CHAPTER THREE

Conditional Probability
and
Independence

Recall some probability from your past:

A probability function assigns a value from $[0, 1]$ to each event in the sample space Ω.   i.e. $P(E) = 0.75$

Furthermore:

$P(\Omega) = 1$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

But what if A and B are disjoint?

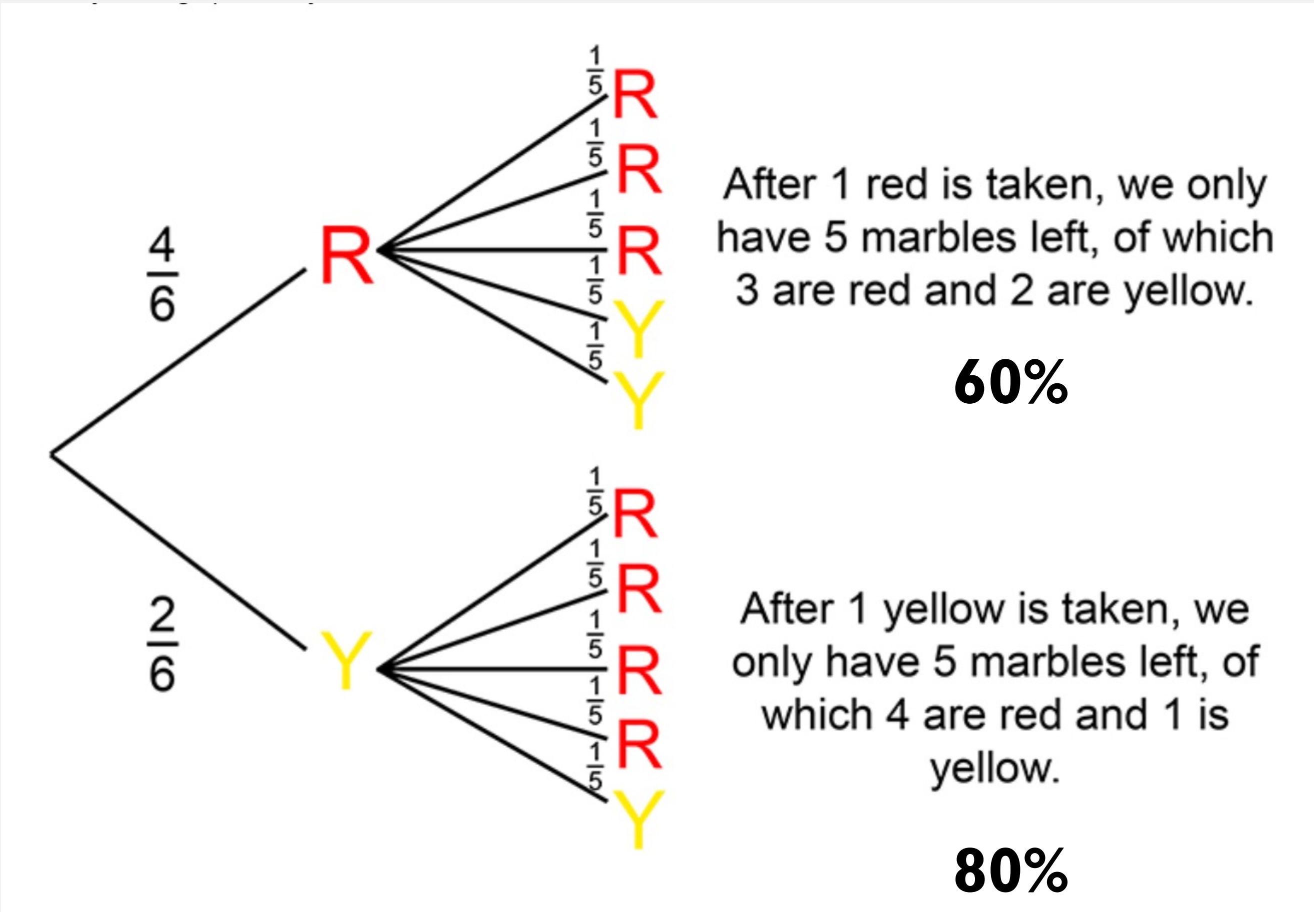
Practice: Suppose you draw a card from a standard deck.

1] What is the probability you draw the queen of hearts?

2] What is the probability that you draw a queen or a heart?

Today we will move a bit deeper into:

- Conditional Probability
- Independence
- Bayes Theorem.



- How likely is it that you pick a **red** marble on your second draw of a marble (without replacement) from the above group of marbles ?

The answer is dependent on the first condition.

What does the following mean?

$$P(E) = \sum_{\omega \in E} P(\omega)$$

Suppose you have a coin (biased or not).

$P(H) = p$

$P(T) = 1 - p$

Now suppose you flip it three times in a row.

1] What is the probability that you get 2 or more tails?

2] What is the probability that you eventually get heads?

1] What is the probability that you get 2 or more tails?

E = getting 2 or more tails on three flips with the probability of tails $1 - p$.

$$P(E) = P(\text{getting two tails}) + P(\text{getting three tails})$$

$$\boxed{P(E) = \sum_{\omega \in E} P(\omega)}$$

$$P(\text{getting two tails}) = 3 \cdot p \cdot (1-p)^2$$
$$P(\text{getting three tails}) = (1-p)^3$$

2] What is the probability that you eventually get heads?

