Rovira e Virgili University

Master in Computer Science and Mathematics

Modelización mediante ecuaciones diferenciales

1. Consider the standard SIR epidemic model

$$\begin{cases} S' = -\beta SI, \\ I' = \beta SI - \gamma I, \end{cases}$$

where $I(t)$ and $S(t)$ are, respectively, the number of infectives and susceptibles at time t.

(a) Prove that, at any t, it holds the formula

$$I(t) = K - S(t) + \frac{\gamma}{\beta}\log S(t),$$

where K is a constant depending only on the initial number of infectives $i_0 := I(0)$ and susceptibles $s_0 := S(0)$.

dS(t) /dt = -βSI

dI(t) /dt = βSI - γI

dI/dS = (dI/dt) / (dS/dt)

dI/dS = (βSI – γI) / -βSI

dI /dS = -1 + γ/βS

dI = (-1 + γ/βS) dS

separable differential equation

∫dI = -∫dS + ∫(γ/βS) dS

∫dI = -∫dS + γ/β ∫ dS /S

$\log e^x = \ln e^x = \ln x$

I(t) = -S(t) + γ/β ln (S(t)) + K

I(t) = -S(t) + γ/β log (S(t)) + K

$I(0) = i_0$

$S(0) = s_0$

t = 0

$i_0 = -s_0 + \gamma/\beta \log(s_0) + K$

$K = i_0 + s_0 - \gamma/\beta \log(s_0)$

(b) If s_f denotes the remaining number of susceptibles when there are no remaining infectives, show that

$$\frac{\beta}{\gamma} = \frac{\log(s_0/s_f)}{s_0 + i_0 - s_f}$$

$I(t) = -S(t) + \gamma/\beta \log(S(t)) + K$

there are no remaining infectives $I(t) = 0$

$0 = -S(t) + \gamma/\beta \log(S(t)) + K$

$K = i_0 + s_0 - \gamma/\beta \log(s_0)$

$0 = -S(t) + \gamma/\beta \log(S(t)) + i_0 + s_0 - \gamma/\beta \log(s_0)$

S_f = the remaining number of susceptibles

$0 = -s_f + \gamma/\beta \log(S_f) + i_0 + s_0 - \gamma/\beta \log(s_0)$

$\gamma/\beta \log(s_0) - \gamma/\beta \log(S_f) = i_0 + s_0 - s_f$

$\gamma/\beta (\log(s_0) - \log(s_f)) = i_0 + s_0 - s_f$

$\log_a x/y = \log_a x - \log_a y$

$\gamma/\beta \log(s_0/s_f) = i_0 + s_0 - s_f$

$\beta/\gamma = \log(s_0/s_f) / (i_0 + s_0 - s_f)$

(c) From your answer to (*a*), find the maximum of *I* regarded as a function of *S* and sketch some typical trajectories. Relate it with the basic reproduction number R_0.

$I(t) = -S(t) + \gamma/\beta \log (S(t)) + K$

$I(0) = i_0 \quad S(0) = s_0 \quad t = 0$

$i_0 = -s_0 + \gamma/\beta \log (s_0) + K$

$K = i_0 + s_0 - \gamma/\beta \log (s_0)$

Imax occurred when dI/dt = 0

$dI(t)/dt = \beta SI - \gamma I$

$\beta SI = \gamma I$

$\beta S = \gamma$

$S = \gamma / \beta$

since

$I_{max}(t) + S(t) - \gamma/\beta \log (S(t)) = i_0 + s_0 - \gamma/\beta \log (s_0)$

$I_{max} = i_0 + s_0 - \gamma/\beta \log (s_0) - S(t) + \gamma/\beta \log (S(t))$

