Finding The Lost Gravitational Constant*

Felipe Tala, Camdem Ruckman†
University of Kansas \\
(Dated: November 28, 2023)

In this experiment, we carried out measurements for three different metals, utilizing a Cavendish device, to measure the gravitational constant. Our result plots suggested accurate measurements, while our final result for the gravitational constant was $G = 2.5117e^-11[m^3/Kg.s^2]$, suggesting calculations or uncertainty inaccuracy for an expected gravitational constant of $G = 6.67430e^-11[m^3/Kg.s^2]$.

I. INTRODUCTION

A.

Many of us go through our lives without stopping and questioning our surroundings. We get caught up in the routine, and next thing we know, we're doing the same thing every single day. By having an analytical perception, we are able to escape the rat race and wonder, Why is the sky blue? How was this computer used to read this article? Why are there different languages?

It is Physicists' duty to stop and question the nature of things. Yet, some intrinsic traits in nature go unseen even by Physicists. One of these is gravity. Is it such an obvious part of our daily life that we never think about how life would be without it?

For Sir Isaac Newton, the gravity question seemed to be an essential question to be answered. After being isolated from society for a few years, he surprised the world by introducing Physics as an area to be studied. His answer to the gravity question was: "Once a body is set in motion, it will remain moving at constant speed in a straight line unless a force acts on it."[1]

Newton understood from an early time that gravity was a force exerted on our bodies constantly, and we were just unaware of it because it did not change. This helped him formulate his famous universal gravitational equation, finding that there was a constant relationship between the force of a two-body system, and the mass and distance between these.

$$F_{grav} = \frac{GmM}{r^2} \tag{1}$$

In Newton's universal gravitational formula, m and M represent the masses of two bodies, and r represents the distance between them, while G is the constant found by Newton, which defined the linear relationship in the equation. Yet, there was yet one experiment to be made to complete the formula. Finding G. In 1798, Mr. Cavendish 'weighted the world". At the time of the experiment, Cavendish was nearly sixty-seven.[2], contradicting Einstein's future famous quote "A person who has not made his great contribution to science before the age of 30 will never do so". Mr. Cavendish had not published an experiment for ten years, and he would not publish another.[2]

The first mention of weighing the world occurred in a correspondence in 1783 between Cavendish and the Yorkshire clergyman John Michell. They were writing to one another about an even more ambitious weighing, that of the stars, though they touched on other subjects as well. Cavendish wrote to Michell, if your health does not allow you to go on with that [grinding a large mirror for a telescope] I hope it may at least permit the easier and less laborious employment of weighing the world."[2].

After a year of working on several improved devices, Mr. Cavendish achieved the ultimate torsion balance that made him the first person to measure the gravitational constant. The goal of this article is to present the steps followed towards measuring the gravitational constant and its results.

B. Formulas

For an underdamped harmonic oscillator, the angle of an object with respect to time can be represented by the following equation:

$$\theta(t) = Acos(\omega t - \phi) * e^{-t/\tau} + C \tag{2}$$

Where θ is the angle of the underdamped harmonic oscillator with respect to time, A is the amplitude of the system, ω is its resonant frequency, t is the time variable, ϕ is the phase of the sinusoidal system, τ is the decaying time constant, and C is a shift constant.

The moment of inertia of the system is defined by the following equation:

$$I = 2mr^2 \tag{3}$$

Where I is the moment of inertia, m is the mass of the small spheres, labelled as q in Fig. 1, and r is the dis-

*

† Also at Physics Department, University of Kansas; felipe.tala@ku.edu

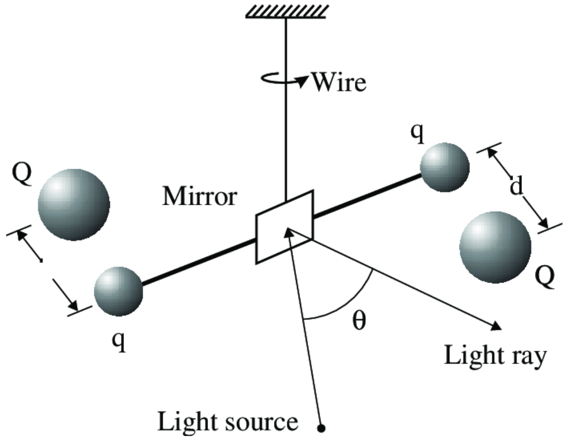

FIG. 1. Experiment Setup

tance between the spheres and the center of mass which is located in the middle of the mirror in Fig. 1.

