Chapter 7. Lagrange's Equations

7.1 Lagrange's Equation for Unconstrained Motion

Consider a system of n particles moving unconstrained (without any boundary condition) in a space. Let

$$\begin{cases} q \,=\, (q_1,\, q_2,\, q_3,\, q_4,\, \dots,\, q_N) \,=\, (x_1,\, y_1,\, z_1,\, x_2,\, y_2,\, z_2,\, \dots) \\ \dot{q} \,=\, (\dot{q}_1,\, \dot{q}_2,\, \dot{q}_3,\, \dot{q}_4,\, \dots,\, \dot{q}_N) \,=\, (\dot{x}_1,\, \dot{y}_1,\, \dot{z}_1,\, \dot{x}_2,\, \dot{y}_2,\, \dot{z}_2,\, \dots) \end{cases} \tag{1}$$

be the coordinates and velocities of particles, where $N = 3n$ if the motion of each particle is in a 3-dimensional space, and T is the kinetic energy of the system where

$$T \,=\, \sum_{i=1}^{N} \frac{1}{2}\, m_i\, \dot{q}_i^2\,, \qquad \text{if } q \text{ is the Cartesian coordinates.}$$

Consider all forces on the particles are derivable from a generalized scalar potential U that may be a function of coordinates, velocities, or time. In general coordinates, both kinetic and potential energy can be a function of coordinates, velocities, or time, *i.e.*

$$T = T(q, \dot{q}, t) \qquad \text{and} \qquad U = U(q, \dot{q}, t) \tag{2}$$

Define:

$$\text{Lagrangian:} \qquad \mathcal{L}(q, \dot{q}, t) \,=\, T(q, \dot{q}, t) - U(q, \dot{q}, t) \tag{3}$$

$$\text{Action:} \qquad S[\mathcal{L}] \,=\, \int_{t_1}^{t_2} \mathcal{L}(q, \dot{q}, t) dt \tag{4}$$

Action Principle:

The motion of the system from t_1 to t_2 is such that the action is at an extrema, i.e.

$$\delta S = \delta \int_{t_1}^{t_2} \mathcal{L}\left(q, \dot{q}, t\right) dt = 0 \tag{5}$$

The extrema of the action satisfies the Euler-Lagrange Equation, which is the equation of motion of the system

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}(q, \dot{q}, t)}{\partial \dot{q}_i}\right] - \frac{\partial \mathcal{L}(q, \dot{q}, t)}{\partial q_i} = 0 \qquad \text{for } i = 1, \dots, N \tag{6}$$

This is the Lagrange's Equation. Note that $\mathcal{L}(q, \dot{q}, t)$ is not uniquely defined. For any function $F(q, t)$,

$$\mathcal{L}\left(q, \dot{q}, t\right) = \mathcal{L}\left(q, \dot{q}, t\right) + \frac{dF(q, t)}{dt}$$

is also a Lagrangian of the same system with the same equation of motion, since

$$\frac{dF(q, t)}{dt} = \frac{\partial F(q, t)}{\partial t} + \frac{\partial F(q, t)}{\partial q}\dot{q}\,, \qquad \frac{\partial}{\partial \dot{q}_i}\left(\frac{dF(q, t)}{dt}\right) = \frac{\partial F(q, t)}{\partial q}$$

and

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}}\left(\frac{dF(q,t)}{dt}\right)\right] = \frac{d}{dt}\left(\frac{\partial F(q,t)}{\partial q}\right) = \frac{\partial^2 F(q,t)}{\partial t \partial q} + \frac{\partial^2 F(q,t)}{\partial q^2}\dot{q} = \frac{\partial}{\partial q}\left(\frac{dF(q,t)}{dt}\right),$$

$$\Longrightarrow \qquad \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}}\left(\frac{dF(q,t)}{dt}\right)\right] - \frac{\partial}{\partial q}\left(\frac{dF(q,t)}{dt}\right) = 0$$

Example 7.1 A particle moving in 3-dimensional space under a force $\vec{F}(\vec{r},t) = -\nabla U(\vec{r},t)$

In the Cartesian coordinates,

$$T(\vec{v}) = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \qquad \text{and} \qquad \mathcal{L} = T(\vec{v}) - U(\vec{r},t)$$

