CSCI 3104: Algorithms

Lecture 4: Asymptotic Notation and Analysis

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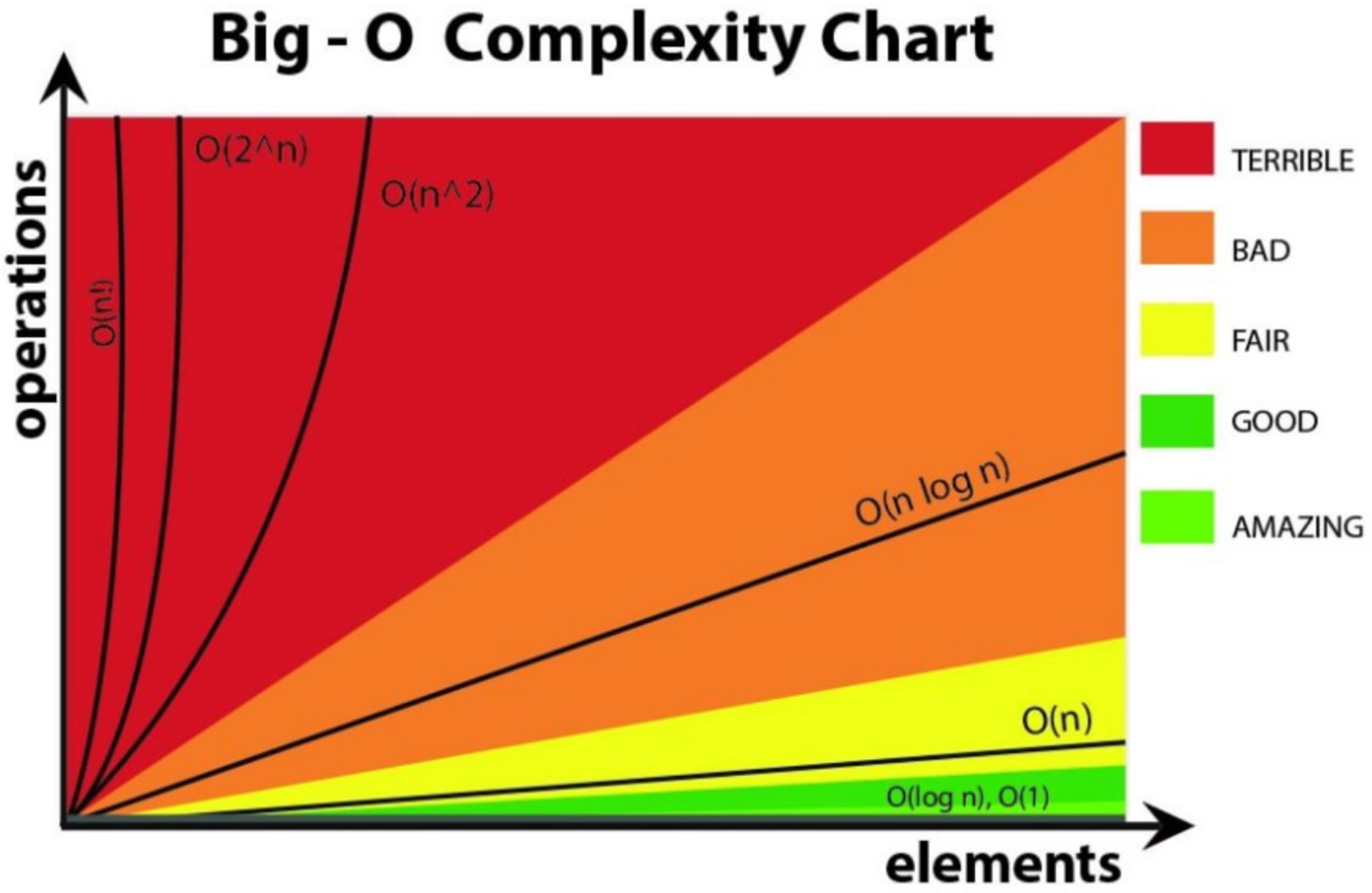