$\boldsymbol{E}$ = eventually flipping a heads

$$P(E) = (\text{prob of H on 1st or H on 2nd or H on 3rd or H on 4th} \ldots)$$

$$P(E) = P(\text{H on 1st}) + P(\text{H on 2nd}) + P(\text{H on 3rd}) + P(\text{H on 4th}) + \cdots$$

$$P(E) = (1-p)^0 p + (1-p)^1 p + (1-p)^2 p + (1-p)^3 p + \cdots$$

$$P(E) = p \sum_{i=0}^{\infty} (1-p)^i = p\left(\frac{1}{1-(1-p)}\right) = 1$$

How likely is it that a person will get sick from a virus? ANS: p_1

How likely is it that a person will get sick from a virus, given that they wash their hands, wear a mask, have a vaccination, and do not hang out in large crowds of people? ANS: p_2

Presented with extra information, sometimes forces us to reassess the probability of another event.

This new probability, p_2 (given the extra information) is called the **conditional probability.**

If the conditional probability equals what the probability was before, $p_2 = p_1$ then the events involved are called **independent**.

How likely is it that a person will get sick from a virus, given that you wear blue shoes? Well, $p_2 = p_1$ So the events are independent.

We know that two events are independent when we can show at least one of the following:

1] $P(A|B) = P(A)$

2] $P(B|A) = P(B)$

3] $P(A \cap B) = P(A) \cdot P(B)$

Similarly, we know that multiple events $A_1, A_2, A_3, \dots, A_m$ are independent when:

$$P(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_m) = P(A_1) \cdot P(A_2) \cdot P(A_3) \cdot \ \dots \ \cdot P(A_m)$$

So, to figure out whether two events are independent,
we need to know what $P(A|B)$ means.

Consider the following sets of months:

Months with more than 30 days, aka 'Long' months

$S_1 = \{Jan, Mar, May, Jul, Aug, Oct, Dec\}, \ |S_1| = 7$

Months with an 'R' in their name

$S_2 = \{Jan, Feb, Mar, Apr, Sep, Oct, Nov, Dec\}, \ \ |S_2| = 8$

Consider the following events:

L = A person was born in a long month

R = A person was born in a month containing the letter 'R'

1] If you randomly pick a person and you wonder whether they were born in a month containing the letter 'R', then

$$P(R) = \frac{8}{12} = \frac{2}{3} = 0.\overline{6} \approx 67\%$$

2] However, if you only ask people from S_1, i.e. months that are *Long*, and wonder whether they were born in a month containing the letter 'R', then

$$P(R|L) = \frac{4}{7} = 0.\overline{571428} \approx 57\%$$

Now you are 'drawing' from a set with $|S_1| = 7$, containing 4 months from S_2.

$S_1 = \{Jan, Mar, May, Jul, Aug, Oct, Dec\}, \ |S_1| = 7$

Months with an 'R' in their name

$S_2 = \{Jan, Feb, Mar, Apr, Sep, Oct, Nov, Dec\}, \ |S_2| = 8$

The probability changed with the newly given information, from 67% to 57%

Months with more than 30 days

$$S_1 = \{Jan, Mar, May, Jul, Aug, Oct, Dec\}, \quad |S_1| = 7$$

Months with an ‘R’ in their name

$$S_2 = \{Jan, Feb, Mar, Apr, Sep, Oct, Nov, Dec\}, \quad |S_2| = 8$$

Notice:

$$S_2 \cap S_1 = \{Jan, Mar, Oct, Dec\}$$

The conditional probability of R given L is written: $P(R|L) = \frac{4}{7}$

Furthermore, note that $P(R \cap L) = P(S_2 \cap S_1) = \frac{4}{12}$

and $P(R|L) = \frac{P(R \cap L)}{P(L)} = \frac{4/12}{7/12} = \frac{4}{12} \cdot \frac{12}{7} = \frac{4}{7}$

In general, the conditional probability of A given C is:

$$P(A|C) = \frac{P(A \cap C)}{P(C)}, \quad \text{as long as } P(C) > 0.$$

Computing the probability of an event A, given that an event C occurs, means finding which fraction of the probability of C is also in the event A.

In general, the conditional probability of A given C is:

$$P(A|C) = \frac{P(A \cap C)}{P(C)}, \quad \text{as long as } P(C) > 0.$$

$\boldsymbol{L} = \{Jan, Mar, May, Jul, Aug, Oct, Dec\}$

$\boldsymbol{R} = \{Jan, Feb, Mar, Apr, Sep, Oct, Nov, Dec\}$

$\boldsymbol{N} = R^c = \{May, Jun, Jul, Aug\}$

$$P(N|L) = \ ?$$

$$\boldsymbol{L} = \{Jan, Mar, May, Jul, Aug, Oct, Dec\}$$

$$\boldsymbol{R} = \{Jan, Feb, Mar, Apr, Sep, Oct, Nov, Dec\}$$

$$\boldsymbol{N} = R^c = \{May, Jun, Jul, Aug\}$$

$$P(N|L) = \frac{P(N \cap L)}{P(L)} = \frac{3/_{12}}{7/_{12}} = \frac{3}{7}$$

Example

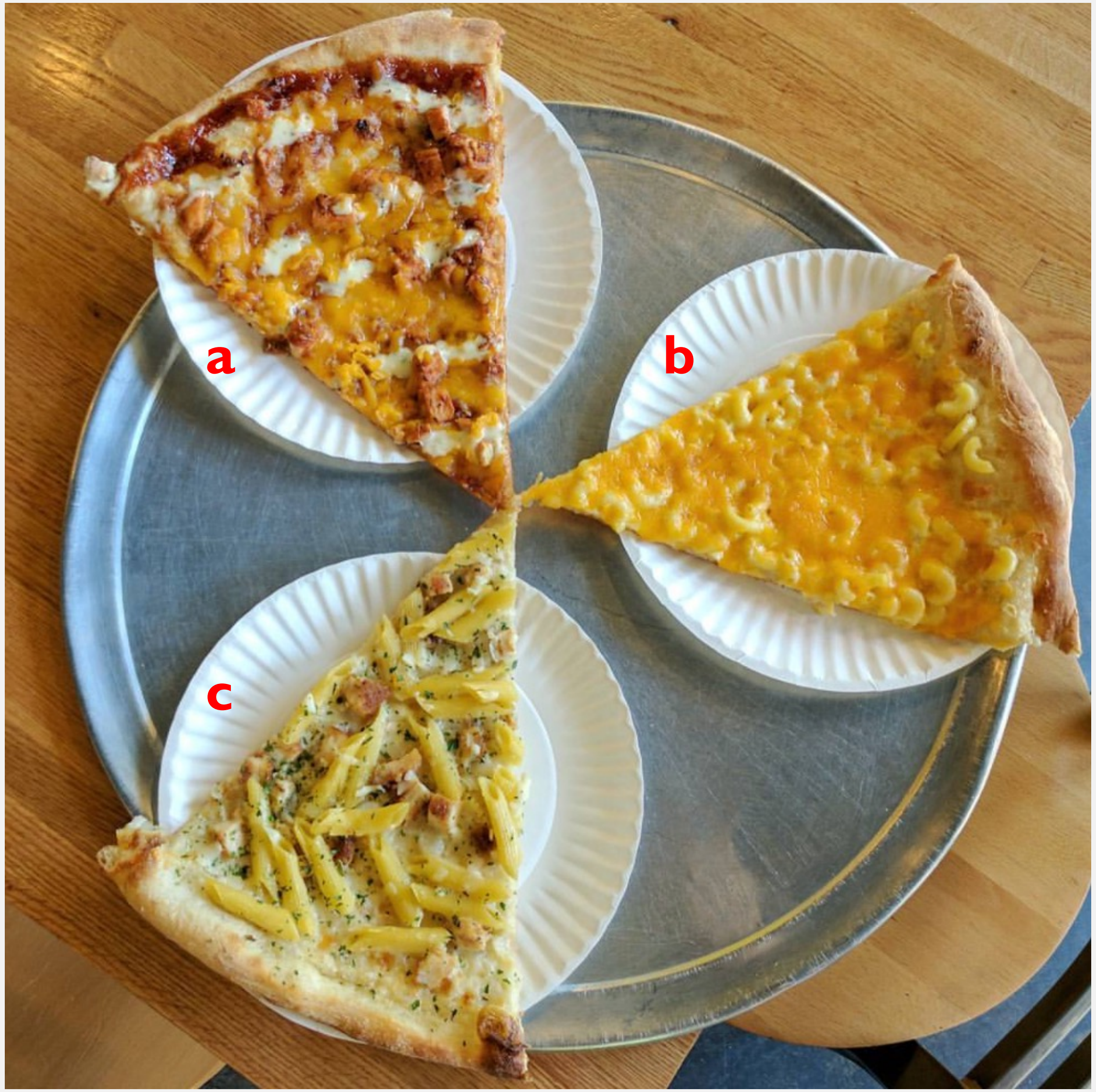
Experiment: You call for pizza deliveries from 3 different locations at the same time in order to see the order in which they arrive.

$$\Omega = \{abc, acb, bac, bca, cab, cba\}$$

Event A = Company 'a' delivers first.

Event C = the deliveries were in either forwards or backwards alphabetical order.

- $P(A) = ?$
- $P(A|C) = ?$



A bit string of length 4 is randomly generated.

What is the probability that you get a string with two 1's in a row, given that the first bit is a 1?

A = two 1's in a row.

C = The first bit is a 1.

$$P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{5/16}{8/16} = \frac{5}{8}$$

$$\{1111, 1110, 1101, 1011, 0111, 1100, 1010, 1001, 0110, 0011, 0101, 1000, 0100, 0010, 0001, 0000\}$$

In general, the conditional probability of A given C is:

$$P(A|C) = \frac{P(A \cap C)}{P(C)}, \quad \text{as long as } P(C) > 0.$$

Notice the definition of conditional probability leads us to the multiplication rule:

In general, the conditional probability of A given C is:

$$P(A|C) = \frac{P(A \cap C)}{P(C)}, \quad \text{as long as } P(C) > 0.$$

$\downarrow$

The multiplication rule:

$$P(A \cap C) = P(A|C) \cdot P(C)$$

What is the probability that you draw 2 red cards in a row from a standard deck?

