# CSCI 3202: Intro to Artificial Intelligence

# Lecture 30: HMM, Forward_backward

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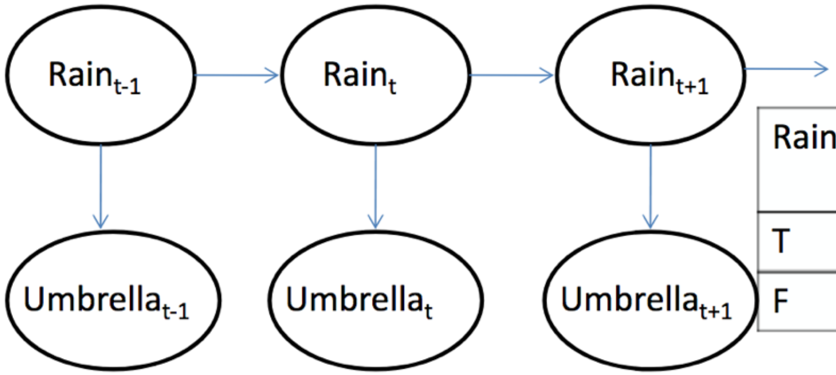
| $Rain_{t-1}$ | $P(Rain_t)$ |
|---|---|
| T | 0.7 |
| F | 0.3 |

$P(Rain_0 = T) = 0.5$

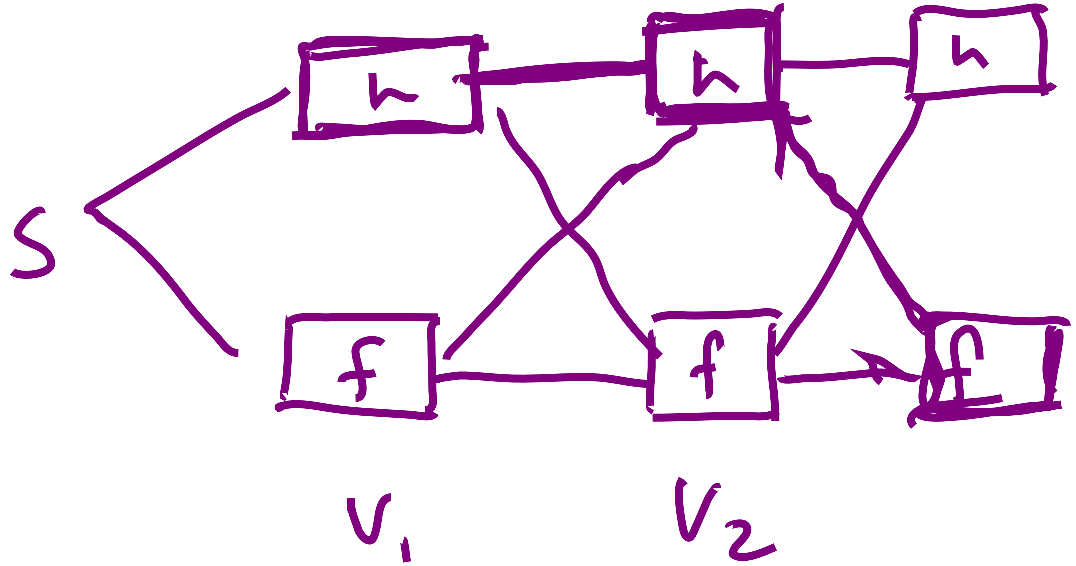



| $Rain_t$ | $P(Umbrella_t)$ |
|---|---|
| T | 0.9 |
| F | 0.2 |

HMMs — Adapted from Russel and Norvig, Chapter 15.

# HMMs – Viterbi

max probability of a sequence of states




Start at $X_T$

Find max prob

follow prev to get previous.

Repeat back to Start.

Sequence $h \to h \to f$ has max pr.

# Hidden Markov Models (HMMs) – smoothing

- **Smoothing:** Compute the belief state given the evidence over a previous state using all evidence up to current state.

  - *$P(X_k \mid e_{1:t})$, where $0 <= k < t$*

  - Revise our belief about the past given what was observed in the future
  - Split computation into two parts: evidence up to *k*, evidence *k+1* to *t*.

$$P(X_k \mid e_{1:t}) = P(X_k \mid e_{1:k}, e_{k+1:t})$$

$$= \alpha P(X_k \mid e_{1:k}) P(e_{k+1:t} \mid X_k)$$

$$= \alpha \underbrace{f_{1:k}}_{\text{forward filtering alg}} \times \underbrace{b_{k+1:t}}_{\text{backward alg.}}$$

# Hidden Markov Models (HMMs) – smoothing

- $b_{k+1:t}$ - runs backward from t

$f_{1:k}$ - runs forward from 1

$$b_{k+1:t} = \sum_{x_{k+1}} \underbrace{P(e_{k+1} \mid x_{k+1})}_{\text{given in model}} \underbrace{P(e_{k+2:t} \mid x_{k+1})}_{\text{we need to calculate this}} \underbrace{P(x_{k+1} \mid x_k)}_{\text{transition, also given in model}}$$

$$b_{k+1:t} = \text{Backward}(b_{k+2:t}, E_{k+1})$$

# HMM example – rain and the umbrella

**Example**: You are curious if it is raining, and the only contact you have with the outside world is through your advisor. If it is raining, she brings her umbrella 90% of the time, and has it just in case on 20% of sunny days. You know that historically, 70% of rainy days were followed by another rainy day, and 30% of sunny days were followed by a rainy day.

If you observe the umbrella two days in a row, what is the P(rain) on each day?