$I_{max} = - S(t) + \gamma/\beta \log (S(t)) + K$

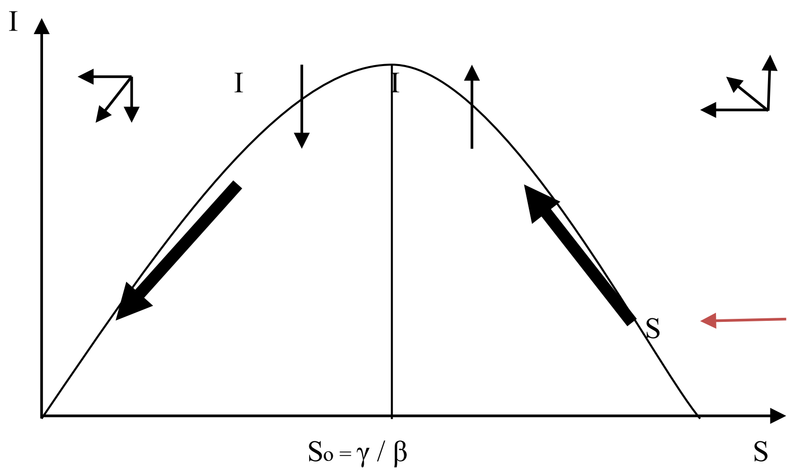
$I_{max} = - \gamma / \beta + \gamma/\beta \log (\gamma / \beta) + K$

$S > 0, \quad I > 0$

$dS(t)/dt = -\beta SI < 0 \quad \forall \quad S > 0, I > 0$

$S = \gamma / \beta$

$dI/dt < 0$ if $S < \gamma / \beta$ and $dI/dt > 0$ if $S > \gamma / \beta$



When $S \to 0, \quad I = - 0 + \gamma/\beta \log (0) + K \quad I \to - \infty$

When $S \to \infty, \quad I = -\infty + \gamma/\beta \log (\infty) + K$ from $dI/dS = -1 + \gamma/\beta S$ then $dI/dS \to -1, \quad I \to - \infty$

Interpretation :

- It is imposible for the disease to infect all the subceptibles
- $S_0 > \gamma / \beta$ epidemic to occur

$S_0 < \gamma / \beta$ for disease died out

$S_0 > \gamma / \beta$

Basic reproduction ratio $R_0 = \beta S_0 / \gamma > 1$

If $R_0 > 1$ an epidemic occurs in the absence of intervention

If R0 < 1 the disease died out

(d) The *intensity* of an epidemic is the proportion of the total number of susceptibles that finally contract the disease. Show that it is equal to $i = (i_0 + s_0 - s_\infty) / s_0$.

As The limit point must be a an equilibrium point

$t \to \infty$ $I(t) = 0$ $t >= 0$

$I(t) = i_0 + s_0 - S + \gamma/\beta \log S/s_0$

$0 = i_0 + s_0 - S + \gamma/\beta \log S/s_0$

$S = io + s_0 + \gamma/\beta \log S(\infty) /s_0$

Solved

$S(\infty) = s_0\ e\ ((so + i_0 - S(\infty)) / (\gamma/\beta)) > 0$

Some susceptible individuals can scape infection, and the epidemic end not by the extinsion of susceptibles, but by the eradication of those infected

For an epidemic starting from the initial point s_0 i_0 $r_0 = 0$

$N(t) = S(t) + I(t) + R(t)$, is actually constant and equal to

$N(0) = s_0 + i_0 + r_0 = s_0 + i_0$

The intensity or final size of the epidemic is

$i = I_{total} = i0 + s0 - s(\infty)$

$i = N0 - s(\infty) / N0$

$i = (s_0 + io - s(\infty))/ (s_0 + i_0)$ the only way I can think about io = 0 is because next sentence

An important implication of this analysis, namely that $I(t) \to 0$ and $S(t) \to S(\infty) > 0$, is that the disease dies out from lack of infectives and not from the lack of susceptibles.

