Get in small groups (about 4 students maximum) and work out these problems on the whiteboard. Ask one of the teaching assistants for help if your group gets stuck. You do **not** need to turn anything in. Since this worksheet contains a lot of problems, a good strategy would be to first skim the worksheet and then discuss and solve the problems which you think are difficult.

1. Let $A = \{1, 2, \{3, 4\}, 5\}$. Determine whether each of the following are true or false, and explain why.
   (a) $\{3, 4\} \subset A$
   (b) $\{3, 4\} \in A$
   (c) $\{\{3, 4\}\} \subset A$
   (d) $1 \in A$
   (e) $1 \subset A$
   (f) $\{1, 2, 5\} \subset A$
   (g) $\{1, 2, 5\} \in A$
   (h) $\{1, 2, 3\} \subset A$
   (i) $\emptyset \in A$

2. Prove that $\{9^n \mid n \in \mathbb{Z}\} \subseteq \{3^n \mid n \in \mathbb{Z}\}$, but $\{9^n \mid n \in \mathbb{Z}\} \neq \{3^n \mid n \in \mathbb{Z}\}$

3. Prove the following if A, B, C are sets.
   (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, using set builder notation
   (b) $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

4. Uncountable/countable sets
   (a) Prove directly from the definition of countable/uncountable that the set of natural numbers that are multiples of 3 or multiples of 4 is countable.
   (b) Prove that the set of real numbers in the interval $[4, 5]$ is uncountable.
   (c) Suppose A and B are both countable sets. Prove whether the Cartesian product $A \times B$ is countable or uncountable.

5. Show that if A, B, C, and D are sets with $|A| = |B|$ and $|C| = |D|$, then $|A \times C| = |B \times D|$.

6. Show that if A and B are sets and $A \subset B$ then $|A| \leq |B|$.

7. Prove or disprove each of these statements about the floor and ceiling functions
   (a) $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$ for all real numbers x and y.
   (b) $\lfloor \sqrt{\lceil x \rceil} \rfloor = \lfloor \sqrt{x} \rfloor$ for all positive real numbers x.

REVIEW:

8. On the Island of Knights and Knaves live two types of people: Knights who always tell the truth and Knaves who always lie. Consider the following situations, and see if you can classify each of the inhabitants as either a knight or a knave **using truth tables**.

(a) Willem says "Tucker is a knave." Tucker says "Willem and I are knights."

(b) Ioana says "Rachel and I are not the same." Rachel says "Of Ioana and I, exactly one is a knight."

(c) Rachel says "Either Tucker is a knight or I am a knight." Tucker says that Rachel is a knave.

(d) Aiden says "I am a knight or Willem is a knave." Willem says "Of Aiden and I, exactly one is a knight."

9. Determine a truth value for these quantifier statements where the domain for all variables consists of all integers. If they are false, then find a counterexample.

   (a) $\forall x \ (x^2 \geq x)$

   (b) $\forall x \ (x > 0 \vee x < 0)$

   (c) $\forall x \ (x = 1)$

10. Prove that, given an integer a, then $a^2 + 4a + 5$ is odd if and only if a is even.

11. Determine the truth value of each of these statements if the domain for n consists of all integers.

   (a) $\forall n \ (n + 1 > n)$

   (b) $\exists n \ (n = -n)$

   (c) $\exists n \ (2n = 3n)$

   (d) $\forall n \ (3n \leq 4n)$

Review other worksheets and past homework for practice.