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CSCI 2824: Discrete Structures

Lecture 15: Functions and Cardinality

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Office Hours on Thursday cancelled

Friday, Monday 10-11am

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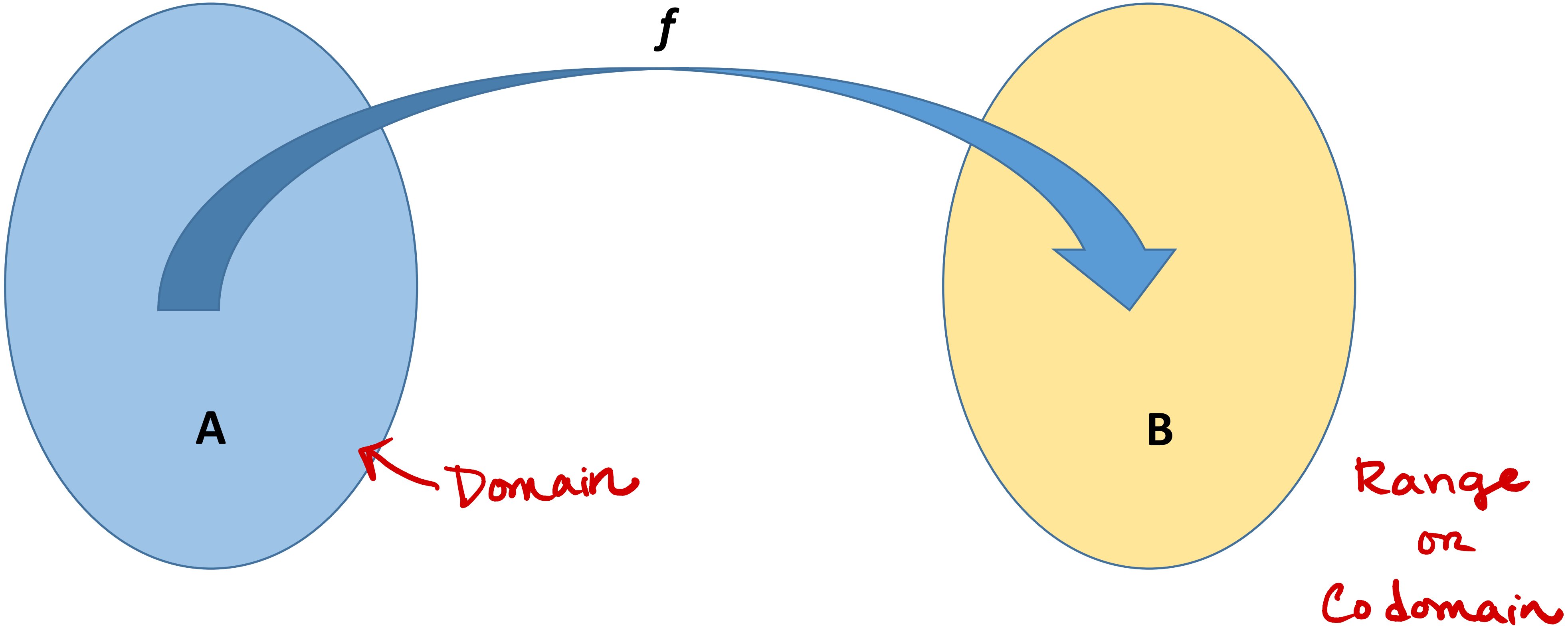
Functions

Let A and B be nonempty sets. A **function** f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f: A \to B$.



Functions

Functions are everywhere in computer science and engineering.

```
In [7]: def Square(x_in):
   ...:
   ...:     x2_out = x_in * x_in
   ...:
   ...:     return (x2_out)
   ...:
```

A function is a routine that takes some kind of input, *does stuff*, and yields some kind of output, as well as possibly some side effects.

Computer science distinction: **function:** takes inputs, produces outputs
vs. **procedure:** takes inputs, produces side effects (but no outputs)

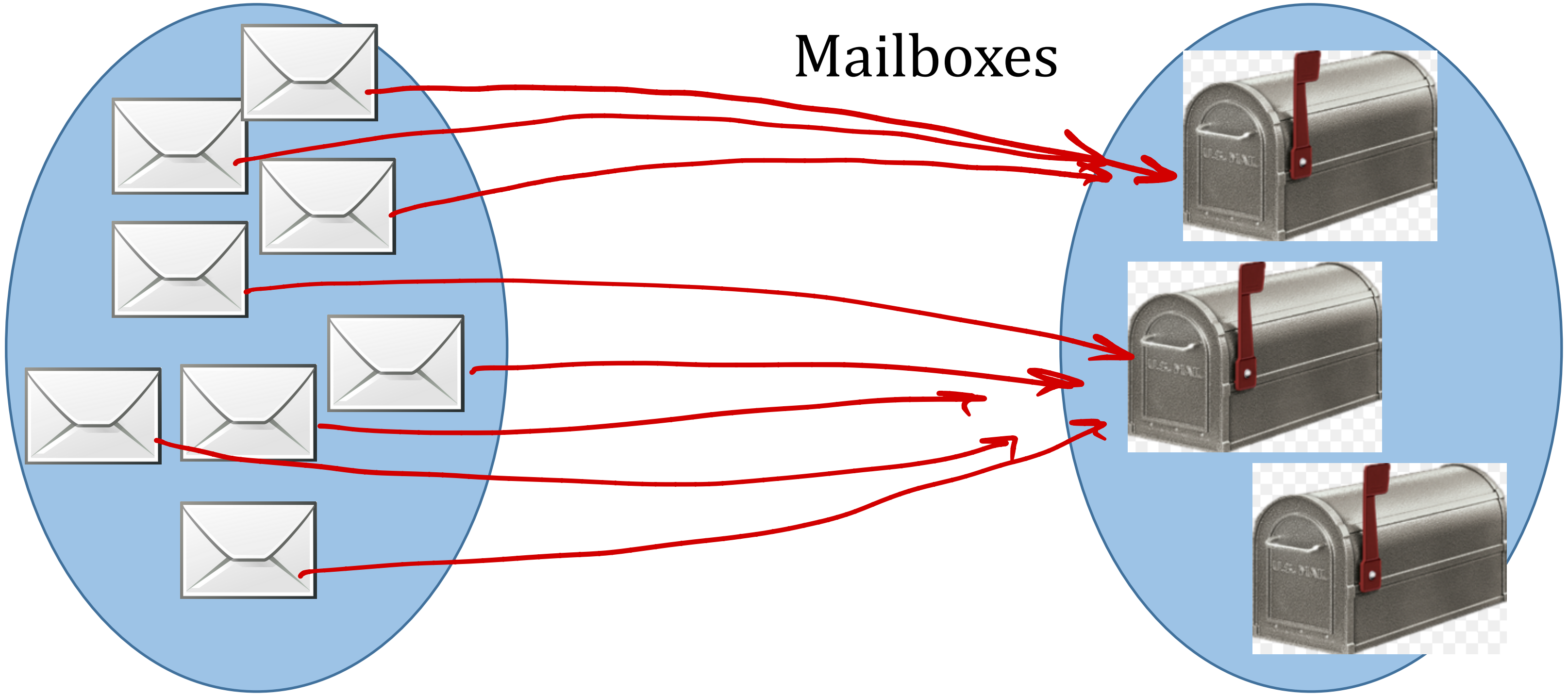
We will use this narrower definition of a function (inputs ⇒ outputs)