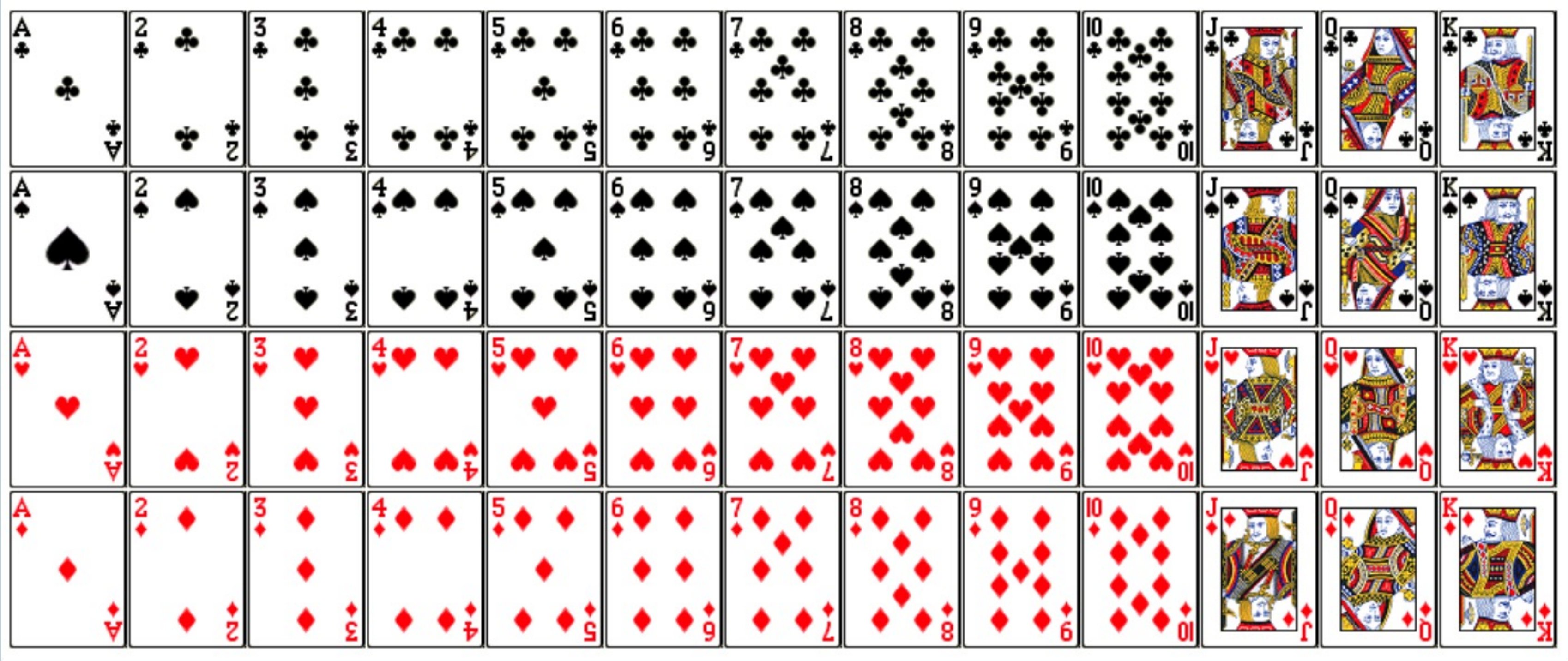
Event A: The second card drawn is red

Event C: The first card drawn is red

The multiplication rule:

$$P(A \cap C) = P(A|C) \cdot P(C)$$

$$P(A \cap C) = P(A|C) \cdot P(C) = \frac{25}{51} \cdot \frac{26}{52}$$



So now we have an idea about what $P(A|B)$ means. Let's get back to independence.

We know events $A_1, A_2, \dots, A_m$ are independent when we can show:

$$P(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_m) = P(A_1) \cdot P(A_2) \cdot P(A_3) \cdot \ \dots \ \cdot P(A_m)$$

Or, we know just two events are independent when we can show at least one of the following:

1] $P(A|B) = P(A)$

2] $P(B|A) = P(B)$

3] $P(A \cap B) = P(A) \cdot P(B)$

Consider the following events after flipping a coin twice: $\Omega = \{HH, HT, TH, TT\}$

A = {Getting Heads on the second flip} = {HH, TH}

B = {Getting Heads on the first flip} = {HH, HT}

C = {Getting the same outcome on both flips of the coin} = {HH, TT}

Q: Are A and B independent?

A: They are independent if you can you show:

$$P(A \cap B) = P(A) \cdot P(B) \quad \text{or}$$

$$P(A|B) = P(A) \quad \text{or}$$

$$P(B|A) = P(B)$$

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}}$$

Q: Are A, B, and C independent?

A: They are if you can you show $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$

Consider the following events after flipping a coin twice: $\Omega = \{HH, HT, TH, TT\}$

A = {Getting Heads on the second flip} = {HH, TH}

B = {Getting Heads on the first flip} = {HH, HT}

C = {Getting the same outcome on both flips of the coin} = {HH, TT}

Q: Are A and B independent? YES!

A: They are independent if you can you show:

$$P(A \cap B) = P(A) \cdot P(B) \qquad \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2}$$

$$P(A|B) = P(A)$$ or

$$P(B|A) = P(B)$$

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}}$$

Q: Are A, B, and C independent? NO.

A: They are if you can you show $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$

$$\frac{1}{4} \neq \frac{2}{4} \cdot \frac{2}{4} \cdot \frac{2}{4}$$

Conditional probability is made easier (usually) with the multiplication rule.

The multiplication rule:

$$P(A \cap C) = P(A|C) \cdot P(C)$$

However, it matters how you condition:

$$P(A \cap C) = P(A|C) \cdot P(C)$$

versus

$$P(A \cap C) = P(C|A) \cdot P(A)$$

Both are valid, but often one of them is easy and the other is NOT!

Conditioning is supposed to lead to easier probabilities;
if not, then it is probably the wrong approach.

Try these calculations before we talk about the next virus example

$$(A \cap C) \cup (A^c \cap C) = \ ?$$

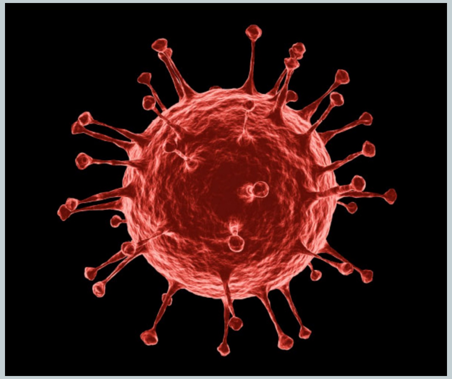
$$P(A \cap C) + P(A^c \cap C) = \ ?$$

$$P(A|C) + P(A^c|C) = \ ?$$

Suppose there is a virus moving thru the population.
It is important to have a test to determine who is infected.
No test is 100% accurate since there is always a chance of a false negative or a false positive.

