Chapter 4. Mechanical Energy

4.0 Phase Space and States of a Mechanics System

In general, the Newton's equation is 3-dimensional 2nd-order ordinary differential equations (ODEs) for a single particle and $3N$-dimensional 2nd-order ODEs for a system of N particles

$$\dot{\vec{p}} = \vec{F}(\vec{r}) \qquad \text{or} \qquad \dot{\vec{p}}_\alpha = \vec{F}_\alpha(\vec{r}_1, \cdots \vec{r}_N) \qquad \text{with } \alpha = 1, \cdot, N$$

For a system of 2nd-order ODEs, the solution that is a trajectory of the motion is guaranteed to exist and be unique for each given set of initial condition $(\vec{r}_1(0)\,, \vec{p}_1(0)\,, \cdots\,, \vec{r}_N(0)\,, \vec{p}_N(0))\,.$ The motion is therefore completely specified at each point in the space of $(\vec{r}_1, \vec{p}_1, \cdots, \vec{r}_N, \vec{p}_N)\,,$ that is called phase space, and each point in the phase space is an unique state of a system. For a system of N particles, the phase space is a $6N$ dimensional space. Since the solution of the Newton's equation is unique for each given initial condition, each point in the phase space belong only to one specific trajectory of a system and different trajectories of a system cannot intersect each other in the phase space, unless at a point in the phase space where the system does not change with time — the equilibrium points.

4.1 Kinetic Energy and Work

The kinetic energy is defined as

$$T = \frac{1}{2} m(\vec{v} \cdot \vec{v}) \tag{1}$$

for a single particle of mass m and moving with a velocity $\vec{v}$, and

$$T = \frac{1}{2} \sum_{\alpha=1}^{N} m_\alpha(\vec{v}_\alpha \cdot \vec{v}_\alpha) \tag{2}$$

for a system of N particles. The kinetic energy of a system is a state function in the phase space, *i.e.* the change of the kinetic energy between two arbitrary points in the phase space depends only on the locations of the two points in the phase space. The kinetic energy is, however, not a state function in the configuration space and the change of the kinetic energy when a system moves the configuration space depends on the path in the configuration space the system took. The change of the kinetic energy as a particle moving in configuration space can be calculated as

$$\frac{dT}{dt} = m\vec{v} \cdot \frac{d\vec{v}}{dt} = \vec{F} \cdot \vec{v} \qquad \text{with} \qquad \vec{F} = m\dot{\vec{v}} \tag{3}$$

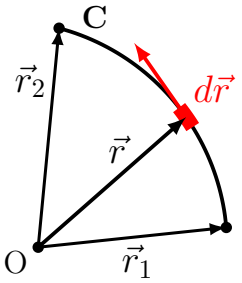
When a particle travels from position $\vec{r}_1$ at $t = t_1$ to position $\vec{r}_2$ at $t = t_2$, the change of the kinetic energy is then

$$\Delta T = T_2 - T_1 = \int_{t_1}^{t_2} \vec{F}(\vec{r}) \cdot \vec{v}\, dt = \int_C \vec{F}(\vec{r}) \cdot d\vec{r} \tag{4}$$

where

$$W = \int_C \vec{F}(\vec{r}) \cdot d\vec{r} = \int_C [F_x(\vec{r})dx + F_y(\vec{r})dy + F_z(\vec{r})dz] \tag{5}$$

is the work done on a particle when it moves from position $\vec{r}_1$ to $\vec{r}_2$ along a path C in the configuration space. The integral in Eq. (5) is a line integral in the configuration space and it depends on the path that the particle followed from $\vec{r}_1$ to $\vec{r}_2$. The path-dependence of the work done is because the states of a system can only be uniquely specified in the phase space.

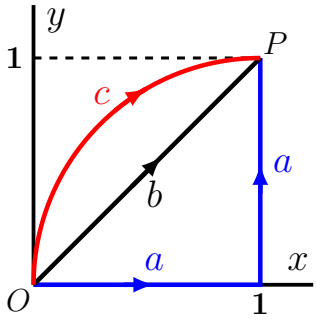


Example 4.1 Three Line Integrals in 2-Dimensional Space

$$\vec{F} = y\,\vec{e}_x + 2x\,\vec{e}_y$$

Along Path a,

$$\begin{aligned} W_a &= \int_a [F_x(x,y)\,dx + F_y(x,y)\,dy] \\ &= \int_0^1 F_x(x,0)dx + \int_0^1 F_y(1,y)dy = 0 + 2\int_0^1 dy = 2 \end{aligned}$$



Along Path b,

$$\begin{aligned} W_b &= \int_b [F_x(x,y)dx + F_y(x,y)dy] \\ &\overset{x=y}{=} \int_0^1 [F_x(x,x) + F_y(x,x)]\,dx = \int_0^1 (x+2x)\,dx = \frac{3}{2} \end{aligned}$$

Along Path c,

$$\vec{r} = (x\,,\,y) = (1-\cos\theta\,,\,\sin\theta) \quad \text{and} \quad d\vec{r} = (\sin\theta\,d\theta\,,\,\cos\theta\,d\theta) \quad \text{where } 0 \le \theta \le \pi/2$$

$$\begin{aligned} W_c &= \int_c [F_x(x,y)dx + F_y(x,y)dy] = \int_0^{\pi/2} [\sin\theta\,(\sin\theta\,d\theta) + 2(1-\cos\theta)(\cos\theta\,d\theta)] \\ &= \int_0^{\pi/2} \left(\sin^2\theta - 2\cos^2\theta + 2\cos\theta\right) d\theta = \int_0^{\pi/2} \left(2\cos\theta - \frac{3}{2}\cos 2\theta - \frac{1}{2}\right) d\theta = 2 - \pi \end{aligned}$$

Therefore, $W_a \neq W_b \neq W_c$. Note that

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \partial_x & \partial_y & \partial_z \\ y & 2x & 0 \end{vmatrix} = (2-1)\vec{e}_z \neq 0$$

$\vec{F} = y\,\vec{e}_x + 2x\,\vec{e}_y$ is not a curl-free vector field and, therefore, not a conservative force.

4.2 Potential Energy and Conservative Force

A force $\vec{F}$ acting on a particle is **conservative** if it depends only on the position vector of the particle and can be expressed as the gradient of a potential function of the interaction,

$$\vec{F}(\vec{r}) = -\nabla U(\vec{r}) \qquad \text{and} \qquad U(\vec{r}) = -\int \vec{F}(\vec{r}) \cdot d\vec{r} + U_0 \tag{6}$$

where $U(\vec{r})$ is the potential energy for the interaction and U_0 is an arbitrary constant. The gradient in the Cartesian coordinates is

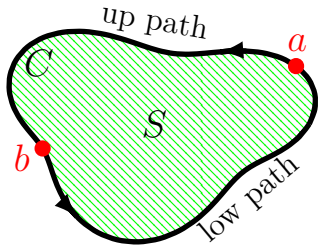
$$\nabla = \vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y} + \vec{e}_z \frac{\partial}{\partial z} \tag{7}$$

A conservative force $\vec{F}(\vec{r})$ is also equivalent to a curl-free vector field,

$$\nabla \times \vec{F} \;=\; -\nabla \times (\nabla U) \;=\; 0 \tag{8}$$

For a conservative force, the work done by the force when a particle moves from one point to another in the configuration space is independent of paths that connect two points. To prove the path-independence of the work done, we consider the work done by a conservative force $\vec{F}$ along an arbitrary closed path C that consists of two sections, up and low path, in the configuration,

$$\begin{aligned} W(a \to b \to a) \;&=\; \oint_C \vec{F}(\vec{r}) \cdot d\vec{r} \\ &=\; -\oint_C \nabla U(\vec{r}) \cdot d\vec{r} \;\overset{\text{Stockes}}{=}\; -\iint_S [\nabla \times (\nabla U)] \cdot \vec{n}\, ds \\ &=\; 0 \end{aligned}$$



where S is the surface enclosed by the closed path C. Since

$$0 \;=\; W(a \to b \to a) \;=\; W_{up}(a \to b) + W_{low}(b \to a) \;=\; W_{up}(a \to b) - W_{low}(a \to b)$$

the work done are the same through the up and low path,

$$W_{up}(a \to b) = W_{low}(a \to b)$$

As both the up and low path are arbitrary, the work done by a conservative force between two arbitrary points in the configuration space is therefore independent of the path. The work done by a conservative force $\vec{F} = -\nabla U(\vec{r})$ can be calculated by the change of the potential energy,

$$W(\vec{r}_1 \to \vec{r}_2) = -\left[U(\vec{r}_2) - U(\vec{r}_1)\right] \tag{9}$$

Stocks' Theorem: For a well-behaved vector function $\vec{A}(\vec{r})$,

$$\int_S (\nabla \times \vec{A}) \cdot \vec{n}\, ds \;=\; \oint_C \vec{A} \cdot d\vec{r}$$

where S is a surface and C is the contour bounding it. The normal $\vec{n}$ of the surface S is defined by the right-hand rule in the relation to the direction of the line integral around C.

Example 4.2

The gravitational or Coulomb electric force are conservative force. Between two particles, the force can be written as

$$\vec{F} \;=\; -\frac{\gamma\,\vec{r}}{|\vec{r}\,|^3}$$

where $\gamma = GmM$ for the gravity and $\gamma = kQ_1Q_2$ for the electric force.

