Lecture Note of Mechanics I

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Chapter 1. Newton's Laws of Motion

1.1 The Foundation of Newtonian Physics

• Reference Frame

Reference frame is a set of the origin and axes for space and time for describing the dynamics of a system.

• Space and Time for Newtonian Mechanics

In Newtonian physics, time is considered to be absolute, *i.e.* independent of space, and space is Euclidean. In three-dimensional Euclidean space, vectors can be written as

$$\vec{A} \;=\; A_1\,\vec{e}_1 + A_2\,\vec{e}_2 + A_3\,\vec{e}_3 \tag{1}$$

where $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ are three perpendicular unit vectors, called based vector, that represent the directions of coordinate axes, *i.e.*

$$|\vec{e}_i| = 1\,, \qquad \vec{e}_i\cdot\vec{e}_j = \delta_{ij} \qquad \text{and} \qquad \vec{e}_i\times\vec{e}_j = \epsilon_{ijk}\,\vec{e}_k$$

where δ_{ij} is the Kronecker delta and ϵ_{ijk} is the permutation (Levi-Civita) symbol. For the Cartesian coordinates,

$$(\,\vec{e}_1\,,\,\vec{e}_2\,,\,\vec{e}_3\,) = (\,\hat{x}\,,\,\hat{y}\,,\,\hat{z}\,)$$

The magnitude or norm of vector $\vec{A}$ in the Euclidean space is defined as

$$A \;=\; |\vec{A}\,| \;=\; \sqrt{\vec{A}\cdot\vec{A}} \;=\; \sqrt{A_1^2+A_2^2+A_3^2} \tag{2}$$

In the Euclidean space, the norm of a vector is independent of the choice of $(\vec{e}_1,\vec{e}_2,\vec{e}_3)$, *i.e.* $|\vec{A}\,|$ is invariant under rotational or translational transformations of the coordinates, if

$$\vec{A} \;=\; A_1\vec{e}_1 + A_2\vec{e}_2 + A_3\vec{e}_3 \;=\; A_1'\vec{e}\,'_1 + A_2'\vec{e}\,'_2 + A_3'\vec{e}\,'_3$$

then

$$\sqrt{A_1^2+A_2^2+A_3^2} = \sqrt{A_1'^2+A_2'^2+A_3'^2}$$

invariance of Euclidean distance

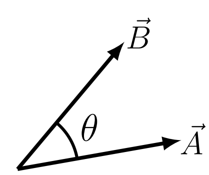
Operations of Vectors

addition and subtraction $\quad \vec{A}\pm\vec{B} \;=\; (A_1\pm B_1)\,\vec{e}_1 + (A_2\pm B_2)\,\vec{e}_2 + (A_3\pm B_3)\,\vec{e}_3$

multiplication of vector with scalar $\quad \alpha\vec{A} \;=\; (\alpha A_1)\,\vec{e}_1 + (\alpha A_2)\,\vec{e}_2 + (\alpha A_3)\,\vec{e}_3$

dot-product $\quad \vec{A}\cdot\vec{B} \;=\; A_1B_1 + A_2B_2 + A_3B_3 \;=\; AB\cos\theta$

cross-product $\quad \vec{A}\times\vec{B} \;=\; \det\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{pmatrix}$



$$= \;(A_2B_3 - A_3B_2)\,\vec{e}_1 + (A_3B_1 - A_1B_3)\,\vec{e}_2 + (A_1B_2 - A_2B_1)\,\vec{e}_3$$

where $\;|\vec{A}\times\vec{B}\,| = AB\sin\theta$

Position, Velocity, and Acceleration

For each point P in the three-dimensional space, the position vector is

$$\vec{r} \;=\; r_1\,\vec{e}_1 + r_2\,\vec{e}_2 + r_3\,\vec{e}_3 \;=\; x\,\hat{x} + y\,\hat{y} + z\,\hat{z} \;=\; (\,x\,,\,y\,,\,z\,) \tag{3}$$

The velocity of point P moving in the space is defined as

$$\begin{aligned} \vec{v} \;&=\; \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} \;=\; \frac{d\vec{r}}{dt} \\ &=\; \dot{r}_1\,\vec{e}_1 + r_1\,\dot{\vec{e}}_1 + \dot{r}_2\,\vec{e}_2 + r_2\,\dot{\vec{e}}_2 + \dot{r}_3\,\vec{e}_3 + r_3\,\dot{\vec{e}}_3 \qquad \longleftarrow \text{in general coordinates} \\ &=\; \dot{x}\,\vec{e}_x + \dot{y}\,\vec{e}_y + \dot{z}\,\vec{e}_z \qquad \longleftarrow \text{in Cartesian coordinates} \end{aligned} \tag{4}$$

where $\dot{\vec{e}}_x = \dot{\vec{e}}_y = \dot{\vec{e}}_z = 0$ in the Cartesian coordinates, and the acceleration is defined as

