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CSCI 2824: Discrete Structures

Lecture 12: Set Theory and Set Operations

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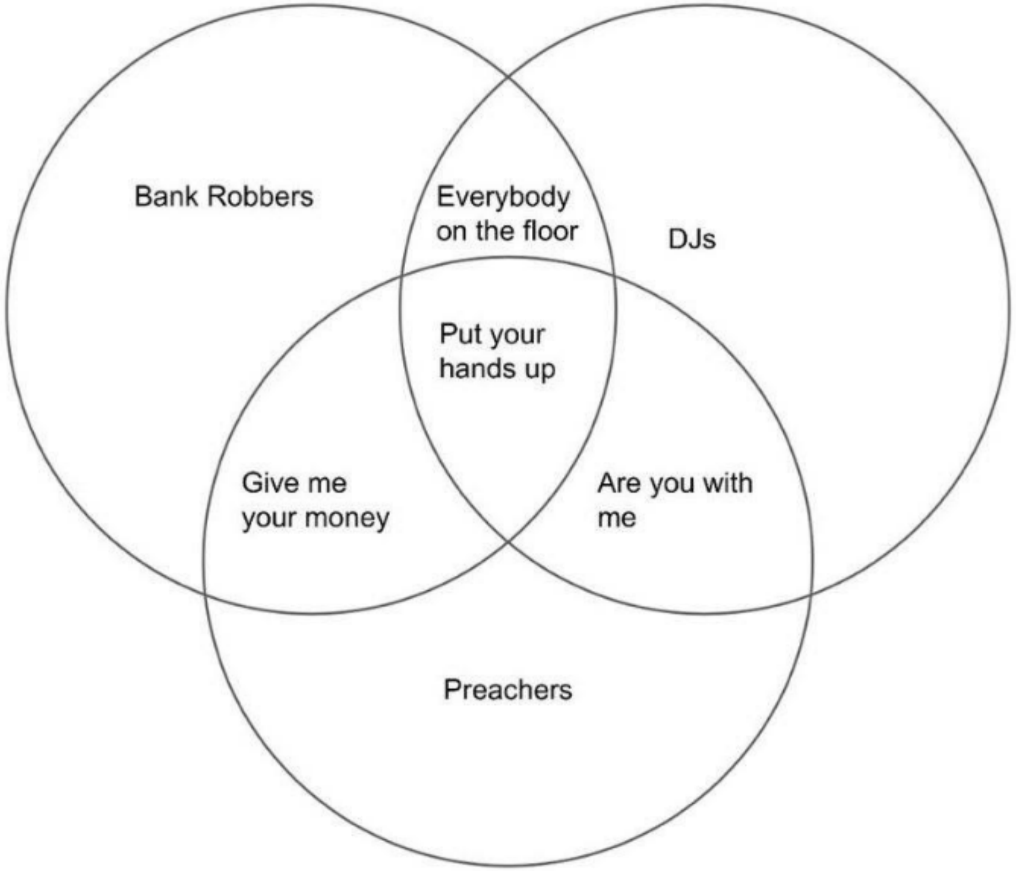

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Set Theory

$\in$ "is an element of"

A **set** is an unordered collection of objects, called *elements* or *members* of the set. A set is said to *contain* its elements. We write $a \in A$ to denote that a is an element of the set A. The notation $a \notin A$ denotes that a is not an element of the set A.

$\mathbb{N}$ $N = \{0, 1, 2, 3, 4, 5, \dots\}$ the set of natural numbers

$\mathbb{Z}$ $Z = \{\dots, -2, -1, 0, 1, 2, \dots\}$ the set of integers

$Z^+ = \{1, 2, 3, \dots\}$ the set of positive integers

$Q = \left\{\frac{p}{q} \;\middle|\; p \in Z, q \in Z, and\ q \neq 0\right\}$ the set of rational numbers

$\emptyset$ aka $\{\ \}$ empty set

$A = \{cat, dog, 1, 8, -7, circle\}$

$V = \{a, e, i, o, u\}$

❖ Typically sets are denoted with uppercase letters while elements of sets are denoted by lowercase letters.

