Chapter 3. Momentum and Angular Momentum

3.1 Conservation of Momentum

Consider a system of N particles. Similar to textbook, we use a Greek index α or β to label individual particle. The Newton's equation can be written as

$$\dot{\vec{p}}_\alpha \; = \; \sum_{\beta \neq \alpha}^{N} \vec{F}_{\alpha\beta} + \vec{F}_\alpha^{ext}\,, \qquad\qquad \alpha = 1, \cdots, N \tag{1}$$

where $\vec{p}_\alpha$ is the momentum of particle α, $\vec{F}_{\alpha\beta}$ is the internal interaction between particle α and particle β, and $\vec{F}_\alpha^{ext}$ is the net external force on particle α. The total momentum of a multiparticle system is defined as

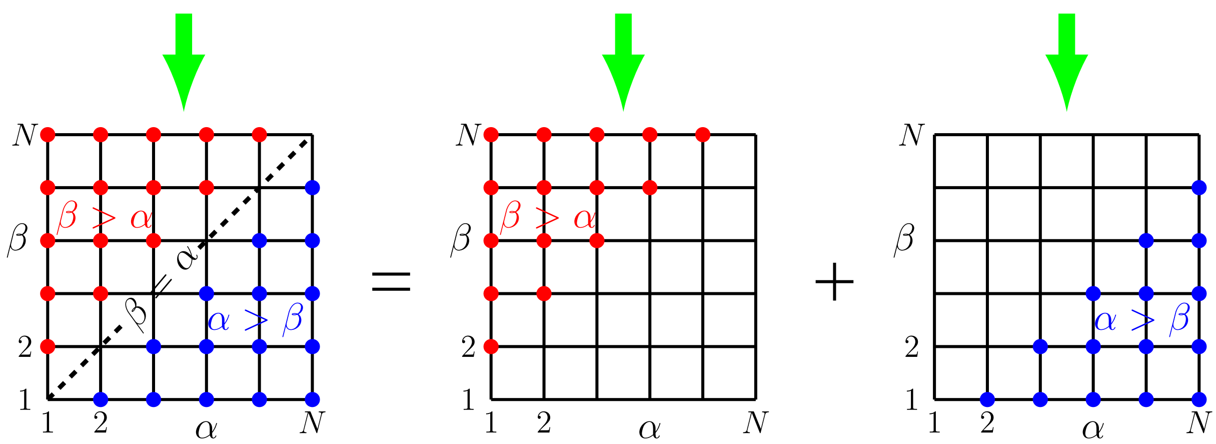
$$\vec{P} \; = \; \sum_{\alpha=1}^{N} \vec{p}_\alpha \tag{2}$$

The equation of motion for the total momentum can be obtained as

$$\dot{\vec{P}} \; = \; \sum_{\alpha=1}^{N} \dot{\vec{p}}_\alpha \; = \; \sum_{\alpha=1}^{N} \sum_{\beta\neq\alpha}^{N} \vec{F}_{\alpha\beta} + \sum_{\alpha=1}^{N} \vec{F}_\alpha^{ext} \tag{3}$$

where the summation of the internal interaction can be rewritten as

$$\sum_{\alpha=1}^{N} \sum_{\beta\neq\alpha}^{N} \vec{F}_{\alpha\beta} \quad = \quad \sum_{\alpha=1}^{N} \sum_{\beta>\alpha}^{N} \vec{F}_{\alpha\beta} \quad + \quad \sum_{\beta=1}^{N} \sum_{\alpha>\beta}^{N} \vec{F}_{\alpha\beta}$$



sum over all nodes with $\alpha \neq \beta$ — sum over red nodes with $\beta > \alpha$ — sum over blue nodes with $\alpha > \beta$

For the 2nd group of the summations for $\alpha > \beta$, we interchange the labelling of the summation indices as

$$\sum_{\beta=1}^{N} \sum_{\alpha>\beta}^{N} \vec{F}_{\alpha\beta} \; = \; \sum_{\alpha=1}^{N} \sum_{\beta>\alpha}^{N} \vec{F}_{\beta\alpha}$$

The summation of the internal interactions in Eq. (3), the equation of motion for $\vec{P}$, can therefore be written as

$$\begin{aligned}\sum_{\alpha=1}^{N}\sum_{\beta\neq\alpha}^{N}\vec{F}_{\alpha\beta} &= \sum_{\alpha=1}^{N}\sum_{\beta>\alpha}^{N}\vec{F}_{\alpha\beta}+\sum_{\beta=1}^{N}\sum_{\alpha>\beta}^{N}\vec{F}_{\alpha\beta}\\ &= \sum_{\alpha=1}^{N}\sum_{\beta>\alpha}^{N}\vec{F}_{\alpha\beta}+\sum_{\alpha=1}^{N}\sum_{\beta>\alpha}^{N}\vec{F}_{\beta\alpha} = \sum_{\alpha=1}^{N}\sum_{\beta>\alpha}^{N}\left(\vec{F}_{\alpha\beta}+\vec{F}_{\beta\alpha}\right) = 0 \end{aligned} \tag{4}$$

where

$$\vec{F}_{\alpha\beta}+\vec{F}_{\beta\alpha} = 0 \tag{5}$$

because $\vec{F}_{\alpha\beta}$ and $\vec{F}_{\beta\alpha}$ are the action and reaction between particle α and particle β, *i.e.*

$$\vec{F}_{\alpha\beta} = -\frac{\partial}{\partial\vec{r}_\alpha}U(|\vec{r}_\alpha-\vec{r}_\beta) = \frac{\partial}{\partial\vec{r}_\beta}U(|\vec{r}_\alpha-\vec{r}_\beta) = -\vec{F}_{\beta\alpha}$$

where $U(|\vec{r}_\alpha-\vec{r}_\beta)$ is the interaction potential between particle α and β. Therefore, the internal interaction does not affect the motion of the total momentum of a multiparticle system. The equation of motion for the total momentum is therefore

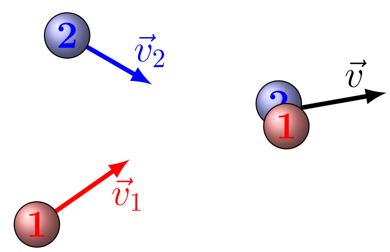
$$\dot{\vec{P}} = \vec{F}^{ext} \qquad \text{where} \qquad \vec{F}^{ext} = \sum_{\alpha=1}^{N}\vec{F}^{ext}_{\alpha} \tag{6}$$

If the net external force on a system is zero, the system total momentum is a constant of motion — **conservation of momentum**.

Example 3.1 An Inelastic Collision of Two Bodies

Two bodies have mass m_1 and m_2 and velocities $\vec{v}_1$ and $\vec{v}_2$ initially. They collide and move together after the collision. The total momentum before the collision is

$$\vec{P}_i = m_1\vec{v}_1 + m_2\vec{v}_2$$



and after the collision is

$$\vec{P}_f = (m_1+m_2)\vec{v}$$

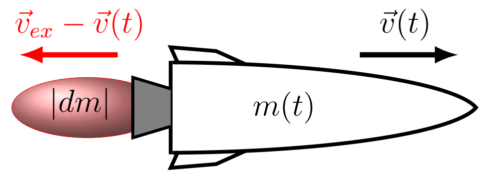
Consider that there is no external force or external force (such as the gravity) is negligible for a very short time interval of the collision. The conservation of the total momentum, $\vec{P}_i = \vec{P}_f$, yields

$$m_1\vec{v}_1 + m_2\vec{v}_2 = (m_1+m_2)\vec{v} \qquad \Longrightarrow \qquad \vec{v} = \frac{m_1\vec{v}_1+m_2\vec{v}_2}{m_1+m_2}$$

3.2 Rocket

A rocket is accelerating forward by ejecting burning fuel backward while the mass of the rocket decreases because of the burning of the fuel. Consider a rocket is traveling in the x direction with velocity $\vec{v} = v(t)\,\vec{e}_x$ and mass $m(t)$ at time t. The total momentum of the rocket is

$$\vec{P}(t) = m(t)v(t)\,\vec{e}_x$$



$$dm = m(t+dt) - m(t) < 0$$
$$\vec{v}(t) = v(t)\,\vec{e}_x$$
$$\vec{v}_{ex} - \vec{v}(t) = -[v_{ex} - v(t)]\,\vec{e}_x$$