$$\begin{aligned} \vec{a} \;&=\; \frac{d\vec{v}}{dt} \;=\; \frac{d^2\vec{r}}{dt^2} \\ &=\; \ddot{r}_1\,\vec{e}_1 + \dot{r}_1\,\dot{\vec{e}}_1 + r_1\,\ddot{\vec{e}}_1 + \ddot{r}_2\,\vec{e}_2 + \dot{r}_2\,\dot{\vec{e}}_2 + r_2\,\ddot{\vec{e}}_2 + \ddot{r}_3\,\vec{e}_3 + \dot{r}_3\,\dot{\vec{e}}_3 + r_3\,\ddot{\vec{e}}_3 \end{aligned} \tag{5}$$

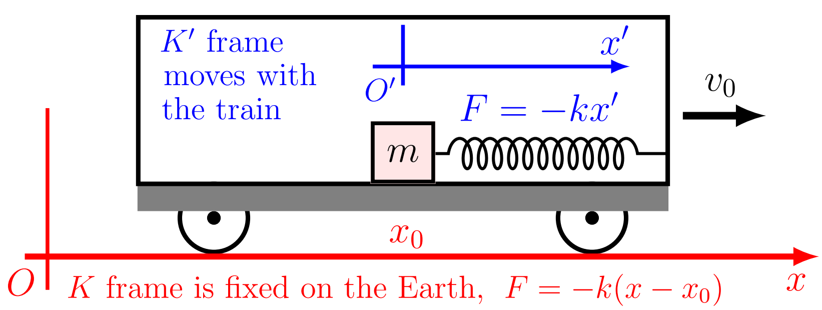
• Inertial Frames

These are the reference frames that move with a constant velocity with respect to each other. Newton's laws are only true in the inertial frames. Newton's law in the inertial frame,

$$\vec{F} \;=\; m\vec{a} \tag{6}$$

where $\vec{F}$ and m are the net force on and the mass of an object, respectively.

Example. You are studying the motion of spring-block in a train that moves with speed v_0



(a) $v_0 = constant$

$$x = x' + x_0 \quad \text{and} \quad x_0 = v_0\,t$$
$$v = v' + v_0$$
$$\frac{dv}{dt} = \frac{dv'}{dt} \quad \Longrightarrow \quad a = a'$$
$$F = ma \quad \text{and} \quad F = ma'$$

(b) $v_0 = a_0\,t$

$$x = x' + x_0 \quad \text{and} \quad x_0 = \int v_0\,dt$$
$$v = v' + v_0$$
$$\frac{dv}{dt} = \frac{dv'}{dt} + a_0 \quad \Longrightarrow \quad a = a' + a_0$$
$$F = ma \quad \text{and} \quad F \neq ma'$$

- **Two Postulates of Galilean Relativity — The Foundation of Newtonian Physics**

  (a) Physics laws are invariant in inertial frames.

  (b) The space geometry of physics is Euclidean space and the Euclidean distance

$$dr^2 = dr_1^2 + dr_2^2 + dr_3^2 \qquad \text{with} \quad d\vec{r} = dr_1\,\vec{e}_1 + dr_2\,\vec{e}_2 + dr_3\,\vec{e}_3 \tag{7}$$

is invariant under physics transformations. The Galilean transformation between two inertial frames K and K' is simply

$$\vec{r}\,' = \vec{r} - \vec{v}_0\,t \qquad \text{and} \qquad t' = t \tag{8}$$

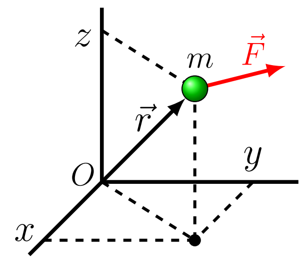
where $\vec{v}_0$ is the constant velocity between the two frames.

HW. Show that the Euclidean distance is invariant for the Galilean transformation.

1.2 Newton's Equation in Cartesian Coordinates

$$\vec{F} = m\frac{d^2\vec{r}}{dt^2} \qquad \Longrightarrow \qquad \begin{cases} F_x = m\ddot{x} = \dot{p}_x \\ F_y = m\ddot{y} = \dot{p}_y \\ F_z = m\ddot{z} = \dot{p}_z \end{cases}$$

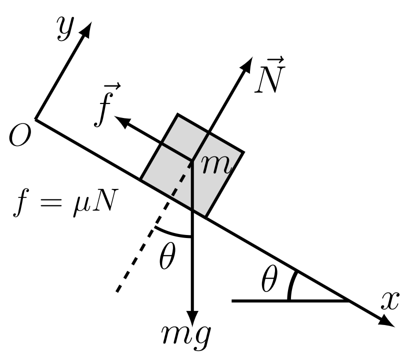
$$\dot{\vec{e}}_x = 0\,, \quad \dot{\vec{e}}_y = 0\,, \quad \dot{\vec{e}}_z = 0$$



where $\vec{p} = m\vec{v}$ is the linear momentum.