"Age-Structured Population Dynamics in Demography and Epidemiology, By Hisashi Inaba "

Hence $i = (s_0 + io - s(\infty))/ s_0$

0

(e) A study at Yale University in 1982 described an influenza epidemic with initial proportions of susceptibles of the student population as 91.1% and final proportion of susceptibles as 51.3%. (Assume, initially, that no one had recovered).

Given that the mean infectious period for influenza γ^{-1} is approximately 3 days, estimate the combination βN , where N is the total population size, and hence estimate R_0.

Data

N = i0 + s0 +ro = i0 +s0 = 0.911S + 0.089S =

s0 = 0.911S

i0 = (1-.911)S = 0.089S rest of student are infected

$S(\infty) = 0.513S$

R0 = 0

$\gamma = 1/3$

βN =?

$dS / dt = -\beta IS,\ \ dI / dt = \beta IS - \gamma I,\ \ dR / dt = \gamma I$

$N = S + I + R$

$dN / dt = dS / dt + dI / dt + dR / dt = 0$

S(0) = 0.911S I(0) = (1-.911)S = 0.089S R(0) =0

$S(0) \approx N$ I(0) = N- $S(0) \approx 0$ R(0) = 0

From c above

The maximum value of the curve occurs at $S = \gamma/\beta$. This means that an epidemic will start and amplify only if $S(0) \approx N$ is larger than γ/β or equivalently if

$R_0 = \beta S_0 / \gamma > 1$

$R_0 = \beta N / \gamma > 1$

$$\int_0^\infty (S(t) + I(t)]\, dt = S0 + I0 - S_\infty = N - S_\infty$$

$$\text{Ln}\ (s_0/s_\infty) = \beta \int_0^\infty I(t)]\, dt = R0[1 - S(\infty)/N]$$

Ln (.911s/.513S) = R0 [1- 0.513S/S] = .574 = R0[0.487] then

R0 = 1.178 per day

$\beta N = R_0 \gamma = 1.178\ (1/3) = 0.392$

$1/\gamma = 3$ days

[4 points]

2. Consider the nonlinear system of differential equations

$$\left\{\begin{aligned} x' &= 2x - x^5 - xy^4, \\ y' &= y - y^3 - x^2 y, \end{aligned}\right.$$

where the differentiation is with respect to time.

(a) Prove that there are no equilibrium points outside the coordinate axes.

$x' = 2x - x^5 - xy^4 = f(x,y) = x(2 - x^4 - y^4)$
$y' = y - y^3 - x^2y = g(x,y) = y(1 - y^2 - x^2)$

Equilibrium points are where x'. y' are simultaneously zero.
For this is at $x = 0$, $y = 0$ Thus $(x,y) = (0,0)$**.**

Alternatives: using matlab

- $x = 0, y \neq 0$, giving $(1-y^2) = 0$ then $y = +-1$ Thus $(x,y) = (0,1), (0,-1)$
  $x \neq 0, y = 0$ giving $(2-x^4) = 0$ then $x^4 = 2$, $x^2 = \pm\sqrt{2}$, $x = \pm 2^{1/4}$,
  two more points imaginary root $x = \pm j(2^{1/4})$

It is seen if x is an imaginary fourth root of 2 the condition is not true, hence the points $(\pm I(2^{1/4}), 0)$ are discarded as equilibrium points, or **only 5 equil points are in the (X,Y) plane.**

For $(xe, ye) = (0,0)$

$$J = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

For $(xe, ye) = (0, \pm 1)$

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

For $(\pm 2^{1/4}, 0)$

$$\mathbf{J} = \begin{bmatrix} -8 & 0 \\ 0 & (1-\sqrt{2}) \end{bmatrix}$$

(b) Prove that there are exactly five equilibrium points in the coordinate axes.

Hence in the coordinate frame (x,y) there are five equilibrium points
(0,0), (0,1),(0,-1), (2 ¼, 0), (- 2 ¼,0)

(c) Linearise the system and establish the classification of each equilibrium point.