During a time interval of $t \longrightarrow t+dt$, the amount of the burning fuel ejected from the rocket is $|dm|$ while the mass of the rocket reduces from $m(t)$ to $m(t+dt)$, where $dm = m(t+dt) - m(t) < 0$. The ejected burning fuel travels with the exhaust speed $-v_{ex}\,\vec{e}_x$ relative to the rocket or $-[v_{ex} - v(t)]\,\vec{e}^{\,\prime}_x$, relative to the ground. The total momentum of the system (rocket and burning fuel) at $t+dt$ is then

$$\begin{aligned}
\vec{P}(t+dt) &= m(t+dt)v(t+dt)\,\vec{e}_x - |dm|\,[v_{ex} - v(t)]\,\vec{e}_x \\
&= \left\{ m(t)v(t) + v(t)\frac{dm}{dt}\,dt + m(t)\frac{dv}{dt}\,dt + dm\,[v_{ex} - v(t)] \right\} \vec{e}_x \\
&= [m(t)v(t) + v_{ex}\,dm + m(t)\,dv]\;\vec{e}_x \\
&= \vec{P}(t) + [v_{ex}\,dm + m(t)\,dv]\;\vec{e}_x
\end{aligned}$$

Since the gravity is insignificant during the infinitesimal time interval dt, the total momentum of the system is constant, *i.e.*

$$\vec{P}(t) = \vec{P}(t+dt) \qquad \Longrightarrow \qquad v_{ex}\,dm + m(t)\,dv = 0 \qquad \Longrightarrow \qquad dv = -v_{ex}\frac{dm}{m}$$

Dividing both sides by dt yields the equation of motion for a rocket when the force from the rocket engine is dominant

$$m\dot{v} \;=\; -\dot{m}\,v_{ex} \qquad \text{where} \quad \text{thrust} = -\dot{m}\,v_{ex} \tag{7}$$

If the speed v_{ex} of the ejected burning fuel is constant, the speed of the rocket after burning $(m - m_0)$ amount of the fuel is

$$v \;=\; v(0) + v_{ex}\ln\left(m_0/m\right)$$

where m_0 is initial mass of the rocket.

3.3 Center of Mass (CM)

The center of mass of a multiparticle system is defined as

$$\vec{R} \;=\; \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \vec{r}_\alpha \;=\; \frac{m_1\vec{r}_1 + \cdots + m_N\vec{r}_N}{m_1 + \cdots + m_N} \tag{8}$$

where $M = m_1 + \cdots + m_N$ is the total mass of the system. The total momentum of the system can then be written as

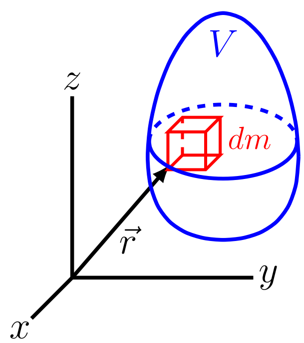
$$\vec{P} \;=\; \sum_{\alpha=1}^{N} \vec{P}_\alpha \;=\; \sum_{\alpha=1}^{N} m_\alpha \dot{\vec{r}}_\alpha \;=\; M\dot{\vec{R}} \tag{9}$$

i.e. the total momentum of a system is in fact the momentum of the center of mass. The Newton's equation for the center of mass of a system is thus

$$\vec{F}^{ext} \;=\; \dot{\vec{P}} \;=\; M\ddot{\vec{R}} \tag{10}$$

The motion of the center of mass of a system depends only on the net external force and is independent of the internal interactions.

Center of Mass of a Rigid Body



For a rigid body with continuously distributed mass, if the mass density of the body is $\mathcal{D}(\vec{r})$, the mass in an infinitesimal volume element is

$$dm \;=\; \mathcal{D}(\vec{r})\, dV$$

and the summation in the definition of the CM becomes a volume integral over the rigid body,

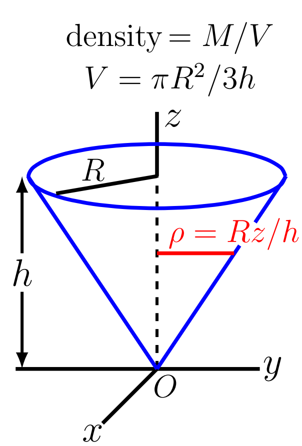
$$\vec{R} \;=\; \frac{1}{M}\sum_{\alpha} \vec{r}_\alpha\,(dm)_\alpha \;\overset{dm\to 0}{=}\; \frac{1}{M}\int\limits_V \vec{r}\, dm \;=\; \frac{1}{M}\int\limits_V \vec{r}\,\mathcal{D}(\vec{r})\, dV \tag{11}$$

where V is the volume of the body and the volume element dV in Cartesian, cylindrical, and spherical coordinates is

$$dV \;=\; dx\,dy\,dz \;=\; \rho\, d\rho\, d\phi\, dz \;=\; r^2 \sin\theta\, dr\, d\theta\, d\phi$$

where $(\,x\,,\,y\,,\,z\,) = (\,\rho\cos\phi\,,\,\rho\sin\phi\,,\,z\,) = (\,r\sin\theta\cos\phi\,,\,r\sin\theta\sin\phi\,,\,r\cos\theta\,)$. For a rigid body with a uniformly distributed mass, $\mathcal{D}(\vec{r}) = M/V$.

Example 3.2 Center of Mass of a Solid Cone with Uniform Mass Distribution



Since the uniform cone is rotational symmetric around the z axis, it is easier to calculate the integral in Eq. (11) in the cylindrical coordinates, where the equation for the surface of the cone is

$$\rho = Rz/h$$

The CM can thus be calculated from

$$\vec{R} \;=\; \frac{1}{V}\int\limits_V \vec{r}\,\rho\, d\rho\, d\phi\, dz \;=\; \frac{1}{V}\int_0^h dz \int_0^{Rz/h} \rho\, d\rho \int_0^{2\pi} \vec{r}\, d\phi$$

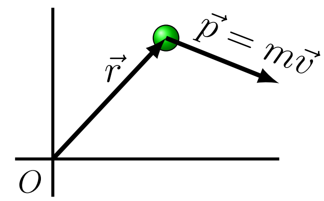
where $\vec{r} = (\,\rho\cos\phi\,,\,\rho\sin\phi\,,z\,)$. Due to the cylindrical symmetry, the x and y coordinate of CM is at $X = Y = 0$, for example,

$$X \;=\; \frac{1}{V}\int_0^h dz \int_0^{Rz/h} \rho\, d\rho \int_0^{2\pi} x\, d\phi \;=\; \frac{1}{V}\int_0^h dz \int_0^{Rz/h} \rho^2 d\rho \int_0^{2\pi} \cos\phi\, d\phi \;=\; 0$$

The z component of the CM is nonzero,

$$Z \;=\; \frac{1}{V}\int_0^h dz \int_0^{Rz/h} \rho\, d\rho \int_0^{2\pi} z\, d\phi \;=\; \frac{2\pi}{V}\int_0^h z dz \int_0^{Rz/h} \rho\, d\rho \;=\; \frac{\pi R^2 h^2}{4V} \;=\; \frac{3}{4}h$$

3.4 Angular Momentum of a Single Particle



The angular momentum of a particle is defined as

$$\vec{L} \;=\; \vec{r}\times\vec{p} \qquad \text{where} \qquad \vec{p} = m\vec{v} \tag{12}$$

Note that the angular momentum depends on the location of the coordinate origin. This is very different from the linear momentum $\vec{p}$ that is independent of the origin in the inertial frames when the equation of motion is concerned. The **origin-dependence of the angular momentum** is important in solving some problems: the coordinate origin in some system can be chosen to make the angular momentum a constant of motion.

The equation of motion for the angular momentum can be obtained as

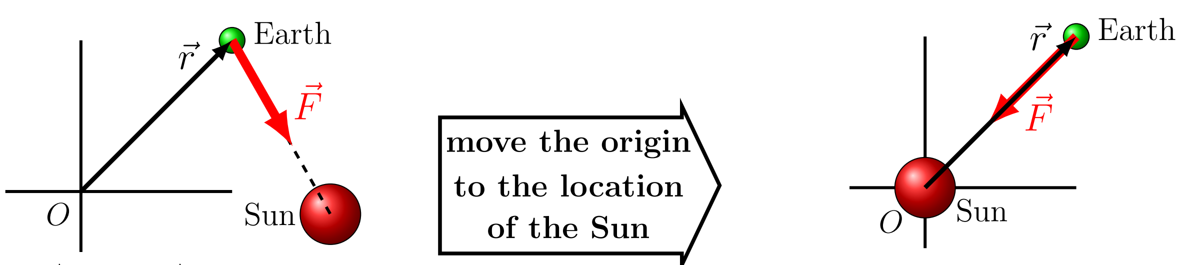
$$\frac{d}{dt}\left(\vec{r}\times\vec{p}\right) \;=\; \dot{\vec{r}}\times\vec{p} + \vec{r}\times\dot{\vec{p}} \;=\; \vec{r}\times\dot{\vec{p}} \;=\; \vec{r}\times\vec{F} \qquad \text{where} \qquad \vec{F} = \dot{\vec{p}}$$

The equation of motion for the angular momentum is therefore

$$\vec{\Gamma} \;=\; \dot{\vec{L}} \qquad \text{where} \qquad \vec{\Gamma} = \vec{r}\times\vec{F} \tag{13}$$

$\vec{\Gamma}$ is the net torque on a particle for the rotational motion around the origin. When $\vec{\Gamma} = 0$, the angular momentum $\vec{L}$ is a constant of motion — **conservation of angular momentum**. Since both the angular momentum and torque depend on the location of the origin, if the interaction is in a fixed direction when viewed in a certain inertial frame, one could choose a location of the origin to make $\vec{\Gamma} = 0$.

