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CSCI 2824: Discrete Structures

Lecture 6: Predicates and Quantifiers

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Predicate Logic

There are deficiencies in propositional logic. We will talk about two new constructs: **Predicates** & **Quantifiers**

From Merriam–Webster:

Definition of PREDICATE

1 a : something that is affirmed or denied of the subject in a proposition in logic
b : a term designating a property or relation

2 : the part of a sentence or clause that expresses what is said of the subject and that usually consists of a verb with or without objects, complements, or adverbial modifiers

Predicate Logic

Example: If we let F be the name of the predicate, then we can think of $F(x)$ as a sentence that asserts an object (or subject) is fast.

$F(x)$ is read as "x is fast." where x represents the domain or "domain of discourse"

Rockets are fast. Usain Bolt is fast.

The predicate can be thought of as "is fast".

Predicate Logic

Example: Consider the statement $x > 3$

$P(x): x > 3$

- x is a variable or a placeholder
- > 3 is the predicate

Let $P(x)$ represent $x > 3$. We call $P(x)$ a propositional function. When we assign a value to x then $P(x)$ becomes a proposition and has a truth value.

$P(4)$ is true. $4 > 3$

$P(1)$ is false. $1 > 3$

Predicate Logic

Example: Rachel makes lunch for Murray.
Anna makes necklaces for Naomi.

Predicate "template" is

_____ makes _____ for _____

The predicate describes a relationship between three variables or objects.

$Makes(x, y, z)$ or $M(x, y, z)$

x: who makes something
y: things being made
z: people that something is being made for

Predicate Logic

Propositional functions can have multiple variables.

Example: Let $Q(x, y)$ represent $x + 1 = y$

- What is the truth value of $Q(1,2)$? $1 + 1 = 2$ True!
- What is the truth value of $Q(3,2)$? $3 + 1 = 2$ False!

Example: Let $R(x, y, z)$ represent $x^2 + y^2 = z^2$

- What is the truth value of $R(1,1,1)$? $1^2 + 1^2 = 1^2$ False!
- What is the truth value of $R(3,4,5)$? $3^2 + 4^2 = 5^2$ True!

Predicate Logic

When using predicates we have to think about what values we input.

The set of values we intend to plug in is called the **domain of discourse** or commonly just the **domain**.

Example: All babies love to sleep. Parker is a baby.

Let $S(x)$ denote: x likes to sleep.

S(Parker) is a true statement.

What is $S(8)$?

Predicate Logic

Suppose we fix a domain for *S(x)*. Let the domain of babies be

$$\{Parker, Madeline, Nocona, Tatum\}$$

Then *S(Parker)*, *S(Madeline)*, *S(Nocona)*, and *S(Tatum)* have truth values and make sense.
S(8), S(★) don't have truth values and can't really be defined.

Example: [with Propositional Logic] "All babies love to sleep." might be represented as

Parker loves to sleep ∧ **Madeline** loves to sleep ∧ **Nocona** loves to sleep ∧ **Tatum** loves to sleep

$S(Parker) \wedge S(Madeline) \wedge S(Nocona) \wedge S(Tatum)$

But what if we re–define our domain to be All Babies in the USA…. using only propositional logic would be pretty inefficient.

Predicate Logic

Universal Quantifier: $\forall$

$\underbrace{\forall x\, P(x)}$ means "For all x in my domain, P(x)." for some general predicate $P(x)$

Example: [with Quantifiers] "All babies love to sleep." might be represented as

$$\underbrace{\forall x\, S(x)}$$

Note that with the quantifier $\forall x\, S(x)$ becomes a proposition and therefore has a truth value.

Predicate Logic

Question: When is the statement $\forall x\, P(x)$ true?

It's true, when $P(x)$ is true for every single element of the domain.

Question: When is the statement $\forall x\, P(x)$ false?

There is at least one x such that $P(x)$ is false.

counterexample.

Predicate Logic

Example: Let the domain be the integers. Our proposition is; $\forall x\,(x^2 \geq 0)$.

Is this true or false?

True.

Example: Let the domain be the integers. Our proposition is; $\forall x\,(x^2 > x)$.

Is this true or false?

False. $x = 0$ $\quad 0 > 0$ No!

$x = 1$ $\quad 1 > 1$ No!

The case that breaks a universal statement is called a **counterexample**.

Predicate Logic

Long-Term Takeaway:

- To **disprove** a universal proposition, all you need is **one** specific counterexample that makes it not work
- To **prove** a universal proposition **one** specific example does **nothing**! Usually you have to work a lot harder ...
- **NEVER** try to prove a universal statement by choosing a specific example and showing $P(x)$ to be true

Predicate Logic

So the **universal quantifier** $\forall$ is how we talk about **all things**.

What if we want to talk about something?

Many mathematical statements claim that there is an element of the domain that has a certain property.

Existential Quantifier: $\exists$

$\exists x\ P(x)$ means "there exists an x in the domain, such that P(x)"

Predicate Logic

Question: When is the statement $\exists x\, P(x)$ true?

