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CSCI 2824: Discrete Structures

Lecture 5: Logical Equivalences

Rachel Cox

Department of Computer Science

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Logical Equivalences

A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a **tautology**. The compound propositions p and q are called **logically equivalent** if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are **logically equivalent**.

A compound proposition that is always false is called a **contradiction**.

A compound proposition that is neither a tautology nor a contradiction is called a **contingency**.

Logical Equivalences

Example of a Tautology:

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

Example of a Contradiction:

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

Example: Show that $\neg(p \wedge q) \equiv \neg p \vee \neg q$

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$\neg(p \wedge q) \leftrightarrow \neg p \vee \neg q$
T	T	T	F	F	F	F	T
T	F	F	T	F	T	T	T
F	T	F	T	T	F	T	T
F	F	F	T	T	T	T	T

tautology! so

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

Example: Show that $\neg(p \vee q) \equiv \neg p \wedge \neg q$

$\mathbf{p}$	$\boldsymbol{q}$	$\boldsymbol{p} \vee \boldsymbol{q}$	$\neg(\boldsymbol{p} \vee \boldsymbol{q})$	$\neg\boldsymbol{p}$	$\neg\boldsymbol{q}$	$\neg\boldsymbol{p} \wedge \neg\boldsymbol{q}$	* $\neg(\boldsymbol{p} \vee \boldsymbol{q}) \leftrightarrow \neg\boldsymbol{p} \wedge \neg\boldsymbol{q}$
T	T	T	F	F	F	F	T
T	F	T	F	F	T	F	T
F	T	T	F	T	F	F	T
F	F	F	T	T	T	T	T

$\therefore \boxed{\neg(p \vee q) \equiv \neg p \wedge \neg q}$

Logical Equivalences

We just Proved:

De Morgan's Laws

$$\neg (p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg (p \vee q) \equiv \neg p \wedge \neg q$$

Other Equivalences (from our book) Section 1.3

TABLE 7 Logical Equivalences Involving Conditional Statements.

★ $p \to q \equiv \neg p \vee q$	Relation by Implication (RBI)
★ $p \to q \equiv \neg q \to \neg p$	Contraposition
$p \vee q \equiv \neg p \to q$	
$p \wedge q \equiv \neg(p \to \neg q)$	
$\neg(p \to q) \equiv p \wedge \neg q$	
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$	
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$	
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$	
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$	

TABLE 8 Logical Equivalences Involving Biconditional Statements.

★ $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$	Biconditional
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$	
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$	
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$	

★ $p \oplus q \equiv (p \vee q) \wedge \neg(p \wedge q)$ Alternate definition of xor

Logical Equivalences

Other Equivalences (from our book)

To show that two compound propositions are logically equivalent:

- Prove it with a Truth Table
- Use Equivalence Rules to go from one to the other.

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

Logical Equivalences

Example: Show that $p \to q \equiv \neg q \to \neg p$ (without a truth table)

$$p \to q \equiv \neg p \vee q \qquad \text{RBI}$$

$$\equiv q \vee \neg p \qquad \text{commutativity}$$

$$\equiv \neg q \to \neg p \qquad \text{RRI}$$

RBI

$$p \to q \equiv \neg p \vee q$$

Logical Equivalences

Associativity $(2+3)+5 = 2+3+5 = 2+(3+5)$

Example: Show that $(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$ (without a truth table)

$(p \to q) \vee (p \to r) \equiv (\neg p \vee q) \vee (\neg p \vee r)$ Relation by Implication (RBI)

$\equiv \neg p \vee q \vee \neg p \vee r$ Associativity

$\equiv \underbrace{\neg p \vee \neg p} \vee q \vee r$ Commutativity

$\equiv \neg p \vee q \vee r$ Idempotent

$\equiv \neg p \vee (q \vee r)$ Associativity

$\equiv p \to (q \vee r)$ RBI

Satisfiability

A compound proposition is **satisfiable** if there is an assignment of truth values to its variables that makes it true. If there is no such case then we say it is **unsatisfiable** (i.e. a contradiction)

Example: Show that $\underbrace{(p \vee \neg q)}_{T} \wedge \underbrace{(\neg p \vee q)}_{T} \wedge \underbrace{(\neg p \vee \neg q)}_{T}$ is satisfiable.