Functions

Examples of Functions:

- Let $f\colon \mathbb{Z} \to \mathbb{Z}$ assign the cube of an integer to this integer.

- Letters

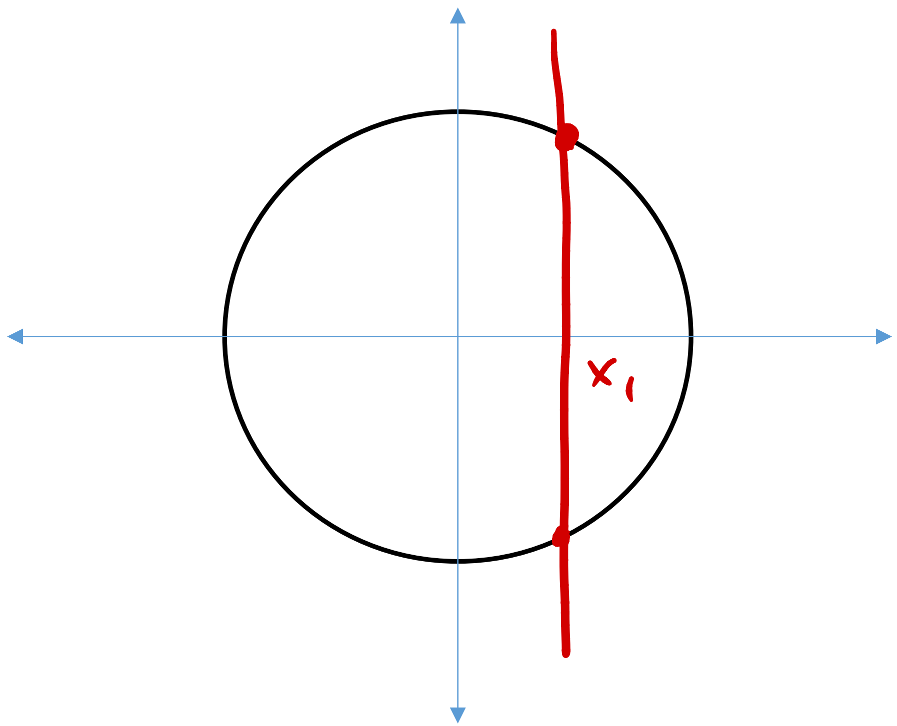


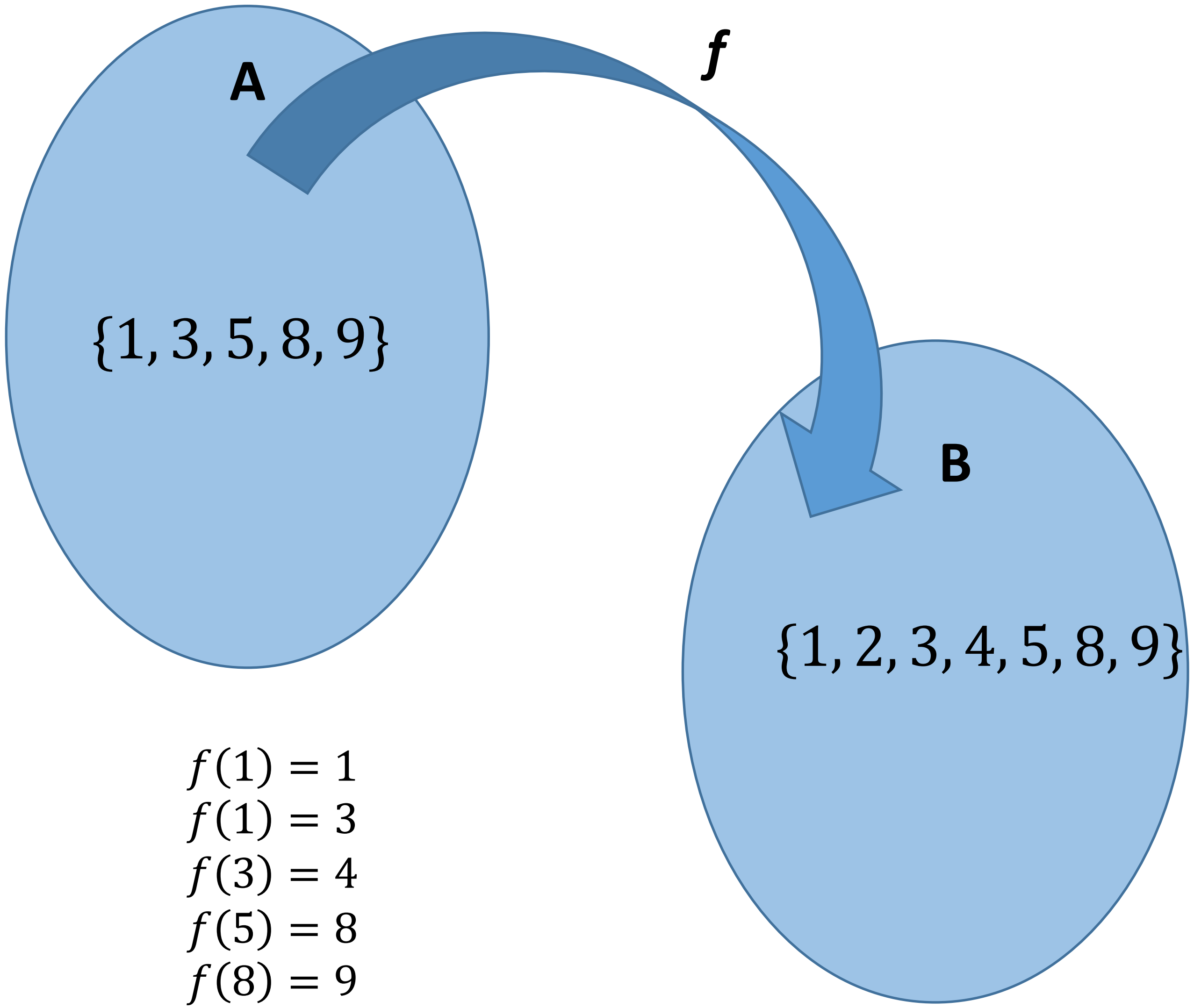
Functions

Examples of **NOT** Functions

$x^2 + y^2 = 16$

vertical line test





$f(1) = 1$
$f(1) = 3$
$f(3) = 4$
$f(5) = 8$
$f(8) = 9$
$f(9) = 9$

Functions

If f is a function from A to B, we say that A is the **domain** of f and B is the **codomain** of f. If $f(a) = b$, we say that b is the **image** of a and a is the **preimage** of b. The **range**, or image, of f is the set of all images of elements of A. Also, if f is a function from A to B, we say f maps A to B.

