Bloch 1

3.2.9	**3.3.5**	**3.4.1**	**4.1.5**	**5.1.4**	**5.3.1**	**7.4.2**
3.2.10	**3.3.9**	**3.4.2**	**4.1.6**	**5.1.5**	**5.3.2**	**7.4.3**
3.2.11	**3.3.10**	**3.4.4**	**4.1.7**	**5.1.7**	**5.3.4**	**7.4.5**
3.2.12	**3.3.11**	**3.4.5**	**4.1.8**	**5.1.8**	**5.3.5**	**7.4.6**
3.2.14	**3.3.12**	**3.4.6**	**4.1.10**	**5.1.9**	**5.3.9**	**7.4.7**
3.2.16	**3.3.13**	**3.4.7**		**5.1.11**	**5.3.10**	**7.4.13**
	3.3.15				**5.3.13**	**7.4.17**
	3.3.16					
	3.3.17					
	3.3.18					
	3.3.19					
	3.3.21					
	3.3.22					

A is the set of all aunts;

C is the set of all people who are children of other people.

Exercise 3.2.8. Among the following sets, which is a subset of which?

$$C = \{n \in \mathbb{Z} \mid \text{there exists } k \in \mathbb{Z} \text{ such that } n = k^4\};$$
$$E = \{n \in \mathbb{Z} \mid \text{there exists } k \in \mathbb{Z} \text{ such that } n = 2k\};$$
$$P = \{n \in \mathbb{Z} \mid n \text{ is a prime number}\};$$
$$N = \{n \in \mathbb{Z} \mid \text{there exists } k \in \mathbb{Z} \text{ such that } n = k^8\};$$
$$S = \{n \in \mathbb{Z} \mid \text{there exists } k \in \mathbb{Z} \text{ such that } n = 6k\};$$
$$D = \{n \in \mathbb{Z} \mid \text{there exists } k \in \mathbb{Z} \text{ such that } n = k - 5\};$$
$$B = \{n \in \mathbb{Z} \mid n \text{ is non-negative}\}.$$

Exercise 3.2.9. Find sets A and B such that $A \in B$ and $A \subseteq B$. (It might appear as if we are contradicting what was discussed after Example 3.2.3; the solution, however, is the "exception that proves the rule.")

Exercise 3.2.10. Let A, B and C be sets. Suppose that $A \subseteq B$ and $B \subseteq C$ and $C \subseteq A$. Prove that $A = B = C$.

Exercise 3.2.11. **[Used in Theorem 3.5.6.]** Let A and B be sets. Prove that it is not possible that $A \subsetneqq B$ and $B \subseteq A$ are both true.

Exercise 3.2.12. Let A and B be any two sets. Is it true that one of $A \subseteq B$ or $A = B$ or $A \supseteq B$ must be true? Give a proof or a counterexample.

Exercise 3.2.13. Let $A = \{x, y, z, w\}$. List all the elements in $\mathcal{P}(A)$?

Exercise 3.2.14. Let A and B be sets. Suppose that $A \subseteq B$. Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Exercise 3.2.15. List all elements of each of the following sets.

(1) $\mathcal{P}(\mathcal{P}(\emptyset))$.

(2) $\mathcal{P}(\mathcal{P}(\{\emptyset\}))$.

Exercise 3.2.16. Which of the following are true and which are false?

(1) $\{\emptyset\} \subseteq G$ for all sets G.

(2) $\emptyset \subseteq G$ for all sets G.

(3) $\emptyset \subseteq \mathcal{P}(G)$ for all sets G.

(4) $\{\emptyset\} \subseteq \mathcal{P}(G)$ for all sets G.

(5) $\emptyset \in G$ for all sets G.

(6) $\emptyset \in \mathcal{P}(G)$ for all sets G.

(7) $\{\{\emptyset\}\} \subseteq \mathcal{P}(\emptyset)$.

(8) $\{\emptyset\} \subseteq \{\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}\}$.

(9) $\mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$.

(1) $G \cup I$.
(2) $G \cap I$.
(3) $G \cap H$.
(4) $J - G$.
(5) $I - H$.
(6) $J \cap (G - H)$.

Exercise 3.3.5. Given two sets A and B, are the sets $A - B$ and $B - A$ necessarily disjoint? Give a proof or a counterexample.

Exercise 3.3.6. **[Used in Theorem 3.3.3.]** Prove Theorem 3.3.3 (1) (2) (3) (6) (7) (8) (9).

Exercise 3.3.7. **[Used in Theorem 3.3.8.]** Prove Theorem 3.3.8 (1) (2) (3) (4) (5) (6).

Exercise 3.3.8. **[Used in Theorem 3.3.12.]** Prove Theorem 3.3.12 (1) (2) (4) (5).

Exercise 3.3.9. **[Used in Theorem 7.6.7.]** Let A and B be sets. Prove that $(A \cup B) - A = B - (A \cap B)$

Exercise 3.3.10. **[Used in Theorem 6.3.6.]** Let A, B and C be sets. Suppose that $C \subset A \cup B$, and that $C \cap A = \emptyset$. Prove that $C \subseteq B$.

Exercise 3.3.11. Let X be a set, and let $A, B, C \subseteq X$ be subsets. Suppose that $A \cap B = A \cap C$, and that $(X - A) \cap B = (X - A) \cap C$. Prove that $B = C$.

Exercise 3.3.12. Let A, B and C be sets. Prove that $(A - B) \cap C = (A \cap C) - B = (A \cap C) - (B \cap C)$.

Exercise 3.3.13. **[Used in Exercise 6.5.15.]** For real numbers a, b and c, we know that $a - (b - c) = (a - b) + c$. Let A, B and C be sets.

(1) Suppose that $C \subseteq A$. Prove that $A - (B - C) = (A - B) \cup C$.
(2) Does $A - (B - C) = (A - B) \cup C$ hold for all sets A, B and C? Prove or give a counterexample for this formula. If the formula is false, find and prove a modification of this formula that holds for all sets.

Exercise 3.3.14. Let A and B be sets. The **symmetric difference** of A and B, denoted $A \triangle B$, is the set $A \triangle B = (A - B) \cup (B - A)$.

Let X, Y and Z be sets. Prove the following statements.

(1) $X \triangle \emptyset = X$.
(2) $X \triangle X = \emptyset$.
(3) $X \triangle Y = Y \triangle X$.
(4) $X \triangle (Y \triangle Z) = (X \triangle Y) \triangle Z$.
(5) $X \cap (Y \triangle Z) = (X \cap Y) \triangle (X \cap Z)$.
(6) $X \triangle Y = (X \cup Y) - (X \cap Y)$.

Exercise 3.3.15. Prove or find a counterexample to the following statement. Let A, B and C be sets. Then $(A - B) \cup C = (A \cup B \cup C) - (A \cap B)$.

