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LIMITES FUNDAMENTAIS

PRIMEIRO LIMITE FUNDAMENTAL

$$\lim_{x \to 0} \frac{\operatorname{sen}(x)}{x} = 1$$

EXEMPLOS

① $$\lim_{x \to 0} \frac{1 - \cos x}{x}$$

$$\frac{1-\cos x}{x} \cdot \frac{(1+\cos x)}{1+\cos x} = \frac{1-\cos^2 x}{x\,(1+\cos x)}$$

$$= \frac{\operatorname{sen}^2 x}{x\,(1+\cos x)}$$

$$\lim_{x \to 0} \frac{1-\cos x}{x} = \lim_{x \to 0} \frac{\operatorname{sen}^2 x}{x\,(1+\cos x)} = \lim_{x \to 0} \frac{\operatorname{sen} x}{x} \cdot \frac{\operatorname{sen} x}{1+\cos x}$$

$$= \lim_{x \to 0} \frac{\operatorname{sen} x}{x} \cdot \lim_{x \to 0} \frac{\operatorname{sen} x}{1+\cos x} = 1 \cdot \frac{\operatorname{sen} 0}{1+\cos 0} = 0$$

(2) $\lim_{x \to 0} \frac{Tg\,x}{x}$

$= \lim_{x \to 0} \frac{sen\,x}{cos\,x \cdot x} = \lim_{x \to 0} \frac{sen\,x}{x} \cdot \frac{1}{cos\,x}$

$\lim \frac{sen\,x}{x} \cdot \lim \frac{1}{cos\,x} = 1$

(3) $\lim_{x \to 0} \frac{sec\,x - 1}{x^2 \cdot sec\,x}$

$= \frac{\frac{1}{cos\,x} - 1}{x^2 \cdot \frac{1}{cos\,x}} = \frac{\frac{1 - cos\,x}{cos\,x}}{\frac{x^2}{cos\,x}} = \frac{1 - cos\,x}{x^2}$

$= \frac{1 - cos^2 x}{x^2(1 + cos\,x)} = \frac{sen^2 x}{x^2(1 + cos\,x)} = \left(\frac{sen\,x}{x}\right)^2 \cdot \frac{1}{1 + cos\,x}$

$\lim_{x \to 0} \frac{sec\,x - 1}{x^2 \cdot sec\,x} = \lim_{x \to 0} \left(\frac{sen\,x}{x}\right)^2 \cdot \lim_{x \to 0} \frac{1}{1 + cos\,x}$

$= 1 \cdot \frac{1}{2} = 1/2$ //

4) $\lim_{x \to 0} \dfrac{sen(5x)}{sen(4x)}$

$$\frac{sen\,5x}{sen\,4x} = \frac{sen(5x) \cdot \frac{5x}{5x}}{sen\,4x \cdot \frac{4x}{4x}}$$

$$= \frac{\frac{sen(5x)}{5x} \cdot 5\cancel{x}}{\frac{sen(4x)}{4x} \cdot 4\cancel{x}} = \frac{5}{4} \cdot \frac{\left(\frac{sen\,5x}{5x}\right)}{\frac{sen\,4x}{4x}}$$

$$\lim \frac{sen(5x)}{sen(4x)} = \frac{5}{4} \frac{\left(\frac{sen\,5x}{5x}\right)}{\left(\frac{sen\,4x}{4x}\right)} = \frac{5}{4} \left[\frac{\lim_{x \to 0} \frac{sen\,5x}{5x}}{\lim_{x \to 0} \frac{sen\,4x}{4x}}\right]$$

* $t = 5x$ // $x \to 0$ então $t \to 0$

* $u = 4x$ // $x \to 0$ então $u \to 0$

$$= \frac{5}{4} \cdot \frac{\lim_{t \to 0} \frac{sen(t)}{t}}{\lim_{u \to 0} \frac{sen\,u}{u}} = \frac{5}{4} \cdot \frac{1}{1} = \boxed{\frac{5}{4}}$$ //

SEGUNDO LIMITE FUNDAMENTAL

$$\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e = \lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x$$

