Get in small groups (about 4 students maximum) and work out these problems on the whiteboard. Ask one of the teaching assistants for help if your group gets stuck. You do **not** need to turn anything in.

Counting

1. How many 11-element RNA sequences consist of 4 As, 3Cs, 2Us, and 2Gs, and end with CAA?

2. A witness to a hit-and-run accident tells the police that the license plate of the car in the accident, which contains three letters followed by three digits, starts with the letters AS and contains both the digits 1 and 2. How many different license plates can fit this description?

3. A professor writes 20 multiple-choice questions, each with the possible answer a, b, c, or d, for a discrete mathematics test. If the number of questions with a, b, c, and d as their answer is 8, 3, 4, and 5, respectively, how many different answer keys are possible, if the questions can be placed in any order?

4. (a) How many different strings can be made from the word PEPPERCORN when all the letters are used?
   Solution:

   (b) How many of these strings start and end with the letter P?
   Solution:

   (c) In how many of these strings are the three letter Ps consecutive?
   Solution:

Binomial Theorem

5. For each of the following binomials, find the coefficient on the given x y multiple:

   (a) $(x+y)^{13} \mid x^5y^8$
   Solution: 1287

   (b) $(5x-y)^{20} \mid x^3y^{17}$
   Solution: -142500

   (c) $(-3x+7y)^{15} \mid x^{10}y^5$
   Solution:

   (d) $(2x-4y)^{17} \mid x^3y^{14}$
   Solution:

   (e) $(9x-6y)^{25} \mid x^{20}y^5$
   Solution:

Probability

6. From a deck of cards you pull out 1 card. What is the probability that the card you pulled out is red? Now, suppose the card you chose is an Ace. What is the probability that the Ace you chose is a red Ace? Are these events independent?

   Solution: Probability of card being red - $P(R) = \frac{26}{52} = 1/2$

   Probability of card being an ace - $P(A) = \frac{4}{52} = 1/13$

   Probability of card being a red ace - $P(A \cap R) = \frac{2}{52} = 1/26$

   Since $P(A \cap R) = P(A)P(R)$, the events are independent.

7. What is the probability of the result of 4 independent coin flips being exactly 1 head and 3 tails?

   Solution: This is:

$$\begin{aligned} P(1H, 3T) &= \frac{\text{number of ways to get exactly 1 Heads in 4 coin flips}}{\text{number of ways to flip 4 coins}} \\ &= \frac{C(4,1)}{2^4} \\ &= \frac{4}{16} \\ &= \frac{1}{4} \end{aligned}$$

8. Which is more likely: rolling a total of 8 when 2 fair six-sided dice are rolled OR rolling a total of 8 when 3 dice are rolled?

   Solution:

   With 2 dice: Combinations to get a sum of 8 are $\{2+6, 3+5, 4+4, 5+3, 6+2\}$, and there are 36 total possible outcomes, so this probability is $5/36 \approx 0.139$.

   We can count up the number of ways to get a sum of 8 when 3 dice are rolled as the number of solutions to the linear equation $x+y+z = 8$ with the constraints that $1 \leq x, y, z \leq 6$ (and all three are integers).

   Note that if $1 \leq x, y, z$, then we do not actually need to impose the constraint that $x, y, z \leq 6$ because (for example) if $x = y = 1$, then there's no way $z > 6$.

   So we want to count up the number of solutions to the linear equation $x + y + z = 8$ subject to the constraint $1 \leq x, y, z$.

   This is stars and bars! There are 8 units to distribute among x, y and z. We can preliminarily distribute 1 to each of them to satisfy the constraint. Our new problem is then to find the number of solutions to $x + y + z = 5$, with no constraints now on the variables (aside from being nonnegative integers).

   So we have 5 stars and 3-1=2 bars. Number of ways is $C(5+3-1, 3-1) = C(7,2) = \dfrac{7!}{2! \cdot 5!} = \dfrac{7 \cdot 6}{2} = 21$

   And the total number of possible outcomes on 3 dice is $6^3 = 216$.

