EDC - 3ª prova individual - tipo A (25 pts)

1. (7 pontos) Calcule a série de Fourier gerada por $f(x)$ e desenhe o gráfico da função para a qual a série converge no intervalo $[-3,3]$:

$$f(x) = x^2 \quad \text{para} \;\; x \in (-1,1), \qquad f(x+2) = f(x).$$

2. (10 pontos) O problema da difusão de calor em uma barra de comprimento L sem isolamento lateral em ambiente a $0\,^o$C é

$$\begin{cases} \dfrac{\partial u}{\partial t} = \alpha^2 \dfrac{\partial^2 u}{\partial x^2} - hu & (0 < x < L,\ t > 0), \\ u(0,t) = u(L,t) = 0 & (t > 0) \end{cases}$$

onde $\alpha^2 > 0$ é a difusividade térmica do material e $h > 0$ é a taxa de fuga lateral de calor da barra. Encontre todas as soluções separáveis desse problema.

3. (7 pontos) Encontre a solução estacionária (independente do tempo) para o problema de valores de contorno abaixo.

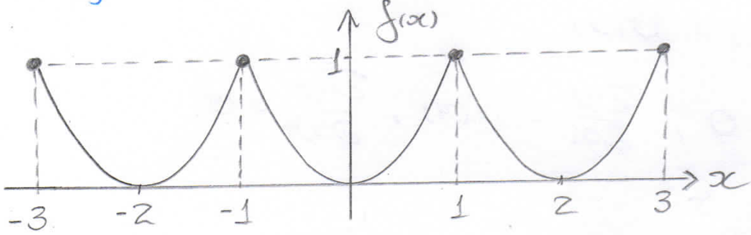
$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = \operatorname{sen} x \quad (0 < x < \pi,\ t > 0), \\ &u(0,t) = 0,\ \frac{\partial u}{\partial x}(\pi,t) = 4 \quad (t > 0) \end{aligned}$$

$$\text{Fourier} \quad \begin{cases} f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} \left[a_n \cos\!\left(\frac{n\pi x}{L}\right) + b_n \operatorname{sen}\!\left(\frac{n\pi x}{L}\right) \right] \\ a_0 = \dfrac{1}{L} \displaystyle\int_{-L}^{L} f(x)\, dx \\ a_n = \dfrac{1}{L} \displaystyle\int_{-L}^{L} f(x) \cos\!\left(\frac{n\pi x}{L}\right)\, dx \quad (n = 1, 2, \ldots) \\ b_n = \dfrac{1}{L} \displaystyle\int_{-L}^{L} f(x) \operatorname{sen}\!\left(\frac{n\pi x}{L}\right)\, dx \quad (n = 1, 2, \ldots) \end{cases}$$

$$\text{Integrais} \quad \begin{cases} \displaystyle\int x \cos(kx)\, dx = \frac{x}{k} \operatorname{sen}(kx) + \frac{1}{k^2} \cos(kx) + C \\ \displaystyle\int x \operatorname{sen}(kx)\, dx = -\frac{x}{k} \cos(kx) + \frac{1}{k^2} \operatorname{sen}(kx) + C \\ \displaystyle\int x^2 \cos(kx)\, dx = \frac{2x}{k^2} \cos(kx) + \left(\frac{x^2}{k} - \frac{2}{k^3} \right) \operatorname{sen}(kx) + C \\ \displaystyle\int x^2 \operatorname{sen}(kx)\, dx = \frac{2x}{k^2} \operatorname{sen}(kx) + \left(-\frac{x^2}{k} + \frac{2}{k^3} \right) \cos(kx) + C \end{cases}$$

EDC - 3ª PROVA INDIVIDUAL

① $f(x) = x^2 \quad x \in (-1,1) \quad , \quad f(x+2) = f(x) \qquad L = 1$



função par $\Rightarrow b_n = 0 \quad (n=1,2,\dots)$

$$a_0 = 2\int_0^1 x^2 dx = \frac{2}{3}\left[x^3\right]_0^1 = \frac{2}{3}$$

$$a_n = 2\int_0^1 x^2 \cos(n\pi x)\,dx = 2\left[\frac{2x}{n^2\pi^2}\cos(n\pi x) + \left(\frac{x^2}{n\pi} - \frac{2}{n^3\pi^3}\right)\operatorname{sen}(n\pi x)\right]_0^1$$

$$= 2\cdot\frac{2}{n^2\pi^2}\cos(n\pi) = \frac{4}{\pi^2}\cdot\frac{(-1)^n}{n^2}$$

$$\Rightarrow \boxed{f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\cos(n\pi x)}$$