① $\lim_{x \to 0} (1 + x)^{1/x}$

TEMOS QUE CALCULAR

$\lim_{x \to 0^-} (1+x)^{1/x}$ e $\lim_{x \to 0^+} (1+x)^{1/x}$

$t = \frac{1}{x}$ ∴ $\begin{cases} x \to 0^- \text{ ENTÃO } t \to -\infty \\ x \to 0^+ \text{ ENTÃO } t \to +\infty \end{cases}$

$$\lim_{x \to 0^-} (1+x)^{1/x} = \lim_{t \to -\infty} \left(1 + \frac{1}{t}\right)^t = e$$

$$\lim_{x \to 0^+} (1+x)^{1/x} = \lim_{t \to +\infty} \left(1 + \frac{1}{t}\right)^t = e$$

Portanto, $\lim_{x \to 0} (1+x)^{1/x} = e$

(2) $\lim_{x \to -\infty} \left(1 + \frac{k}{x}\right)^x \qquad k \in \mathbb{N}^*$

$\frac{1}{t} = \frac{k}{x} \therefore x = k \cdot t$

1º caso $k > 0 \Rightarrow x \to -\infty$ então $t \to -\infty$

$\lim_{x \to -\infty} \left(1 + \frac{k}{x}\right)^x = \lim_{t \to -\infty} \left(1 + \frac{1}{t}\right)^{k \cdot t} =$

$= \left[\lim_{t \to -\infty} \left(1 + \frac{1}{t}\right)^t\right]^k = e^k$

2º caso $k < 0 \Rightarrow x \to -\infty$ então $t \to +\infty$

$\lim_{x \to -\infty} \left(1 + \frac{k}{x}\right)^x = \lim_{x \to +\infty} \left(1 + \frac{1}{t}\right)^{k \cdot t} =$

$\left[\lim_{t \to +\infty} \left(1 + \frac{1}{t}\right)^t\right]^k = e^k$

Portanto, para $k \in \mathbb{N}^*$

$\lim_{x \to -\infty} \left(1 + \frac{k}{x}\right)^x = e^k$

③ $\lim_{x \to +\infty} \left(1 + \frac{5}{x-1} \right)^{x+7}$

$\Rightarrow \frac{5}{x-1} = \frac{1}{t} \quad ; \quad 5t = x-1 \quad ; \quad x = 5t+1$

$x \to +\infty \qquad t \to +\infty$

$\Rightarrow \lim_{t \to +\infty} \left(1 + \frac{1}{t} \right)^{(5t+1+7)}$

$= \lim_{t \to +\infty} \left(1 + \frac{1}{t} \right)^{5t+8}$

$= \lim_{t \to +\infty} \left(1 + \frac{1}{t} \right)^{5t} \cdot \left(1 + \frac{1}{t} \right)^{8}$

$= \lim_{t \to +\infty} \left(1 + \frac{1}{t} \right)^{5t} \cdot \lim_{t \to +\infty} \left(1 + \frac{1}{t} \right)^{8}$

$\lim_{t \to +\infty} \left[\left(1 + \frac{1}{t} \right)^{t} \right]^{5} \cdot \left[\lim \left(1 + \frac{1}{t} \right) \right]^{8}$

$= \left[\lim_{t \to +\infty} \left(1 + \frac{1}{t} \right)^{t} \right]^{5} \cdot \left(\lim_{t \to +\infty} 1 + \lim_{t \to +\infty} \frac{1}{t} \right)^{8}$

$= e^{5} \, (1+0)^{8} = e^{5}$ //

TERCEIRO LIMITE FUNDAMENTAL

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \ln(a)$$

$a > 0, \quad a \neq 1$

DEMONSTRAÇÃO:

$$\lim_{x \to 0} \frac{a^x - 1}{x} \qquad \begin{matrix} a > 0 \\ a \neq 1 \end{matrix}$$

$$a^x = 1 + \frac{1}{u} \Leftrightarrow x = \log_a \left(1 + \frac{1}{u}\right)$$

$$a^x = 1 + \frac{1}{u} \Rightarrow a^x - 1 = \frac{1}{u} \Rightarrow u = \frac{1}{a^x - 1}$$

$x \to 0^+ \Rightarrow a^x \to 1 \Rightarrow a^x - 1 \to 0$ e $a^x - 1 > 0$
$\Rightarrow u \to +\infty$