(a) Calculate Potential Energy from a Conservative Force

$$\begin{aligned} U(\vec{r}) - U_0 \;&=\; -\int \vec{F}\cdot d\vec{r} \\ &=\; -\int (F_x dx + F_y dy + F_z dz) \;=\; \gamma \int \frac{1}{r^3}\,(x dx + y dy + z dz) \\ &=\; \frac{\gamma}{2}\int \frac{d(x^2+y^2+z^2)}{r^3} \;=\; \frac{\gamma}{2}\int \frac{d(r^2)}{r^3} \;=\; \gamma \int \frac{dr}{r^2} \;=\; -\frac{\gamma}{r} \end{aligned}$$

It is convenient to choose $U(\infty) = 0$ and, therefore, $U_0 = 0$. Note that with this $1/r^2$-force, a particle cannot be at $r = 0$.

(b) Calculate Force from Potential Energy $U(r) = -\dfrac{\gamma}{r}$

$$\begin{aligned} \vec{F} \;=\; -\nabla U(r) \;&=\; \gamma\left(\vec{e}_x\,\frac{\partial}{\partial x} + \vec{e}_y\,\frac{\partial}{\partial y} + \vec{e}_z\,\frac{\partial}{\partial z}\right)\frac{1}{r} \\ &=\; -\frac{\gamma}{r^2}\left(\vec{e}_x\,\frac{\partial r}{\partial x} + \vec{e}_y\,\frac{\partial r}{\partial y} + \vec{e}_z\,\frac{\partial r}{\partial z}\right) \;=\; -\frac{\gamma\vec{r}}{r^3} \end{aligned}$$

where

$$\frac{\partial r}{\partial x} = \frac{\partial\sqrt{x^2+y^2+z^2}}{\partial x} = \frac{x}{r}\,, \qquad \frac{\partial r}{\partial y} = \frac{y}{r}\,, \qquad \frac{\partial r}{\partial z} = \frac{z}{r}$$

(c) Checking the Path-Independence of Work Done by a Conservative Force

A particle moves from $\vec{r}_1$ to $\vec{r}_2$ following a certain path, the work done is

$$W(\vec{r}_1 \to \vec{r}_2) = \int\limits_{\vec{r}_1\to\vec{r}_2} \vec{F}(\vec{r})\cdot d\vec{r} = \int\limits_{\vec{r}_1\to\vec{r}_2} [F_x(x,y,z)\,dx + F_y(x,y,z)\,dy + F_z(x,y,z)\,dz]$$

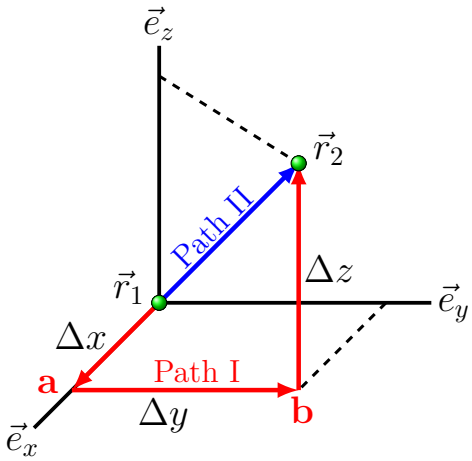
where

$$\vec{F}(\vec{r}) = -\gamma\,\frac{\vec{r}}{r^3} = -\frac{\gamma}{(x^2+y^2+z^2)^{3/2}}\;(x\,\vec{e}_x + y\,\vec{e}_y + z\,\vec{e}_z)$$

Along Path I: $\vec{r}_1 \longrightarrow a \longrightarrow b \longrightarrow \vec{r}_2$

where $\vec{r}_1 = (x_1,\, y_1,\, z_1)\,, \qquad \vec{r}_2 = (x_2,\, y_2,\, z_2)$

$a = (x_2,\, y_1,\, z_1)\,, \qquad b = (x_2,\, y_2,\, z_1)$



$$\begin{aligned} W_I &= W(\vec{r}_1 \to a) + W(a \to b) + W(b \to \vec{r}_2) \\ &= \int_{x_1}^{x_2} F_x(x, y_1, z_1)\, dx + \int_{y_1}^{y_2} F_y(x_2, y, z_1)\, dy \\ &+ \int_{z_1}^{z_2} F_z(x_2, y_2, z)\, dz \end{aligned}$$

where

$$\begin{aligned} \int_{x_1}^{x_2} F_x(x, y_1, z_1)\, dx &= -\gamma \int_{x_1}^{x_2} \frac{x dx}{(x^2 + y_1^2 + z_1^2)^{3/2}} \\ &= \gamma \left[\frac{1}{(x_2^2 + y_1^2 + z_1^2)^{1/2}} - \frac{1}{(x_1^2 + y_1^2 + z_1^2)^{1/2}} \right] \\ \int_{y_1}^{y_2} F_y(x_2, y, z_1)\, dy &= -\gamma \int_{y_1}^{y_2} \frac{y dy}{(x_2^2 + y^2 + z_1^2)^{3/2}} \\ &= \gamma \left[\frac{1}{(x_2^2 + y_2^2 + z_1^2)^{1/2}} - \frac{1}{(x_2^2 + y_1^2 + z_1^2)^{1/2}} \right] \\ \int_{z_1}^{z_2} F_z(x_2, y_2, z)\, dz &= -\gamma \int_{z_1}^{z_2} \frac{z dz}{(x_2^2 + y_2^2 + z^2)^{3/2}} \\ &= \gamma \left[\frac{1}{(x_2^2 + y_2^2 + z_2^2)^{1/2}} - \frac{1}{(x_2^2 + y_2^2 + z_1^2)^{1/2}} \right] \end{aligned}$$

Therefore

$$W_I = \frac{\gamma}{(x_2^2 + y_2^2 + z_2^2)^{1/2}} - \frac{\gamma}{(x_1^2 + y_1^2 + z_1^2)^{1/2}} = -\left[U(\vec{r}_2) - U(\vec{r}_1)\right]$$

Along Path II: $\vec{r}_1 \longrightarrow \vec{r}_2$

$$W_{II} = \int\limits_{\vec{r}_1 \to \vec{r}_2} \vec{F} \cdot d\vec{r} = -\gamma \int_{\vec{r}_1}^{\vec{r}_2} \frac{\vec{r} \cdot d\vec{r}}{r^3} = -\gamma \int_{\vec{r}_1}^{\vec{r}_2} \frac{dr}{r^2} = \gamma \left(\frac{1}{r_2} - \frac{1}{r_1} \right)$$

$$= -\left[U(\vec{r}_2) - U(\vec{r}_1)\right]$$

where $d\vec{r} = \vec{e}_r\, dr$ and $\vec{r} = r\, \vec{e}_r$.

The Gradient Operator in Cylindrical and Spherical Coordinates

The gradient operator in other coordinates can be obtained from the gradient operator in the Cartesian coordinates using the transformation matrix of the base vectors (in Chapter

WE GET THE CYLINDRICAL COORDINATE EQUIVALENCE FROM CHPT 1

In Chapter 1 we had the expression (en vectores) of:
(e-x, e-y, e-x) = (cos, sin, 0; sin, cos, 0; 0, 0, 1) * (e-r, e-phi, e-theta)
CHECK IT AND COMPARE IT TO THE EQUATION BELOW

1) and the chain rule for derivatives. For cylindrical coordinates, for example

$$\nabla = \vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y} + \vec{e}_z \frac{\partial}{\partial z} = \begin{pmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} = \begin{pmatrix} \vec{e}_\rho & \vec{e}_\phi & \vec{e}_z \end{pmatrix} (\mathbf{T}^{-1})^T \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}$$

Here you need to put the derivative operators on the most right position of the equation so that the derivatives will not operator on the base vectors or matrix $\mathbf{T}$. The transformation matrix $\mathbf{T}$ from Chapter 1 is

To get this, we first change to polar coordinates, and then get the partial derivatives

$$\mathbf{T}^{-1} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{for} \qquad \begin{pmatrix} \vec{e}_x \\ \vec{e}_y \\ \vec{e}_z \end{pmatrix} = \mathbf{T}^{-1} \begin{pmatrix} \vec{e}_\rho \\ \vec{e}_\phi \\ \vec{e}_z \end{pmatrix}$$

and

$$\begin{pmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \end{pmatrix} = \begin{pmatrix} \vec{e}_\rho & \vec{e}_\phi & \vec{e}_z \end{pmatrix} (\mathbf{T}^{-1})^T$$

The derivatives with respect to (x, y, z) can be converted to the derivatives with respect to (ρ, ϕ, z) using the chain rule. With

$$x = \rho\cos\phi\,, \qquad y = \rho\sin\phi\,, \qquad z = z$$

$$\left.\begin{aligned} \partial_\rho &= \frac{\partial x}{\partial\rho}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\rho}\frac{\partial}{\partial y} = \cos\phi\,\frac{\partial}{\partial x} + \sin\phi\,\frac{\partial}{\partial y} \\ \partial_\phi &= \frac{\partial x}{\partial\phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\phi}\frac{\partial}{\partial y} = -\rho\sin\phi\,\frac{\partial}{\partial x} + \rho\cos\phi\,\frac{\partial}{\partial y} \end{aligned}\right\} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\rho\sin\phi & \rho\cos\phi \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}$$

Therefore

$$\begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} = \begin{pmatrix} \cos\phi & -\rho^{-1}\sin\phi & 0 \\ \sin\phi & \rho^{-1}\cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \partial_\rho \\ \partial_\phi \\ \partial_z \end{pmatrix}$$

The gradient in the cylindrical coordinate is then

$$\begin{aligned} \nabla &= \begin{pmatrix} \vec{e}_\rho & \vec{e}_\phi & \vec{e}_z \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\phi & -\rho^{-1}\sin\phi & 0 \\ \sin\phi & \rho^{-1}\cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \partial_\rho \\ \partial_\phi \\ \partial_z \end{pmatrix} \\ &= \begin{pmatrix} \vec{e}_\rho & \vec{e}_\phi & \vec{e}_z \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \partial_\rho \\ \partial_\phi \\ \partial_z \end{pmatrix} = \vec{e}_\rho \frac{\partial}{\partial\rho} + \vec{e}_\phi \frac{1}{\rho}\frac{\partial}{\partial\phi} + \vec{e}_z \frac{\partial}{\partial z} \end{aligned}$$

Extra HW 1. Derive the gradient operator in the spherical coordinates.

