CSCI 3104: Algorithms

Lecture 16: Max Flows, Min Cuts

Rachel Cox

Department of Computer Science

What will we learn today?

- ❑ Maximum Flow, Minimum Cut
- ❑ Ford-Fulkerson Algorithm

Intro to Algorithms, CLRS:
Sections 26.1, 26.2, 26.3

Acknowledgements & Citations

- Material, examples, and definitions taken from CSCI 3104 Lecture 18 & 19 by Sam Molnar and Princeton University

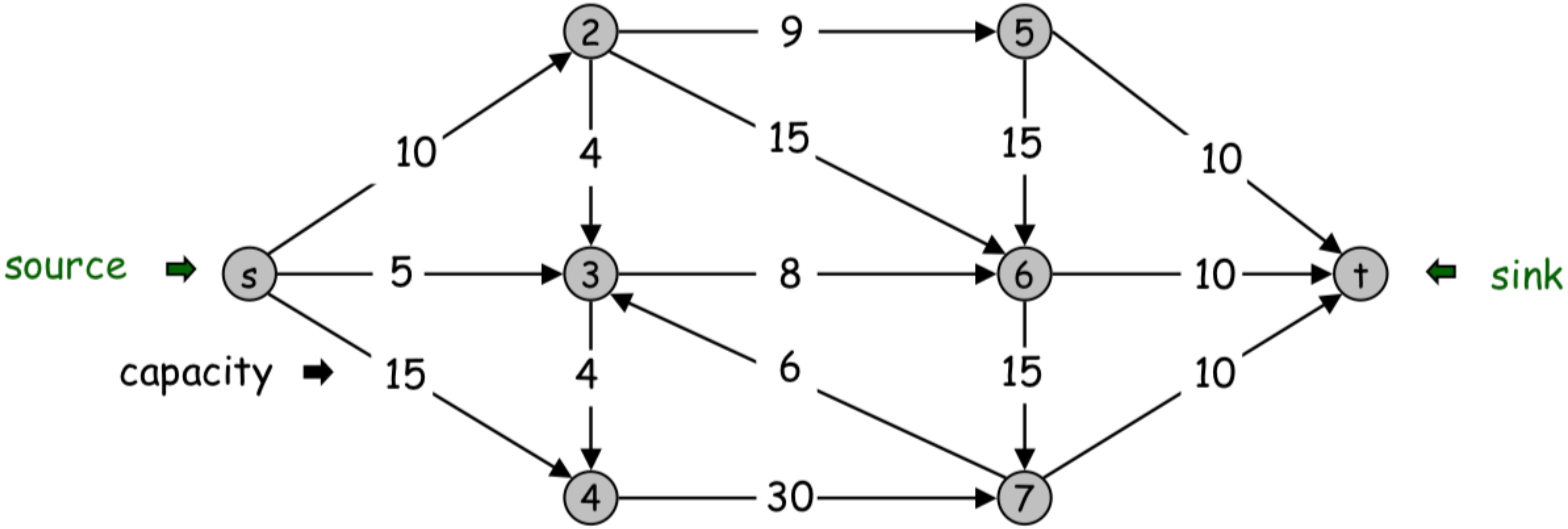
Flow Networks

Given a Graph $G = (V, E)$.
The edges represent "capacities"

flow - the amount of capacity being used on the edge.

source - where flow originates – only having outgoing edges

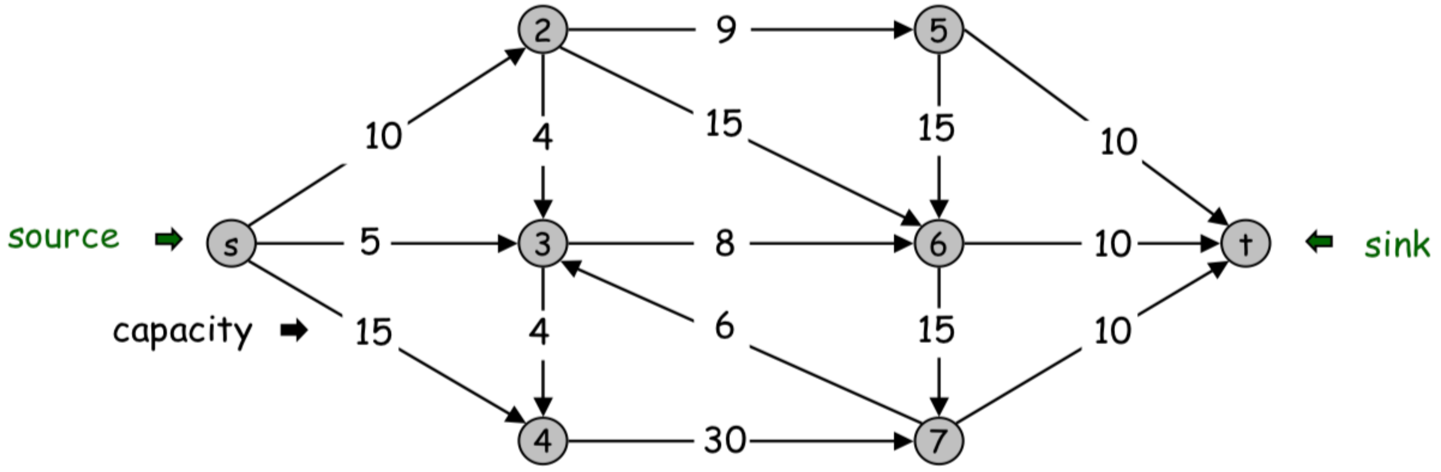
sink - where flow terminates – only having incoming edges



Flow Networks

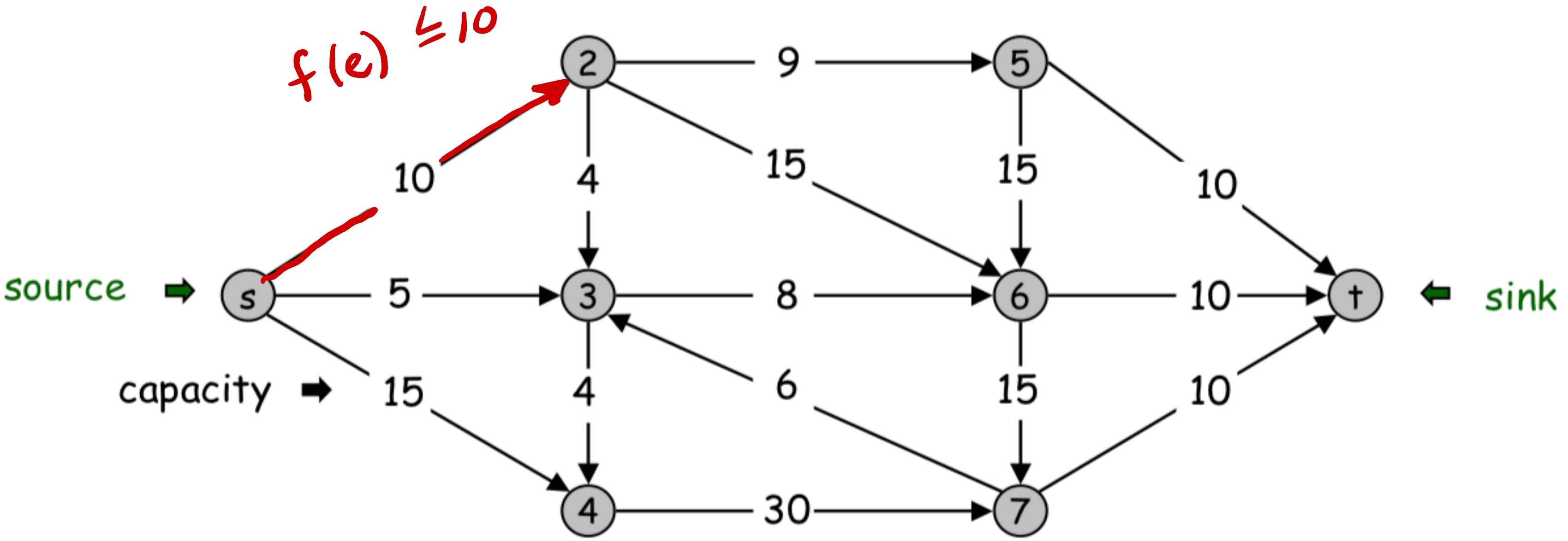
set of nodes

- A flow network is a directed graph $G = (V, E)$ set of edges
- Each edge has non-negative integer capacity, c_e e.g. $c_{(2,5)} = 9$
- There is a single source vertex, $s \in V$
- There is a single sink vertex, $t \in V$
- Vertices that are not source or sink are internal vertices.
- No edge enters the source
- No edge leaves the sink
- Graph is connected



Maximum Flow

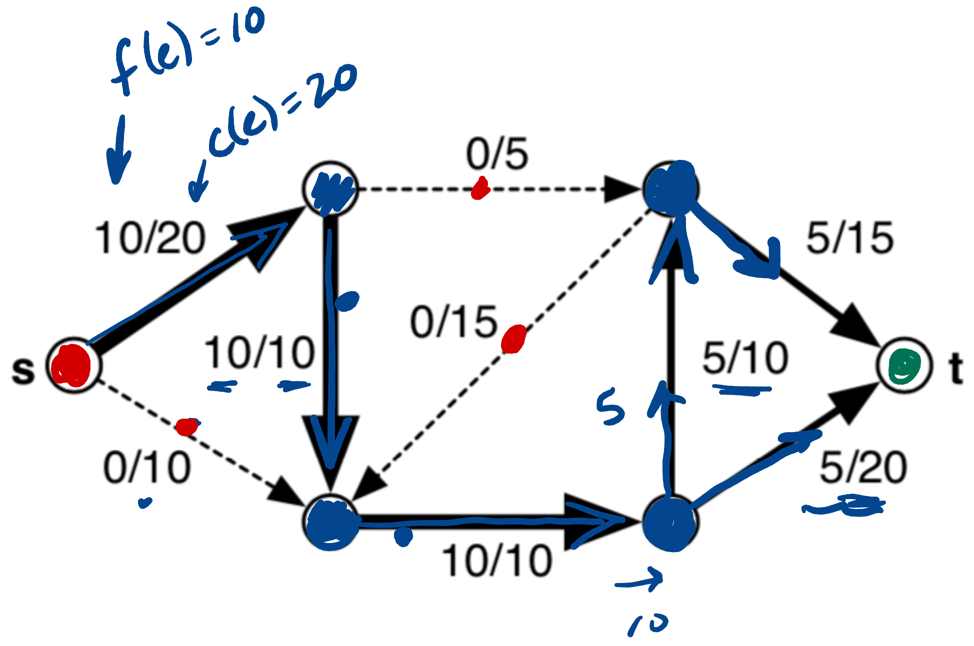
The maximum flow problem: Given a graph G, capacity function c, and choices s, t, compute a feasible $(s,t) - flow$ on G whose value $|f|$ is maximized.



