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CSCI 2824: Discrete Structures

Lecture 25: Permutations, Combinations, and the Binomial Theorem

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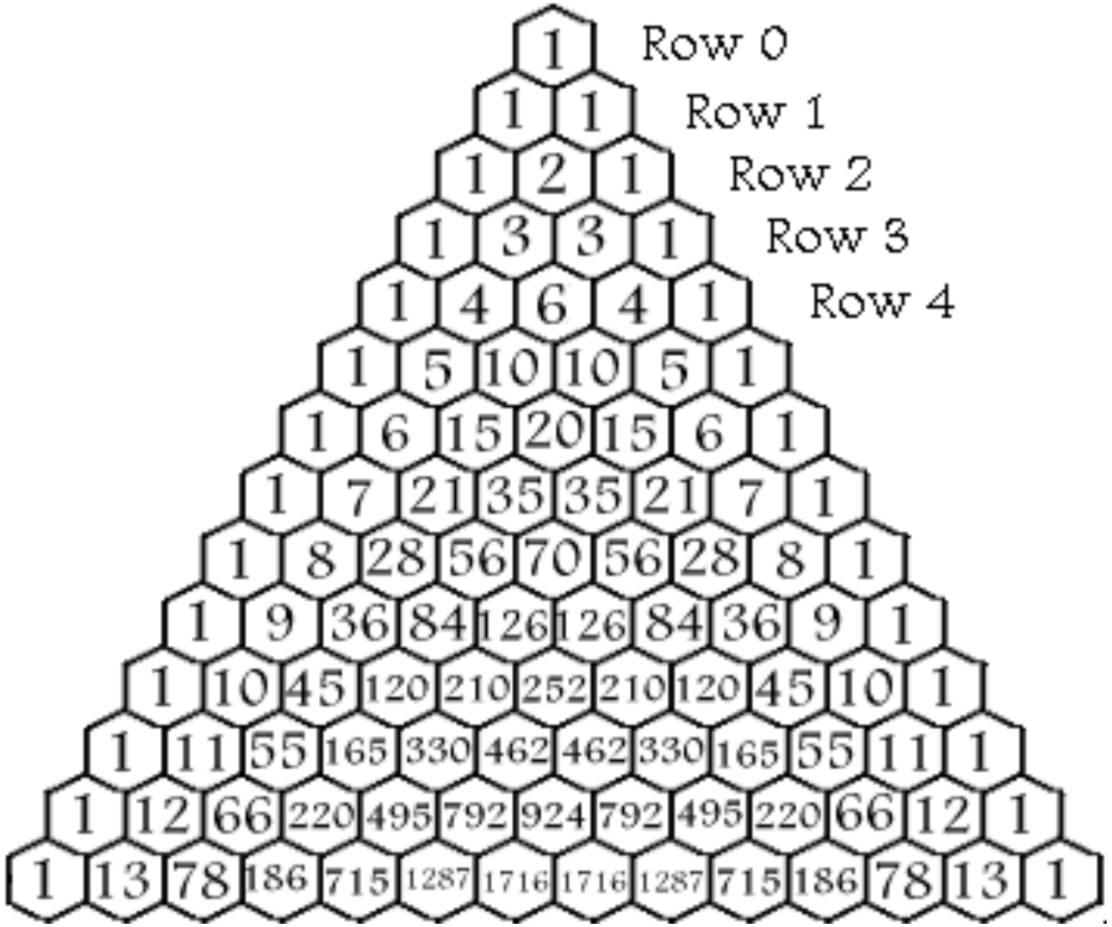

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Permutations and Combinations

❖ When the order of the r objects is important we call the selection a **permutation**.

The number of permutations is given by:

permutation without Repetition

$$P(n,r) = n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

❖ When the order of the r objects is not important we call the selection a **combination**.

The number of combinations is given by:

$$C(n,r) = \frac{n!}{(n-r)!\,r!}$$

combination without repetition

Permutations and Combinations

Example: You're planning a summer road trip and you have 10 national parks that you'd like to visit. Unfortunately, you only have time to visit 3 of them. How many possible routes are there?

permutations

$n = 10$ parks

$r = 3$

$$P(10,3) = \frac{10!}{(10-3)!} = 10 \cdot 9 \cdot 8 \boxed{= 720 \text{ routes}}$$

Permutations and Combinations

Example: A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to choose a crew of 6 people to go on the mission?