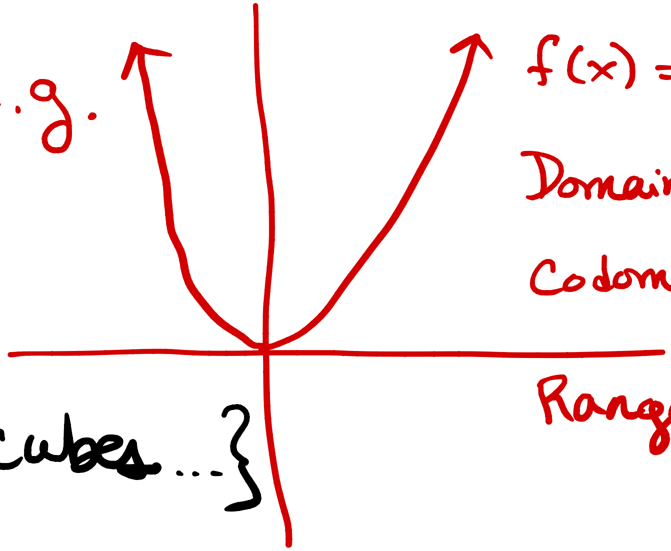
Example: Let $f\colon \mathbb{Z}^+ \to \mathbb{Z}^+$ assign the cube of an integer to this integer. What is the domain, the codomain, and the range of this function?

domain: $\mathbb{Z}^+$

Codomain: $\mathbb{Z}^+$

Range: $\{1, 8, 27 \ldots$ all perfect cubes$\ldots\}$

e.g.



$f(x) = x^2$

Domain: $(-\infty, \infty)$

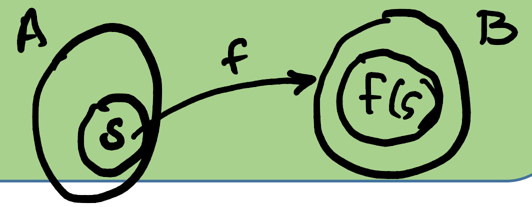
Codomain: $(-\infty, \infty)$

Range: $[0, \infty)$

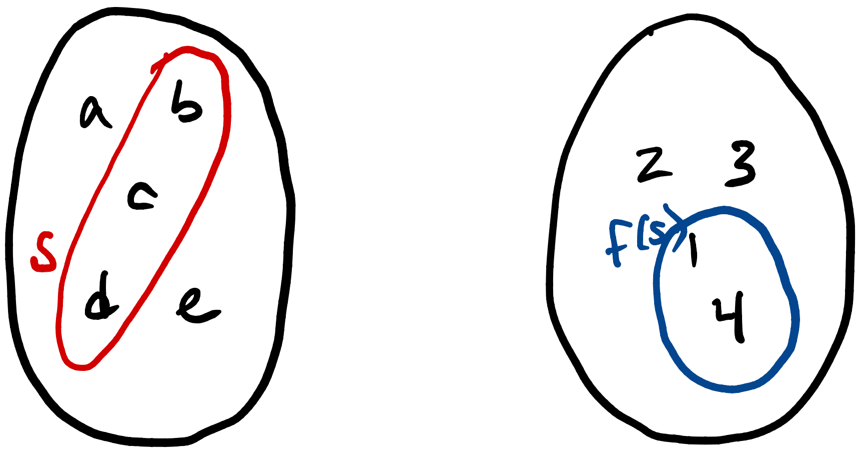
Functions

Let f be a function from A to B and let S be a subset of A. The image of S under the function f is the subset of B that consists of the images of the elements of S. We denote the image of S by $f(S)$, so

