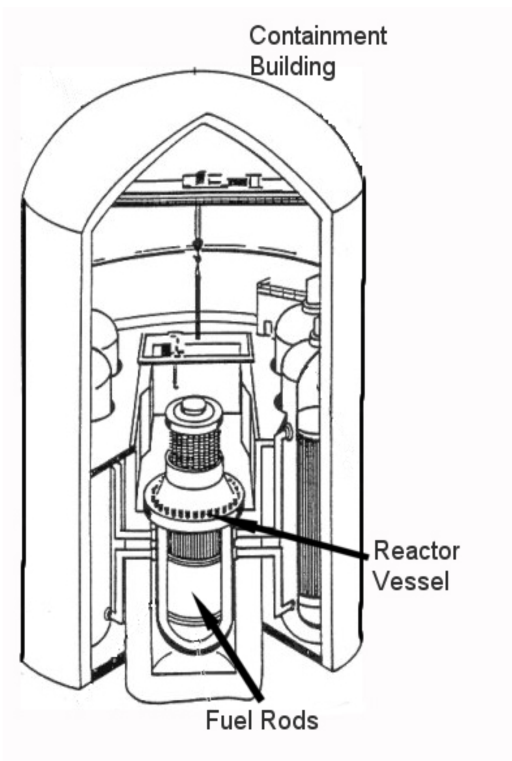
1. Consider the shielding wall of a nuclear reactor of width l and assume that heat flows through the wall in the x-direction from the inside to the outside. Moreover, shielding walls of nuclear reactors are known to generate heat internally due to the interaction of gamma rays with the walls. This internal heat generation (per unit volume) $q(x)$, can be modeled by

$$q(x) = q_0 e^{-ax}$$

where q_0 and a are positive constants.



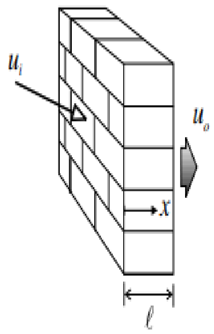
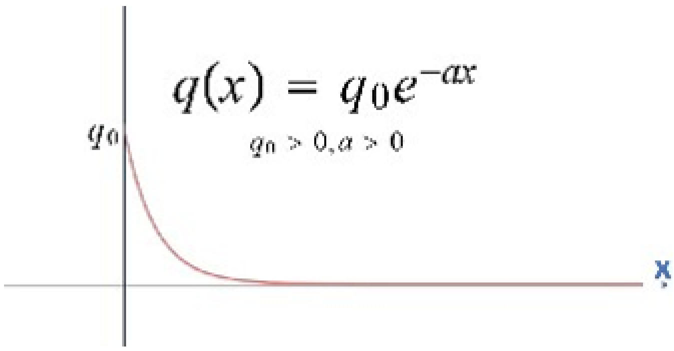


Figure 11.1: Heat flow through the wall of a house, $0 \leq x \leq \ell$.

(a) Sketch $q(x)$ as a function of x

Since $q_0 > 0, a > 0$ then $e^{-ax} = \frac{1}{e^{ax}} > 0 \rightarrow q(x) = q_0 e^{-ax} > 0$
It is a definite positive function further $\lim_{x \to \infty} q(x) = \lim_{x \to \infty} \frac{q0}{e^{ax}} = 0$ so
the positive direction of the x-axis is a horizontal asymptote of q(x)
the function is decreasing in the interval $(0, \infty)$ since
$\frac{d}{dx} q(x) = -aqe^{-ax} < 0$ in $(0, \infty)$ further $q(0) = q_0$, so the graph of the function has the following path



(b) Deduce that the equilibrium temperature inside the wall satisfies the differential equation

$$k\frac{d^2U}{dx^2} + q(x) = 0$$

where k is the conductivity. Justify your answer.

k: The positive constant k is called the conductivity, which will be different for different materials. The minus sign is necessary to ensure a positive rate for the conduction of heat. This means that when the temperature is decreasing (i.e., when the temperature gradient is negative), the heat flows in the positive direction. When the temperature is increasing (i.e., when the temperature gradient is positive), the heat flows in the negative direction. This reflects the fact that heat flows from regions of higher temperature to regions of cooler temperatures, and not the other way around.