Combinations

$$C(30,6) = \frac{30!}{(30-6)! \cdot 6!} = \boxed{\frac{30!}{24! \cdot 6!}}$$

Permutations and Combinations

What about permutations and combinations with repetition?

Theorem: The number of r −permutations of n objects with repetition is n^r.

Example: How many strings of length 4 can be formed using the uppercase letter of the English alphabet?

26^4

AAAA

Permutations and Combinations

Theorem: There are $C(n+r-1, r) = C(n+r-1, n-1)$ r −combinations from a set with n elements where repetition of elements is allowed.

Example: Suppose that a cookie shop has 4 different kinds of cookies. How many different ways can 6 cookies be chosen?

$n = 4$

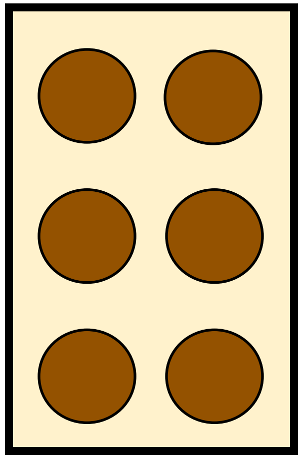
$r = 6$

$$C(4+6-1, 6) = \binom{9}{6} = \frac{9!}{3!\,6!} = \frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = 3 \cdot 4 \cdot 7$$

$$\boxed{= 84 \text{ ways}}$$

Permutations and Combinations –"Stars and Bars"

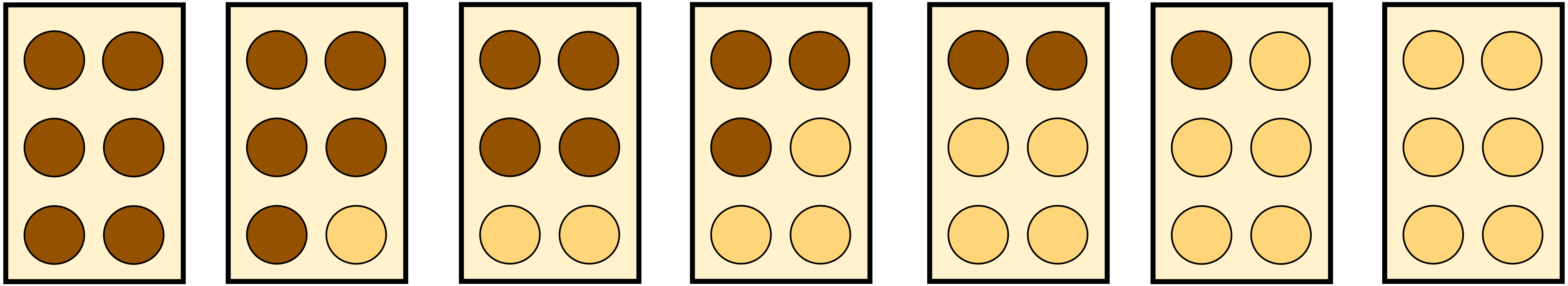
How many ways are there to buy 6 cookies given only <u>1 type</u> to choose from?



1 way

Permutations and Combinations –"Stars and Bars"

How many ways are there to buy 6 cookies given 2 types to choose from?

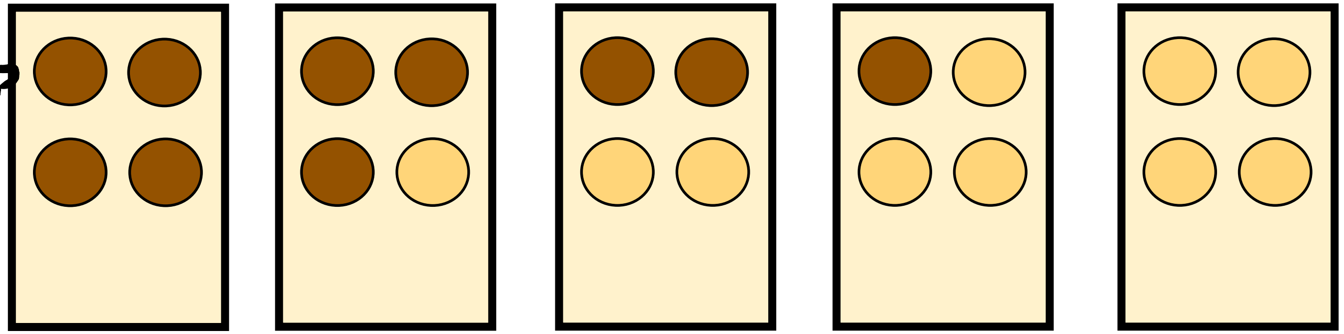


6+1 = 7 ways

Permutations and Combinations –"Stars and Bars"