Exercise 3.3.16. Prove or find a counterexample to the following statement. Let A, B and C be sets. Then $(A \cup C) - B = (A - B) \cup (C - B)$.

Exercise 3.3.17. Let A, B and C be sets. Prove that $A \subseteq C$ if and only if $A \cup (B \cap C) = (A \cup B) \cap C$.

Exercise 3.3.18. Prove or give a counterexample for each of the following statements.

(1) Let A and B be sets. Then $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$.
(2) Let A and B be sets. Then $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

Exercise 3.3.19. Let A, B and C be sets. Prove that $A \times (B - C) = (A \times B) - (A \times C)$.

Exercise 3.3.20. Let A and B be sets. Suppose that $B \subseteq A$. Prove that $A \times A - B \times B = [(A - B) \times A] \cup [A \times (A - B)]$.

Exercise 3.3.21. Let A and B be sets. Suppose that $A \neq B$. Suppose that E is a set such that $A \times E = B \times E$. Prove that $E = \emptyset$.

Exercise 3.3.22. Let X be a set. Suppose that X is finite. Which of the two sets $\mathcal{P}(X \times X) \times \mathcal{P}(X \times X)$ and $\mathcal{P}(\mathcal{P}(X))$ has more elements?

3.4 Families of Sets

So far we have dealt with unions and intersections of only two sets at a time. We now want to apply these operations to more than two sets.

For the sake of comparison, let us look at addition of real numbers. Formally, addition is what is called a binary operation, which takes pairs of numbers as input and produces single numbers as output. We will see a rigorous treatment of binary operations in Section 7.1, but for now it is sufficient to take an informal approach to this concept. In particular, we see that in principle it is possible to add only two numbers at a time. Of course, in practice it is often necessary to add three or more numbers together, and here is how it is done. Suppose that we want to compute $2+3+9$. We would proceed in one of two ways, either first computing $2+3=5$ and then computing $5+9=14$, or first computing $3+9=12$ and then computing $2+12=14$. As expected, we obtained the same answer both ways, and this common answer is what we would call the sum of the three numbers $2+3+9$. Another way of writing these two ways of computing the sum is as $(2+3)+9$ and $2+(3+9)$. It turns out that there is a general rule about addition, called the Associative Law, that says that in all cases of three numbers that are being added, the same result is obtained from either way of positioning the parentheses. This property of addition is stated in Theorem A.1 (1) in the Appendix. Hence, for any three numbers a, b and c, we can define the sum $a+b+c$ to be the number that results from computing either $(a+b)+c$ or $a+(b+c)$.

Intuitively, a similar approach would work for the sum of any finite collection of numbers, though to do so formally would require definition by recursion, a topic we will see in Section 6.4; see Example 6.4.4 (2) for the use of recursion for adding finitely many numbers. Sums of infinite collections of numbers are much trickier. The reader has most likely encountered the notion of a series of numbers, for example $\sum_{n=1}^{\infty} \frac{1}{n^2}$, in a calculus course. Not all such series actually add up to a real number, and the question of figuring out for which series that happens is somewhat tricky,

(6). Let $a \in B - (\bigcap_{X \in \mathcal{A}} X)$. Then $a \in B$ and $a \notin \bigcap_{X \in \mathcal{A}} X$. Then $a \notin Y$ for some $Y \in \mathcal{A}$. Then $a \in B - Y$. Hence $a \in \bigcup_{X \in \mathcal{A}} (B - X)$ by Part (2) of this theorem. It follows that $B - (\bigcap_{X \in \mathcal{A}} X) \subseteq \bigcup_{X \in \mathcal{A}} (B - X)$.

Now let $b \in \bigcup_{X \in \mathcal{A}} (B - X)$. Then $b \in B - Z$ for some $Z \in \mathcal{A}$. Then $b \in B$ and $b \notin Z$. Hence $b \notin \bigcap_{X \in \mathcal{A}} X$. It follows that $b \in B - \bigcap_{X \in \mathcal{A}} X$. Therefore $\bigcup_{X \in \mathcal{A}} (B - X) \subseteq B - (\bigcap_{X \in \mathcal{A}} X)$.

We conclude that $B - (\bigcap_{X \in \mathcal{A}} X) = \bigcup_{X \in \mathcal{A}} (B - X)$. □

It can be verified that all the parts of Theorem 3.4.5 that involve union but not intersection hold also when $\mathcal{A} = \emptyset$; the parts of the theorem that involve intersection are not defined when $\mathcal{A} = \emptyset$.

It is interesting to compare the proof of Theorem 3.4.5 (3) with the proof of Theorem 3.3.3 (5). Though Theorem 3.4.5 (3) is a generalization of Theorem 3.3.3 (5), the proof of the generalized statement is slightly more concise than the proof of the simpler statement. The proof of Theorem 3.4.5 (3) is more concise precisely because it is phrased explicitly in terms of quantifiers, which allows us to avoid the need for cases as in the proof of Theorem 3.3.3 (5).

In addition to defining the union and intersection of families of sets, it is also possible to form the product of a family of sets, though doing so requires the use of functions, and hence we will wait until the end of Section 4.5 for the definition.

Exercises

Exercise 3.4.1. In each of the following parts, we are given a set B_k for each $k \in \mathbb{N}$. Find $\bigcup_{k \in \mathbb{N}} B_k$ and $\bigcap_{k \in \mathbb{N}} B_k$.

(1) $B_k = \{0, 1, 2, 3, \ldots, 2k\}$.
(2) $B_k = \{k-1, k, k+1\}$.
(3) $B_k = [\frac{3}{k}, \frac{5k+2}{k}) \cup \{10+k\}$.
(4) $B_k = [-1, 3 + \frac{1}{k}] \cup [5, \frac{5k+1}{k})$.
(5) $B_k = (-\frac{1}{k}, 1] \cup (2, \frac{3k-1}{k}]$.
(6) $B_k = [0, \frac{k+1}{k+2}] \cup [7, \frac{7k+1}{k})$.

Exercise 3.4.2. In each of the following parts, you need to find a family of sets $\{E_k\}_{k \in \mathbb{N}}$ such that $E_k \subseteq \mathbb{R}$ for each $k \in \mathbb{N}$, that no two sets E_k are equal to each other and that the given conditions hold.