Newton's Equation with Conservative force in Cylindrical Coordinates

The components of a conservative force and the Newton's equation with the conservative force in the cylindrical coordinates are then U stands for potential energy

$$\begin{cases} F_\rho &= -\dfrac{\partial U(\vec{r})}{\partial \rho} \\ F_\phi &= -\dfrac{1}{\rho}\dfrac{\partial U(\vec{r})}{\partial \phi} \\ F_z &= -\dfrac{\partial U(\vec{r})}{\partial z} \end{cases} \quad \text{and} \quad \begin{cases} m(\ddot{\rho} - \rho\dot{\phi}^2) &= -\dfrac{\partial U(\vec{r})}{\partial \rho} \\ m(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}) &= -\dfrac{1}{\rho}\dfrac{\partial U(\vec{r})}{\partial \phi} \\ m\,\ddot{z} &= -\dfrac{\partial U(\vec{r})}{\partial z} \end{cases} \tag{10}$$

4.3 Conservation of Mechanical Energy

The mechanical energy is defined by the sum of the kinetic and potential energy. In general, the potential could be a function of position vector and time. For a single particle

$$E = \frac{1}{2}m(\vec{v}\cdot\vec{v}) + U(\vec{r},t) = \frac{1}{2m}(\vec{p}\cdot\vec{p}) + U(\vec{r},t) \tag{11}$$

The mechanical energy could thus be a function of $\vec{v}$ (or $\vec{p}$), $\vec{r}$, and t. If the force on a system is conservative, *i.e.*

$$\dot{\vec{p}} = \vec{F} = -\nabla U(\vec{r},t)$$

$\frac{\partial E}{\partial \vec{r}} = \nabla E$

$\frac{\partial E}{\partial \vec{\rho}} = \nabla_\rho E = \vec{e}_x \frac{\partial E}{\partial \rho_x} + \vec{e}_y \frac{\partial E}{\partial \rho_y} + \vec{e}_z \frac{\partial E}{\partial \rho_z}$

the change of the mechanical energy with time is

$$\frac{dE}{dt} = \frac{\partial E}{\partial t} + \dot{\vec{p}}\cdot\frac{\partial E}{\partial \vec{p}} + \dot{\vec{r}}\cdot\frac{\partial E}{\partial \vec{r}} = \frac{\partial E}{\partial t} - \frac{\vec{p}}{m}\cdot\nabla U(\vec{r},t) + \vec{v}\cdot\frac{\partial U(\vec{r},t)}{\partial \vec{r}} = \frac{\partial E}{\partial t}$$

Therefore

$$\frac{\partial E}{\partial t} = 0 \quad \Longrightarrow \quad \frac{dE}{dt} = 0 \quad \Longrightarrow \quad E = const \tag{12}$$

If the mechanical energy does not depend on t **explicit**, *i.e.* the mass and the potential energy of a system are independent of t, the mechanical energy is a constant of motion. This constant of energy is the most common constraints to the dynamics of mechanics systems.

Graphs of Potential Energy

For a conservative system in which the energy is a constant of motion, the potential-energy plot is useful in analyzing the dynamics of the system.

With $E = \frac{1}{2}mv^2 + U(\vec{r})$

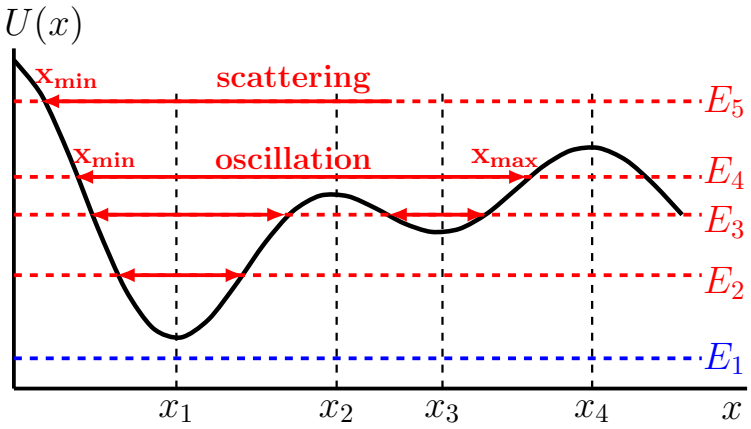
$E = E_1 < \min(U(\vec{r}))$, forbidden

$E = E_2$, one oscillation state

$E = E_3$, two oscillation states

$E = E_4$, one oscillation state

$E = E_5$, scattering



Equilibriums

The equilibriums are at either maxima (e.g. x_2 and x_4) or minima (e.g. x_1 and x_3) of the potential energy, where the net force on a system is zero, *i.e.*

$$\frac{\partial U(\vec{r})}{\partial \vec{r}} = 0 \tag{13}$$

A system will stay at an equilibrium state forever if it is at the equilibrium initially. The equilibriums can be categorized into the stable equilibrium, where the potential function is minimal in all directions in space, and the unstable equilibrium, where the potential function is at a maximum in at least one direction in space. The stability of an equilibrium can be determined by the second-order derivatives of a potential function. For a n-variable function $V(\vec{x}) = V(x_1, \cdots, x_n)$ where $\vec{x} = (x_1, x_2, \cdots, x_n)$, the second-order derivatives form a matrix, called the Hessian matrix,

$$\begin{pmatrix} V''_{11} & V''_{12} & \cdots & V''_{1n} \\ V''_{21} & V''_{22} & \cdots & V''_{2n} \\ \vdots & \vdots & & \vdots \\ V''_{n1} & V''_{n2} & \cdots & V''_{nn} \end{pmatrix} \qquad \text{where} \qquad V''_{ij} = \frac{\partial^2 V(\vec{x})}{\partial x_i \partial x_j}$$

The minima of $V(\vec{x})$ occurs where all the first-order derivatives vanish while the Hessian matrix is semi-positive definite, *i.e.* all the leading principal minors of the Hessian matrix of $V(\vec{x})$ are either positive or zero (but not all zero). For a one-dimensional system, the stability of the equilibriums can easily be determined by a single 2nd-order derivative as

$$\text{For} \quad \frac{dU(x_0)}{dx_0} = 0\,, \qquad \begin{cases} \left.\dfrac{d^2U(x)}{dx^2}\right|_{x=x_0} > 0\,, & \text{stable equilibrium} \\ \left.\dfrac{d^2U(x)}{dx^2}\right|_{x=x_0} < 0\,, & \text{unstable equilibrium} \end{cases} \tag{14}$$

Small Motion Near a Stable Equilibrium

For the small motion near a stable equilibrium, we can expand the potential energy at the equilibrium,

$$U(x) = U(x_0) + \frac{dU(x_0)}{dx_0}\delta x + \frac{1}{2}\frac{d^2U(x_0)}{dx_0^2}(\delta x)^2 + \cdots \tag{15}$$

Since $x = x_0$ is a stable equilibrium, $dU(x_0)/dx_0 = 0$ and $d^2U(x_0)/dx_0^2 > 0$. For a small motion, $\delta x << 1$, we can keep the lowest-order nonzero term in the expansion,

$$U(\delta x) \simeq U(x_0) + \frac{1}{2}\frac{d^2U(x_0)}{dx_0^2}(\delta x)^2 = U(x_0) + \frac{1}{2}k(\delta x)^2 \tag{16}$$

where $k = d^2U(x_0)/dx_0^2 > 0$ and the constant term $U(x_0)$ has no any observable effect on the motion and can be dropped. The mechanical energy for the small motion near a stable equilibrium is therefore

$$E = \frac{1}{2}mv^2 + \frac{1}{2}k(\delta x)^2 \tag{17}$$

This is the energy for a harmonic oscillator and the motion can be easily solved as,

$$\begin{cases} \delta x(t) &= \delta x(0)\cos(\omega t) + \dfrac{v(0)}{m\omega}\sin(\omega t) \\ v(t) &= m\omega\delta x(0)\sin(\omega t) + v(0)\cos(\omega t) \end{cases} \qquad \text{with} \quad \omega = \sqrt{k/m} \tag{18}$$

Extra HW 2. Solve the equation of motion for harmonic oscillator.