Since

$$\frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}} = m\dot{\vec{r}} = \vec{p} \qquad \text{and} \qquad \frac{\partial \mathcal{L}}{\partial \vec{r}} = -\nabla U(\vec{r},t) = \vec{F}(\vec{r},t)$$

the Lagrange's equation yields the Newton's equation,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}}\right) - \frac{\partial \mathcal{L}}{\partial \vec{r}} = 0 \qquad \Longrightarrow \qquad \dot{\vec{p}} = \vec{F}(\vec{r},t)$$

7.2 Invariance of Lagrange's Equation in Coordinate Transformation

Consider a transformation between two coordinates that are independent of motion of the system,

$$q = (q_1, q_2, ..., q_N) \qquad \longrightarrow \qquad q' = (q'_1, q'_2, ..., q'_N)$$

Cosider the q-coordinate be a Cartesian coordinate for an inertial reference frame. The q'-coordinates, in general, can depend on t and the transformation can therefore depend on t. With the transformation, the coordinate q can be expressed as a function of q' and vice versa,

$$q'_i = q'_i\,(q_1,\, ...,\, q_N,\, t\,) = q'_i\,(q,\, t) \qquad \text{and} \qquad q_i = q_i\,(q'_1,\, ...,\, q'_N,\, t\,) = q_i\,(q',\, t) \tag{7}$$

where $i = 1, \cdots, N$. If

$$\frac{\partial q'}{\partial t} \neq 0$$

The q'-coordinate is a non-inertial reference frame. The Lagrange's equation in the q-coordinate is

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0, \qquad \forall i = 1, \cdots, N$$

we now derive the Lagrange's equation in the q'-coordinate. From the coordinate transformation in Eq. (**??**),

$$\dot{q}_i = \frac{\partial q_i(q',t)}{\partial t} + \sum_{j=1}^{N} \frac{\partial q_i(q',t)}{\partial q'_j}\,\dot{q}'_j\,,$$

Therefore,

$$\frac{\partial \dot{q}_i}{\partial \dot{q}'_k} = \frac{\partial q_i}{\partial q'_k} \qquad \text{and} \qquad \frac{\partial \dot{q}_i}{\partial q'_k} = \frac{\partial^2 q_i(q',t)}{\partial q'_k \partial t} + \sum_{j=1}^{N} \frac{\partial^2 q_i(q',t)}{\partial q'_k \partial q'_j}\,\dot{q}'_j = \frac{d}{dt}\left(\frac{\partial q_i}{\partial q'_k}\right)$$

Since the Lagrangian $\mathcal{L}$ is a scalar function, the transformation of $\mathcal{L}$ between q-coordinate to q'-coordinate is simply

$$\mathcal{L}'(\,q',\,\dot{q}',\,t\,) = \mathcal{L}(\,q(q',\,t),\,\dot{q}(q',\,\dot{q}',\,t),\,t\,) \qquad \text{or} \qquad \mathcal{L}(\,q,\,\dot{q},\,t\,) = \mathcal{L}'(\,q'(q,\,t),\,\dot{q}'(q,\,\dot{q},\,t),\,t\,)$$

(Question: what is the difference in the coordinate transformation of a scalar function and the coordinate transformation of a vector function?) Therefore,

$$\begin{aligned}
\frac{\partial \mathcal{L}'(q',\dot{q}',t)}{\partial q'_k} &= \sum_{i=1}^{N}\left[\frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial q_i}\,\frac{\partial q_i}{\partial q'_k} + \frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial \dot{q}_i}\,\frac{\partial \dot{q}_i}{\partial q'_k}\right] \\
&= \sum_{i=1}^{N}\left[\frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial q_i}\,\frac{\partial q_i}{\partial q'_k} + \frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial \dot{q}_i}\,\frac{d}{dt}\left(\frac{\partial q_i}{\partial q'_k}\right)\right] \\
\frac{\partial \mathcal{L}'(q',\dot{q}',t)}{\partial \dot{q}'_k} &= \sum_{i=1}^{N}\frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial \dot{q}_i}\,\frac{\partial \dot{q}_i}{\partial \dot{q}'_k} = \sum_{i=1}^{N}\frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial \dot{q}_i}\,\frac{\partial q_i}{\partial q'_k} \\
\frac{d}{dt}\left[\frac{\partial \mathcal{L}'(q',\dot{q}',t)}{\partial \dot{q}'_k}\right] &= \sum_{i=1}^{N}\left[\frac{d}{dt}\left[\frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial \dot{q}_i}\right]\frac{\partial q_i}{\partial q'_k} + \frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial \dot{q}_i}\,\frac{d}{dt}\left(\frac{\partial q_i}{\partial q'_k}\right)\right]
\end{aligned}$$