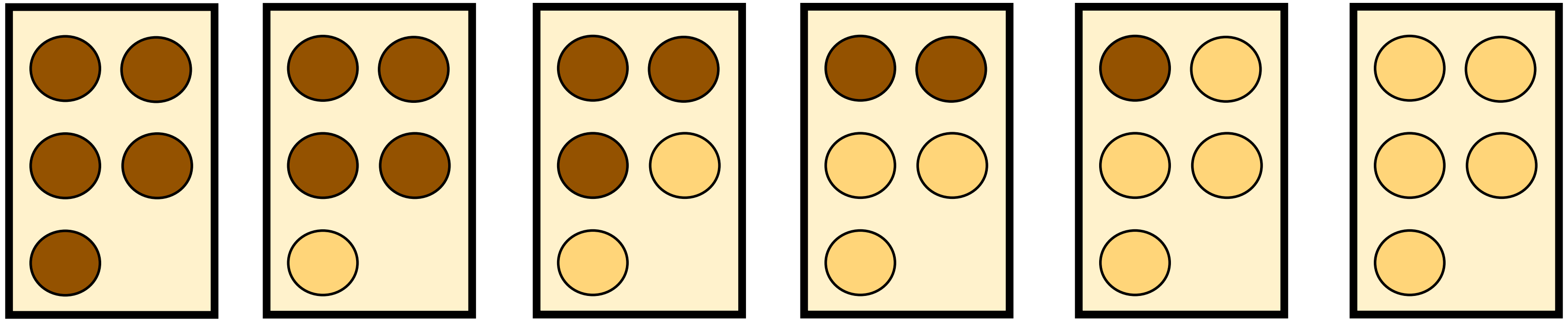
IN GENERAL: How many ways are there to buy **x cookies** given 2 types to choose from?

Buying 4 cookies



4+1 = 5 ways

5 cookies



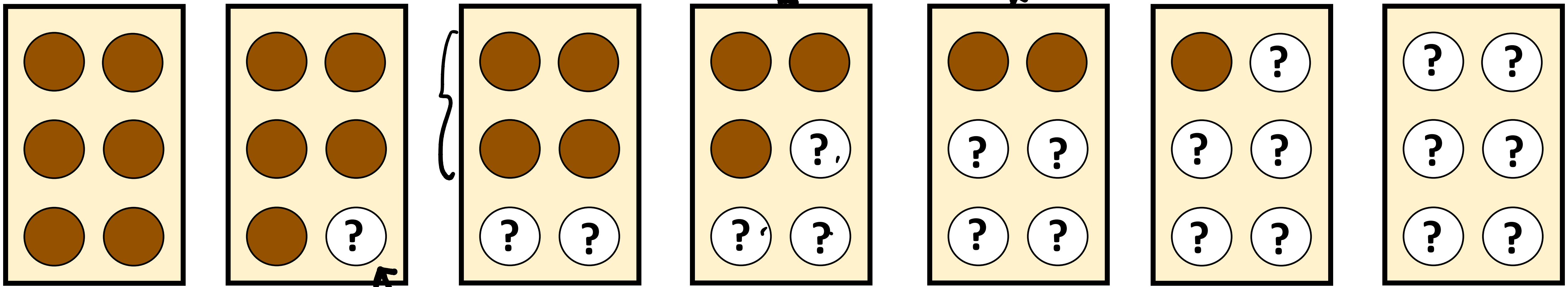
5+1 = 6 ways

In general, if we are buying x cookies given 2 types to choose from, there are **x+1 ways to choose**.

Permutations and Combinations –"Stars and Bars"

How many ways are there to buy 6 cookies given 3 types to choose from?

- We can reduce this question to buying x cookies given 2 types of cookies. Suppose we choose the first type of cookie in the ways pictured below.



1 types of cookies to fill 6 spots $= 1$	2 types of cookies to fill 1 spot $= 1 + 1$ $= 2$	2 types of cookies to fill 2 spots $= 2 + 1$ $= 3$	2 types of cookies to fill 3 spots $= 3 + 1$ $= 4$	2 types of cookies to fill 4 spots $= 4 + 1$ $= 5$	2 types of cookies to fill 5 spots $= 5 + 1$ $= 6$	2 types of cookies to fill 6 spots $= 6 + 1$ $= 7$

Permutations and Combinations –"Stars and Bars"

IN GENERAL: How many ways are there to buy **x cookies** given 3 types to choose from?