Source

Asymptotic Analysis

Example: Suppose that $T(n) = \frac{1}{2}n^2 - 3n$. Prove that $T(n)$ is $\Theta(n^2)$.

n_0

Need to determine c_1, c_2, K such that $c_1 n^2 \le T(n) \le c_2 n^2$

Specifically $\underbrace{c_1 n^2 \le \frac{1}{2}n^2 - 3n} \le c_2 n^2$

$\frac{1}{2}n^2 - 3n$

Look at this term by term

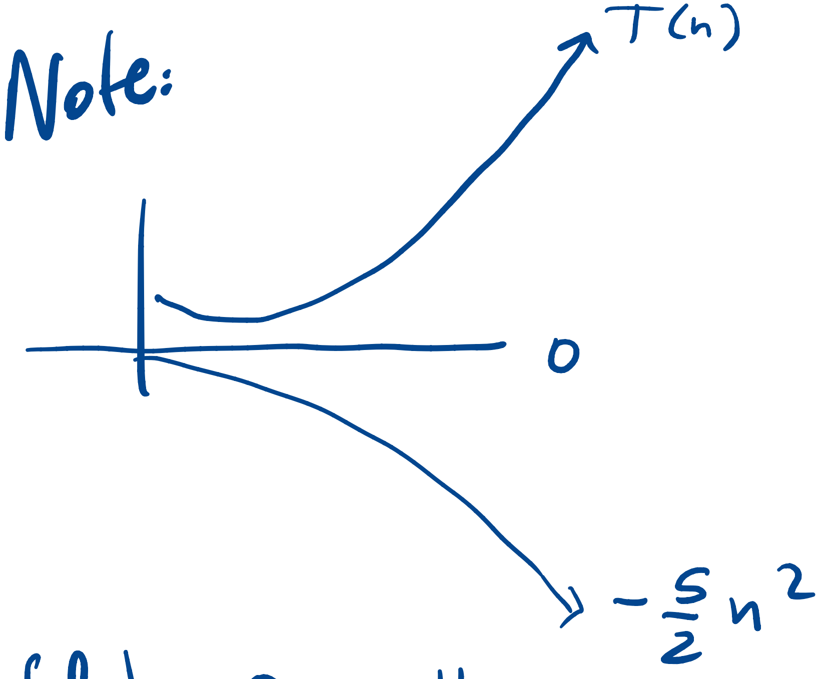
$\frac{1}{2}n^2 \ge \frac{1}{2}n^2$

$3n \le 3n^2$ when $n \ge 1$

$-3n \ge -3n^2$

$\Rightarrow \frac{1}{2}n^2 - 3n \ge \frac{1}{2}n^2 - 3n^2 = -\frac{5}{2}n^2$

Note:



unhelpful! 0 would be better.

Asymptotic Analysis

Example: Suppose that $T(n) = \frac{1}{2}n^2 - 3n$. Prove that $T(n)$ is $\Theta(n^2)$.