The fundamental frequency of the gravitational wave is given by the following equation:

$$\omega_o = 2\pi\sqrt{\frac{k}{I}} \tag{4}$$

For this equation, I is the moment of inertia of the system, k is the spring constant of the oscillator, and ω_1 is the fundamental frequency. The fundamental frequency relates to the time constant τ by the following equation:

$$\omega_o = \sqrt{{\omega_1}^2 + (\frac{1}{\tau})^2} \tag{5}$$

The energy of the underdamped oscillator system is given by:

$$E = \frac{1}{2}kA^2 \tag{6}$$

Here A is the amplitude of the underdamped oscillator, k is the spring constant, and E is the total energy of the system.

Finally, we used the equation for the total energy of a two-body gravitational system, given by:

$$E = \Delta F = -2GMm[(\frac{1}{r_1} + \frac{1}{r_2}) - (\frac{1}{r_3} + \frac{1}{r_4})] \tag{7}$$

Where E is the total energy of the system, which is equal to ΔF, the change in force of the system.

G is the gravitational constant, m is the small ball mass for this experiment, M is the tungsten big ball mass, and r is the distance between the small and big ball. For this experiment, we have 4 bodies. Because the small balls are fixed and evenly distributed in the same axis, we only have 4 different radius components. Looking at Fig. 1, calling the left-side balls q_1 and Q_1, and the right-side balls q_2 and Q_2, these radius are defined as:

r_1: distance between q_1 and Q_1
r_2: distance between q_1 and Q_2
r_3: distance between q_2 and Q_1
r_4: distance between q_2 and Q_2

II. METHOD

This experiment consisted of two main instruments used to make measurements. The first instrument was a torsion balance. The second instrument used was a measurement tape, which was stuck to a wall in the lab. The torsion balance consisted of 5 main parts. The main case, a laser, two tungsten spheres, a mirror, and two "'aluminum spheres"'. These last two parts were inside the main case. The tungsten spheres were placed in a longitudinal pendulum with two circular bases at its ends, where the spheres fit. at a distance of $25cm$ from the main case.

When the laser was set in the right direction, it would shoot into the mirror aligned with the inner "'aluminum"' spheres and reflect. The reflection of the laser beam appeared in the wall where the metric tape was set.

The weight of the tungsten spheres was measured using a scale. These were both found to be $3096g$.
The distance from the torsion balance to the wall was measured with a metric tape, as well as the distance from the floor to the mirror in the torsion balance, and the distance from the floor to the initial laser reflection.

We started our measurements by setting up a 30 seconds beeping timer. We made a mark for the initial position of the laser beam in the wall. Following, we changed the position of the tungsten spheres, by rotating 180° them around the mirror once. Once the tungsten spheres were back in touch with the wall where the tungsten balls were, the tungsten spheres started to attract and repel from the tungsten balls. This behavior made the whole device move in oscillation. The mirror started to oscillate as well because it was centered between the tungsten balls. This oscillation could be observed in the laser reflection in the wall.

To keep track of the change of the laser reflection on the wall, one of us would make a mark in the position of the laser beam in the wall board aligned with the measurement tape every 30 seconds and shout out loud the position of the mark with respect to the metric tape. The other student would write down the position of the measurement in an Excel sheet to keep precise track of the measurements and observe whether the behavior of the laser beam reflection was oscillating and was unaffected by any external disturbances.

Finally, we utilized the following figure to develop formulas for finding the values of r_1, r_2, r_3, and r_4.

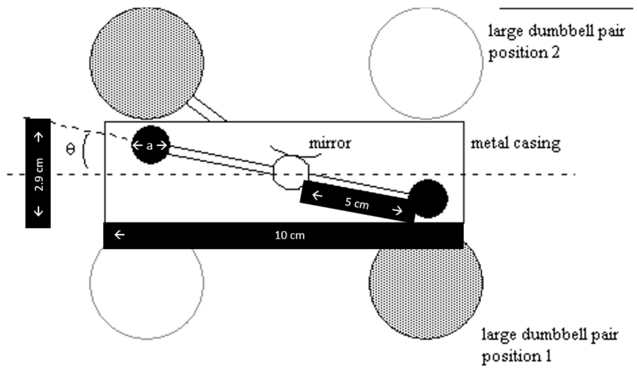

FIG. 2. Experiment's 2D setup seen from above

The distance between the two big balls is fixed to $10cm$, and the distance between the small balls and their center of mass is fixed to $5cm$. Finally, the distance between the each big ball and the in-parallel small balls has a maximum fixed distance of 2.9 cm. The distance a is dependent on the size of the big ball. The radius of the big balls was labelled as r_z. The value for the radius of the small balls had a fix value of $r_{small} = 0.69cm$.