When there is at least one x in the domain such that P(x) is True.

Question: When is the statement $\exists x\, P(x)$ false?

When there is not an x in the entire domain such that P(x) is true.

Predicate Logic

Example: Let the domain be the integers. Our proposition is; $\exists x\ (x^2 \geq 0)$.

Is this true or false?

True!

Example: Let the domain be the integers. Our proposition is; $\exists x\ (x = x + 1)$.

Is this true or false?

False.

Predicate Logic

Special case to consider. **Empty Domain**

- $\forall x\, P(x)$ is true vacuously true.
- $\exists x\, P(x)$ is false no x's exist.

Example: All of the Olympic running medals I have won are gold.

Predicate Logic

$- 2 + x$

$-(2 + x)$

Scope of Quantifiers

Quantifiers have the narrowest scope of all logical operands.

<u>**Example:**</u> $\forall x \left(P(x) \land Q(x)\right)$ is not the same as $\forall x\, P(x) \land Q(x)$

* Takeaway – pay attention to parentheses

Predicate Logic

Logical Equivalence involving Quantifiers

Statements involving predicates and quantifiers are logically equivalent if and only if they have the same truth value no matter which predicates you use and which domain they're defined over.

Example: Are these equivalent?

$$\forall x \left(P(x) \wedge Q(x)\right) \equiv \forall x\, P(x) \wedge \forall x\, Q(x)$$

yes!

Predicate Logic

let this be "arbitrary"

Example: Let our domain be $\{a, b, c\}$. Prove that the following is true:

$$\forall x\,(P(x) \wedge Q(x)) \equiv \forall x\, P(x) \wedge \forall x\, Q(x)$$

$$\begin{aligned}
\forall x\,(P(x) \wedge Q(x)) &\equiv (P(a) \wedge Q(a)) \wedge (P(b) \wedge Q(b)) \wedge (P(c) \wedge Q(c)) \\
&\equiv P(a) \wedge Q(a) \wedge P(b) \wedge Q(b) \wedge P(c) \wedge Q(c) && \text{Associativity Law} \\
&\equiv P(a) \wedge P(b) \wedge P(c) \wedge Q(a) \wedge Q(b) \wedge Q(c) && \text{Commutativity Law} \\
&\equiv (P(a) \wedge P(b) \wedge P(c)) \wedge (Q(a) \wedge Q(b) \wedge Q(c)) && \text{Associativity} \\
&\equiv \forall x\, P(x) \wedge \forall x\, Q(x)
\end{aligned}$$

Predicate Logic

No! See counterexample

Example: Are the statements below equivalent?

$$\forall x\,(P(x) \vee Q(x)) \equiv \forall x\, P(x) \vee \forall x\, Q(x)$$

Suppose again we have a domain $\{a, b, c\}$

$$\forall x\,(P(x) \vee Q(x)) \equiv (P(a) \vee Q(a)) \wedge (P(b) \vee Q(b)) \wedge (P(c) \vee Q(c))$$

Look for a counterexample. Domain: All integers.

$P(x)$: x is even.

$Q(x)$: x is odd.

$$\forall x\,(P(x) \vee Q(x)) \equiv T \quad \text{vs.} \quad \forall x\, P(x) \vee \forall x\, Q(x) \equiv F$$

Predicate Logic

Example: Are the statements below equivalent? yes!

$$\exists x\,(P(x) \lor Q(x)) \equiv \exists x\, P(x) \lor \exists x\, Q(x)$$

Suppose we have an arbitrary domain $\{a, b, c\}$

$$\exists x\,(P(x) \lor Q(x)) \equiv (P(a) \lor Q(a)) \lor (P(b) \lor Q(b)) \lor (P(c) \lor Q(c))$$

$$\equiv (P(a) \lor P(b) \lor P(c)) \lor (Q(a) \lor Q(b) \lor Q(c))$$

$$\equiv \exists x\, P(x) \lor \exists x\, Q(x)$$

Predicate Logic

Example: Are the statements below equivalent?

$$\exists x \left(P(x) \wedge Q(x)\right) \equiv \exists \underline{x}\ \underline{P(x)} \wedge \exists \underline{x}\ \underline{Q(x)}$$

Domain: All integers

$P(x)$: x is even

$Q(x)$: x is odd

Not logically equivalent.

Predicate Logic

Question: What is the negation of $\forall x\, P(x)$?

$$\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$$

Example: What is the negation of the statement: All babies like to sleep?

¬("All babies like to sleep")

Not all babies like to sleep.

There is at least one baby that doesn't like sleep.

Predicate Logic

Question: What is the negation of $\exists x\, P(x)$?

$$\neg\exists x\, P(x) \quad \equiv \quad \forall x \; \neg P(x)$$

Example: What is the negation of the statement: There exists a baby that loves sleep?

It is not the case that there exists a baby that loves sleep.

All babies do not love sleep.

No babies love sleep.

