Integrais

$$\int \frac{dx}{x^3\sqrt{x^2-16}}$$

$$\left\{\begin{array}{l} x = 4\sec(\theta) \Rightarrow dx = 4\sec(\theta)\cdot tg(\theta)\,d\theta \\ \sqrt{x^2-16} = 4tg(\theta) \quad \text{para } 0 \le \theta \le \frac{\pi}{2} // \\ \qquad \text{ou } \pi \le \theta \le \frac{3\pi}{2} // \end{array}\right\}$$

$$\int \frac{dx}{x^3\sqrt{x^2-16}} = \int \frac{4\sec(\theta)\,\cancel{tg(\theta)}\,d\theta}{(4\sec(\theta))^3\cdot \cancel{4tg(\theta)}} = \int \frac{\sec(\theta)\,d\theta}{64\sec^3(\theta)}$$

$$= \frac{1}{64}\int \frac{d\theta}{\sec^2(\theta)} = \frac{1}{64}\int \frac{d\theta}{\frac{1}{\cos^2(\theta)}}$$

$$= \frac{1}{64}\int \cos^2(\theta)\,d\theta$$

$$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$$

$$= \frac{1}{64} \int \left(\frac{1 + \cos(2\theta)}{2} \right) d\theta$$

$$= \frac{1}{128} \int (1 + \cos(2\theta))\, d\theta$$

$$\boxed{\begin{array}{l} \int \cos(2\theta)\, d\theta \\ u = 2\theta \Rightarrow du = 2d\theta \\ \int \cos(u) \frac{du}{2} \\ = \frac{\operatorname{sen}(u)}{2} = \frac{\operatorname{sen}(2\theta)}{2} \end{array}}$$

$$= \frac{1}{128} \left(\theta + \frac{\operatorname{sen}(2\theta)}{2} \right) + C$$

2º Passo: Devolver para a variável x.

$$\operatorname{sen}(\theta) = \frac{\sqrt{x^2 - 16}}{x} \quad ; \quad \cos(\theta) = \frac{x}{4}$$

$$\frac{\operatorname{sen}(2\theta)}{2} = \operatorname{sen}(\theta)\cos(\theta)$$

$$\frac{\operatorname{sen}(2\theta)}{2} = \frac{\sqrt{x^2 - 16}}{x} \cdot \frac{x}{4} = \frac{\sqrt{x^2 - 16}}{4}$$

$$\theta = \operatorname{arc\,sen}\left(\frac{x}{4}\right)$$

Observação: Para substituir os valores de θ, temos que tomar cuidado pois neste caso o domínio da função integrando é definido para $x > 4$ e $x < -4$.

Para $x > 4 \Rightarrow \sec(\theta) = \frac{x}{4} > 1 \Rightarrow \theta = \operatorname{arc\,sec}\left(\frac{x}{4}\right) > 1$

$$0 \leq \theta < \frac{\pi}{2}$$

Para $x < -4 \Rightarrow \sec(\theta) = \dfrac{x}{4} < -1$

$$\Rightarrow \theta = \operatorname{arc\,sec}\left(\frac{x}{4}\right) < -1$$

$$\frac{\pi}{2} < \theta < \pi$$

Quando fixamos a substituição

$$x = 4\sec(\theta)$$

assumimos que $\pi \leq \theta \leq \dfrac{3\pi}{2}$

De $\sec(2\pi - \theta) = \sec(\theta)$.

Para $x < -4$ podemos reescrever

$$\theta = 2\pi - \operatorname{arc\,sec}\left(\frac{x}{4}\right)$$

$$\pi \leq \theta < \frac{3\pi}{2}$$

Portanto a solução de

$$\int \frac{dx}{x^3\sqrt{x^2-16}}$$ divide em dois casos.

1º Para $x > 4$

$$\int \frac{dx}{x^3\sqrt{x^2-16}} = \frac{1}{128}\left(\theta + \frac{\operatorname{sen}(2\theta)}{2}\right) + C$$

onde $\theta = \operatorname{arc\,sec}\left(\frac{x}{4}\right)$

$$e \cdot \frac{\operatorname{sen}(2\theta)}{2} = \frac{\sqrt{x^2-16}}{4}$$

$$= \frac{1}{128}\left(\operatorname{arc\,sec}\left(\frac{x}{4}\right) + \frac{\sqrt{x^2-16}}{4}\right) + C$$