Set Theory

Roster Method: set notation where all members of the set are listed between braces.

e.g. $B = \{bison, mountain\ lion, deer, rabbit, coyote, prairie\ dog\}$

Set Builder Notation: members of a set are characterized by stating the property or properties they must have to be members.

e.g. $C = \{x \mid x > 3\}$

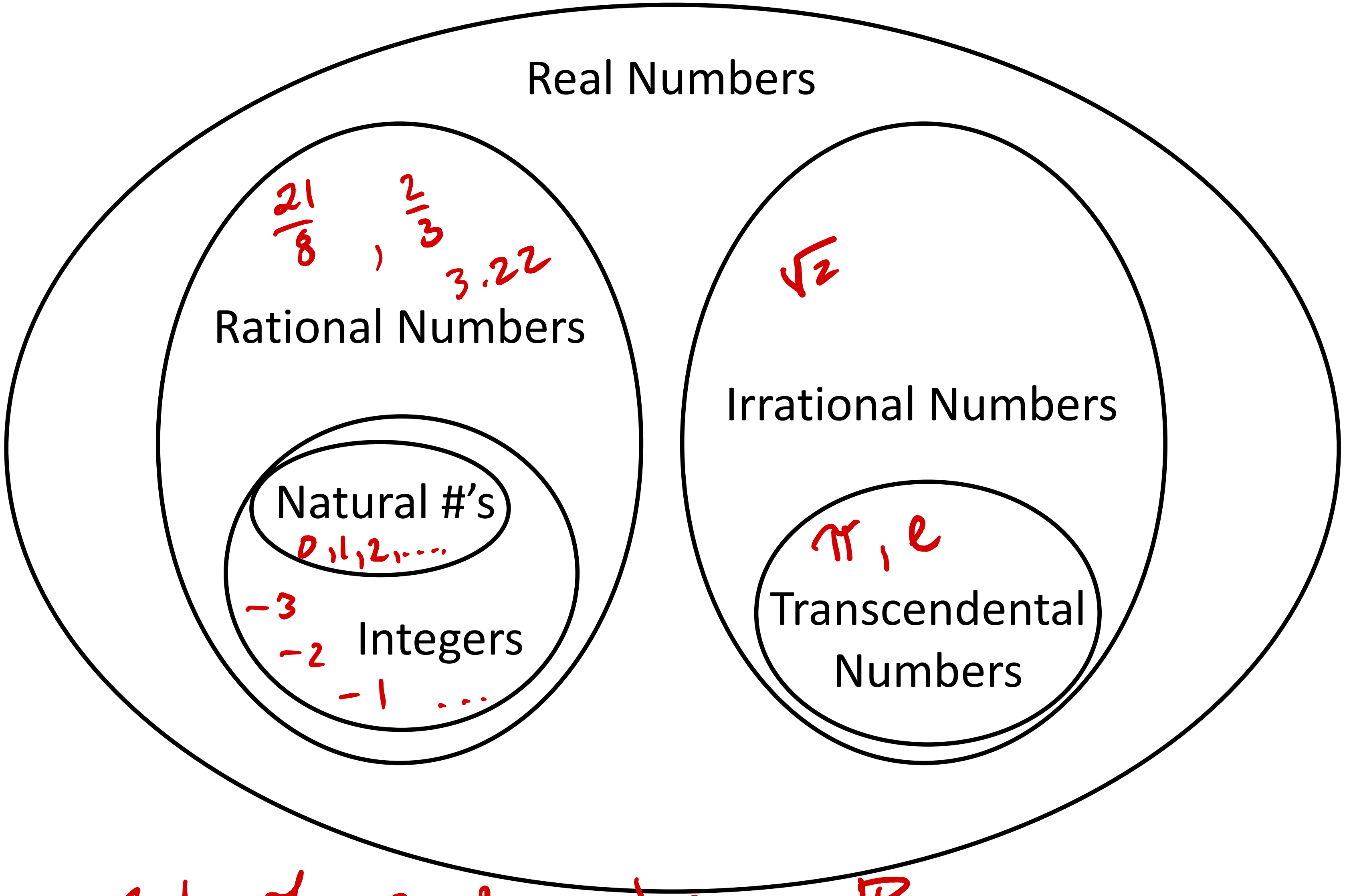
$A = [1, 3]$

set of Real numbers

$C = \{x \in \mathbb{Z} \mid x > 3\}$

$C = \{4, 5, 6, \ldots\}$

Set Theory



Set of Real numbers: $\mathbb{R}$

The set A is a **subset** of B if and only if every element of A is also an element of B.