U(x) denote the equilibrium temperature
J(x) flux of heat at x

rate heat conducted in at x = J(x)A

rate heat conducted out at x + Δx = J(x + Δx)A

heat source producing heat a rate = $q(x) = q_0e^{-ax}$

rate of change of heat in section = [J(x)A - J(x + Δx)A] + q(x)

making Δx →0 and applying Fourier law

J = -k dU/dx

From the derivative definition

[J(x)A - J(x + Δx)A /Δx] = dJ/dx = d/dx(-kdU/dx) = d^2U/dx^2

So by substitution

rate of change of heat in section = [J(x)A - J(x + Δx)A] + q(x) = 0 when reach equilibrium so

The temperature U(x) inside the wall reaches equilibrium when

$$k\frac{d^2}{dx^2}U(x) + \mathrm{q(x)} = 0$$

them the differential equation that is satisfied by the equilibrium temperature inside the wall is

$$\mathrm{k}\frac{d^2}{dx^2}U(x) + q_0e^{-ax} = 0$$ where k is the conductivity

(c) Find the general solution of the differential equation in (b).

$$k\,\frac{d^2U}{dx^2} + q_0 e^{-ax} = 0$$

$$\frac{d^2U}{dx^2} = -\frac{q(x)}{k} = -\frac{q_0}{k}e^{-ax}$$

To find the general solution we must integrate twice with respect to x so

$$\frac{dU}{dx} = \int -\frac{q_0}{k}e^{-ax}\,dx = -\frac{q_0}{k}\int e^{-ax}\,dx = \frac{q_0}{ak}\,e^{-ax} + C1$$

General solution of b

U(x) = $\int(\frac{q_0}{ak}\,e^{-ax} + C1)dx = -\frac{q_0}{k\,a^2}\,e^{-ax} + C1x + C2$

(d) Suppose that the temperature of the wall in its inside part x = 0 is always kept at a given value u_i and that in the outside part of the wall heat is lost to the surroundings according to Newton's law of cooling. Assuming that the temperature in the surroundings of the wall is u_o, find an expression for the equilibrium temperature inside the wall. Justify your answer.

Observation: Apply boundary condition at x =0 and x = l

Since U(x) = U_i at x = 0 applying to general solution in b and solve for C2

$$-\frac{q_0}{k\,a^2}\,e^{-ax} + C1x + C2 = -\frac{q_0}{k\,a^2}e^{-a0} + C1(0) + C2 = -\frac{q_0}{k\,a^2} + C2 = U_i \text{ so}$$

$$C2 = U_i + \frac{q_0}{k\,a^2}$$

Using newton law of cooling and Fourier law in order to find C1

$$\frac{dU}{dx} = \int -\frac{q_0}{k}e^{-ax}\,dx = -\frac{q_0}{k}\int e^{-ax}\,dx = \frac{q_0}{ak}\,e^{-ax} + C1$$

Fourier law

$$J = -k\frac{dU}{dx} = -k\,\frac{q_0}{ak}\,e^{-ax} - kC1$$

At boundary condition with x = l, k the conductivity and h the convective heat transfer

J(l) = $-k\frac{dU}{dx}(l) = h(U(l) - u_0)$

Solve for C1

$$-k\frac{q_0}{ak}e^{-al} - kC1 = h(U(l) - u_0) = h\left(-\frac{q_0}{ka^2}e^{-al} + C1l + u_i + \frac{q_0}{ka^2} - u_0\right)$$

$$-k\,\frac{q_0}{ak}\,e^{-al} + h\,\frac{q_0}{ka^2}e^{-al} - h\frac{q_0}{ka^2} + hu_0 - hu_i = hC1l + kC1) = \text{C1(hl + k)}$$

$$C1 = \frac{q_0}{k(hl+k)a^2}\left(ae^{-al} - hake^{-al} - h\right) + hu_0 - hu_i$$

From c

U(x) = $\int(\frac{q_0}{ak}\,e^{-ax} + C1)dx = -\frac{q_0}{k\,a^2}\,e^{-ax} + C1x + C2$

