Name: Felipe Lima

ID: 109290055

CSCI 3104, Algorithms **Due Feb 19, 2021**

Problem Set 05 (50 points) **Spring 2021, CU-Boulder**

Advice 1: For every problem in this class, you must justify your answer: show how you arrived at it and why it is correct. If there are assumptions you need to make along the way, state those clearly.
Advice 2: Verbal reasoning is typically insufficient for full credit. Instead, write a logical argument, in the style of a mathematical proof.

Instructions for submitting your solution:

- The solutions **should be typed** and we cannot accept hand-written solutions. Here's a short intro to Latex.
- You should submit your work through **Gradescope** only.
- The easiest way to access Gradescope is through our Canvas page. There is a Gradescope button in the left menu.
- Gradescope will only accept **.pdf** files.
- It is vital that you match each problem part with your work. Skip to 1:40 to just see the matching info.

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1. *(16 pts) Suppose in quicksort, we have access to an algorithm which chooses a pivot such that, the ratio of the size of the two subarrays divided by the pivot is a* ***constant*** *k. i.e an array of size n is divided into two arrays, the first array is of size $n_1 = \frac{nk}{k+1}$ and the second array is of size $n_2 = \frac{n}{k+1}$ so that the ratio $\frac{n_1}{n_2} = k$ a constant.*

   (a) *(3 pts) Given an array, what value of k will result in the best partitioning?*

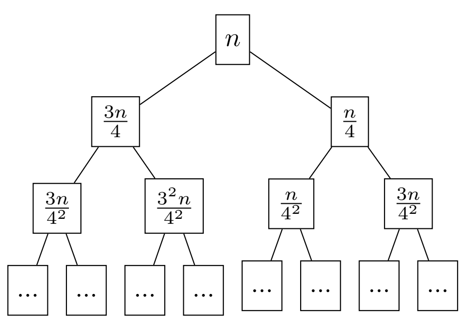
   (b) *(10 pts) Write down a recurrence relation for this version of QuickSort, and solve it asymptotically using* ***recursion tree*** *method to come up with a big-O notation. For this part of the question assume $k = 3$. Show your work, write down the first few levels of the tree, identify the pattern and solve. Assume that the time it takes to find the pivot is $\Theta(n)$ for lists of length n. Note: Remember that a big-O bound is just an upper bound. So come up with an expression and make arguments based on the big-O notation definition.*

   (c) *(3 pts) Does the value of k affect the running time?*

Solution:

(a) The best possible partitioning would be if the array were divided into two subawarrays of the same length. Meaning $n_1 = n_2$ and therefore $\frac{n_1}{n_2} = k = 1$.
The value of k that would result in the best partitioning is $k = 1$.

(b) Recurrence relation for this version of QuickSort: $T(n) = T(\frac{nk}{k+1}) + T(\frac{n}{k+1}) + Cn$
If $k = 3$
Recusion Tree:



For $k_1 : \frac{n}{\frac{4}{3}^{k_1}} = 1 \rightarrow \frac{4}{3}^{k_1} = n \rightarrow k_1 = log_{\frac{4}{3}}(n)$
For $k_2 : \frac{n}{4^{k_2}} = 1 \rightarrow 4^{k_2} = n \rightarrow k_2 = log_4(n)$
$k_1 = log_{\frac{4}{3}}(n)$ $k_2 = log_4(n)$
Since we have that $k_1 > k_2$
$\therefore T(n) = O(nlog_{\frac{4}{3}})$

(c) The value of k does not affect the running time because for any value of k, it will be a constant. Therefore, the function will be always asymptotically equal for any value of k.

2. *(10 pts) Consider a chaining hash table A with b slots that holds data from a fixed, finite universe U.*

   (a) *(3 pts) State the simple uniform hashing assumption.*

   (b) *(7 pts) Consider the worst case analysis of hash tables. Suppose we start with an empty hash table, A. A* ***collision*** *occurs when an element is hashed into a slot where there is another element already. Assume that $|U|$ represents the size of the universe and b represents the number of slots in the hash table. Let us assume that $|U| \leq b$. Suppose we intend to insert n elements into A* ***Do not assume the simple uniform hashing assumption for this subproblem.***

      i. *What is the worst case for the number of collisions? Express your answer in terms of n.*

      ii. *What is the load factor for A in the previous question?*

      iii. *How long will a successful search take, on average? Give a big-Theta bound.*

Solution:

(a) Simple uniform hasing assumption states that each of the hash slots has the same probability of being chosen. That is, there is a hash function that will uniformly distrubute all values into the hash table slots.

(b)

   i. The worst case is if each of the n items are assigned to the same bucket, meaning there are n collisions and search takes $\Theta(n)$ time.

   ii. The load factor of a has table is (number or elements)/(number of slots) $\therefore$ the load factor for A is $\frac{n}{b}$.

   iii. A successful search takes time $\Theta(1 + \alpha)$, on average.

3. *(12 pts) Consider a hash table of size 100 with slots from 1 to 100. Consider the hash function* $h(k) = \lfloor 100k \rfloor$ *for all keys* k *for a table of size 100. You have three applications.*

   - ***Application 1****: Keys are generated uniformly at random from the interval* $[0.3, 0.8]$.
   - ***Application 2****: Keys are generated uniformly at random from the interval* $[0.1, 0.4] \cup [0.6, 0.9]$.
   - ***Application 3****: Keys are generated uniformly at random from the interval* $[0, 1]$.

   (a) *(3 pts) Suppose you have* n *keys in total chosen for each application. What is the resulting load factor* α *for each application?*

   (b) *(3 pts) Which application will yield the worst performance?*

   (c) *(3 pts) Which application will yield the best performance?*

   (d) *(3 pts) Which application will allow the uniform hashing property to apply?*

Solution:

(a) **Application 1:**
The keys can only be within the 30 to 80 interval, meaning there are 50 possible keys on this application
$\therefore \alpha = \frac{n}{50}$

Application 2:
The keys can only be within the 10 to 40 and 60 to 90 interval, meaning there are 60 possible keys on this application
$\therefore \alpha = \frac{n}{62}$

Application 3:
The keys can be within the 0 to 100 interval, meaning there are 100 possible keys on this application
$\therefore \alpha = \frac{n}{100}$

(b) Application 1 will yield to the worst performance because it has the smallest range for key generation and the largest load factor.

(c) Application 1 will yield to the best performance because it has the largest range for key generation and the smallest load factor.

(d) Application 3 will allow the uniform hashing property to be applied because the key generation interval allows for every slots in the hashtable to have an equal chance to be picked.

4. *(12 pts) Median of Medians Algorithm*

   (a) *(4 pts) Illustrate how to apply the QuickSelect algorithm to find the $k = 4$th smallest element in the given array:* `A = [5, 3, 4, 9, 2, 8, 1, 7, 6]` *by showing the recursion call tree. Refer to Sam's Lecture 10 for notes on QuickSelect algorithm works*

   (b) *(4 pt) Explain in 2-3 sentences the purpose of the Median of Medians algorithm.*

   (c) *(4 pts)Consider applying Median of Medians algorithm (A Deterministic QuickSelect algorithm) to find the 4th largest element in the following array:* `A = [6, 10, 80, 18, 20, 82, 33, 35, 0, 31, 99, 22, 56, 3, 32, 73, 85, 29, 60, 68, 99, 23, 57, 72, 25]`*.Illustrate how the algorithm would work for the first two recursive calls and indicate which sub array would the algorithm continue searching following the second recursion. Refer to Rachel's Lecture 8 for notes on Median of Medians Algorithm*

Solution:

(a) Let k be the index to be searched on the in the sorted array. Also, let pivot be the last element of the given array and let p represent the Pivot Index.
Array: $A = [5, 3, 4, 9, 2, 8, 1, 7, 6]$
Left Subarray Right Subarray

$k = 3$ and $p = 5$ Move to left sub tree $k = 3$
$LeftA = [3, 4, 2, 1]$ Using a 5 as the pivot $RightA = [9, 7, 8]$

$k = 3$ and $p = 0$ Move to right sub tree $k = k - 1 = 2$
$LeftA = []$ Using a 1 as the pivot $RightA = [3, 4, 2, 5]$

$k = 2$ and $p = 4$ Move to left sub tree $k = 2$
$LeftA = [1, 3, 2, 4]$ Using a 5 as the pivot $RightA = []$

$k = 2$ and $p = 1$ Move to right sub tree $k = k - 1 = 1$
$LeftA = [1]$ Using a 2 as the pivot $RightA = [4, 3]$

$k = 1$ and $p = 0$ Move to right sub tree $k = k - 1 = 0$
$LeftA = []$ Pick a 3 as the pivot $RightA = [4]$

$k = 0$ and $p = 0$ Return the pivot value. (K+1)th min =4
$LeftA = []$ Pick a 4 as the pivot $RightA = []$

Thus 4 is the 4^{th} smallest element in the given array.

(b) The purpose of the median of medians algorithm is to find a pivot for the sorting algorithm that is not too far from the true median. By doing so, it reduces the worst-case complexity of a quickselect algorithm and builds an asymptotically optimal sorting algorithm.

(c) First step is to by divide the array into groups of five elements:

$A_1 = [6, 10, 80, 18, 20], A_2 = [82, 33, 35, 0, 31], A_3 = [99, 22, 56, 3, 32], A_4 = [73, 85, 29, 60, 68], A_5 = [99, 23, 57, 72, 25]$

Then, sort each of the five groups:

$A_1 = [6, 10, 18, 20, 80] A_2 = [0, 31, 33, 35, 82] A_3 = [3, 22, 32, 56, 99] A_4 = [29, 60, 68, 73, 85] A_5 = [23, 25, 57, 72, 99]$

Take the median of each group and put these values into a list M.

$$M = [18, 33, 32, 68, 57]$$

Sort the list M:

$$M = [18, 32, 33, 57, 68]$$

Choose the median point of the M list as the pivot:

$$P = 33$$

Place all the elements that are smaller than P on the left and all elements that are greater than P on the right. We then get a list A:

$$A = [6, 10, 18, 20, 0, 31, 3, 22, 32, 29, 23, 25, 33, 80, 35, 82, 56, 99, 60, 68, 73, 85, 57, 72, 99]$$

Select all the elements greater than 33 and create a list out of them. Then sort this list. We then have:

$$A'' = [35, 56, 57, 60, 68, 72, 73, 80, 82, 85, 99, 99]$$

Then we have that the 4^{th} largest value is 82.