Predicate Logic

Collectively, these are called **DeMorgan's Law for Quantifiers**

- $\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$
- $\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$

And then we had the distribution laws

- $\forall x\, (P(x) \land Q(x)) \equiv \forall x\, P(x) \land \forall x\, Q(x)$
- $\exists x\, (P(x) \lor Q(x)) \equiv \exists x\, P(x) \lor \exists x\, Q(x)$

With these rules, and the logical equivalences we found for regular propositions, we can prove all kinds of equivalences of quantifier propositions

Predicate Logic

Example: Prove that $\neg\forall x\,(P(x) \rightarrow Q(x)) \equiv \exists x(P(x) \wedge \neg Q(x))$

$$\neg\forall x(P(x) \rightarrow Q(x)) \equiv \exists x\,\neg(P(x) \rightarrow Q(x))$$ Demorgans Law for Quant.

$$\equiv \exists x\,\neg(\neg P(x) \vee Q(x))$$ by RBI

$$\equiv \exists x\,(\neg\neg P(x) \wedge \neg Q(x))$$ by Demorgans

$$\equiv \exists x\,(P(x) \wedge \neg Q(x))$$ by negation Law

Predicate Logic

Translating from English into Logical Expressions

Example: Translate the following into symbols: "Every student in CSCI 2824 has passed Calculus 1." Let the domain be all the students in CSCI 2824.

$C(x)$: x has passed Calc 1.

$\forall x\, C(x)$

Example: Translate the following into symbols: "Every student in CSCI 2824 has passed Calculus 1." but use the domain, all students at CU.

$C(x)$: x has passed Calc 1

$D(x)$: x is in CSCI 2824

$\forall x\,(D(x) \to C(x))$

Predicate Logic

Translating from English into Logical Expressions

Example: Let the domain be the set of all CU students, and translate:
"Every student $\underbrace{\text{in CSCI 2824}}_{D(x)}$ is either $\underbrace{\text{taking Data Structures}}_{DS(x)}$, or $\underbrace{\text{has already passed it}}_{P(x)}$."

$\forall$

$D(x)$: x is in CSCI 2824.

$DS(x)$: x is in Data Structures.

$P(x)$: x has passed Data Structures.

$$\boxed{\forall x\,\big(D(x) \rightarrow (DS(x) \oplus P(x))\big)}$$

Extra Practice

Example 1: What is the truth value of $\forall x\,(x^2 \geq x)$ when the domain is all real numbers?

Example 2: Think of a domain, and specific propositional functions $P(x)$ and $Q(x)$ to illustrate this equivalence

$$\neg\forall x\,(P(x) \rightarrow Q(x)) \equiv \exists x\,(P(x) \wedge \neg Q(x))$$

Example 3: Translate the following and then find the negation.

Let the domain be of all CSCI 2824 students, and translate:

- "There exists a student in CSCI 2824 that has taken Calculus 3 but not Differential Equations"

Example 4: Translate the following and then find the negation.

Let the domain be the set of all CU students, and translate:

- "There exists a student in CSCI 2824 that has taken Calculus 3 but not Differential Equations"

Solutions

Example 1: What is the truth value of $\forall x\,(x^2 \geq x)$ when the domain is all real numbers?

Solution: The statement is false. A counter example is $x = \frac{1}{2}$ because $\left(\frac{1}{2}\right)^2 = \frac{1}{4} < \frac{1}{2}$

Example 2: Think of a domain, and specific propositional functions $P(x)$ and $Q(x)$ to illustrate this equivalence

$$\neg\forall x\,(P(x) \rightarrow Q(x)) \equiv \exists x\,(P(x) \wedge \neg Q(x))$$

Solution: Let x be the set of all shapes, $P(x)$ mean x is a polygon, and $Q(x)$ mean x is a rectangle.

The first proposition says it's not the case that if x is a polygon then it is necessarily a rectangle

The second proposition says that there exists a shape that is a polygon and is not a rectangle (e.g. a triangle)

Example 3: Translate the following and then find the negation.

Let the domain be of all CSCI 2824 students, and translate:

- "There exists a student in CSCI 2824 that has taken Calculus 3 but not Differential Equations"

Solution: Let $C(x)$ mean x has taken Calc 3 and $E(x)$ mean x has taken DiffEq. In symbols the statement as $\exists x(C(x) \wedge \neg E(x))$

The negation is $\neg \exists x(C(x) \wedge \neg E(x)) \equiv \forall x(\neg C(x) \vee E(x))$

In English,the negation is

- "All 2824 students have taken DiffEq or not taken Calc 3"

Example 4: Translate the following and then find the negation.

Let the domain be the set of all CU students, and translate:

- "There exists a student in CSCI 2824 that has taken Calculus 3 but not Differential Equations"

Solution Let $D(x)$ mean x is a 2824 student. In symbols, we have

$$\exists x\ [D(x) \wedge (C(x) \wedge \neg E(x)]$$

It's negation is $\forall x\ [\neg D(x) \vee (\neg C(x) \vee E(x))]$, which in English is

- "All CU students either aren't 2824 students or haven't taken Calc 3 or have taken Differential Equations"