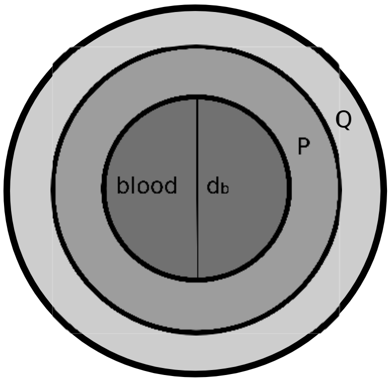
$$U(x) = -\frac{q_0}{ka^2}e^{-ax} + \left[\frac{q_0}{k(hl+k)a^2}\left(ae^{-al} - hake^{-al} - h\right) + hu_0 - hu_i\right]x + u_i + \frac{q_0}{ka^2}$$

Rearranging the expression for equilibrium temperature inside the wall is

$$U(x) = \frac{q_0}{ka^2}(1 - e^{-ax} - \frac{x}{hl+k}\,(h - ae^{-al} + hake^{-al})) + (u_0 - u_i)hx + u_i$$

[3,5 points]

2. We consider a model for tumor cells that are growing around a cylindrical blood vessel of diameter d_b as shown in the figure below. In this figure P and Q respectively denote the proliferating and the quiescent layer. We suppose that oxygen diffuses into the tumor from the surface of the blood vessel in the radial direction. To simplify the problem we also assume that the oxygen concentration quickly reaches equilibrium.



(a) If live cells inside the tumor consume oxygen at a constant rate of M kg/m^3/s write down the differential equation that is satisfied by the equilibrium oxygen concentration at the proliferating layer. Justify your answer.

Consider the diffusion equation which relates the changes in any senation with time to the spatial distribution

Oxygen molecules diffuse outward from high concentrations on the surface of the blood vessel towards lower concentrations at its tumor perimeter. So the governing equation will be a diffusion equation for a **cylindrical geometry** with

M(r) < o cell consume oxygen

M(r) = -Ao constant oxigen comsuption rate

Diffusion is outward from blood vessel along the radius

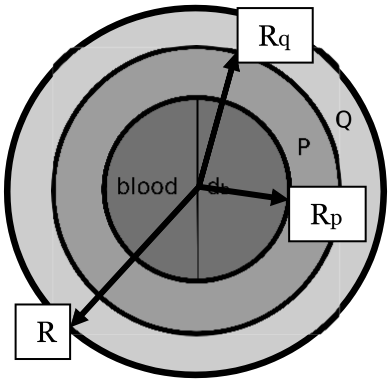
C(r) will be a function of the distance from the center of the blood vessel

$M(r) = -A_0$ for $R_p(t) < r < R_q(t)$. proliferating

$$\frac{d}{dr}\left(r\frac{dc_p}{dr}\right) + M = 0$$

Using product rule we get

$$r\,\frac{d^2C_p}{dr^2} + \frac{dC_p}{dr} + M = 0 \qquad R_p(t) < r < R_q(t)$$



(b) Solve the equation in (a) assuming that oxygen concentration at the blood vessel surface is a fixed value c_b.

Solve the next equation where M = -A_0

$\frac{d}{dr}\left(r\frac{dc}{dr}\right) + M = 0$

oxygen concentration at the blood vessel surface is a fixed value c_b imply the following boundary condition $C\left(\frac{d_b}{2}\right) = c_b$,

Rewritten the ODE we have $\frac{d}{dr}\left(r\frac{dC}{dr}\right) = A_0$ integrating this equation we obtain,

$$r\frac{dC}{dr} = A_0 r + C_1,$$

$$\frac{dC}{dr} = A_0 + \frac{C_1}{r},$$

And integrating again we obtain

$C(r) = A_0 r + C_1 \ln(r) + C_2,$

The boundary condition $C\left(\frac{d_b}{2}\right) = c_b$ imply

$$C(r) = C\left(\frac{d_b}{2}\right) = c_b = \frac{A_0 d_b}{2} + C_1 \ln\left(\frac{d_b}{2}\right) + C_2,$$

The values of C_1 and C_2 are determined by the value of C(r) at the common border of the regions P and Q in the figure.

(c) Write down the differential equation that is satisfied by the equilibrium oxygen concentration at the quiescent layer. Justify your answer.