- $denoted$: $A \subseteq B$

$means$: $\forall x(x \in A \rightarrow x \in B)$

is in

$\mathbb{N} \subseteq \mathbb{R}$

$\mathbb{N} \subseteq \mathbb{Z}$

$\mathbb{N} \subseteq \mathbb{Q}$

Set Theory

$\subseteq$ vs. $\subset$

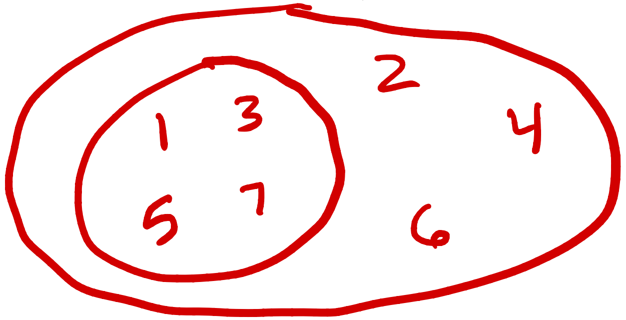
analogous $x \leq 3$ vs. $x < 3$

To show that $A \subseteq B$ you have to show that every element of A is also in B.

To show that $A \nsubseteq B$ you have to find just one element in A that is not in B.

Proper Subset: In order for A to be a proper subset of B we must have that A is a subset of B but that $A \neq B$.

- denoted: $A \subset B$ $\mathbb{N} \subset \mathbb{Z}$ $\mathbb{N} \subseteq \mathbb{Z}$
- means: $\forall x(x \in A \rightarrow x \in B) \land \exists x(x \in B \land x \notin A)$



Set Theory

Two sets are **equal** if and only if they have the same elements. Therefore, if A and B are sets, then A and B are equal if and only if $\underbrace{\forall x(x \in A \leftrightarrow x \in B)}$. We write $A = B$ if A and B are equal sets.

Example: Consider the sets $A = \{ black, gold, ralphie, 2018\}$ and $B = \{gold, black, 2018, 2018, ralphie, black\}$.
Are these sets equal?

yes!

Set Theory

- Sets can have pretty much anything in them, including other sets.

  e.g. $S = \{0, 1, \{0, 1, 2\}, \boldsymbol{Z}, \text{lamp}\}$

- Sets have no ordering. e.g. $\{1, 2, 3, 4\} = \{4, 3, 2, 1\}$

- Repeated elements don't matter. e.g. $\{cat, dog\} = \{cat, cat, dog\}$

Theorem: For every set S, $\emptyset \subseteq S$ and $S \subseteq S$

Set Theory

Three Special Sets

The Empty Set: The set that has no elements $\emptyset \ \ or \ \ \{\,\}$

The Singleton Set: A set with only one element

The Power Set: set of all subsets of a set S $P(S)$

$P(S)$

Example: $P(\{0,1,2\}) =$

$\{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$

➢ If S has n elements then $P(S)$ has 2^n elements.

Set Theory

The number of elements in a set is called the sets **cardinality**. If a set's cardinality is a finite number, then we say the set is finite.

Example: What is the cardinality of the english alphabet?

26

A

$|A|$ cardinality of A

Example: What is the cardinality of $\mathbb{Z}$?

infinite

will discuss size of infinity next week.

Set Theory

Let A and B be sets. The **Cartesian product** of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a, b) | a \in A \land b \in B\}$$

The Cartesian product of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is the set of ordered n–tuples $(a_1, a_2, \ldots, a_n)$, where a_i belongs to A_i for $i = 1, 2, \ldots, n$.

