H.$$

Bilinearity and $H$ being closed for multiplication implies that this definition respects the equivalence classes. When $b(u,v) = u_0v_0 + u_1v_1$, the graph $F(q,s,b)$ is equal to the original Füredi graph.

**Theorem 1.** *Let $q$ be an arbitrary prime power (WE MUST Carefully CHECK THE PROOF FOR EVEN CHARACTERISTIC!!).*

*(a) For $b(u,v) = u_0v_1 - u_1v_0$ and any integer $t \geq 2$ we have*

$$NG(q,t+1) \subseteq BP(q,t+1) \cong F\left(q^t, \frac{q^t-1}{q-1}, b\right)$$

*(b) For an arbitrary non-degenerate symmetric bilinear form $b$ over $\mathbb{F}_{q^t}^2$ and integer $s|(q-1)$*

$$F(q,s,b) \subseteq Pol(q,s).$$

For the case of $F\left(q^t, \frac{q^{t-1}}{q-1}\right)$ is isomorphic to the geometric description of Ball and Pepe [2] of projective norm graph $NG(q,t+1)$.

Note that the order of the number of vertices is $q^{t+1}$ for all these graphs, hence $F\left(q^t, \frac{q^t-1}{q-1}\right)$ and $NG(q,t+1)$ intersect in an induced subgraph that contains almost all of their vertices. The more precise situation depends on the parameters. As we have seen above, when $q^t \equiv 1 \pmod 4$, $F(q^t,s)$ contains $NG(q,t)$ as an induced subgraph. In the rest of the cases the two graphs have a common induced subgraph on the same order of vertices.

**Proposition 1.** *Let $q^t$ be congruent to 3 modulo 4. Then the induced subgraph of $NG(q,t+1)$ on the vertex set $V_1 = \{(X,x) \in \mathbb{F}_{q^{t+1}} \times \mathbb{F}_q^* : ...\}$ is isomorphic to the induced subgraph of $F\left(q^t, \frac{q^{t-1}}{q-1}\right)$ on the vertex set $V_2 := \{\langle u,v \rangle \in V(F\left(q^t, \frac{q^{t-1}}{q-1}\right))\}$.*

## 2 Case $t = 2$

In this section we will analyse the case when $t = 2$. We start by defining the Füredi graphs: In 1996 [3] Füredi gave a construction of a graph on $n = q^2 - 1$ vertices (for $q$ a prime power ) and at least $\frac{qn}{2} \sim \frac{1}{2}n^{3/2}$ edges containing no $K_{2,r+1}$. The edge density of this construction match the upper bound given by Kövari-Sós-Turán in [6]. The graph $F(q,b,2)$ is defined as follows: Let $q$ a prime power and $b$ a symmetric billinear form over $\mathbb{F}_q^2$, define the vertex set of the graph $V = (\mathbb{F}_q \times \mathbb{F}_q) \setminus \{(0,0)\}$ and two $u,v$ are adjacent if and only if

$$b(u,v) = 1.$$

The norm graph presented in [1] for $t=2$ gives a very similar construction. In this case the vertex set is $\mathbb{F}_q \times \mathbb{F}_q^*$ and two vertices $(x,\lambda)$ and $(y,\mu)$ are adjacent if and only if $x+y=\lambda\mu$. By taking the billinear form $b(u,v)=u_0v_1+u_1v_0$ the norm graph can be embedded in the subgraph $F(q,b,2)$ by the map:

$$\mathbb{F}_q \times \mathbb{F}_q^* \hookrightarrow \mathbb{F}_q \times \mathbb{F}_q \setminus \{(0,0)\}$$

$$(x,\lambda) \mapsto \left(\frac{x}{\lambda}, \frac{1}{\lambda}\right).$$

Because $x+y=\lambda\mu$ if and only if $\frac{x}{\lambda\mu}+\frac{y}{\lambda\mu}=1$.
All of this constructions can be embedded in the orthogonal polarity graph described by Füredi in [4]. The **Orthogonal Polarity Graph** Pol $(q,2)$ is classically defined as follows. The vertex set will be the projective plane $PG(2,q)$ and two vertices $u=[x,y,z]$ and $v=[a,b,c]$ will be adjacent if and only if