A flow is ***feasible*** with respect to the edge capacities c if for every edge $e \in E$, the capacity of e is not exceeded by the flow on e; $f(e) \leq c(e)$

Max_Flow Min_Cut

- If $f(e) = c(e)$, then we say that f ***saturates*** the edge e.
- If $f(e) = 0$, then we say that f ***avoids*** the edge e



conservation of flow

flow in = flow out

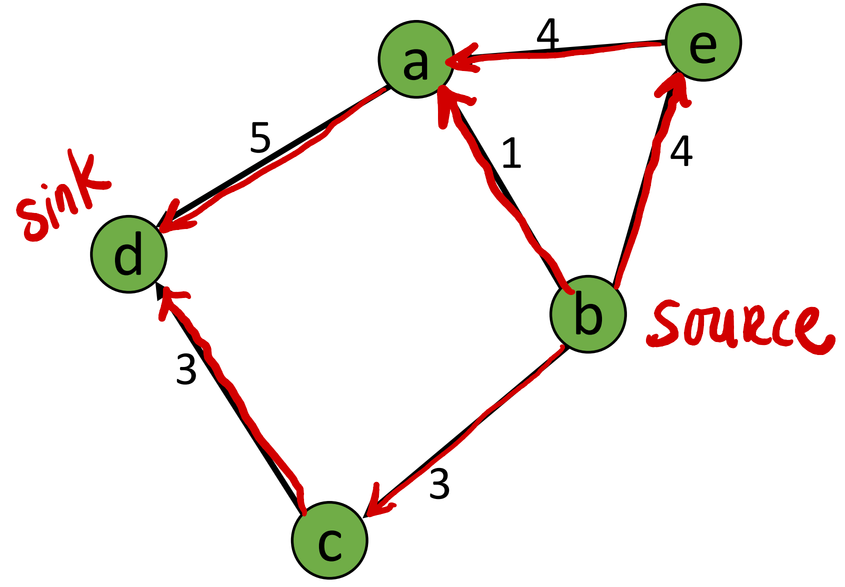
Max Flow – Min Cuts

Flow is conserved for each $v \in V$, other than s, t.

$$\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$$

- The flow into a vertex = the flow out of a vertex

· here we saturate every edge + maintaining conservation of flow and feasibility



Max Flow – Min Cuts

Example: What is the maximum feasible flow from source to target.

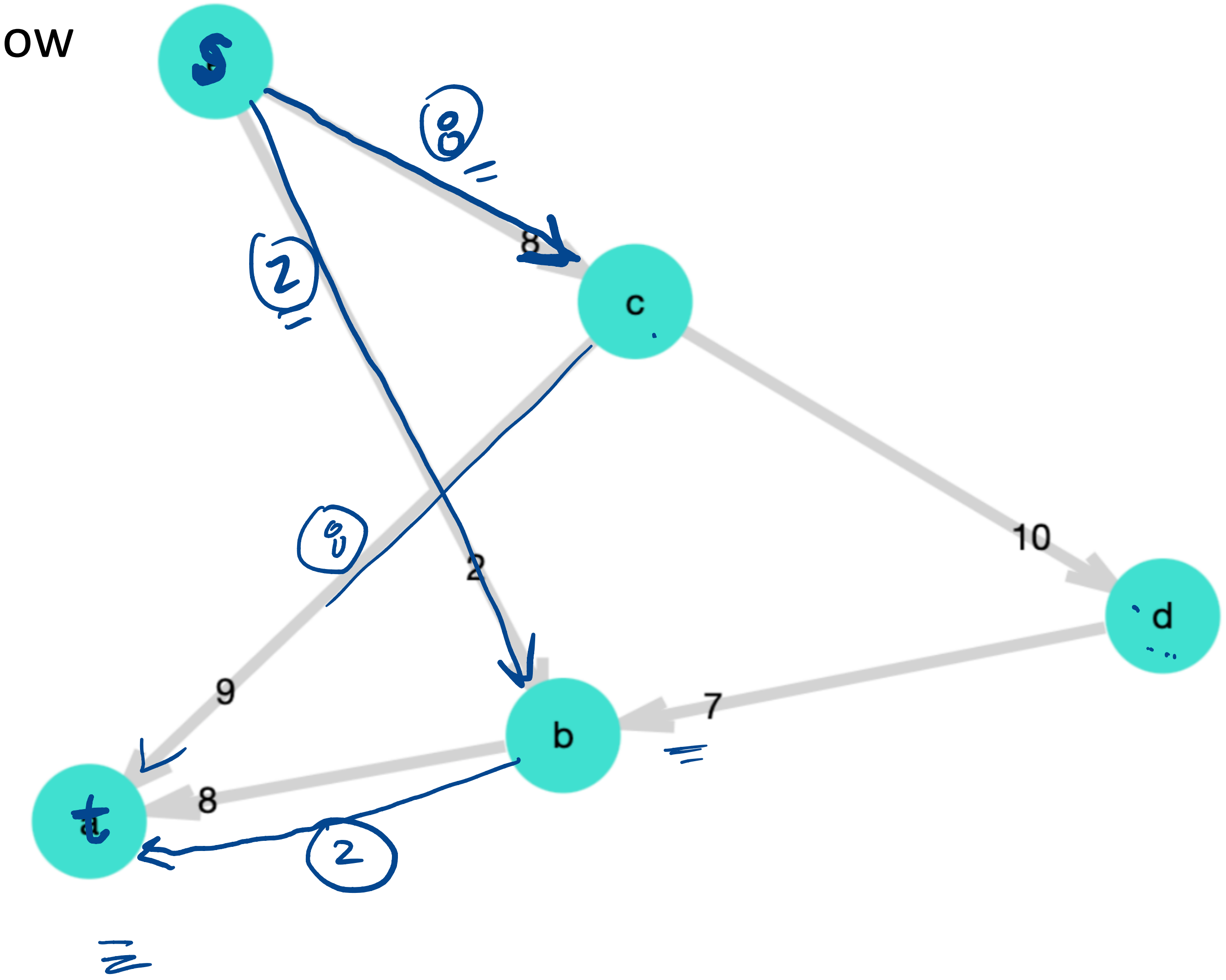
10

sent 8 units of flow

s → c → t

sent 2 units of flow

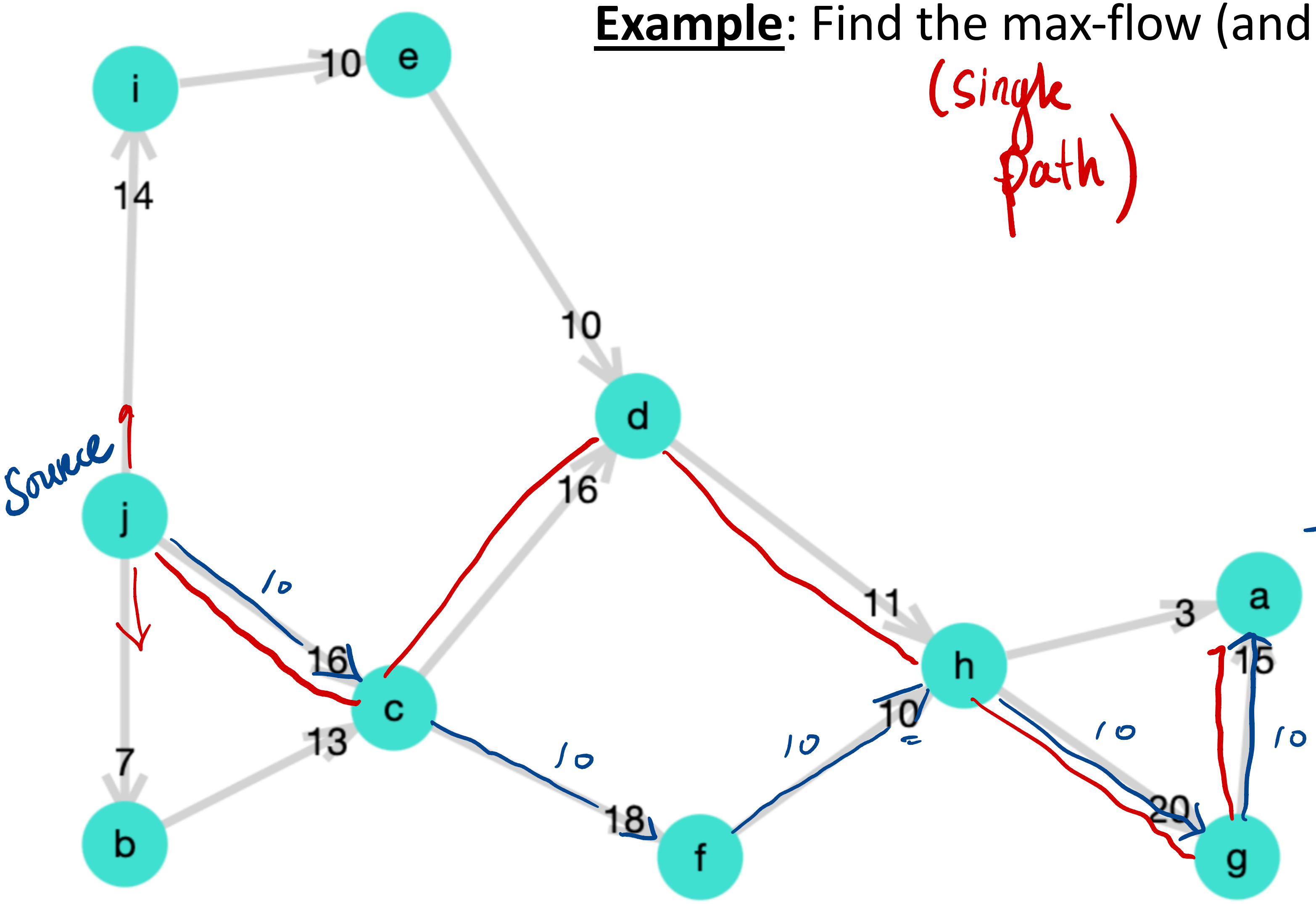
s → b → t



Max Flow – Min Cuts

Example: Find the max-flow (and the path) for the following graph.

