This assignment is due on Friday October 11 to Gradescope by Noon. You are expected to write or type up your solutions neatly. Remember that you are encouraged to discuss problems with your classmates, but you must work and write your solutions on your own.

Important: Make sure to clearly write your full name, the lecture section you belong to (001 or 002), and your student ID number at the top of your assignment. You may **neatly** LaTex your solutions for +1 extra credit on the assignment.

1. (a) Suppose P, Q, and R are non-empty sets. Prove that $P \times (Q \cap R) = (P \times Q) \cap (P \times R)$ by showing that each side of this equation must be a **subset** of the other side, and concluding that the two sides must therefore be equal.

   (b) Suppose that P, Q, and R are non-empty sets. Prove that $P \times (Q \cap R) = (P \times Q) \cap (P \times R)$ by using **set builder notation** and set identities and definitions.

2. (a) Give an example of two uncountable sets A and B with a nonempty intersection, such that $A - B$ is

      i. finite

      ii. countably infinite

      iii. uncountably infinite

   (b) Use the Cantor diagonalization argument to prove that the number of real numbers in the interval $[3, 4]$ is uncountable.

   (c) Use a proof by contradiction to show that the set of irrational numbers that lie in the interval $[3, 4]$ is uncountable. (You can use the fact that the set of rational numbers ($\mathbb{Q}$) is countable and the set of reals ($\mathbb{R}$) is uncountable). Show all work.

3. (a) Find a closed form for the recurrence relation: $a_n = 2a_{n-1} - 2, a_0 = -1$

   (b) Find a closed form for the recurrence relation: $a_n = (n + 2)a_{n-1}, a_0 = 3$

   (c) Show that $a_n = 5(-1)^n - n + 2$ is a solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2} + 2n - 9$.

4. (a) Consider the function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ where $f(m, n) = 2m - n$. Is this function onto?

   (b) Consider the function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ where $f(m, n) = m^2 - n^2$. Is this function onto?

   (c) Define the set C = the set of all residents of Colorado. Define in words a function $f : C \to \mathbb{Z}$. Is your function one-to-one? Is it onto? Be sure that the f you defined is indeed a **function**. Be creative and have fun!

   (d) Again, define the set C = the set of all residents of Colorado. Define in words a function $f : C \to \mathbb{Z}$. However this time, make sure that your function is one-to-one. (Make sure to give a different example from part (c)).

5. This problem is EXTRA CREDIT! It is worth 5 points of EC. These 5 points of EC will only apply to the homework portion of your grade.

   Cantor's theorem is an important result in set theory, and hence in the theory of computation, among other significant areas of Computer Science. The theorem states that for any set A, $|A| < |\mathcal{P}(A)|$. That is, the cardinality of a set is strictly less than its power set.

   (a) Provide an example which shows Cantor's theorem to be true for a finite set. It will suffice to define a set, write its power set, and write the cardinality of each.

   (b) Prove Cantor's theorem for an arbitrary infinite set A. The general outline is as follows:

      i. Define a mapping $f : A \to \mathcal{P}(A)$ which is injective (one-to-one), and show that it is so.

         Hint: To define a mapping, simply show where elements in your domain will go in your codomain. To show that a mapping is injective, you must show that for arbitrary elements a and b in the domain, $f(a) = f(b)$ implies $a = b$.

         What does this say about the cardinality of A as it compares to the cardinality of $\mathcal{P}(A)$?

      ii. Assume that there is a mapping $g : A \to \mathcal{P}(A)$ which is surjective (onto), and then reach a contradiction by defining a subset of A which cannot be mapped to $\mathcal{P}(A)$, and show that it cannot be so.

         Important hint: Consider the following subset of A: $B = \{x \in A : x \notin g(x)\}$

         What would g being surjective say about the cardinality of A as it compares to the cardinality of $\mathcal{P}(A)$? Since at this point we have shown that there cannot be a surjective function between A and $\mathcal{P}(A)$, what can we now say about the cardinalities of the two as they relate to each other?