② Para $x < -4$

$$\int \frac{dx}{x^3\sqrt{x^2-16}} = \frac{1}{128}\left(\theta + \frac{\operatorname{sen}(2\theta)}{2}\right) + C$$

onde $\theta = 2\pi - \operatorname{arc\,sec}\left(\frac{x}{4}\right)$

$$\frac{\operatorname{sen}(2\theta)}{2} = \frac{\sqrt{x^2-16}}{4}$$

$$= \frac{1}{128}\left(2\pi - \operatorname{arc\,sec}\left(\frac{x}{4}\right) + \frac{\sqrt{x^2-16}}{4}\right) + C$$

$$= \frac{1}{128}\left[-\operatorname{arc\,sec}\left(\frac{x}{4}\right) + \frac{\sqrt{x^2-16}}{4}\right] + \frac{\pi}{64} + C$$

$$= \frac{\left[-\operatorname{arc\,sec}\left(\frac{x}{4}\right) + \frac{\sqrt{x^2-16}}{4}\right]}{128} + C_1$$

onde $C_1 = \frac{\pi}{64} + C$

$$\sqrt{x^2 - \underset{4^2}{16}} \cong \sqrt{x^2 - a^2}$$

$$\int_{-\infty}^{+\infty} e^{-|x|}\,dx$$

$$= \int_{-\infty}^{0} e^{-|x|}\,dx + \int_{0}^{+\infty} e^{-|x|}\,dx$$

$$= \int_{-\infty}^{0} e^{-(-x)}\,dx + \int_{0}^{+\infty} e^{-x}\,dx$$

$$= \int_{-\infty}^{0} e^{x}\,dx + \int_{0}^{+\infty} e^{-x}\,dx$$

$$= \lim_{a \to -\infty} \int_{a}^{0} e^{x}\,dx + \lim_{b \to +\infty} \int_{0}^{b} e^{-x}\,dx$$

$$= \lim_{a \to -\infty} \left[e^{x} \Big|_{a}^{0} \right] + \lim_{b \to +\infty} \left[-e^{-x} \Big|_{0}^{b} \right]$$

$$= \lim_{a \to -\infty} \left[e^{0} - e^{a} \right] + \lim_{b \to +\infty} \left[-e^{-b} + e^{-0} \right]$$

$$= \lim_{a \to -\infty} \left[1 - e^{a} \right] + \lim_{b \to +\infty} \left[-e^{-b} + 1 \right]$$

$a \to -\infty \Rightarrow e^{a} \to 0 \Rightarrow -e^{a} \to 0$

Colocar a justificação

$b \to +\infty \Rightarrow -b \to -\infty \Rightarrow -e^{-b} \to 0$

$$= 1 - \lim_{a \to -\infty} e^{a} + \lim_{b \to +\infty} -e^{-b} + 1$$

$$= 1 - 0 + 0 + 1 = 2$$

$$\int^{+\infty} e^{-x^2}\,dx \quad \text{é convergente.}$$

$$1$$

$$\left\{ \begin{array}{l} [1, +\infty) \Rightarrow e^{-x^2} > 0 \\ 0 < e^{-x^2} \leq x e^{-x^2} \text{ para } x \geq 1 \end{array} \right\}$$

$$\int_1^{+\infty} 0\,dx < \int_1^{+\infty} e^{-x^2}\,dx \leq \int_1^{+\infty} x e^{-x^2}\,dx$$

$$0 < \int_1^{+\infty} e^{-x^2}\,dx \leq \int_1^{+\infty} x e^{-x^2}\,dx = \frac{1}{2e}$$

$$\int_1^{+\infty} x e^{-x^2}\,dx = \lim_{b \to +\infty} \int_1^{b} x e^{-x^2}\,dx -$$

$$\int x e^{-x^2}\,dx = \int -\frac{e^u\,du}{2} = -\frac{e^u}{2} = -\frac{e^{-x^2}}{2}$$

$u = -x^2$

$du = -2x\,dx \Rightarrow x\,dx = \frac{du}{-2}$

$$= \lim_{b \to +\infty} \left(-\frac{e^{-x^2}}{2} \Big|_1^b \right)$$

$$= \lim_{b \to +\infty} \left[-\frac{e^{-b^2}}{2} + \frac{e^{-1}}{2} \right]$$

$$= \lim_{b \to +\infty} \cancel{-\frac{e^{-b^2}}{2}}^{\,0} + \frac{1}{2e} = \frac{1}{2e}$$

Como $b \to +\infty \Rightarrow b^2 \to +\infty \Rightarrow -b^2 \to -\infty \Rightarrow e^{-b^2} \to 0$

Portanto

$$0 < \int_1^{+\infty} e^{-x^2} dx \leq \int_1^{+\infty} x e^{-x^2} dx = \frac{1}{2e},$$

mostrando que a integral do enunciado é convergente

Dica: $\displaystyle\int_1^{+\infty} \frac{1 + e^{-x}}{x} dx$

é divergente.

$$\frac{1}{x} < \frac{1}{x} + \frac{e^{-x}}{x} \qquad x \geq 1$$

$$\frac{1}{x} < \frac{1+e^{-x}}{x}$$

Calcule $\int_{1}^{+\infty} \frac{1}{x}\,dx$ verifique se é divergente.