(Single path)



- one possibility

$j \to c \to f \to h \to g \to a$

min capacity = 10

10

- $j \to c \to d \to h \to g \to a$

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Max Flow – Min Cuts

A cut is a node partition (S, T) such that s is in S and t is in T.

- capacity(S, T) = sum of weights of edges leaving S.

(s,t) - cuts

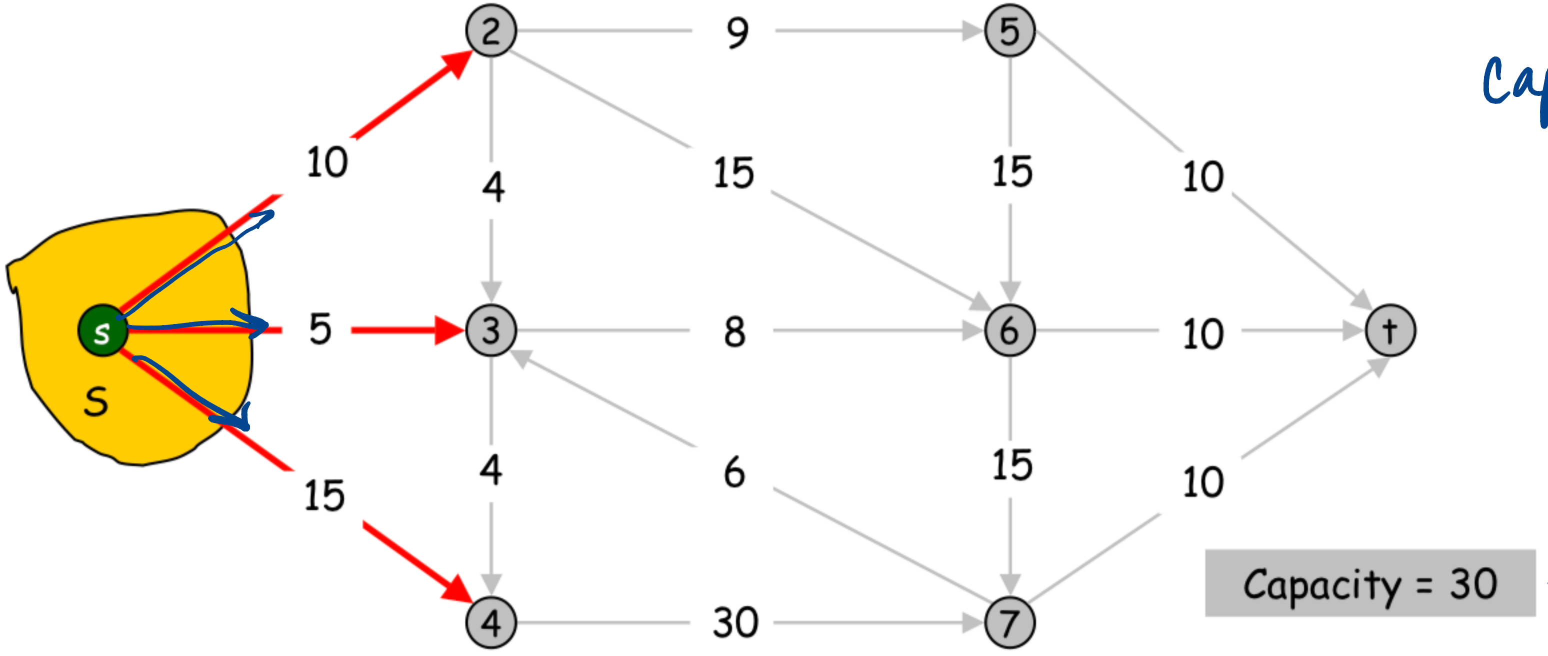
source in S (aka A)

target/sink in T (aka B)

$S = \{s\}$

$T = \{2,3,4,5,6,7,t\}$

$\boxed{S \cup T = V}$



capacity of (s,t)-cut

$= 10 + 5 + 15$

$\boxed{= 30}$

Max Flow – Min Cuts

A cut is a node partition (S, T) such that s is in S and t is in T.

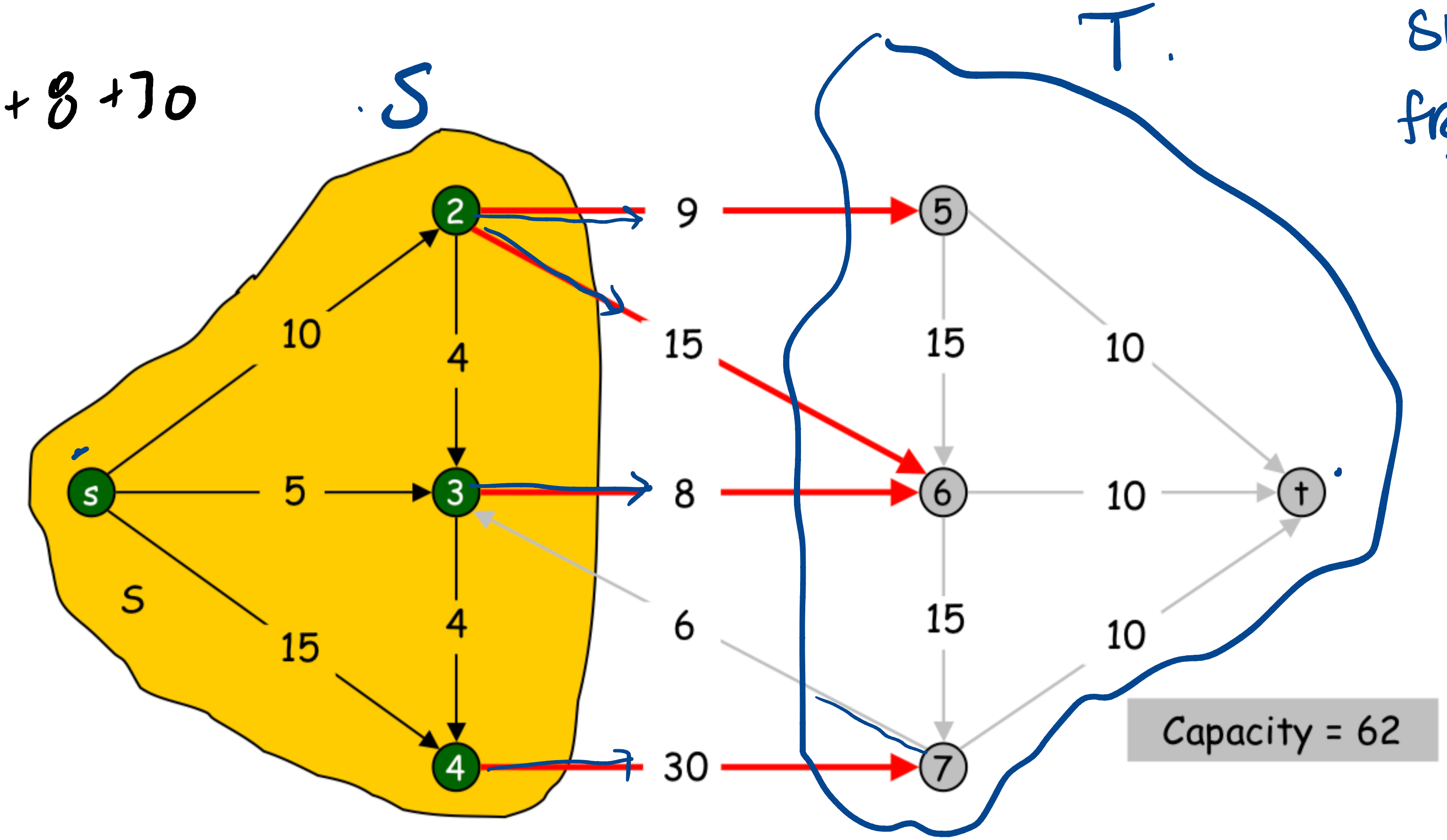
- capacity(S, T) = sum of weights of edges leaving S.

capacity of cut
=
sum of edges
from S to T
= 9 + 15 + 8 + 30
= 62
= ||S, T||

Notation

$||S, T|| = 9 + 15 + 8 + 30$
$= 62$

$||T, S|| = 6$



Max Flow – Min Cuts

A cut is a node partition (S, T) such that s is in S and t is in T.

- capacity(S, T) = sum of weights of edges leaving S.