$c_1 n^2 \le \frac{1}{2}n^2 - 3n \le c_2 n^2$

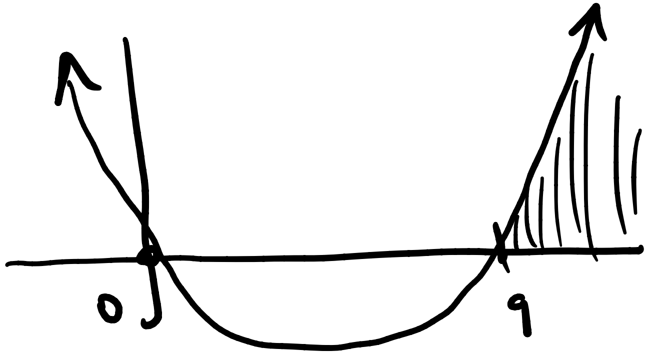
$\frac{1}{2}n^2 \geq \frac{1}{2}n^2$

$3n \le \frac{1}{3}n^2$ $\quad n \geq 9$

$0 \le \frac{1}{3}n^2 - 3n$

$0 \le \frac{1}{3}n(n-9)$. $\quad n \geq 9$

Scratch work



Let's put this altogether

$\frac{1}{2}n^2 \geq \frac{1}{2}n^2$

$-3n \geq -\frac{1}{3}n^2$

$\Rightarrow \frac{1}{2}n^2 - 3n \geq \frac{1}{2}n^2 - \frac{1}{3}n^2$ for $n \geq 9$

$= \frac{1}{6}n^2$

We've shown that

$\frac{1}{6}n^2 \le \frac{1}{2}n^2 - 3n$ for $n \geq 9$

Asymptotic Analysis

Example: Suppose that $T(n) = \frac{1}{2}n^2 - 3n$. Prove that $T(n)$ is $\Theta(n^2)$.

Now, we consider $\frac{1}{2}n^2 - 3n \leq c_2 n^2$

$\frac{1}{2}n^2 \leq \frac{1}{2}n^2$ for $\underline{\underline{n \geq 0}}$

$-3n \leq 0$

$\Rightarrow \frac{1}{2}n^2 - 3n \leq \frac{1}{2}n^2$

Putting everything together, $\frac{1}{6}n^2 \leq \frac{1}{2}n^2 - 3n \leq \frac{1}{2}n^2$

This shows that $T(n)$ is $\Theta(n^2)$ with $c_1 = 1/6$, $c_2 = 1/2$ and $n_0 = 9$.

Asymptotic Analysis

Example: Suppose we run an algorithm to determine whether an array A contains the integer t. What is the running time?

Algorithm 1

```
1: for i = 1 to n do
2:     if A[i] == t then
3:         Return TRUE
4: Return FALSE
```

costs	number of times
c_1	$n+1$
c_2	n
c_3	1
c_4	1

$$T(n) = c_1(n+1) + c_2 n + c_3 + c_4$$

$$= (c_1 + c_2)\, n + (c_1 + c_3 + c_4)$$

$$= an + b$$

$$\leq an + bn \quad \text{for } n > 1$$

$$\Rightarrow T(n) \leq dn \qquad \Rightarrow \boxed{T(n) \text{ is } O(n)}$$

Asymptotic Analysis

Two consecutive for loops

Example: Suppose we run an algorithm to determine whether arrays A or B contain the integer t. What is the running time?

Assume both A and B have length n

Algorithm 2

```
for i = 1 to n do
    if A[i] == t then
        Return TRUE
for i = 1 to n do
    if B[i] == t then
        Return TRUE
Return FALSE
```

Lines 1–3: $O(n)$

Lines 4–6: $O(n)$

Line 7: 1

$$T_1(n) \le d_1 n$$

$$T_2(n) \le d_2 n$$

$$T(n) = T_1(n) + T_2(n)$$
$$\le d_1 n + d_2 n$$
$$= (d_1 + d_2) n$$
$$= d_3 n$$

$\Rightarrow$ This running time is also $O(n)$.

Asymptotic Analysis

Nested for loops

Example: Suppose we run an algorithm to determine whether arrays A and B have a number in common. What is the running time?

Algorithm 3

```
for i = 1 to n do
    for j = 1 to n do
        if A[i] == B[j] then
            Return TRUE
Return FALSE
```

costs	times
c_1	$n+1$
c_2	$n \cdot (n+1)$
c_3	$n \cdot n$
c_4	1
c_5	1

number of times the outer for loop is executed.

$$
\begin{aligned}
T(n) &= c_1(n+1) + c_2 n(n+1) + c_3 n^2 + c_4 + c_5 \\
&= (c_2+c_3)n^2 + (c_1+c_2)n + (c_4+c_5+c_1) \\
&= an^2 + bn + c \\
&\leq an^2 + bn^2 + cn^2 \qquad \text{for } n \geq 1 \\
&= dn^2
\end{aligned}
$$

$$\boxed{\Rightarrow T(n) \text{ is } \Theta(n^2)}$$

Asymptotic Analysis

Example: Let $f(n) = 3n^2 + 17nlog_2(n) + 1000$. Which of the following are true?