   So our probability of rolling an 8 on 3 dice is $21/216 \approx 0.0972$

9. Which is more likely: rolling a total of 10 when 3 fair dice are rolled, or rolling a total of 15 when 4 fair dice are rolled?

   Solution: This solution follows a similar methodology to Problem 8's solution. Please read that solution for a simpler example.

   Each of these can be broken down into a Stars and Bars problem, using the dice as the bins and the total you want as the number of objects to place.

   Finding the probability of rolling 10 with 3 fair die:
   Because no die can be less than 1, distribute 1 to each die to start. This leaves 3 dice with 7 "ones" to place. Now, following Stars and Bars, there are 7 "ones" + 3-1 = 2 bars. Therefore, we have:

$$C(9,2) = \tfrac{9!}{7!2!} = 9 \cdot 4 = 36$$

ways to distribute the 7 ”ones” to any of the die. However, no die can total greater than 6. Therefore, examples with a single die totaling more than 6 must be removed. 1 ”one” has already been distributed to each die, so any additional 6 must be distributed to any of them to break the condition. Since each die can receive the 6, there are 3 ways to distribute those 6 ”ones”. After distributing the 6 ”ones”, there is 1 ”ones” remaining. With the same number of bars, we now have:

$$3 \cdot C(3,2) = 3 \cdot (\tfrac{3!}{1!2!}) = 3 \cdot 3 = 9$$

ways of getting 10 total where a die has rolled more than 6.

Simplifying, there are $36 - 9 = 27$ ways to roll 10 on 3 dice. Because there are $6^3 = 216$, the probability of rolling 10 on 3 dice is $\frac{27}{216} = .125$.

Finding the probability of rolling 15 with 4 fair die:
Because no die can be less than 1, distribute 1 to each die to start. This leaves 4 dice with 11 ”ones” to place. Now, following Stars and Bars, there are 11 ”ones” + 4-1 = 3 bars. Therefore, we have:

$$C(14,3) = \tfrac{14!}{11!3!} = \tfrac{14 \cdot 13 \cdot 12}{3 \cdot 2} = 14 \cdot 13 \cdot 2 = 364$$

ways to distribute the 11 ”ones” to any of the dice. This includes situations with a single die rolling greater than 6, so those instances need to be removed. To do this, give one die 6 more ”ones” (making the die have 7 ”ones” total). There are 4 dice, so there are 4 ways to give the 6 ”ones” to a single die. This leaves 5 ”ones” to distribute among the 4 dice. Therefore, we have:

$$4 \cdot C(8,3) = 4 \cdot \tfrac{8!}{5!3!} = 4 \cdot (8 \cdot 7) = 224$$

ways to distribute 11 ”ones” to any of the dice where one die is guaranteed to be greater than 7.

Simplifying, there are $364 - 224 = 140$ ways to get 15 when rolling 4 dice. With $6^4 = 1296$ total ways to roll 4 fair dice, the probability of rolling 15 on 4 dice is $\frac{140}{1296} = .108$.

Because $.125 > .108$, we can conclude that rolling 10 on 3 fair dice is more likely than rolling 15 on 4 fair dice.

10. There are many lotteries now that award enormous prizes to people who correctly choose a set of six numbers out of the first n positive integers, where n is usually between 30 and 60. What is the probability that a person picks the correct six numbers out of 50?

    Solution:

11. In a lottery, a player selects 7 numbers out of the first 80 positive integers. What is the probability that a person wins the grand prize by picking 7 numbers that are among the 11 numbers selected at random by a computer?

    Solution:

12. In roulette, a wheel with 38 numbers is spun. Of these, 18 are red, and 18 are black. The other two numbers, which are neither black nor red, are 0 and 00. The probability that when the wheel is spun it lands on any particular number is 1/38.

    (a) What is the probability that the wheel lands on a red number?
    Solution:

    (b) What is the probability that the wheel lands on a black number twice in a row?
    Solution:

(c) What is the probability that the wheel lands on 0 or 00?
Solution:

(d) What is the probability that in five spins the wheel never lands on either 0 or 00?
Solution:

(e) What is the probability that the wheel lands on one of the first six integers on one spin, but does not land on any of them on the next spin?
Solution: