Chapter 5. Oscillations

5.1-5.2 Hooke's Law and Simple Harmonic Motion

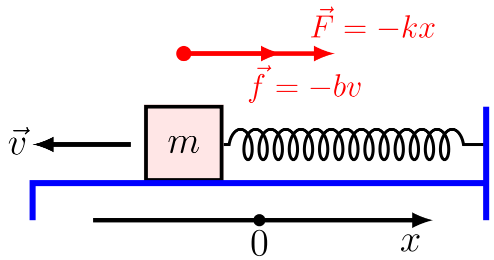
5.3 Two-Dimensional Oscillations

5.4 Damped Oscillations

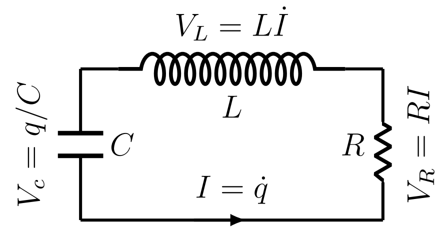
In general, the equation of motion for one-dimensional damped harmonic oscillator can be written as

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \tag{1}$$

This equation of motion could come from many different physical systems. Here are two examples,



$$m\ddot{x} = -kx - b\dot{x}$$



$$V_L + V_R + V_C = 0$$
$$L\ddot{q} + R\dot{q} + q/C = 0$$

To solve Eq. (1), we try a solution in the form of

$$x(t) = e^{rt}\,, \qquad \dot{x}(t) = re^{rt}\,, \qquad \ddot{x}(t) = r^2 e^{rt}$$

where r is a constant. The characteristic equation of Eq. (1) is

$$r^2 + 2\beta r + \omega_0^2 = 0 \qquad \Longrightarrow \qquad r_{1,2} = -\beta \pm \sqrt{\beta^2 - \omega_0^2} \tag{2}$$

The general solution of the damped harmonic oscillator is then

$$x(t) = e^{-\beta t}\left(C_1\, e^{\sqrt{\beta^2-\omega_0^2}\,t} + C_2\, e^{-\sqrt{\beta^2-\omega_0^2}\,t}\right) \tag{3}$$

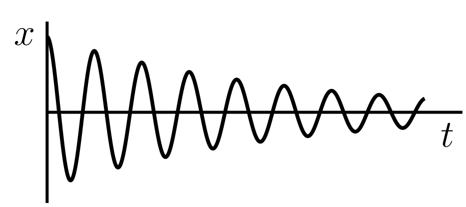
Undamped Oscillation $\beta = 0$

$$x(t) = C_1\, e^{i\omega_0 t} + C_2\, e^{-i\omega_0 t} \tag{4}$$

Weak Damping $\beta < \omega_0$

$$x(t) = e^{-\beta t}\left(C_1\, e^{i\omega t} + C_2\, e^{-i\omega t}\right).$$

where $\omega = \sqrt{\omega_0^2 - \beta^2}$ and $\tau = 1/\beta$ is damping time scale.

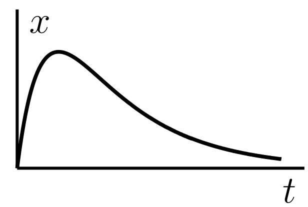


Strong Damping $\beta > \omega_0$

$$x(t) = C_1\, e^{-\left(\beta-\sqrt{\beta^2-\omega_0^2}\right)t} + C_2\, e^{-\left(\beta+\sqrt{\beta^2-\omega_0^2}\right)t}$$

In this case, both terms decay with time and there is no oscillation at all.

Example. An oscillator with $\beta > \omega_0$ is initially kicked, *i.e.* $x(0) = 0$ and $\dot{x}(0) = v_0$. The solution of motion is



$$\begin{aligned} x(t) &= \frac{v_0}{2\sqrt{\beta^2-\omega_0^2}}\left[e^{-\left(\beta-\sqrt{\beta^2-\omega_0^2}\right)t} - e^{-\left(\beta+\sqrt{\beta^2-\omega_0^2}\right)t}\right] \\ &= \frac{v_0}{2\sqrt{\beta^2-\omega_0^2}}\, e^{-\beta t}\sinh\left(\sqrt{\beta^2-\omega_0^2}\, t\right) \end{aligned}$$

Critical Damping $\beta = \omega_0$

The solution in Eq. (3) at $\beta = \omega_0$ is

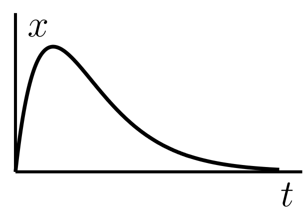
$$x(t) = C\, e^{-\beta t}$$

but it can be easily checked that

$$x(t) = C\, t\, e^{-\beta t}$$

is also a solution. The general solution in this case is therefore

$$x(t) = C_1\, e^{-\beta t} + C_2\, t\, e^{-\beta t} = (C_1 + C_2\, t)\, e^{-\beta t}$$



For an initially kicked oscillator with $x(0) = 0$ and $\dot{x}(0) = v_0$, $C_1 = 0$ and $C_2 = \dot{x}(0) = v_0$, and the solution is

$$x(t) = v_0 t\, e^{-\beta t}$$

5.5 Driven Damped Oscillations

With a driven force, a time-dependent external force, on a damped harmonic oscillator, the equation of motion becomes

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t) \tag{5}$$

where β is the damping constant, ω_0 is the natural frequency of the oscillator, and $f(t)$ is the driven force. This is an inhomogeneous 2nd-order PDE. The solution of Eq. (5) is the sum of a particular solution $x_p(t)$ of Eq. (5) and a general solution $x_h(t)$ in Eq. (3) of the homogeneous equation of Eq. (5) with $f(t) = 0$,

$$x(t) = x_h(t) + x_p(t) \tag{6}$$

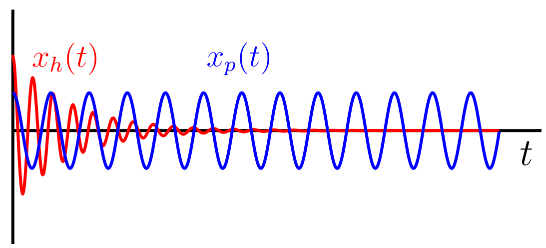
where

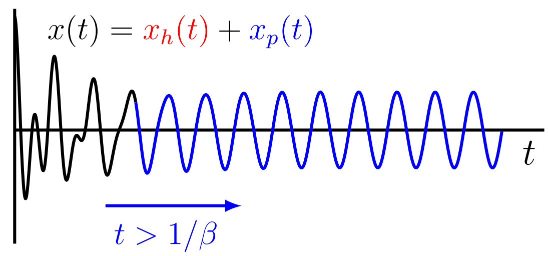
$$x_h(t) = e^{-\beta t}\left(C_1\, e^{\sqrt{\beta^2-\omega_0^2}\, t} + C_2\, e^{-\sqrt{\beta^2-\omega_0^2}\, t}\right)$$

For a periodic or a pulsed driven force $f(t)$, the particular solution $x_p(t)$ can be obtained in general using the Fourier series. Since the general solution $x_h(t)$ decays exponentially, the driven damped harmonic oscillation is dominated by the particular solution $x_p(t)$ after the damping time of the general solution, *i.e.*

$$x(t) \overset{t>1/\beta}{=} x_p(t) \tag{7}$$

For example, if $x_p(t) = A\cos(\omega t + \delta)$ and $x_h(t) = Ce^{-\beta t}\cos(\omega_0 t)$ for $\beta < \omega_0$,





5.7 Fourier Series of a Periodic Function

Any continuous function that is periodic with period τ and frequency $\omega = 2\pi/\tau$,

$$f(t+\tau) = f(t)$$

can be expanded into a Fourier series

$$f(t) = \sum_{n=0}^{\infty} \left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right] \tag{8}$$

where a_n and b_n are constant expansion coefficients.

Orthogonality of Sine and Cosine Functions

For arbitrary integer n and m with $n \neq m$,

$$\begin{aligned}
\int_{-\tau/2}^{\tau/2} \cos(n\omega t)\cos(m\omega t)dt &= \frac{1}{2}\int_{-\tau/2}^{\tau/2} \left[\cos((n-m)\omega t) + \cos((n+m)\omega t)\right] dt \\
&= \frac{1}{2\omega}\left[\frac{1}{(n-m)}\sin((n-m)\omega t) + \frac{1}{(n+m)}\sin((n+m)\omega t)\right]_{-\tau/2}^{\tau/2} = 0 \\
\int_{-\tau/2}^{\tau/2} \sin(n\omega t)\sin(m\omega t)dt &= \frac{1}{2}\int_{-\tau/2}^{\tau/2} \left[\cos((n-m)\omega t) - \cos((n+m)\omega t)\right] dt \\
&= \frac{1}{2\omega}\left[\frac{1}{(n-m)}\sin((n-m)\omega t) - \frac{1}{(n+m)}\sin((n+m)\omega t)\right]_{-\tau/2}^{\tau/2} = 0 \\
\int_{-\tau/2}^{\tau/2} \sin(n\omega t)\cos(m\omega t)dt &= \frac{1}{2}\int_{-\tau/2}^{\tau/2} \left[\sin((n+m)\omega t) + \sin((n-m)\omega t)\right] dt \\
&= -\frac{1}{2\omega}\left[\frac{1}{(n+m)}\cos((n+m)\omega t) + \frac{1}{(n-m)}\cos((n-m)\omega t)\right]_{-\tau/2}^{\tau/2} = 0
\end{aligned}$$

where

$$\begin{aligned}\sin\left(\pm(n\pm m)\frac{\omega\tau}{2}\right) &= \pm\sin\left((n\pm m)\pi\right) = 0 \\ \cos\left(\pm(n\pm m)\frac{\omega\tau}{2}\right) &= \cos\left((n\pm m)\pi\right) = (-1)^{(n\pm m)}\end{aligned}$$

and

$$\begin{aligned}\frac{2}{\tau}\int_{-\tau/2}^{\tau/2}\cos(n\omega t)\cos(n\omega t)dt &= \frac{1}{\tau}\int_{-\tau/2}^{\tau/2}\left[1+\cos(2\omega t)\right]dt = 1+\left.\frac{\sin(2\omega t)}{2\tau\omega}\right|_{-\tau/2}^{\tau/2} = 1 \\ \frac{2}{\tau}\int_{-\tau/2}^{\tau/2}\sin(n\omega t)\sin(n\omega t)dt &= \frac{1}{\tau}\int_{-\tau/2}^{\tau/2}\left[1-\cos(2\omega t)\right]dt = 1-\left.\frac{\sin(2\omega t)}{2\tau\omega}\right|_{-\tau/2}^{\tau/2} = 1 \\ \int_{-\tau/2}^{\tau/2}\sin(n\omega t)\cos(n\omega t)dt &= \left.\frac{1}{n\omega}\sin^2(n\omega t)\right|_{-\tau/2}^{\tau/2} = 0\end{aligned}$$

The sine and cosine functions are therefore orthogonal functions,

$$\frac{2}{\tau}\int_{-\tau/2}^{\tau/2}\cos(n\omega t)\cos(m\omega t)dt = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2}\sin(n\omega t)\sin(m\omega t)dt = \delta_{nm} \tag{9}$$

$$\int_{-\tau/2}^{\tau/2}\sin(n\omega t)\cos(m\omega t)dt = 0 \tag{10}$$

With the orthogonal condition of the sine and cosine functions, the expansion coefficients of the Fourier expansion in Eq. (8) can be obtained by an integral of a sine or cosine function on the both sides of Eq. (8),

$$\begin{aligned}&\frac{1}{\tau}\int_{-\tau/2}^{\tau/2}dt\left\{f(t) = \sum_{n=0}^{\infty}\left[a_n\cos(n\omega t)+b_n\sin(n\omega t)\right]\right\} \\ &\frac{2}{\tau}\int_{-\tau/2}^{\tau/2}dt\,\cos(m\omega t)\left\{f(t) = \sum_{n=0}^{\infty}\left[a_n\cos(n\omega t)+b_n\sin(n\omega t)\right]\right\}, \qquad \text{for } m\ge 1 \\ &\frac{2}{\tau}\int_{-\tau/2}^{\tau/2}dt\,\sin(m\omega t)\left\{f(t) = \sum_{n=0}^{\infty}\left[a_n\cos(n\omega t)+b_n\sin(n\omega t)\right]\right\}, \qquad \text{for } m\ge 1\end{aligned}$$

which yield

$$a_0 = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2}f(t)\,dt \qquad \text{and} \qquad a_m = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2}f(t)\,\cos(m\omega t)\,dt\,, \qquad \text{for } m\ge 1$$

$$b_m = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2}f(t)\,\sin(m\omega t)\,dt\,, \qquad \text{for } m\ge 1$$

5.8 Fourier Series for Particular Solution of Driven Oscillator

We now use the Fourier expansion to solve the driven damped harmonic oscillator

$$\ddot{x}+2\beta\,\dot{x}+\omega_0^2\,x = f(t) \tag{11}$$

where $f(t) = f(t+\tau)$ is any periodic driving force with period τ and frequency $\omega = 2\pi/\tau$. Let

$$f(t) \;=\; \sum_{n=0}^{\infty} [f_n \cos(n\omega t) + g_n \sin(n\omega t)]$$

be the Fourier expansion of $f(t)$, where the expansion coefficients f_n and g_n can be calculated for a given function $f(t)$. The Fourier expansion of the particular solution of the oscillator can be solved from the equation of motion. Let

$$\begin{aligned}
x(t) &= \sum_{n=0}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)] \\
\dot{x}(t) &= \omega \sum_{n=0}^{\infty} n\; [-a_n \sin(n\omega t) + b_n \cos(n\omega t)] \\
\ddot{x}(t) &= -\omega^2 \sum_{n=0}^{\infty} n^2\; [a_n \cos(n\omega t) + b_n \sin(n\omega t)]
\end{aligned}$$

With the Fourier expansion of $x(t)$ and $f(t)$, the equation of motion becomes

$$\sum_{n=0}^{\infty} \left[\left(-\omega^2 n^2 a_n + 2\beta\omega\, n b_n + \omega_0^2 a_n\right)\cos(n\omega t) + \left(-\omega^2 n^2 b_n - 2\beta\omega\, n a_n + \omega_0^2 b_n\right)\sin(n\omega t)\right] = \sum_{n=0}^{\infty} [f_n \cos(n\omega t) + g_n \sin(n\omega t)]$$

The orthogonal condition of the sine and cosine functions yields,

$$\left\{ \begin{aligned} -\omega^2 n^2 a_n + 2\beta\omega\, n b_n + \omega_0^2 a_n &= f_n \\ -\omega^2 n^2 b_n - 2\beta\omega\, n a_n + \omega_0^2 b_n &= g_n \end{aligned} \right. \tag{12}$$

From the 2nd equation,

$$b_n = \frac{g_n + 2\beta\omega\, n\, a_n}{\omega_0^2 - n^2\omega^2}$$

and substituting it into the 1st equation yields

$$a_n = \frac{(\omega_0^2 - n^2\omega^2) f_n - 2\beta\omega\, n\, g_n}{(\omega_0^2 - n^2\omega^2)^2 + (2\beta\omega\, n)^2} \qquad \text{and} \qquad b_n = \frac{(\omega_0^2 - n^2\omega^2) g_n + 2\beta\omega\, n\, f_n}{(\omega_0^2 - n^2\omega^2)^2 + (2\beta\omega\, n)^2}$$