$A.$ $f(n)$ is $\mathcal{O}(n^2)$.

$B.$ $f(n)$ is $\mathcal{O}(n^3)$.

C. Both A and B.

D. Neither A nor B.

Asymptotic Analysis

Example: Which is an equivalent definition of big Omega notation?

A. $f(n)$ is $\Omega(g(n))$ iff $g(n)$ is $\mathcal{O}(f(n))$.

B. $f(n)$ is $\Omega(g(n))$ iff there exist constants $c > 0$ such that $f(n) \geq c \cdot g(n)$ for infinitely many n.

$\boxed{\text{C. Both A and B.}}$

D. Neither A nor B.

Suppose $g(n)$ is $O(f(n))$

$\Rightarrow \quad g(n) \leq c f(n)$

$\frac{1}{c} g(n) \leq f(n)$

$f(n) \geq \frac{1}{c} g(n)$

$\Rightarrow \quad f(n)$ is $\Omega(g(n))$

Asymptotic Analysis

Example: Which is an equivalent definition of big Theta notation?

$\Theta: \quad \underbrace{c_1 g(n) \leq f(n)}_{\Omega} \underbrace{\leq c_2 g(n)}_{O}$

A. $f(n)$ is $\Theta(g(n))$ iff $f(n)$ is both $O(g(n))$ and $\Omega(g(n))$.

B. $f(n)$ is $\Theta(g(n))$ iff $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c$ for some constant $0 < c < \infty$.

C. Both A and B.

D. Neither A nor B.

Asymptotic Analysis

$\mathcal{O}(1)$: Functions bounded above by a constant.

$\Rightarrow$ no dependence on n

- good!

- constant running time regardless of input

e.g. hashing

$\Omega(1)$: Functions bounded below by a constant.

$\Rightarrow f(n) \geq c \cdot 1$ $\rightarrow$ not saying much

$o(1)$: Function that converges to 0.

$f(n) < c \cdot 1$

fantastic!

As $n \to \infty$, Run time $\to 0$

$\omega(1)$: Function that converges to ∞.

$f(n) > c$

$\Theta(n)$, $\Theta(n^2)$

Limits at Infinity – Review!

$\frac{\infty}{\infty}$ or $\frac{0}{0}$

$$\log^q(n) < n^p < a^n < n! < n^n$$

L'hospital's Rule: $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$

- $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0 \Rightarrow f(n) < o(g(n))$

  f(n) is o(g(n))

  · means g(n) → ∞ much faster than f(n)

  e.g. $f(n) = \log^2(n)$, $g(n) = n$

  $\lim_{n\to\infty} \frac{\log^2(n)}{n} \overset{L'H}{=} \lim_{n\to\infty} \frac{2\log n \cdot \frac{1}{n}}{1} = 2\lim_{n\to\infty}\frac{\log n}{n} \overset{L'H}{=} 2\lim_{n\to\infty} \frac{1}{n} = 0$

- $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty \Rightarrow f(n) > \omega(g(n))$

  · means f(n) → ∞ much faster than g(n)

  e.g. $f(n) = n!$, $g(n) = 4^n$

  $\lim_{n\to\infty} \frac{n!}{4^n} = \lim_{n\to\infty} \frac{n(n-1)(n-2)(n-3)\cdots 4\cdot 3\cdot 2\cdot 1}{4\cdot 4\cdot 4\cdot 4 \cdots 4\cdot 4\cdot 4\cdot 4} = \frac{6}{64}\lim_{n\to\infty} \frac{n}{4}\cdot\frac{n-1}{4}\cdot\frac{n-2}{4}\cdots\frac{5}{4} = \infty$