$$f(S) = \{t \mid \exists s \in S\ (t = f(s))\}$$



Example: Let $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4\}$ with $f(a) = 2, f(b) = 1, f(c) = 4$, $f(d) = 1$, and $f(e) = 1$. The image of the subset $S = \{b, c, d\}$ is the set $f(S) = \{1, 4\}$.



Functions: One–to–One, Onto

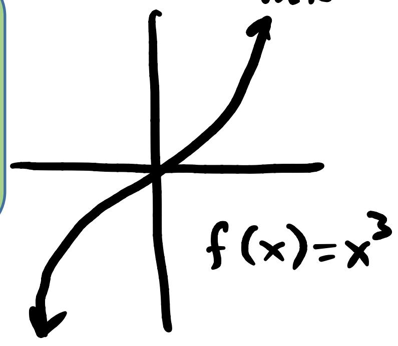
A function f is said to be **one-to-one**, or an injunction, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be ***injective*** if it is one-to-one.

A function f from A to B is said to be **onto**, or a surjection, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function is said to be ***surjective*** if it is onto.

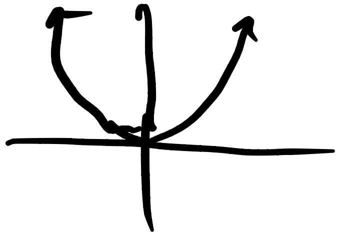
The function f is a one-to-one correspondence, or a **bijection**, if it is both one-to-one and onto.

$f: \mathbb{R} \to \mathbb{R}$

e.g. 1-1 onto



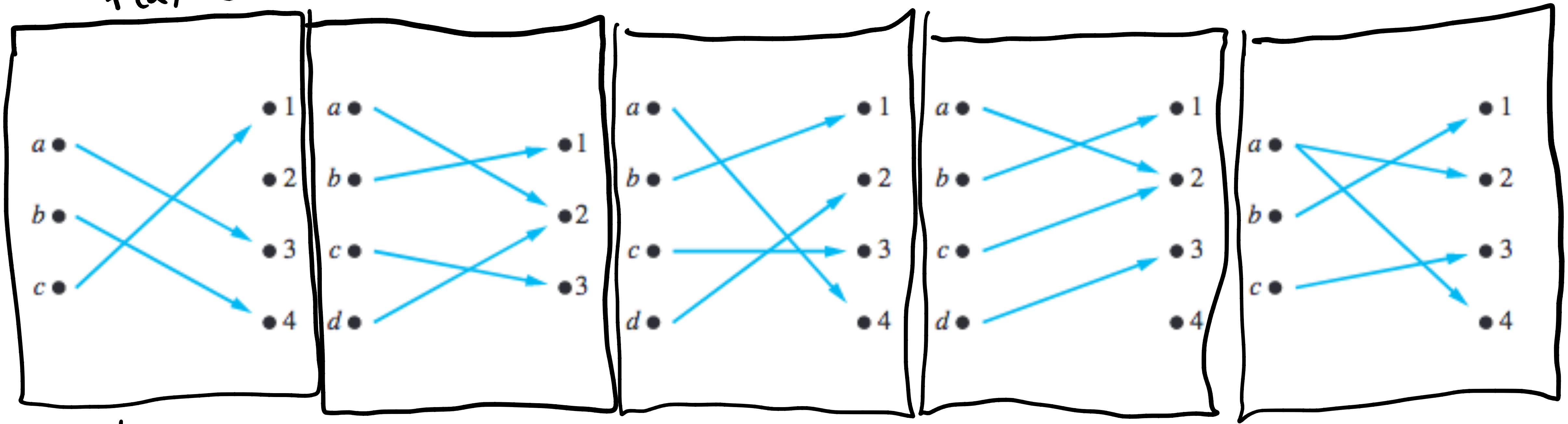
not 1-1
not onto
$f: \mathbb{R} \to \mathbb{R}$



Functions: One–to–One, Onto

Example: Classify each of these functions as one-to-one, onto, both, or neither.

$f(a) = 3$



one-to-one
not onto

not one-to-one
onto

one-to-one
AND
onto

not 1-1
AND
not onto

Not
a
function

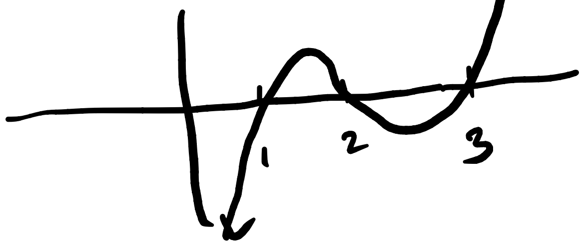
Functions

A function f whose domain and codomain are subsets of the set of real numbers is called ***increasing*** if $f(x) \leq f(y)$, and ***strictly increasing*** if $f(x) < f(y)$, whenever $x < y$ and x and y are in the domain of f. Similarly, f is called ***decreasing*** if $f(x) \geq f(y)$, and ***strictly decreasing*** if $f(x) > f(y)$, whenever $x < y$ and x and y are in the domain of f.