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_1 \in A_i \text{ for } i = 1, 2, \ldots, n\}$$

Example: Think about ordered pairs of integers representing points in the xy-plane. $(3, 5)$ is very different from $(5, 3)$

Set Theory

Example: What is the Cartesian product $A \times B \times C$ where $A = \{a, b\}, B = \{x, y\}$, and $C = \{m, n\}$

$A = \{a, b\}$

$B = \{x, y\}$

$C = \{m, n\}$

$$A \times B \times C = \{(a, x, m), (a, x, n), (a, y, m), (a, y, n), (b, x, m), (b, x, n), (b, y, m), (b, y, n)\}$$

Cardinality $|A \times B \times C| = |A| \times |B| \times |C| = 2 \cdot 2 \cdot 2 = 8$

Set Theory

Example: Suppose that $A = \{1, 2\}$. Find $A \times A$ (aka A^2) and find $A \times A \times A$ (aka A^3)

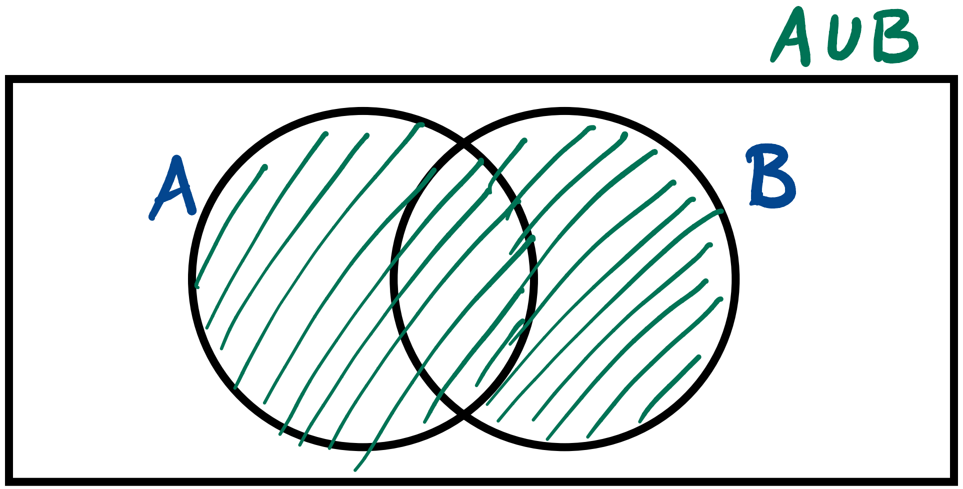
$A \times A = \{(1,1), (1,2), (2,1), (2,2)\}$

exercise on own $A \times A \times A$

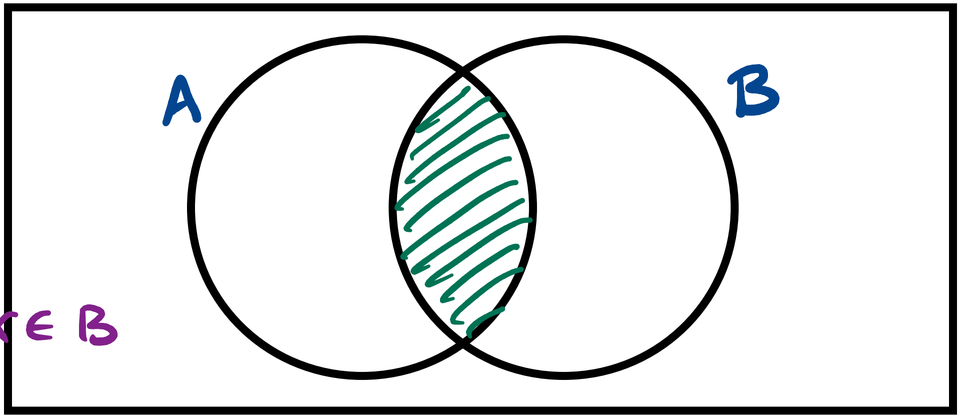
Set Operations

Venn diagrams can be useful when trying to understand and represent set operations.

Let A and B be sets. The **union** of the sets A and B, denoted $\boldsymbol{A \cup B}$, is the set that contains those elements that are either in A or in B, or in both.



Let A and B be sets. The **intersection** of the sets A and B, denoted $\boldsymbol{A \cap B}$, is the set that contains those elements that are in both A and B. $x \in A \cap B \Rightarrow x \in A \wedge x \in B$



* $|A \cup B| = |A| + |B| - |A \cap B|$

Set Operations

Example: Consider the sets: $A = \{1, 3, 5\}$ and $B = \{1, 2, 3\}$. Find the union and the intersection of these two sets.