From Fig.2, we derived the following formulas to solve for the radius of the experiment's gravitational system.

$$r_1 = \sqrt{(10-a)^2 + (r_{small} + r_z)^2} \tag{8}$$

$$r_2 = \sqrt{(a^2 + (2.9 - r_{small} + r_z)^2} \tag{9}$$

$$r_3 = \sqrt{a^2 + r_{small} + r_z} \tag{10}$$

$$r_4 = \sqrt{(10-a)^2 + (2.9 - r_{small} + r_z)^2} \tag{11}$$

For these equations, using Fig.2 as reference, and the large dumbbell position 2, r_1 is the distance between the small ball on the right and the big ball on the left of the sketch. r_2 is the distance between the small ball on the left and the big ball on the left of the sketch. r_3 is the distance between the the small ball on the right and the big ball on the right side of the sketch. r_4 is the distance between the small ball on the left and the big ball on the right of the sketch.

III. RESULTS

Once we finished gathering the positional measurements for the laser reflections, we used Eq.IV to turn our positional data into angular data. We continued to plot the angular measurements with respect to time. These plots can be observed for each metal's measurements in Fig. 3.

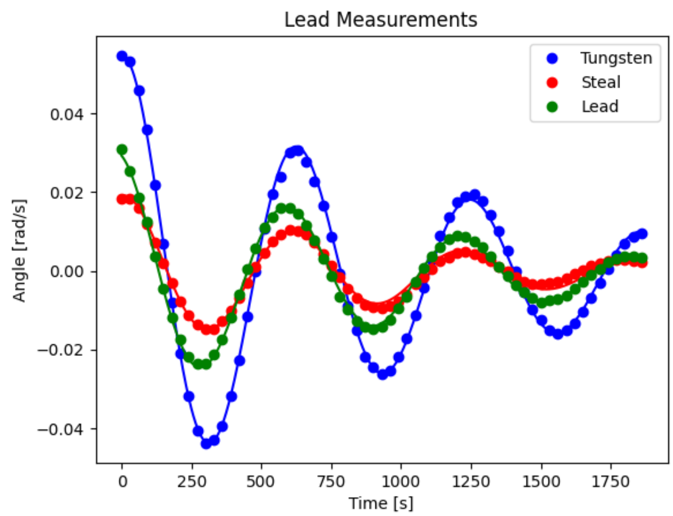

FIG. 3. Gravity Measurements

In the figure above we can observe the sinusoidal fitted measurements for all three metals: tungsten, steal and lead. The plot represents the laser reflection position measurements taken in the lab wall, as a function of time. We can observe that tungsten has the largest amplitude, and therefore under takes the longest time to become damped. On the other hand, lead damps the fastest as a consequence of having the smallest amplitude.

Through plotting, we found the characteristic parameters for the damped harmonic oscillation equation describing the behavior of the three metals' position with respect to time. We can observe the specific values for each experiment in the following chart:

Parameter	Tungsten	Steal	Lead
$M[Kg]$	3.096	1.090	1.545
A	$5.6108e-02$	$1.9941e-02$	$3.0901e-02$
ω	$1.0111e-02$	$1.0374e-02$	$1.0187e-02$
ϕ	6.1758	6.1379	6.4643
τ	$1.1654e+03$	$9.7570e+03$	$1.0377e+03$
C	$-1.0965e-03$	$-5.1970e-03$	$3.1085e-03$

TABLE I. Measurement Parameters

These parameters can be observed in Eq. I B. In these measurements we can observe how some parameters behave as a function mass. In Fig. 4 we can observe the relationship between the amplitude, decaying time constant, and resonant frequency with respect to mass.

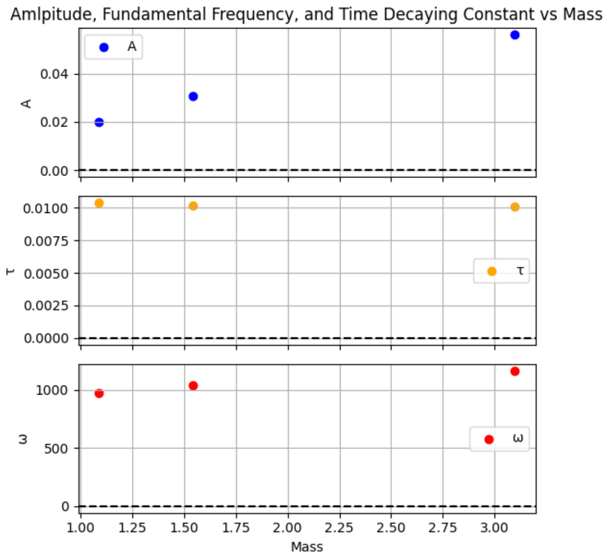

FIG. 4. Mass relationships to resonant frequency, amplitude and decaying exponential

In Fig. 4 we can observe the linear relationship between the amplitude of the gravitational wave and the mass of the sphere. There is no relationship between the mass of the big balls and their decaying exponential. Finally, there seems to be a relationship between the mass and the resonant frequency of the gravitational wave, but due to the experiments' total uncertainty, this relationship is not strong enough to conclude such a relationship.