(1) $\bigcup_{k \in \mathbb{N}} E_k = [0, \infty)$ and $\bigcap_{k \in \mathbb{N}} E_k = [0, 1]$.
(2) $\bigcup_{k \in \mathbb{N}} E_k = (0, \infty)$ and $\bigcap_{k \in \mathbb{N}} E_k = \emptyset$.
(3) $\bigcup_{k \in \mathbb{N}} E_k = \mathbb{R}$ and $\bigcap_{k \in \mathbb{N}} E_k = \{3\}$.
(4) $\bigcup_{k \in \mathbb{N}} E_k = (2, 8)$ and $\bigcap_{k \in \mathbb{N}} E_k = [3, 6]$.
(5) $\bigcup_{k \in \mathbb{N}} E_k = [0, \infty)$ and $\bigcap_{k \in \mathbb{N}} E_k = \{1\} \cup [2, 3)$.
(6) $\bigcup_{k \in \mathbb{N}} E_k = \mathbb{Z}$ and $\bigcap_{k \in \mathbb{N}} E_k = \{\ldots, -2, 0, 2, 4, 6, \ldots\}$.
(7) $\bigcup_{k \in \mathbb{N}} E_k = \mathbb{R}$ and $\bigcap_{k \in \mathbb{N}} E_k = \mathbb{N}$.

Exercise 3.4.3. [Used in Theorem 3.4.5.] Prove Theorem 3.4.5 (1) (2) (4) (5). Do some in the indexed notation and some in the non-indexed notation.

Exercise 3.4.4. Let $\mathcal{A}$ and $\mathcal{B}$ be non-empty families of sets. Suppose that $\mathcal{A} \subseteq \mathcal{B}$.

(1) Prove that $\bigcup_{X\in\mathcal{A}} X \subseteq \bigcup_{Y\in\mathcal{B}} Y$.
(2) Prove that $\bigcap_{X\in\mathcal{A}} X \subseteq \bigcap_{Y\in\mathcal{B}} Y$.

Exercise 3.4.5. Let I be a non-empty set, and let $\{A_i\}_{i\in I}$ and $\{B_i\}_{i\in I}$ be families of sets indexed by I. Suppose that $A_i \subseteq B_i$ for all $i \in I$.

(1) Prove that $\bigcup_{i\in I} A_i \subseteq \bigcup_{i\in I} B_i$.
(2) Prove that $\bigcap_{i\in I} A_i \subseteq \bigcap_{i\in I} B_i$.

Exercise 3.4.6. Let $\mathcal{A}$ be a non-empty family of sets and let B be a set.

(1) Prove that $(\bigcup_{X\in\mathcal{A}} X) - B = \bigcup_{X\in\mathcal{A}}(X - B)$.
(2) Prove that $(\bigcap_{X\in\mathcal{A}} X) - B = \bigcap_{X\in\mathcal{A}}(X - B)$.

Exercise 3.4.7. Let I be a non-empty set, let $\{A_i\}_{i\in I}$ be a family of sets indexed by I and let B be a set.

(1) Prove that $B \times (\bigcup_{i\in I} A_i) = \bigcup_{i\in I}(B \times A_i)$.
(2) Prove that $B \times (\bigcap_{i\in I} A_i) = \bigcap_{i\in I}(B \times A_i)$.

Exercise 3.4.8. Suppose that $\mathcal{W}$ is some property of subsets of $\mathbb{R}$ (for example, being finite). A subset $X \subseteq \mathbb{R}$ is called **co-**$\mathcal{W}$ if $\mathbb{R} - X$ has property $\mathcal{W}$.

Let $\mathcal{A}$ be a non-empty family of sets. Suppose that X is a co-$\mathcal{W}$ subset of $\mathbb{R}$ for all $X \in \mathcal{A}$. For each of the properties $\mathcal{W}$ listed below, either prove that $\cup_{X\in\mathcal{A}} X$ is co-$\mathcal{W}$, or give a counterexample. Try to figure out a general rule for deciding when $\cup_{X\in\mathcal{A}} X$ is co-$\mathcal{W}$ for a given property $\mathcal{W}$.

(1) A subset of $\mathbb{R}$ has property $\mathcal{W}$ if and only if it is finite.
(2) A subset of $\mathbb{R}$ has property $\mathcal{W}$ if and only if it has at most 7 elements.
(3) A subset of $\mathbb{R}$ has property $\mathcal{W}$ if and only if it has precisely 7 elements.
(4) A subset of $\mathbb{R}$ has property $\mathcal{W}$ if and only if it contains only integers.
(5) A subset of $\mathbb{R}$ has property $\mathcal{W}$ if and only if it is finite, and has an even number of elements.

3.5 Axioms for Set Theory

Set theory is a very remarkable idea that works so very well, and is so broadly useful, that it is used as the basis for modern mathematics. Unfortunately, however, it does not work quite as nicely as we might have made it appear in the previous sections of this chapter. Early in the development of set theory, a number of "paradoxes" were discovered, the most well-known of which is Russell's Paradox, which is as follows.

Suppose that we could form the set of all sets; let S denote this set. Observe that $S \in S$. We then define the set $T = \{A \in S \mid A \notin A\}$. Is T a member of itself? Suppose first that $T \notin T$. Then $T \in T$. Now suppose that $T \in T$. Then $T \notin T$. There is something wrong here. The problem is that we are trying to use a set of all sets, and more generally the problem is that we have to be more careful how we quantify over sets. See [GG94, Section 5.3] for further comments on the paradoxes of set theory.

(1) $\{(b,1),(c,2),(a,3)\}$.
(2) $\{(a,3),(c,2),(a,1)\}$.
(3) $\{(c,1),(b,1),(a,2)\}$.
(4) $\{(a,1),(b,3)\}$.
(5) $\{(c,1),(a,2),(b,3),(c,2)\}$.
(6) $\{(a,3),(c,3),(b,3)\}$.

Exercise 4.1.2. Let X denote the set of all people. Which of the following descriptions define functions $X \to X$?

(1) $f(a)$ is the mother of a.
(2) $g(a)$ is a brother of a.
(3) $h(a)$ is the best friend of a.
(4) $k(a)$ is the firstborn child of a if she is a parent, and is the father of a otherwise.
(5) $j(a)$ is the sibling of a if she has siblings, and is a otherwise.

Exercise 4.1.3. Which of the diagrams in Figure 4.1.4 represent functions?

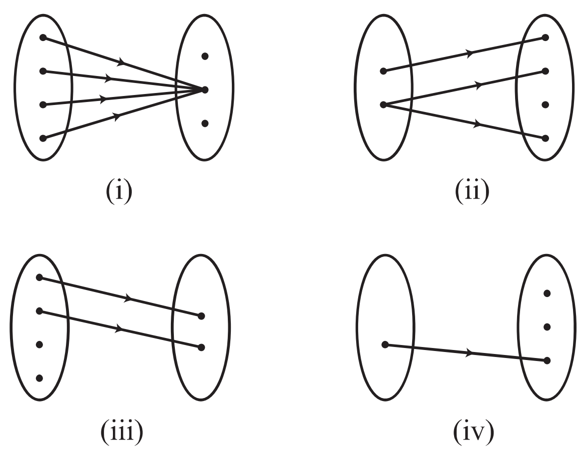

Fig. 4.1.4.