$A \cup B = \{1, 2, 3, 5\}$

$A \cap B = \{1, 3\}$

Example: Consider the sets: $A = \{1, 3, 5\}$ and $B = \{2, 4, 6\}$. Find the union and the intersection of these two sets.

$A \cup B = \{1, 2, 3, 4, 5, 6\}$

$A \cap B = \emptyset$

- Two sets are called disjoint if their intersection is the empty set.

Set Operations

$I - I = \{\}$

$\phi \subseteq I$

Let A and B be sets. The **difference of A and B**, denoted $A - B$, (or $A \backslash B$), is the set containing those elements that are in A but not in B. The difference of A and B is also called the complement of B with respect to A.

$A - B = \forall x\ (\{x \mid (x \in A \wedge x \notin B)\})$ *

Example: What is the difference of the set of positive integers less than 10 and the set of prime numbers?

$I = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$P = \{2, 3, 5, 7, 11, 13, 17, 19, \ldots\}$

$I - P = \{1, 4, 6, 8, 9$ $\}$ we are left with composite #'s less than 10

Set Operations

$S = \{1, \mathbb{N}, \text{'cat'}, \{1, \text{'cat'}\}\}$

Definition: The **universal set**, denoted typically by U, is the set containing all elements within the domain of discourse.

- You can think about the universal set as the set containing all elements under consideration.

Example: if the domain of discourse is all CU students, then U is the set of all CU students.

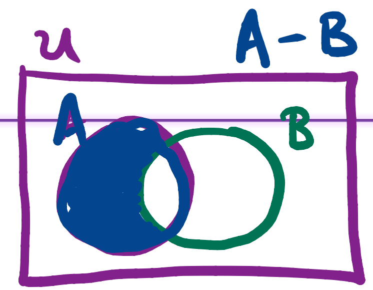
Definition: Let U be the universal set. The **complement** of the set A, denoted $\bar{A}$, is the set $U - A$.

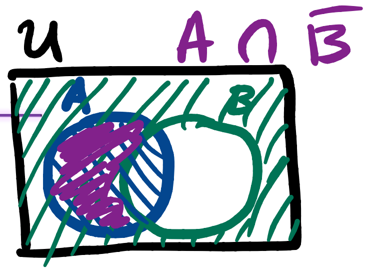
An element belongs to $\bar{A}$ if and only if $x \notin A$, so $\bar{A} = \{x \in U \mid x \notin A\}$, or just $\{x \mid x \notin A\}$

complement notation: $\bar{A}$ or A^c

Set Operations

intuitive picture





Example: Show that $A - B = A \cap \overline{B}$

- In general: To show that two sets R and S are equal, we will show that $R \subseteq S$ and $S \subseteq R$

Proof: First we want to show that $A - B \subseteq A \cap \overline{B}$

Let $x \in A - B$ be arbitrary.

$\Rightarrow x \in A \wedge x \notin B$ by definition of difference of sets

Since $x \notin B \Rightarrow x \in \overline{B}$ by definition of complement

$\Rightarrow x \in A \wedge x \in \overline{B}$

$\Rightarrow x \in A \cap \overline{B}$ by definition of intersection.

$\Rightarrow A - B \subseteq A \cap \overline{B}$ by definition of subset.

Now we want to show that $A \cap \overline{B} \subseteq A - B$.

Let $x \in A \cap \overline{B}$ be arbitrary.

$\Rightarrow x \in A \wedge x \in \overline{B}$ def. of intersect.