False negative: The test indicates you don't have the virus when you actually do.

False positive: The test indicates you have the virus when you actually don't.



Consider the events:

V = Virus is in the person.

T = Test is positive (i.e. test claims person has virus)

Now suppose a new test for the virus is being studied; call it **test A**.
After some investigation it is discovered that an infected person has an 85% chance of testing positive and a healthy person just 5%.

What does it mean to say this test is 85% effective?

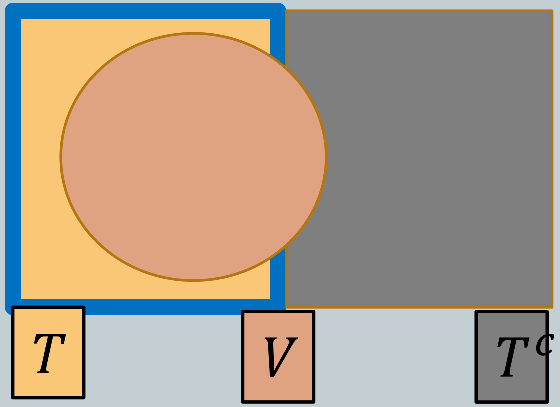
Test A: An infected person has an 85% chance of testing positive
A healthy person has a 5% chance of testing positive

$P(T|V) = 0.85$
$P(T|V^c) = 0.05$

Choose a random person from the population of people that are being tested.
We want to determine $P(T)$.
$P(T)$ is the probability that this arbitrary person tests positive.

We know the person is either infected or is not infected (V or V^c) and event T occurs in combination with V or V^c. There are no other possibilities.

That is to say that $(T \cap V)$ and $(T \cap V^c)$ are disjoint.
This means $T = (T \cap V) \cup (T \cap V^c)$,
therefore $\mathrm{P}(T) = P(T \cap V) \cup P(T \cap V^c)$



$$\mathrm{P}(T) = P(T \cap V) \cup P(T \cap V^c)$$

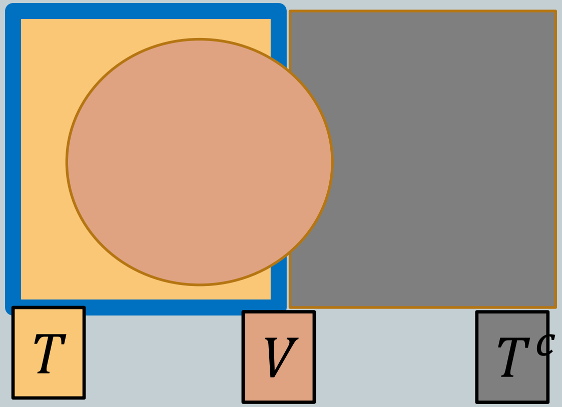
Now apply the multiplication rule:

$$P(T \cap V) = P(T|V) \cdot P(V)$$

$$P(T \cap V^c) = P(T|V^c) \cdot P(V^c)$$

$$P(T) = P(T|V) \cdot P(V) + P(T|V^c) \cdot P(V^c)$$

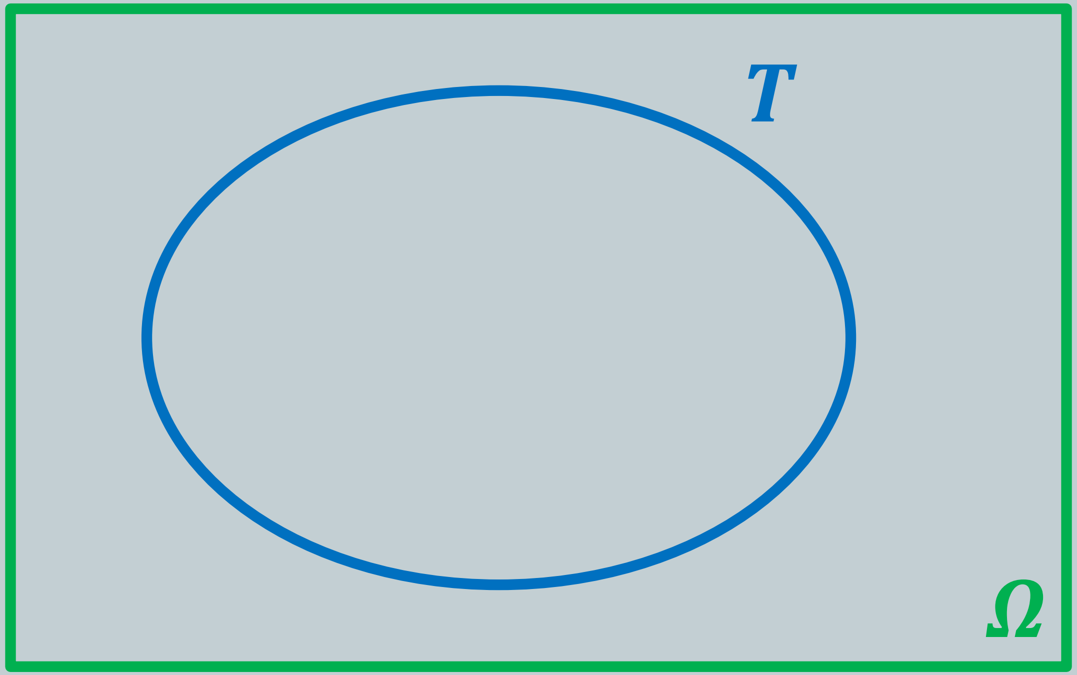
This is an application of the law of total probability: computing a probability through conditioning on several disjoint events that make up the whole sample space.