3.4(a) Interaction between a Moving Particle and a Fixed Object (Central-Force)



$\vec{\Gamma} = \vec{r}\times\vec{F} \neq 0$ $\qquad$ $\vec{\Gamma} = \vec{r}\times\vec{F} = 0 \longrightarrow$ Kepler's Laws

3D Newton's Equation $\qquad$ Reduced to 1D Newton's Equation

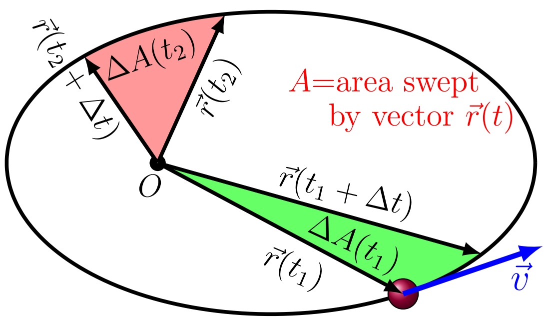
In the central-force problems, if the origin of the coordinate is chosen to be at the source of the force, the equation of motion for a single particle is simply

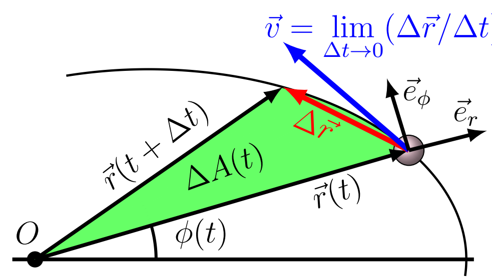
$$m\ddot{\vec{r}} \;=\; F(r)\,\vec{e}_r \tag{14}$$

Moreover, the torque from the central force is zero and the angular momenta, $\vec{p}_\theta$ and $\vec{p}_\phi$ in the spherical coordinates, are the constants of motion. The three-dimensional Newton's equations in Eq. (14) can be reduced to a one-dimensional Newton's equation (Example 4 in Chapter 1 of Lecture Note), that is a motion on a plane and always solvable. The planetary motion is one example of central-force problem in which the Kepler's 2nd law is the consequence of the conservation of the angular momentum.

Kepler's 2nd Law of Planetary Motion:

"In a planetary motion, the radius vector $\vec{r}(t)$ joining the planet to the Sun sweeps out equal areas in equal lengths of time."





Kepler's 2nd Law: $\dfrac{\Delta A(t_1)}{\Delta t} = \dfrac{\Delta A(t_2)}{\Delta t}$

$$\Longrightarrow \qquad \frac{dA}{dt} = \frac{1}{2m}|\vec{L}\,| = constant$$

$$\Delta\vec{r} = \vec{r}(t+\Delta t) - \vec{r}(t) = \vec{v}(t)\Delta t$$

$$\Delta A = \frac{1}{2}|\vec{r}\times\Delta\vec{r}\,| = \frac{1}{2}|\vec{r}\times\vec{v}\,|\Delta t$$

$$\Longrightarrow \quad \frac{dA}{dt} = \frac{1}{2}|\vec{r}\times\vec{v}\,| = \frac{1}{2m}|\vec{L}\,|$$

In the cylindrical (polar in 2D) coordinate, $\vec{r} = r\,\vec{e}_r$ and

$$\vec{v} = r\dot{\phi}\,\vec{e}_\theta + \dot{r}\,\vec{e}_r \qquad \text{(in our Chapter 1 notation: } \vec{v} = \rho\,\dot{\phi}\,\vec{e}_\theta + \dot{\rho}\,\vec{e}_\rho\text{)}$$

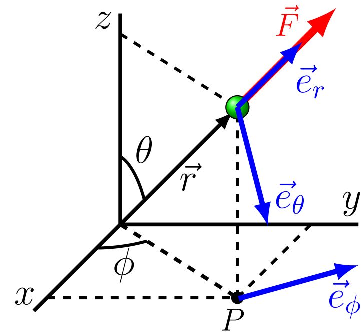
Therefore, in the Kepler problem

$$L = m|\vec{r}\times\vec{v}\,| = mr^2\omega = constant \qquad \text{where} \quad \omega = \dot{\phi}$$

3.4(b) General Calculation of Angular Momentum in Central-Force Problem

The central-force problem such as the planetary motion, hydrogen atom, ..., in general, considers a particle moving in a 3-dimensional space with a force $\vec{F} = F(r)\,\vec{e}_r$ that is originated from the origin of the coordinate and parallel or anti-parallel to the position vector $\vec{r}$ of the particle. Because the force on the particle is independent of the angular variables

θ and ϕ in the spherical coordinate (the spherical symmetry of the force), the spherical coordinates is the most appropriate coordinates for studying the central-force problem. In the spherical coordinates, the position and velocity of a particle are



$$\begin{cases} \vec{r} \;=\; r\,\vec{e}_r \\ \vec{v} \;=\; \dot{r}\,\vec{e}_r + r\dot{\theta}\,\vec{e}_\theta + (r\dot{\phi}\sin\theta)\,\vec{e}_\phi \end{cases}$$

The angular momentum of a particle can easily be calculated as

$$\begin{aligned} \vec{L} \;=\; m(\vec{r}\times\vec{v}) \;&=\; (mr^2\dot{\theta})\,(\vec{e}_r\times\vec{e}_\theta) + (mr^2\dot{\phi}\sin\theta)\,(\vec{e}_r\times\vec{e}_\phi) \\ &=\; (mr^2\dot{\theta})\,\vec{e}_\phi - (mr^2\dot{\phi}\sin\theta)\,\vec{e}_\theta \end{aligned}$$

where $\vec{e}_r\times\vec{e}_\theta = \vec{e}_\phi$ and $\vec{e}_r\times\vec{e}_\phi = -\vec{e}_\theta$. In the spherical coordinates, the angular momentum of a single particle has thus two components

$$L_\phi = mr^2\dot{\theta} \qquad \text{and} \qquad L_\theta = -mr^2\dot{\phi}\sin\theta$$

Because the torque of the force is zero the angular momentum of the particle is a constant of motion,

$$\vec{F} = F(r)\,\vec{e}_r \qquad \Longrightarrow \qquad \vec{\Gamma} = \vec{r}\times\vec{F} = 0 \qquad \Longrightarrow \qquad \vec{L} = \mathit{constant}$$

The components of the angular momentum L_θ and L_ϕ in the spherical coordinates are, however, not necessarily constant since the directions of base vectors $\vec{e}_\theta$ and $\vec{e}_\phi$ changes with the motion. To obtain the constant components of a constant angular momentum, we have to transfer the angular momentum in the spherical coordinate back to the Cartesian coordinates. The transformation of $\vec{e}_\theta$ and $\vec{e}_\phi$ to the Cartesian coordinates is

$$\begin{cases} \vec{e}_\phi \;=\; -\vec{e}_x\,\sin\phi + \vec{e}_y\,\cos\phi \\ \vec{e}_\theta \;=\; \vec{e}_x\,\cos\theta\cos\phi + \vec{e}_y\,\cos\theta\sin\phi - \vec{e}_z\,\sin\theta \end{cases} \tag{15}$$

The Cartesian components of the angular momentum calculated using the spherical coordinates can be obtained as

$$\begin{aligned} \vec{L} \;&=\; L_\theta\,\vec{e}_\theta + L_\phi\,\vec{e}_\phi \\ &=\; \vec{e}_x\,(L_\theta\,\cos\theta\cos\phi - L_\phi\,\sin\phi) + \vec{e}_y\,(L_\theta\,\cos\theta\sin\phi + L_\phi\,\cos\phi) - \vec{e}_z\,L_\theta\,\sin\theta \end{aligned} \tag{16}$$