6 cookies: # of ways = $1+2+3+4+5+6+7 = \frac{(6+1)(6+2)}{2}$

$\frac{7(7+1)}{2}$

7 cookies: # of ways = $\underbrace{1+2+3+4+5+6+7}+8 = \frac{(7+1)(7+2)}{2}$

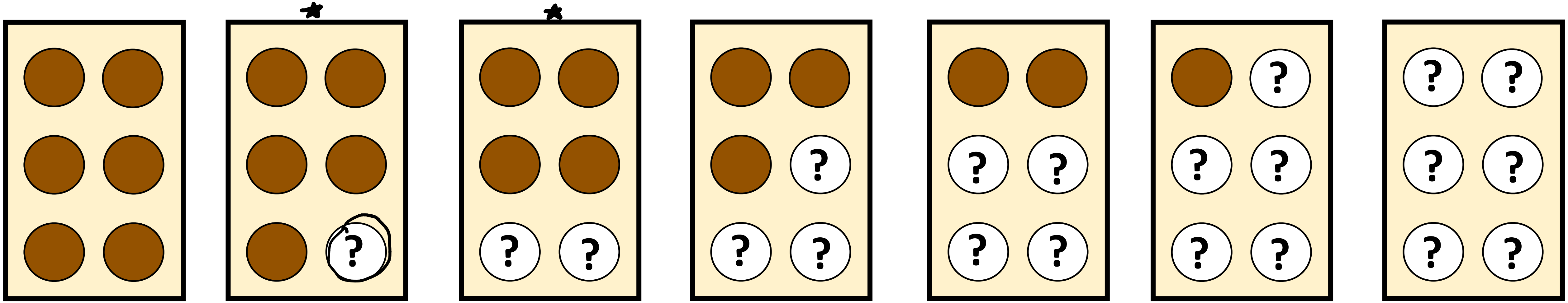
⋮

★ x cookies: # of ways = $1+2+3+4+5+\cdots+x = \frac{(\boldsymbol{x}+\mathbf{1})(\boldsymbol{x}+\mathbf{2})}{\mathbf{2}}$

Permutations and Combinations –"Stars and Bars"

Back to our original question: How many ways are there to buy 6 cookies given <u>4 types</u> to choose from?

➢ We can reduce this question to buying x cookies given 3 types of cookies. Suppose we choose the first type of cookie in the ways pictured above.



1 types of cookies to fill 6 spot $= 1$

3 types of cookies to fill 1 spot $\frac{(1+1)(1+2)}{2}$ $= \frac{2 \cdot 3}{2} = 3$

3 types of cookies to fill 2 spots $\frac{(2+1)(2+2)}{2}$ $= \frac{3 \cdot 4}{2} = 6$

3 types of cookies to fill 3 spots $= \frac{4 \cdot 5}{2} = 10$

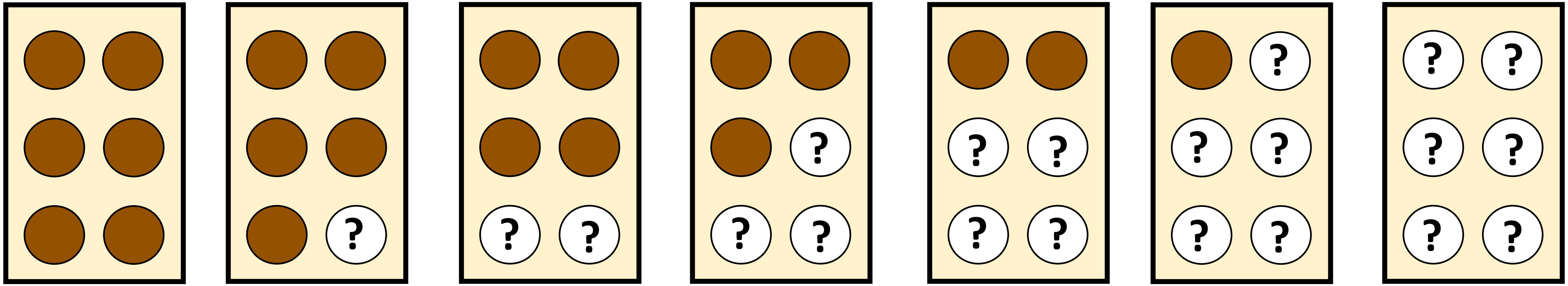
3 types of cookies to fill 4 spots $= \frac{5 \cdot 6}{2} = 15$

3 types of cookies to fill 5 spots $= \frac{6 \cdot 7}{2} = 21$

3 types of cookies to fill 6 spots $= \frac{7 \cdot 8}{2} = 28$

Permutations and Combinations –"Stars and Bars"

Back to our original question: How many ways are there to buy 6 cookies given 4 types to choose from?