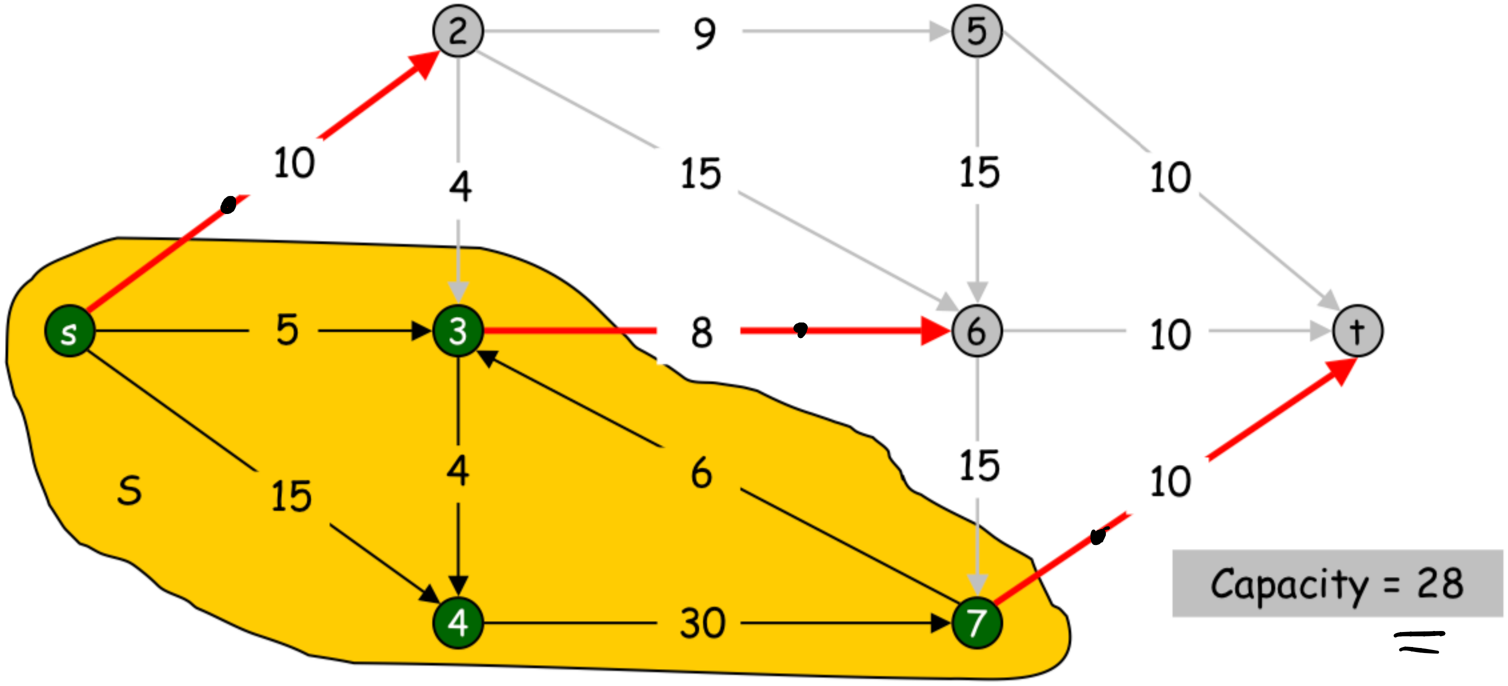
Min cut problem. Find an s-t cut of minimum capacity.

$\|S,T\| = 10 + 8 + 10$

$= 28$

$S = \{s, 3, 4, 7\}$

$T = \{2, 5, 6, t\}$



Max Flow – Min Cuts

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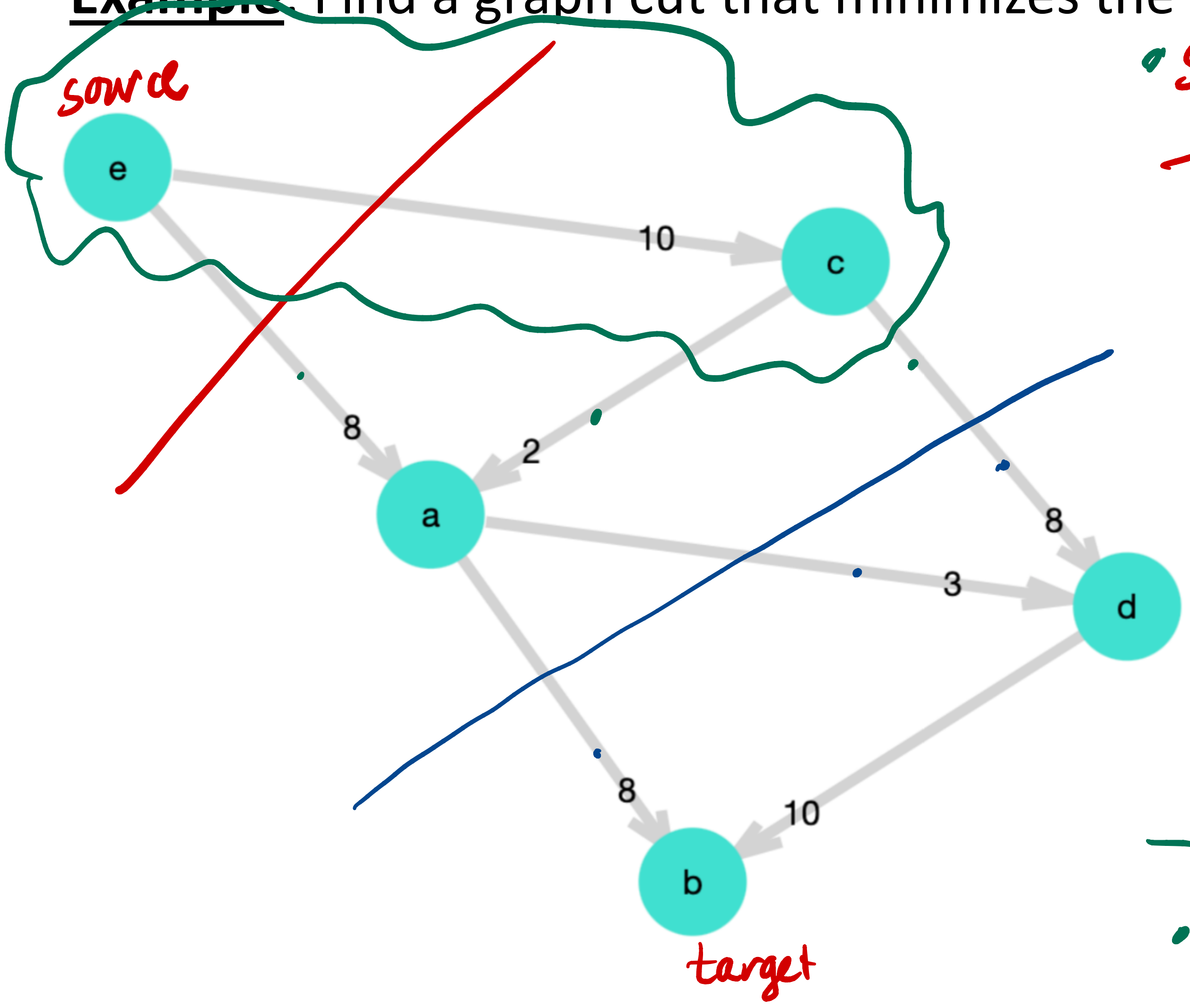
Example: Find a graph cut that minimizes the capacity of the cut.



- $S = \{e\}$
  $T = \{a, c, b, d\}$
  capacity of cut $= 8 + 10 = 18$

- $S = \{a, c, e\}$
  $T = \{d, b\}$
  Capacity of cut $= 8 + 3 + 8 = 19$

- $S = \{e, c\}$ $T = \{a, d, b\}$
  capacity of cut $= 8 + 2 + 8 = 18$

Max Flow – Min Cuts

"pushing flow" forward - push forward on edges that have remaining capacity

"pushing flow" backward - push backward on edges already carrying flow to direct to a different edge

- Need a new graph to represent forward and backward edges.
- Graph of capacities is not sufficient by itself to solve problem

Residual graph

helpful to find max flow

Max Flow – Min Cuts

Residual capacity = 2

Residual Flow Graph

Given a flow network G and a feasible flow f the residual flow graph G_f is defined as:

- The vertex set of G_f is the same as G
- For each edge $e = (u, v)$ of G where $f(e) < c(e)$, there are $c_e - f(e)$ remaining units of capacity that can handle additional flow units (residual capacity)
  - These edge $e = (u, v) \in G_f$ are called forward edges with capacity $c_e - f(e)$
- For each edge $e = (u, v) \in G$, where $f(e) > 0$, there are $f(e)$ that we can push backward
  - In G_f, we have edges $e' = (v, u)$ with capacity $f(e)$. Direction is reversed from G, the flow is the capacity.