- $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c \Rightarrow f(n) = \Theta(g(n))$

  · means that f(n) and g(n) are the same "order"

  e.g. $f(n) = 3n^2 + 7n - 2$, $g(n) = n^2$

  $\lim_{n\to\infty} \frac{3n^2+7n-2}{n^2} \boxed{= 3}$

Logs and Exponents– Review!

$$(x^y)^z = x^{yz}$$

$$x^y x^z = x^{y+z}$$

$$\log_x y = z \text{ iff } y = x^z$$

$$x^{\log_x y} = y$$

$$\log xy = \log x + \log y$$

$$\log(x^c) = c \log x$$

$$\log \frac{x}{y} = \log x - \log y$$

$$\log_c x = \frac{\log x}{\log c}$$

change of base formula

Common Functions

function	description
1	constant, independent of n
$\log n$	sublinear, specifically logarithmic
n^c for $c < 1$	sublinear $n^{1/2}$
n	linear
$n \log n$	super-linear, but much less than quadratic
n^2	quadratic
n^3	cubic
n^c for $c > 1$	polynomial, super-linear
c^n for $c > 1$	exponential
$n!$ or n^n	extremely fast-growing functions

Asymptotic Analysis

notation: means $n\log n + n$ is $o(n^2)$

Example: Show that $nlogn + n \leq o(n^2)$.

$$\lim_{n\to\infty} \frac{n\log n + n}{n^2} \overset{L'H}{=} \lim_{n\to\infty} \frac{n\left(\frac{1}{n}\right) + \log n \cdot 1 + 1}{2n}$$

$$= \lim_{n\to\infty} \frac{2 + \log n}{2n}$$

$$\overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{2}$$

$$= \lim_{n\to\infty} \frac{1}{2n}$$

$$= 0$$

$$\boxed{\Rightarrow n\log n + n \text{ is } o(n^2)}$$

Order of Growth

change of base formula: $\log_a n = \dfrac{\log_b(n)}{\log_b(a)}$

Example: Put the growth rates in order, from slowest growing to fastest.

$$\log_4^2(n) \quad \log_5(n) \quad \log_7(n) \quad \log_{3.1}(n) \quad \log_4(n^3) \quad \sqrt[5]{n}$$

$$\log_4(n^3) = 3\log_4(n)$$

Look at: $\log_7(n), \log_5(n), \log_4(n^3), \log_{3.1}(n)$

- $\lim\limits_{n\to\infty} \dfrac{\log_7(n)}{\log_5(n)} = \lim\limits_{n\to\infty} \dfrac{\frac{\log_5(n)}{\log_5(7)}}{\log_5(n)} = \lim\limits_{n\to\infty} \dfrac{1}{\log_5(7)} = \dfrac{1}{\log_5(7)}$ ← some constant

$\Rightarrow \log_7(n) = \Theta(\log_5(n))$

Calc review: $\dfrac{d}{dx}\log_b x = \dfrac{1}{\ln(b)\cdot x}$

- $\lim\limits_{n\to\infty} \dfrac{\log_5(n)}{\log_4(n^3)} = \lim\limits_{n\to\infty} \dfrac{\log_5(n)}{3\log_4(n)} \overset{L'H}{=} \lim\limits_{n\to\infty} \dfrac{\frac{1}{\ln 5}\cdot\frac{1}{n}}{3\cdot\frac{1}{\ln 4}\cdot\frac{1}{n}} = \dfrac{\ln(4)}{3\ln(5)}$

$\Rightarrow \log_5(n) = \Theta(\log_4(n^3))$

by transitivity, $\log_7(n) = \Theta(\log_4(n^3))$

Order of Growth

Example: Put the growth rates in order, from slowest growing to fastest.