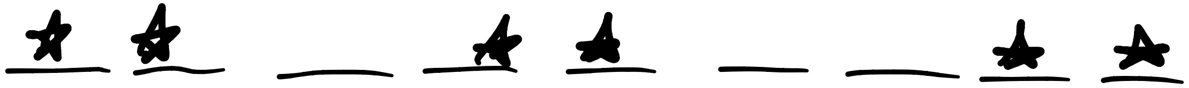


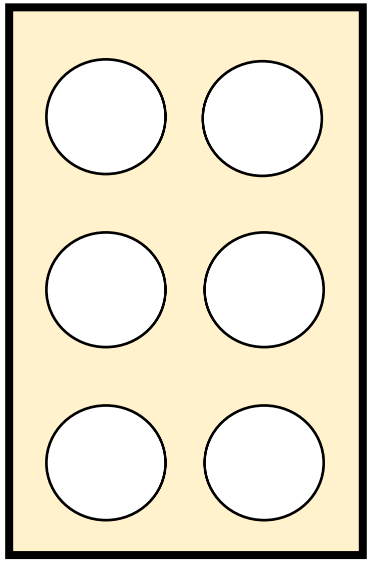
Therefore, the total number of ways to choose 6 cookies given 4 types

$$= 1 + 3 + 6 + 10 + 15 + 21 + 28$$

$$= 84 \text{ ways}$$

Permutations and Combinations –"Stars and Bars"





Instead of a cookie box to visualize the different types of cookies, let's visualize it like this:

$C(9, 3) = C(9, 6)$

This represents 1 cookie of Type 1, 2 cookies of Type 2, 2 cookies of Type 3, and 1 cookie of Type 4

We only need 3 partitions to separate our 4 types of cookies. And we can represent the types of cookies by stars!

Why does this theorem work?

Proof: We can represent the distribution of the r items from n choices as an arrangement of r $\star$'s and $n-1$ vertical lines

For example, for the cookie distribution, one possibility is

$$\star\star||\star\star\star|\star$$

where this indicates 2 cookies of type 1, 0 cookies of type 2, 3 cookies of type 3, and 1 cookie of type 4

We can now reduce the problem of asking how many different ways we can arrange r stars and $n-1$ lines

Permutations and Combinations

Think of the number of arrangements of stars and lines as the number of ways we can place r stars is $n + r - 1$ possible positions

The order of choosing the r positions for the stars does not matter, so this is the number of combinations we can choose the position of r things from $n + r - 1$ possible positions, which is exactly

$$C(n + r - 1, r)$$

Similarly, we could also think of the number of positions we could place the $n - 1$ vertical lines in the $n + r - 1$ slots, which gives

$$C(n + r - 1, n - 1)$$

Permutations and Combinations

$C(9,6) = C(9,3)$

And of course, these two quantities are equal, since

$r=6$

$$\star\ C(n+r-1,r) = \frac{(n+r-1)!}{(n+r-1-(r))!r!} = \frac{(n+r-1)!}{(n-1)!r!}$$

and

cookies

$n-1 = 4-1$

3

$$\star\ C(n+r-1,n-1) = \frac{(n+r-1)!}{(n+r-1-(n-1))!(n-1)!}$$
$$= \frac{(n+r-1)!}{r!(n-1)!}$$

So clearly, $C(n+r-1,r) = C(n+r-1,n-1)$

Permutations and Combinations

final answer:

$$\binom{13}{2} = \frac{13!}{11! \cdot 2!} \boxed{= 78}$$

Example: How many solutions does the equation

$$x_1 + x_2 + x_3 = 11$$

have where x_1, x_2, and x_3 are nonnegative integers?

$1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 11$

one solution: $x_1 = 3$, $x_2 = 2$, $x_3 = 6$

$n = 3$

$r = 11$

$C(3+11-1, 3-1) = C(3+11-1, 11) \Longrightarrow C(13,2)$ or $C(13,11)$

$C(13,2)$ note: _ _ _ _ _ _ _ _ _ _ _ _ _

13 spaces
place 2 partitions
to solve.