- The quiescent core expands as more cells in the proliferating layer starve and become quiescent.
- Rq(t) be the time-dependent radius of the inner quiescent perimeter.
- R(t) be the outer radius of the tumor
- Cp(r) be the equilibrium oxygen concentration in the proliferating layer
- $C_q(r)$ the equilibrium of oxygen concentration in the quiescent layer.
- In the proliferating layer, cells consume oxygen, but in the quiescent layer no consumption takes place even though.

oxygen can diffuse into the region with Cq(r) < cq.
So M(r) = 0 for $R_q(t) < r < R(t)$ quiescent

the diffusion equations for quiescent is

$$\frac{d}{dr}\left(r\frac{dc}{dr}\right) = 0 \qquad R_q(t) < r < R(t)$$

[3,5 points]

3. We consider a polluted lake, estimated by a rectangular shape of length 12 and constant cross-sectional area A, as illustrated in Figure 12.4 of the textbook. Under the following assumptions (i)–(vi)

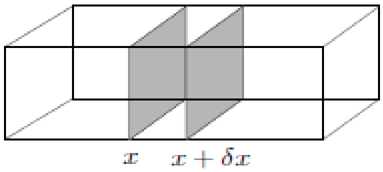


Figure 12.4: Diagram of a 'rectangular' lake, from $x = 0$ to $x = b$.

(i) The volume V is constant.
(ii) A unidirectional flow of freshwater is entering the lake and flowing to its exit at a constant rate $F\,\mathrm{m}^3/\mathrm{s}$.
(iii) The concentration of pollutants depends only on x (horizontal distance from the entry point of the freshwater) and time t, $C(x, t)$.
(iv) Pollution travels through the lake only by water flow, that is, by advection.
(v) The initial concentration of the pollutant in the lake is $C(x, 0) = \sin^2 x$.
(vi) $F/A = v = 1$.

(a) Calculate which is the equation in t whose solution gives the time required to reduce the pollution concentration in the lake to 10% of the initial one.

Solution:
Rectangular shape lake
L = 12 units
Area = A constant
Boundary conditions at x = a and t = 0 are :

- Pollution entering the lake = Freshwater entering the lake mean at x = a and t > 0
  g(t) = 0 so that C(a,t) = $\frac{g(t)}{F}$ =0
- Concentration at x location when t = 0 Initial Pollutant concentration
  C(x,0) =P(x) = $sin^2 x$

$$\left\{\begin{array}{l} C(a, t) = g(t) = 0 \\ \\ C(x, 0) = P(x) = \sin^2 x \end{array}\right\}$$

10% of the current level = 10% X C(x,t) at x =a
V =Volume is constant

We will solve PDE for pollutant concentration

$$\frac{\partial C}{\partial t} + \frac{F}{A}\frac{\partial C}{\partial x} = 0$$

assumption $\frac{F}{A} = v = 1$ then

$$\frac{\partial C}{\partial t} + \frac{\partial C}{\partial x} = 0$$

Solving the equation we got

$$\frac{\partial C}{\partial t} = -\frac{\partial C}{\partial x}$$

The characteristic curves are givent by x = t + K
Setting x = a and t = 0 in x = t + K
Associate characteristic t = x – a
Considering the pollutant concentration C is a constant and no pollutant is entering the lake

$$C(x, t) = \begin{cases} 0 & \text{if } t > x - a \\ \sin^2 (x - t) & \text{if } t < x - a \end{cases}$$

Since the volume V is constant and lake length b = 12
the equation in t whose solution gives the time required to reduce the pollution concentration in the lake to 10% of the initial values is :