Conservation of Momentum: If $F_i = 0$, $p_i = constant$

Example 1. A block sliding down an incline



The Newton's equation is

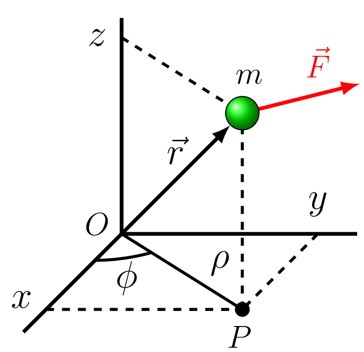
$$\begin{cases} F_x = mg\sin\theta - \mu N = m\ddot{x} \\ F_y = N - mg\cos\theta \; = \; 0 \end{cases}$$

which yields $\quad \ddot{x} = g(\sin\theta - \mu\cos\theta)$

The solution of the equation of motion is

$$x(t) = x(0) + v(0)t + \frac{1}{2}g(\sin\theta - \mu\cos\theta)t^2$$

1.3 Newton's Equation in Cylindrical Coordinates



The transformation between the Cartesian (x, y, z) and cylindrical (ρ, ϕ, z) coordinates is

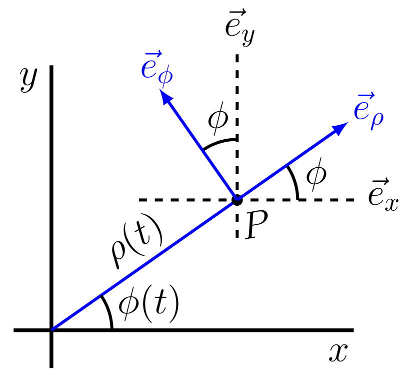
$$\begin{cases} x = \rho\cos\phi \\ y = \rho\sin\phi \end{cases} \qquad \text{and} \qquad \begin{cases} \rho = \sqrt{x^2 + y^2} \\ \phi = \arctan{(y/x)} \end{cases} \tag{9}$$

The transformation of the base vectors

$$(\,\vec{e}_x\,,\,\vec{e}_y\,,\,\vec{e}_z\,) \longleftrightarrow (\,\vec{e}_\rho\,,\,\vec{e}_\phi\,,\,\vec{e}_z\,)$$

between the Cartesian and cylindrical coordinates is

$$\begin{cases} \vec{e}_\rho = \quad \vec{e}_x \cos\phi + \vec{e}_y \sin\phi \\ \vec{e}_\phi = -\vec{e}_x \sin\phi + \vec{e}_y \cos\phi \\ \vec{e}_z = \vec{e}_z \end{cases}$$



$$|\vec{e}_\rho| = |\vec{e}_\phi| = 1$$
$$|\vec{e}_x| = |\vec{e}_y| = |\vec{e}_z| = 1$$

which can be written as

$$\begin{pmatrix} \vec{e}_\rho \\ \vec{e}_\phi \\ \vec{e}_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \vec{e}_x \\ \vec{e}_y \\ \vec{e}_z \end{pmatrix} = \mathbf{T} \begin{pmatrix} \vec{e}_x \\ \vec{e}_y \\ \vec{e}_z \end{pmatrix}$$

and

$$\begin{pmatrix} \vec{e}_x \\ \vec{e}_y \\ \vec{e}_z \end{pmatrix} = \mathbf{T}^{-1} \begin{pmatrix} \vec{e}_\rho \\ \vec{e}_\phi \\ \vec{e}_z \end{pmatrix} \qquad \text{where} \qquad \mathbf{T}^{-1} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

With $\dot{\vec{e}}_x = \dot{\vec{e}}_y = \dot{\vec{e}}_z = 0$, the time-derivatives of the base vectors of the cylindrical coordinates can be calculated as

$$\begin{cases} \dot{\vec{e}}_\rho = \dot{\phi}\,(-\vec{e}_x \sin\phi + \vec{e}_y \cos\phi) = \quad \dot{\phi}\,\vec{e}_\phi \\ \dot{\vec{e}}_\phi = \dot{\phi}\,(-\vec{e}_x \cos\phi - \vec{e}_y \sin\phi) = -\dot{\phi}\,\vec{e}_\rho \end{cases} \tag{10}$$

Since

$$\begin{aligned} \vec{r} &= x\,\vec{e}_x + y\,\vec{e}_y + z\,\vec{e}_z = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} \vec{e}_x \\ \vec{e}_y \\ \vec{e}_z \end{pmatrix} = \begin{pmatrix} x & y & z \end{pmatrix} \mathbf{T}^{-1} \begin{pmatrix} \vec{e}_\rho \\ \vec{e}_\phi \\ \vec{e}_z \end{pmatrix} \\ &= (x\cos\phi + y\sin\phi)\;\vec{e}_\rho + (-x\sin\phi + y\cos\phi)\;\vec{e}_\phi + z\,\vec{e}_z \\ &= \rho\,\vec{e}_\rho + z\,\vec{e}_z \end{aligned} \tag{11}$$

the velocity and acceleration in the cylindrical coordinates can then be calculated as

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{\rho}\,\vec{e}_\rho + \rho\,\dot{\vec{e}}_\rho + \dot{z}\,\vec{e}_z = \dot{\rho}\,\vec{e}_\rho + \rho\dot{\phi}\,\vec{e}_\phi + \dot{z}\,\vec{e}_z \tag{12}$$

i.e.

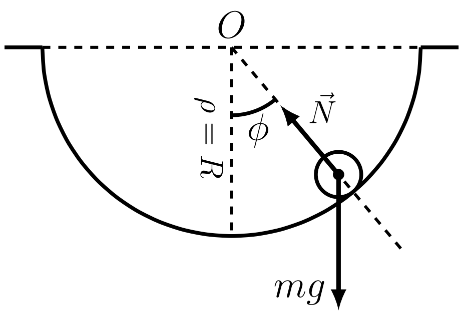
$$(\,v_\rho\,,\,v_\phi\,,\,v_z\,) = (\,\dot{\rho}\,,\,\rho\dot{\phi}\,,\,\dot{z}\,)$$

and

$$\begin{aligned} \frac{d\vec{v}}{dt} &= \ddot{\rho}\,\vec{e}_\rho + \dot{\rho}\,\dot{\vec{e}}_\rho + \dot{\rho}\dot{\phi}\,\vec{e}_\phi + \rho\ddot{\phi}\,\vec{e}_\phi + \rho\dot{\phi}\,\dot{\vec{e}}_\phi + \ddot{z}\,\vec{e}_z \\ &= \left(\ddot{\rho} - \rho\dot{\phi}^2\right)\,\vec{e}_\rho + \left(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}\right)\,\vec{e}_\phi + \ddot{z}\,\vec{e}_z \end{aligned} \implies \begin{cases} F_\rho = m(\ddot{\rho} - \rho\dot{\phi}^2) \\ F_\phi = m(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}) \\ F_z = m\,\ddot{z} \end{cases} \tag{13}$$

Example 2. An Oscillating Skateboard

For an object sliding inside a half cylinder, the problem is two-dimensional in the polar coordinates (ρ, ϕ) with $\rho = const$. The Newton's equation is



$$\begin{cases} F_\rho &= mg\cos\phi - N &= m(\ddot{\rho} - \rho\dot{\phi}^2) \\ F_\phi &= -mg\sin\phi &= m(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}) \end{cases}$$

With $\rho = R$, the equation of motion is reduced to the equation of a pendulum,

$$\ddot{\phi} = -\omega^2 \sin\phi \qquad \text{where} \quad \omega = \sqrt{g/R}$$

For a small-angle oscillation $\phi << 1$, $\sin\phi \simeq \phi$,

$$\ddot{\phi} = -\omega^2 \phi \qquad \longleftarrow \text{Solve this PDE Using Characteristic Equation}$$

Let's try $\phi = e^{\lambda t}$

$\Longrightarrow \quad \lambda^2\phi = -\omega^2\phi$

$\Longrightarrow \quad \lambda = \pm i\omega$

$\phi = Ae^{-i\omega t} + Be^{i\omega t}$

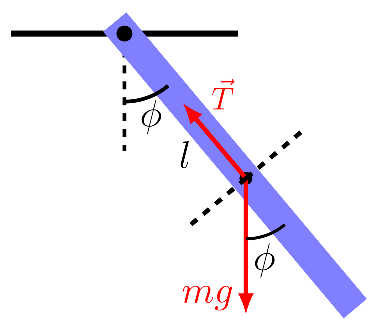
This is the equation for a harmonic oscillator and its solution is

$$\phi(t) = A\sin(\omega t) + B\cos(\omega t)$$

The normal force can be calculated from the Newton's eq. as

$$N(t) = mg\cos\phi(t) + mR\dot{\phi}^2$$

Example 3. Pendulum



The equation of motion for a pendulum is

$$\ddot{\phi} \;=\; -\omega^2 \sin\phi \qquad \text{where} \;\; \omega = \sqrt{g/l}$$

which can be solved as

$$\dot{\phi}\frac{d\dot{\phi}}{dt} \;=\; -\left(\omega^2 \sin\phi\right)\dot{\phi} \qquad \Longrightarrow \qquad \frac{1}{2}\frac{d\dot{\phi}^2}{dt} \;=\; \omega^2 \frac{d\cos\phi}{dt}$$

Therefore

$$\frac{d}{dt}\left(\frac{1}{2}\dot{\phi}^2 - \omega^2\,\cos\phi\right) = 0 \qquad \Longrightarrow \qquad \frac{1}{2}\dot{\phi}^2 - \omega^2\,\cos\phi \;=\; const.$$

where the constant of motion is the mechanic energy,

$$\text{Energy} \;=\; \frac{1}{2}m(l\dot{\phi})^2 + mgl(1-\cos\phi) \;=\; \frac{1}{2}mv_{tan}^2 + mg(l-y)$$

with $v_{tan} = l\dot{\phi}$ and $y = l\cos\phi$. Therefore, the trajectory of a pendulum satisfies

$$E = \frac{1}{2}\dot{\phi}^2 - \omega^2\,\cos\phi = \frac{1}{2}\dot{\phi}^2(0) - \omega^2\,\cos\phi(0) \qquad \Longrightarrow \qquad \dot{\phi} = \pm\sqrt{2(E + \omega^2\,\cos\phi)}$$

where $E < \omega^2$ is for the libration, $E > \omega^2$ is for the rotation, and $E = \omega^2$ is the separatrix that divides two phases of motion. For the libration, the maximal oscillation angle (turning point) is at

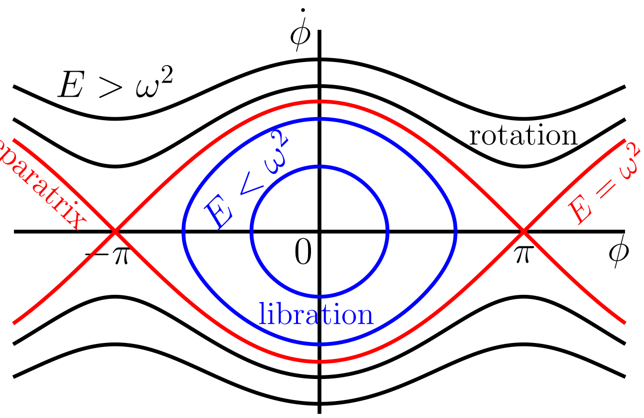
$$\dot{\phi} = 0 \quad \Longrightarrow \quad E = -\omega^2 \cos\phi_{max}$$

which yields

$$\phi_{max} = \cos^{-1}\left(E/\omega^2\right)$$

The t-dependence of ϕ can be obtained from

$$t = \int_{\phi(0)}^{\phi} \frac{d\phi}{\sqrt{2(E+\omega^2\,\cos\phi)}}$$



1.4 Newton's Equation in Spherical Coordinates

The transformation between the Cartesian (x, y, z) and spherical (r, θ, ϕ) coordinates is

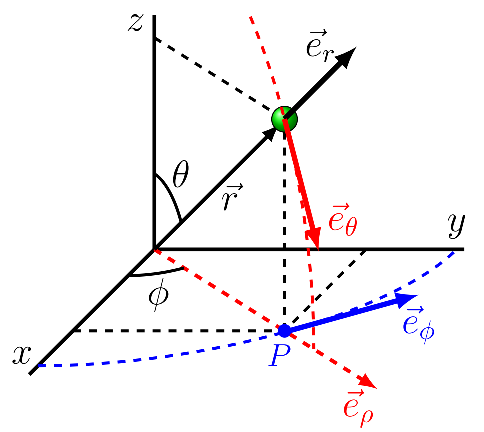
$$x = r\sin\theta\cos\phi\,, \quad y = r\sin\theta\sin\phi\,, \quad z = r\cos\theta$$

where $r^2 = x^2 + y^2 + z^2$ and

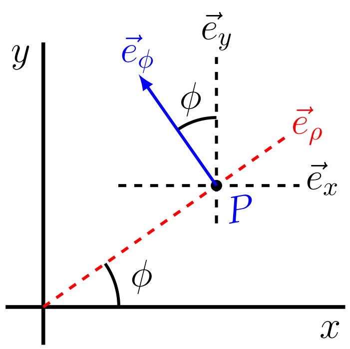
$$\vec{r} = x\,\vec{e}_x + y\,\vec{e}_y + z\,\vec{e}_z = r\,\vec{e}_r$$

Therefore

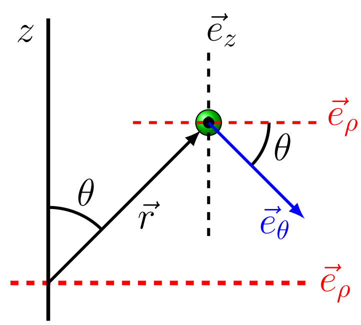
$$\vec{e}_r = \vec{e}_x\,\sin\theta\cos\phi + \vec{e}_y\,\sin\theta\sin\phi + \vec{e}_z\,\cos\theta$$



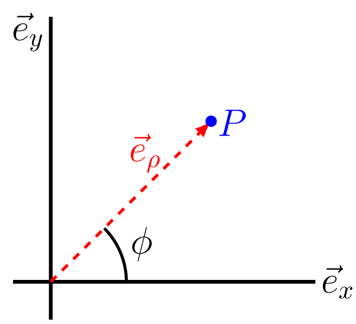
Transformation of Base Vectors $(\,\vec{e}_x\,,\,\vec{e}_y\,,\,\vec{e}_z\,) \longleftrightarrow (\,\vec{e}_r\,,\,\vec{e}_\theta\,,\,\vec{e}_\phi\,)$



$$\vec{e}_\phi = -\vec{e}_x\sin\phi + \vec{e}_y\cos\phi$$



$$\vec{e}_\theta = \vec{e}_\rho\cos\theta - \vec{e}_z\sin\theta$$



$$\vec{e}_\rho = \vec{e}_x\cos\phi + \vec{e}_y\sin\phi$$

The transformations of the base vectors are therefore

$$\left\{\begin{array}{rcl} \vec{e}_r &=& \vec{e}_x\,\sin\theta\cos\phi + \vec{e}_y\,\sin\theta\sin\phi + \vec{e}_z\,\cos\theta \\ \vec{e}_\phi &=& -\vec{e}_x\,\sin\phi + \vec{e}_y\,\cos\phi \\ \vec{e}_\theta &=& \vec{e}_\rho\,\cos\theta - \vec{e}_z\,\sin\theta \;=\; \vec{e}_x\,\cos\theta\cos\phi + \vec{e}_y\,\cos\theta\sin\phi - \vec{e}_z\,\sin\theta \end{array}\right. \tag{14}$$

which can be written as

$$\begin{pmatrix} \vec{e}_r \\ \vec{e}_\theta \\ \vec{e}_\phi \end{pmatrix} = \mathbf{T} \begin{pmatrix} \vec{e}_x \\ \vec{e}_y \\ \vec{e}_z \end{pmatrix} \qquad \text{where} \qquad \mathbf{T} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \tag{15}$$

and

$$\begin{pmatrix} \vec{e}_x \\ \vec{e}_y \\ \vec{e}_z \end{pmatrix} = \mathbf{T}^{-1} \begin{pmatrix} \vec{e}_r \\ \vec{e}_\theta \\ \vec{e}_\phi \end{pmatrix} \qquad \text{where} \qquad \mathbf{T}^{-1} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \tag{16}$$

The time-derivatives of the base vectors $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$ can then be calculated as

$$\begin{aligned}
\dot{\vec{e}}_r &= (\dot\theta\cos\theta\cos\phi - \dot\phi\sin\theta\sin\phi)\,\vec{e}_x + (\dot\theta\cos\theta\sin\phi + \dot\phi\sin\theta\cos\phi)\,\vec{e}_y - (\dot\theta\sin\theta)\,\vec{e}_z \\
&= \dot\theta\,(\vec{e}_x\cos\theta\cos\phi + \vec{e}_y\cos\theta\sin\phi - \vec{e}_z\sin\theta) + \dot\phi\,(-\vec{e}_x\sin\theta\sin\phi + \vec{e}_y\sin\theta\cos\phi) \\
&= \dot\theta\,\vec{e}_\theta + (\dot\phi\,\sin\theta)\,\vec{e}_\phi \\
\dot{\vec{e}}_\theta &= -(\dot\theta\sin\theta\cos\phi + \dot\phi\cos\theta\sin\phi)\,\vec{e}_x - (\dot\theta\sin\theta\sin\phi - \dot\phi\cos\theta\cos\phi)\,\vec{e}_y - (\dot\theta\cos\theta)\,\vec{e}_z \\
&= -\dot\theta(\vec{e}_x\sin\theta\cos\phi + \vec{e}_y\sin\theta\sin\phi + \vec{e}_z\cos\theta) - \dot\phi(\vec{e}_x\cos\theta\sin\phi - \vec{e}_y\cos\theta\cos\phi) \\
&= -\dot\theta\,\vec{e}_r + (\dot\phi\,\cos\theta)\,\vec{e}_\phi \\
\dot{\vec{e}}_\phi &= -(\dot\phi\cos\phi)\,\vec{e}_x - (\dot\phi\sin\phi)\,\vec{e}_y \;=\; -(\dot\phi\sin\theta)\,\vec{e}_r - (\dot\phi\cos\theta)\,\vec{e}_\theta
\end{aligned}$$

The velocity and acceleration can be calculated from the position $\vec{r} = r\vec{e}_r$ as

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot r\,\vec{e}_r + r\,\dot{\vec{e}}_r = \dot r\,\vec{e}_r + r\dot\theta\,\vec{e}_\theta + (r\dot\phi\sin\theta)\,\vec{e}_\phi \tag{17}$$

