Graph Coloring

What is the color of the 3rd vertex?
How many vertices will be colored with the color Red?

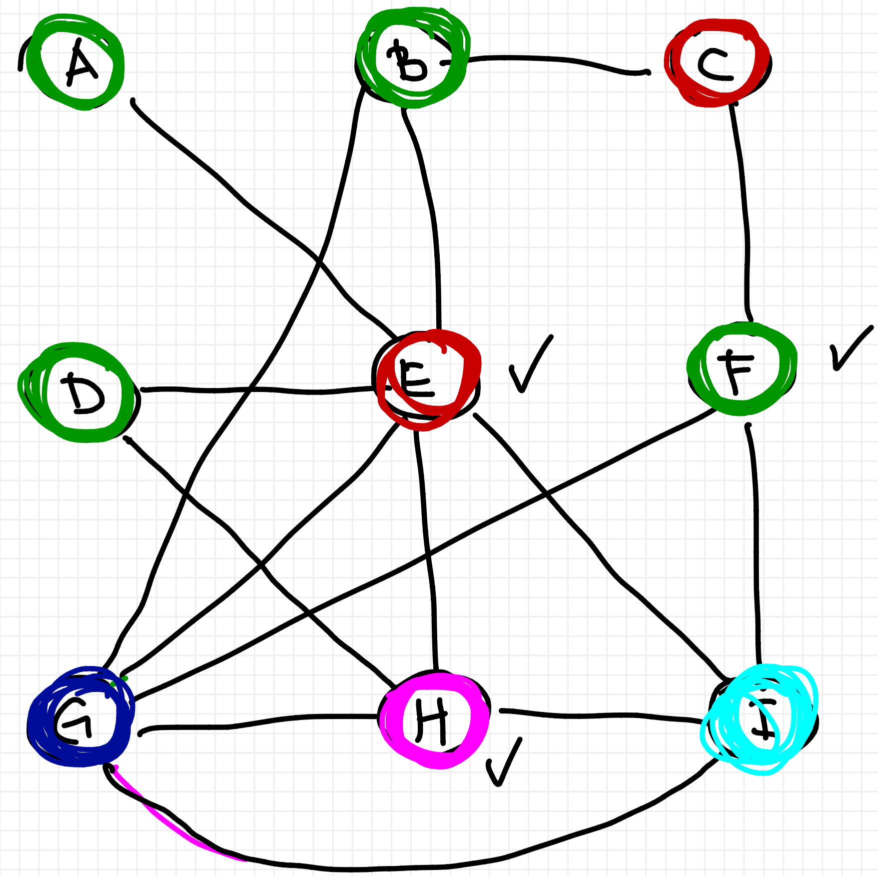
Remember the Greedy Coloring Algorithm:
The procedure requires us to number the colors **consecutively**

1. Order the vertices in some (arbitrary) way
2. Color the first vertex using Color 1
3. Pick the next uncolored vertex v.
   - Color it using the lowest-numbered color that has not been used on any of the vertices adjacent to v.
   - If you run out of colors, then add a new color
4. Repeat Step 3 until all vertices are colored

1- Green
2- Red
3- Blue
4- Pink

~~A~~, ~~B~~, ~~C~~, ~~D~~, E, F, G, H, I

1) How many Green vertices?
2) What color is vertex I?



Recurrence relations

Example: an old exam problem

a) Solve for the general solution $a_n^{(h)}$ to the homogeneous recurrence relation $a_n = 2a_{n-1} + 8a_{n-2}$

b) Find a particular solution $a_n^{(p)}$ to the nonhomogeneous recurrence relation $a_n = 2a_{n-1} + 8a_{n-2} + 3^n$ – NH

c) What is the full general solution to the nonhomogeneous recurrence relation in (b)? – don't solve!