Exercise 4.1.4. Which of the following descriptions properly describe functions?

(1) Let $f(x) = \cos x$.
(2) To every person a, let $g(a)$ be the height of a in inches.
(3) For every real number, assign the real number that is the logarithm of the original number.
(4) Let $g\colon \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = e^x$.

Exercise 4.1.5. Which of the following formulas define functions $\mathbb{R} \to \mathbb{R}$?

(1) $f(x) = \sin x$ for all $x \in \mathbb{R}$.

(2) $p(x) = \frac{x^2+3}{x+5}$ for all $x \in \mathbb{R}$.

(3) $q(x) = \ln(x^4+1)$ for all $x \in \mathbb{R}$.

(4) $r(x) = \begin{cases} e^x, & \text{if } x \geq 0 \\ \cos x, & \text{if } x \leq 0. \end{cases}$

(5) $s(x) = \begin{cases} x^2, & \text{if } x \geq 1 \\ x^3, & \text{if } x \leq 0. \end{cases}$

(6) $t(x) = \begin{cases} x^3 - 2, & \text{if } x \geq 1 \\ |x|, & \text{if } x \leq 1. \end{cases}$

(7) $g(x) = \begin{cases} \sin x, & \text{if } x \geq \pi \\ x, & \text{if } x < \pi. \end{cases}$

Exercise 4.1.6. For each of the following formulas, find the largest subset $X \subseteq \mathbb{R}$ such that $g\colon X \to \mathbb{R}$ is a function.

(1) $g(x) = \frac{1}{x^4-3}$ for all $x \in X$.
(2) $g(x) = \sqrt{1-x^2}$ for all $x \in X$.
(3) $g(x) = 3\ln(\sin x)$ for all $x \in X$.
(4) $g(x) = \begin{cases} \sqrt{x}, & \text{if } x \in X \text{ and } x \geq 0 \\ x+1, & \text{if } x \in X \text{ and } x \leq 0. \end{cases}$
(5) $g(x) = \begin{cases} \tan \pi x + 4, & \text{if } x \in X \text{ and } x \geq 1 \\ 3x^2+1, & \text{if } x \in X \text{ and } x \leq 1. \end{cases}$

Exercise 4.1.7. Let A and B be sets, let $S \subseteq A$ be a subset and let $f\colon A \to B$ be a function. Let $g\colon A \to B$ be an extension of $f|_S$ to A. Does g equal f? Give a proof or a counterexample.

Exercise 4.1.8. [Used in Theorem 4.5.4 and Section 8.7.] Let X be a non-empty set, and let $S \subseteq X$ be a subset. The **characteristic map** for S in X, denoted χ_S, is the function $\chi_S\colon X \to \{0,1\}$ defined by

$$\chi_S(y) = \begin{cases} 1, & \text{if } y \in S \\ 0, & \text{if } y \in X - S. \end{cases}$$

Let $A, B \subseteq X$ be subsets. Prove that $\chi_A = \chi_B$ if and only if $A = B$. (Observe that "$\chi_A = \chi_B$" is a statement of equality of functions, whereas "$A = B$" is a statement of equality of sets.)

Exercise 4.1.9. [Used in Section 4.1.] Restate Theorem 4.1.5 in a non-indexed version.

Exercise 4.1.10. [Used in Exercise 4.1.11.] Let A and B be sets. A **partial function** from A to B is a function of the form $f_J\colon J \to B$, where $J \subseteq A$. We can think of partial functions from A to B as subsets of $A \times B$ that satisfy a certain condition.

Let f_J and g_K be partial functions from A to B. Prove that $f_J \subseteq g_K$ if and only if $J \subseteq K$ and $g_K|_J = f_J$.

Exercise 4.1.11. [Used in Section 3.5.] The purpose of this exercise is to prove that Zorn's Lemma (Theorem 3.5.6) implies the Axiom of Choice. Given that we used

(3) $\bar{O} = \{(1,1),(2,2),(1,2)\}$.
(4) $\bar{P} = \{(1,1),(2,2),(3,3)\}$.
(5) $\bar{Q} = \{(1,2),(2,1),(1,3),(3,1),(1,1)\}$.
(6) $\bar{R} = \{(1,2),(2,3),(3,1)\}$.
(7) $\bar{T} = \{(1,1),(1,2),(2,2),(2,3),(3,3),(1,3)\}$.

Exercise 5.1.4. Is each of the following relations reflexive, symmetric and/or transitive?

(1) Let S be the relation on $\mathbb{R}$ defined by $x\,S\,y$ if and only if $x = |y|$, for all $x, y \in \mathbb{R}$.
(2) Let P be the set of all people, and let R be the relation on P defined by $x\,R\,y$ if and only if x and y were not born in the same city, for all $x, y \in P$.
(3) Let T be the set of all triangles in the plane, and let G be the relation on T defined by $s\,G\,t$ if and only if s has greater area than t, for all triangles $s, t \in T$.
(4) Let P be the set of all people, and let M be the relation on P defined by $x\,M\,y$ if and only if x and y have the same mother, for all $x, y \in P$.
(5) Let P be the set of all people, and let N be the relation on P defined by $x\,N\,y$ if and only if x and y have the same color hair or the same color eyes, for all $x, y \in P$.
(6) Let D be the relation on $\mathbb{N}$ defined by $a\,D\,b$ if and only if $a|b$, for all $a, b \in \mathbb{N}$.
(7) Let T be the relation on $\mathbb{Z} \times \mathbb{Z}$ defined by $(x,y)\ T\ (z,w)$ if and only if there is a line in $\mathbb{R}^2$ that contains (x,y) and (z,w) and has slope an integer, for all $(x,y),(z,w) \in \mathbb{Z} \times \mathbb{Z}$.

Exercise 5.1.5. Let A be a set, and let R be a relation on A. Suppose that R is defined by the set $\bar{R} \subseteq A \times A$. Let R' be the relation on A defined by the set $(A \times A) - \bar{R}$.

(1) If R reflexive, is R' necessarily reflexive, necessarily not reflexive or not necessarily either?
(2) If R symmetric, is R' necessarily symmetric, necessarily not symmetric or not necessarily either?
(3) If R transitive, is R' necessarily transitive, necessarily not transitive or not necessarily either?

Exercise 5.1.6. Let A be a set, and let R be a relation on A. Suppose that R is symmetric and transitive. Find the flaw in the following alleged proof that this relation is necessarily reflexive; there must be a flaw by Example 5.1.6 (4). "Let $x \in A$. Choose $y \in A$ such that $x\,R\,y$. By symmetry know that $y\,R\,x$, and then by transitivity we see that $x\,R\,x$. Hence R is reflexive."

