This assignment is to help you prepare for questions on the new material on the Final Exam. It is NOT TO BE TURNED IN! Solutions will be posted on Piazza closer to the actual final.

1. Determine whether each of the following relations $R \subseteq A \times A$, where A is the set of all CU students, is reflexive, symmetric, transitive, and/or an equivalence relation. Briefly justify each conclusion.

   (a) $(a, b) \in R$ if and only if a shares at least one class with b.

   (b) $(a, b) \in R$ if and only if a has a higher GPA than b.

   (c) $(a, b) \in R$ if and only if a is roommates with b.

   Solution:

   (a) **Reflexive**: The relation **is** reflexive. To see this note that $(a, a) \in R$ since a shares at least one class with herself (in fact, she shares all of her classes with herself).

   Symmetric: The relation **is** symmetric. To see this note that if $(a, b) \in R$, meaning that a shares at least one class with b then certainly b shares at least one class with a.

   Transitive: The relation **is NOT** transitive. As a counterexample, suppose that a and b are both in Discrete Structures, and b and c are both in Data Science, this does not guarantee that a and c share any classes.

   Equivalence Relation: The relation **is NOT** an equivalence relation because it is not transitive.

   (b) **Reflexive**: The relation **is NOT** reflexive. To see this note that a person cannot have a higher GPA than themselves, thus $(a, a) \notin R$.

   Symmetric: The relation **is NOT** symmetric. To see this note that if a has a higher GPA than b than necessarily b has a lower GPA than a. Thus if $(a, b) \in R$ then $(b, a) \notin R$.

   Transitive: The relation **is** transitive. To see this note that if a has a higher GPA than b and b has a higher GPA than c, then a has a higher GPA than c. Thus if $(a, b) \in R$ and $(b, c) \in R$ it must be true that $(a, c) \in R$.

   Equivalence Relation: The relation **is NOT** an equivalence relation because it is neither reflexive nor symmetric.

   (c) In this problem we will assume that *roommates with* is synonymous with *lives in the same home as*.

   Reflexive: The relation **is** reflexive. To see this note that $(a, a) \in R$ since a lives in the same place as herself.

   Symmetric: The relation **is** symmetric. To see this note that if $(a, b) \in R$, meaning that a is roommates with b then certainly b is roommates with a.

   Transitive: The relation **is** transitive. To see this note that if a is roommates with b and b is roommates with c then a must be roommates with c as well.

   Equivalence Relation: The relation **is** an equivalence relation because it is reflexive, symmetric, and transitive.

2. Consider the relation $R = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (2, 0), (2, 1), (2, 2), (3, 3)\}$, where $R \subseteq A \times A$, with $A = \{0, 1, 2, 3\}$.

(a) Draw the graph of R. **Note**: If possible, it is good practice to organize your graph such that all directed edges are non-intersecting.

(b) Is the relation R reflexive? Symmetric? Transitive? An equivalence relation? Fully justify your responses.

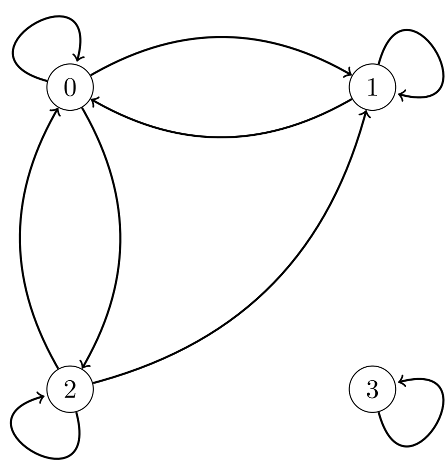
(c) The **complement** of a relation $R \subseteq A \times A$ is defined as $\overline{R} = (A \times A) - R$.

i. What is the set $\overline{R}$ for R as defined in this problem?

ii. Draw the graph of $\overline{R}$. Again, organize your graph such that all directed edges are non-intersecting.

iii. Is the following statement true or false? Briefly justify your conclusion. "A relation R is symmetric if and only if its complement $\overline{R}$ is symmetric."

Solution:



(a)

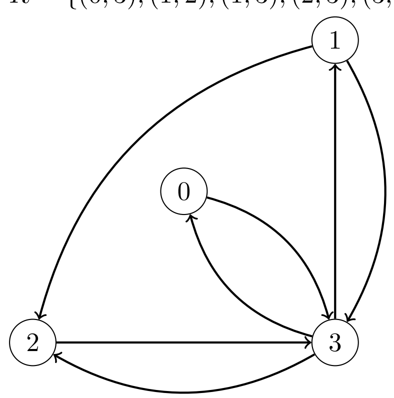
(b) **Reflexive**: The relation is reflexive because each vertex in its graph has a self-loop.

Symmetric: The relation is **NOT** symmetric because there is a directed edge from 2 to 1 but there is not a directed edge from 1 to 2.

Transitive: The relation is **NOT** transitive. This is because there is an edge from 1 to 0 and from 0 to 2, but there is not an edge from 1 directly to 2.

Equivalence Relation: The relation is **NOT** an equivalence relation because it is not symmetric.

(c) i. $\overline{R} = \{(0,3),(1,2),(1,3),(2,3),(3,0),(3,1),(3,2)\}$



ii.

iii. The statement is **true**. If R is not symmetric, then there is some pair of vertices, a and b, such that there is an edge from a to b but not from b to a. This means that in the graph of $\overline{R}$ there will be an edge from b to a but not from a to b. Thus $\overline{R}$ is not symmetric either. Since this argument works in the reverse direction as well, we conclude that the statement is true.

3. For each of the degree sequences shown below, determine whether they represent a valid undirected graph with no self-loops. If they do, draw the graph. If they do not, explain why.

   (a) $4, 3, 2, 1, 0$

   (b) $2, 2, 2, 2, 2$

   (c) $1, 1, 1, 1, 1$

   (d) $4, 4, 3, 2, 1$

   Solution:

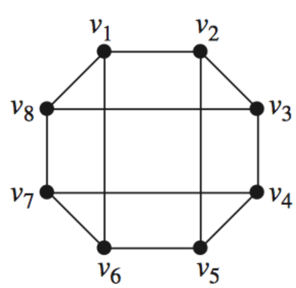
   (a) This is not a valid graph because the last vertex has degree zero (meaning it's not connected to anyone) but the first vertex has degree four, which means it's connected to each other vertex. This is a contradiction. Additionally, we proved in class that there must always be two vertices with the same degree, which is not the case in this degree sequence.

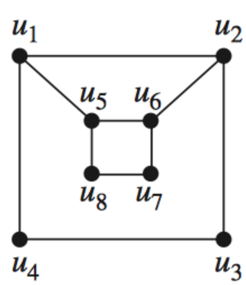
   (b) This is a valid graph. Arrange the vertices in a circle and connect each vertex to it's two closest neighbors.

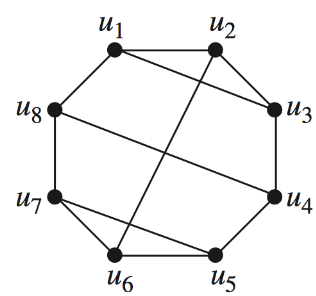
   (c) This is not a valid graph. Notice that the degree sequence sums to 5, which is odd.

   (d) This is not a valid graph. Notice that vertices v_1 and v_2 have degree 4. This means that they must each be connected to every other vertex in the graph, including v_5. But v_5 only has degree 1, which is a contradiction.

The next 2 questions involve the undirected graphs below.







4. For each of the graphs above, determine if the graph has an Eulerian Tour. If it does, give one such tour. If it does not, explain why. For graphs that do not contain Eulerian tours, can you **add** a small number of edges so that they do contain one?

   Solution: Note that determining the specific Eulerian Tours in this problem was pretty hard. If you are asked to do this on the exam it will be easier.

   (a) The first graph does not have an Eurlerian Tour as it is, because every vertex in the graph has odd degree. To fix this, add the edges $v_1 - v_4$, $v_3 - v_6$, $v_5 - v_8$, and $v_7 - v_2$. One Eulerian Tour is then

   $$1 \to 2 \to 5 \to 4 \to 1 \to 6 \to 5 \to 8 \to 3 \to 6 \to 7 \to 4 \to 3 \to 2 \to 7 \to 8 \to 1$$

   (b) The second graph does not have an Eurlerian Tour as it is, because u_1, u_2, u_5, and u_6 all have odd degree. To fix this, add the edges $u_2 - u_5$ and $u_1 - u_6$. An Eulerian Tour is then

   $$1 \to 6 \to 5 \to 2 \to 6 \to 7 \to 8 \to 5 \to 1 \to 4 \to 3 \to 2 \to 1$$

   (c) The third graph does not have an Eurlerian Tour as it is because each vertex has degree 3, which is odd. To fix this, add the edges $u_1 - u_5$, $u_2 - u_8$, $u_3 - u_7$, and $u_4 - u_6$. An Eulerian Tour is then

   $$1 \to 5 \to 7 \to 3 \to 1 \to 2 \to 6 \to 4 \to 8 \to 2 \to 3 \to 4 \to 5 \to 6 \to 7 \to 8 \to 1$$

5. For each of the graphs above, determine if the graph is Bipartite. If it is, specify a two-coloring of the vertices. If it is not, explain why. For graphs that are not bipartite, can you **remove** a small number of edges so that they become bipartite?

