As we already have Newton's equation, why do we need new approaches for the mechanics?

Problem 1 of Newtonian mechanics:

Newton's laws depends on the reference frame.

In the inertial frames:

$$\vec{F} = m\vec{a} = \frac{d\vec{p}}{dt}$$

In the non-inertial frames:

Consider a reference frame K' that is moving at a nonconstant $\vec{v}_0(t)$ and $\vec{a}_0 = \dot{\vec{v}}_0(t)$ with respect to an inertial frame K. The transformation from K to K' is

$$\vec{x}\,' = \vec{x} - \int \vec{v}_0(t)dt\,, \qquad \vec{v}\,' = \vec{v} - \vec{v}_0\,, \qquad \vec{a}\,' = \vec{a} - \vec{a}_0$$

and the equation of motion in K' is thereofre

$$\vec{F}\,' = \frac{d\vec{p}\,'}{dt} + m\vec{a}_0$$

• The Newton's equation is not invariant in different reference frames. We need a formalism for the mechanics that is invariant and can be easily used in all reference frames.

Problem 2 of Newtonian mechanics:

Newton's equation cannot be easily used in general coordinates.

In the Cartesian coordinate (x, y):

$$\vec{F} = \frac{d\vec{p}}{dt} \quad \Longrightarrow \quad \begin{cases} F_x &= \dot{p}_x \\ F_y &= \dot{p}_y \end{cases}$$

In the polar coordinates (r, ϕ):

$$\begin{cases} F_r = m(\ddot{r} - r\dot{\phi}^2) \neq \dot{p}_r \\ F_\phi = \dfrac{d}{dt}(mr^2\dot{\phi}) = \dot{p}_\phi \end{cases}$$

• The Newton's equation is not invariant in different coordinates. We need a formalism for the mechanics that is invariant and can be easily used in all coordinates.

Solutions:

Lagrange's and Hamilton's equations are independent (invariant) of the reference frame and coordinate system. In addition, the Lagrange's and Hamilton's equations more clearly reflect the symplectic symmetry that is the fundamental symmetry of physics and the classical and quantum mechanics can be connected naturally. The foundation of of the Lagrange's and Hamilton's equations is the variational principle.

Chapter 6. Calculus of Variations
Extrema of Functional — Variational Principle

Functionals

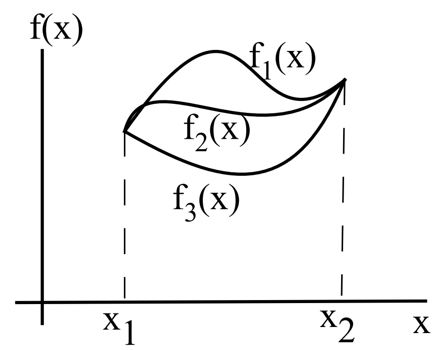
Function: $y = f(x)$

Functional: $S = S[f]$

Example of Functional:

$$S[f] = \int_{x_1}^{x_2} f(x)\, dx$$

If $f(x)\, dx = dF(x)$ is an exact differential, $S[f]$ depends only on the function $F(x)$ at two end points.



The Lagrangian and Hamiltonian mechanics are based on the extrema of functionals in the form of

$$S[f] = \int_{x_1}^{x_2} f(y(x), y'(x), x)\, dx \tag{1}$$

where $y(x)$ is an arbitrary function of x, $y'(x) = dy(x)/dx$ and, in general, $f(y(x), y'(x), x)\, dx$ is not an exact differential and $S[f]$ is path-dependent.

Variation of a Function

A variation of a function $y(x)$ is a change of the function at every x,

$$y(x) \longrightarrow y(x) + \delta y(x) \qquad \text{for} \quad x \in [x_1, x_2] \tag{2}$$

where $\delta y(x)$ is an arbitrary function.

