Get in small groups (about 4 students maximum) and work out these problems on the whiteboard. Ask one of the teaching assistants for help if your group gets stuck. You do **not** need to turn anything in. Since this worksheet contains a lot of problems, a good strategy would be to first skim the worksheet and then discuss and solve the problems which you think are difficult.

Complexity

1. Fill in the following table by providing a tight big-O and big-Ω bound for each function. Then, determine if it can be said that the function is big-Θ of any function.

Function	$O(??)$	$\Omega(??)$	$\Theta(??)$
$2n^2 + 3n + 1$			
$n^7 + \log n^8$			
$-1000n^4 \log(n^6) + n^5 + 22n^3 - 6n^2$			

2. Suppose the complexity of two algorithms, `FrancescaFunction` and `GregoryGoTo`, are given by the functions $f(n) = n^2 - 2n + 10$ and $g(n) = 4n + 10$. Sketch these two functions on the same set of axes and identify which function dominates the other, and for which values of n. Only consider $n \geq 1$. Provide a statement of your result in terms of big-O and big-Ω.

3. Show that $1^k + 2^k + \ldots + n^k$ is $O(n^{k+1})$.

Matricies/Matrix Operations

4. **Matrix-vector multiplication** Suppose you have a matrix A of size $m \times n$ (m rows and n columns) and a vector x of size $p \times 1$ (p rows, and 1 column). What is the **necessary condition** in order to be able to multiply A by x on the right, as Ax? State this as a conditional "if-then" statement like back in our good ol' days of propositional logic!

5. Consider the following matrices.

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 3 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}, \quad \text{and } C = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}$$

For this problem, you must calculate the product $P = ABC$. But wait! Don't start yet!

Matrix multiplication is associative, which means we can calculate the product P by first computing the matrix (AB), then multiplying this by C to obtain $P = ABC$. *Or* we could first compute the matrix (BC), then multiply it by A to obtain $P = ABC$. Furthermore, to multiply an $m \times n$ matrix by an $n \times k$ matrix requires $m \times n \times k$ multiplications. *Note: we are **only** counting multiplications here.*

(a) How many multiplications are needed to calculate P in the order $(AB)C$?

(b) How many multiplications are needed to calculate P in the order $A(BC)$?

(c) Calculate $P = ABC$ using whichever order you prefer.

Induction

6. For the following problem, we will be proving that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. Fill in the blanks to complete the proof.

   For $n =$ ________________ (any positive integer, usually 1 or 2), $\dfrac{n(n+1)}{2} =$ ________________.
   (*HINT*: Use the value of n you just defined.) This works!

   Let's assume the equation holds for $n =$ ______________. The equation is now ______________. (1)

   We have to prove the equation holds for $n =$ ____________. The equation is now ____________. (2)

   We now add ________________ to the equation using the (1) ________________ hypothesis. (3)

   After rearranging the terms in (3), we show that this equation is exactly equal to $n =$ ____________ (from (2)). QED

   Finally, go back through the proof and label the **base case** and the **inductive hypothesis**.

7. Use a **direct proof** to show that for every positive integer n, it is true that $\displaystyle\sum_{i=1}^{n}(8i-5) = 4n^2 - n$

   Then, use a **proof by induction** to prove this as well. Did you use strong or weak induction?

   Potentially useful fact: $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

8. Use induction to prove that you can tile any checkerboard with an even number of rows.

9. **Challenge Problem:** Professor Snape has n magical widgets that are supposedly both identical and capable of testing each other's correctness. Snape's test apparatus can hold two widgets at a time. When it is loaded, each widget tests the other and reports whether it is good or bad. A good widget always reports accurately whether the other widget is good or bad, but the answer of a bad widget cannot be trusted. Thus, the four possible outcomes of a test are as follows:

Widget A says	Widget B says	Conclusion
B is good	A is good	both are good/bad
B is good	A is bad	at least one is bad
B is bad	A is good	at least one is bad
B is bad	A is bad	at least one is bad

   Prove that if n/2 or more widgets are bad, Snape cannot necessarily determine which widgets are good using any strategy based on this kind of pairwise test. Assume a worst-case scenario in which the bad widgets are intelligent and conspire to fool Snape.