$$b(u,v) := ax+by+cz=0$$

Note that here we used the symmetrical billinear form $b(u,v)=u_0v_0+u_1v_1+u_2v_2$. By considering another symmetric billinear form $b'$ which is equivalent to $b$ via an automorphism $\phi$ of the vector space of $\mathbb{F}_q^3$. Then $\phi$ also defines a collineation of $PG(2,q)$ which is an isomorphism from Pol $(q,2)$ to the graph on the same vertex set and edge relation defined by $b'(u,v)=0$. Therefore we can consider $b$ to be written in the canonical form, i.e.

$$b_\beta(u,v)=\frac{1}{2}(u_0v_1+u_1v_0)+\beta u_2v_2$$

where $\beta$ is either 1 or a non-square. If $\beta$ is a non square then the collineation defined by the linear trasformation

$$\phi : \mathbb{F}_q^3 \to \mathbb{F}_q^3$$

$$(u_0,u_1,u_2) \mapsto (\beta u_0, u_1, u_2)$$

defines an isomorphism of graphs since

$$b_\beta(\psi(u),\psi(v))=\frac{1}{2}(\beta u_0v_1+\beta u_1v_0)+\beta u_2v_2=\beta\cdot b_1(u,v)$$

and therefore $u,v$ are adjacent in the graph defined by $b_1$ if and only if $\psi(u),\psi(v)$ are adjacent on the graph defined by $b_\beta$. We obtained that no matter which symmetric billinear form we take the orthogonal polarity graph is the same.
Now we will embed all the graphs $F(q,b,2)$ into Pol$(q,2)$ by considering the embedding

$$\mathbb{F}_q \times \mathbb{F}_q \setminus \{(0,0)\} \hookrightarrow PG(2,q)$$

$$(a,b) \mapsto [a,b,1]$$

Note that $b((a,b),(x,y)) := ax+\theta by=1$. For a pair of points $[x,y,z]$ and $[a,b,c]$ in $PG(2,q)$ the relation $ax+\theta by-cz=0$ (comming from a symmetrical billinear form) defines the graph Pol$(2,q)$. Therefore we obtained that all the graphs a $F(q,b,2)$ and of course also the norm graph $N(q,2)$ are induced subgraphs of Pol$(2,q)$.

## 3 General case

For a general $t$ we will define 3 different graphs: the norm graph $N(q,t)$ the Ball-Pepe description of the norm graph $\Gamma(q,t)$, the Livinsky graph $F(q,b,t)$. Our aim is to prove, as in the previous section that, all this three graphs are induced subgraphs the orthogonal polarity graph $\mathrm{Pol}(q,t)$.
The norm graph $H(q,t)$ presented in [1] has vertex set $\mathbb{F}_{q^{t-1}} \times \mathbb{F}_q^*$ and two vertices $(X,\lambda)$ and $(Y,\mu)$ are adjacent if and only if

$$N(X+Y)=\lambda\mu.$$

In the paper [2] Ball and Pepe give a description of the norm graph as an induced subgraph of a finite geometric construction. We will work on $\overline{\mathbb{F}_q}$ (the algebraic closure of $\mathbb{F}_q$) and here we consider $\mathbb{P}^{2^{t-1}}$, the projective space over $\overline{\mathbb{F}_q}$, where the first $2^{t-1}$ coordinates will be index by a subset $T \subseteq \{0,\dots,t-2\}$ and the last coordinate is simply index by the number $2^{t-1}$. On this setting, let $p$ be the point $[0,\dots,0,1]$ (the last element of the canonical basis). The point $p$ will be the apex of our cone and it is non isotropic with respect to the following billinear form:

$$b(u,v)=\sum_T u_T v_{T^c} - u_{2^{t-1}} v_{2^{t-1}}.$$

The hyperplane $p^\perp \cong \mathbb{P}^{2^{t-1}-1}$ is defined by the elements of $\mathbb{P}^{2^{t-1}}$ for which the last coordinate is 0. In such hyperplane we embed the Segre Product $\Sigma_{2,t-1}$. In $p^\perp \cong \mathbb{P}^{2^{t-1}-1}$ we consider the subgeometry formed by the elements whose entries are in $\mathbb{F}_{q^{t-1}}$, this subgeometry is precisely $PG(2^{t-1}-1, q^{t-1})$. Therefore the we can embed the algebraic set $\mathcal{V}_{2,t-1}$ inside the Segre Product $\Sigma_{2,t-1}$. (Take into account that $\Sigma_{2,t-1}$ is the product of $t-1$ lines $\mathbb{P}^1$ over $\overline{\mathbb{F}_q}$ which is a larger variety than just the product of the lines $PG(1,q^{t-1})$. However it contains all subproducts for any $t$ ).
Define the cone $\mathcal{C}$ whose base is $\mathcal{V}_{2,t-1}$ and its apex is $p$. and inside it define the following cone graph $\Gamma=\Gamma(t,q)=(V,E)$ whose vertex set is

$$V := \{\sigma(a)+\lambda p : \quad \sigma(a)\in\mathcal{V}_{2,t-1}\subseteq p^\perp \quad \lambda\in\mathbb{F}_q^*\}$$

and the edges are define by the relation:

$$xy\in E \iff b(x,y)=0.$$

Note that the norm graph $H(t,q)$ presented in [5] is the subgraph of $\Gamma$ induced by the vertex set $\{\sigma(a)+\lambda p : \sigma(a) \quad \text{with } a\neq\infty \quad \lambda\in\mathbb{F}_q^*\}$ if we consider the embedding $(X,\lambda)\mapsto\sigma(X)+\lambda p$ then $\sigma(X)+\lambda p$ and $\sigma(Y)+\mu p$ are adjacent if and only if

$$0=\sum_T X^T Y^{T^c} - \lambda\mu$$

or in other words $N(X+Y)=\lambda\mu$. (Here by $X^T$ we mean $X^{\sum_{i\in T} q^i}$).
In a recent preprint [7] Livinsky described a construction inspired in the Füredi

graph $F(q,b,2)$. We define the general case $F(q,b,t)$ as follows. Let $H \leq \mathbb{F}^*_{q^{t-1}}$ be the subgroup of order $\dfrac{q^{t-1}-1}{q-1} = q^{t-2} + \cdots + q + 1$. For $q$ a prime power and $b$ be a non*degenerate symmetric bilinear form over the vector space $\mathbb{F}^2_{q^{t-1}}$ we define $F(q,b,t)$ to be the graph whose vertex set is

$$(\mathbb{F}_{q^{t-1}} \times \mathbb{F}_{q^{t-1}}) \setminus \{(0,0)\}/\sim$$

where $\sim$ is the equivalence relation defined by $(x,y) \sim (x',y')$ if and only if $(x',y') = h(x,y)$ for $h \in H$. Two vertices $u = \langle u_0, u_1\rangle$ and $v = \langle v_0, v_1\rangle$ are adjacent if and only if

$$b(u,v) \in H \iff N(b(u,v)) = 1.$$

Where $N$ is the norm function associated to the field extension $[\mathbb{F}_{q^{t-1}} : \mathbb{F}_q]$. The adjacency relation is symmetric and well defined because if we take another representatives $(u_0', u_1') = (hu_0, hu_1)$ and $(v_0', v_1') = (h'v_0, h'v_1)$ then $b((u_0', u_1'), (v_0', v_1')) = hh' \cdot b(u,v) \in H$. Livinsky defined his graph as $F(q,t) := F(q,b,t)$ where $b(u,v) = u_0v_0 + u_1v_1$. Note that if we take the symmetric billinear form $b(u,v) = u_0v_1 + u_1v_0$ then the graph $F(q,b,t)$ is precisely the Ball-Pepe construction $\Gamma(t,q)$ simply by taking the map

$$\langle a,b\rangle \mapsto \sigma(a/b) + N(b)^{-1}p \quad \text{for } b \neq 0$$