Variation of Functional $S[f]$ with Fixed End Points

For an infinitesimal variation of $y(x)$ with fixed points at x_1 and x_2, the change of $S[f]$ is

$$\begin{aligned}
\delta S[f] &= \int_{x_1}^{x_2} \left[f(y(x) + \delta y(x),\, y'(x) + \delta y'(x),\, x) - f(y(x),\, y'(x),\, x) \right] dx \\
&= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \delta y' \right) dx \;=\; \frac{\partial f}{\partial y'} \delta y \bigg|_{x_1}^{x_2} + \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y'} \right) \right] \delta y\, dx \\
&= \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y'} \right) \right] \delta y\, dx
\end{aligned} \tag{3}$$

where $\delta y(x)$ is an arbitrary infinitesimal function with

$$\delta y(x_1) = \delta y(x_2) = 0 \qquad \text{and} \qquad \delta y'(x) = \frac{d}{dx} \delta y(x)$$

Extrema of Functional $S[f]$ with Fixed End Points

Similar to the condition for a maximal or minimal point of a function $F(x)$, $F'(x) = 0$, the extremum of $S[f]$ satisfies

$$\delta S[f] \;=\; \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y'} \right) \right] \delta y \, dx \;=\; 0 \tag{4}$$

For an arbitrary and unconstrainted variation of $\delta y(x)$, the extremum of $S[f]$ requires function $f(x)$ satisfies the Eular-Lagrange Equation,

$$\frac{\partial f}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y'} \right) = 0 \tag{5}$$

Example 1. Shortest distance between two points on a plane

The infinitesimal distance on a 2-dimensional plane along an arbitrary curve $y = y(x)$ is

$$ds \;=\; \sqrt{dx^2 + dy^2} \;=\; \sqrt{1 + [dy(x)/dx]^2}\, dx$$

The distance along a curve that connects point 1 and point 2 is then

$$S[f] \;=\; \int_1^2 ds \;=\; \int_1^2 f(y'(x))\, dx \qquad \text{where} \quad f(y'(x)) = \sqrt{1 + (dy/dx)^2}$$

The shortest distance between point 1 and 2 is thus an extrema of $S[f]$ that satisfies the Eular-Lagrange Equation. Since

$$\frac{\partial f}{\partial y} = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1 + y'^2}}$$

The Eular-Lagrange Equation yields

$$\frac{\partial}{\partial x}\left(\frac{y'}{\sqrt{1 + y'^2}} \right) = 0 \qquad \Longrightarrow \qquad \frac{y'}{\sqrt{1 + y'^2}} = const. \qquad \Longrightarrow \qquad \begin{cases} y'(x) = C_1 \\ y(x) \;= C_1 x + C_2 \end{cases}$$

The straight line is therefore the shortest path on a plane — geodesic on a plane.

Example 2. Shortest distance between two points on a spherical surface

The infinitesimal arc length on sphere is

$$ds^2 \;=\; dx^2 + dy^2 + dz^2 \;=\; r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \;=\; r^2 \left[1 + \left(\frac{d\phi}{d\theta} \sin\theta \right)^2 \right] d\theta^2$$

where $x^2 + y^2 + z^2 = r^2 = \text{constant}$,

$$x = r \sin\theta \cos\phi \,, \qquad y = r \sin\theta \sin\phi \,, \qquad z = r \cos\theta \,,$$

and

$$\begin{cases} dx &= -r\sin\theta\sin\phi\,d\phi + r\cos\theta\cos\phi\,d\theta \\ dy &= r\sin\theta\cos\phi\,d\phi + r\cos\theta\sin\phi\,d\theta \\ dz &= -r\sin\theta\,d\theta \end{cases}$$

The distance between two points on a spherical surface is then

$$S[f] \;=\; \int_1^2 ds \;=\; r\int_1^2 \left[1+\left(\frac{d\phi}{d\theta}\sin\theta\right)^2\right]^{1/2} d\theta \;=\; \int_1^2 f(\phi,\phi',\theta)\,d\theta$$

where $\phi' = d\phi/d\theta$ and

$$f(\phi,\phi',\theta) = r\sqrt{1+(\phi'\sin\theta)^2}$$

Since

$$\frac{\partial f}{\partial\phi} = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial\phi'} = \frac{r\,\phi'\sin^2\theta}{[1+(\phi'\sin\theta)^2]^{1/2}}$$

the Eular-Lagrange Equation is

$$\frac{d}{d\theta}\left(\frac{\partial f}{\partial\phi'}\right) = \frac{d}{d\theta}\left(\frac{r\,\phi'\sin^2\theta}{[1+(\phi'\sin\theta)^2]^{1/2}}\right) = 0 \qquad\Longrightarrow\qquad \frac{\phi'\sin^2\theta}{[1+(\phi'\sin\theta)^2]^{1/2}} = C_1$$