Example 4.3 Continue on Central-Force Problem with $(1/r)$-Potential

In Chapter 1, by using the conservation of the angular momenta, the original 3D Newton's equation of the central-force problem has be reduced to a 1D Newton's equation,

$$\begin{cases} \ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta = F(r)/m \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta = 0 \\ r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\phi}\dot{\theta}\cos\theta = 0 \end{cases} \quad \Longrightarrow \quad \begin{cases} m\ddot{r} = \dfrac{p_\theta^2}{mr^3} + F(r) \\ \dot{\theta} = p_\theta/(mr^2) \quad \text{or} \quad p_\theta = mr^2\dot{\theta} \end{cases}$$

where p_θ is a constant angular momentum and $F(r)$ is the central force.

We now consider $F(r)$ from a $(1/r)$-potential, that is the case of both the gravity and the Coulomb interaction $F = -k/r^2$. To solve the differential equation for $r = r(t)$, we introduce an effective potential,

$$V(r) = -\int\left(\frac{p_\theta^2}{mr^3} - \frac{k}{r^2}\right)dr = k\left(\frac{r_0}{2r^2} - \frac{1}{r}\right)$$

where $r_0 = p_\theta^2/(km)$. The differential equation for r can then be written as

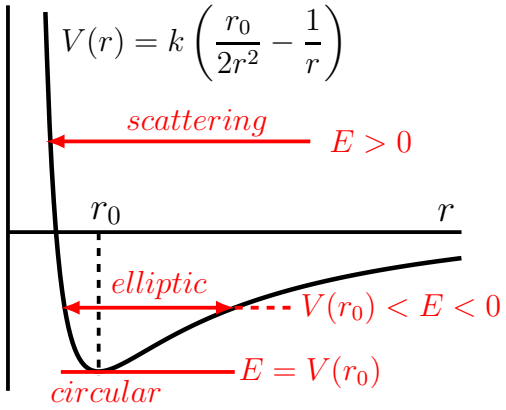
$$m\frac{d\dot{r}}{dt} = -\frac{\partial V(r)}{\partial r}$$

Multiplying $\dot{r}$ to both sides of the equation yields

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2\right) = -\frac{\partial V(r)}{\partial r}\frac{dr}{dt} = -\frac{dV(r)}{dt}$$

The mechanical energy for the motion in r is then

$$E = \frac{1}{2}m\dot{r}^2 + V(r) = constant$$



(a) Circular Orbit, $E = V(r_0)$

When $E = V(r_0)$, $\dot{r} = 0$ and $r = r_0$ where

$$\frac{dV(r)}{dr} = -k\left(\frac{r_0}{r^3} - \frac{1}{r^2}\right) = 0 \qquad \Longrightarrow \qquad r = r_0$$

Since r is a constant in this case, $\dot{\theta}$ is the constant angular velocity,

$$\omega = \dot{\theta} = \frac{p_\theta}{mr_0^2} = \frac{(kmr_0)^{1/2}}{mr_0^2} \qquad \Longrightarrow \qquad |F(r_0)| = \frac{k}{r_0^2} = m\,\omega^2 r_0 = m\,\frac{v^2}{r_0}$$

where $v = \omega r_0$.

(b) Elliptic Orbit, $V(r_0) < E < 0$

The solve the elliptic orbit, it is easier to integrate θ as a function of r. From the constant energy E and constant angular momentum p_θ,

$$\begin{cases} E = \frac{1}{2}m\dot{r}^2 + V(r) \\ \dot{\theta} = \frac{p_\theta}{mr^2} \end{cases} \qquad \Longrightarrow \qquad \begin{cases} dt = \pm\frac{1}{\sqrt{2(E - V(r))/m}}\,dr \\ d\theta = \left(\frac{p_\theta}{mr^2}\right)\,dt \end{cases} \tag{19}$$

We can use either $+$ or $-$ branch for the calculation of trajectory (time-reversal symmetry). Combining two equations in Eq. (19) yields

$$d\theta = -\frac{p_\theta}{r^2\sqrt{2m(E - V(r))}}\,dr$$

and

$$\theta - \theta(0) = -\frac{p_\theta}{\sqrt{m}}\int_{r(0)}^{r}\frac{dr}{r^2\sqrt{2(E - V(r))}} \tag{20}$$

where for the Kepler problem, the effective potential is

$$V(r) = \frac{p_\theta^2}{2mr^2} - \frac{k}{r}$$

In order to simplify the integral in Eq. (20) into a known integral, we change the variable

$$x = \frac{p_\theta}{\sqrt{m}\, r} \qquad \text{and} \qquad dx = -\frac{p_\theta}{\sqrt{m}\, r^2}\, dr$$

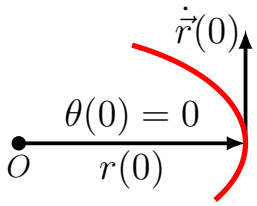
The effective potential in x is

$$w(x) \;=\; V\left(p_\theta/(\sqrt{m}\, x)\right) \;=\; \frac{1}{2}x^2 - \lambda x \qquad \text{with} \quad \lambda = k\sqrt{m}/p_\theta$$

and the integral in Eq. (20) becomes

$$\begin{aligned} \theta - \theta(0) \;&=\; \int_{x(0)}^{x} \frac{dx}{\sqrt{2(E - w(x))}} \;=\; \int_{x(0)}^{x} \frac{dx}{\sqrt{2E + \lambda^2 - (x-\lambda)^2}} \\ &=\; \arcsin\left(\frac{x-\lambda}{\sqrt{2E+\lambda^2}}\right) - \arcsin\left(\frac{x(0)-\lambda}{\sqrt{2E+\lambda^2}}\right) \end{aligned} \tag{21}$$

For convenience, we can choose the initial condition



$$\theta(0) = 0 \qquad \text{and} \qquad \dot{r}(0) = 0$$

i.e. at $\theta = 0$, $\vec{v} \perp \vec{r}$. From the constant energy, we have

$$E = w(x(0)) = \frac{1}{2}x(0)^2 - \lambda x(0) \qquad \Longrightarrow \qquad 2E + \lambda^2 = (x(0) - \lambda)^2$$

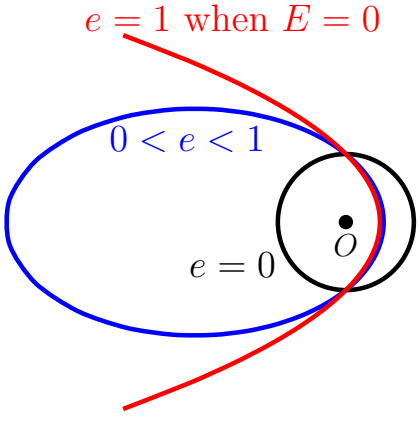
$$\Longrightarrow \qquad \arcsin\left(\frac{x(0)-\lambda}{\sqrt{2E+\lambda^2}}\right) = \frac{\pi}{2}$$

The solution of the trajectory is then

$$\theta + \frac{\pi}{2} = \arcsin\left(\frac{x-\lambda}{\sqrt{2E+\lambda^2}}\right) \qquad \Longrightarrow \qquad x \;=\; \sqrt{2E+\lambda^2}\,\cos\theta + \lambda \;=\; \frac{p_\theta}{\sqrt{m}\, r}$$

Therefore, the elliptic planetary motion is at a trajectory of

$$r \;=\; \frac{p_\theta}{\sqrt{m(2E+\lambda^2)}\,\cos\theta + \sqrt{m}\lambda} \;=\; \frac{p'}{e\,\cos\theta + 1}$$



where $p' = p_\theta^2/(km)$,

$$0 \;\le\; e = \sqrt{1 + \frac{2Ep_\theta^2}{k^2 m}} \;\le\; 1$$

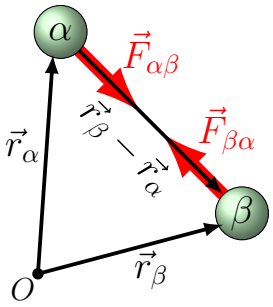
is the eccentricity of an ellipse orbit, and

$$r_{min} = r(0) = \frac{p'}{1+e}, \qquad r_{max} = r(\pi) = \frac{p'}{1-e}$$

4.4 Energy of a Multiparticle System

Potential for Interactions Between Two Particles

For all the interactions we know so far, the interactions between two particles can always be written as a derivative of a potential function. Let $U(\vec{r}_\alpha, \vec{r}_\beta)$ be the interaction potential between particle α and particle β, where



$$\vec{F}_{\alpha\beta} = -\frac{\partial U(\vec{r}_\alpha, \vec{r}_{beta})}{\partial \vec{r}_\alpha}, \qquad \vec{F}_{\beta\alpha} = -\frac{\partial U(\vec{r}_\alpha, \vec{r}_\beta)}{\partial \vec{r}_\beta}$$

and

$$\vec{F}_{\alpha\beta} = -\vec{F}_{\beta\alpha} \qquad \Longrightarrow \qquad \frac{\partial U(\vec{r}_\alpha, \vec{r}_\beta)}{\partial \vec{r}_\alpha} + \frac{\partial U(\vec{r}_\alpha, \vec{r}_\beta)}{\partial \vec{r}_\beta} = 0$$

We now show that the interaction potential $U(\vec{r}_\alpha, \vec{r}_\beta)$ depends only on $\vec{r}_\alpha - \vec{r}_\beta$, not on $\vec{r}_\alpha$ and $\vec{r}_\beta$ individually. The two independent variable $\vec{r}_\alpha$ and $\vec{r}_\beta$ in $U(\vec{r}_\alpha, \vec{r}_\beta)$ can be replaced by another pair of independent vectors,

$$\vec{r}_\pm = \vec{r}_\beta \pm \vec{r}_\alpha \qquad \text{and} \qquad U(\vec{r}_\alpha, \vec{r}_\beta) = w(\vec{r}_-, \vec{r}_+)$$