The conservation of the angular momentum, $\vec{L} = \mathit{constant}$, yields

$$\begin{cases} L_x \;=\; L_\theta\,\cos\theta\cos\phi - L_\phi\,\sin\phi \;=\; -mr^2\left(\dot{\phi}\sin\theta\cos\theta\cos\phi + \dot{\theta}\sin\phi\right) \;=\; \mathit{constant} \\ L_y \;=\; L_\theta\,\cos\theta\sin\phi + L_\phi\,\cos\phi \;=\; -mr^2\left(\dot{\phi}\sin\theta\cos\theta\sin\phi - \dot{\theta}\cos\phi\right) \;=\; \mathit{constant} \\ L_z \;=\; -L_\theta\,\sin\theta \;=\; mr^2\dot{\phi}\,\sin^2\theta \;=\; \mathit{constant} \end{cases}$$

where L_x and L_y are not two independent constants because they can be switched into each other by redefining the zero point of angular variable ϕ as $\sin(\phi\pm\pi/2) = \pm\cos\phi$

and $\cos(\phi \pm \pi/2) = \mp \sin\phi$. The zero point of ϕ here is arbitrary because the system is rotationally symmetric. These three constant components of the angular momentum yields two linearly-independent constants of motion $L_z = constant$ and $(L_x \text{ or } L_y) = constant$. Since

$$L^2 = L_\phi^2 + L_\theta^2 = L_x^2 + L_y^2 + L_z^2 = m^2 r^4 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) \tag{17}$$

is also a constant but not independent of $(L_x, L_y, L_z) = constants$, for the central-force problem, the two linearly-independent constants of motion from the conservation of the angular momentum are commonly chosen as the total angular momentum and the z-component angular momentum,

$$\begin{cases} L^2 = m^2 r^4 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) = constant \\ L_z = m r^2 \dot{\phi} \sin^2\theta = constant \end{cases} \tag{18}$$

Integrable Systems

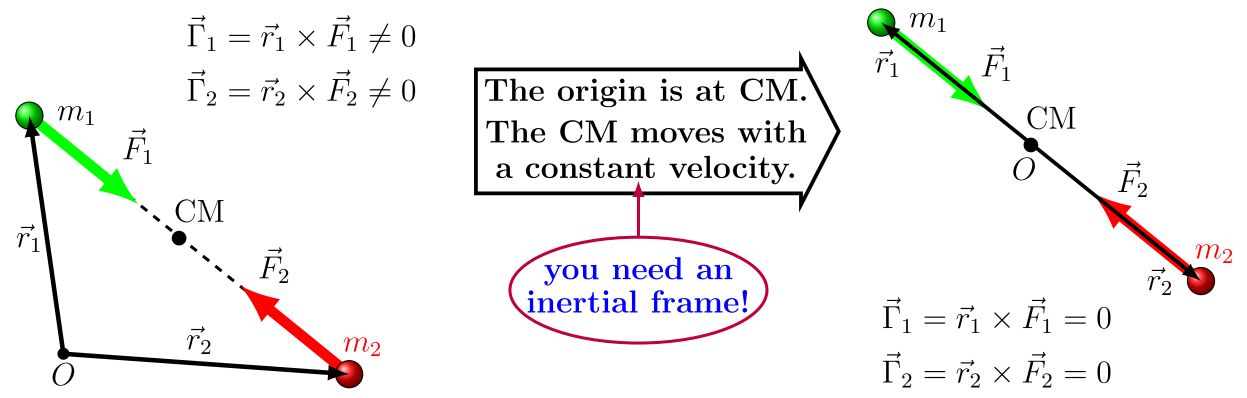
Mathematically, a system of N-dimensional Newton's equation (2nd-order ODE) is solvable, called integrable, if there are N linearly-independent constants of motion. The constant of motion for the Newton's equation is also called the first integral for ODE. The equation of motion for the central-force problem is a 3-dimensional Newton's equation. Because of the two constants of motion from the conservation of the angular momentum and an additional constant of motion from the conservation of the mechanical energy, the central-force problem is a solvable system. In quantum mechanics, each quantum number for labeling a quantum state of a system corresponds to a constant of motion of the system in the classical mechanics. For the hydrogen atom that is a central-force problem in quantum mechanics, there are three quantum numbers (n, l, m) for each state of the electron orbital motion. These three quantum number correspond to three constants of motion, n for the energy, l for the total angular momentum, and m for the z-component of the angular momentum.

3.4(c) Two-Body Problem
— Interaction between Two Moving Objects without External Force

For a system of two objects with only interaction between two objects, the equations of motion in an arbitrary coordinates is

$$\begin{cases} m_1 \ddot{\vec{r}}_1 = -F(|\vec{r}_2 - \vec{r}_1|)\, \vec{e}_r \\ m_2 \ddot{\vec{r}}_2 = F(|\vec{r}_2 - \vec{r}_1|)\, \vec{e}_r \end{cases} \quad \text{with} \quad \vec{e}_r = \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|} \tag{19}$$

where $\vec{r}_1$ and $\vec{r}_2$ are the position vectors of two objects. Because the interaction F is a function of $\vec{r}_1$ and $\vec{r}_2$, these six 2nd-order differential equations are in general coupled and cannot be solved in an arbitrary coordinate.



Two-Body Problem in Center-of-Mass Coordinate

The CM coordinate is a coordinate with its origin at and moving with the CM. Without the net external force, the CM momentum of two objects is a constant of motion and the CM coordinate is an inertial frame. With three constant components of the CM momentum, the original six-dimensional coupled Newton's equation can be reduced to a three-dimensional Newton's equation in the CM coordinate. In the CM coordinate, the torque of the interaction between two objects is zero, the angular momentum is another constant of motion and the three-dimensional Newton's equation in the CM coordinate is further reduced to a one-dimensional Newton's equation, in which Kepler's problem is an example. To derive the equation of motion from an arbitrary to the CM coordinate, we introduce two new vector variables for $\vec{r}_1$ and $\vec{r}_2$,

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \qquad \text{and} \qquad \vec{r} = \vec{r}_2 - \vec{r}_1 \tag{20}$$

where $\vec{R}$ is the position vector of the CM in an arbitrary coordinate while $\vec{r}$ is the vector connecting two objects and a position vector in the CM coordinate. Note that $\vec{r} = 0$ corresponds to the origin of the CM coordinate. The transformation from vectors $(\vec{R}\,,\,\vec{r}\,)$ to $(\vec{r}_1\,,\,\vec{r}_2)$ is

$$\vec{r}_1 = \vec{R} - \frac{m_2}{m_1 + m_2}\vec{r} \qquad \text{and} \qquad \vec{r}_2 = \vec{R} + \frac{m_1}{m_1 + m_2}\vec{r} \tag{21}$$

The equation of motion for two objects in terms of $(\vec{R}\,,\,\vec{r}\,)$ can be obtained as

$$\ddot{\vec{R}} \;=\; \frac{1}{m_1 + m_2}\left(m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2\right) \;=\; 0 \qquad \Longrightarrow \qquad \dot{\vec{R}} = constant \tag{22}$$

$$\ddot{\vec{r}} \;=\; \ddot{\vec{r}}_2 - \ddot{\vec{r}}_1 \;=\; \frac{1}{m_2}F(r)\,\vec{e}_r + \frac{1}{m_1}F(r)\,\vec{e}_r \;=\; \left(\frac{m_1 + m_2}{m_1 m_2}\right)F(r)\,\vec{e}_r \tag{23}$$

With the CM coordinate, the two-body problem is reduced to a three-dimensional central-force problem of an object with mass μ,

$$\mu\,\ddot{\vec{r}} \;=\; F(r)\,\vec{e}_r \qquad \text{where} \quad \mu = \frac{m_1 m_2}{m_1 + m_2} \tag{24}$$

and μ is called reduced mass. For the system of the earth and the sun, $m_{sun}/m_{earth} \sim 3\times 10^5$ and $\mu = m_{earth}$. For two oxygen atoms in an oxygen molecule or two similar stars in a binary star, on the other hand, $m_1 = m_2$ and $\mu = m_1/2$.