Exercise 5.1.7. Let A be a set, and think of $\subseteq$ as defining a relation on $\mathcal{P}(A)$, as stated in Example 5.1.2 (5). Is this relation reflexive, symmetric and/or transitive?

Exercise 5.1.8. Let A be a set, and let R be a relation on A.

(1) Suppose that R is reflexive. Prove that $\bigcup_{x \in A} [x] = A$.
(2) Suppose that R is symmetric. Prove that $x \in [y]$ if and only if $y \in [x]$, for all $x, y \in A$.
(3) Suppose that R is transitive. Prove that if $x\,R\,y$, then $[y] \subseteq [x]$, for all $x, y \in A$.

Exercise 5.1.9. Let A and B be sets, let R be a relation on A and let $f: A \to B$ be a function. The function f **respects** the relation R if $x\,R\,y$ implies $f(x) = f(y)$, for all $x, y \in A$. Which of the following functions respects the given relation?

(1) Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^6$ for all $x \in \mathbb{R}$; let S be the relation on $\mathbb{R}$ defined by $x\,S\,y$ if and only if $|x| = |y|$, for all $x, y \in \mathbb{R}$.
(2) Let $g: \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = \cos x$ for all $x \in \mathbb{R}$; let W be the relation on $\mathbb{R}$ defined by $x\,W\,y$ if and only if $x - y = \frac{\pi k}{2}$ for some $k \in \mathbb{Z}$, for all $x, y \in \mathbb{R}$.
(3) Let $h: \mathbb{R} \to \mathbb{R}$ be defined by $h(x) = \lfloor x \rfloor$ for all $x \in \mathbb{R}$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x; let T be the relation on $\mathbb{R}$ defined by $x\,T\,y$ if and only if $|x - y| < 1$, for all $x, y \in \mathbb{R}$.
(4) Let $k: \mathbb{R}^2 \to \mathbb{R}$ be defined by $k((x,y)) = 3x^2 + 6xy + 3y^2$ for all $(x,y) \in \mathbb{R}^2$; let M be the relation on $\mathbb{R}^2$ defined by $(x,y)\,M\,(z,w)$ if and only if $x + y = z + w$, for all $(x,y), (z,w) \in \mathbb{R}^2$.

Exercise 5.1.10. Let A and B be sets, let R be a relation on A and let $f: A \to B$ be a function. Suppose that f is injective, and that it respects the relation R, as defined in Exercise 5.1.9. What, if anything, can be proved about the relation R?

Exercise 5.1.11. Let A and B be sets, let R and S be relations on A and B, respectively, and let $f: A \to B$ be a function. The function f is **relation preserving** if $x\,R\,y$ if and only if $f(x)\,S\,f(y)$, for all $x, y \in A$.

(1) Suppose that f is bijective and relation preserving. Prove that f^{-1} is relation preserving.
(2) Suppose that f is surjective and relation preserving. Prove that R is reflexive, symmetric or transitive if and only if S is reflexive, symmetric or transitive, respectively.

5.2 Congruence

In this section we discuss a very important type of relation on the set of integers, which will serve to illustrate the general topic discussed in the next section, and is also a valuable tool in various parts of mathematics and its applications, for example number theory, cryptography and calendars. See [Ros05, Chapters 4 and 5] for further discussion of congruence and its applications, and see [Kob87] for a treatment of congruence and cryptography.

The idea of congruence is based upon the notion of "clock arithmetic," a term sometimes used in elementary mathematics. (For the reader who has not seen "clock arithmetic," it will be sufficient to have seen a clock). For the sake of uniformity, we will make all references to time using the American 12-hour system (ignoring a.m. vs. p.m.), as opposed to the 24-hour system used many places around the world, and in the U.S. military.

Suppose that it is 2 o'clock, and you want to know what time it will be in 3 hours. Clearly the answer is $2 + 3 = 5$ o'clock. Now suppose that it is 7 o'clock, and you want to know what time it will be in 6 hours. A similar calculation would yield and transitivity of $\sim$ it follows that $x \sim y$. Now suppose that $x \sim y$. Then $y \in [x]$. By the reflexivity of $\sim$, we know that $x \in [x]$. The definition of Φ implies that $[x] \in \mathcal{D}$. Hence, by the definition of Ψ, it follows that $x \approx y$. Therefore $x \approx y$ if and only if $x \sim y$. We conclude that $\approx = \sim$.

Second, we prove that $\Phi \circ \Psi = 1_{\mathcal{T}_A}$. Let $\mathcal{D} \in \mathcal{T}_A$ be a partition of A. Let $\mathcal{F} = \Phi(\Psi(\mathcal{D}))$. We will show that $\mathcal{F} = \mathcal{D}$, and it will then follow that $\Phi \circ \Psi = 1_{\mathcal{T}_A}$. For convenience let $\Bumpeq = \Psi(\mathcal{D})$, so that $\mathcal{F} = \Phi(\Bumpeq)$.

Let $B \in \mathcal{F}$. Then by the definition of Φ we know that B is an equivalence class of $\Bumpeq$, so that $B = [z]$ for some $z \in A$. Because $\mathcal{B}$ is a partition of A, then there is a unique $P \in \mathcal{B}$ such $z \in P$. Let $w \in A$. Then by the definition of Ψ we see that $z \Bumpeq w$ if and only if $w \in P$. It follows that $w \in [z]$ if and only if $w \in P$, and hence $P = [z]$. Hence $B = [z] = P \in \mathcal{B}$. Therefore $\mathcal{F} \subseteq \mathcal{B}$.

Let $C \in \mathcal{B}$. Let $y \in C$. As before, it follows from the definition of Ψ that $C = [y]$. Therefore by the definition of Φ we see that $C \in \Phi(\Bumpeq) = \mathcal{F}$. Hence $\mathcal{B} \subseteq \mathcal{F}$. We conclude that $\mathcal{F} = \mathcal{B}$. □

Exercises

Exercise 5.3.1. Which of the following relations is an equivalence relation?

(1) Let M be the relation on $\mathbb{R}$ defined by $x\, M\, y$ if and only if $x - y$ is an integer, for all $x, y \in \mathbb{R}$.
(2) Let S be the relation on $\mathbb{R}$ defined by $x\, S\, y$ if and only if $x = |y|$, for all $x, y \in \mathbb{R}$.
(3) Let T be the relation on $\mathbb{R}$ defined by $x\, T\, y$ if and only if $\sin x = \sin y$, for all $x, y \in \mathbb{R}$.
(4) Let P be the set of all people, and let Z be the relation on P defined by $x\, Z\, y$ if and only if x and y are first cousins, for all $x, y \in P$.
(5) Let P be the set of all people, and let R be the relation on P defined by $x\, R\, y$ if and only if x and y have the same maternal grandmother, for all $x, y \in P$.
(6) Let L be the set of all lines in the plane, and let W be the relation on L defined by $\alpha\, W\, \beta$ if and only if α and β are parallel, for all $\alpha, \beta \in L$.