$$\int_a^b c(x,t)dx = 0.1\int_0^{12} sin^2 x\, dx$$

Simplify to

$$\int_a^b sin^2(x-t)dx = 0.1\int_a^b sin^2 x\, dx$$

$$\int_a^b sin^2(x-t)dx = 0.1\int_0^{12} sin^2 x\, dx$$

The equation we are looking for

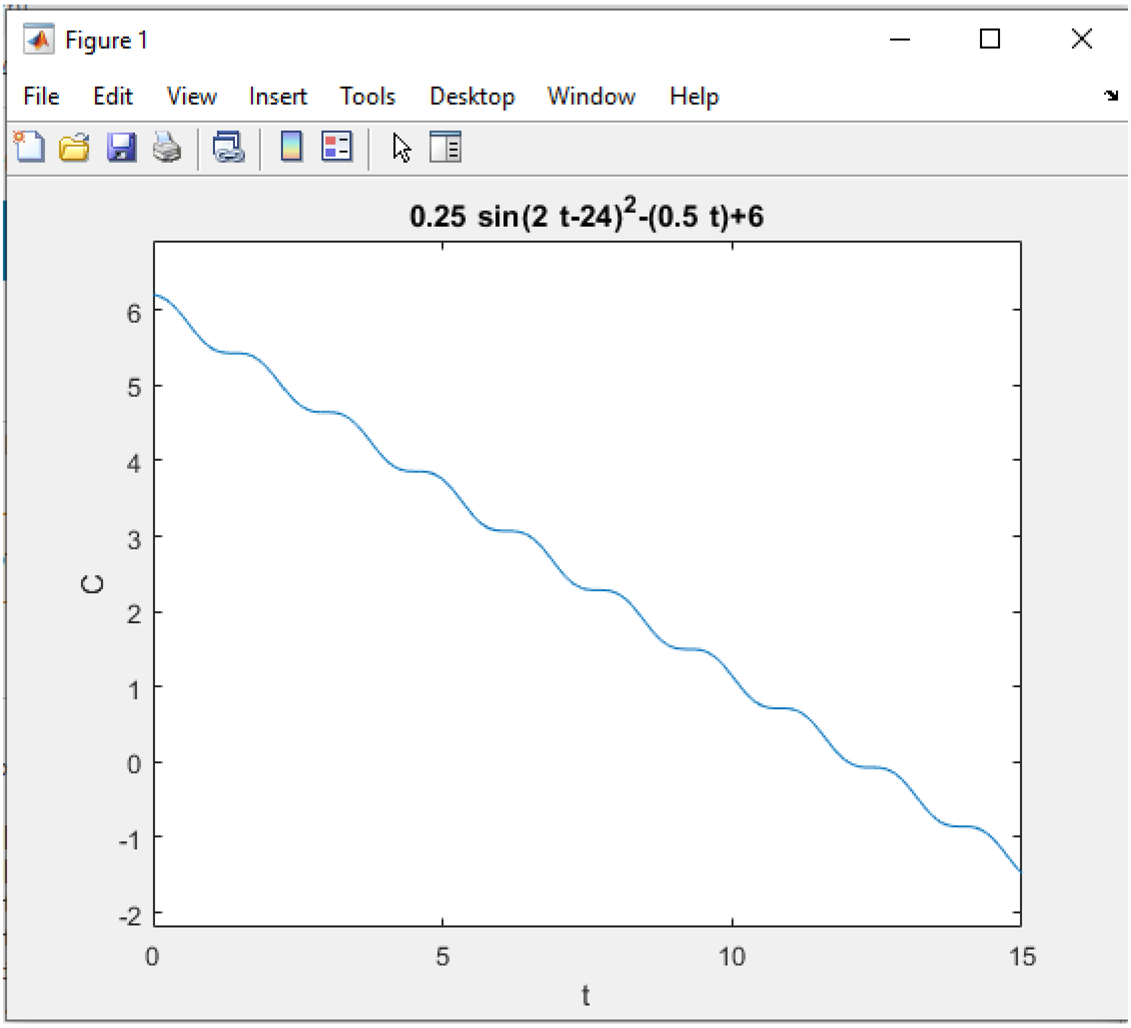
$$\frac{1}{4}sin(2t-24) - \frac{1}{2}t + 6 = 0.622639459$$