② $\dfrac{\partial u}{\partial t} = \alpha^2\dfrac{\partial^2 u}{\partial x^2} - hu \quad (0<x<L), (t>0) \quad e \quad u(0,t) = u(L,t) = 0 \quad (t>0)$

Soluções separáveis: $u(x,t) = X(x)T(t) \Rightarrow \begin{cases} u(0,t) = X(0)T(t) = 0 \Rightarrow X(0) = 0 \\ u(L,t) = X(L)T(t) = 0 \Rightarrow X(L) = 0 \end{cases}$

EDP: $XT' = \alpha^2 X''T - hXT$

$\div XT \Rightarrow \dfrac{T'}{T} = \alpha^2\dfrac{X''}{X} - h \Rightarrow \underbrace{\dfrac{1}{\alpha^2}\left(\dfrac{T'}{T} + h\right)}_{\text{depende de } t} = \underbrace{\dfrac{X''}{X}}_{\text{depende de } x} = \lambda$ constante de separação

(I) $T' = (\alpha^2\lambda - h)T \Rightarrow T(t) = \text{const.}\, e^{(\alpha^2\lambda - h)t}$

(II) $X'' = \lambda X \quad , \quad X(0) = X(L) = 0$

$\lambda = 0 \rightarrow X(x) = Ax + B \quad , \quad X(0) = B = 0 \quad , \quad X(L) = AL = 0 \Rightarrow A = 0 \Rightarrow X(x) \equiv 0$

$\lambda > 0 \quad (\lambda = \omega^2, \omega > 0) \rightarrow X(x) = Ae^{\omega x} + Be^{-\omega x}$

$X(0) = A + B = 0 \rightarrow B = -A \quad , \quad X(L) = A(\underbrace{e^{\omega L} - e^{-\omega L}}_{\neq 0}) = 0 \Rightarrow A = 0$

$\Rightarrow X(x) \equiv 0$

$\lambda < 0 \quad (\lambda = -\omega^2, \omega > 0) \rightarrow X(x) = A\cos(\omega x) + B\operatorname{sen}(\omega x)$

$X(0) = A = 0 \quad , \quad X(L) = B\operatorname{sen}(\omega L) = 0$

Portanto $\omega L = n\pi \quad (n = 1, 2, \dots) \rightarrow \omega_n = \dfrac{n\pi}{L} \quad , \quad \lambda_n = -\dfrac{n^2\pi^2}{L^2}$

Soluções separáveis $\boxed{u_n(x,t) = \text{const.}\, e^{-\left(\frac{n^2\pi^2\alpha^2}{L^2} + h\right)t}\operatorname{sen}\left(\frac{n\pi x}{L}\right) \quad (n = 1, 2, \dots)}$

(3) $\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = \operatorname{sen} x \qquad (0 < x < \pi,\ t > 0)$

$u(0,t) = 0\ , \quad \frac{\partial u}{\partial x}(\pi,t) = 4 \qquad (t > 0)$

Solução estacionária: $\quad u(x,t) = v(x)$

$$\frac{\partial u}{\partial t} = 0\ , \quad \frac{\partial u}{\partial x} = v'(x)\ , \quad \frac{\partial^2 u}{\partial x^2} = v''$$

Assim $\quad 0 + v'' = \operatorname{sen} x \quad \Rightarrow \quad v' = -\cos x + A$

$$v(x) = -\operatorname{sen} x + Ax + B$$

$u(0,t) = 0 \Rightarrow v(0) = 0 \rightarrow B = 0$

$\frac{\partial u}{\partial x}(\pi,t) = 4 \Rightarrow v'(\pi) = 4 \rightarrow -\cos\pi + A = 4 \rightarrow A = 3$

Assim $\quad \boxed{v(x) = -\operatorname{sen} x + 3x}$