The particular solution for the driven damped harmonic oscillation is then

$$\begin{aligned}
x(t) &= \sum_{n=0}^{\infty} \frac{f_n}{(\omega_0^2 - n^2\omega^2)^2 + (2\beta\omega\, n)^2} \left[(\omega_0^2 - n^2\omega^2)\, \cos(n\omega t) + (2\beta\omega\, n)\, \sin(n\omega t)\right] \\
&+ \sum_{n=0}^{\infty} \frac{g_n}{(\omega_0^2 - n^2\omega^2)^2 + (2\beta\omega\, n)^2} \left[(\omega_0^2 - n^2\omega^2)\, \sin(n\omega t) - (2\beta\omega\, n)\, \cos(n\omega t)\right] \\
&= \sum_{n=0}^{\infty} \frac{1}{\sqrt{(\omega_0^2 - n^2\omega^2)^2 + (2\beta\omega\, n)^2}} \left[f_n\, \cos(n\omega t - \delta_n) + g_n\, \sin(n\omega t - \delta_n)\right]
\end{aligned} \tag{13}$$

where

$$\delta_n = \arctan\left(\frac{2\beta\omega\, n}{\omega_0^2 - n^2\omega^2}\right)$$

5.6 A Single Sinusoidal Driving Force $f(t) = f_1 \cos(\omega t)$

In this case, $g_n = 0$ for all n and $f_n = 0$ for $n \neq 1$. The solution of the oscillation is then

$$x(t) = \frac{f_1}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}} \cos(\omega t - \delta_1) \tag{14}$$

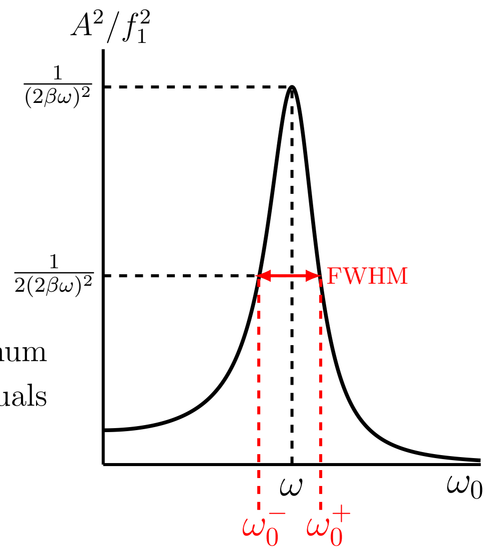
where the phase shift is

$$\delta_1 = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)$$

and the amplitude of the oscillation is

$$A^2 \;=\; \frac{f_1^2}{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}$$

The amplitude of the driven oscillation reaches its maximum when the natural frequency of the harmonic oscillator equals the driven frequency — **resonance**.



Width of the Resonance and the Q Factor

FWHM = full width at half maximum

The maximum of the oscillation amplitude is at the resonance frequency $\omega_0 = \omega$ and the maximal amplitude is $A^2_{max} = f_1^2/(2\beta\omega)^2$. The FWHM is at the values of ω_0 that are solved from $A^2 = A^2_{max}/2$, *i.e.*

$$\frac{f_1^2}{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2} = \frac{f_1^2}{2(2\beta\omega)^2} \qquad \Longrightarrow \qquad (\omega_0^2 - \omega^2)^2 = (2\beta\omega)^2$$

The FWHM are thus at

$$(\omega_0^2 - \omega^2)^{\pm} \;=\; \pm 2\beta\omega \qquad \Longrightarrow \qquad \omega_0^{\pm} \;=\; \omega\sqrt{1 \pm 2\beta/\omega} \;\simeq\; \omega(1 \pm \beta/\omega)$$

Therefore

$$\text{FWHM} \;=\; \omega_0^+ - \omega_0^- \;\simeq\; 2\beta$$

The sharpness of the resonance peak is measured by the quality factor Q that is the ratio of the resonance frequency $\omega_0 = \omega$ to the width of the resonance peak 2β,

$$Q \;=\; \frac{\omega_0}{2\beta}$$

The larger the Q factor, the sharper (narrower) the resonance peak.