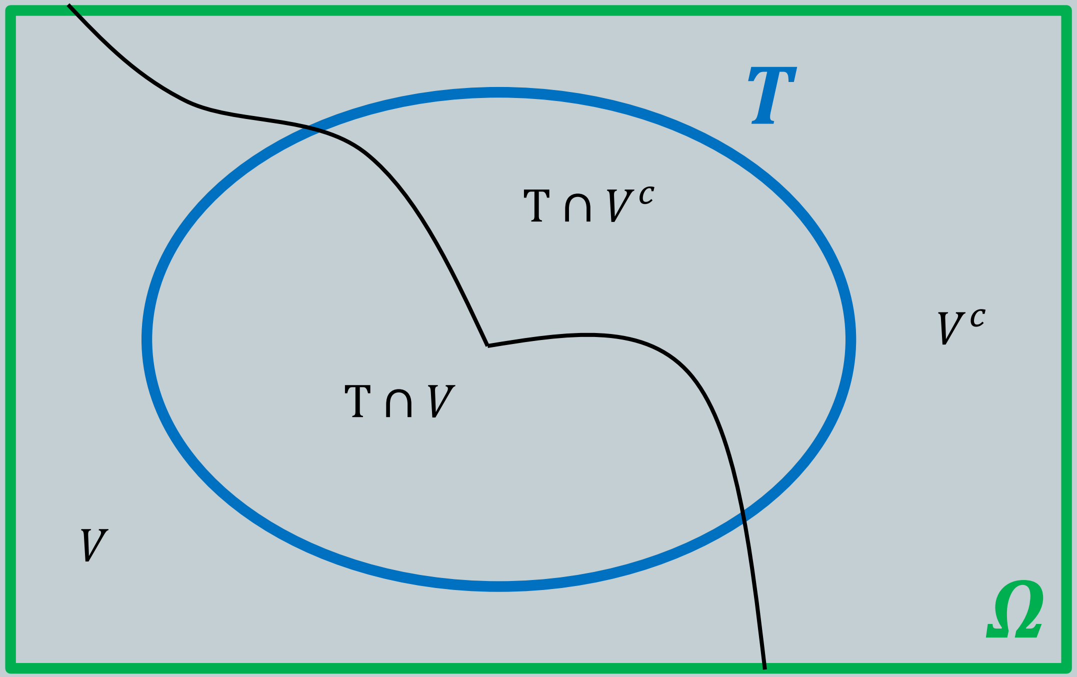


The law of total probability graphically: What is $P(T)$?

$$P(T) = \underline{P(T|V) \cdot P(V)} + \underline{P(T|V^c) \cdot P(V^c)}$$

$$\mathrm{P}(T) = P(T \cap V) \cup P(T \cap V^c)$$





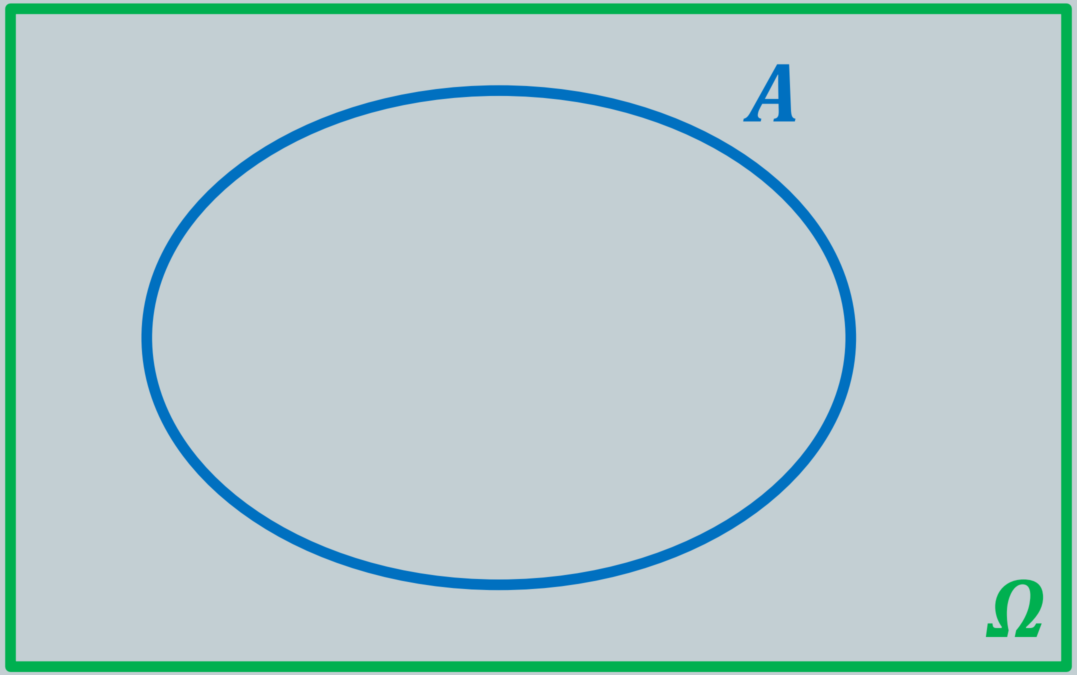
The law of total probability: What is $P(A)$?