Exercise 5.3.2. For each of the following equivalence relations on $\mathbb{R}$, find the equivalence classes $[0]$ and $[3]$.

(1) Let R be the relation defined by $a\, R\, b$ if and only if $|a| = |b|$, for all $a, b \in \mathbb{R}$.
(2) Let S be the relation defined by $a\, S\, b$ if and only if $\sin a = \sin b$, for all $a, b \in \mathbb{R}$.
(3) Let T be the relation defined by $a\, T\, b$ if and only if there is some $n \in \mathbb{Z}$ such that $a = 2^n b$, for all $a, b \in \mathbb{N}$.

Exercise 5.3.3. For each of the following equivalence relations on $\mathbb{R}^2$, give a geometric description of the equivalence classes $[(0,0)]$ and $[(3,4)]$.

(1) Let Q be the relation defined by $(x,y)\, Q\, (z,w)$ if and only if $x^2 + y^2 = z^2 + w^2$, for all $(x,y), (z,w) \in \mathbb{R}^2$.
(2) Let U be the relation defined by $(x,y)\, U\, (z,w)$ if and only if $|x| + |y| = |z| + |w|$, for all $(x,y), (z,w) \in \mathbb{R}^2$.

(3) Let V be the relation defined by $(x,y)\ V\ (z,w)$ if and only if $\max\{|x|,|y|\} = \max\{|z|,|w|\}$, for all $(x,y),(z,w) \in \mathbb{R}^2$.

Exercise 5.3.4. Let A and B be sets, and let $f\colon A \to B$ be a function. Let $\sim$ be the relation on A defined by $x \sim y$ if and only if $f(x) = f(y)$, for all $x,y \in A$.

(1) Prove that $\sim$ is an equivalence relation.
(2) What can be proved about the equivalence classes of $\sim$? Does the answer depend upon whether f is injective and/or surjective?

Exercise 5.3.5. Let A be a set, and let $\asymp$ be a relation on A. Prove that $\asymp$ is an equivalence relation if and only if the following two conditions hold.

(1) $x \asymp x$, for all $x \in A$.
(2) $x \asymp y$ and $y \asymp z$ implies $z \asymp x$, for all $x,y,z \in A$.

Exercise 5.3.6. **[Used in Theorem 5.3.4.]** Prove Theorem 5.3.4 (1).

Exercise 5.3.7. **[Used in Lemma 5.3.9.]** Prove Lemma 5.3.9.

Exercise 5.3.8. Which of the following families of subsets of $\mathbb{R}$ are partitions of $[0,\infty)$?

(1) $\mathcal{H} = \{[n-1,n)\}_{n\in\mathbb{N}}$.
(2) $\mathcal{G} = \{[x-1,x)\}_{x\in[0,\infty)}$.
(3) $\mathcal{F} = \{\{x\}\}_{x\in[0,\infty)}$.
(4) $\mathcal{I} = \{[n-1,n+1)\}_{n\in\mathbb{N}}$.
(5) $\mathcal{J} = \{[0,n)\}_{n\in\mathbb{N}}$.
(6) $\mathcal{K} = \left\{[2^{n-1}-1,2^n-1)\right\}_{n\in\mathbb{N}}$.

Exercise 5.3.9. For each of the following equivalence relations, describe the corresponding partition. Your description of each partition should have no redundancy, and should not refer to the name of the relation.

(1) Let P be the set of all people, and let $\asymp$ be the relation on P defined by $x \asymp y$ if and only if x and y have the same mother, for all $x,y \in P$.
(2) Let $\sim$ be the relation on $\mathbb{R}-\{0\}$ defined by $x \sim y$ if and only if $xy > 0$, for all $x,y \in \mathbb{R}-\{0\}$.
(3) Let $\approx$ be the relation on $\mathbb{R}^2$ defined by $(x,y) \approx (z,w)$ if and only if $(x-1)^2 + y^2 = (z-1)^2 + w^2$, for all $(x,y),(z,w) \in \mathbb{R}^2$.
(4) Let L be the set of all lines in $\mathbb{R}^2$, and let $\doteqdot$ be the relation on L defined by $l_1 \doteqdot l_2$ if and only if l_1 is parallel to l_2 or is equal to l_2, for all $l_1,l_2 \in L$.

Exercise 5.3.10. For each of the following partitions, describe the corresponding equivalence relation. Your description of each equivalence relation should not refer to the name of the partition.

(1) Let $\mathcal{E}$ be the partition of $A = \{1,2,3,4,5\}$ defined by $\mathcal{E} = \{\{1,5\},\{2,3,4\}\}$.
(2) Let $\mathcal{Z}$ be the partition of $\mathbb{R}$ defined by $\mathcal{Z} = \{T_x\}_{x\in\mathbb{R}}$, where $T_x = \{x,-x\}$ for all $x \in \mathbb{R}$.
(3) Let $\mathcal{D}$ be the partition of $\mathbb{R}^2$ consisting of all circles in $\mathbb{R}^2$ centered at the origin (the origin is considered a "degenerate" circle).
(4) Let $\mathcal{W}$ be the partition of $\mathbb{R}$ defined by $\mathcal{W} = \{[n,n+2) \mid n \text{ is an even integer}\}$.

Exercise 5.3.11. **[Used in Lemma 5.3.16.]** Prove Item (2) in the proof of Lemma 5.3.16.

Exercise 5.3.12. Let X and Y be sets, and let $h\colon X \to Y$ be a function. Let $\asymp$ be the relation on X defined by $s \asymp t$ if and only if $h(s) = h(t)$, for all $s, t \in X$.

(1) Prove that $\asymp$ is an equivalence relation on X.
(2) Let $\gamma\colon X \to X/\!\asymp$ be the canonical map. Let $j\colon h(X) \to Y$ be the inclusion map. Prove that there is a unique bijective function $\hat{h}\colon X/\!\asymp\, \to h(X)$ such that $h = j \circ \hat{h} \circ \gamma$. This last condition is represented by the following commutative diagram (as discussed in Section 4.3).

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\gamma"'] & Y \\ X/\!\asymp \arrow[r, "\hat{h}"'] & h(X) \arrow[u, "j"'] \end{tikzcd}$$

Observe that γ is surjective (because $\asymp$ is reflexive), that $\hat{h}$ is bijective and that j is injective. Hence any function can be written as a composition of a surjective function, a bijective function and an injective function.

Exercise 5.3.13. **[Used in Exercise 5.3.14.]** Let A be a non-empty set. Let $\mathcal{R}(A)$ denote the set of all relations on A, and let $\mathcal{S}_A$ denote the set of all families of subsets of A.