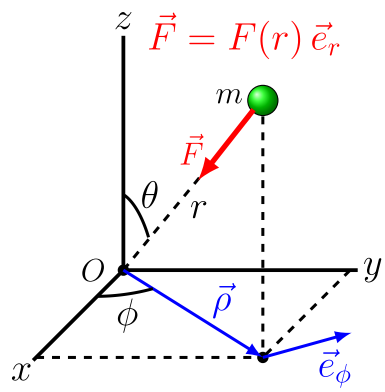
i.e. $v_r = \dot r$, $v_\theta = r\dot\theta$, $v_\phi = r\dot\phi\sin\theta$, and

$$\begin{aligned}
\frac{d\vec{v}}{dt} &= \ddot r\,\vec{e}_r + (\dot r\dot\theta + r\ddot\theta)\vec{e}_\theta + (\dot r\dot\phi\sin\theta + r\ddot\phi\sin\theta + r\dot\phi\dot\theta\cos\theta)\vec{e}_\phi + \dot r\dot{\vec{e}}_r + r\dot\theta\dot{\vec{e}}_\theta + (r\dot\phi\sin\theta)\dot{\vec{e}}_\phi \\
&= \ddot r\,\vec{e}_r + (\dot r\dot\theta + r\ddot\theta)\,\vec{e}_\theta + (\dot r\dot\phi\sin\theta + r\ddot\phi\sin\theta + r\dot\phi\dot\theta\cos\theta)\,\vec{e}_\phi \\
&\quad + \dot r\left[\dot\theta\,\vec{e}_\theta + (\dot\phi\,\sin\theta)\,\vec{e}_\phi\right] + r\dot\theta\,\left[-\dot\theta\,\vec{e}_r + (\dot\phi\,\cos\theta)\,\vec{e}_\phi\right] \\
&\quad + (r\dot\phi\sin\theta)\,\left[-(\dot\phi\sin\theta)\,\vec{e}_r - (\dot\phi\cos\theta)\,\vec{e}_\theta\right] \\
&= \vec{e}_r\left(\ddot r - r\dot\theta^2 - r\dot\phi^2\sin^2\theta\right) + \vec{e}_\theta\left(2\dot r\dot\theta + r\ddot\theta - r\dot\phi^2\sin\theta\cos\theta\right) \\
&\quad + \vec{e}_\phi\left(2\dot r\dot\phi\sin\theta + r\ddot\phi\sin\theta + 2r\dot\phi\dot\theta\cos\theta\right)
\end{aligned} \tag{18}$$

The Newton's equation in the spherical coordinates is then

$$\left\{\begin{aligned}
F_r &= m\left(\ddot r - r\dot\theta^2 - r\dot\phi^2\sin^2\theta\right) \\
F_\theta &= m\left(r\ddot\theta + 2\dot r\dot\theta - r\dot\phi^2\sin\theta\cos\theta\right) \\
F_\phi &= m\left(r\ddot\phi\sin\theta + 2\dot r\dot\phi\sin\theta + 2r\dot\phi\dot\theta\cos\theta\right)
\end{aligned}\right. \tag{19}$$

Example 4. Motion Under a Central Force



In the central-force problem, the net force on an object is either parallel or anti-parallel to $\vec{e}_r$ and depends only on r. The equations of motion in the spherical coordinates becomes

$$\left\{\begin{array}{rcl} \ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2 \sin^2\theta & = & F(r)/m \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2 \sin\theta\cos\theta & = & 0 \\ r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\phi}\dot{\theta}\cos\theta & = & 0 \end{array}\right. \tag{20}$$

where $F_\theta = F_\phi = 0$. Multiplying r to the θ-component of the Newton's equation (2nd equation) yields

the θ-component: $$r\left(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta\right) = \frac{d}{dt}(r^2\dot{\theta}) - r^2\dot{\phi}^2\sin\theta\cos\theta$$

The θ-component of the Newton's equation can thus be written as

the θ-component: $$\frac{d}{dt}(r^2\dot{\theta}) - r^2\dot{\phi}^2\sin\theta\cos\theta = 0 \tag{21}$$

Multiplying $r\sin\theta$ to the ϕ-component of the Newton's equation (3rd equation) yields

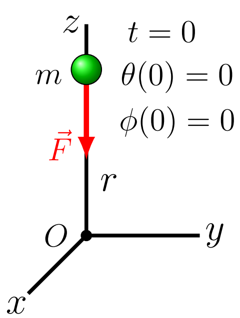
the ϕ-component: $$r\sin\theta\left(r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\phi}\dot{\theta}\cos\theta\right) = \frac{d}{dt}\left(r^2\dot{\phi}\sin^2\theta\right) = 0$$