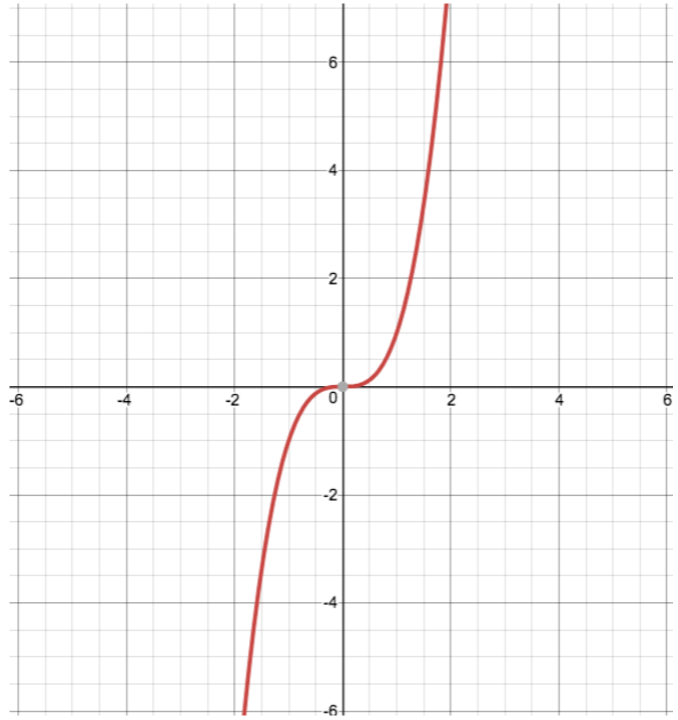
- A function that is either strictly increasing or strictly decreasing must be one-to-one.

Not 1-1

$f(x) = (x-1)(x-2)(x-3)$



$f(x) = x^3$



Functions

Example: Prove that $f(n) = n^3$ is one-to-one. (implicit: $f\colon \mathbb{R} \to \mathbb{R}$)

We must show that $f(a) = f(b) \Rightarrow a = b$

Proof: Assume $f(a) = f(b)$ for arbitrary real numbers a, b.

$$a^3 = b^3$$

$$a = b$$

e.g $f(x) = x^3$

$f'(x) = 3x^2 > 0$

Functions

Example: Prove that $f(n) = n^2$ is NOT one-to-one. (implicit: $f\colon\ \mathbb{R} \to (\mathbb{R} \geq 0)$)

Consider $n = -2$ and $m = 2$

Then $f(-2) = 4$ and $f(2) = 4$

We have $f(-2) = f(2)$

$$-2 \neq 2$$

So this is not one-to-one.

Functions

domain

codomain

Example: Prove that $f(n) = n^2$ is NOT onto. (where: $f\colon \mathbb{Z} \to \mathbb{N}$)

$f(-5) = 25$

$f(-1) = 1$

$f(0) = 0$

$f(1) = 1$

$f(2) = 4$

Consider the number 3

$3 \in \mathbb{N}$

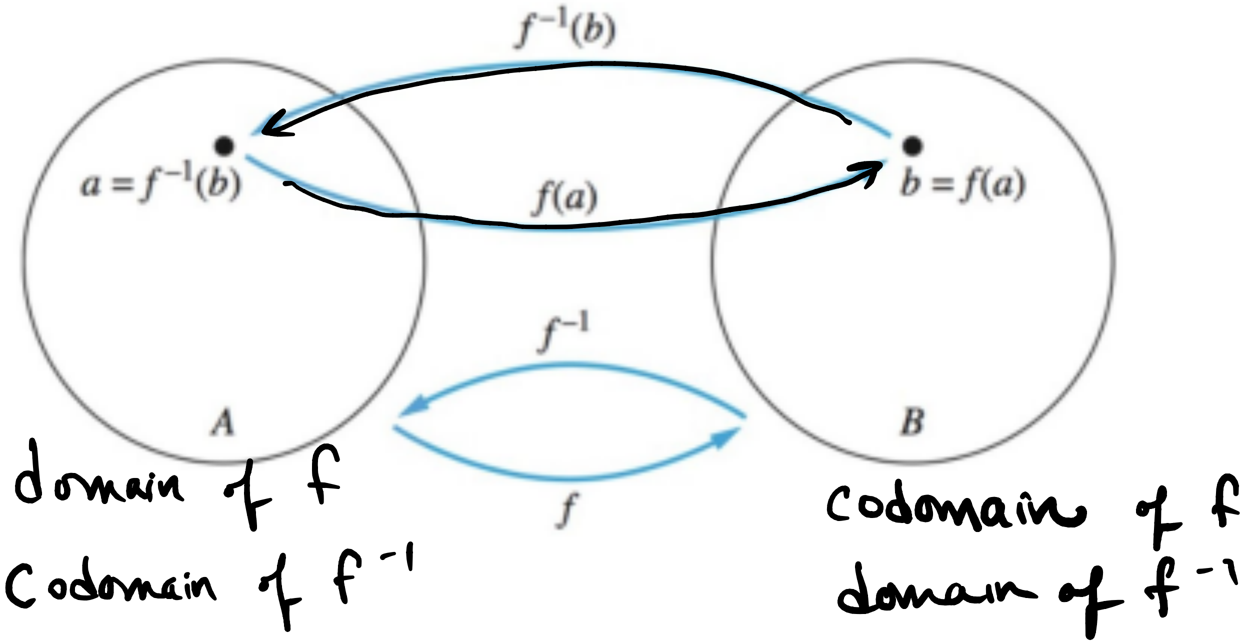
$n^2 = 3$

There is not an <u>integer</u> that makes this true.