$$\langle a,0\rangle \to \sigma(\infty) + N(a)^{-1}p.$$

First it is clear that if we have another representative $\langle ha, hb\rangle$ then the image is the same because $N(h) = 1$ and $h$ is cancelled in the division $ha/hb$. The map is clearly bijective, to check this we describe the inverse function. Take $\sigma(x) + \lambda p$ a vertex of $\Gamma(t,q)$ and send it to either $\langle a, 0\rangle$ (where $N(a)^{-1} = \lambda$) for $x = \infty$ or to $\langle xy, y\rangle$ (for $N(y)^{-1} = \lambda$) otherwise. Two vertices $\langle x,y\rangle$ and $\langle c,d\rangle$ with $y \neq 0$ and $d \neq 0$ are adjacent if and only if

$$N(xd + yc) = 1 \iff N\left(\frac{x}{y} + \frac{c}{d}\right) = N(y)^{-1}N(d)^{-1}.$$

Note that when $y$ is zero then

$$\sigma(\infty) + N(x)^{-1}p = [0, 0, \cdots, 1, N(x)^{-1}]$$

and therefore the relation

$$0 = B(\sigma(\infty) + N(x)^{-1}p, \sigma(c/d) + N(d)^{-1}p) =$$

$$1 - N(d)^{-1}N(x)^{-1}$$

happens precisely when $N(xd) = 1$ (i.e. when $\langle x, 0\rangle$ and $\langle c, d\rangle$ are adjacent). Finally when $d$ is also 0 the relation

$$0 = B(\sigma(\infty) + N(x)^{-1}p, \sigma(\infty) + N(d)^{-1}p) =$$

$$N(d)^{-1}N(x)^{-1}$$

or in other words $N(xc) = 0$ which is not possible (also $\langle x, 0\rangle$ and $\langle c, 0\rangle$ are not adjacent).

Now we are ready define the **Orthogonal Polarity graph** $\mathrm{Pol}(q,t)$ that unifies all the previous construction. Let $H \leq \mathbb{F}^*_{q^{t-1}}$ be again the subgroup of order $\dfrac{q^{t-1}-1}{q-1} = q^{t-2} + \cdots + q + 1$. Consider over the projective plane $PG(2, q^{t-1})$ the equivalence relation where

$$[a, b, 1] \sim [a', b', 1] \iff a' = ha \quad \text{and} \quad b' = hb, \quad h \in H$$

. Here the classes $\langle a, b, 1\rangle$ have exactly $|H|$ elements and the classes $\langle a, b, 0\rangle$ and $\langle 0, 0, 1\rangle$ have exactly one element. The relation is an equivalence relation because $H$ is group. The edge set of our graph will be defined by the pairs $u = \langle a, b, c\rangle, v = \langle x, y, z\rangle$ for which

$$ax + by + hcz = 0. \quad \text{for } h \in H$$

This edge relation is symmetric and well-defined over the equivalence classes, because if $[h_1a, h_1b, 1]$ and $[h_2x, h_2y, 1]$ are tow other representatives then

$$h_1a \cdot h_2x + h_1b \cdot h_2y + h_1h_2h \cdot cz =$$

$$h_1h_2(ax + by + hcz) = 0$$

and clearly $h_1h_2h \in H$.
In general given a non-degenerate billinear form $b$ over $\mathbb{F}^3_{q^{t-1}}$ we pick a non-isotropic point $p \in PG(2, q^{t-1})$ (and we call $p$ the **origin**) and let $l_\infty = p^\perp$ be a line a infinity. Consider the family of billinear forms $\{b_h\}_{h\in H}$ where $b_h(p, p) = h \cdot b(p, p)$ and $b_h \equiv b$ over $p^\perp$ . Note that every point outside $l_\infty$ can be written as $p + \lambda q$ where $q \in l_\infty$. We define two elements $p + \lambda q \sim p + \lambda' q'$outside $l_\infty$ to be equivalent if and only if $q = q'$ and $\lambda\lambda'^{-1} \in H$ (i.e. they point in the same direction by scalars which are multiple of each other by a factor in $H$ ). Now we define the graph whose vertex set is $PG(2, q^{t-1})/\sim$ and two vertices $u$ and $v$ are adjacent if and only if there is an $h$ for which $b_h(u, v) = 0$. Note that this relation is symmetric and respect the definition of the equivalent classes because if

$$0 = b_h(p + \lambda q, p + \mu q')$$

then

$$b_{h_1h_2h}(p + h_1\lambda q, p + h_2\mu q') = h_1h_2 \cdot b_h(p + \lambda q, p + \mu q') = 0.$$