and

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}'(q',\dot{q}',t)}{\partial \dot{q}'_k}\right] - \frac{\partial \mathcal{L}'(q',\dot{q}',t)}{\partial q'_k} = \sum_{i=1}^{N}\left\{\frac{d}{dt}\left[\frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial \dot{q}_i}\right] - \frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial q_i}\right\}\frac{\partial q_i}{\partial q'_k}$$

Since the Lagrange's equation in the q-coordinate is

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial \dot{q}_i}\right] - \frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial q_i} = 0$$

the Lagrange's equation in the q'-coordinate has the same form of

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}'(q',\dot{q}',t)}{\partial \dot{q}'_i}\right] - \frac{\partial \mathcal{L}'(q',\dot{q}',t)}{\partial q'_i} = 0$$

Lagrange's equation is therefore invariant under the transformation between arbitrary coordinates that are independent of motion of the system.

Comments:

a. Lagrange's equation has the same form in **non-inertial frames** where $\partial \dot{q}_i/\partial t \neq 0$.

b. Because Lagrangian $\mathcal{L}(q,\dot{q},t)$ is a scalar function, the coordinator transformation of $\mathcal{L}(q,\dot{q},t)$ is much easier as compared with the Newton's equation.

Lagrangian in General Coordinates

Consider a coordinate transformation from the Cartesian coordinate $(\vec{r}_1, \vec{r}_2, ..., \vec{r}_n)$ to a general coordinate $q = (q_1, q_2, ..., q_N)$, where $N = 3n$,

$$\vec{r}_i = \vec{r}_i(q,t) \qquad \text{and} \qquad \dot{\vec{r}}_i = \sum_{k=1}^{3n} \frac{\partial \vec{r}_i}{\partial q_k}\,\dot{q}_k + \frac{\partial \vec{r}_i}{\partial t} = \dot{\vec{r}}_i(q,\,\dot{q},\,t) \tag{8}$$

The kinetic energy in a general coordinate can be obtained from the kinetic energy in the Cartesian coordinate as

$$\begin{aligned} T(q,\dot{q},t) &= \frac{1}{2}\sum_{i=1}^{n} m_i(\dot{\vec{r}}_i \cdot \dot{\vec{r}}_i\,) \\ &= \frac{1}{2}\sum_{i=1}^{n} m_i \left(\sum_{k=1}^{3n} \frac{\partial \vec{r}_i}{\partial q_k}\,\dot{q}_k + \frac{\partial \vec{r}_i}{\partial t}\right) \cdot \left(\sum_{k=1}^{3n} \frac{\partial \vec{r}_i}{\partial q_k}\,\dot{q}_k + \frac{\partial \vec{r}_i}{\partial t}\right) \;=\; T_0 + T_1 + T_2 \end{aligned} \tag{9}$$

with

$$\begin{aligned} T_0(q,t) &= \frac{1}{2}\sum_{i=1}^{n} m_i \left(\frac{\partial \vec{r}_i}{\partial t} \cdot \frac{\partial \vec{r}_i}{\partial t}\right) \\ T_1(q,\dot{q},t) &= \sum_{j=1}^{3n} M_j \dot{q}_j\,, \qquad\qquad M_j(q,t) \;=\; \sum_{i=1}^{n} m_i \left(\frac{\partial \vec{r}_i}{\partial t} \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right) \\ T_2(q,\dot{q},t) &= \frac{1}{2}\sum_{j,k=1}^{3n} M_{jk} \dot{q}_j \dot{q}_k\,, \qquad M_{jk}(q,t) \;=\; \sum_{i=1}^{n} m_i \left(\frac{\partial \vec{r}_i}{\partial q_j} \cdot \frac{\partial \vec{r}_i}{\partial q_k}\right) \end{aligned}$$