Solution:

(a) The first graph is Bipartite. Notice that odd points are only connected to even points, and vice versa. Thus we have the two-coloring $V_1 = \{1, 3, 5, 7\}$ and $V_2 = \{2, 4, 6, 8\}$.

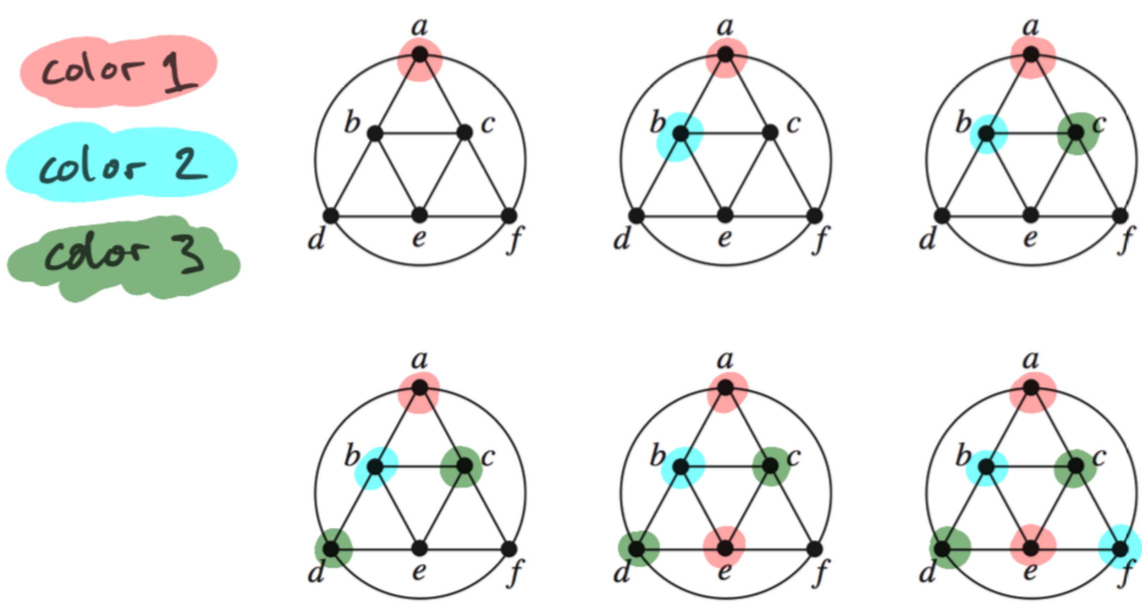
(b) The second graph is also Bipartite. Notice that opposite edges within the two squares are not connected, thus if we color opposite corners the same we'll end up with a two coloring. The one hitch is that u_1 is connected to u_5 and u_2 is connected to u_6, thus we must color u_5 differently from u_1 and u_6 differently from u_2. The following two-coloring accomplishes this: $V_1 = \{u_1, u_3, u_6, u_8\}$ and $V_2 = \{u_2, u_4, u_5, u_7\}$.

(c) The third graph is not Bipartite. Remember that a graph is Bipartite if and only if it has no odd length cycles. But the third graph has odd length cycles all over the place, e.g. $u_1 - u_2 - u_3 - u_1$ and $u_2 - u_3 - u_4 - u_5 - u_6 - u_1$. Notice that each edge that goes through the center of the octagon participates in an odd length cycle, thus edges $(1, 3)$, $(4, 8)$, $(5, 7)$, and $(2, 6)$ must be removed. After that the graph is easily two-colorable by taking $V_1 = \{1, 3, 5, 7\}$ and $V_2 = \{2, 4, 6, 8\}$.

6. Use the Greedy Coloring Algorithm to find a coloring of the vertices of the graphs below. To start out, consider the vertices in alphabetical order. Then choose your own vertex order and repeat the coloring process.

Solution:

For the first graph, note that there are odd length cycles, so at the very least we will need three colors to color the graph. Proceeding from left to right, then top to bottom, and coloring the vertices in the order a, b, c, d, e, f, we have



For the second graph, note that there are again odd length cycles, so at the very least we will need three colors to color the graph. Proceeding from left to right, then top to bottom, and coloring the vertices in the order $a, b, c, d, e, f, h, i, j, k, l$, we have