a) Guess r^n → plug it in: $r^n = 2 * r^{n-1} + 8 * r^{n-2}$

Divide by r^{n-2} ⟹ $r^2 = 2r + 8$

Charcteristic Polynomial

$r^2 - 2r - 8 = 0$ $(\quad)(\quad)$

Recurence relations: close form solution

$$r^2 - 2r - 8 = 0$$

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4+32}}{2}$$

$$= \frac{2 \pm 6}{2} = \begin{cases} 8/2 = 4 \\ -4/2 = -2 \end{cases}$$

$$r_1 = 4$$

$$r_2 = -2$$

$$(r-4)(r+2) = 0$$

$$a_n^{(h)} = A(-2)^n + B(4)^n$$

If:

$$\left.\begin{array}{l} r_1 = 3 \\ r_2 = 3 \end{array}\right\} \quad a_n^{(h)} = A * 3^n + \cancel{B * 3^n}$$

$$B \cdot n \cdot 3^n$$

Important!

$$\boxed{r_1 = r_2 = r}$$

$$a_n^{(h)} = A * \underbrace{r^n} + B * n * \underbrace{r^n}$$

3) $n * 4^n \longrightarrow$ Guess: $4^n(Cn + D) = C*n*4^n + D*4^n$

$3n^2 * 2^n + 3n*3^n \rightarrow$ Guess: $2^n(Cn^2 + Dn + E) + 3^n(Fn + G)$

Recurence relations: close form solution

	NH part of recurrence	Guess for $a_n^{(p)}$	Example
1)	$F_n = c_k n^k + c_{k-1} n^{k-1} + \ldots + c_1 n + c_0$	$d_k n^k + d_{k-1} n^{k-1} + \ldots + d_1 n + d_0$	$F_n = n^2 \Rightarrow a_n^{(p)} = A \cdot n^2 + B \cdot n + C$
2)	$F_n = b \cdot c^n$	$d \cdot c^n$	$F_n = 3 \cdot 2^n \Rightarrow a_n^{(p)} = A \cdot 2^n$
3)	$F_n = (c_k n^k + \ldots c_1 n + c_0)(b \cdot e^n)$	$(d_k n^k + \ldots d_1 n + d_0)(e^n)$	$F_n = 3n \cdot 2^n \Rightarrow a_n^{(p)} = (An+B) \cdot 2^n$

2) If NH is $3^n \longrightarrow$ Guess is $C * 3$

If NH is $7 * 4^n \longrightarrow$ Guess is $C * 4^n$

Could overlap

If the term 4^n already exists:

$*n \longrightarrow C * n * 4^n$

Again

$*n \longrightarrow C * n^2 * 4^n$

1) $n \longrightarrow$ Guess: $Cn + D$

$n^2 + 2n \longrightarrow$ Guess: $Cn^2 + Dn + E$

$3n^2 + 5 \longrightarrow$ Guess: $Cn^2 + Dn + E$

No possibility of overlap

If you don't need to solve for C

$$a_n = a_n^{(h)} + a_n^{(p)} = \underline{A*(-2)^n + B*4^n} + \underbrace{C*3^n}_{NH}$$

Solving for C:

Recurence relations: close form solution

$$a_n = 2a_{n-1} + 8a_{n-2} + 3^n$$

Guess for NH : $\boxed{\underline{C*3^n}}$ → plug it in

$$a_n = C*3^n$$
$$a_{n-1} = C*3^{n-1}$$
$$a_{n-2} = C*3^{n-2}$$

$$C*3^n = 2C*3^{n-1} + 8C*3^{n-2} + 3^n$$

Divide by 3^{n-2}

$$C*3^2 = 2*C*3 + 8*C + 3^2$$

$$9C = 6C + 8C + 9 \Rightarrow -5C = 9 \Rightarrow$$

$$\Rightarrow C = -\frac{9}{5}$$

Now we have : $a_n = a_n^{(h)} + a_n^{(p)} = A(-2)^n + B*4^n - \frac{9}{5}*3^n$

Use initial conditions $(a_0, a_1, \ldots)$ to find A and B

$P(r_5) = P(r_5 \mid R) * P(R) + P(r_5 \mid M) * P(M) =$

$= \frac{1}{6} * \frac{1}{5} + \frac{2}{6} * \frac{2}{5} = \frac{5}{30} = \frac{1}{6}$

Probability

Suppose you have a bag of 5 dice:

- One is a regular, fair **6-sided** die; let **R** represent the event that you draw this die from the bag.
- Two are fair **4-sided** dice; let **F** represent the event that you draw one of these dice from the bag. No 5, 6
- Two are **6-sided** dice, with an extra pip drawn on the four-pip side so that each die has two 5's but no 4's (so they are fair, but numbered {1, 2, 3, 5, 5, 6}); let **M** represent the event that you draw one of these marked-up dice from the bag. → No 4

Suppose you draw a die at random from the bag and then you roll it. Define r_3 as the event that you roll a 3.