When lhs = 0.622639459 pollution concentration is 10 % of initial concentration

$$\frac{1}{4}sin(2t-24) - \frac{1}{2}t + 5.37736054 = 0$$

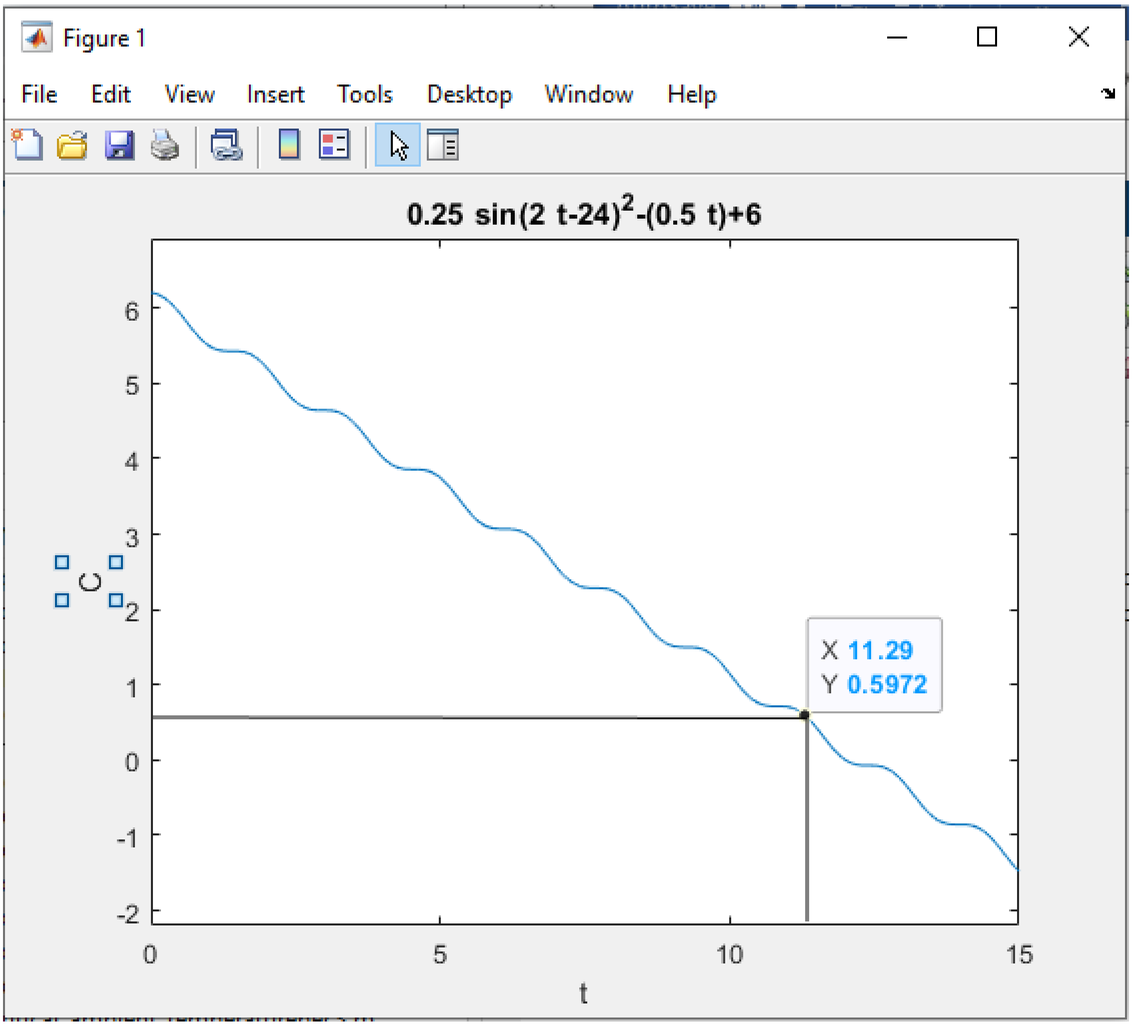
(b) Generate a plot that allows you to estimate the solution of the equation in (a). The following plot shows the graph

We generate a plot using ezplot matlab (no code required) for
C(x,t) = $\frac{1}{4}sin(2t-24)-\frac{1}{2}t+6 = 0.622639459$ in the domain [0,12]



(c) Using the plot generated in (b) gives a numerical estimation for the time required to reduce the pollution concentration in the lake to 10% of the initial one.

From b we know that C(x,t) = 0.1C(initial) = 0.622639459 approximatly then looking at C(x,t) =0.622639459 in the plot



The time estimation for reach 10% of the initial value will be 11.3 months approximately

[3 points]

Additional problem: to be done *only* for those students that *started the master the current academic year* (19-20)

We consider a polluted lake, estimated by a rectangular shape with cross–sectional area A, as illustrated in Figure 12.4 of the textbook. Under the following assumptions

(i) The volume V is constant.

(ii) The concentration of pollutant depends only on x (horizontal distance from the entry point of the fresh water) and time t, $C(x, t)$.

(iii) Pollution travels through the lake only by water flow, that is, by advection.

(iv) An unidirectional flow of fresh water is entering the lake and flowing to its exit at a non–constant rate $F(x, t) = Av(x, t)$.

Write down the partial differential equation that is satisfied by the pollutant concentration $C(x, t)$. Justify your answer.

[2 points]