Note that $C_1 < 1$. If the coordinate is such that $\theta = \theta_1 = 0$ at point 1, then $C_1 = 0$ and

$$\frac{\phi'\sin^2\theta}{[1+(\phi'\sin\theta)^2]^{1/2}} = 0 \qquad\Longrightarrow\qquad \frac{d\phi}{d\theta} = 0 \qquad\Longrightarrow\qquad \phi = \phi_2$$

where (r,ϕ_2,θ_2) is the coordinate of point 2. The shortest distance is thus the big arc,

$$r = \text{constant}\,, \qquad \phi = \phi_1 = \phi_2\,, \qquad \theta_1 \le \theta \le \theta_2$$

In a general coordinate,

$$\frac{\phi'\sin^2\theta}{[1+(\phi'\sin\theta)^2]^{1/2}} = C_1 \qquad\Longrightarrow\qquad \phi' = \frac{C_1}{\sin\theta\,(\sin^2\theta - C_1^2)^{1/2}} = \frac{C_1\csc^2\theta}{(1-C_1^2\csc^2\theta)^{1/2}}$$

Since $d\cot\theta = -\csc^2\theta\,d\theta$ and $\csc^2\theta = 1+\cot^2\theta$,

$$\phi(\theta) = -\int \frac{C_1\,d\xi}{[1-C_1^2(1+\xi^2)]^{1/2}} = -\int \frac{d\eta}{\sqrt{1-\eta^2}} = -\sin^{-1}\eta + C_2$$

where $\xi = \cot\theta$ and $\eta = C_1\xi/\sqrt{1-C_1^2} = C\,\xi$. Therefore, the curve on a spherical surface that has a shortest distance between two point is

$$\eta = C\,\frac{\cos\theta}{\sin\theta} \;=\; \sin(\phi - C_2) = \sin\phi\cos C_2 - \cos\phi\sin C_2$$

which yields

$$\sin C_2\sin\theta\cos\phi - \cos C_2\sin\theta\sin\phi + C\cos\theta = 0 \qquad\Longrightarrow\qquad Ax + By + Cz = 0$$

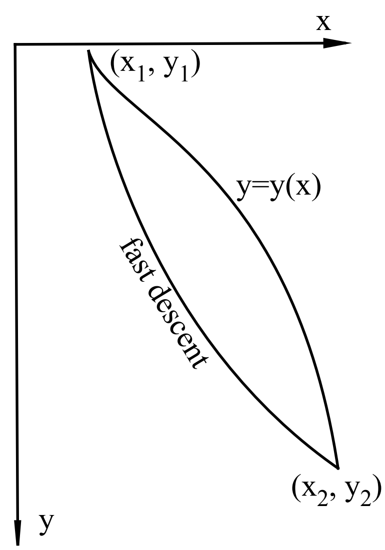
This is the equation of a plane passing through the origin. The intersection of this plane and the sphere is a big circle — geodesic on sphere.

Example 3. The Brachistochrone problem: Curve of fast descent.

This problem is to find the path of least time for a particle traveling under the gravity from a high point at $(x_1, y_1) = (0, 0)$ to a low point at $(x_2, -y_2)$, where $y_2 > 0$.

We first consider the case that the particle is initially at rest, *i.e.* $v_1 = 0$. The speed of a particle at any position on a path is

$$\frac{1}{2}mv^2 - mgy = 0 \qquad \Longrightarrow \qquad v = \sqrt{2gy}$$



where $y_2 > y > y_1 = 0$. The infinitesimal distance for a particle traveling along a path of $y = y(x)$ is

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2}\, dx$$

The time needed for a particle traveling along a path $y = y(x)$ from (x_1, y_1) to (x_2, y_2) is then

$$t_{12} = \int_1^2 \frac{ds}{v} = \int_1^2 f(y(x), y'(x), x)\, dx$$

where

$$f(y(x), y'(x), x) = \left(\frac{1 + y'^2}{2gy}\right)^{1/2}$$

Since

$$\frac{\partial f}{\partial y} = -\frac{1}{2y^{3/2}}\left(\frac{1 + y'^2}{2g}\right)^{1/2} = -\frac{1}{2y}\left(\frac{1 + y'^2}{2gy}\right)^{1/2} \qquad \text{and} \qquad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{2gy(1 + y'^2)}}$$

the Eular-Lagrange Equation for the least time is

$$-\frac{1}{2y}\sqrt{\frac{1 + y'^2}{2gy}} - \frac{d}{dx}\left[\frac{y'}{\sqrt{2gy\,(1 + y'^2)}}\right] = 0$$

which can be rewritten as

$$-\frac{y'}{y^2} = 2\left[\frac{y'}{\sqrt{y(1 + y'^2)}}\right]\frac{d}{dx}\left[\frac{y'}{\sqrt{y(1 + y'^2)}}\right] \qquad \Longrightarrow \qquad \frac{d}{dx}\left(\frac{1}{y}\right) = \frac{d}{dx}\left[\frac{y'^2}{y(1 + y'^2)}\right]$$

