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CSCI 2824: Discrete Structures

Lecture 16: Sequences

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Moodle HW5 - Due at Noon

Written HW6 - Posted - Due Friday at noon

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Sequences

A **sequence** is a function from a subset of the set of integers (usually either the set {0, 1, 2, . . .} or the set {1, 2, 3, . . .}) to a set S. We use the notation a_n to denote the image of the integer n. We call a_n a *term* of the sequence.

function notation:

$$f(n) = \frac{1}{n+5}$$

Example: Write out the first 5 terms of the sequence $a_n = \frac{1}{n+5}$

$$a_0 = \frac{1}{0+5} = \frac{1}{5}$$

$$a_1 = \frac{1}{1+5} = \frac{1}{6}$$

$$a_2 = \frac{1}{2+5} = \frac{1}{7}$$

$$a_3 = \frac{1}{3+5} = \frac{1}{8}$$

$$a_4 = \frac{1}{4+5} = \frac{1}{9}$$

Sequences

To find "closed form" we need a, r

A **geometric sequence** (or geometric progression) is a sequence of the form:

$$a, ar, ar^2, ar^3, \dots, ar^n, \dots$$

where the initial term a and the common ratio r are real numbers.

Example: Find the n^{th} term of the following sequence: $1, -\frac{3}{4}, \frac{9}{16}, -\frac{27}{64}, \dots$

common ratio in general: $\frac{ar^{n+1}}{ar^n} = r$

common ratio for this example: $\frac{\frac{9}{16}}{-\frac{3}{4}} = \frac{9}{16} \cdot -\frac{4}{3} = -\frac{3}{4}$

general form: ar^n

$\boxed{n^{th} \text{ term: } a_n = \left(-\frac{3}{4}\right)^n}$

Sequences

An **arithmetic sequence** (or arithmetic progression) is a sequence of the form:

$$a, a+d, a+2d, a+3d, \ldots, a+nd, \ldots$$

where the initial term a and the common difference d are real numbers.

Example: Find the n^{th} term of the following sequence: $8, 3, -2, -7, -12, \ldots$

common difference : -5

first term : $a = 8$

$a_n = a + nd$

$$\boxed{a_n = 8 - 5n}$$

assuming an index scheme that starts at 0.

Re-index to start $k = 1$

$a_1 = 8$
$a_2 = 3$
$\vdots$

$$\boxed{a_n = 8 - 5(n-1)}$$

Sequences

A recurrence relation for a sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer. A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.

Example: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}$$

for $n = 3, 4, 5, \ldots$ and suppose that $\underbrace{a_0 = 3, a_1 = -1, a_2 = 4}_{\text{initial conditions}}$. What are a_3, a_4, and a_5?

$$\begin{aligned} a_3 &= a_{3-1} + a_{3-2} + a_{3-3} \\ &= a_2 + a_1 + a_0 \\ &= 4 + -1 + 3 \\ &= 6 \end{aligned}$$

$$\begin{aligned} a_4 &= a_3 + a_2 + a_1 \\ &= 6 + 4 + -1 \\ &= 9 \end{aligned}$$

$$\begin{aligned} a_5 &= a_4 + a_3 + a_2 \\ &= 9 + 6 + 4 \\ &= 19 \end{aligned}$$

Sequences – Fibonacci

The Fibonacci sequence, $f_0, f_1, f_2, \ldots$, is defined by the initial conditions $f_0 = 0, f_1 = 1$, and the recurrence relation

$$f_n = f_{n-1} + f_{n-2}$$

for $n = 2, 3, 4, \ldots$

not closed form

Example: Write out the first 10 terms of the Fibonacci sequence.

$f_0 = 0$

$f_1 = 1$

$f_2 = 0 + 1 = 1$

$f_3 = 1 + 1 = 2$

$f_4 = 2 + 1 = 3$

$f_5 = 3 + 2 = 5$

$f_6 = 5 + 3 = 8$

$f_7 = 8 + 5 = 13$

$f_8 = 13 + 8 = 21$

$f_9 = 34$

$0, 1, 1, 2, 3, 5, 8, 13, 21, 34$

Sequences

Forward progression

We say that we have solved the recurrence relation together with the initial conditions when we find an explicit formula, called a **closed formula**, for the terms of the sequence.

Example: Let $a_0 = 1, a_1 = 3,$ and $a_n = 2a_{n-1} - a_{n-2}$. Find a closed-form version of a_n.

$a_0 = 1$

$a_1 = 3$

$a_2 = 2a_1 - a_0 = 2 \cdot 3 - 1 = 5$

$a_3 = 2a_2 - a_1 = 2 \cdot 5 - 3 = 7$

$a_4 = 2a_3 - a_2 = 2 \cdot 7 - 5 = 9$

$\vdots$

$\boxed{a_n = 2n + 1 \quad \text{for } n = 0, 1, \ldots}$

vs.