If two billinear forms $b$ and $b'$ are equivalent via an automorphism $\phi$ of $\mathbb{F}^3_q$. If we also have that $\phi(p) = p'$ then the graph with origin $p$ and billinear form $b$ is isomorphic to the graph with origin $p'$ and billinear form $b'$. First simply note that $p'$ is not isotropic and that its orthogonal complement is precisely the line $l'_\infty = \phi(p^\perp) = \phi(p)^{\perp'}$. Second $\phi$ respects the equivalence classes because if we pick a class $\{p+\lambda hq\}_{h\in H}$ is sent to the class $\{p'+\lambda h\phi(q)\}_{h\in H}$ (clearly $\phi(q)\in l'_\infty$). Finally pick two vertices $v$ and $w$, if $v=p+\lambda q_1$ and $w=p+\mu q_2$ then

$$b_h(v,w)=h\cdot b(p,p)+\lambda\mu b(q_1,q_2)$$
$$=h\cdot b'(p',p')+\lambda\mu b'(\phi(q_1),\phi(q_2))=b'_h(p'+\lambda\phi(q_1),p'+\mu\phi(q_2))$$

which implies that $v$ and $w$ are adjacent if and only if $\phi(v)$ and $\phi(w)$ are adjacent. If $v=q_1\in l_\infty$ and $w=p+\mu q_2$ then $v$ and $w$ are adjacent

$$0=b_h(v,w)=\mu b(q_1,q_2)$$
$$=\mu b'(\phi(q_1),\phi(q_2))=b'_h(\phi(q_1),p'+\lambda\phi(q_2))$$

if and only if $\phi(v)$ and $\phi(w)$ are adjacent.
Every billinear form over $\mathbb{F}_q^3$ can be written canonically as $\frac{1}{2}(u_0v_1+u_1v_0)+\theta u_2v_2$ where $\theta$ is either one or a non-square (all non-square give equivalent forms). Let $b=\frac{1}{2}(u_0v_1+u_1v_0)+u_2v_2$, $p=[0,0,1]$ and $b'=\frac{1}{2}(u_0v_1+u_1v_0)+\theta u_2v_2$ (where $\theta$ is a non-square in $\mathbb{F}_q$), $p'=[0,0,1]$; we will prove that they define equivalent graphs. Consider the automorphism of $\mathbb{F}_q^3$ defined by

$$[x,y,z]\mapsto[\theta x,y,z].$$

Note that $\phi(p)=p'$, the classes $\{[hx,hy,1]\}_{h\in H}$ is sent to the class $\{[\theta hx,hy,1]\}_{h\in H}$ and finally $[\theta x,y,1]$ and $[\theta a,b,1]$ are adjacent and the second graph if and only if

$$0=\frac{1}{2}(\theta ax+\theta by)+\theta cz=$$
$$\theta\cdot b([x,y,1],[a,b,1])$$

which happens precisely when $[x,y,1]$ and $[a,b,1]$ are adjacent and the first graph. We have just prove that regardless of which symmetrical billinear form $b$ and which origin $p$ we select all graphs are the same.
In order to embed Livinsky graph $F(q,b,t)$ inside $\mathrm{Pol}(q,t)$ simply send the class

$$\langle u_0,u_1\rangle\mapsto\{[hu_0,hu_1,1]\}_{h\in H}$$

and consider the polarity on $PG(2,q^{t-1})$ defined by the point $p=[0,0,1]$ billinear form $\hat{b}=b((u_0,u_1),(v_0,v_1))-u_2v_2$. Then we have that for $u=\langle u_0,u_1\rangle$ and $v=\langle v_0,v_1\rangle$ are two vertices of $F(q,b,t)$ then they are adjacent if and only if