The solution of the Eular-Lagrange Equation is then

$$\frac{1}{y} = \frac{y'^2}{y(1 + y'^2)} + c_1 \qquad \Longrightarrow \qquad x + c_2 = \int \sqrt{\frac{c_1\, y}{1 - c_1\, y}}\, dy$$

To calculation the integration, let $c_1 y = \sin^2\theta$,

$$x + c_2 = \int \sqrt{\frac{c_1\, y}{1 - c_1\, y}}\, dy = \frac{2}{c_1}\int \sin^2\theta d\theta = \frac{1}{c_1}\int (1 - \cos 2\theta) d\theta = \frac{1}{2c_1}(2\theta - \sin 2\theta)$$

The solution parameterized by $\phi = 2\theta$ is then

$$\begin{cases} x &= a(\phi - \sin\phi) - c_2 \\ y &= a(1 - \cos\phi) \end{cases}$$

where $a = 1/(2c_1)$. For $y = 0$ at $x = 0$, $c_2 = 0$. This is a cycloid passing through the origin.

For the case that the particle has an initially speed of v_0, the speed of a particle at any position on a path becomes

$$\frac{1}{2}mv^2 - mgy = \frac{1}{2}mv_0^2 = 2mgC \qquad \Longrightarrow \qquad v = \sqrt{2gy + 2gC} = \sqrt{2gz}$$

where $C = v_0^2/(2g)$, $z(x) = y(x) + C$ and $z'(x) = y'(x)$. The time needed for a particle traveling along a path $y = y(x)$ from (x_1, y_1) to (x_2, y_2) is then

$$t_{12} = \int_1^2 \left(\frac{1 + z'^2}{2gz}\right)^{1/2} dx$$

which will yield the same solution as the case of $v_0 = 0$.

Multi-Dimensional Eular-Lagrange Equation

Consider the extrema of functionals in the N-dimensional function space

$$S[f] = \int_{x_1}^{x_2} f(\,\vec{y}(x),\, \vec{y}\,'(x),\, x\,)\, dx \tag{6}$$

where $\vec{y} = (y_1(x), y_2(x), ..., y_N(x))$ with $\vec{y}\,'(x) = (y_1'(x), y_2'(x), ..., y_N'(x))$ is an arbitrary N-dimensional vector function. For an infinitesimal variation of $\vec{y}(x)$ with fixed points at x_1 and x_2,

$$\vec{y}(x) \Longrightarrow \vec{y}(x) + \delta\vec{y}(x) \qquad \text{where} \quad \delta\vec{y}(x_1) = \delta\vec{y}(x_2) = 0\,,$$

the change of $S[f]$ is

$$\begin{aligned} \delta S[f] &= \int_{x_1}^{x_2} [f\,(\,\vec{y}(x) + \delta\vec{y}(x),\, \vec{y}\,'(x) + \delta\vec{y}\,'(x),\, x\,) - f(\,\vec{y}(x),\, \vec{y}\,'(x),\, x\,)]\, dx \\ &= \int_{x_1}^{x_2} \sum_{i=1}^{N} \left(\frac{\partial f}{\partial y_i}\delta y_i + \frac{\partial f}{\partial y_i'}\delta y_i'\right) dx = \sum_{i=1}^{N} \left\{ \frac{\partial f}{\partial y_i}\delta y_i \Big|_{x_1}^{x_2} + \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y_i} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y_i'}\right)\right] \delta y_i\, dx \right\} \\ &= \sum_{i=1}^{N} \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y_i} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y_i'}\right)\right] \delta y_i\, dx \end{aligned} \tag{7}$$

For arbitrary and unconstrainted variations of $\{\,\delta y_i(x)|\, i = 1, ...N\}$, an extremum of $S[f]$ requires function $f(x)$ satisfies the N-dimensional Eular-Lagrange Equation,

$$\frac{\partial f}{\partial y_i} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y_i'}\right) = 0\,, \qquad i = 1, ..., N$$