residual

amount currently being put down edge

Max Flow – Min Cuts

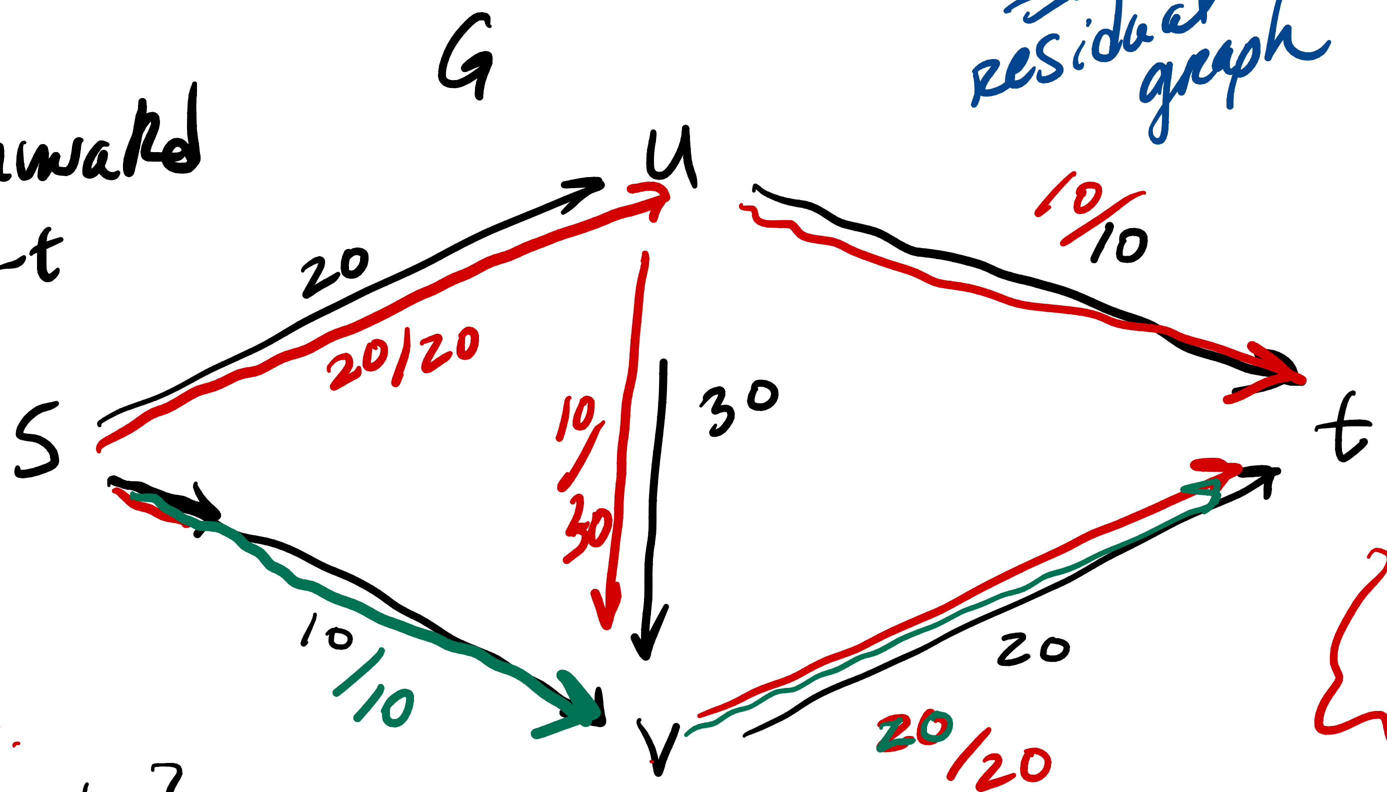
~~Residual~~ Flow Graph

flow graph

. not residual graph

Residual graph

- helps us visualize where we can send flow



· push flow forward along s-u-v-t path

$f(e) = 20$

↑ this is valid, but not max.

max flow = 30

is this valid ?

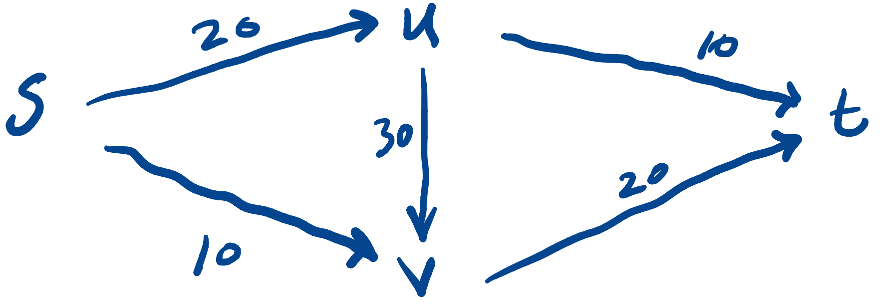
· $f(e) \leq c(e)$ ✓

· conservation holds ✓

Max Flow – Min Cuts

Residual Flow Graph

G



* Suppose we push 30

$s-u = 20$

$s-v = 10$

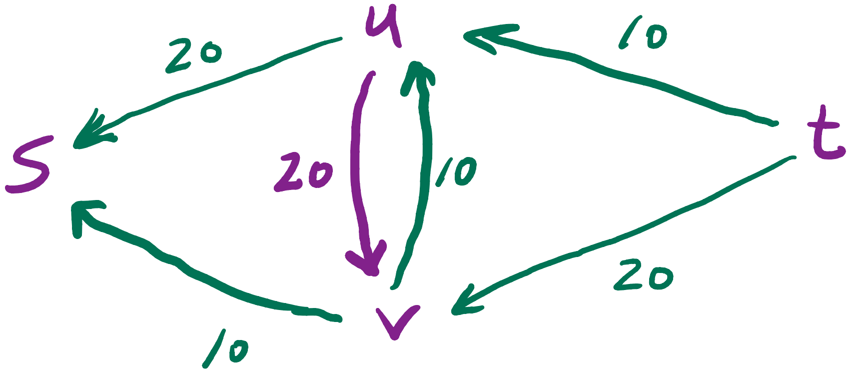
$u-v = 10$ } Residual capacity = 20

$v-t = 20$

$u-t = 10$

- This is what residual graph looks like!

G_f



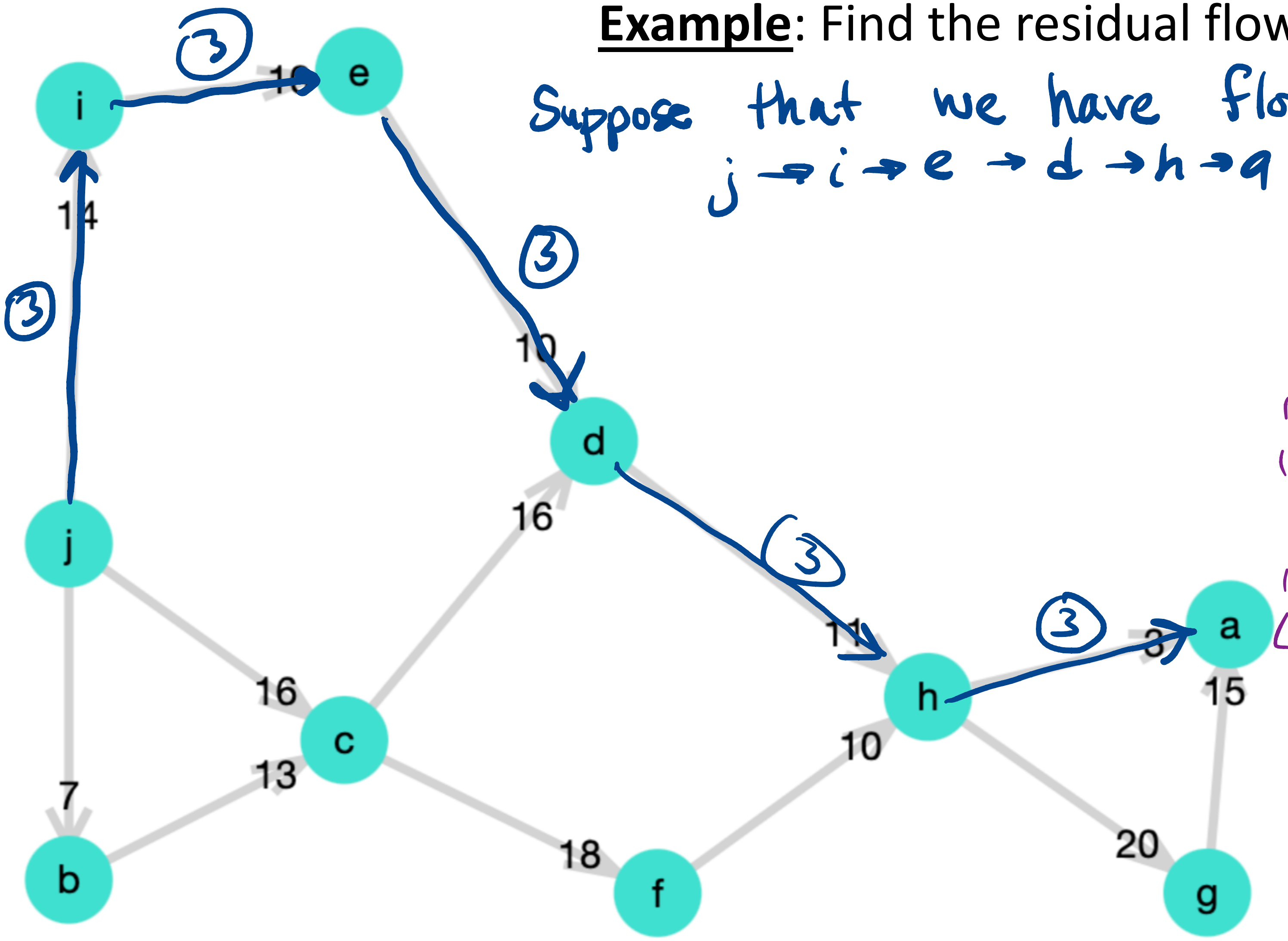
(interesting note)

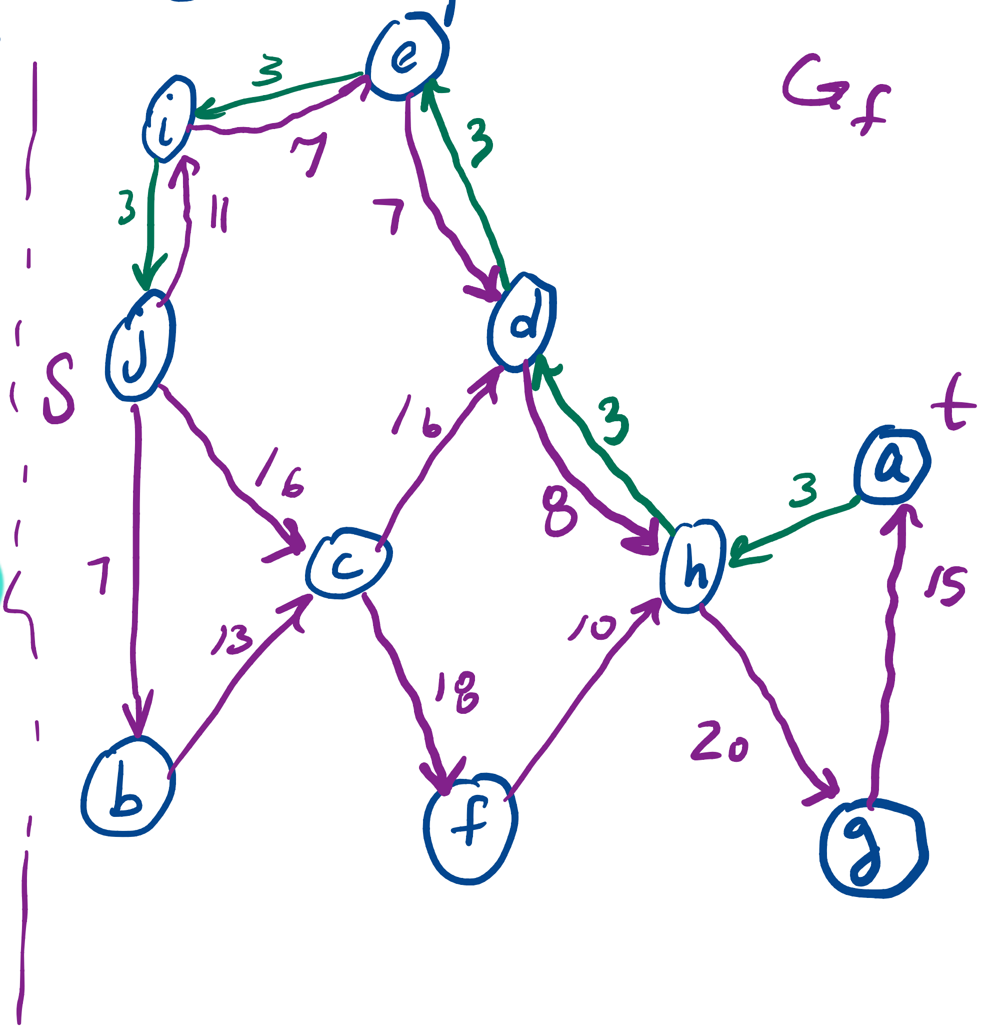
- no simple path from $s \to t$ on residual graph, we've achieved max flow!

