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CSCI 3104, Algorithms **Due January 29, 2021**

Problem Set 2 (50 points) **Spring 2021, CU-Boulder**

Advice 1: For every problem in this class, you must justify your answer: show how you arrived at it and why it is correct. If there are assumptions you need to make along the way, state those clearly.
Advice 2: Verbal reasoning is typically insufficient for full credit. Instead, write a logical argument, in the style of a mathematical proof.

Instructions for submitting your solution:

- The solutions **should be typed** and we cannot accept hand-written solutions. Here's a short intro to Latex.
- You should submit your work through **Gradescope** only.
- The easiest way to access Gradescope is through our Canvas page. There is a Gradescope button in the left menu.
- Gradescope will only accept **.pdf** files.
- It is vital that you match each problem part with your work. Skip to 1:40 to just see the matching info.

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1. The following problems are a review of logarithm and exponent topics.

   (a) Solve for x.
      i. $3^{2x} = 81$
      ii. $3(5^{x-1}) = 375$
      iii. $\log_3 x^2 = 4$

   (b) Solve for x.
      i. $x^2 - x = \log_5 25$
      ii. $\log_{10}(x+3) - \log_{10} x = 1$

   (c) Answer each of the following with a TRUE or FALSE.
      i. $a^{\log_a x} = x$
      ii. $a^{\log_b x} = x$
      iii. $a = b^{\log_b a}$
      iv. $\log_a x = \frac{\log_b x}{\log_b a}$
      v. $\log b^m = m \log b$

Solution:

(a) Solve for x.

i. $3^{2x} = 81$
$3^{2x} = 3^4$
$2x = 4$
$x = 2$

ii. $3(5^{x-1}) = 375$
$5^{x-1} = 125$
$5^{x-1} = 5^3$
$x - 1 = 3$
$x = 4$

iii. $\log_3 x^2 = 4$
$3^4 = x^2$
$x = \pm\sqrt{3^4}$
$x = \pm 9$

(b) Solve for x

i. $x^2 - x = \log_5 25$
$x^2 - x = \log_5(5^2)$
$x^2 - x = 2\log_5 5$
$x^2 - x = 2$
$x^2 - x - 2 = 0$
$(x-2)(x+1)$
$x = 2, x = -1$

ii. $\log_{10}(x+3) - \log_{10} x = 1$
$\log_{10}(x+3) = 1 + \log_{10}(x)$
$x + 3 = 10$
$9x = 3$
$x = \frac{1}{3}$

(c) TRUE or FALSE

i. TRUE ii. FALSE iii. TRUE iv. TRUE v. TRUE

2. Compute the following limits at infinity. Show all work and justify your answer.

(a) $\lim_{x\to\infty} \frac{3x^3+2}{9x^3-2x^2+7}$

(b) $\lim_{x\to\infty} \frac{x^3}{e^{x/2}}$

(c) $\lim_{x\to\infty} \frac{\ln x^4}{x^3}$

Solution:

(a) $\lim_{x\to\infty} \frac{3x^3+2}{9x^3-2x^2+7}$

$$= \lim_{x\to\infty} \frac{3x^3+2}{9x^3-2x^2+7}$$

$$= \lim_{x\to\infty} \frac{3+\frac{2}{x^3}}{9-\frac{2}{x^2}+\frac{7}{x^3}}$$

$$= \frac{3}{9}$$

$$= \frac{1}{3}$$

(b) $\lim_{x\to\infty} \frac{x^3}{e^{x/2}}$

$= \lim_{x\to\infty} \frac{3x^2}{e^{x/2}\frac{1}{2}}$ L'hopital

$= \lim_{x\to\infty} \frac{6x^2}{e^{x/2}}$

$= \lim_{x\to\infty} \frac{12x}{e^{x/2}\frac{1}{2}}$ L'hopital

$= \lim_{x\to\infty} \frac{24x}{e^{x/2}}$

$= \lim_{x\to\infty} \frac{24}{e^{x/2}\frac{1}{2}}$ L'hopital

$= \lim_{x\to\infty} \frac{48}{e^{x/2}}$

$= \frac{48}{\infty}$

$= 0$

(c) $\lim_{x\to\infty} \frac{\ln x^4}{x^3}$

$= \lim_{x\to\infty} \frac{\frac{4}{x}}{3x^2}$ L'hopital

$= \lim_{x\to\infty} \frac{4}{3x^3}$

$= \frac{4}{3} \lim_{x\to\infty} \frac{1}{3x^3}$

$= \frac{4}{3} * \frac{1}{\infty}$

$= 0$

3. Compute the following limits at infinity. Show all work and justify your answer.

   (a) For real numbers $m, n > 0$ compute $\lim_{x \to \infty} \frac{x^m}{e^{nx}}$

   (b) What does this tell us about the rate at which e^{nx} approaches infinity relative to x^m? A brief explanation is fine for this part.