(1) Clearly $\mathcal{E}(A) \subseteq \mathcal{R}(A)$ and $\mathcal{T}_A \subseteq \mathcal{S}_A$. Are these inclusions proper?
(2) Express the sets $\mathcal{R}(A)$ and $\mathcal{S}_A$ in terms of products of sets and power sets.
(3) Let $A = \{1, 2\}$. What are $\mathcal{R}(A)$ and $\mathcal{S}_A$?
(4) Suppose that A is a finite set. Express $|\mathcal{R}(A)|$ and $|\mathcal{S}_A|$ in terms of $|A|$. Do $\mathcal{R}(A)$ and $\mathcal{S}_A$ have the same number of elements? Use Example 3.2.9 (2) and Example 3.3.10 (3).

Exercise 5.3.14. This exercise makes use of the definitions given at the start of Exercise 5.3.13. We generalize the functions Φ and Ψ given in Definition 5.3.15 as follows. Let A be a non-empty set. Let $\widetilde{\Phi}\colon \mathcal{R}(A) \to \mathcal{S}_A$ be defined as follows. If $\propto$ is a relation on A, let $\Phi(\propto)$ be the family of all relation classes of A with respect to $\propto$. Let $\widetilde{\Psi}\colon \mathcal{S}_A \to \mathcal{R}(A)$ be defined as follows. If $\mathcal{D}$ is a family of subsets of A, let $\Psi(\mathcal{D})$ be the relation on A defined by $x\,\Psi(\mathcal{D})\,y$ if and only if there is some $D \in \mathcal{D}$ such that $x, y \in D$, for all $x, y \in A$. (There is a distinct function $\widetilde{\Phi}$ and a distinct function $\widetilde{\Psi}$ for each non-empty set A, but we will assume that the set A is always known from the context.)

(1) Find a set B and an element $\mathcal{D} \in \mathcal{S}_B$ such that $\widetilde{\Psi}(\mathcal{D})$ is not reflexive. Find a set C and an element $\mathcal{E} \in \mathcal{S}_C$ such that $\widetilde{\Psi}(\mathcal{E})$ is not transitive.
(2) Suppose that A is finite and has at least two elements. Prove that each of $\widetilde{\Phi}$ and $\widetilde{\Psi}$ is neither injective nor surjective. Is it necessary to restrict our attention to sets with at least two elements?

Proof. We follow [KR83a]. We prove the result by induction on n. When $n = 1$ the result is trivial. Now assume that $n \geq 2$. Suppose that the result holds for $n-1$.

By Theorem 7.4.9 the poset A has a maximal element, say $r \in A$. Let $x \in A$. Because $\preccurlyeq$ is a total ordering, we know that $x \preccurlyeq r$ or $r \preccurlyeq x$. If it were the case that $r \preccurlyeq x$, then by hypothesis on r we would know that $r = x$. Hence $x \preccurlyeq r$.

Let $B = A - \{r\}$. By Exercise 7.4.8 we know that $(B, \preccurlyeq)$ is a poset. Because $|B| = n-1$, it follows from the inductive hypothesis that there is an order isomorphism from $(B, \preccurlyeq)$ to $(\{1, 2, \ldots, n-1\}, \leq)$, say $f \colon B \to \{1, 2, \ldots, n-1\}$. Let $F \colon A \to \{1, 2, \ldots, n\}$ be defined by $F(x) = f(x)$ for all $x \in B$, and $F(r) = n$.

Because f is bijective, it is straightforward to see that F is bijective as well; we omit the details. To see that F is an order isomorphism, it suffices by Lemma 7.4.16 to show that $x \preccurlyeq y$ if and only if $F(x) \leq F(y)$, for all $x, y \in A$. First, let $x, y \in B$. Then $x \preccurlyeq y$ if and only if $f(x) \leq f(y)$ because f is an order isomorphism. Because $F(x) = f(x)$ and $F(y) = f(y)$, then $x \preccurlyeq y$ if and only if $F(x) \leq F(y)$. Now let $z \in B$. We know that $z \preccurlyeq r$, and we also know that $F(z) \leq n = F(r)$, because $F(z) \in \{1, 2, \ldots, n-1\}$. Hence $z \preccurlyeq r$ if and only if $F(z) \leq F(r)$, because both these statements are true. It follows that F is an order isomorphism. □

The analog of Theorem 7.4.18 for infinite sets is not true. For example, as the reader is asked to show in Exercise 7.4.16, there is no order isomorphism from the totally ordered set $(\mathbb{N}, \leq)$ to the totally ordered $(\mathbb{N}^-, \leq)$, where $\mathbb{N}^-$ denotes the set of negative integers, even though both sets have the same cardinality.

Exercises

Exercise 7.4.1. Is each of the relations given in Exercise 5.1.3 antisymmetric, a partial ordering and/or a total ordering?

Exercise 7.4.2. Is each of the following relations antisymmetric, a partial ordering and/or a total ordering?

(1) Let F be the set of people in France, and let M be the relation on F defined by $x \, M \, y$ if and only if x eats more cheese annually than y, for all $x, y \in F$.

(2) Let W be the set of all people who ever lived and ever will live, and let A be the relation on W defined by $x \, A \, y$ if and only if y is an ancestor of x or if $y = x$, for all $x, y \in W$.

(3) Let T be the set of all triangles in the plane, and let L be the relation on T defined by $s \, L \, t$ if and only if s has area less than or equal to t, for all triangles $s, t \in T$.

(4) Let U be the set of current U.S. citizens, and let Z be the relation on U defined by $x \, Z \, y$ if and only if the Social Security number of x is greater than the Social Security number of y, for all $x, y \in U$.

Exercise 7.4.3. **[Used in Exercise 7.4.4, Exercise 7.4.15 and Example 7.5.2.]** Let $A \subset \mathbb{N}$ be a subset, and let $\preccurlyeq$ be the relation on A defined by $a \preccurlyeq b$ if and only if $b = a^k$ for some $k \in \mathbb{N}$, for all $a, b \in A$. Prove that $(A, \preccurlyeq)$ is a poset. Is $(A, \preccurlyeq)$ a totally ordered set?

Exercise 7.4.4. Draw a Hasse diagram for each of the following posets.

(1) The set $A = \{1,2,3,\ldots,15\}$, and the relation $a|b$.
(2) The set $B = \{1,2,3,4,6,8,12,24\}$, and the relation $a|b$.
(3) The set $C = \{1,2,4,8,16,32,64\}$, and the relation $a|b$.
(4) The set $C = \{1,2,4,8,16,32,64\}$, and the relation $\preccurlyeq$ defined by $a \preccurlyeq b$ if and only if $b = a^k$ for some $k \in \mathbb{N}$, for all $a,b \in C$. (It was proved in Exercise 7.4.3 that $(C,\preccurlyeq)$ is a poset.)
(5) The set $\mathcal{P}(\{1,2,3\})$, and the relation $\subseteq$.