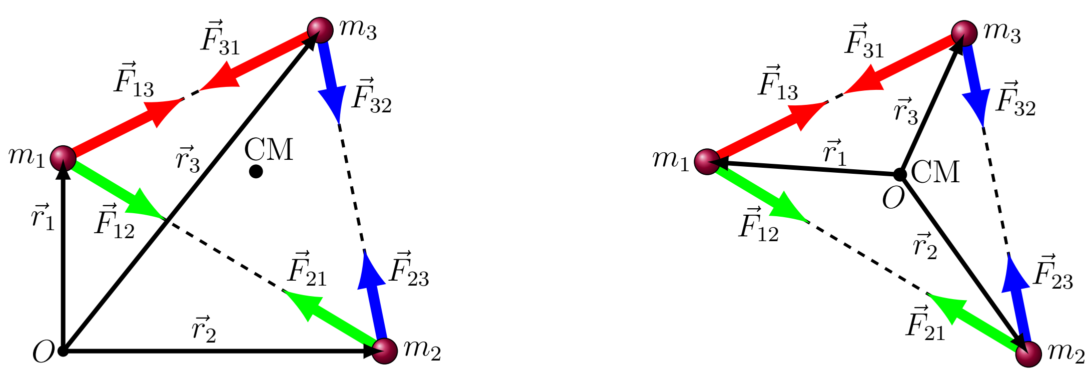
With the conservation of the angular momenta because $\vec{\Gamma} = \vec{r}\times\vec{F} = 0$, the 3-dimensional Newton's equation in Eq. (24) is further reduced to a 1-dimensional Newton's equation, that is always solvable.

3.4(d) Three-Body Problem — Interactions in a System of Three Bodies

For a system of three particles with only internal interaction between the particles, the equations of motion in an arbitrary coordinates is

$$\begin{cases} m_1\ddot{\vec{r}}_1 &= \vec{F}_{12}(|\vec{r}_1-\vec{r}_2|)+\vec{F}_{13}(|\vec{r}_1-\vec{r}_3|) \\ m_2\ddot{\vec{r}}_2 &= \vec{F}_{21}(|\vec{r}_2-\vec{r}_1|)+\vec{F}_{23}(|\vec{r}_2-\vec{r}_3|) \\ m_3\ddot{\vec{r}}_3 &= \vec{F}_{31}(|\vec{r}_3-\vec{r}_1|)+\vec{F}_{32}(|\vec{r}_3-\vec{r}_2|) \end{cases} \tag{25}$$

where $\vec{r}_1$, $\vec{r}_2$, and $\vec{r}_3$ are the position vectors of three particles. Because the interaction $\vec{F}_{ij}$ is a function of $\vec{r}_i$ and $\vec{r}_j$, these nine 2nd-order differential equations are in general coupled and cannot be solved in any coordinate.



The origin is not at the CM. The origin is at the CM.

No matter where the origin is, in general, the torques on the particles do not vanish in the three-body problem,

$$\begin{cases} \vec{\Gamma}_1 &= \vec{r}_1\times\vec{F}_{12}+\vec{r}_1\times\vec{F}_{13} &= \vec{r}_1\times(\vec{F}_{12}+\vec{F}_{13}) &\neq 0 \\ \vec{\Gamma}_2 &= \vec{r}_2\times\vec{F}_{21}+\vec{r}_2\times\vec{F}_{23} &= \vec{r}_2\times(\vec{F}_{21}+\vec{F}_{23}) &\neq 0 \\ \vec{\Gamma}_3 &= \vec{r}_3\times\vec{F}_{31}+\vec{r}_3\times\vec{F}_{32} &= \vec{r}_3\times(\vec{F}_{31}+\vec{F}_{32}) &\neq 0 \end{cases}$$

The total angular momentum

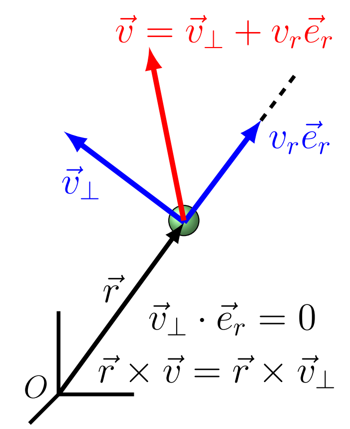
$$\vec{L} = \vec{L}_1+\vec{L}_2+\vec{L}_3 \tag{26}$$

is constant since

$$\begin{aligned}\dot{\vec{L}} \;=\; \dot{\vec{L}}_1+\dot{\vec{L}}_2+\dot{\vec{L}}_3 \;&=\; \vec{r}_1\times(\vec{F}_{12}+\vec{F}_{13})+\vec{r}_2\times(\vec{F}_{21}+\vec{F}_{23})+\vec{r}_3\times(\vec{F}_{31}+\vec{F}_{32})\\ &=\; (\vec{r}_1-\vec{r}_2)\times\vec{F}_{12}+(\vec{r}_1-\vec{r}_3)\times\vec{F}_{13}+(\vec{r}_2-\vec{r}_3)\times\vec{F}_{23} \;=\; 0\end{aligned}$$

where $\vec{F}_{ij}=-\vec{F}_{ji}$ and $(\vec{r}_i-\vec{r}_j)\times\vec{F}_{ij}=0$ because the interaction between two particles is along the line connecting two particles. With three constants of motion for the CM momentum, two linearly-independent constants of motion from the total angular momentum, and additional one from the conservation of energy, the nine coupled Newton's equations in the three-body problem can be reduced to three coupled nonlinear 2nd-order differential equations. Without knowing any other constant of motion, we don't know how to reduce those equations to a one-dimensional problem. The three-body problem is therefore unsolvable.

3.4(e) Angular Velocity

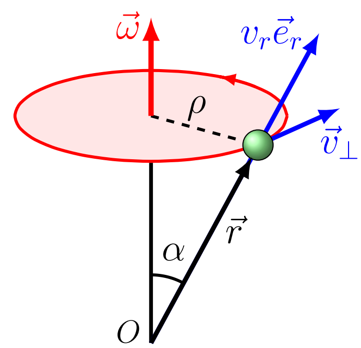


For a particle at position $\vec{r}$ moving with velocity $\vec{v}$, in general, the velocity $\vec{v}$ can always be decomposed into two vectors, one is the velocity component in $\vec{r}$ direction and the other is a velocity vector perpendicular to $\vec{r}$,

$$\vec{v} \;=\; v_r\,\vec{e}_r+\vec{v}_\perp \qquad \text{where} \;\; \vec{v}_\perp\cdot\vec{e}_r=0 \tag{27}$$

For example, in the spherical coordinates,

$$\begin{cases}\vec{v} \;=\; v_r\,\vec{e}_r+v_\theta\,\vec{e}_\theta+v_\phi\,\vec{e}_\phi \;=\; v_r\,\vec{e}_r+\vec{v}_\perp\\ \vec{v}_\perp \;=\; v_\theta\,\vec{e}_\theta+v_\phi\,\vec{e}_\phi\end{cases} \tag{28}$$



With the perpendicular velocity $\vec{v}_\perp$, one can introduce an angular velocity $\vec{\omega}$ that is perpendicular to $\vec{v}_\perp$,

$$\vec{v}_\perp \;=\; \vec{\omega}\times\vec{r} \tag{29}$$

Note that the angular velocity is not uniquely defined as the magnitude ω depends on the choice of the direction of $\vec{\omega}$ that is parameterized by the angle α between $\vec{\omega}$ and $\vec{r}$,

$$v_\perp=(r\sin\alpha)\,\omega=\rho\,\omega \qquad \text{with} \quad \rho=r\sin\alpha$$

Relationship between Angular Momentum and Angular Velocity

With the velocity written in the component in the $\vec{r}$ direction and a velocity vector perpendicular to $\vec{r}$, the relationship between the angular momentum and the angular velocity of a particle can be obtained as

$$\vec{L} \;=\; m(\vec{r}\times\vec{v}\,) \;=\; m\,[\vec{r}\times(v_r\vec{e}_r+\vec{v}_\perp)] \;=\; m\,[\vec{r}\times\vec{v}_\perp] \;=\; m\,[\vec{r}\times(\vec{\omega}\times\vec{r}\,)] \tag{30}$$

3.5 Angular Momentum of a Multiparticle System

For a system of N particles, the total angular momentum of the system is

$$\vec{L} = \sum_{\alpha=1}^{N} \vec{L}_\alpha = \sum_{\alpha=1}^{N} (\vec{r}_\alpha \times \vec{p}_\alpha) \tag{31}$$

where $\vec{L}_\alpha = \vec{r}_\alpha \times \vec{p}_\alpha$ is the angular momentum of particle α. The equation of motion for $\vec{L}$ is

$$\frac{d\vec{L}}{dt} = \sum_{\alpha=1}^{N} \frac{d\vec{L}_\alpha}{dt} = \sum_{\alpha=1}^{N} \vec{r}_\alpha \times \vec{F}_\alpha = \sum_{\alpha=1}^{N} \sum_{\beta\neq\alpha}^{N} (\vec{r}_\alpha \times \vec{F}_{\alpha\beta}) + \sum_{\alpha=1}^{N} (\vec{r}_\alpha \times \vec{F}_\alpha^{ext}) \tag{32}$$

where $\vec{F}_\alpha = \vec{F}_{\alpha\beta} + \vec{F}_\alpha^{ext}$ is the net force on particle α, $\vec{F}_{\alpha\beta}$ is the internal interaction between particle α and particle β, and $\vec{F}_\alpha^{ext}$ is the external force on particle α. For the double summations in the 1st term, it has been shown in Section 3.1,