Suppose $C_1, C_2, C_3, \ldots, C_m$ are disjoint events such that $C_1 \cup C_2 \cup C_3 \cup \cdots \cup C_m = \Omega$

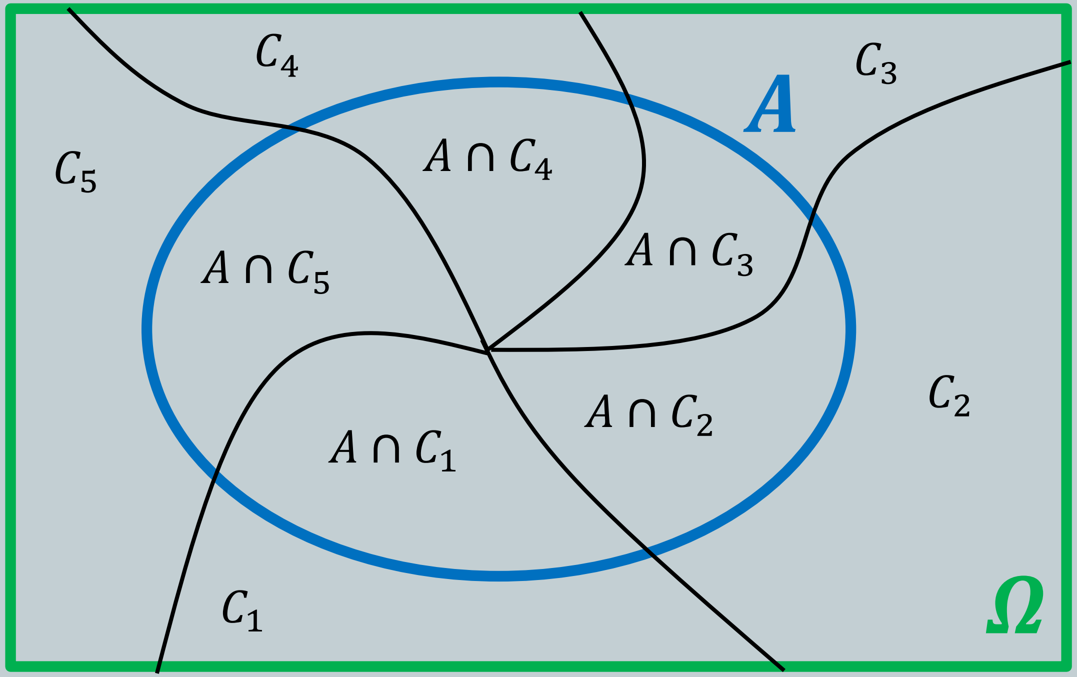
The probability of an arbitrary event A can be expressed as:

$$P(A) = P(A|C_1)P(C_1) + P(A|C_2)P(C_2) + \cdots + P(A|C_m)P(C_m)$$

$$P(A) = P(A \cap C_1) + P(A \cap C_2) + \cdots + P(A \cap C_m)$$



For $m = 5$



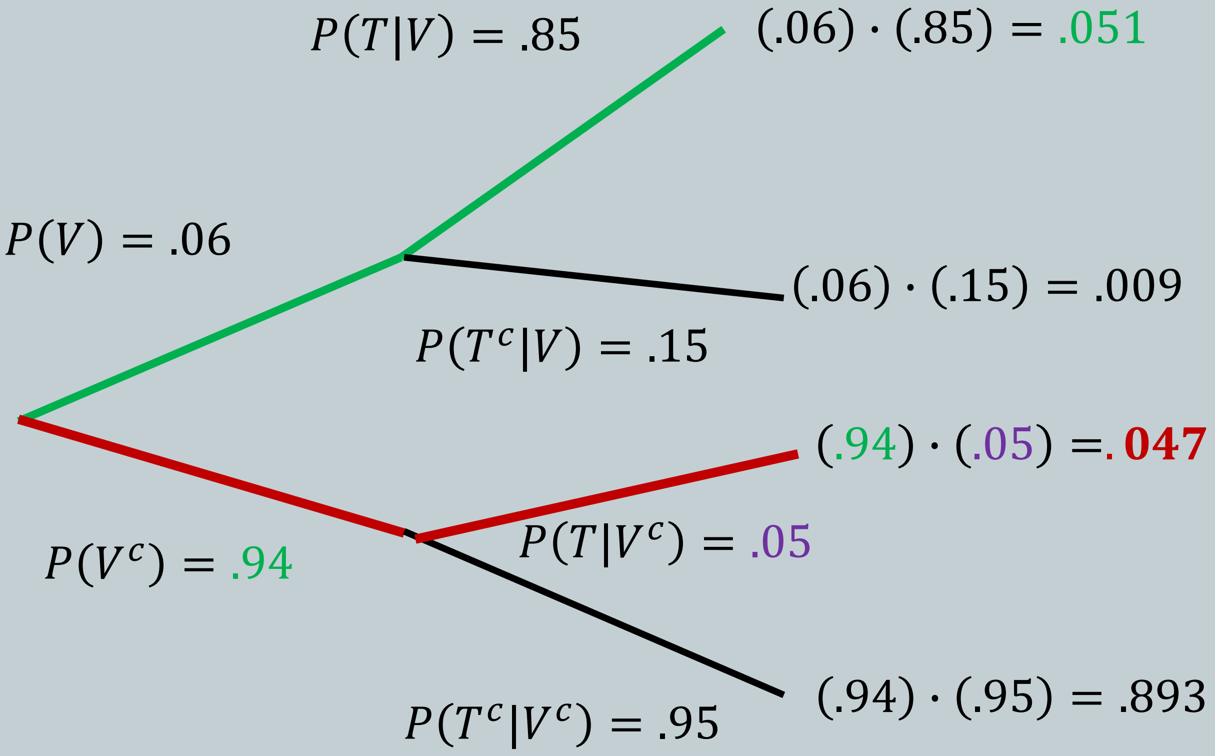
$$P(T) = P(T|V) \cdot P(V) + P(T|V^c) \cdot P(V^c)$$

Now suppose the likelihood (probability) of a person having the virus is 6%. The we say that $P(V) = 0.06$.