where if $\partial q_i/\partial t = 0$, $T_0 = T_1 = 0$. For a potential energy $U = U(\vec{r}_1, \cdots, \vec{r}_n,\, t)$ in the Cartesian coordinate, the potential energy in a general coordinate can be obtained by substituting the coordinate transformation into the scalar function U as

$$U(\vec{r}_1, \cdots, \vec{r}_n,\, t) \;=\; U(\vec{r}_1(q,t), \cdots, \vec{r}_n(q,t),\, t) \;=\; V(q,t)$$

where $V(q,t)$ is the potential energy in a general coordinate. The Lagrangian in a general coordinate is therefore

$$\mathcal{L}(q,\dot{q},t) \;=\; T_0(q,t) + T_1(q,\dot{q},t) + T_2(q,\dot{q},t) - V(q,t)$$

and the equation of motion in a general coordinate can be easily obtained from the Lagrange's equation as

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \qquad \Longrightarrow \qquad \frac{d}{dt}\left[\frac{\partial T(q,\dot{q},t)}{\partial \dot{q}_i}\right] - \frac{\partial T(q,\dot{q},t)}{\partial q_i} = Q_i$$

with

$$Q_i = -\frac{\partial V}{\partial q_i} = -\sum_{k=1}^{n} \frac{\partial U}{\partial \vec{r}_k} \cdot \frac{\partial \vec{r}_k}{\partial q_i} = \sum_{k=1}^{n} \vec{F}_k \cdot \frac{\partial \vec{r}_k}{\partial q_i}$$

where $\vec{F}_k$ is the force on the particle k in the Cartesian coordinates and Q_i is the general force which could be forces or torques in the q-coordinate, and $\partial T/\partial q_i$ is the additional force due to the kinetic effect of a general coordinate.

Example 7.2 Equation of Motion in an Accelerated Reference Frame

Consider a non-inertial reference frame K' moves with respect to an inertial frame K with a velocity $\vec{v}_0(t)$ and an acceleration $a_0(t) = \dot{\vec{v}}_0$. The coordinate and velocity of a particle are $(\vec{r}, \dot{\vec{r}})$ in the K-frame and $(\vec{r}', \dot{\vec{r}}')$ in the K'-frame, respectively. The transformation between the inertial and non-inertial frame is

$$\vec{r}' = \vec{r} - \int_{t_0}^{t} \vec{v}_0(t)\, dt \qquad \text{and} \qquad \dot{\vec{r}} = \dot{\vec{r}}' + \vec{v}_0(t)$$

The kinetic and potential energy in the K'-frame can be obtained by the transformation from the K-frame as

$$T(\dot{\vec{r}}) = \frac{1}{2}m(\dot{\vec{r}}\cdot\dot{\vec{r}}) = \frac{1}{2}m[\dot{\vec{r}}' + \vec{v}_0(t)]\cdot[\dot{\vec{r}}' + \vec{v}_0(t)] = T'(\dot{\vec{r}}', t) \tag{10}$$

$$U(\vec{r}) = U\left(\vec{r}' + \int_{t_0}^{t} \vec{v}_0(t)\, dt\right) \tag{11}$$

The Lagrangian in an non-inertial frame is then

$$\mathcal{L}(\vec{r}', \dot{\vec{r}}', t) = \frac{1}{2}m[\dot{\vec{r}}' + \vec{v}_0(t)]\cdot[\dot{\vec{r}}' + \vec{v}_0(t)] - U\left(\vec{r}' + \int_{t_0}^{t} \vec{v}_0(t)\, dt\right) \tag{12}$$

and the equation of motion can be easily obtained from the Lagrange's equation as

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\vec{r}}'}\right) - \frac{\partial\mathcal{L}}{\partial\vec{r}'} = m\,\ddot{\vec{r}}' + m\,\vec{a}_0(t) - \vec{F}' = 0 \tag{13}$$

where the original force is the same in both the inertial and non-inertial frame for the case of an uniform non-inertial frame,

$$\vec{F}'\left(\vec{r}' + \int_{t_0}^{t} \vec{v}_0(t)\, dt\right) = -\frac{\partial}{\partial\vec{r}'}U\left(\vec{r}' + \int_{t_0}^{t} \vec{v}_0(t)\, dt\right) = -\frac{\partial U(\vec{r})}{\partial\vec{r}} = \vec{F}(\vec{r}) \tag{14}$$

But an additional "kinetic force", $-m\vec{a}_0(t)$, in the equation of motion due to the acceleration of the reference frame.