$$1=b(u,v)\iff\hat{b}([u_0,u_1,1],[v_0,v_1,1])=0.$$

**$K_{t,(t-1)!+1}$-freeness** Now we will prove that the graph $\mathrm{Pol}(q,t)$ contains no copy of $K_{t,(t-1)!+1}$. As Livinsky graph $F(q,t)$ is an induced subgraph covering all the classes $\{[hu_0,hu_1,1]\}_{h\in H}$ when we pick $b(u,v)=u_0v_0+u_1v_1-u_2v_2$ an the origin to be $p=[0,0,1]$. The neigbourhood of the origin $p$ is equal to the line at infinity $l_\infty=\{[a,b,0]\in PG(2,q^{t-1})\}$ and the neigbourhood of a point $q\in l_\infty$ is a line through the origin. Let $T$ be a subset of the vertices of $\mathrm{Pol}(q,t)$ with cardinality $t$ we will give bounds on the size of $N(T)$.

- If $T$ contains two points $q_1, q_2$ from the line $l_\infty$, then $N(T) \subseteq N(q_1) \cap N(q_2) = \{p\}$.
- If $T \cap l_\infty = \{q\}$. We have that $N(T)$ is contained in a line $l_q := N(q)$ through 0. Pick any other point $u$ in $T$ as $u$ is not in $l_\infty$ the neighbourhood of $u$ is a line $l_u$ not through 0 (in particular different from $l_q$). Therefore we obtain that $|N(T)| \leq |l_q \cap l_u| = 1$.
- In the final case if $T$ and $l_\infty$ are disjoint then since the graph spanned by the vertices outside $l_\infty$ is the Livinsky graph with an extra isolated vertex (the origin $p$) then $|N(T)| \leq (t-1)! + 1$.

By now we have establish that that:

- $H(q,t) \subseteq \Gamma(q,t) \cong F(q,b,t)$ when we take $b(u,v) = u_0 v_1 - u_1 v_0$.
- All the Livinsky graphs $F(q,b,t)$ are an induced subgraph of $\mathrm{Pol}(q,t)$.

# References


[1] Noga Alon, Lajos Rónyai, and Tibor Szabó. "Norm-Graphs: Variations and Applications". In: *Journal of Combinatorial Theory, Series B* 76.2 (1999), pp. 280–290. ISSN: 0095-8956. DOI: https://doi.org/10.1006/jctb.1999.1906. URL: https://www.sciencedirect.com/science/article/pii/S0095895699919068.

[2] Simeon Ball and Valentina Pepe. "Forbidden Subgraphs in the Norm Graph". In: *Discrete Math.* 339.4 (Apr. 2016), pp. 1206–1211. ISSN: 0012-365X. DOI: 10.1016/j.disc.2015.11.010. URL: https://doi.org/10.1016/j.disc.2015.11.010.

[3] Zoltán Füredi. "New asymptotics for bipartite Turán numbers". In: *Journal of Combinatorial Theory, Series A* 75.1 (1996), pp. 141–144.

[4] Zoltán Füredi. "On the Number of Edges of Quadrilateral-Free Graphs". In: *J. Comb. Theory, Ser. B* 68.1 (1996), pp. 1–6. DOI: 10.1006/jctb.1996.0052. URL: https://doi.org/10.1006/jctb.1996.0052.

[5] János Kollár, Lajos Rónyai, and Tibor Szabó. "Norm-graphs and bipartite Turán numbers". In: *Combinatorica* 16.3 (1996), pp. 399–406.

[6] P Kővári, Vera T Sós, and Pál Turán. "On a problem of Zarankiewicz". In: *Colloquium Mathematicum.* Vol. 3. Polska Akademia Nauk. 1954, pp. 50–57.

[7] Ivan Livinsky. *A construction for bipatite Turán numbers.* 2021. arXiv: 2101.06726 [math.CO].