Range: Perfect Squares

Functions – Inverse Functions

Let f be a one-to-one and onto function from A to B. Then there exists an inverse function, f^{-1}, such that $f^{-1}(b) = a$ when $f(a) = b$.



If $f: A \to B$ is 1-1 and onto, then:

- f maps to each of the elements of B (because f is onto)
- But f is 1-1 as well, so each element of B has a unique element in A that maps to it.

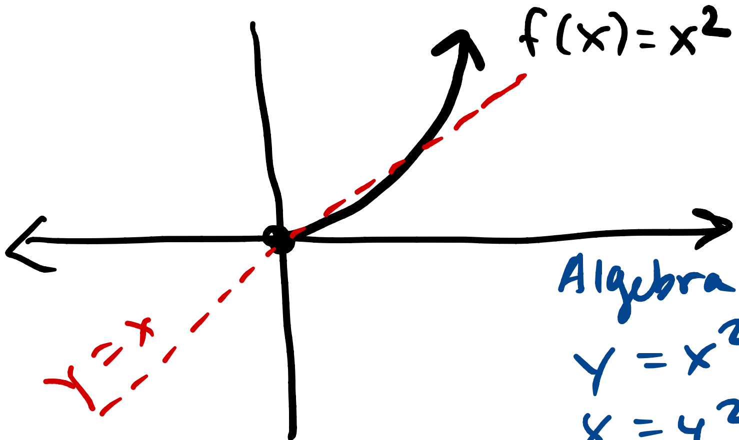
There is a unique one-to-one correspondence between elements in A and elements in B. When this happens, we can go back and forth between A and B via f and f^{-1}.

Functions – Inverse Functions

Example: The inverse of $f(x) = x^3$, where $f\colon \mathbb{R} \to \mathbb{R}$ is $f^{-1}(y) = y^{1/3}$.

Example: The $f(x) = x^2$, where $f\colon \mathbb{R} \to \mathbb{R}$ does not have an inverse. Could we redefine f such that it does have an inverse?

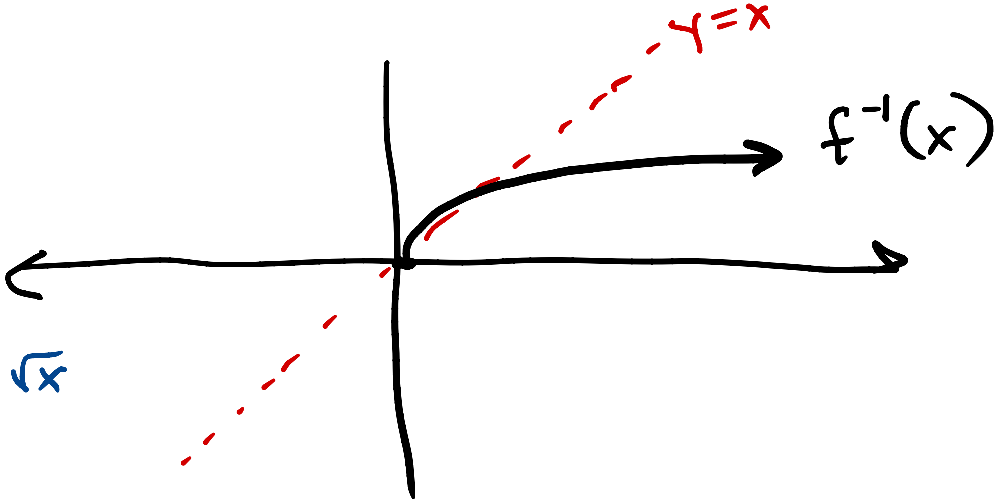
Suppose we Restrict the domain: $f\colon \mathbb{R}^+ \to \mathbb{R}$