$x \in \overline{B} \Rightarrow x \notin B$ def. of complement

$\Rightarrow x \in A \wedge x \notin B$

$\Rightarrow x \in A - B$ def. of difference

$\Rightarrow A \cap \overline{B} \subseteq A - B$

Since $A - B \subseteq A \cap \overline{B}$ and $A \cap \overline{B} \subseteq A - B$, it must be that $A - B = A \cap \overline{B}$

$\blacksquare$

Set Operations

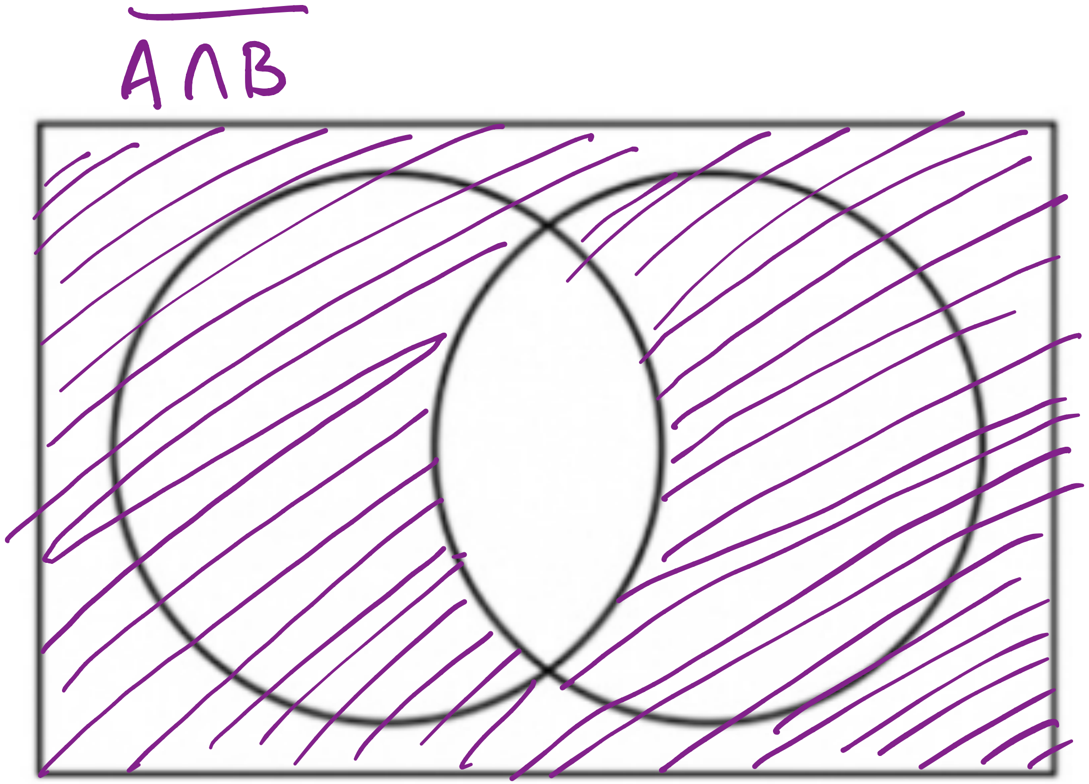
When sets are combined using only ∪,∩, and complements, there is a set of **Set Identities** that completely mirrors the logical equivalences from last chapter.

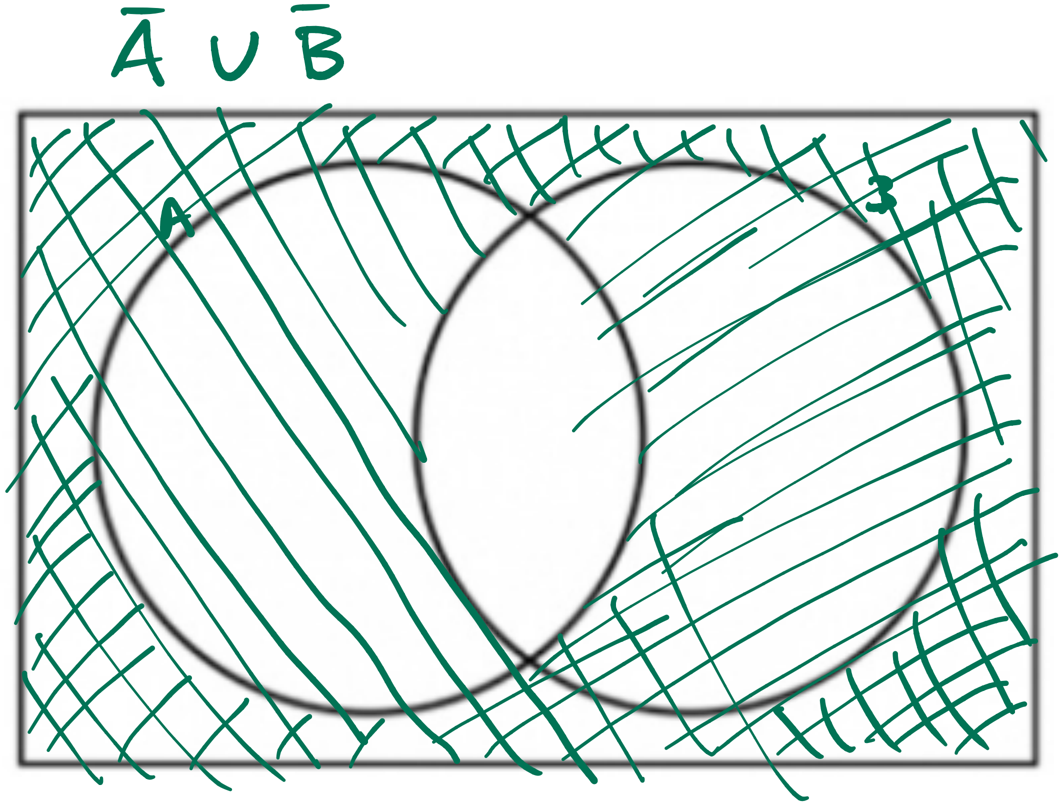
TABLE 1 Set Identities.