   (c) For real numbers $m, n > 0$ compute $\lim_{x \to \infty} \frac{(\ln x)^n}{x^m}$

   (d) What does this tell us about the rate at which $(\ln x)^n$ approaches infinity relative to x^m? A brief explanation is fine for this part.

Solution:

(a) $\lim_{x \to \infty} \frac{x^m}{e^{nx}}$

$= \lim_{x \to \infty} \frac{x^{m-1} * m}{e^{nx} * n}$ L'hopital

$= \lim_{x \to \infty} \frac{x^{m-2} * m * (m-1)}{e^{nx} * n^2}$ L'hopital

The numerator eventually eliminates the variable x while the denominator always approaches ∞.

$\therefore \lim_{x \to \infty} \frac{x^m}{e^{nx}} = \frac{m}{\infty} = 0$

(b) The limit being 0 tells us the denominator (e^{nx}) grows faster than the numerartor (x^m), that is, it tells us that e^{nx} approaches infinity much faster relative to x^m.

(c) $\lim_{x \to \infty} \frac{(\ln x)^n}{x^m}$

$= \lim_{x \to \infty} \frac{\frac{n \ln^{n-1}(x)}{x}}{mx^{m-1}}$ L'hopital

$= \lim_{x \to \infty} \frac{n \ln^{n-1}(x)}{mx^{m-1}}$

The numerator eventually becomes a constant m (eliminating the variable x) while the denominator always approaches ∞.

$\therefore \lim_{x \to \infty} \frac{(\ln x)^n}{x^m} = \frac{m}{\infty} = 0$

(d) The limit being 0 tells us the denominator (x^m) grows faster than the numerartor $(\ln x)^n$, that is, it tells us that (x^m) approaches infinity much faster relative to $(\ln x)^n$.

4. The problems in this question deal with the Root and the Ratio Tests. Determine the convergence or divergence of the following series. State which test you used.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$

   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$

   (c) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^n$

Solution:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$

$$L = \lim_{n\to\infty} \sqrt[n]{\left|\frac{e^{2n}}{n^n}\right|} = \lim_{n\to\infty} \frac{e^2}{n} = 0$$

Seeing that $L = 0$, by the root test if $L < 1$, the series convergers, thus this series is absolutely convergent.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$

Let $a_n = \frac{2^n}{n!}$ hence $a_{n+1} = \frac{2^{n+1}}{(n+1)!}$

$L = \displaystyle\lim_{n\to\infty} \left(\frac{2^{n+1}}{(n+1)!} * \frac{n!}{2^n}\right) = \lim_{n\to\infty} \frac{2 * n!}{(n+1) * n!}$

$= \lim_{n\to\infty} \frac{2}{(n+1)} = 0$

$L = 0 < 1$

By the ratio test, since $L < 1$ this series is absolutely convergent.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$

Let $a_n = \frac{n^2 2^{n+1}}{3^n}$ hence $a_{n+1} = \frac{(n+1)^2 2^{n+2}}{3^{n+1}}$

$L = \displaystyle\lim_{n\to\infty} \frac{(n+1)^2 * 2^n * 4}{3^n * 3} * \frac{3^n}{n^2 * 2^n * n}$

$= \displaystyle\lim_{n\to\infty} \frac{6n+2}{3} = \infty$

$\infty > 1$

By the ratio test, since $L > 1$ this series is divergent.

(d) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^n$

$$L = \lim_{n\to\infty} \sqrt[n]{\left|\frac{\ln n}{n}\right|^n} = \lim_{n\to\infty} \frac{\ln n}{n}$$

$= \lim_{n\to\infty} \left(\frac{\frac{1}{n}}{1}\right)$ L'hopital

$= \frac{1}{\infty} = 0$

By the root test $L < 1$, therefore this series is absolutely convergent.