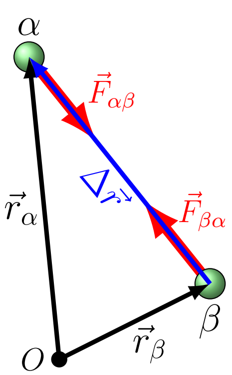
$$\sum_{\alpha=1}^{N} \sum_{\beta\neq\alpha}^{N} (\cdots) = \sum_{\alpha=1}^{N} \sum_{\beta>\alpha}^{N} (\cdots) + \sum_{\beta=1}^{N} \sum_{\alpha>\beta}^{N} (\cdots)$$

where $(\cdots)$ can be any function for the summations. Therefore, the summation of the internal interactions in Eq. (32) is

$$\begin{aligned}
\sum_{\alpha=1}^{N} \sum_{\beta\neq\alpha}^{N} (\vec{r}_\alpha \times \vec{F}_{\alpha\beta}) &= \sum_{\alpha=1}^{N} \sum_{\beta>\alpha}^{N} (\vec{r}_\alpha \times \vec{F}_{\alpha\beta}) + \sum_{\beta=1}^{N} \sum_{\alpha>\beta}^{N} (\vec{r}_\alpha \times \vec{F}_{\alpha\beta}) \\
&= \sum_{\alpha=1}^{N} \sum_{\beta>\alpha}^{N} (\vec{r}_\alpha \times \vec{F}_{\alpha\beta}) + \sum_{\alpha=1}^{N} \sum_{\beta>\alpha}^{N} (\vec{r}_\beta \times \vec{F}_{\beta\alpha}) \\
&= \sum_{\alpha=1}^{N} \sum_{\beta>\alpha}^{N} (\vec{r}_\alpha \times \vec{F}_{\alpha\beta}) - \sum_{\alpha=1}^{N} \sum_{\beta>\alpha}^{N} (\vec{r}_\beta \times \vec{F}_{\alpha\beta}) \\
&= \sum_{\alpha=1}^{N} \sum_{\beta>\alpha}^{N} (\vec{r}_\alpha - \vec{r}_\beta) \times \vec{F}_{\alpha\beta} = 0
\end{aligned}$$

exchange index labels

$\vec{F}_{\beta\alpha} = -\vec{F}_{\alpha\beta}$



where

$$(\vec{r}_\alpha - \vec{r}_\beta) \times \vec{F}_{\alpha\beta} = 0$$

because the distance vector $\Delta\vec{r} = \vec{r}_\alpha - \vec{r}_\beta$ between particle α and particle β is parallel to the interaction between the two particles. The equation of motion for the total angular momentum of a multiparticle system is thus

$$\frac{d\vec{L}}{dt} = \vec{\Gamma}^{ext} \tag{33}$$

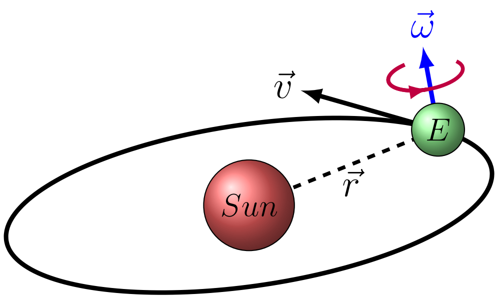
where $\vec{\Gamma}^{ext}$ is the net external torque of all the external forces on the system,

$$\vec{\Gamma}^{ext} = \sum_{\alpha=1}^{N} (\vec{r}_\alpha \times \vec{F}_\alpha^{ext}) \tag{34}$$

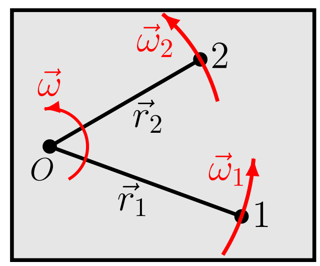
If $\vec{\Gamma}^{ext} = 0$, the system's total angular momentum $\vec{L}(t)$ is a constant of motion.

3.6 Angular Momentum of a Rigid Body

A **rigid body** is a system of particles whose relative distances are absolutely fixed. For many objects, if the changes in size and shape can be neglected during their motion, the objects can be treated as rigid bodies. In general, the motion of a rigid body can be considered to be the sum of two independent or coupled motions, the motion of a reference point, such as the center of mass, plus the motion with respect to that reference point.The motion of the reference point can be treat as a motion of a particle while the motion with respect to the reference point is a rotation around that point as the shape of a rigid body cannot be changed during the motion. Note that the angular velocity of the rotational motion at different points on a rigid body have to be the same, otherwise the shape of the body would be changed during the rotation.



Motion of E = motion around Sun
+ rotation of E around its CM



For a rotating rigid body, if $\vec{\omega}_1 \neq \vec{\omega}_2$, the area between point 1 and point 2 will be deformed.

We now consider a rigid body that is composed of N particles of mass m_α, with $\alpha = 1, 2, \cdots, N$, and rotates around an arbitrary point O with an angular velocity $\vec{\omega}$. For convenience, point O is chose as the origin of the coordinates. The linear velocity of the αth particle with respect to the origin O is

$$\vec{v}_\alpha = \vec{\omega} \times \vec{r}_\alpha \tag{35}$$

where $\vec{r}_\alpha$ is the position vector of the αth particle from the origin O. The total angular momentum of the body for the rotation with respect to the origin O can be written as

$$\vec{L} = \sum_\alpha^N m_\alpha \left[\vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha)\right] = \sum_\alpha^N m_\alpha \left[r_\alpha^2 \, \vec{\omega} - (\vec{r}_\alpha \cdot \vec{\omega}) \, \vec{r}_\alpha \right] \tag{36}$$

which yields

$$L_x = \sum_\alpha m_\alpha \left[\, (y_\alpha^2 + z_\alpha^2) \, \omega_x - x_\alpha y_\alpha \, \omega_y - x_\alpha z_\alpha \, \omega_z \right] \tag{37}$$

$$L_y = \sum_\alpha m_\alpha \left[-y_\alpha x_\alpha \, \omega_x + (x_\alpha^2 + z_\alpha^2) \, \omega_y - y_\alpha z_\alpha \, \omega_z \right] \tag{38}$$

$$L_z \;=\; \sum_\alpha m_\alpha \left[-z_\alpha x_\alpha\,\omega_x - z_\alpha y_\alpha\,\omega_y + (x_\alpha^2 + y_\alpha^2)\,\omega_z\right] \tag{39}$$

where $\vec{\omega} = \omega_x\vec{e}_x + \omega_y\vec{e}_y + \omega_z\vec{e}_z$ is the angular velocity of a rigid body rotating around the origin. We now introduce a 3×3 matrix $\mathbf{I}$, called inertia tensor. Equations (37)–(39) can then be written as a matrix form

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \tag{40}$$

where the inertia matrix $\mathbf{I}$ is a symmetric matrix, $I_{ij} = I_{ji}$, and three diagonal elements are the moment of inertia,

$$\begin{cases} I_{11} = \sum_\alpha m_\alpha (r_\alpha^2 - x_\alpha^2) \\ I_{22} = \sum_\alpha m_\alpha (r_\alpha^2 - y_\alpha^2) \\ I_{33} = \sum_\alpha m_\alpha (r_\alpha^2 - z_\alpha^2) \end{cases} \quad \text{and} \quad \begin{cases} I_{12} = I_{21} = -\sum_\alpha m_\alpha x_\alpha y_\alpha \\ I_{13} = I_{31} = -\sum_\alpha m_\alpha x_\alpha z_\alpha \\ I_{23} = I_{32} = -\sum_\alpha m_\alpha x_\alpha z_\alpha \end{cases} \tag{41}$$