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Set Operations

Example: Prove DeMorgan's Law for Sets: $\overline{A \cap B} = \bar{A} \cup \bar{B}$





Set Operations

Example: Prove DeMorgan's Law for Sets: $\overline{A \cap B} = \overline{A} \cup \overline{B}$

$\overline{A \cap B} = \{x \mid x \notin A \cap B\}$ Def. of Complement

$= \{x \mid \neg(x \in A \cap B)\}$ Def. of "Not In"

$= \{x \mid \neg(x \in A \wedge x \in B)\}$ Def. of Intersection

$= \{x \mid \neg(x \in A) \vee \neg(x \in B)\}$ DeMorgan's Law

$= \{x \mid x \notin A \vee x \notin B\}$ "Not In"

$= \{x \mid x \in \overline{A} \vee x \in \overline{B}\}$ Def. of complement.

$= \{x \mid x \in \overline{A} \cup \overline{B}\} = \overline{A} \cup \overline{B}$

End of Sets and Set Operations!

Next Up: More Set Examples!

Extra Practice

Example 1: Show that if a, b, and c are real numbers with $a \neq 0$ then there **exists** a **unique** solution x to the equation $ax + b = c$

Example 2: Prove this identity using set builder notation $\overline{A \cup B} = \bar{A} \cap \bar{B}$

Solutions

Example 1: Show that if a, b, and c are real numbers with $a \neq 0$ then there **exists** a **unique** solution x to the equation $ax + b = c$

Existence: Solve for x

$ax + b = c \Rightarrow x = \dfrac{b - c}{a}$ (where here we know that $a \neq 0$)

Uniqueness: Assume x and y are both solutions to the system, then

$$ax + b = c = ay + b \Rightarrow ax + b = ay + b \Rightarrow ax = ay \Rightarrow x = y$$

Since x and y are necessarily the same number, it follows that our solution x is unique

Example 2: Prove this identity using set builder notation $\overline{A \cup B} = \bar{A} \cap \bar{B}$

$$
\begin{aligned}
\overline{A \cup B} &= \{x \mid x \notin A \cup B\} && \text{(def. complement)} \\
&= \{x \mid \neg(x \in A \cup B)\} && \text{(def. not in)} \\
&= \{x \mid \neg(x \in A \vee x \in B)\} && \text{(def. intersection)} \\
&= \{x \mid \neg(x \in A) \wedge \neg(x \in B)\} && \text{(DeMorgan's)} \\
&= \{x \mid x \notin A \wedge x \notin B\} && \text{(def. not in)} \\
&= \{x \mid x \in \overline{A} \wedge x \in \overline{B}\} && \text{(def. complement)} \\
&= \{x \mid x \in \overline{A} \cap \overline{B}\} && \text{(def. union)} \\
&= \overline{A} \cap \overline{B}
\end{aligned}
$$