Then

$$\begin{aligned} 0 &= \frac{\partial U(\vec{r}_\alpha, \vec{r}_\beta)}{\partial \vec{r}_\alpha} + \frac{\partial U(\vec{r}_\alpha, \vec{r}_\beta)}{\partial \vec{r}_\beta} \\ &= \frac{\partial w(\vec{r}_-, \vec{r}_+)}{\partial \vec{r}_-} \cdot \frac{\partial \vec{r}_-}{\partial \vec{r}_\alpha} + \frac{\partial w(\vec{r}_-, \vec{r}_+)}{\partial \vec{r}_+} \cdot \frac{\partial \vec{r}_+}{\partial \vec{r}_\alpha} + \frac{\partial w(\vec{r}_-, \vec{r}_+)}{\partial \vec{r}_-} \cdot \frac{\partial \vec{r}_-}{\partial \vec{r}_\beta} + \frac{\partial w(\vec{r}_-, \vec{r}_+)}{\partial \vec{r}_+} \cdot \frac{\partial \vec{r}_+}{\partial \vec{r}_\beta} \\ &= 2\frac{\partial w(\vec{r}_-, \vec{r}_+)}{\partial \vec{r}_+} \end{aligned}$$

i.e. $w(\vec{r}_-, \vec{r}_+)$ is independent of $\vec{r}_+$, where

$$\frac{\partial \vec{r}_-}{\partial \vec{r}_\alpha} = -1 \qquad \text{and} \qquad \frac{\partial \vec{r}_-}{\partial \vec{r}_\beta} = \frac{\partial \vec{r}_+}{\partial \vec{r}_\alpha} = \frac{\partial \vec{r}_+}{\partial \vec{r}_\beta} = 1$$

The interaction potential between two particles therefore depends only on $\vec{r}_- = \vec{r}_\beta - \vec{r}_\alpha$, the vector connecting the two particles. To simplify the notation, let

$$\vec{r} = \vec{r}_- = \vec{r}_\beta - \vec{r}_\alpha$$

In the case that the interaction between two particles is along the direction of $\vec{r} = \vec{r}_\beta - \vec{r}_\alpha$ (Newton's 3rd law), the potential energy has to be a function of $r = |\vec{r}_\beta - \vec{r}_\alpha|$. This can be shown as following. Since the potential energy is an scalar function, it can only be a function of scalar variables. From vector variable $\vec{r}$, scalar variables can be constructed in two forms,

$$\vec{r} \cdot \vec{r} = r^2 = |\vec{r}_\beta - \vec{r}_\alpha|^2 \qquad \text{and} \qquad \vec{b} \cdot \vec{r} = \vec{b} \cdot (\vec{r}_\beta - \vec{r}_\alpha)$$

where $\vec{b}$ can be any vector. We therefore can consider the interaction potential as a function of r and $(\vec{b}\cdot\vec{r})$, *i.e.*

$$U(\vec{r}) \;=\; U(r,\,(\vec{b}\cdot\vec{r}))$$

The force on the two particles are then

$$\begin{aligned}\vec{F}_{\alpha\beta} \;=\; -\frac{\partial U(r,\,(\vec{b}\cdot\vec{r}))}{\partial\vec{r}_\alpha} \;&=\; -\left.\frac{\partial U(r,\,(\vec{b}\cdot\vec{r}))}{\partial r}\right|_{(\vec{b}\cdot\vec{r})}\frac{\partial r}{\partial\vec{r}_\alpha} - \left.\frac{\partial U(r,\,(\vec{b}\cdot\vec{r}))}{\partial(\vec{b}\cdot\vec{r})}\right|_{r}\frac{\partial(\vec{b}\cdot\vec{r})}{\partial\vec{r}_\alpha}\\ &=\; \left(\frac{\vec{r}}{r}\right)\left.\frac{\partial U(r,\,(\vec{b}\cdot\vec{r}))}{\partial r}\right|_{(\vec{b}\cdot\vec{r})} + \vec{b}\left.\frac{\partial U(r,\,(\vec{b}\cdot\vec{r}))}{\partial(\vec{b}\cdot\vec{r})}\right|_{r}\end{aligned}$$

where

$$\frac{\partial r}{\partial\vec{r}_\alpha} \;=\; -\frac{\vec{r}}{r} \qquad \text{and} \qquad \frac{\partial(\vec{b}\cdot\vec{r})}{\partial\vec{r}_\alpha} = -\vec{b}$$

Since $\vec{b}$ is an arbitrary vector that could be along a different direction of $\vec{r}$ while $\vec{F}_{\alpha\beta}$ and $\vec{r}/r$ are along the direction of $\vec{r}$, the 2nd term that is not in the direction of $\vec{r}$ has to be zero,

$$\left.\frac{\partial U(r,\,(\vec{b}\cdot\vec{r}))}{\partial(\vec{b}\cdot\vec{r})}\right|_{r} \;=\; 0$$

i.e. $U(r,\,(\vec{b}\cdot\vec{r}))$ cannot be a function of $(\vec{b}\cdot\vec{r})$ and the interaction potential has to be a function r. To simplify the notation, we use

$$U(\vec{r}_\alpha,\vec{r}_\beta) \;=\; U(|\vec{r}_\alpha-\vec{r}_\beta|) \;=\; U_{\alpha\beta}(r) \tag{22}$$

as the interaction potential between two particles.

Mechanical Energy of a System of N Particles

Consider a system of N particles with conservative forces, in general, the total mechanical energy is

$$E \;=\; \sum_{\alpha=1}^{N}\frac{1}{2}m_\alpha(\vec{v}_\alpha\cdot\vec{v}_\alpha) + \sum_{\alpha=1}^{N}\sum_{\beta>\alpha}^{N}U_{\alpha\beta} + \sum_{\alpha=1}^{N}U_\alpha^{ext} \tag{23}$$

where $U_{\alpha\beta}$ is the potential for the interactions between particles α and β and U_α^{ext} is the external interaction from the outside of the system on particle α. The double summations for the internal interactions can also be written into another form,

$$\sum_{\alpha=1}^{N}\sum_{\beta>\alpha}^{N}U_{\alpha\beta} \;=\; \frac{1}{2}\sum_{\alpha=1}^{N}\sum_{\beta\neq\alpha}^{N}U_{\alpha\beta}$$

where $\alpha>\beta$ in the 1st form (left side) and a factor of 1/2 in the 2nd form (right side) is to eliminate the double counting of the interaction potential between particle α and particle β as

$$U_{\alpha\beta} \;=\; U(|\vec{r}_\alpha-\vec{r}_\beta|) \;=\; U(|\vec{r}_\beta-\vec{r}_\alpha|) \;=\; U_{\beta\alpha}$$

Conservation of Mechanical Energy

The equation of motion for a system of N particles is

$$\begin{cases} m_1\ddot{\vec{r}}_1 &= -\dfrac{\partial E}{\partial \vec{r}_1} = -\dfrac{\partial}{\partial \vec{r}_1}\left(\displaystyle\sum_{\beta\neq 1}^{N} U_{1\beta} + U_1^{ext}\right) \\ m_2\ddot{\vec{r}}_2 &= -\dfrac{\partial E}{\partial \vec{r}_2} = -\dfrac{\partial}{\partial \vec{r}_2}\left(\displaystyle\sum_{\beta\neq 2}^{N} U_{2\beta} + U_2^{ext}\right) \\ &\vdots \\ m_N\ddot{\vec{r}}_N &= -\dfrac{\partial E}{\partial \vec{r}_N} = -\dfrac{\partial}{\partial \vec{r}_N}\left(\displaystyle\sum_{\beta\neq N}^{N} U_{N\beta} + U_N^{ext}\right) \end{cases} \tag{24}$$

When the forces are nonlinear as they usually are, these equations are coupled $3N$-dimensional nonlinear 2nd-order ODEs and nobody know how to solve this system. The time dependence of the mechanical energy is

$$\begin{aligned} \frac{d}{dt}E(\dot{\vec{r}}_1, \cdots, \dot{\vec{r}}_N, \vec{r}_1, \cdots, \vec{r}_N, t) &= \frac{\partial E}{\partial t} + \sum_{\alpha=1}^{N}\left(\frac{\partial E}{\partial \dot{\vec{r}}_\alpha}\cdot\frac{d\dot{\vec{r}}_\alpha}{dt} + \frac{\partial E}{\partial \vec{r}_\alpha}\cdot\frac{d\vec{r}_\alpha}{dt}\right) \\ &= \frac{\partial E}{\partial t} + \sum_{\alpha=1}^{N}\left(m_\alpha\frac{d\dot{\vec{r}}_\alpha}{dt} + \frac{\partial E}{\partial \vec{r}_\alpha}\right)\cdot\dot{\vec{r}}_\alpha = \frac{\partial E}{\partial t} \end{aligned}$$

When the energy of the system does not depend on time explicitly, the energy is a constant of motion, *i.e.*

$$E(\dot{\vec{r}}_1, \cdots, \dot{\vec{r}}_N, \vec{r}_1, \cdots, \vec{r}_N) = \textit{constant}$$

but this constant of motion has little use for solving the motion as the motion is in the $6N$-dimensional space and a constant of motion only reduce the dimensionality to $6N-1$. Note that the energy conservation is the key to solve the one-dimensional Newton's equation.