For a continuously mass distributed body, the summations over particles become integrals over the volume of a body,

$$\begin{cases} I_{11} = \int (y^2 + z^2)\,\varrho(\vec{r})\,dV \\ I_{22} = \int (x^2 + z^2)\,\varrho(\vec{r})\,dV \\ I_{33} = \int (x^2 + y^2)\,\varrho(\vec{r})\,dV \end{cases} \quad \text{and} \quad \begin{cases} I_{12} = I_{21} = -\int xy\,\varrho(\vec{r})\,dV \\ I_{13} = I_{31} = -\int xz\,\varrho(\vec{r})\,dV \\ I_{23} = I_{32} = -\int yz\,\varrho(\vec{r})\,dV \end{cases} \tag{42}$$

where $\varrho(\vec{r})$ is the mass density of the body. The volume integral in the Cartesian coordinates is a triple integral

$$\int (\cdots)\,dV = \int_{x_{min}}^{x_{max}} \int_{y_{min}}^{y_{max}} \int_{z_{min}}^{z_{max}} (\cdots)\,dxdydz$$

In the cylindrical coordinates, the volume integral is calculated from

$$\iiint (\cdots)\,dxdydz = \iiint (\cdots)\,|J(\rho,\phi,z)|\,d\phi\,d\rho\,dz = \iiint (\cdots)\,\rho\,d\phi\,d\rho\,dz \tag{43}$$

where $J(\rho,\phi,z)$ is the Jacobian of the transformation from the Cartesian to the cylindrical coordinates, $x = \rho\cos\phi$ and $y = \rho\sin\phi$,

$$J(\rho,\phi,z) \;=\; \det\left[\frac{\partial(x,y,z)}{\partial(\rho,\phi,z)}\right] \;=\; \begin{vmatrix} \cos\phi & \sin\phi & 0 \\ -\rho\sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} \;=\; \rho \tag{44}$$

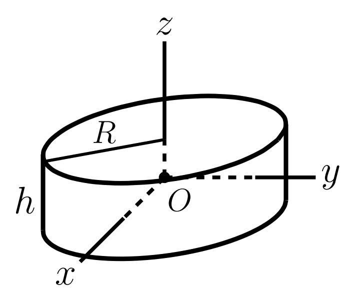
In the spherical coordinates, the volume integral is calculated from

$$\iiint (\cdots)\, dxdydz = \iiint (\cdots)\, |J(r,\theta,\phi)|\, d\theta\, d\phi\, dr = \iiint (\cdots)\, r^2 \sin\theta\, d\theta\, d\phi\, dr \qquad (45)$$

where $J(\rho, \phi, z)$ is the Jacobian of the transformation from the Cartesian to the spherical coordinates, $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$,

$$J(r,\theta,\phi) \;=\; \det\left[\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}\right] \;=\; \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{vmatrix} \;=\; r^2\sin\theta \qquad (46)$$

Example 3.3 A Rotating Uniform Disk or Cylinder Around Its Center



The disk has cylindrical symmetry around it center axis, it is easier to use the cylindrical coordinates for the integrals for inertia matrix. For the uniform density of mass, the mass density of the disk is

$$\varrho \;=\; m/V \;=\; m/(\pi R^2 h)$$

Due to the symmetry of the disk with respect to x-, y-, and z-axis

$$I_{23} \;=\; I_{13} \;=\; I_{23} \;=\; 0$$

This can be easily shown as

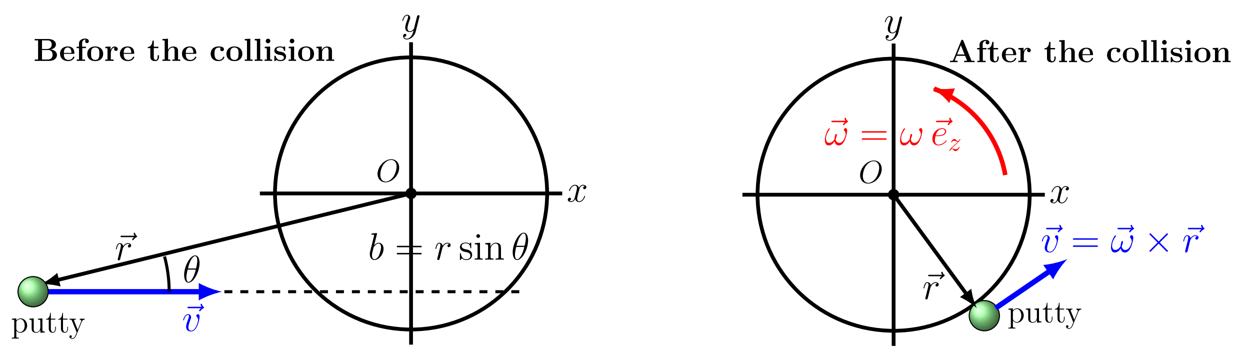
$$I_{12} \;=\; -\frac{m}{\pi R^2 h}\int xy\, dV \;=\; -\frac{m}{\pi R^2 h}\int_{-h/2}^{h/2} dz \int_0^R \rho^3 d\rho \int_0^{2\pi} \sin\theta\cos\theta\, d\theta \;=\; 0$$

The the moment of inertia can be calculated as

$$\begin{aligned}
I_{11} &= \frac{m}{\pi R^2 h}\int (y^2+z^2)\, dV \\
&= \frac{m}{\pi R^2 h}\int_{-h/2}^{h/2} dz \int_0^R \rho\, d\rho \int_0^{2\pi} (\rho^2\sin^2\theta + z^2)\, d\theta \;=\; \frac{\pi}{4}\left(R^2 + \frac{1}{3}h^2\right) \\
I_{22} &= \frac{m}{\pi R^2 h}\int (x^2+z^2)\, dV \\
&= \frac{m}{\pi R^2 h}\int_{-h/2}^{h/2} dz \int_0^R \rho\, d\rho \int_0^{2\pi} (\rho^2\cos^2\theta + z^2)\, d\theta \;=\; \frac{\pi}{4}\left(R^2 + \frac{1}{3}h^2\right) \\
I_{33} &= \frac{m}{\pi R^2 h}\int (x^2+y^2)\, dV \\
&= \frac{m}{\pi R^2 h}\int_{-h/2}^{h/2} dz \int_0^R \rho\, d\rho \int_0^{2\pi} \rho^2\, d\theta \;=\; \frac{\pi}{4}\left(R^2 + \frac{1}{3}h^2\right) \;=\; \frac{1}{2}mR^2
\end{aligned}$$

Because the inertia matrix $\mathbf{I}$ is a diagonal matrix, in this case, I_{11} is the moment of inertia for the rotation with respect to the axis of a cylinder and I_{22} is the moment of inertia for the rotation around an axis passing through the cylinder center along the radial direction.

Example 3.4 A Lump of Putty Collides with and Sticks on a Turntable



A turntable rotates in the x-y plane and a lump of putty is thrown at the turntable with velocity $\vec{v}$ along the x direction. The turntable does not rotate initially. The only external force, gravity, is along the z direction and does not affect the motion in the x-y plane. Since there is no z-component of the torque of the gravity, the total angular momentum of the turntable plus putty is constant. After the putty hits on the turntable, the putty sticks on and rotates with the turntable.

Before the collision,

$$\begin{cases} \text{Putty:} \quad \vec{L}_{1i} = m(\vec{r} \times \vec{v}) \\ \qquad\qquad = (mrv\sin\theta)\,\vec{e}_z = mbv\,\vec{e}_z \\ \text{Table:} \quad \vec{L}_{2i} = 0 \end{cases}$$

After the collision,

$$\begin{cases} \text{Putty:} \quad \vec{L}_{1f} = m(\vec{r} \times \vec{v}) \\ \qquad\qquad = m[\vec{r} \times (\vec{\omega} \times \vec{r})] = mR^2\omega\,\vec{e}_z \\ \text{Table:} \quad \vec{L}_{2f} = I_z\,\vec{\omega} = (MR^2/2)\,\vec{\omega} \end{cases}$$

The conservation of the total angular momentum:

$$mbv = mR^2\omega + \frac{MR^2}{2}\omega \qquad \Longrightarrow \qquad \omega = \frac{2bv}{(2 + M/m)R^2}$$

3.7 How to Describe Motion of a Rigid Body

The motion of a rigid body can be conveniently described as a combination of the motion of its center of mass and a rotational motion around the center of the mass. As shown in Section 3.3, the equation of motion for the center of mass is

$$\frac{d\vec{P}}{dt} = \vec{F}^{ext} \qquad \text{where} \quad \vec{P} = M\dot{\vec{R}} \tag{47}$$

where $\vec{R}$ is the position vector of the center of mass and $\vec{P}$ is the momentum of the center of mass in an inertial frame and $\vec{F}^{ext}$ is the net external force. When $\vec{F}^{ext} = 0$, the momentum of the center of mass , $\vec{P}$ is a constant of motion.