The first integral of the ϕ-component of the Newton's equation is then

$$r^2\dot{\phi}\sin^2\theta = constant \qquad \Longrightarrow \qquad p_\phi = m|\vec{\rho}\times(v_\phi\,\vec{e}_\phi)| = constant \tag{22}$$

where $\rho = r\sin\theta$, $v_\phi = r\dot{\phi}\sin\theta$, and p_ϕ is the angular (canonical) momentum conjugate to ϕ (we will study a lot of the canonical momentum late.) Equation (22) is the conservation of angular momentum p_ϕ because $F_\phi = 0$. Since the orientation of the coordinate, the directions of base vectors $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$, can be chosen arbitrarily for the initial condition of the motion, this constant of the motion can be determined by chosen a special direction of $\vec{e}_z$ such that

$$\text{at } t = 0,\ \theta(0) = 0 \qquad \Longrightarrow \qquad p_\phi(t) \;=\; r^2(t)\,\dot{\phi}(t)\,\sin^2\theta(t) \;=\; 0 \tag{23}$$



where $p_\phi(t) = 0$ requires $r(t) = 0$, $\dot{\phi}(t) = 0$, or $\theta(t) = 0$. It can be seen from the original Newton's equation that $r(t)$ or $\theta(t)$ cannot be zero at all t, otherwise $\dot{\theta}(t) = 0$ and there would be no planetary motion at all. Therefore, $p_\phi(t) = 0$ requires

$$\dot{\phi}(t) = 0 \qquad \text{and} \qquad \phi(t) = constant$$

Since $\theta(0) = 0$, the initial value of ϕ has no physical meaning and we can simply choose $\phi(0) = \phi(t) = 0$. The motion is then limited in the

x-z plane — a two-dimensional motion. With $\dot{\phi} = 0$ and $\ddot{\phi} = 0$, the original Newton's equations in Eq. (20) becomes

$$\begin{cases} \ddot{r} - r\dot{\theta}^2 &= F(r)/m \\ \dfrac{d}{dt}(r^2\dot{\theta}) &= 0 \end{cases} \qquad \Longrightarrow \qquad \begin{cases} \ddot{r} - \dfrac{p_\theta^2}{m^2r^3} = \dfrac{1}{m}F(r) \\ r^2\dot{\theta} = \dfrac{p_\theta}{m} = constant \end{cases} \tag{24}$$

where

$$p_\theta = mr^2\dot{\theta} = m|\vec{r} \times (v_\theta \vec{e}_\theta)| = constant \qquad \text{with} \qquad v_\theta = r\dot{\theta}$$

is the angular momentum conjugate to θ and conserved because of $F_\theta = 0$. Because $\phi = 0$ during the motion, the motion in the central-force problem is 2-dimensional on the x-z plane in the coordinates that Newton's equation in Eq. (20) is based on and p_θ is the angular momentum with the y-axis as the rotation axis. Since the motion is 2-dimensional, we can relabel the z-axis as the y-axis and then the motion is in the x-y plane and p_θ is the traditionally-called angular momentum in the z direction.

The final one-dimensional equation for the r component in Eq. (24) can easily be solved as the following,

$$m\ddot{r} = \frac{p_\theta^2}{mr^3} + F(r) = Q(r) \qquad \Longrightarrow \qquad m\dot{r}\frac{d\dot{r}}{dt} = \frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2\right) = Q(r)\frac{dr}{dt}$$

Let

$$V(r) = -\int Q(r)dr \qquad \text{or} \qquad dV(r) = -Q(r)dr$$

then

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2\right) = -\frac{dV(r)}{dt} \qquad \Longrightarrow \qquad \frac{1}{2}m\dot{r}^2 + V(r) = E = constant$$

The solution of the trajectory of the motion under a central-force can be obtained as

$$\begin{aligned} \frac{dr}{dt} &= \sqrt{2[E - V(r)]/m} \qquad &\Longrightarrow \qquad t &= \int_{r_0}^{r} \frac{dr}{\sqrt{2[E - V(r)]/m}} \\ \frac{d\theta}{dt} &= \frac{M}{mr^2} \qquad &\Longrightarrow \qquad \theta(t) &= \frac{M}{m}\int_0^t \frac{dt}{r^2(t)} \end{aligned} \tag{25}$$

or

$$\theta = \frac{M}{\sqrt{2m}} \int_{r_0}^{r} \frac{dr}{r^2\sqrt{E - V(r)}} \tag{26}$$

where $r_0 = r(0)$ and $\theta(0) = 0$ are the initial condition of the motion.

Homework for Chapter 1

Assig.	Problem	Covered Subject
1.1	nine problems from 1.1 to 1.9	vector calculation
1.2	1.10, 1.12, 1.24, 1.25, 1.26	circular and helix motion, ODE, inertial frame
1.3	1.36, 1.37, 1.40, 1.43, 1.44, 1.45	2nd law in Cartesian: incline, projectile motion in polar coordinate
1.4	1.47, 1.48, 1.49	cylindrical coordinate