Read these instructions!

This assignment is **due on Friday, September 27 to Gradescope by 12 PM noon.** You are expected to write up your solutions neatly, with **full explanations and justifications** when necessary. Remember that you are encouraged to discuss problems with your classmates, but you must work and write your solutions on your own. **Important**: On the **FRONT** of your assignment clearly write your full name and the lecture section you belong to (001 or 002). You may upload your assignment as a pdf or as images of your work. Make sure that your images/scans are clear or you will lose points/possibly be given a 0.

1. Consider the following argument. Let the domain for x be all dogs.

Premise i: All dogs either go to "The Good Place" or have a rampant gambling addiction (possibly both).
Premise ii: All dogs either do not have a rampant gambling addiction or they like bacon (possibly both).
Premise iii: Every dog that enjoys rolling in the grass does not like bacon.
Premise iv: There is a dog that will go to "The Bad Place".
Conclusion: There is a dog that does not like rolling in the grass.

[Note: We are assuming that "The Good Place" and "The Bad Place" are the only options for dogs, i.e. there is no puppy purgatory.]

(a) Translate each of the premises and the conclusion into logical statements using propositional functions, connectives and quantifiers. Be sure to define all quantities that you use.

(b) Use rules of inference and logical equivalences to show that the argument is valid. Be sure to cite which rules of inference or logical equivalences you use on each line of your proof, and indicate which line numbers are involved in each step.

Remember that you are only allowed to use the logical equivalences given in Table 6 on page 27 of the Rosen textbook, plus the four named equivalences from the lecture slides, and the rules of inference in Table 1 on page 72.

2. For each claim below, prove or disprove the claim. Remember that to prove a claim, you must prove it *in general* (i.e., for all cases), and to disprove a claim, you should present a counterexample. If you prove a claim, be sure to indicate whether you are using a Direct Proof, a Contrapositive Proof, a Proof by Cases, a Proof by Contradiction, or a Proof by Construction (existence proof). If you use a Proof by Cases, be sure to indicate what each case is, and in each case, which type of proof you are using.

(a) If $a, b, c, d, e,$ and f are real numbers such that $ad - bc \neq 0$, then the system of two equations $ax + by = e$ and $cx + dy = f$ can be solved for the real numbers x and y.

(b) Every positive integer can be expressed as the sum of two perfect squares. (A perfect square is the square of an integer. 0 may be used in the sum.)

(c) Let a and b be integers. If a and b are expressible in the form $4n + 1$ where n is an integer, then $a \cdot b$ is also expressible in that form.

3. Prove each of the following. Be sure to indicate whether you are using a Direct Proof, a Contrapositive Proof, a Proof by Cases, or a Proof by Contradiction. If you use a Proof by Cases, be sure to indicate what each case is, and in each case, which type of proof you are using.

(a) Let a and b be integers and define $c = a \times b + a + b$. Prove that c is even if and only if a and b are both even.

(b) Suppose n is a positive integer. Prove that if 5 divides $4n$, then 5 divides n.

Two more problems on the next page!

4. (a) Write down in roster notation the power set $\mathcal{P}(A)$, where $A = \{\emptyset, 1, \{cat, dog, 7\}\}$.

   (b) Draw a Venn diagram to represent the relationship between the following three sets: C is the set of all University of Colorado faculty, N is the set of all people whose names contain the letter "o", and F is the set of all people who work at any university. Which, if any, of these sets are subsets of one another? Justify your answer.

   (c) Is $1 \in \{1, \{1\}\}$? How do you know?
   Is $1 \subseteq \{1, \{1\}\}$? How do you know?

5. (a) For sets A and B, the cardinality of their union is given by: $|A \cup B| = |A| + |B| - |A \cap B|$. Derive the following analogous rule for the union of three sets:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

   [Hint: Start by treating $(B \cup C)$ as one set and then apply the two-set rule given above, along with set identities from Table 1 in Section 2.2.]

   (b) 40 surfers all participated in three surfing competitions on the following planets: Earth (E), Neptune (N), and Bird World (B). The following table shows how many surfers wiped out in each contest and in their various combinations (e.g. 5 people wiped out in both the contests on Earth and Bird World, and 4 people wiped out in all three contests).

B	E	N	B and E	B and N	E and N	B and E and N
13	11	22	5	10	8	4

   How many people did not wipe out at all in any of these three contests?

Potentially useful facts

You may find these fun facts potentially useful in your homework proofs. There are often multiple methods to prove a single statement, so it is not necessarily the case that you *need* to use these facts.

1. For integers a, b and e, if e is even and $ab = e$, then at least one of a or b must be even.

2. If an integer m divides another integer n, then there exists some integer k such that $n = mk$.

3. For real numbers a and b, $(a + b)(a - b) = a^2 - b^2$. (This is known as a "difference of two squares.")