Only case we know how to solve a system of N particles is the small oscillations near the stable equilibriums. Our whole physics complex has basically been built based on the knowledge of the small oscillations near the equilibriums.

4.5 Small Oscillation near a Stable Equilibrium of a Multiparticle System

For a multiparticle system, it is convenient to label the position and velocity components of N particles sequentially,

$$\begin{cases} \vec{q} = (q_1, q_2, q_3, \cdots, q_{3\alpha-2}, q_{3\alpha-1}, q_{3\alpha}, \cdots, q_{3N-2}, q_{3N-1}, q_{3N}) \\ \qquad\qquad = (x_1, y_2, y_3, \cdots, x_\alpha, y_\alpha, z_\alpha, \cdots, x_N, y_N, z_N) \\ \dot{\vec{q}} = (\dot{q}_1, \dot{q}_2, \dot{q}_3, \cdots, \dot{q}_{3\alpha-2}, \dot{q}_{3\alpha-1}, \dot{q}_{3\alpha}, \cdots, \dot{q}_{3N-2}, \dot{q}_{3N-1}, \dot{q}_{3N}) \\ \qquad\qquad = (\dot{x}_1, \dot{y}_2, \dot{y}_3, \cdots, \dot{x}_\alpha, \dot{y}_\alpha, \dot{z}_\alpha, \cdots, \dot{x}_N, \dot{y}_N, \dot{z}_N) \end{cases}$$

The energy of the system can be written as

$$E = \sum_{\alpha=1}^{3N} \frac{1}{2} m_\alpha \dot{q}_\alpha^2 + U(\vec{q}) \tag{25}$$

and

$$U(\vec{q}) \;=\; U(q_1, \cdots\; q_{3N}) \;=\; \sum_{\alpha=1}^{N} \sum_{\beta>\alpha}^{N} U_{\alpha\beta}(|\vec{r}_\alpha - \vec{r}_\beta|) + \sum_{\alpha=1}^{N} U_\alpha^{ext}(\vec{r}_\alpha) \tag{26}$$

where $\vec{r}_\alpha = (q_{3\alpha-2},\, q_{3\alpha-1},\, q_{3\alpha})$ and $\vec{r}_\beta = (q_{3\beta-2},\, q_{3\beta-1},\, q_{3\beta})$. The stable equilibrium is at a minima of the potential function U, where the first-order derivatives of the potential vanish

$$\frac{\partial U(\vec{q}_0)}{\partial q_{0i}} \;=\; 0 \qquad\qquad \text{for } i = 1, \cdots, 3N \tag{27}$$

where $\vec{q}_0 \;=\; (q_{01}, q_{02}, \cdots, q_{03N})$, and the second-order derivatives of the potential forms a Hessian matrix,

$$\begin{pmatrix} U''_{11} & U''_{12} & \cdots & U''_{1n} \\ U''_{21} & U''_{22} & \cdots & U''_{2n} \\ \vdots & \vdots & & \vdots \\ U''_{n1} & U''_{n2} & \cdots & U''_{nn} \end{pmatrix} \qquad \text{where} \qquad U''_{ij} \;=\; \frac{\partial^2 U(\vec{q}_0\,)}{\partial q_{0i} \partial q_{0j}}$$

that is semi-positive definite.

Linearized Motion near a Stable Equilibrium

We now consider a small motion near a stable equilibrium point $\vec{q}_0$. Let

$$\vec{x} = \vec{q} - \vec{q}_0 \qquad \text{and} \qquad \dot{\vec{x}} = \dot{\vec{q}}$$

The expansion of the potential energy $U(\vec{q})$ at $\vec{q}_0$ for a small $\vec{x}$ yields

$$\begin{aligned} U(\vec{q}) \;&=\; U(\vec{q}_0) \;+\; \sum_{i=1}^{3N} \frac{\partial U(\vec{q}_0)}{\partial q_{0i}}\, x_i \;+\; \frac{1}{2} \sum_{i=1}^{3N} \sum_{j=1}^{3N} \frac{\partial^2 U(\vec{q}_0)}{\partial q_{0i} \partial q_{0j}}\, x_i x_j \;+\; \cdots \\ &=\; U(\vec{q}_0) \;+\; \frac{1}{2} \sum_{i=1}^{3N} \sum_{j=1}^{3N} \frac{\partial^2 U(\vec{q}_0)}{\partial q_{0i} \partial q_{0j}}\, x_i x_j \;+\; \cdots \end{aligned}$$

where the 0th-order term $U(\vec{q}_0)$ is a constant potential at the equilibrium that does not curry any physics and the first-order terms are all zero because of the equilibrium condition at $\vec{q}_0$. If we only keep the lowest-order nonzero terms that curry physics, the potential energy for a small motion near a stable equilibrium is

$$\begin{aligned} U(\vec{x}) \;=\; U(x_1, \cdots, x_{3N}) \;&\simeq\; U(\vec{q}_0) \;+\; \frac{1}{2} \sum_{i=1}^{3N} \sum_{j=1}^{3N} \frac{\partial^2 U(\vec{q}_0)}{\partial q_{0i} \partial q_{0j}}\, x_i x_j \\ &=\; U(\vec{q}_0) \;+\; \frac{1}{2} \sum_{i=1}^{3N} \sum_{j=1}^{3N} k_{ij} x_i x_j \end{aligned} \tag{28}$$

where

$$k_{ij} \;=\; \frac{\partial^2 U(\vec{q}_0)}{\partial q_{0i}\partial q_{0i}} \;=\; \frac{\partial^2 U(\vec{q}_0)}{\partial q_{0j}\partial q_{0i}} \;=\; k_{ji}$$

and the equation of motion for the small motion near an equilibrium is

$$\left\{\begin{array}{rcl} m_1\ddot{x}_1 &=& -k_{11}x_1 - k_{12}x_2 - \cdots - k_{1,3N}x_{3N} \\ m_2\ddot{x}_2 &=& -k_{21}x_1 - k_{22}x_2 - \cdots - k_{2,3N}x_{3N} \\ \vdots & & \\ m_{3N}\ddot{x}_{3N} &=& -k_{3N,1}x_1 - k_{3N,2}x_2 - \cdots - k_{3N,3N}x_{3N} \end{array}\right. \tag{29}$$

This is a system of linear ODEs.

Solving Normal-Mode Oscillations for Linearized Newton's Equation

It is easier to solve the linearized equation of motion in Eq. (29) using its matrix form,

$$\mathbf{M}\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{3N} \end{pmatrix} = -\mathbf{K}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{3N} \end{pmatrix} \qquad \text{or} \qquad \frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{3N} \end{pmatrix} = -\mathbf{B}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{3N} \end{pmatrix} \tag{30}$$

where

$$\mathbf{M} \;=\; \begin{pmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & m_{3N} \end{pmatrix}, \qquad \mathbf{K} \;=\; \begin{pmatrix} k_{11} & k_{12} & \cdots & k_{1,3N} \\ k_{21} & k_{22} & \cdots & k_{2,3N} \\ \vdots & \vdots & & \vdots \\ k_{3N,1} & k_{3N,2} & \cdots & k_{3N,3N} \end{pmatrix} \tag{31}$$

and

$$\mathbf{B} \;=\; \mathbf{M}^{-1}\mathbf{K} \;=\; \begin{pmatrix} k_{11}/m_1 & k_{12}/m_1 & \cdots & k_{1,3N}/m_1 \\ k_{21}/m_2 & k_{22}/m_2 & \cdots & k_{2,3N}/m_2 \\ \vdots & \vdots & & \vdots \\ k_{3N,1}/m_{3N} & k_{3N,2}/m_{3N} & \cdots & k_{3N,3N}/m_{3N} \end{pmatrix} \tag{32}$$

The matrix $\mathbf{K}$ is the $3N \times 3N$ Hessian matrix of the potential $U(\vec{q})$ at the equilibrium $\vec{q} = \vec{q}_0$ and is semi-positive definite, the matrix $\mathbf{M}$ is also a positive definite matrix. The matrix $\mathbf{B}$ can therefore be diagonalized and the eigenvalues are non-negative,

$$\mathbf{\Gamma}^{-1}\,\mathbf{B}\,\mathbf{\Gamma} \;=\; \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_{3N} \end{pmatrix} \;=\; \mathbf{\Lambda} \tag{33}$$

where $\mathbf{\Lambda}$ is the diagonalized matrix of matrix $\mathbf{B}$ and $\lambda_i \geq 0$ for $i = 1, \cdots, 3N$. To diagonalize a matrix $\mathbf{B}$, we need to solve the eigenvalue equation,

$$\det(\mathbf{B} - \lambda \mathbf{I}) = 0 \tag{34}$$

which yields $3N$ roots of λ for the eigenvalues, and also solve the eigenvector equation,

$$\mathbf{B} \begin{pmatrix} \gamma_1^{(i)} \\ \gamma_2^{(i)} \\ \vdots \\ \gamma_{3N}^{(i)} \end{pmatrix} = \lambda_i \begin{pmatrix} \gamma_1^{(i)} \\ \gamma_2^{(i)} \\ \vdots \\ \gamma_{3N}^{(i)} \end{pmatrix} \tag{35}$$