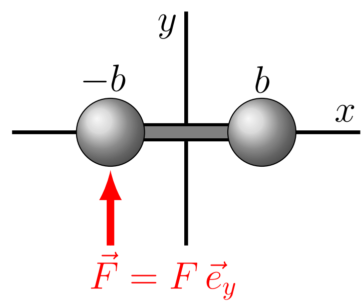
For the rotational motion around the center of mass, we consider the case of $\vec{F}^{ext} = 0$. The reference frame attached on the center of mass is then an inertial frame and the equation of motion for the rotation around the center of mass is simply given Section 3.5 as

$$\frac{d\vec{L}}{dt} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \begin{pmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{pmatrix} = \vec{\Gamma}^{ext} \qquad (48)$$

where $\vec{L}$ and $\vec{\omega}$ are the angular momentum and angular velocity, respectively, and $\vec{\Gamma}^{ext}$ is the net external torque of all the external forces with respect to the center of mass. Note that Eq. (48) needs to be modified when the center of mass coordinates is not an inertial frame in the case of $\vec{F}^{ext} \neq 0$.

Example 3.5 A Sliding and Spinning Dumbbell

A dumbbell is initially at rest on a frictionless table, lying on the x axis and center (center of mass) on the origin. The distance between two weights is $2b$ and the rigid rod can be approximated as massless. At time $t = 0$, the left weight is given a sharp tap, a force in the y direction, $\vec{F} = F\,\vec{e}_y$, for a short time Δt. To calculate the motion of the dumbbell, we consider the motion in two segments: form $t = 0$ to $t = \Delta t$ and then for $t > \Delta t$.



Dumbbell Motion From $t = 0$ to $t = \Delta t$

During this interval, the changes in the CM momentum and the angular momentum of the dumbbell can be calculated as

$$\begin{aligned} \dot{\vec{P}} &= \vec{F}^{ext} \qquad \Longrightarrow \qquad \Delta\vec{P} = \vec{P}(\Delta t) - \vec{P}(0) = \vec{F}\Delta t \\ \dot{\vec{L}} &= \vec{\Gamma}^{ext} \qquad \Longrightarrow \qquad \Delta\vec{L} = \vec{L}(\Delta t) - \vec{L}(0) = \vec{\Gamma}\Delta t \end{aligned}$$

where $\vec{\Gamma}$ is the torque of the tapping force $\vec{F}$, $(\vec{P}(0), \vec{L}(0))$ and $(\vec{P}(\Delta t), \vec{L}(\Delta t))$ are the CM momentum and angular momentum right before and after the tap, respectively. The CM velocity at $t = \Delta t$ is then

$$\vec{v}_{CM}(\Delta t) = \vec{v}_{CM}(0) + \left(\frac{F\Delta t}{2m}\right)\vec{e}_y = \left(\frac{F\Delta t}{2m}\right)\vec{e}_y$$

For the rotation around the CM, we consider Δt is a very short period of time and during Δt the change in the dumbbell position is negligible. During Δt, the left and right weight stay at their initial positions at

$$\vec{r}_l(0) = -b\,\vec{e}_x \qquad \text{and} \qquad \vec{r}_r(0) = b\,\vec{e}_x$$

and the torque and angular momentum are

$$\vec{\Gamma} = \vec{r}_l(0) \times \vec{F} = -bF\,\vec{e}_z\,, \qquad \vec{L}(0) = 0\,, \qquad \vec{L}(\Delta t) \;=\; m\,[\vec{r}_l(0) \times \vec{v}_l] + m\,[\vec{r}_r(0) \times \vec{v}_r]$$

where v_l and v_r are the velocity of the left and right weight from the rotation around the center of mass due to the tap. Since the dumbbell can only move on the x-y plane, the angular velocity of the dumbbell has only component in the z direction, *i.e.*

$$\vec{\omega} = -\omega\,\vec{e}_z \qquad \text{and} \qquad \vec{v}_l = \vec{\omega} \times \vec{r}_l\,, \qquad \vec{v}_r = \vec{\omega} \times \vec{r}_r \qquad \text{where } \ \vec{\omega} \perp \vec{r}$$

The angular momentum of the dumbbell right after Δt can then be calculated as

$$\vec{L}(\Delta t) \;=\; m\,[\vec{r}_l(0) \times (\vec{\omega} \times \vec{r}_l)] + m\,[\vec{r}_r(0) \times (\vec{\omega} \times \vec{r}_r)] \;=\; -2mb^2\omega\,\vec{e}_z$$

and

$$\vec{L}(\Delta t) \;=\; \vec{\Gamma}\Delta t \;=\; -bF\Delta t\,\vec{e}_z \qquad \Longrightarrow \qquad \omega = \frac{F\Delta t}{2mb}$$

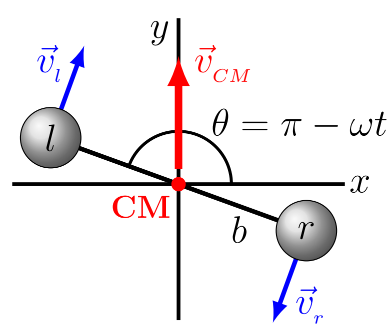
Summary of the motion of the dumbbell at $t = \Delta t$**:**

$$\begin{cases} \text{The center of mass moves with} & \vec{v}_{CM}(\Delta t) \;=\; [F\Delta t/(2m)]\,\vec{e}_y \\ \text{The rotation around the CM has} & \vec{L}(\Delta t) \;=\; -2mb^2\omega\,\vec{e}_z \end{cases}$$

Dumbbell Motion for $t > \Delta t$

After the initial tap, there is no any external force or torque affecting the motion of the dumbbell on the table (x-y plane). The CM momentum and the angular momentum with respect to the CM are the constants of motion. Therefore, both the CM velocity and the angular velocity of the rotation around the CM are constant,

$$\begin{cases} \vec{v}_{CM}(t) \;=\; \vec{v}_{CM}(\Delta t) \;=\; [F\Delta t/(2m)]\;\vec{e}_y \;=\; v_{CM}\,\vec{e}_y \\ \;\vec{\omega}(t) \;\;=\;\; \vec{\omega}(\Delta t) \;=\; -\,[F\Delta t/(2mb)]\;\vec{e}_z \;=\; -\omega\,\vec{e}_z \end{cases}$$



where $v_{CM} \;=\; b\,\omega \;=\; F\Delta t/(2m)$. The t-dependence of the rotation angle around the z axis is simply

$$\theta(t) = \theta(0) + \int_0^t \omega_z(t)dt = \pi - \omega t$$

The instantaneous positions of the weights in the coordinate that moves with the CM are

$$\begin{cases} \vec{r}_l(t) \;=\; b\cos\theta\,\vec{e}_x + b\sin\theta\,\vec{e}_y \;=\; -b\cos(\omega t)\,\vec{e}_x + b\sin(\omega t)\,\vec{e}_y \\ \vec{r}_l(t) \;=\; b\cos(\theta+\pi)\,\vec{e}_x + b\sin(\theta+\pi)\,\vec{e}_y \;=\; b\cos(\omega t)\,\vec{e}_x - b\sin(\omega t)\,\vec{e}_y \end{cases}$$

and the velocity of the weights, $\vec{u}_l$ and $\vec{u}_r$, in the Lab frame (coordinate fixed on the table) are

$$\begin{aligned}
\vec{u}_l(t) \;=\; \vec{v}_{CM} + \vec{\omega} \times \vec{r}_l(t) \;&=\; v_{CM}\vec{e}_y + b\,\omega\cos(\omega t)\,(\vec{e}_z \times \vec{e}_x) - b\,\omega\sin(\omega t)\,(\vec{e}_z \times \vec{e}_y) \\
&=\; v_{CM}\vec{e}_y + b\,\omega\cos(\omega t)\,\vec{e}_y + b\,\omega\sin(\omega t)\,\vec{e}_x \\
&=\; v_{CM}\left\{[1+\cos(\omega t)]\;\vec{e}_y + \sin(\omega t)\,\vec{e}_x\right\} \\
\vec{u}_r(t) \;=\; \vec{v}_{CM} + \vec{\omega} \times \vec{r}_r(t) \;&=\; v_{CM}\vec{e}_y - b\,\omega\cos(\omega t)\,(\vec{e}_z \times \vec{e}_x) + b\,\omega\sin(\omega t)\,(\vec{e}_z \times \vec{e}_y) \\
&=\; v_{CM}\left\{[1-\cos(\omega t)]\;\vec{e}_y - \sin(\omega t)\,\vec{e}_x\right\}
\end{aligned}$$

With respect to the table, right after the tap, the right weight does not move while the left weight moves at a speed of $2v_{CM}$.

Homework for Chapter 3

Assig.	Problem	Covered Subject
3.1	3.4, 3.5, 3.7, 3.11, 3.16, 3.20	momentum, CM