Extra-HW 1

Obtain the equation of motion for a particle in a non-uniform accelerated reference frame K' where the acceleration of the reference frame $\vec{a}_0$ depends on the coordinate of the particle in K', *i.e.*

$$\dot{\vec{r}} = \dot{\vec{r}}' + \vec{v}_0(\vec{r}', t) \qquad \text{and} \qquad \vec{a}_0 = \vec{a}_0(\vec{r}', t)$$

Example 7.3 Equation of Motion of a Particle in Cylindrical Coordinate

With the transformation between the Cartesian (x, y, z) and cylindrical coordinates (ρ, ϕ, z), the velocity $\vec{v} = (\dot{x}, \dot{y}, \dot{z})$ expressed with the cylindrical coordinates can be obtained as

$$\begin{cases} x = \rho\cos\phi \\ y = \rho\sin\phi \\ z = z \end{cases} \quad \Longrightarrow \quad \begin{cases} \dot{x} = \dot{\rho}\cos\phi - \rho\dot{\phi}\sin\phi \\ \dot{y} = \dot{\rho}\sin\phi + \rho\dot{\phi}\cos\phi \\ \dot{z} = \dot{z} \end{cases} \tag{15}$$

The kinetic and potential energy expressed in the cylindrical coordinates can easily be obtained as

$$\begin{aligned} T \;=\; \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \;&=\; \frac{1}{2}\left[(\dot{\rho}\cos\phi - \rho\dot{\phi}\sin\phi)^2 + (\dot{\rho}\sin\phi + \rho\dot{\phi}\cos\phi)^2 + \dot{z}^2\right] \\ &=\; \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) && (16) \\ U \;=\; U(x, y, z, t) \;&=\; U(\rho\cos\phi, \rho\sin\phi, z, t) && (17) \end{aligned}$$

and the Lagrangian for a single-particle motion in the Cylindrical coordinates is then

$$\mathcal{L}(\rho, \phi, z, \dot{\rho}, \dot{\phi}, \dot{z}, t) \;=\; \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) - U(\rho\cos\phi, \rho\sin\phi, z, t) \tag{18}$$

Since

$$\frac{\partial\mathcal{L}}{\partial\dot{\rho}} = m\dot{\rho}\,, \qquad \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = m\rho^2\dot{\phi}\,, \qquad \frac{\partial\mathcal{L}}{\partial\dot{z}} = m\dot{z}$$

and

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial\rho} \;&=\; m\rho\dot{\phi}^2 - \frac{\partial U}{\partial\rho} \;=\; m\rho\dot{\phi}^2 - \frac{\partial U}{\partial x}\frac{\partial x}{\partial\rho} - \frac{\partial U}{\partial y}\frac{\partial y}{\partial\rho} \\ &\qquad\qquad\qquad =\; m\rho\dot{\phi}^2 + F_x\cos\phi + F_y\sin\phi \;=\; m\rho\dot{\phi}^2 + Q_\rho && (19) \\ \frac{\partial\mathcal{L}}{\partial\phi} \;&=\; -\frac{\partial U}{\partial\phi} \;=\; -\frac{\partial U}{\partial x}\frac{\partial x}{\partial\phi} - \frac{\partial U}{\partial y}\frac{\partial y}{\partial\phi} \;=\; -F_x\rho\sin\phi + F_y\rho\cos\phi \;=\; Q_\phi && (20) \\ \frac{\partial\mathcal{L}}{\partial z} \;&=\; -\frac{\partial U}{\partial z} \;=\; F_z \;=\; Q_z && (21) \end{aligned}$$

where

$$\begin{cases} Q_\rho \;=\; -(\partial U/\partial\rho) \;=\; F_x\cos\phi + F_y\sin\phi \\ Q_\phi \;=\; -(\partial U/\partial\phi) \;=\; \rho(-F_x\sin\phi + F_y\cos\phi) \\ Q_z \;=\; -(\partial U/\partial z) \;=\; F_z \end{cases}$$