Algebra

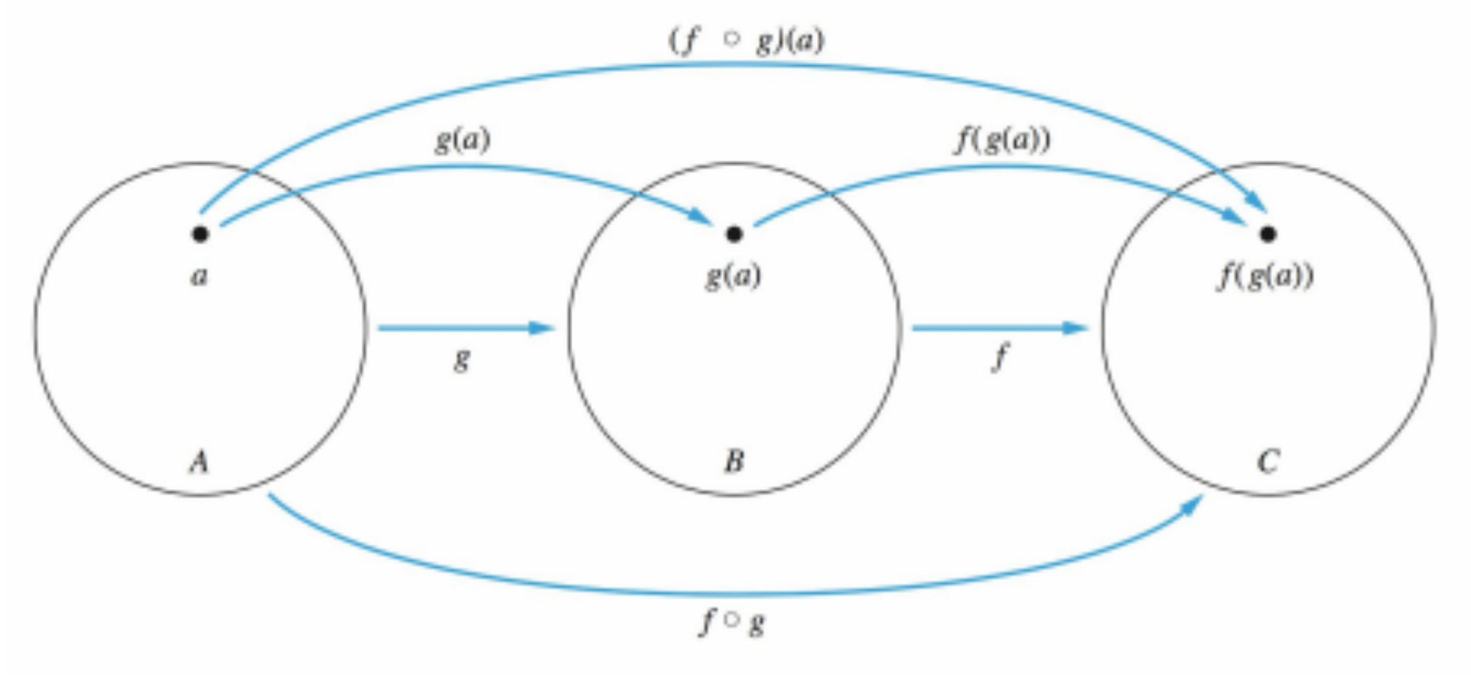
$y = x^2$

$x = y^2 \quad \rightarrow \quad y = \pm\sqrt{x}$



Functions – Composition of Functions

Let g be a function from set A to set B, and let f be a function from set B to set C. The composition of f and g, denoted $f \circ g$, is defined for $a \in A$ by $(f \circ g)(a) = f\big(g(a)\big)$.



Functions – Composition of Functions

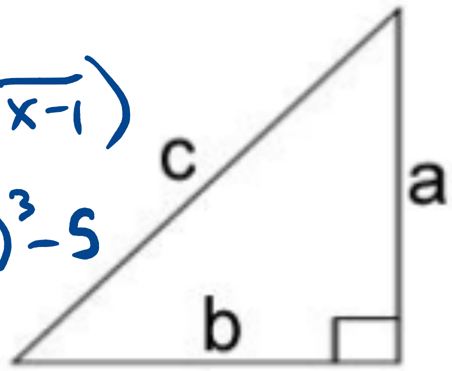
Example: $c = \sqrt{a^2 + b^2}$

```
In [14]: # a function for adding
    ...: def Add(x_in, y_in):
    ...:
    ...:     sum_out = x_in + y_in
    ...:
    ...:     return (sum_out)
    ...:
    ...: # a function for squaring
    ...: def Square(x_in):
    ...:
    ...:     x2_out = x_in * x_in
    ...:
    ...:     return (x2_out)
    ...: c = pow( Add( Square(3), Square(4) ) , 0.5)
    ...: print(c)
    ...:
5.0
```