$x \to 0^- \Rightarrow a^x \to 1 \Rightarrow a^x - 1 \to 0$ e $a^x - 1 < 0$
$\Rightarrow u \to -\infty$

$$\lim_{x\to 0^+}\left(\frac{a^x-1}{x}\right) = \lim_{u\to+\infty} \frac{1+\frac{1}{u}-1}{\log_a\left(1+\frac{1}{u}\right)} = \lim_{u\to+\infty}\left(\frac{1}{u}\cdot\frac{1}{\log_a\left(1+\frac{1}{u}\right)}\right)$$

$$= \lim_{u\to+\infty}\left(\frac{1}{u\cdot\log_a\left(1+\frac{1}{u}\right)}\right) = \lim_{u\to+\infty}\frac{1}{\log_a\left[\left(1+\frac{1}{u}\right)^u\right]}$$

$$= \frac{1}{\lim_{u\to+\infty}\log_a\left[\left(1+\frac{1}{u}\right)^u\right]} = \frac{1}{\log_a\left[\lim_{u\to+\infty}\left(1+\frac{1}{u}\right)^u\right]}$$

$$= \frac{1}{\log_a(e)} = \ln a$$

OBSERVE QUE SE $\log_a e = x$

E $\ln(a) = y$, TEMOS $a^x = e$ E

$e^y = a$, E CONSEQUENTEMENTE

$$e = a^x = (e^y)^x = e^{y\cdot x}$$

$\Rightarrow y\cdot x = 1 \Rightarrow \log_a e \cdot \ln(a) = 1$

$\Rightarrow \ln a = \dfrac{1}{\log_a e}$

$$\lim_{x\to 0^+}\frac{a^x-1}{x} = \ln a$$

ANALOGAMENTE:

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \ln a$$

EXEMPLOS:

1) $\lim_{x \to 0} \frac{a^x - b^x}{x}$ $\quad a, b > 0$; $\quad a, b \neq 1$

$$\frac{a^x - b^x}{x} = \frac{b^x \left(\frac{a^x}{b^x} - 1 \right)}{x}$$

$$= b^x \left[\frac{\left(\frac{a}{b}\right)^x - 1}{x} \right]$$

$$\lim_{x \to 0} \frac{a^x - b^x}{x} = \lim_{x \to 0} \left[b^x \cdot \left(\frac{\left(\frac{a}{b}\right)^x - 1}{x} \right) \right]$$

$$= \lim_{x \to 0} b^x \cdot \lim_{x \to 0} \left(\frac{\left(\frac{a}{b}\right)^x - 1}{x} \right)$$

$$= b^0 \cdot \ln\left(\frac{a}{b}\right) = \ln\left(\frac{a}{b}\right)$$

② $\lim_{x \to 1} \left[\frac{e^{x-1} - a^{x-1}}{x^2 - 1} \right]$

$$\frac{e^{x-1} - a^{x-1}}{x^2-1} = \frac{e^{x-1} - 1 - a^{x-1} + 1}{(x+1)(x-1)}$$

$$= \frac{e^{x-1} - 1 - (a^{x+1} - 1)}{(x+1)(x-1)}$$

$$= \frac{1}{(x+1)} \cdot \left[\frac{e^{x-1} - 1}{(x-1)} - \frac{a^{x-1} - 1}{(x-1)} \right]$$

$$\lim_{x \to 1} \left[\frac{e^{x-1} - a^{x-1}}{x^2-1} \right] = \lim_{x \to 1} \left(\frac{1}{x+1} \left[\frac{e^{x-1} - 1}{x-1} - \frac{(a^{x-1} - 1)}{x-1} \right] \right)$$

$$= \lim_{x \to 1} \left(\frac{1}{x+1} \right) \cdot \left[\lim_{x \to 1} \left(\frac{e^{x-1} - 1}{x-1} \right) - \lim_{x \to 1} \left(\frac{a^{x-1} - 1}{x-1} \right) \right]$$

$t = x - 1$, $x \to 1$ com $x \neq 1$

TOMOS $t \to 0$ com $t \neq 0$

$$= \frac{1}{1+1} \left[\lim_{t \to 0} \left(\frac{e^{t} - 1}{t} \right) - \lim_{t \to 0} \left(\frac{a^{t} - 1}{t} \right) \right]$$

$$= \frac{1}{2} \left[\ln e - \ln a \right]$$

$$= \frac{1}{2} \left[1 - \ln a \right] \quad //$$