$a_n = 2n - 1 \quad \text{for } n = 1, 2, \ldots$

Sequences

exponent review

$\frac{a^m}{a^n} = a^{m-n}$

$a^m a^n = a^{m+n}$

Example: Show that $a_n = 4^n$ is a solution to $a_n = 8a_{n-1} - 16a_{n-2}$.

$$
\begin{aligned}
8a_{n-1} - 16a_{n-2} &= 8(4^{n-1}) - 16(4^{n-2}) \\
&= 2 \cdot 4^1 \cdot 4^{n-1} - 4^2 \cdot 4^{n-2} \\
&= 2(4^n) - (4^n) \\
&= 4^n \\
&= a_n
\end{aligned}
$$

Sequences

$f_n = f_{n-1} + f_{n-2}$

Example: Determine a recurrence relation for the even Fibonacci numbers.

f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}
0	1	1	2	3	5	8	13	21	34	55	89	144

$$f_n = f_{n-1} + f_{n-2}$$

$$= f_{n-2} + f_{n-3} + f_{n-3} + f_{n-4}$$

$$= f_{n-2} + 2f_{n-3} + f_{n-4}$$

$$= f_{n-3} + f_{n-4} + 2f_{n-3} + f_{n-4}$$

$$= 3f_{n-3} + f_{n-4} + f_{n-4}$$

$$= 3f_{n-3} + f_{n-4} + f_{n-5} + f_{n-6}$$

$$= 3f_{n-3} + f_{n-3} + f_{n-6}$$

$$= 4f_{n-3} + f_{n-6}$$

$$\boxed{f_n = 4f_{n-3} + f_{n-6}}$$

Check: $f_0 = 0$, $f_3 = 2$, $f_6 = 8$

$f_9 = 4f_6 + f_3 = 4 \cdot 8 + 2$

$= 34$ ✓

Backwards plugging in

Example: Find the solution to the recurrence relation: $a_n = a_{n-1} + 3,\ a_0 = 1$

aka closed form.

$$a_n = a_{n-1} + 3$$
$$= a_{n-2} + 3 + 3$$
$$= a_{n-3} + 3 + 3 + 3$$
$$= a_{n-4} + 3 + 3 + 3 + 3$$
$$\vdots$$
$$= a_{n-n} + 3 + 3 + 3 \cdots + 3$$
$$= a_0 + 3n$$

$$\boxed{a_n = 1 + 3n}$$

Extra Practice

Ex. 1: Determine a recurrence relation with solution $a_n = n + (-1)^n$

Ex. 2: Determine a recurrence relation with solution $a_n = n^2 + n$

Ex. 3: Determine a recurrence relation with solution $a_n = n!$

Solutions

Ex. 1: Determine a recurrence relation with solution $a_n = n + (-1)^n$

Solution: Let's write out a bunch of terms and look for a pattern

$$\begin{aligned}
a_0 &= 0 + (-1)^0 = 1 \\
a_1 &= 1 + (-1)^1 = 0 \\
a_2 &= 2 + (-1)^2 = 3 \\
a_3 &= 3 + (-1)^3 = 2 \\
a_4 &= 4 + (-1)^4 = 5 \\
a_5 &= 5 + (-1)^5 = 4 \\
a_6 &= 6 + (-1)^6 = 7
\end{aligned}$$

Ex. 1: Determine a recurrence relation with solution $a_n = n + (-1)^n$

Solution: So the first several terms in the sequence are

$$1,\ 0,\ 3,\ 2,\ 5,\ 4,\ 7,\ \ldots$$

Notice the following:

$$\begin{aligned} a_2 = 3 &= 1 \cdot 1 + 0 \cdot 0 + 2 = 3 \\ a_3 = 2 &= 1 \cdot 0 + 0 \cdot 3 + 2 = 2 \\ a_4 = 5 &= 1 \cdot 3 + 0 \cdot 2 + 2 = 5 \end{aligned}$$

So a recurrence relation that works is as follows

$$a_0 = 1,\ a_1 = 0\ ,\ a_n = a_{n-2} + 2$$

Ex. 2: Determine a recurrence relation with solution $a_n = n^2 + n$

Solution: Notice that

$$\begin{aligned} a_{n-1} &= (n-1)^2 + (n-1) = n^2 - n \\ a_{n-2} &= (n-2)^2 + (n-2) = n^2 - 3n + 2 \end{aligned}$$

Now, if we take $2a_{n-1}$ and subtract a_{n-2} we get

$$2a_{n-1} - a_{n-2} = 2n^2 - 2n - n^2 + 3n - 2 = n^2 + n - 2$$

If we add 2 to this result we get back a_n. So we have

$$a_0 = 0,\ a_1 = 2,\ a_n = 2a_{n-1} - a_{n-2} + 2$$

Ex. 3: Determine a recurrence relation with solution $a_n = n!$

Solution: Notice that we have

$$a_n = n(n-1)(n-2)\cdots 2 \cdot 1 = n(n-1)! = na_{n-1}$$

So we could define the recurrence relation as

$$a_1 = 1,\ a_n = na_{n-1} \text{ for } n > 1$$

Or, if you remember the weird definition that $0! = 1$ we could do

$$a_0 = 1,\ a_n = na_{n-1} \text{ for } n > 0$$