$P(R) = 1/5$

$P(F) = 2/5$

$P(M) = 2/5$

$P(r_3) = P(r_3 \mid R) * P(R) + P(r_3 \mid F) * P(F) + P(r_3 \mid M) * P(M) =$

$= \frac{1}{6} \cdot \frac{1}{5} + \frac{1}{4} * \frac{2}{5} + \frac{1}{6} * \frac{2}{5} =$

$= \frac{1}{30} + \frac{1}{10} + \frac{1}{15} = \frac{1}{5}$

Probability

(a) What is the probability that you roll a 3?

Probability

(b) Suppose you roll a 3. Given this information, what is the probability that you drew one of the 4-sided dice?

$$P(F \mid r_3) = \frac{P(r_3 \mid F) * P(F)}{P(r_3)} = \frac{\frac{1}{4} * \frac{2}{5}}{\frac{1}{5}} =$$

$$= \frac{1}{10} * 5 = \frac{1}{2}$$

Probability

(c) Are the events r3 and F independent? Fully justify your answer using math.

If $P(r_3 \mid F) = P(r_3) \Rightarrow$ independent

$1/4 \neq 1/5 \Rightarrow$ not independent

If $P(F \mid r_3) = P(F) \Rightarrow$ independent

Binomial Theorem.

Consider the linear equation

$$x_1 + x_2 + x_3 + x_4 = 17$$

where all the x_i are non-negative integers, $x_i \geq 0$.

(a) How many solutions does that equation have? Fully justify your answer.

(b) How many solutions does that equation have, subject to the additional constraint that all the $x_i \leq 9$? Fully justify your answer.

a) 17 stars, # bars: $4-1 = 3$

$* \; * \; | \; * \; * \; * \; | \; * \; * \; | \; *$

$x_1 \quad x_2 \quad x_3 \quad x_4$

positions: $17 + 3 = 20$

$(17+4-1)$

Answer:

$C(20,3) = C(20,17)$

Binomial Theorem. **"Stars and bars" problem**

Consider the linear equation

$$x_1 + x_2 + x_3 + x_4 = 17$$

where all the x_i are non-negative integers, $x_i \geq 0$.

(a) How many solutions does that equation have? Fully justify your answer.

Binomial Theorem. **"Stars and bars" problem**

Consider the linear equation

$$x_1 + x_2 + x_3 + x_4 = 17$$

where all the x_i are non-negative integers, $x_i \geq 0$.
(b) How many solutions does that equation have, subject to the additional constraint that all the $x_i \leq 9$? Fully justify your answer.

Same as $x_i \geq 10$

$\circledast \; \circledast \mid \circledast \; \circledast \; \circledast \mid \circledast \; \circledast \mid \circledast \; \circledast \; \circledast$

i) Find how many solutions if at least one $x_i > 9$

→ Side note: how many x_i could be > 9? Only one

$17 - 10 = 7$ stars left

x_1 has ≥ 10

$7 + 3 = 10$ total positions $\Rightarrow C(10, 3)$

$\downarrow$

$4 * C(10, 3)$

$x_1 \geq 10$
$x_2 \geq 10$
$x_3 \geq 10$
$x_4 \geq 10$

ii) subtract from a): Answer $= C(20,3) - 4 * C(10,3)$

Binomial Theorem. **“Stars and bars” problem**

Consider the linear equation

$$x_1 + x_2 + x_3 + x_4 = 17$$

where all the x_i are non-negative integers, $x_i \geq 0$.
(b) How many solutions does that equation have, subject to the additional constraint that all the $x_i \leq 9$? Fully justify your answer.