The transformation matrix $\mathbf{\Gamma}$ is constructed with the eigenvectors as

$$\mathbf{\Gamma} = \begin{pmatrix} \gamma_1^{(1)} & \gamma_1^{(2)} & \cdots & \gamma_1^{(3N)} \\ \gamma_2^{(1)} & \gamma_2^{(2)} & \cdots & \gamma_2^{(3N)} \\ \vdots & \vdots & & \vdots \\ \gamma_{3N}^{(1)} & \gamma_{3N}^{(2)} & \cdots & \gamma_{3N}^{(3N)} \end{pmatrix} \tag{36}$$

With the transformation matrix $\mathbf{\Gamma}$, the linearized equation of motion in Eq. (30) can be decoupled as

$$\frac{d^2}{dt^2} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{3N} \end{pmatrix} = -\mathbf{B} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{3N} \end{pmatrix} \quad \Longrightarrow \quad \mathbf{\Gamma}^{-1} \frac{d^2}{dt^2} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{3N} \end{pmatrix} = -\mathbf{\Gamma}^{-1} \mathbf{B}\, \mathbf{\Gamma} \mathbf{\Gamma}^{-1} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{3N} \end{pmatrix}$$

which yields

$$\frac{d^2}{dt^2} \mathbf{\Gamma}^{-1} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{3N} \end{pmatrix} = -\mathbf{\Lambda}\, \mathbf{\Gamma}^{-1} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{3N} \end{pmatrix} \tag{37}$$

where $\mathbf{\Gamma}^{-1}$ is a constant matrix that can exchange the order with the time derivative and $\mathbf{\Gamma}^{-1} \mathbf{B}\, \mathbf{\Gamma} = \mathbf{\Lambda}$. Now we introduce the normal modes for the small motion near a stable equilibrium,

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{3N} \end{pmatrix} = \mathbf{\Gamma}^{-1} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{3N} \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{3N} \end{pmatrix} = \mathbf{\Gamma} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{3N} \end{pmatrix} \tag{38}$$

The equation of motion is thus decoupled in the normal modes,

$$\frac{d^2}{dt^2}\begin{pmatrix}\xi_1\\ \xi_2\\ \vdots\\ \xi_{3N}\end{pmatrix} = -\mathbf{\Lambda}\begin{pmatrix}\xi_1\\ \xi_2\\ \vdots\\ \xi_{3N}\end{pmatrix} = -\begin{pmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & \lambda_{3N}\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\\ \vdots\\ \xi_{3N}\end{pmatrix} \tag{39}$$

Since $\lambda_i \geq 0$, let $\omega_i^2 = \lambda_i$ and the solution for each normal mode is

$$\frac{d^2\xi_i}{dt^2} = -\omega_i^2\,\xi_i \qquad \Longrightarrow \qquad \xi_i(t) = \xi_i(0)\cos(\omega_i t) + \frac{\dot{\xi}_i(0)}{\omega_i}\sin(\omega_i t) \tag{40}$$

where $i = 1, 2, \cdots, 3N$. Each normal mode ξ_i is therefore a harmonic oscillation with the eigen frequency of $\omega_i = \sqrt{\lambda_i}$.

The Final Solution of a Small Oscillation in a Multiparticle System

The initial condition of the normal modes need to be determined through the initial condition of the motion using

$$\begin{pmatrix}\xi_1(0)\\ \xi_2(0)\\ \vdots\\ \xi_{3N}(0)\end{pmatrix} = \mathbf{\Gamma}^{-1}\begin{pmatrix}x_1(0)\\ x_2(0)\\ \vdots\\ x_{3N}(0)\end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix}\dot{\xi}_1(0)\\ \dot{\xi}_2(0)\\ \vdots\\ \dot{\xi}_{3N}(0)\end{pmatrix} = \mathbf{\Gamma}^{-1}\begin{pmatrix}\dot{x}_1(0)\\ \dot{x}_2(0)\\ \vdots\\ \dot{x}_{3N}(0)\end{pmatrix} \tag{41}$$

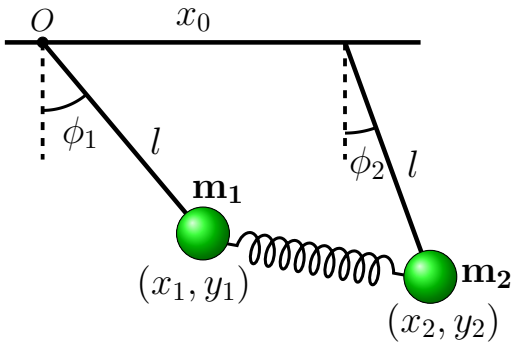
After the oscillations of the normal modes in Eq. (40) are obtained, the solution for the small oscillation near an equilibrium can be calculated from

$$\begin{pmatrix}x_1(t)\\ x_2(t)\\ \vdots\\ x_{3N}(t)\end{pmatrix} = \mathbf{\Gamma}\begin{pmatrix}\xi_1(t)\\ \xi_2(t)\\ \vdots\\ \xi_{3N}(t)\end{pmatrix} = \mathbf{\Gamma}\begin{pmatrix}\xi_1(0)\cos(\omega_1 t) + [\dot{\xi}_1(0)/\omega_1]\sin(\omega_1 t)\\ \xi_2(0)\cos(\omega_2 t) + [\dot{\xi}_2(0)/\omega_2]\sin(\omega_2 t)\\ \vdots\\ \xi_{3N}(0)\cos(\omega_{3N} t) + [\dot{\xi}_{3N}(0)/\omega_{3N}]\sin(\omega_{3N} t)\end{pmatrix} \tag{42}$$

which yields

$$x_i(t) = \sum_{j=1}^{3N}\gamma_i^{(j)}\left[\xi_j(0)\cos(\omega_j t) + \frac{\dot{\xi}_j(0)}{\omega_j}\sin(\omega_j t)\right], \qquad \forall i = 1, \cdots, 3N \tag{43}$$

Example 4.4 Coupled Pendulums



Two pendulum are connected by a spring with a spring constant k and natural length x_0. The distance between two pivots of the pendulums is also x_0. The forces for the Newton's equation of two pendulums can be calculated from the potential energy. To obtain the potential energy as a function of θ_1 and θ_2, we start with the coordinates of the two masses

$$\begin{cases} x_1 = \quad l\sin\phi_1 \\ y_1 = -l\cos\phi_1 \end{cases} \qquad \text{and} \qquad \begin{cases} x_2 = x_0 + l\sin\phi_2 \\ y_2 = -l\cos\phi_2 \end{cases}$$

where the origin O is at the pivot of pendulum m_1. The distance between m_1 and m_2 is

$$d \;=\; \sqrt{(x_0 + l\sin\phi_2 - l\sin\phi_1)^2 + (l\cos\phi_2 - l\cos\phi_1)^2} \;=\; x_0 + l(\phi_2 - \phi_1) + O(2)$$

where the 2nd equation is the Taylor expansion of d with small angle $\phi_1 << 1$ and $\phi_2 << 1$, and $O(2)$ are the terms with the expansion order equal and higher than 2. The potential energy of the double pendulum at small angle oscillation is then

$$\begin{aligned} U(\phi_1,\phi_2) &= m_1 g(l - l\cos\phi_1) + m_1 g(l - l\cos\phi_1) - \frac{1}{2}k(d - x_0)^2 \\ &= \frac{1}{2}m_1 g l\,\phi_1^2 + \frac{1}{2}m_1 g l\,\phi_1^2 + \frac{1}{2}kl^2(\phi_1 - \phi_2)^2 \end{aligned} \tag{44}$$

In the polar coordinates $(\rho_1, \phi_1, z = 0)$ and $(\rho_2, \phi_2, z = 0)$, the motion of m_1 and m_2 are the oscillation of $\phi_1(t)$ and $\phi_2(t)$ while $\rho_1 = \rho_2 = l$. The Newton's equation is

$$\begin{cases} m(\ddot{\rho} - \rho\dot{\phi}^2) \;=\; -\dfrac{\partial U(\vec{r})}{\partial\rho} \\ \\ m(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}) \;=\; -\dfrac{1}{\rho}\dfrac{\partial U(\vec{r})}{\partial\phi} \end{cases} \quad\Longrightarrow\quad \begin{cases} m_1 l\ddot{\phi}_1 \;=\; -\dfrac{1}{l}\dfrac{\partial U(\phi_1,\phi_2)}{\partial\phi_1} \\ \\ m_2 l\ddot{\phi}_2 \;=\; -\dfrac{1}{l}\dfrac{\partial U(\phi_1,\phi_2)}{\partial\phi_2} \end{cases} \tag{45}$$

where $\ddot{\rho}_1 = \ddot{\rho}_2 = 0$ and $\partial U(\phi_1,\phi_2)/\partial\rho_1 = \partial U(\phi_1,\phi_2)/\partial\rho_2 = 0$. The equation of motion for the small oscillation of the double pendulum is then

$$\begin{cases} m_1 l^2\ddot{\phi}_1 \;=\; -m_1 g l\phi_1 - kl^2(\phi_1 - \phi_2) \\ m_2 l^2\ddot{\phi}_2 \;=\; -m_2 g l\phi_2 + kl^2(\phi_1 - \phi_2) \end{cases} \tag{46}$$

which can be written as

$$\frac{d^2}{dt^2}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \;=\; -\mathbf{B}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

where

$$\mathbf{B} = \begin{pmatrix} 1/(m_1 l^2) & 0 \\ 0 & 1/(m_2 l^2) \end{pmatrix} \begin{pmatrix} m_1 gl + kl^2 & kl^2 \\ kl^2 & m_2 gl + kl^2 \end{pmatrix} = \begin{pmatrix} g/l + k/m_1 & k/m_1 \\ k/m_2 & g/l + k/m_2 \end{pmatrix}$$