Max Flow – Min Cuts

Example: Find the residual flow graph for the following graph.

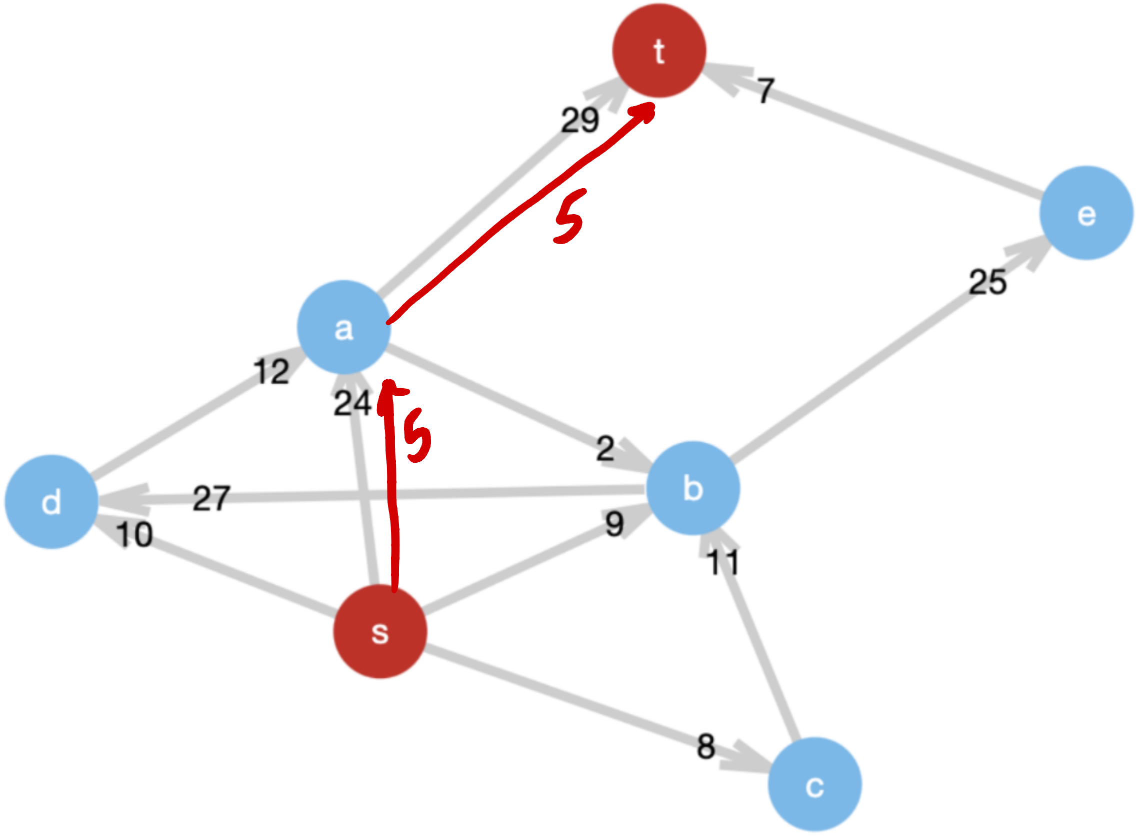
Suppose that we have flow = 3 on path
$j \to i \to e \to d \to h \to a$

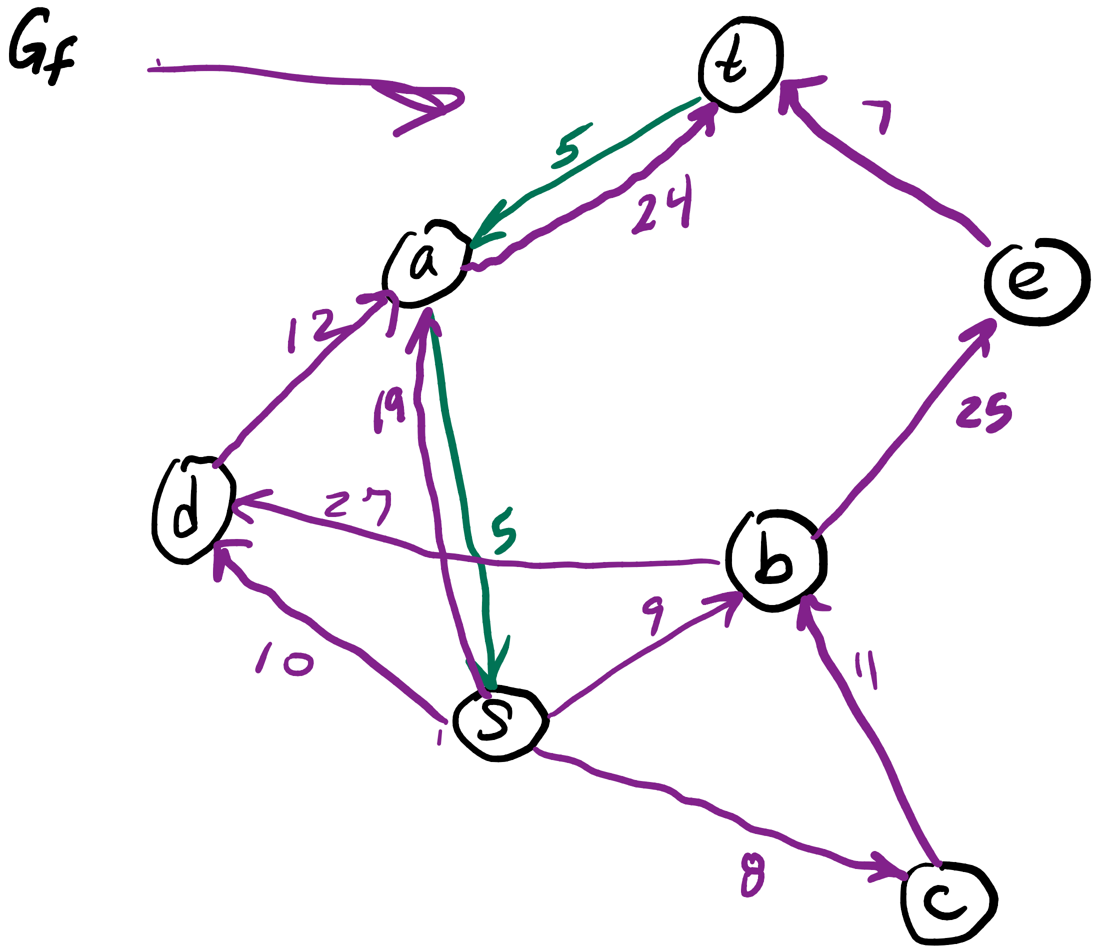




Max Flow – Min Cuts

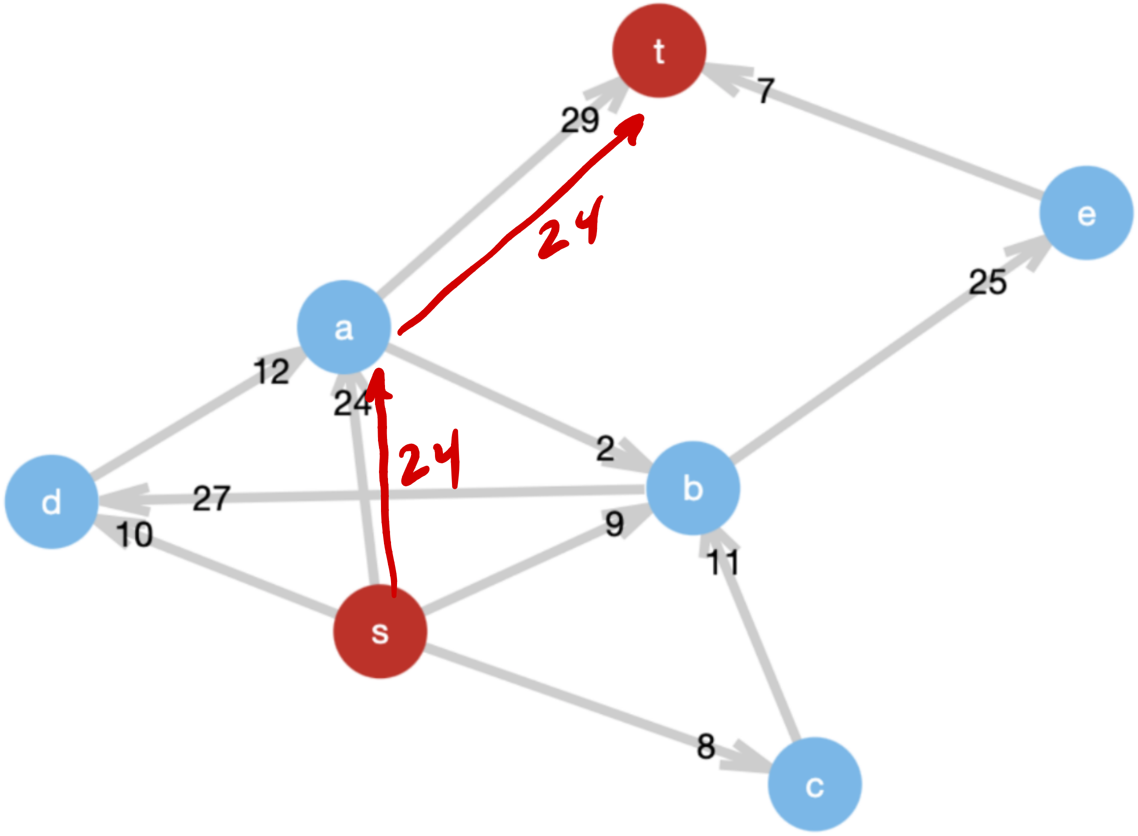
Example: Assume 5 units of flow on the path $s \to a \to t$. Draw the residual graph.