Now we can say $P(T) = (0.85) \cdot (0.06) + (0.05) \cdot (1 - 0.06) = \boxed{\mathbf{0.098}}$.

So, with an 85% accurate test, 10 out of 100 people (about 9.8%) will be reported as 'positive' for the virus when in reality only 6 out of 100 actually have the virus.

You may also find it helpful to find $P(T)$ by creating a tree



$$.051 + .047 = \boxed{.098}$$

Perhaps a more pertinent question to a given individual is:

"Suppose I test positive. what is the probability I actually have the virus?"

Translated, that is $P(V|T) = ?$

The problem is that we solved for $P(T|V)$, we need to switch the T and V.

$$P(V|T) = \frac{P(T \cap V)}{P(T)} = \frac{P(T|V) \cdot P(V)}{P(T|V) \cdot P(V) + P(T|V^c) \cdot P(V^c)}$$

With $P(V) = 0.06$ we get $P(V|T) = \frac{(0.85)\cdot(0.06)}{(0.85)\cdot(0.06)+(0.05)(1-0.06)} = 0.52$

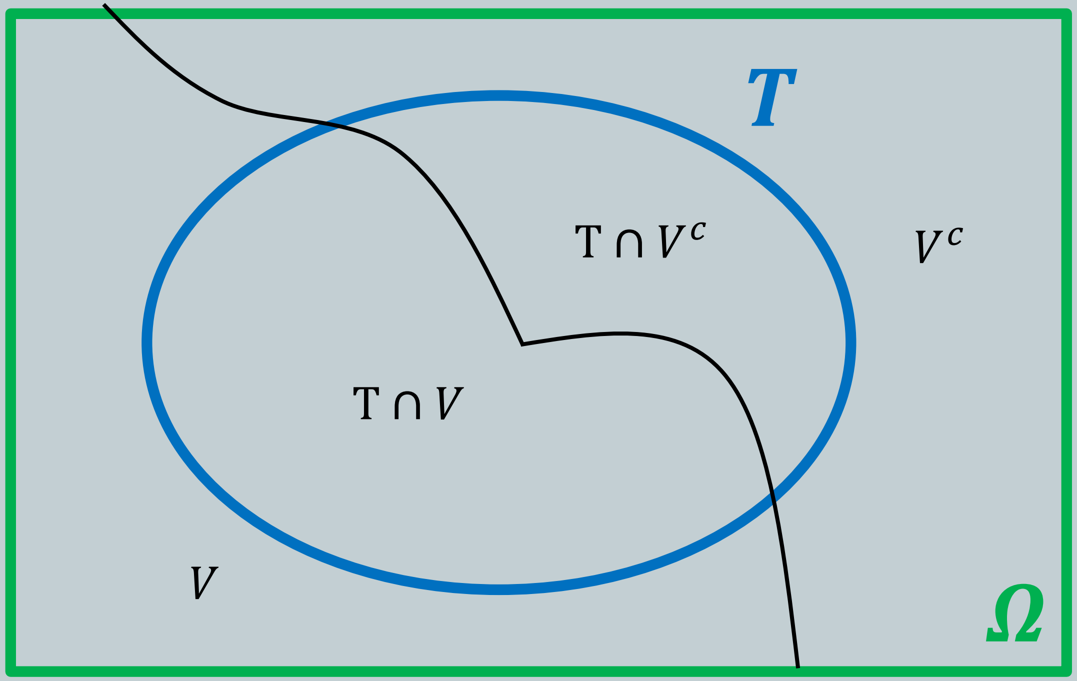
So, with a test that is 85% accurate, when you get a 'positive' test result, this translates into a 52% chance of you actually having the virus.

Bayes Rule: Suppose the events $C_1, C_2, C_3, \ldots, C_m$ are disjoint events such that $C_1 \cup C_2 \cup C_3 \cup \cdots \cup C_m = \Omega$. The conditional probability of C_i, given an arbitrary event A, can be expressed as:

$$P(C_i|A) = \frac{P(A|C_i) \cdot P(C_i)}{P(A|C_1) \cdot P(C_1) + P(A|C_2)P(C_2) + \cdots + P(A|C_m) \cdot P(C_m)}$$

Bayes rule for the virus situation would be:

$$P(V|T) = \frac{P(T|V) \cdot P(V)}{P(T)} = \frac{P(T|V) \cdot P(V)}{P(T|V) \cdot P(V) + P(T|V^c) \cdot P(V^c)}$$



We could also find out $P(V|T^c)$ with similar calculations

$$P(V|T^c) = \frac{P(T^c \cap V)}{P(T^c)} = \frac{P(T^c|V) \cdot P(V)}{P(T^c|V) \cdot P(V) + P(T^c|V^c) \cdot P(V^c)}$$

$$P(V|T^c) = \frac{(0.15) \cdot (0.06)}{(0.15) \cdot (0.06) + (0.95) \cdot (0.94)} = 0.00998$$

So, the probability that you actually have the virus, given that your test came back 'negative' is less than 1%.

False positives are more prevalent than false negatives in this scenario.
So $P(T)$ means something different than $P(V|T)$.

From cdc.gov

Based on evidence from clinical trials in people 16 years and older, the Pfizer-BioNTech vaccine was 95% effective at preventing laboratory-confirmed infection with the virus that causes COVID-19 in people who received two doses and had no evidence of being previously infected.

Learn more on cdc.gov