the equations of motion derived from the Lagrange's equations are

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\rho}}\right) - \frac{\partial\mathcal{L}}{\partial\rho} \;&=\; 0 \quad\Longrightarrow\quad m\ddot{\rho} - m\rho\,\dot{\phi}^2 \;= Q_\rho =\; F_x\cos\phi + F_y\sin\phi && (22) \\ \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right) - \frac{\partial\mathcal{L}}{\partial\phi} \;&=\; 0 \quad\Longrightarrow\quad m\rho\,\ddot{\phi} + 2m\dot{\rho}\,\dot{\phi} \;=\; \frac{Q_\phi}{\rho} \;=\; -F_x\sin\phi + F_y\cos\phi && (23) \\ \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{z}}\right) - \frac{\partial\mathcal{L}}{\partial z} \;&=\; 0 \quad\Longrightarrow\quad m\ddot{z} \;=\; F_z && (24) \end{aligned}$$

These three equations of motion are the same as the ones we derived in Chapter 1, but the derivation from the Lagrange's equation is much easier. For the 2-dimensional central-force problem, $U = U(\rho)$,

$$Q_\rho = -\frac{\partial U(\rho)}{\partial \rho} \qquad \text{and} \qquad Q_\phi = -\frac{\partial U(\rho)}{\partial \phi} = 0$$

and the equations of motion reduce to

$$\frac{d(m\rho^2\dot{\phi})}{dt} = 0 \qquad \Longrightarrow \qquad m\rho^2\dot{\phi} = constant = p_\phi \tag{25}$$

$$\frac{d(m\dot{\rho})}{dt} = m\rho\dot{\phi}^2 + Q_\rho \qquad \Longrightarrow \qquad m\ddot{\rho} = \frac{p_\phi^2}{m\rho^3} - \frac{\partial U(\rho)}{\partial \rho} = V(\rho) \tag{26}$$

where $V(\rho)$ is the effective potential for the central-force motion.

Example 7.4 Equation of Motion of a Particle in Spherical Coordinate

With the transformation between the Cartesian (x, y, z) and spherical coordinates (r, θ, ϕ), the velocity $\vec{v} = (\dot{x}, \dot{y}, \dot{z})$ expressed with the spherical coordinates can be obtained as

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \Longrightarrow \begin{cases} \dot{x} = \dot{r}\sin\theta\cos\phi + r\dot{\theta}\cos\theta\cos\phi - r\dot{\phi}\sin\theta\sin\phi \\ \dot{y} = \dot{r}\sin\theta\sin\phi + r\dot{\theta}\cos\theta\sin\phi + r\dot{\phi}\sin\theta\cos\phi \\ \dot{z} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \end{cases} \tag{27}$$

and the kinetic and potential energy in the spherical coordinates can then be calculated as

$$\begin{aligned} T &= \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\ &= \frac{1}{2}m\Big[(\dot{r}\sin\theta\cos\phi + r\dot{\theta}\cos\theta\cos\phi - r\dot{\phi}\sin\theta\sin\phi)^2 \\ &\qquad +(\dot{r}\sin\theta\sin\phi + r\dot{\theta}\cos\theta\sin\phi + r\dot{\phi}\sin\theta\cos\phi)^2 + (\dot{r}\cos\theta - r\dot{\theta}\sin\theta)^2\Big] \\ &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta\right) \qquad (28) \\ U &= U(x, y, z, t) = U(r\sin\theta\cos\phi\,,\ r\sin\theta\sin\phi\,,\ r\cos\theta\,,\ t) \qquad (29) \end{aligned}$$

The Lagrangian for a single-particle motion in the spherical coordinates is thus

$$\mathcal{L}(r, \theta, \phi, \dot{r}, \dot{\theta}, \dot{\phi}, t) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta) - U(r\sin\theta\cos\phi,\ r\sin\theta\sin\phi,\ r\cos\theta,\ t) \tag{30}$$