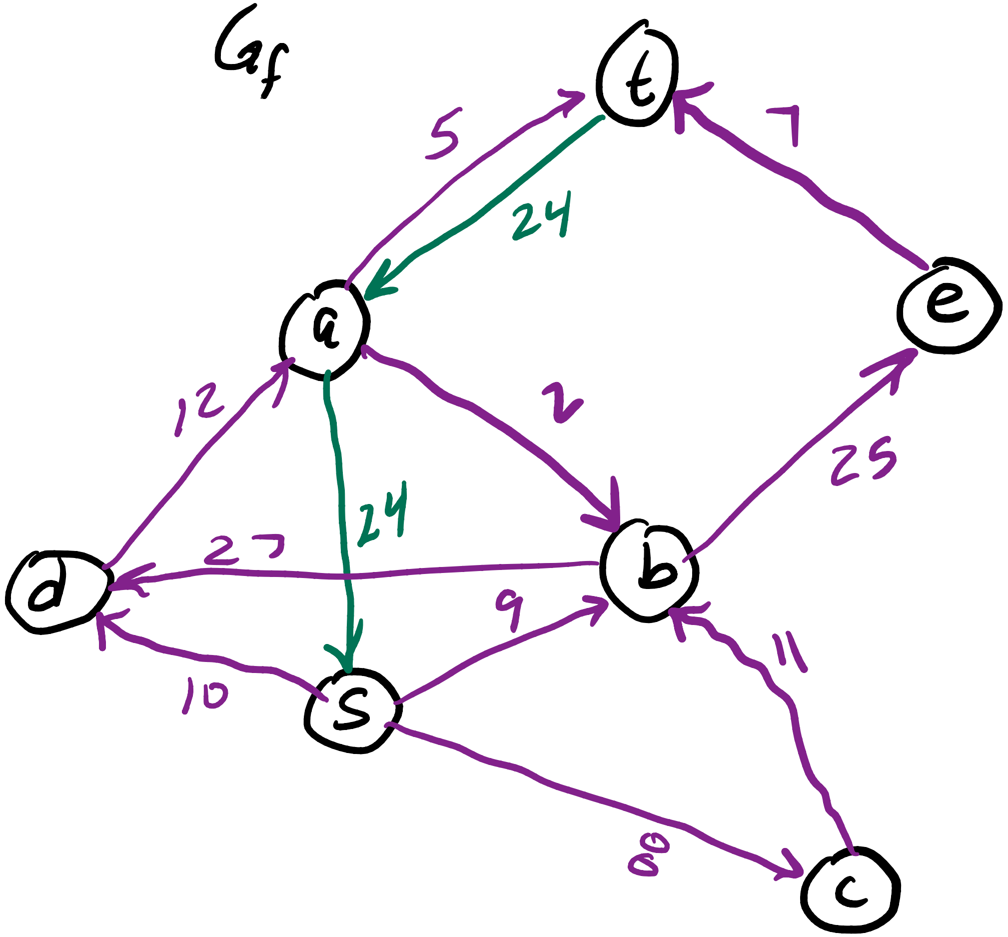




Max Flow – Min Cuts

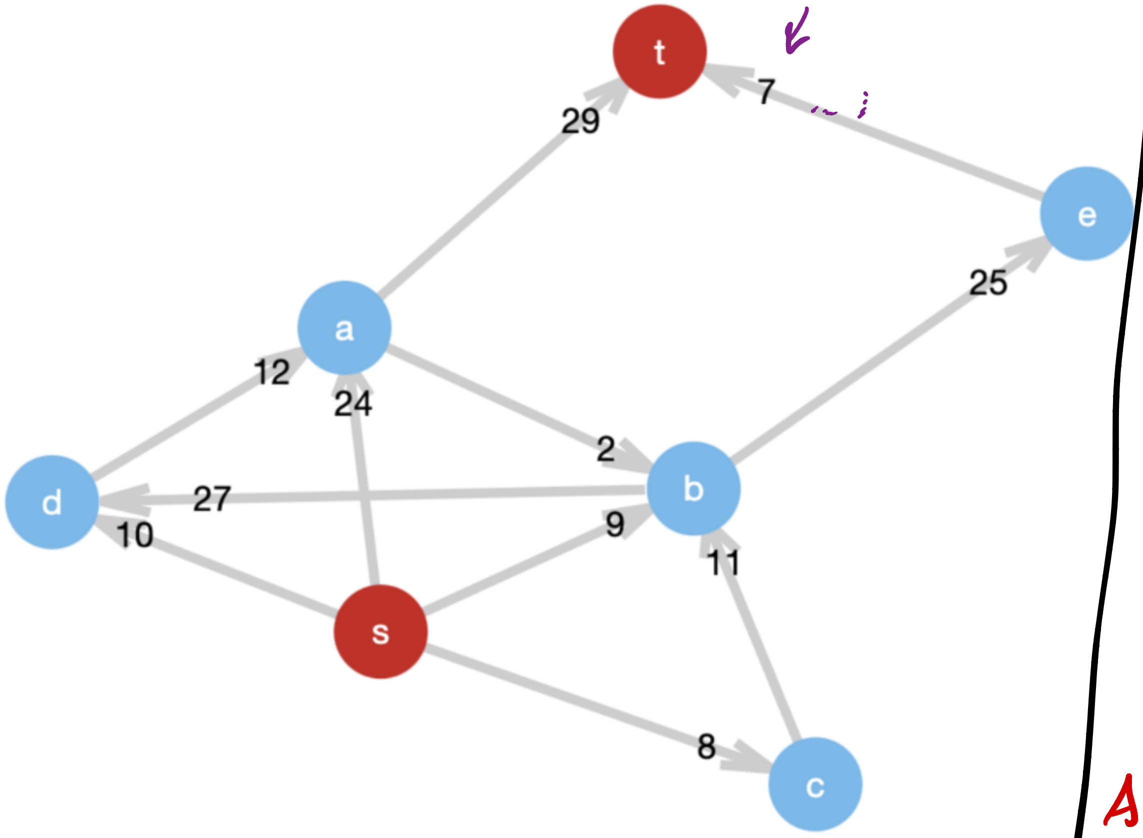
Example: Assume 24 units of flow on the path $s \to a \to t$. Draw the residual graph.

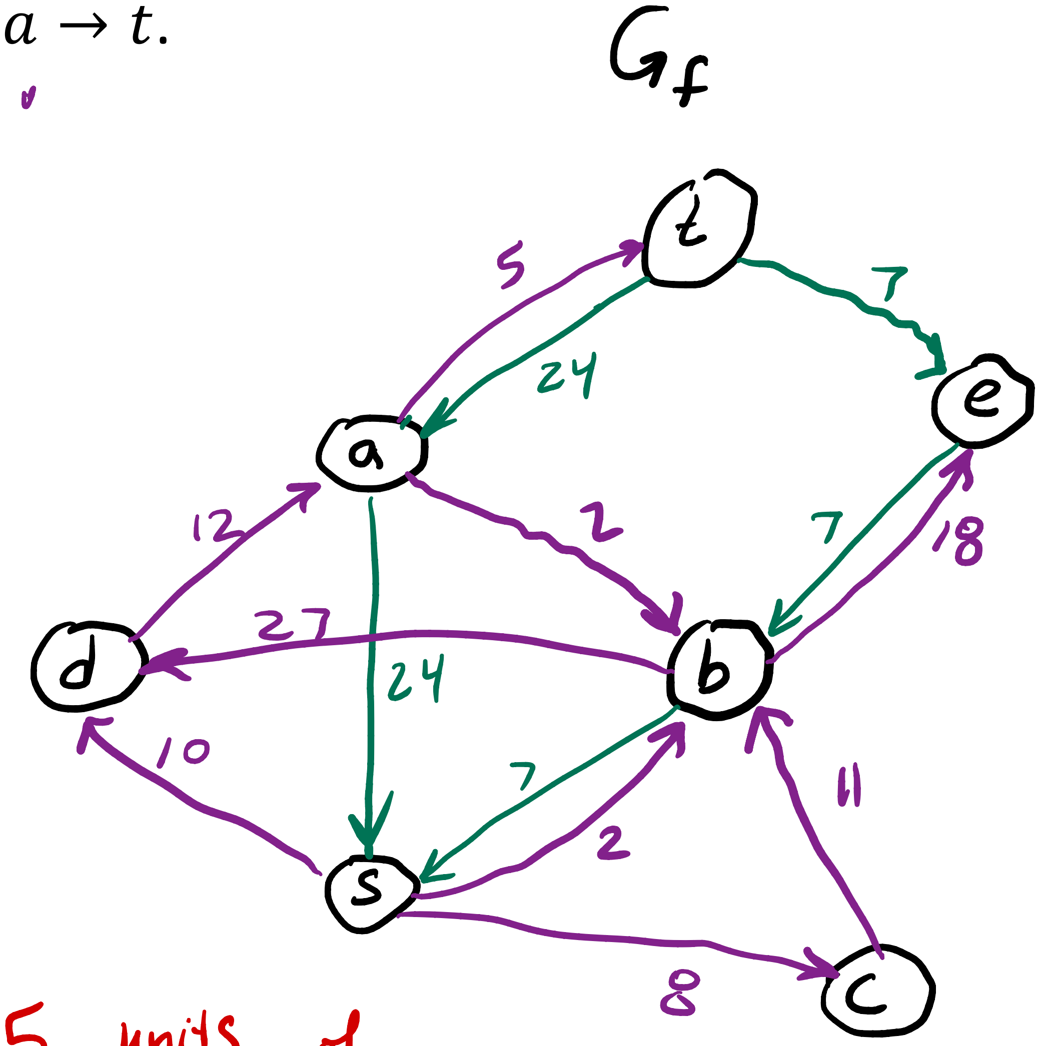




Max Flow – Min Cuts

Example: Assume 24 units of flow on the path $s \to a \to t$.
Assume 7 units of flow on the path $s \to b \to e \to t$
Draw the residual graph.