No!

Suppose p is True, that requires q to be True.

Suppose p is False, $\Rightarrow$ $\neg q$ is True $\Rightarrow$ q is False

$\boxed{\text{So if } p \equiv F \text{, and } q \equiv F \text{ then the expression is satisfiable}}$

p	q	$\neg p$	$\neg q$	$p \vee \neg q$	$\neg p \vee q$	$\neg p \vee \neg q$

Satisfiability

Example: Show that $\underbrace{(p \to q)}_{T} \wedge \underbrace{(p \to \neg q)}_{T} \wedge \underbrace{(\neg p \to q)}_{T} \wedge \underbrace{(\neg p \to \neg q)}_{T}$ is not satisfiable.

- Suppose $p \equiv T$, impossible! Because q, $\neg q$ cannot both be True. $T \to F$ is $\underline{\underline{F}}$.

Suppose $p \equiv F$. That implies that $\neg p \equiv T$.
Since q and $\neg q$ cannot both True and $T \to F \equiv F$, both of $(\neg p \to q)$ and $(\neg p \to \neg q)$ cannot be T.
Therefore, this compound proposition is not satisfiable.

Exercise: Show with a truth table.

Example: Sudoku puzzles can be written (and solved) as a satisfiability problems

Solving this:

1) First chain together the propositions with provided values: $p(1,1,5) \wedge p(1,2,3) \wedge p(1,5,7) \wedge \ldots$
2) Assert that every row contains every number: $\bigwedge_{i=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{j=1}^{9} p(i,j,n)$
3) Assert that every column contains every number:

$$\bigwedge_{j=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{i=1}^{9} p(i,j,n)$$

4) Assert that every 3x3 block contains every number:

$$\bigwedge_{r=0}^{2} \bigwedge_{s=0}^{2} \bigwedge_{n=1}^{9} \bigvee_{i=1}^{3} \bigvee_{j=1}^{3} p(3r+i, 3s+j, n)$$

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

Let $p(i,j,n)$ denote the proposition that a number n is in the cell in row i and column j

9 rows, 9 columns, 9 numbers $= 9 \times 9 \times 9 =$ 729 propositions

5) Assert that no cell contains more than one number:

$$\bigwedge_{i=1}^{9} \bigwedge_{j=1}^{9} \bigwedge_{n=1}^{9} \bigwedge_{m=1, m \neq n}^{9} (p(i,j,n) \rightarrow \neg p(i,j,m))$$

6) String 1-5 together with conjunctions

[Good Read on this problem](#)!

Extra Practice

Example 1: Work out the truth table to show $\neg(p \vee q) \equiv \neg p \wedge \neg q$

Example 2: Work out the truth table to show $p \rightarrow q \equiv \neg p \vee q$

Example 3: Show that $(p \to q) \land (p \to r) \equiv p \to (q \land r)$

Example 4: Show that the following proposition is **not** satisfiable

$$(p \leftrightarrow q) \wedge (\neg p \leftrightarrow q)$$

Solutions

Example 1: Work out the truth table to show $\neg(p \vee q) \equiv \neg p \wedge \neg q$

Solution:

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Example 2: Work out the truth table to show $p \rightarrow q \equiv \neg p \vee q$

p	q	$\neg p$	$p \rightarrow q$	$\neg p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Example 1: Show that $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$

Solution: This one is actually easier if we start from the second proposition

$$
\begin{aligned}
p \to (q \wedge r) &\equiv \neg p \vee (q \wedge r) && \text{(RBI)} \\
&\equiv (\neg p \vee q) \wedge (\neg p \vee r) && \text{(distribution)} \\
&\equiv (p \to q) \wedge (p \to r) && \text{(that one rule in reverse)}
\end{aligned}
$$

Example 1: Show that the following proposition is **not** satisfiable

$$(p \leftrightarrow q) \land (\neg p \leftrightarrow q)$$

Solution: OK, this ones pretty easy

1. From the first conjunct we know that p and q must have the same truth values
2. From the second conjunct we know that p and q must have different truth values
3. This is a contradiction, thus the proposition is **not** satisfiable