Day 1 – $u_1 = T$
Day 2 – $u_2 = T$

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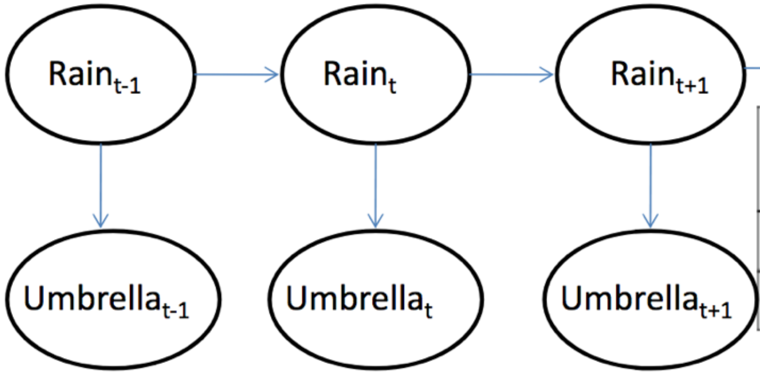


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# Rain and the umbrella – filtering

Day 1 - filtering $= [.818, .182]$

$P(R_1 | u_1)$

Day 2 $= [.883, .117]$

# Rain and the umbrella – filtering

# Rain and the umbrella – backwards

Given observation of umbrella on Day 1 and 2, compute the smoothed estimate for rain on Day 1.

Does seeing an umbrella on Day 2 change our belief about rain on day 1.

Initialize

evidence at $t+1$ is empty.

$$b_{t+1:t} = P(e_{t+1:t} \mid X_2) = [1, 1]$$

$$P(R_1 \mid u_1, u_2) = \alpha \underbrace{P(R_1 \mid u_1)}_{[.818,\ .182]} \underbrace{P(u_2 \mid R_1)}_{b_{t+1:t}}$$

# Rain and the umbrella – backwards

$$P(u_2 | R_1) = \underbrace{\sum_{r_2} P(u_2 | r_2) P( | r_2) P(r_2 | r_1)}$$

$$\sum_{x_{k+1}} P(e_{k+1} | x_{k+1}) P(e_{k+2:t} | x_{k+1}) P(x_{k+1} | x_k)$$

$$= [\underbrace{.9 \times 1 \times .7 + .2 \times 1 \times .3}_{R_1 = T},\ \underbrace{.9 \times 1 \times .3 + .2 \times 1 \times .7}_{R_1 = F}]$$

$$= [.63 + .06 = .69,\ .27 + .14 = .41]$$

$$P(u_2 | R_1) = [\underset{T}{.69},\ \underset{F}{.41}]$$

Probability of seeing umbrella on day 2 given $R_1$ is T, F.

# Forward Backward Algorithm

Suppose we want to smooth the entire sequence of estimates.

❖ Run the Forward algorithm (filter it)

❖ Then run the Backward algorithm (smooth it)

$$P(R_1 \mid u_1, u_2) = \alpha P(X_k \mid e_{1:t}) \, \overbrace{P(e_{k+1:t} \mid X_k)}$$

$$= \boxed{\alpha \, [.818, .182]} \, [.69, .41]$$

$$= \alpha \, [.564, .075]$$

$$= \boxed{[.883, .117]}$$

# Rain and the umbrella – forward – backward

5 days

umbrella

T, T, F, T, T

$P(X_t \mid e_{1:t})$

1: [.818, .182]

2: [.883, .117]

3: [.1907, .8093]

4: [.7308, .2692] ←

5: [.8673, .1327]

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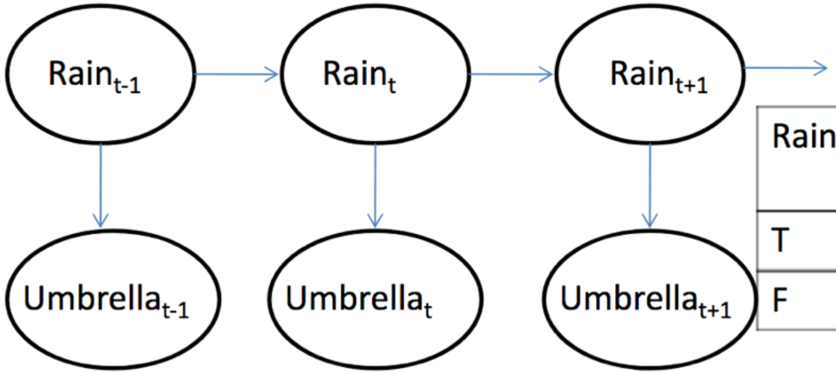


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# Rain and the umbrella – forward – backward

Day 5

$$P(u_5 \mid R_4) = \sum_{r_5} P(u_5 \mid r_5)\, P(\mid r_5)\, P(r_5 \mid r_4)$$

$$= [.69, \ .41]$$

$$P(R_4 \mid u_4, u_5) = \alpha P(X_k \mid e_{1:k})\, P(e_{k+1:t} \mid X_k)$$

$$= \alpha P(R_4 \mid e_{1:4})\, P(u_5 \mid R_4)$$

$$= \alpha [.7308, \ .2692][.69, \ .41]$$

$$= \alpha [.5, \ .11]$$

$$= [.819, \ .18]$$

$\alpha = .5 + .11$

# Rain and the umbrella – forward – backward

Day 4

$$P(u_4 \mid R_3) = \sum_{r_4} P(u_4 \mid r_4) P(u_5 \mid r_4) P(r_4 \mid R_3)$$

$$= [\underbrace{.9 \times .69 \times .7 + .2 \times .41 \times .3}_{R_3 = T},\ \underbrace{.9 \times .69 \times .3 + .2 \times .41 \times .7}_{R_3 = F}]$$

$$= [.459,\ .243]$$

# Rain and the umbrella – forward – backward

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