Add 5 units of flow on $s \to d \to a \to t$

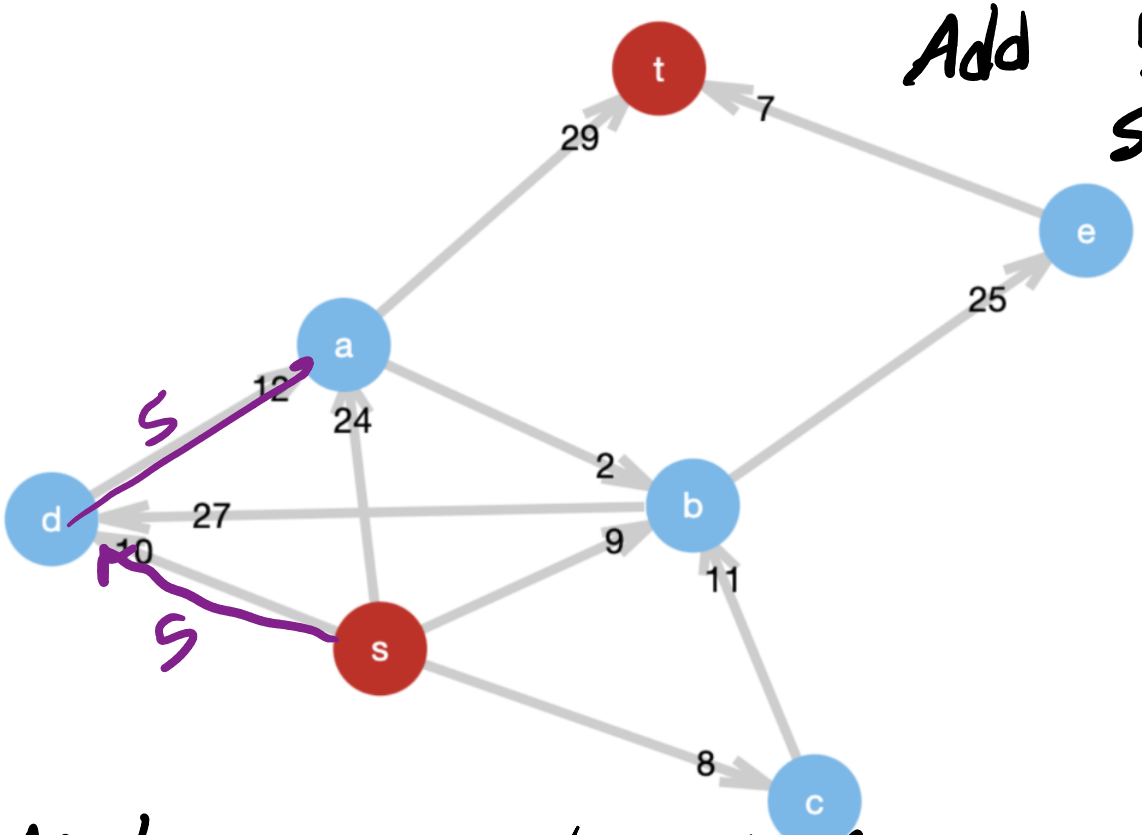
update G_f

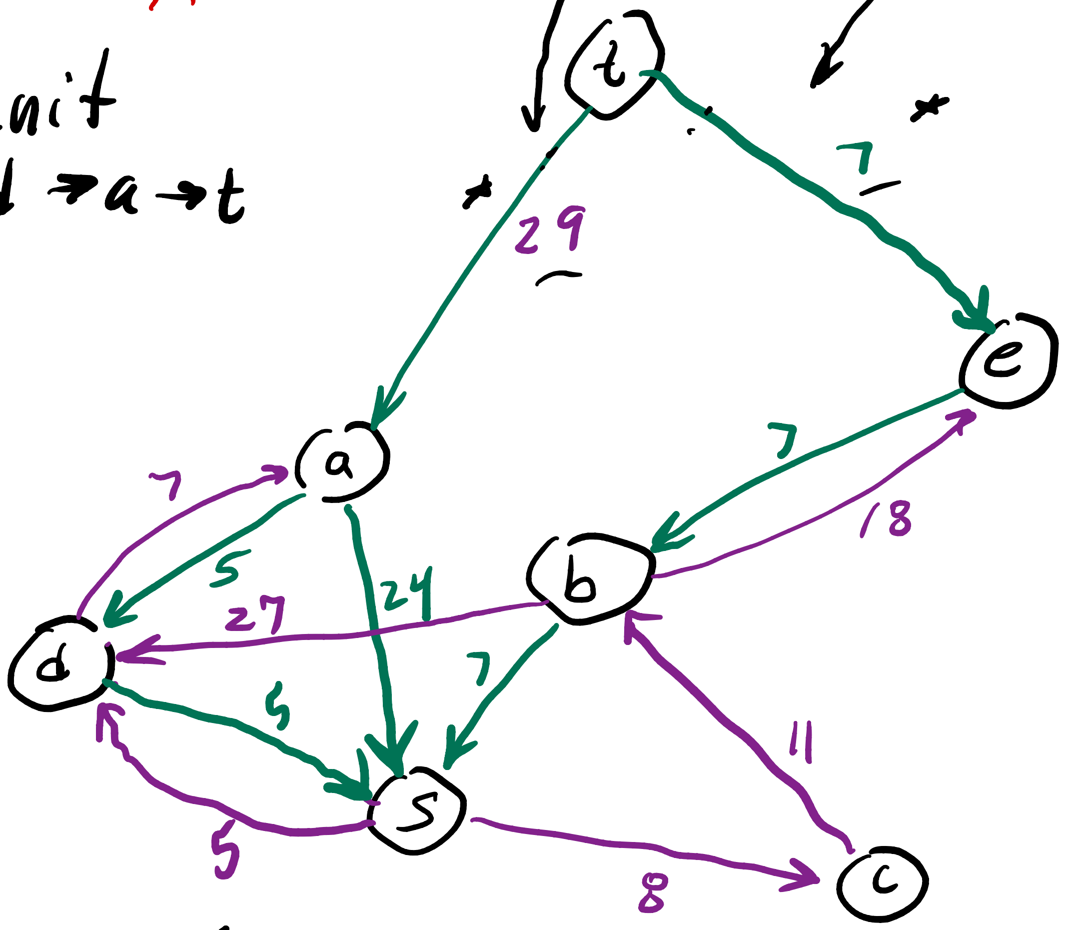
Max Flow – Min Cuts

Example: Assume 24 units of flow on the path $s \to a \to t$.
Assume 7 units of flow on the path $s \to b \to e \to t$
Draw the residual graph.

s→d→a→t
Add 5

Add 5 unit
s→d→a→t





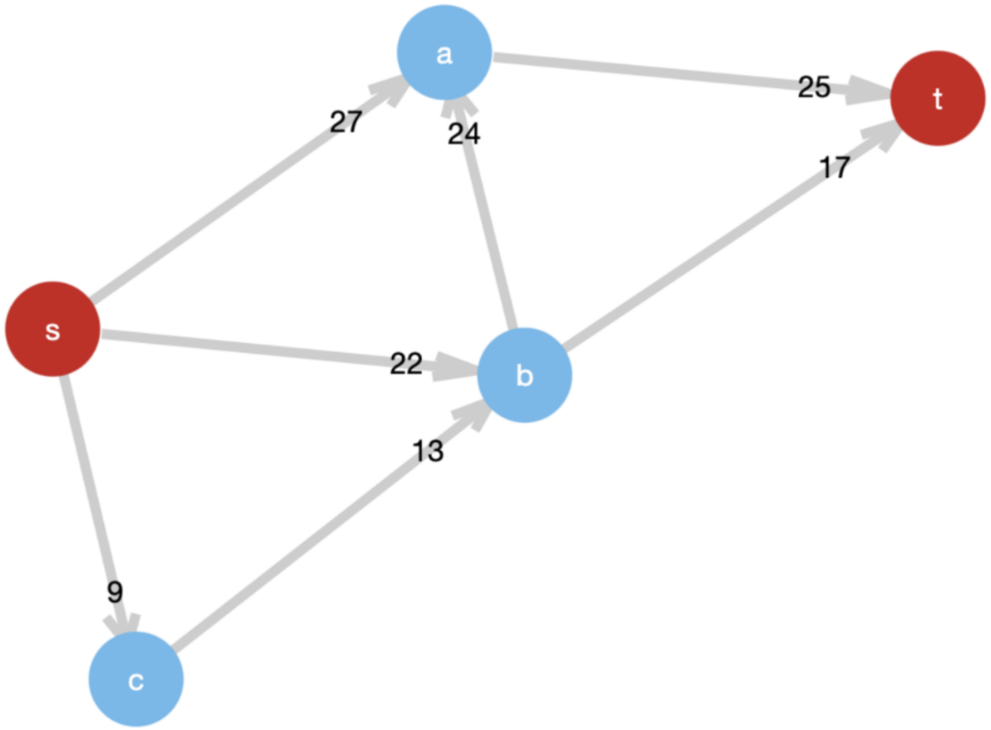
• No longer a simple path from s→t in Gf. ⇒ achieved max flow!

max flow = sum of out edges of t in Gf

Ford–Fulkerson

FORD-FULKERSON-METHOD(G, s, t)

```
1  initialize flow f to 0
2  while there exists an augmenting path p in the residual network G_f
3      augment flow f along p
4  return f
```



- Every edge of G with edge e having a capacity $c(e) - f(e)$
- For every edge $(u, v) \in G$ with $f(u, v) > 0$, the residual graph has edge (v, u) with capacity $f(u, v)$

Ford–Fulkerson

FORD-FULKERSON-METHOD(G, s, t)

```
initialize flow f to 0
while there exists an augmenting path p in the residual network G_f
    augment flow f along p
return f
```