$$\begin{cases} f(x) = x^3 - 5 \\ g(x) = \sqrt{x-1} \end{cases}$$

$$f(g(x)) = f(\sqrt{x-1})$$

$$= (\sqrt{x-1})^3 - 5$$



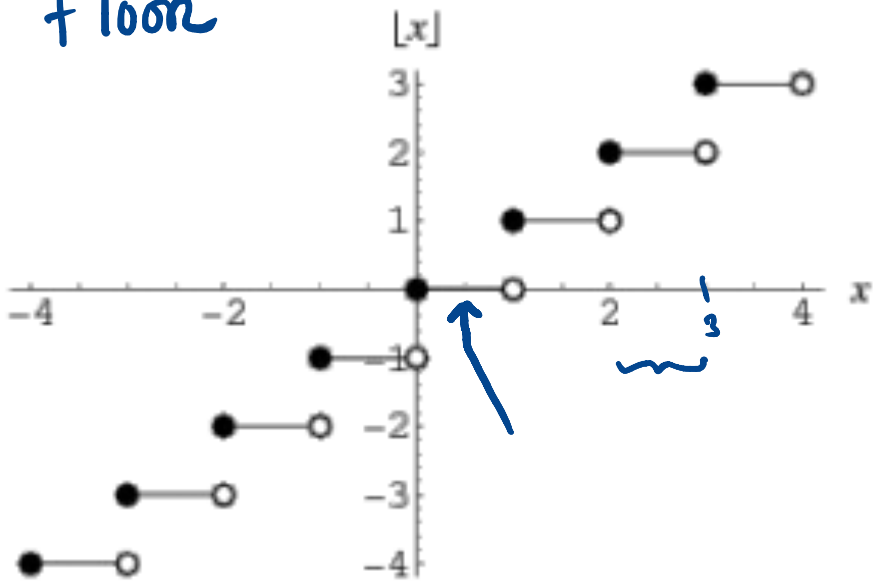
$a^2 + b^2 = c^2$

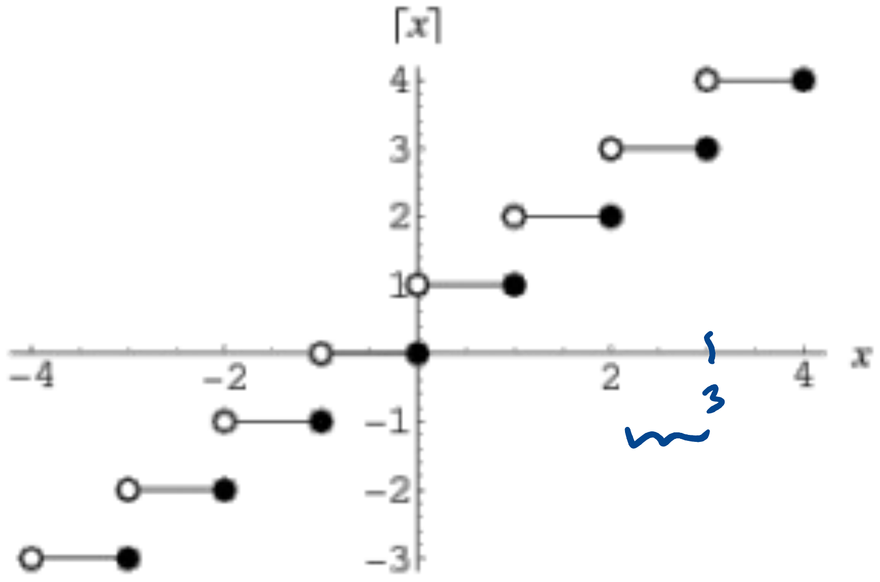
Functions

Definition: The **floor** function, denoted $\lfloor x \rfloor$, assigns to the real number x the largest integer that is less than or equal to x. The **ceiling** function, denoted $\lceil x \rceil$, assigns to the real number x the smallest integer that is greater than or equal to x.

Remark: Both of these come up frequently in CS applications.

floor





Functions

Example: Being able to vote is an example of a floor function.

More examples:

$\lfloor 3.5 \rfloor = 3$	$\lfloor 5 \rfloor = 5$	$\lfloor -3.5 \rfloor = -4$
$\lceil 3.5 \rceil = 4$	$\lceil 5 \rceil = 5$	$\lceil -3.5 \rceil = -3$

Functions

Example: Prove or disprove that $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$.

Let's try an example: $x = 4.01$

$y = 5.01$

$x + y = 9.02$

$\lceil 9.02 \rceil = 10$

$\lceil 4.01 \rceil = 5$

$\lceil 5.01 \rceil = 6$

$10 \neq 5 + 6$

This is disproven with a counterexample.

Cardinality of Sets

A set A is called **countable** or **countably infinite** if it is not finite and there is a one-to-one function between each element of A and the natural numbers. A is called **uncountable** if it is infinite and not countable.

- ***Countably infinite***: we could count up each member of the set if we had infinite time.

- ***Uncountably infinite***: we could never count or list each element of the set, even if we had infinite time.

Cardinality of Sets

Example: Is the set of positive even integers $\boxed{\text{countably infinite}}$? or uncountably infinite?

$$
\begin{array}{lcccccccc}
\text{Even integers:} & 0, & 2, & 4, & 6, & 8, & 10, & \ldots\ldots \\
 & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
\mathbb{N}: & 0, & 1, & 2, & 3, & 4, & 5, & \ldots\ldots
\end{array}
$$

$f(n) = 2n$ for $n \in \mathbb{N}$

Cardinality of Sets

Example: Is the set of all integers countably infinite? or uncountably infinite?

Integers: $0, 1, -1, 2, -2, 3, -3, 4, -4, 5, -5, \ldots$

$\updownarrow \; \updownarrow \; \updownarrow \; \updownarrow \; \updownarrow \; \updownarrow \; \updownarrow \; \updownarrow$

$\mathbb{N}$: $0, 1, 2, 3, 4, 5, 6, 7, 8, \ldots$

$$f(n) = \begin{cases} -\dfrac{n}{2} & n \text{ even} \quad (0 \text{ and negative integers}) \\ \dfrac{n+1}{2} & n \text{ odd} \quad (\text{positive integers}) \end{cases}$$

Cardinality of Sets

Example: Is the set of positive rational numbers countably infinite? or uncountably infinite?

$1/1$	$1/2$	$1/3$	$1/4$	$1/5$	$1/6$	$\cdots$
$2/1$	$2/2$	$2/3$	$2/4$	$2/5$	$2/6$	$\cdots$
$3/1$	$3/2$	$3/3$	$3/4$	$3/5$	$3/6$	$\cdots$
$4/1$	$4/2$	$4/3$	$4/4$	$4/5$	$4/6$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Thm:

A countable number of countable sets is countable.

e.g. each row is countable and each row is countable.

Cardinality of Sets

Cantor's Diagonalization Argument

Example: Is the set of real numbers countably infinite? or uncountably infinite?

Suppose the Real numbers are countable. Specifically that [0, 1] is countable.

K	
1	0.2 5 7 8 9 1 3 2 5 4 ......
2	0.3 4 1 2 3 4 7 7 8 9 ......
3	0.4 7 8 9 2 1 0 0 2 3 ......
4	0.5 6 7 8 9 1 2 3 4 5 ......
5	0.1 1 1 1 1 1 1 1 1 1 ....
6	0.1 2 1 2 1 3 4 5 6 7 ....
⋮	⋮
K	

Make / construct a number m such that the k^{th} decimal of m = 1 if your k^{th} number doesn't have a 1 in that position and the k^{th} decimal of m = 0 if your k^{th} number = 1 in that same position.

m = 0.111101 .....

- We just learned about functions: one-to-one, onto, injective, surjective, bijective functions, inverse functions, composition of functions

- We talked about different sizes of infinity and the cardinality of sets.

Next: Sequences!

Extra Practice

Example 1: Prove or disprove that $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$

Solution

Example 1: Prove or disprove that $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$

DisProof: This equality turns out to not be true. To prove this, we just need to find a counterxample that breaks it.

Notice that on the RHS, $\lceil x \rceil$ and $\lceil y \rceil$ will each round if x and y are slightly bigger than a whole integer

On the other hand, if x and y are only slightly bigger than whole integers then $\lceil x + y \rceil$ will only round up one integer

Counterexample: Let $x = 2.1$ and $y = 3.1$. Then

$$\lceil x + y \rceil = \lceil 5.2 \rceil = 6 \quad \text{but} \quad \lceil x \rceil + \lceil y \rceil = \lceil 3 \rceil + \lceil 4 \rceil = 7$$