Exercise 7.4.5. **[Used in Example 7.4.2.]**

(1) Give an example of a relation on $\mathbb{R}$ that is transitive and antisymmetric but neither symmetric nor reflexive.
(2) Let A be a non-empty set, and let R be a relation on A. Suppose that R is both symmetric and antisymmetric. Prove that every element of A is related at most to itself.

Exercise 7.4.6.

(1) Prove that if the poset has a greatest element, then the greatest element is unique, and if a poset has a least element, then the least element is unique. unique.
(2) Find an example of a poset that has both a least element and a greatest element, an example that has a least element but not a greatest element, an example that has a greatest element but not a least element and an example that has neither.

Exercise 7.4.7. Prove that a greatest element of a poset is a maximal element, and that a least element of a poset is a minimal element.

Exercise 7.4.8. **[Used in Theorem 7.4.9, Theorem 7.4.13 and Theorem 7.4.18.]** Let $(A,\preccurlyeq)$ be a poset, and let $B \subseteq A$ be a subset. The relation $\preccurlyeq$ is defined by a subset $\bar{R} \subseteq A \times A$. Then $\bar{R} \cap B \times B$ defines a relation on B, which can be thought of as the restriction of $\preccurlyeq$ to B; for convenience, because no confusion arises, we will also denote this relation on B by $\preccurlyeq$. Prove that $(B,\preccurlyeq)$ is a poset.

Exercise 7.4.9. **[Used in Theorem 7.4.13.]** Complete the missing step in the proof of Theorem 7.4.13. That is, let $\preccurlyeq'$ be as defined in the proof of the theorem. Prove that $\preccurlyeq'$ is a total order on A, and that if $x \preccurlyeq y$ then $x \preccurlyeq' y$, for all $x,y \in A$.

Exercise 7.4.10. Let A be a non-empty set, and let R be a relation on A. The relation R is a **quasi-ordering** if it is reflexive and transitive.

Suppose that R is a quasi-ordering. Let $\sim$ be the relation on A defined by $x \sim y$ if and only if $x\,R\,y$ and $y\,R\,x$, for all $x,y \in A$.

(1) Prove that $\sim$ is an equivalence relation.
(2) Let $x,y,a,b \in A$. Prove that if $x\,R\,y$, and $x \sim a$, and $y \sim b$, then $a\,R\,b$.
(3) Form the quotient set $A/\!\sim$, as defined in Definition 5.3.6. Let S be the relation on $A/\!\sim$ defined by $[x]\,S\,[y]$ if and only if $x\,R\,y$. Prove that S is well-defined.

(4) Prove that $(A/\sim, S)$ is a poset.

Exercise 7.4.11. Let $(A, \preccurlyeq)$ be a poset. For each $X \subseteq A$, let $Prec(X)$ be the set defined by

$$Prec(X) = \{w \in A \mid w \preccurlyeq x \text{ and } w \neq x \text{ for all } x \in X\}.$$

Let $C, D \subseteq A$. Prove that $Prec(C \cup D) = Prec(C) \cap Prec(D)$.

Exercise 7.4.12. Let $(A, \preccurlyeq)$ be a poset. Let $f : A \to \mathcal{P}(A)$ be defined by $f(x) = \{y \in A \mid y \preccurlyeq x\}$ for all $x \in A$.

(1) Let $x, z \in A$. Prove that $x \preccurlyeq z$ if and only if $f(x) \subseteq f(z)$.
(2) Prove that f is injective.

Exercise 7.4.13. Let $(A, \preccurlyeq)$ be a poset, let X be a set and let $h : X \to A$ be a function. Let $\preccurlyeq'$ be the relation on X defined by $x \preccurlyeq' y$ if and only if $h(x) \preccurlyeq h(y)$, for all $x, y \in X$. Prove that $(X, \preccurlyeq')$ is a poset.

Exercise 7.4.14. Let $(A, \preccurlyeq)$ be a poset, and let X be a set. Let $\mathcal{F}(X, A)$ be as defined in Section 4.5. Let $\preccurlyeq'$ be the relation on $\mathcal{F}(X, A)$ defined by $f \preccurlyeq' g$ if and only if $f(x) \preccurlyeq g(x)$ for all $x \in X$, for all $f, g \in \mathcal{F}(X, A)$. Prove that $(\mathcal{F}(X, A), \preccurlyeq')$ is a poset.

Exercise 7.4.15. Let $\preccurlyeq$ denote the relation $a|b$ on $\mathbb{N}$, and let $\preccurlyeq'$ be the relation on $\mathbb{N}$ defined by $a \preccurlyeq' b$ if and only if $b = a^k$ for some $k \in \mathbb{N}$, for all $a, b \in \mathbb{N}$. (It was proved in Exercise 7.4.3 that $(\mathbb{N}, \preccurlyeq')$ is a poset.) Prove that the identity map $1_{\mathbb{N}} : \mathbb{N} \to \mathbb{N}$ is an order homomorphism from $(\mathbb{N}, \preccurlyeq')$ to $(\mathbb{N}, \preccurlyeq)$, but that it is not an order isomorphism.

Exercise 7.4.16. **[Used in Section 7.4.]** Let $\mathbb{N}^-$ be the set of negative integers. Prove that there is no order isomorphism from the poset $(\mathbb{N}, \leq)$ to the poset $(\mathbb{N}^-, \leq)$.

Exercise 7.4.17. Let $(A, \preccurlyeq)$ and $(B, \preccurlyeq')$ be posets, and let $f : A \to B$ be an order isomorphism. Prove that if $\preccurlyeq$ is a total order, then so is $\preccurlyeq'$.

Exercise 7.4.18. **[Used in Section 3.5.]** The Well-Ordering Theorem states that for any set A, there is a total ordering on the set A such that every subset of A has a least element. Prove that the Well-Ordering Theorem implies the Axiom of Choice (use the version given in Theorem 3.5.3). Recall that the Axiom of Choice is not needed when there is a specific procedure for selecting elements from sets.

7.5 Lattices

In this section we turn our attention to a special type of poset, in which certain least upper bounds and greatest lower bounds exist.

Definition 7.5.1. Let $(A, \preccurlyeq)$ be a poset.

1. Let $a, b \in A$. The **join** of a and b, denoted $a \vee b$, is the least upper bound of $\{a, b\}$, if the least upper bound exists; the join is not defined if the least upper bound does not exist. The **meet** of a and b, denoted $a \wedge b$, is the greatest lower bound of $\{a, b\}$, if the greatest lower bound exists; the meet is not defined if the greatest lower bound does not exist.