Permutations and Combinations

Example: How many solutions does the equation

$$x_1 + x_2 + x_3 = 11$$

have where x_1, x_2, and x_3 are nonnegative integers satisfying

$$x_1 \geq 1, x_2 \geq 2, \text{ and } x_3 \geq 3$$

one possible solution: $\underline{\star}\ \underline{\star}\ \underline{\ }\ \underline{\star}\ \underline{\star}\ \underline{\ }\ \underline{\star}\ \underline{\star}\ \underline{\star}\ \underline{\star}\ \underline{\star}\ \underline{\star}\ \underline{\star}$

$n = 3$

$r = 5$

$$\binom{5+3-1}{5} = \binom{7}{5} = \boxed{21 \text{ solutions}}$$

Permutations and Combinations

TABLE 1 **Combinations and Permutations With and Without Repetition.**

Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$\dfrac{n!}{(n-r)!}$
r-combinations	No	$\dfrac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\dfrac{(n+r-1)!}{r!\,(n-1)!}$

Binomial Theorem

Example: Expand $(x + y)^3$

$$
\begin{aligned}
(x + y)^3 &= (x + y)(x + y)(x + y) \\
&= (x + y)(x^2 + 2xy + y^2) \\
&= (x^3 + 2x^2y + xy^2 + x^2y + 2xy^2 + y^3) \\
&= \mathbf{1}x^3 + \mathbf{3}x^2y + \mathbf{3}xy^2 + \mathbf{1}y^3
\end{aligned}
$$

Pascal's Triangle

1

1 1

1 2 1

$\boxed{1 \quad 3 \quad 3 \quad 1}$

1 4 6 4 1

$(x+y)^5 \longrightarrow$ 1 5 10 10 5 1

1 6 15 20 15 6 1

Binomial Theorem

Pascal's Triangle with Binomial Coefficients

$(x+y)^3 = \mathbf{1}x^3 + \mathbf{3}x^2y + \mathbf{3}xy^2 + \mathbf{1}y^3$

Recall: $C(n,r) = \binom{n}{r}$

Binomial coefficient

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\boxed{\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

Binomial Theorem

$\binom{n}{k} = C(n,k)$

<u>The Binomial Theorem</u>: Let x and y be variables, and let n be a nonnegative integer. Then:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} \mathrm{x}^{\mathrm{n-k}}\mathrm{y}^{\mathrm{k}}$$

But why?

$$(x+y)^3 = (x+y)(x+y)(x+y)$$

1 way to choose $\boldsymbol{x^3}$ term: $x \cdot x \cdot x$

3 ways to choose $\boldsymbol{x^2 y}$ term: $x \cdot x \cdot y,\ \ x \cdot y \cdot x,\ \ y \cdot x \cdot x$

3 ways to choose $\boldsymbol{x y^2}$ term: $x \cdot y \cdot y,\ \ y \cdot x \cdot y, y \cdot y \cdot x$

1 way to choose $\boldsymbol{y^3}$ term: $y \cdot y \cdot y$

Binomial Theorem

Example: Expand $(x+y)^4$

$$(x+y)^4 = \sum_{k=0}^{4} \binom{4}{k} x^{4-k} y^k$$

$$(x+y)^4 = \binom{4}{0} x^{4-0} y^0 + \binom{4}{1} x^{4-1} y^1 + \binom{4}{2} x^{4-2} y^2 + \binom{4}{3} x^{4-3} y^3 + \binom{4}{4} x^{4-4} y^4$$

$$= \frac{4!}{(4-0)!\,0!} x^4 + \frac{4!}{(4-1)!\,1!} x^3 y + \frac{4!}{(4-2)!\,2!} x^2 y^2 + \frac{4!}{(4-3)!\,3!} x y^3 + \frac{4!}{(4-4)!\,4!} y^4$$

$$= 1x^4 + 4x^3 y + \frac{4!}{2!\,2!} x^2 y^2 + 4 x y^3 + 1 y^4$$

$$= 1x^4 + 4x^3 y + 6x^2 y^2 + 4xy^3 + 1y^4$$

Binomial Theorem

Example: What is the coefficient on $x^{11}y^{14}$ in the expansion of $(x+y)^{25}$

Thm: $(x+y)^{25} = \sum_{k=0}^{25} \binom{25}{k} x^{25-k} y^{k}$

The coefficient: $\binom{25}{14} x^{25-14} y^{14}$

$= \binom{25}{14} \underline{x^{11} y^{14}}$

$\boxed{= \frac{25!}{11!\,14!}} = 4,457,400$

Binomial Theorem

Recall $(a \cdot b)^n = a^n b^n$

Example: What is the coefficient on $x^{11}y^{14}$ in the expansion of $(2x - 3y)^{25}$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$(2x-3y)^{25} = \sum_{k=0}^{25} \binom{25}{k} (2x)^{25-k} (-3y)^k$$

$n=25, \quad k=14$

$$\binom{25}{14} 2^{11} (-3)^{14} = \binom{25}{14} 2^{11} 3^{14}$$

$$\boxed{= \frac{25!}{11!\,14!} 2^{11} 3^{14}}$$

Binomial Theorem

<u>Pascal's Identity</u>: Let n and k be positive integers with $n \geq k$.

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

1
1 1
1 2 1
1 3 3 1

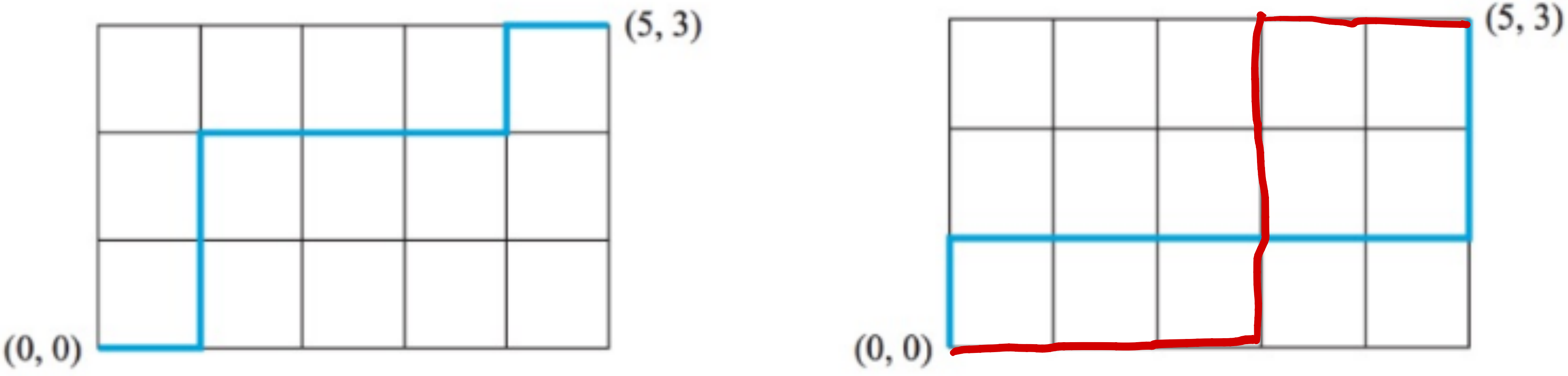
$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

Interesting thing to note.
This comes from the construction of Pascal's triangle.

Binomial Theorem

Example: How many paths in the xy −plane are there between the origin $(0,0)$ and the point (m, n), where m and n are positive integers, and each step in the path is a move one unit up or one unit to the right?

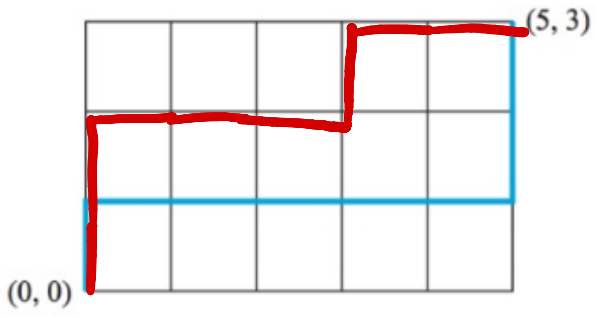
E.g. Two possible paths from $(0,0)$ to $(5,3)$



Binomial Theorem

Example: (continued) How many paths in the xy −plane are there between the origin $(0,0)$ and the point (m, n), where m and n are positive integers, and each step in the path is a move one unit up or one unit to the right?

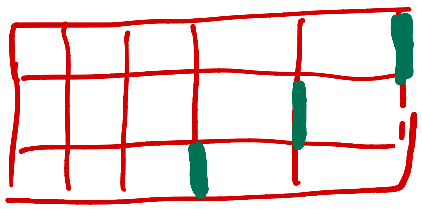
We will show that the answer is $\binom{m+n}{n}$.



moves : m+n

Claim: Each path can be represented as a bit string consisting of m 0s and n 1s where a 0 represents a move one unit to the right and a 1 represents a move one unit up.

11000100



Ex: The path shown above can be represented by 10000011

Question: How does this help us count them?

Binomial Theorem

So if we change this into a bitstring problem, we could solve this by choosing the positions of the n 1's

This could be done in

$$C(m+n, n) = \binom{m+n}{n} \text{ ways}$$

Binomial Theorem

Example: How many paths of this type are there from the origin to the point $(5,3)$?

$$\binom{8}{3} = 56 \text{ paths.}$$

Binomial Theorem

Example: How many paths are there in xyz −space that start at the origin $(0,0,0)$ and end at $(4,3,5)$ if each move in the path is one unit in the positive x, y, or z direction?

0 is a move in the positive x-direction.
1 is a move in the positive y-direction.
2 is a move in the positive z-direction.

$$C(4+3+5,4)\cdot C(3+5,3)\cdot \overset{=1}{C(5,5)} = C(12,4)\cdot C(8,3)\cdot C(5,5)$$

$$=\frac{12!}{8!4!}\cdot\frac{8!}{5!3!}\cdot\frac{5!}{5!0!}$$

$$=27,720$$

Extra Practice

EX. 1 How many ways are there to choose eight coins from a piggy bank containing 100 identical pennies and 80 identical nickels?

EX. 2 How many different ways are there to buy 7 donuts from a shop that sells 4 different types of donuts?

EX. 3 How many different ways are there to buy 7 donuts from a shop that sells 4 different types of donuts if your coworkers have asked that you get at least 1 of each type?

EX. 4 What is the coefficient of x^9 in $(2 - x)^{19}$

EX. 5 There are 10 questions on a Discrete Structures final exam. How many ways are there to assign scores to the problems if the sum of the scores is 100 and each question is worth at least 5 points?

Solutions

EX. 1 How many ways are there to choose eight coins from a piggy bank containing 100 identical pennies and 80 identical nickels?

Solution: This is like the cookie problem. Since we're only taking 8 coins, the fact that there are 100 pennies and 80 nickels is irrelevant. There are $n = 2$ types of coins and we're looking for an 8-combination of them. So we have

$$C(n + r - 1, r) = C(9, 8) = \frac{9!}{8!1!} = 9$$

So there are 9 different ways to choose 8 coins.

EX. 2 How many different ways are there to buy 7 donuts from a shop that sells 4 different types of donuts?

Solution: There are $n = 4$ types of donuts and we want an $r = 7$ combination of them. Thus there are

$$C(n + r - 1, r) = C(10, 7) = \frac{10!}{7!3!} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2} = 120$$

ways to buy 7 donuts.

EX. 3 How many different ways are there to buy 7 donuts from a shop that sells 4 different types of donuts if your coworkers have asked that you get at least 1 of each type?

Solution: If we have to buy 1 of each of the $n = 4$ types then that eliminates 4 of the 7 donuts. We then choose the remaining $r = 3$ donuts from the $n = 4$ types. Thus there are

$$C(n + r - 1, r) = C(6, 3) = \frac{6!}{3!3!} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2} = 20$$

different ways to choose the 7 donuts if we must buy 1 of each type.

EX. 4 What is the coefficient of x^9 in $(2-x)^{19}$

Solution: By the Binomial Thm, we have

$$(2+(-x))^{19} = \sum_{k=0}^{19} \binom{19}{k} (2)^{19-k}(-x)^k$$

The coefficient of x^9 comes from choosing $k = 9$. We have

$$\binom{19}{9}(2)^{19-9}(-x)^9 = -2^{10}\binom{19}{9}x^9 = -94595072x^9$$

EX. 5 There are 10 questions on a Discrete Structures final exam. How many ways are there to assign scores to the problems if the sum of the scores is 100 and each question is worth at least 5 points?

Solution: This is equivalent to asking for the solutions to

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} = 100$$

where each x_i must satisfy $x_i \geq 5$. Since each x_i is at least 5 this accounts for $10 \cdot 5 = 50$ points. We must distribute the remaining 50 points among the x_i's. With $r = 50$ and $n = 10$, we have

$$C(n + r - 1, r) = C(59, 50) = \frac{59!}{50!9!} = 12,565,671,261$$