$$\log_4^2(n) \quad \log_5(n) \quad \log_7(n) \quad \log_{3.1}(n) \quad \log_4(n^3) \quad \sqrt[5]{n}$$

$$\cdot \lim_{n\to\infty} \frac{\log_4(n^3)}{\log_{3.1}(n)} = \lim_{n\to\infty} \frac{3\log_4(n)}{\log_{3.1}(n)} = 3 \lim_{n\to\infty} \frac{\frac{1}{\ln(4)\cdot n}}{\frac{1}{\ln(3.1)\cdot n}} = 3 \cdot \frac{\ln(3.1)}{\ln(4)}$$

$$\Rightarrow \log_4(n^3) = \Theta(\log_{3.1}(n))$$

Thus $\log_7(n)$, $\log_5(n)$, $\log_4(n^3)$, $\log_{3.1}(n)$

all have the same growth rate!

Now consider $\log_4^2(n) = (\log_4(n))^2$

$$\cdot \lim_{n\to\infty} \frac{\log_4(n^3)}{\log_4^2(n)} = \lim_{n\to\infty} \frac{3\log_4(n)}{(\log_4(n))^2} = \lim_{n\to\infty} \frac{3}{\log_4(n)} = 0$$

This tells us that $\log_4(n^3)$ is $o(\log_4^2(n))$

Order of Growth

Example: Put the growth rates in order, from slowest growing to fastest.

$$\log_4^2(n) \quad \log_5(n) \quad \log_7(n) \quad \log_{3.1}(n) \quad \log_4(n^3) \quad \sqrt[5]{n}$$

That tells us that $\log_4^2(n)$ grows faster than

$\{\log_5(n), \log_7(n), \log_{3.1}(n), \log_4(n^3)\}$

- Lastly $\lim_{n\to\infty} \frac{\log_4^2(n)}{\sqrt[5]{n}} = \lim_{n\to\infty} \frac{(\log_4(n))^2}{n^{1/5}}$

$$\overset{L'H}{=} \lim_{n\to\infty} \frac{2\log_4(n) \cdot \frac{1}{\ln 4 \cdot n}}{\frac{1}{5} n^{-4/5}}$$

$$= 10 \lim_{n\to\infty} \frac{\frac{\ln(n)}{\ln(4)} \cdot \frac{1}{\ln 4 \cdot n}}{\frac{1}{n^{4/5}}} = \frac{10}{(\ln 4)^2} \lim_{n\to\infty} \frac{\ln(n)}{n} \cdot n^{4/5}$$

$$= {}^{10}\!/_{(\ln 4)^2} \lim_{n\to\infty} \ln(n)/n^{1/5}$$

Order of Growth

Example: Put the growth rates in order, from slowest growing to fastest.

$$\log_4^2(n) \quad \log_5(n) \quad \log_7(n) \quad \log_{3.1}(n) \quad \log_4(n^3) \quad \sqrt[5]{n}$$

$$\lim_{n\to\infty} \frac{\log_4^2(n)}{\sqrt[5]{n}} = \frac{10}{(\ln 4)^2} \lim_{n\to\infty} \frac{\ln(n)}{n^{1/5}}$$

$$\overset{L'H}{=} \frac{10}{(\ln 4)^2} \lim_{n\to\infty} \frac{1/n}{\frac{1}{5} n^{-4/5}}$$

$$= \frac{50}{(\ln 4)^2} \lim_{n\to\infty} \frac{1}{n^{1/5}} \to 0$$

$$= 0$$

$$\Rightarrow \log_4^2(n) \text{ is } o(\sqrt[5]{n})$$

$$\Rightarrow \boxed{\{\log_7(n), \log_5(n), \log_4(n^3), \log_{3.1}(n)\}, \log_4^2 n, \sqrt[5]{n}}$$

Next Time

- More Asymptotic Analysis
- Begin Divide and Conquer