The eigenvalue equation is

$$\begin{aligned} \det(\mathbf{B} - \omega^2 \mathbf{I}) &= \left(\frac{g}{l} + \frac{k}{m_1} - \omega^2\right)\left(\frac{g}{l} + \frac{k}{m_2} - \omega^2\right) - \frac{k^2}{m_1 m_2} \\ &= \frac{1}{m_1 m_2}\left[(\alpha_1 - m_1\omega^2)(\alpha_2 - m_2\omega^2) - k^2\right] = 0 \end{aligned}$$

where $\alpha_i = m_i g/l + k$. For a simple case, $m_1 = m_2 = m$, the two eigen frequencies are

$$\omega_1 = \sqrt{\frac{g}{l}} \qquad \text{and} \qquad \omega_2 = \sqrt{\frac{g}{l} + \frac{2k}{m}}$$

The equation for two eigenvectors, $(\gamma_1^{(1)}, \gamma_2^{(1)})$ and $(\gamma_1^{(2)}, \gamma_2^{(2)})$, are

$$\mathbf{B}\begin{pmatrix} \gamma_1^{(i)} \\ \gamma_2^{(i)} \end{pmatrix} = \begin{pmatrix} g/l + k/m_1 & k/m_1 \\ k/m_2 & g/l + k/m_2 \end{pmatrix}\begin{pmatrix} \gamma_1^{(i)} \\ \gamma_2^{(i)} \end{pmatrix} = \omega_i^2 \begin{pmatrix} \gamma_1^{(i)} \\ \gamma_2^{(i)} \end{pmatrix}$$

The solution for this linear-algebraic equation is

$$\begin{pmatrix} \gamma_1^{(1)} \\ \gamma_2^{(1)} \end{pmatrix} = \frac{1}{l\sqrt{2m}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \text{for} \quad \omega_1 = \sqrt{\frac{g}{l}}$$

and

$$\begin{pmatrix} \gamma_1^{(2)} \\ \gamma_2^{(2)} \end{pmatrix} = \frac{1}{l\sqrt{2m}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{for} \quad \omega_2 = \sqrt{\frac{g}{l} + \frac{2k}{m}}$$

The transformation matrix $\mathbf{\Gamma}$ constructed from the eigenvectors is

$$\mathbf{\Gamma} = \frac{1}{l\sqrt{2m}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{\Gamma}^{-1} = \frac{l\sqrt{2m}}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

You can check that matrix $\mathbf{\Gamma}$ diagonalizes matrix $\mathbf{B}$ as,

$$\mathbf{\Gamma}^{-1}\mathbf{B}\mathbf{\Gamma} = \begin{pmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{pmatrix}$$

With $\mathbf{\Gamma}$, the equation of motion for the coupled pendulum in Eq. (46) is decoupled into two one-dimensional equation for the normal modes,

$$\frac{d^2}{dt^2}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = -\begin{pmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \tag{47}$$

which yields two harmonic oscillator solution

$$\begin{cases} \xi_1(t) = \xi_1(0)\cos(\omega_1 t) + \dfrac{\dot{\xi}_1(0)}{\omega_1}\sin(\omega_1 t) \\ \xi_2(t) = \xi_2(0)\cos(\omega_2 t) + \dfrac{\dot{\xi}_2(0)}{\omega_2}\sin(\omega_2 t) \end{cases}$$

The transformation between the normal modes and the oscillation angles is

$$\begin{pmatrix} \phi_1(t) \\ \phi_2(t) \end{pmatrix} = \mathbf{\Gamma} \begin{pmatrix} \xi_1(t) \\ \xi_2(t) \end{pmatrix} \qquad \Longrightarrow \qquad \begin{cases} \phi_1(t) = \dfrac{1}{l\sqrt{2m}}\left[\ \xi_1(t) + \xi_2(t)\right] \\ \phi_2(t) = \dfrac{1}{l\sqrt{2m}}\left[-\xi_1(t) + \xi_2(t)\right] \end{cases} \tag{48}$$

and

$$\begin{pmatrix} \xi_1(t) \\ \xi_2(t) \end{pmatrix} = \mathbf{\Gamma}^{-1} \begin{pmatrix} \phi_1(t) \\ \phi_2(t) \end{pmatrix} \qquad \Longrightarrow \qquad \begin{cases} \xi_1(t) = \dfrac{l\sqrt{2m}}{2}\left[\phi_1(t) - \phi_2(t)\right] \\ \xi_2(t) = \dfrac{l\sqrt{2m}}{2}\left[\phi_1(t) + \phi_2(t)\right] \end{cases} \tag{49}$$

where the initial condition of the normal modes is determined by the initial condition of the system using this transformation. If the initial condition of the pendulum oscillation is

$$\phi_1(0) \,=\, \phi_2(0) \,=\, \dot{\phi}_1(0) \,=\, 0\,, \qquad \dot{\phi}_2(0) \,=\, C$$

the initial condition for the normal modes is

$$\begin{cases} \xi_1(0) = \dfrac{l\sqrt{2m}}{2}\left[\phi_1(0) - \phi_2(0)\right] = 0 \\ \xi_2(0) = \dfrac{l\sqrt{2m}}{2}\left[\phi_1(0) + \phi_2(0)\right] = 0 \end{cases} \quad \text{and} \quad \begin{cases} \dot{\xi}_1(0) = \dfrac{l\sqrt{2m}}{2}\left[\dot{\phi}_1(0) - \dot{\phi}_2(0)\right] = -\dfrac{l\sqrt{2m}}{2}C \\ \dot{\xi}_2(0) = \dfrac{l\sqrt{2m}}{2}\left[\dot{\phi}_1(0) + \dot{\phi}_2(0)\right] = \dfrac{l\sqrt{2m}}{2}C \end{cases}$$

The time-dependence of the normal mode is then

$$\begin{cases} \xi_1(t) \,=\, \xi_1(0)\cos(\omega_1 t) + \dfrac{\dot{\xi}_1(0)}{\omega_1}\sin(\omega_1 t) \,=\, -\dfrac{Cl\sqrt{2m}}{2\omega_1}\sin(\omega_1 t) \\ \xi_2(t) \,=\, \xi_2(0)\cos(\omega_2 t) + \dfrac{\dot{\xi}_2(0)}{\omega_2}\sin(\omega_2 t) \,=\, \dfrac{Cl\sqrt{2m}}{2\omega_2}\sin(\omega_2 t) \end{cases}$$

The small oscillation of the double pendulum is then

$$\begin{cases} \phi_1(t) \,=\, \dfrac{1}{l\sqrt{2m}}\left[\ \xi_1(t) + \xi_2(t)\right] \,=\, \dfrac{C}{2}\left[\dfrac{1}{\omega_2}\sin(\omega_2 t) - \dfrac{1}{\omega_1}\sin(\omega_1 t)\right] \\ \phi_2(t) \,=\, \dfrac{1}{l\sqrt{2m}}\left[-\xi_1(t) + \xi_2(t)\right] \,=\, \dfrac{C}{2}\left[\dfrac{1}{\omega_1}\sin(\omega_1 t) + \dfrac{1}{\omega_2}\sin(\omega_2 t)\right] \end{cases}$$

When $\omega_1 \simeq \omega_2$

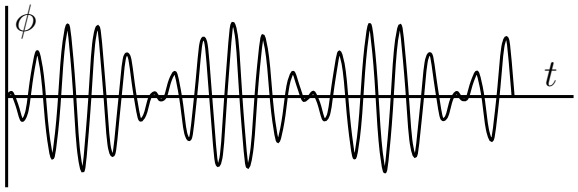
$$\omega_2 \,=\, \omega_1\sqrt{1 + \frac{2kl}{gm}} \,\simeq\, \omega_1 + \alpha\omega_1 \qquad \text{where} \quad \alpha = \frac{kl}{gm} << 1$$

then

$$\begin{aligned}
\phi_1(t) &\sim \frac{1}{\omega_2}\sin(\omega_1 t + \alpha\omega_1 t) - \frac{1}{\omega_1}\sin(\omega_1 t) \\
&= \frac{1}{\omega_2}\left[\sin(\omega_1 t)\cos(\alpha\omega_1 t) + \cos(\omega_1 t)\sin(\alpha\omega_1 t)\right] - \frac{1}{\omega_1}\sin(\omega_1 t) \\
&\simeq \frac{1}{\omega_2}\sin(\alpha\omega_1 t)\cos(\omega_1 t) + \frac{1}{\omega_2}\sin(\omega_1 t) - \frac{1}{\omega_1}\sin(\omega_1 t) \\
&\simeq \frac{1}{\omega_2}\sin(\alpha\omega_1 t)\cos(\omega_1 t)
\end{aligned}$$

Similarly

$$\phi_2(t) \sim -\frac{1}{\omega_2}\cos(\alpha\omega_1 t)\sin\omega_2 t$$



Extra-1. Derive the gradient operator in the spherical coordinates.

Extra-2. Solve the equation of motion for harmonic oscillator.

Homework for Chapter 4

Assig.	Problem	Covered Subject
4.1	4.2, 4.5, 4.6, 4.9, 4.17, 4.21, 4.23, Extra-1	Work, Potential