Table I values were used to solve for the gravitational constant, through the following steps:

We calculated the moment of inertia using Eq.I B, with a given mass for the small spheres, labeled q in 1, being $m = 0.015Kg$, and a distance from the sphere to the center of mass of $r = 6.9mm$. For these values, the moment of inertia is $I = 1.428e + 03g.mm^2$.

We continued by calculating the fundamental frequency of the system. By plugging the values for the resonant frequency, ω_o, and the time decaying constant, τ, into Eq.I B, we found a fundamental frequency of 0.00855362828 Hz.

Having solved for the fundamental frequency and the moment of inertia, we used Eq.I B to find that the system's spring constant k is 3.67206753 $times10^{-12}$.

We used the calculated spring constant and the amplitude found by plot fitting our data, and were able to calculate the total energy of the system using Eq. I B. The total energy of the system was $E = 8.39343e - 11$ J.

To solve for the gravitational constant, we had to work through trigonometric calculations to find $r1$, $r2$, $r3$, and $r4$ for our system. We utilized the Fig.2 derived equations Eq.8, Eq.9, Eq.10, and Eq.11 to find the different values for r_1, r_2, r_3, and r_4 for each big ball case. Having solved for the r values, we plugged these numbers into I B to solve for G.

To get the most accurate approximation of the gravitational constant, we carried out this equation-solving procedure three times for each metal measurement and made an average of each gravitational constant value found. We found an average gravitational constant of $G = 2.5117e - 08[m^3/Kg.s^2]$.

IV. UNCERTAINTIES

The uncertainties in this experiment are given by the following:

- Item 1: Metric tape used to measure the distance between the gravity device and the wall where the laser beam was reflected

- Item 2: Scale used to measure the weight of the big balls

- Item 3: Metric tape used to measure the position of the laser beam reflection

Item 1 provided an uncertainty of $\pm 0.79375mm$, Item 2 provided an uncertainty of $\pm 0.015g$, and Item 3 provided an uncertainty of $\pm 0.002m$.

We first used the error propagation equation **??** to find the total error in the measured angle.

$$z = \frac{x - x_0}{D} \rightarrow \theta = \arctan z \tag{12}$$

$$\delta_z = \sqrt{(\frac{\partial z}{\partial(x - x_0)})^2 * (\delta x - x_0)^2 + (\frac{\partial z}{\partial D})^2 * (\delta D)^2} \tag{13}$$

Where z is the distance between the laser and the reflection of the laser beam on the wall, $x - x_0$ is the distance between the center of the reflection of the laser beam and the farthest point measured. We find the total angle error to be 0.01055°.

We continued to apply IV again, this time for the total error for the gravitational constant, which depends on the angle error and the error from the scale used to weight the mass of the tungsten balls.

$$\delta_G = \sqrt{(\frac{\partial G}{\partial \theta})^2 * (\delta\theta)^2 + (\frac{\partial G}{\partial M})^2 * (\delta M)^2} \tag{14}$$

Here G is the gravitational constant, θ is the angle between the center of the beam laser reflection and its furthest point measured, and M is the mass of the tungsten balls. This yields a total error of $\pm 0.696601e - 06[m^3/Kg.s^2]$ for the gravitational constant.

V. SUMMARY

We found the gravitational constant to be $G = 2.5117e - 11[m^3/Kg.s^2]$, with a total experimental error of $\pm 0.696601e - 11[m^3/Kg.s^2]$, meaning the exact gravitational constant we measured is inside the range $1.8151e - 11[m^3/Kg.s^2] \leq G \leq 3.2083e - 11[m^3/Kg.s^2]$.

In conclusion, the experiment was a partial success, given that the data look accurate with the expected underdamped harmonic behavior of gravity, but the expected gravitational constant value is not inside the uncertainty range compared to the calculated gravitational constant. This could be a consequence of the scale used to measure the mass of the big balls, as this was provided by word and not in the lab manual or measured.

Because of the precise data taken in Fig. 3, we can further conclude that gravity behaved as a semi-harmonic underdamped oscillator, because the data matches the expected behavior for gravity with greater accuracy for our first half of the measurements for every metal. This means that gravity behaves like a box attached to a spring moving in a curved ramp instead of a straight ramp.

Appendix A: Appendixes

[1] Schutz2003gravity, Gravity from the ground up: An introductory guide to gravity and general relativity, Schutz, Bernard, 2003, Cambridge university press

[2] mccormmach1998mr, Mr. Cavendish weighs the world, McCormmach, Russell, Proceedings of the American Philosophical Society, volume 142, number 3, pages 355-366, 1998, JSTOR