Since

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r}\,, \qquad \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta}\,, \qquad \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mr^2\dot{\phi}\sin^2\theta$$

and

$$\frac{\partial \mathcal{L}}{\partial r} = mr\dot{\theta}^2 + mr\dot{\phi}^2\sin^2\theta + Q_r\,, \qquad \frac{\partial \mathcal{L}}{\partial \theta} = mr^2\dot{\phi}^2\cos\theta\sin\theta + Q_\theta\,, \qquad \frac{\partial \mathcal{L}}{\partial \phi} = Q_\phi$$

where

$$\begin{aligned}
Q_r &= -\frac{\partial U(\vec{r},t)}{\partial r} &=& -\frac{\partial U}{\partial x}\frac{\partial x}{\partial r} - \frac{\partial U}{\partial y}\frac{\partial y}{\partial r} - \frac{\partial U}{\partial z}\frac{\partial z}{\partial r} \\
& &=& F_x \sin\theta\cos\phi + F_y \sin\theta\sin\phi + F_z\cos\theta \qquad (31)\\
Q_\theta &= -\frac{\partial U(\vec{r},t)}{\partial \theta} &=& -\frac{\partial U}{\partial x}\frac{\partial x}{\partial \theta} - \frac{\partial U}{\partial y}\frac{\partial y}{\partial \theta} - \frac{\partial U}{\partial z}\frac{\partial z}{\partial \theta} \\
& &=& r(F_x \cos\theta\cos\phi + F_y \cos\theta\sin\phi - F_z\sin\theta) \qquad (32)\\
Q_\phi &= -\frac{\partial U(\vec{r},t)}{\partial \phi} &=& -\frac{\partial U}{\partial x}\frac{\partial x}{\partial \phi} - \frac{\partial U}{\partial y}\frac{\partial y}{\partial \phi} - \frac{\partial U}{\partial z}\frac{\partial z}{\partial \phi} \\
& &=& r\sin\theta\,(-F_x \sin\phi + F_y\cos\phi) \qquad (33)
\end{aligned}$$

the equations of motion derived from the Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) - \frac{\partial \mathcal{L}}{\partial r} = 0 \qquad \Longrightarrow \qquad m\ddot{r} - mr\dot{\theta}^2 - mr\dot{\phi}^2\sin^2\theta \;=\; Q_r \qquad (34)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0 \qquad \Longrightarrow \qquad \frac{d(mr^2\dot{\theta})}{dt} - mr^2\dot{\phi}^2\cos\theta\sin\theta \;=\; Q_\theta \qquad (35)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \qquad \Longrightarrow \qquad \frac{d(mr^2\dot{\phi}\sin^2\theta)}{dt} \;=\; Q_\phi \qquad (36)$$

These three equations are the same as the ones we derived in Chapter 1, but the calculation from the Lagrange's equation is much easier. For the 3-dimensional central-force problem (two-body problem), $U(\vec{r}) = U(r)$,

$$Q_r = -\frac{\partial U(r)}{\partial r}\,, \qquad Q_\theta = -\frac{\partial U(r)}{\partial \theta} = 0\,, \qquad Q_\phi = -\frac{\partial U(r)}{\partial \phi} = 0$$

and the equation of motion for the ϕ-direction reduces to

$$\frac{d(mr^2\dot{\phi}\sin^2\theta)}{dt} = 0 \qquad \Longrightarrow \qquad mr^2\dot{\phi}\sin^2\theta \;=\; constant$$

If we choose $\theta(0) = 0$,

$$mr^2\dot{\phi}\sin^2\theta \;=\; 0 \qquad \Longrightarrow \qquad \dot{\phi} = 0 \qquad \Longrightarrow \qquad \phi = constant$$

The the equation of motion for the θ-direction is then

$$\frac{d(mr^2\dot{\theta})}{dt} - mr^2\dot{\phi}^2\cos\theta\sin\theta = 0 \qquad \Longrightarrow \qquad mr^2\dot{\theta} = p_\theta = constant$$

The final 1-dimensional equation of motion for the 3-dimensional central-force problem is

$$\left\{\begin{array}{l} mr\dot{\phi}^2\sin^2\theta = 0 \\ mr\dot{\theta}^2 = \dfrac{p_\theta^2}{mr^3} \end{array}\right. \qquad \Longrightarrow \qquad m\ddot{r} \;=\; \frac{p_\theta^2}{mr^3} - \frac{\partial U(r)}{\partial r} \;=\; V(r) \qquad (37)$$

where $V(r)$ is the effective potential of the 1-dimension motion in the r-direction.