F(X,Y) = $2x - x^5 - xy^4$ G(X,Y) = $y - y^3 - x^2y,$

To linearise, need the derivatives of F(X,Y) and G(X,Y) to get the Jacobian matrix

Jacobian matrix

$$J = \begin{bmatrix} Fx & Fy \\ Gx & Gy \end{bmatrix} = \begin{bmatrix} (2-5x^4 - y^4) & -4xy^3 \\ -2xy & 1-3y^2 - x^2 \end{bmatrix}$$

Linearised system becomes (x’, y’) = J [(x-xe), (y-ye)]

The value of **J** at the coordinates of equilibrium determine the nature of the equilibrium points.
The value of J has to be real and positive. If > 0 is stable if < 0 unstable.

Det(J) = $(2-5x^4-y^4)(1-3y^2-x^2)$

Construct J for the equilibrium points

For (xe , ye) = (0,0)

$$J = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Eigenvalues are (2, 1) which are positive (2,1) > 0 so **unstable equilibrium.**

For (xe , ye) = (0 , ± 1)

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

Eigenvalues are (1, -2) where 1 > 0 positive and -2 < 0 negative . this is a **Saddle point**

For (± 2 ¼, 0)

$$\mathbf{J} = \begin{bmatrix} \mathbf{-8} & \mathbf{0} \\ \mathbf{0} & \mathbf{(1-\sqrt{2})} \end{bmatrix}$$

Hence eigenvalues are (-8 , (1-√2)) < 0 negative so **Stable equilibrium**

[3 points]

3. Consider the predator-prey model

$$\begin{cases} x' = x(\beta_1 - c_1 y), \; y' \\ = y(-\alpha_2 + c_2 x), \end{cases} \tag{1}$$

where β_1, α_2, c_1 and c_2 are positive real numbers.

a. Find the linearised system at the non-zero equilibrium point $(x, y) = (\alpha_2/c_2, \beta_1/c_1)$.

The linearization of a first-order differential equation can be achieved using the first-order Taylor series. The linearized equations can be expressed as:

$$F(x,y) \approx f(\overline{x},\overline{y}) + \frac{\partial f(\overline{x},\overline{y})}{\partial x}(x-\overline{x}) + \frac{\partial f(\overline{x},\overline{y})}{\partial y}(y-\overline{y})$$

Where $F(x,y)$ is the linearized function and $f(\overline{x},\overline{y})$ is the original equation evaluated at the linearization point. Then, the linearized equations are:

$$\dot{x}^* = \bar{x}(\beta_1 - c_1\bar{y}) + (\beta_1 - c_1\bar{y})(x-\bar{x}) + (-c_1\bar{x})(y-\bar{y})$$
$$\dot{y}^* = \bar{y}(-a_2 + c_2\bar{x}) + (c_2\bar{y})(x-\bar{x}) + (-a_2 + c_2\bar{x})(y-\bar{y})$$

By substitution, we get:

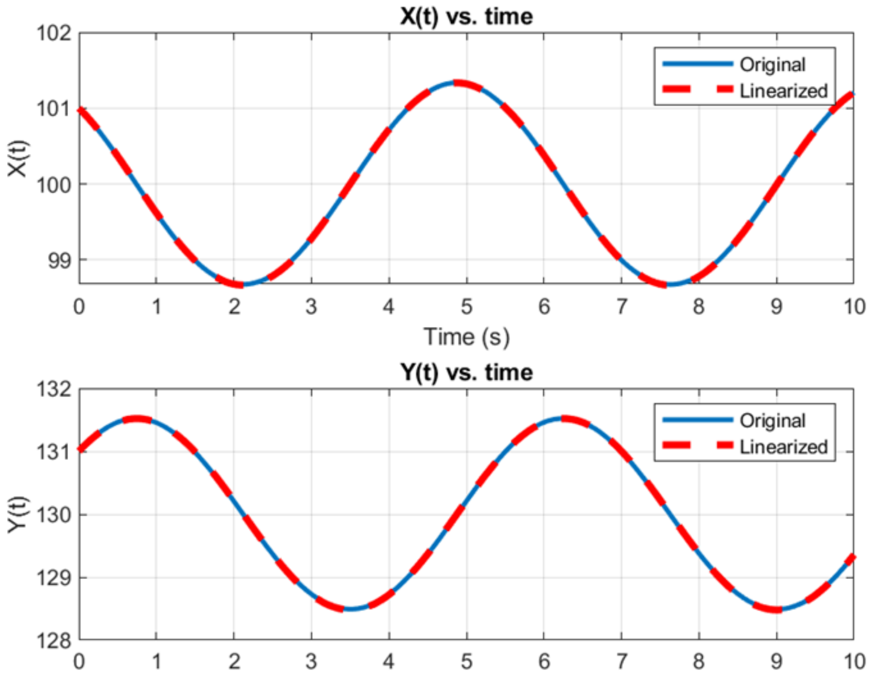
$$\dot{x}^* = \left(-\frac{c_1 a_2}{c_2}\right)\left(y - \frac{\beta_1}{c_1}\right)$$
$$\dot{y}^* = \left(\frac{c_2\beta_1}{c_1}\right)\left(x - \frac{a_2}{c_2}\right)$$

b. In order to compare the nonlinear system (1) with the linearised system obtained in (a), we can draw them both on the same system of axes and compare them directly. With this aim, taking $\beta_1 = 1.3$, $c_1 = 0, 01$, $\alpha_2 = 1$ and $c_2 = 0.01$, use MATLAB or Maple to draw the numerical solution of both systems with the same initial condition $(x_1, y_1) = (101, 131)$. Display them in the same diagram with different colour.

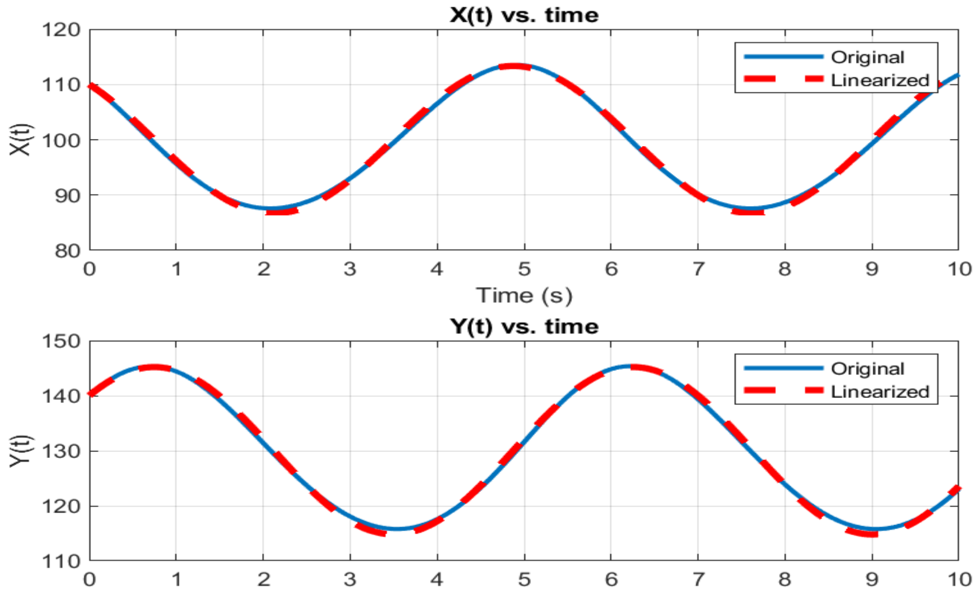
Do the same with $(x_2, y_2) = (110, 140)$ and $(x_3, y_3) = (160, 150)$.

Attached matlab code:

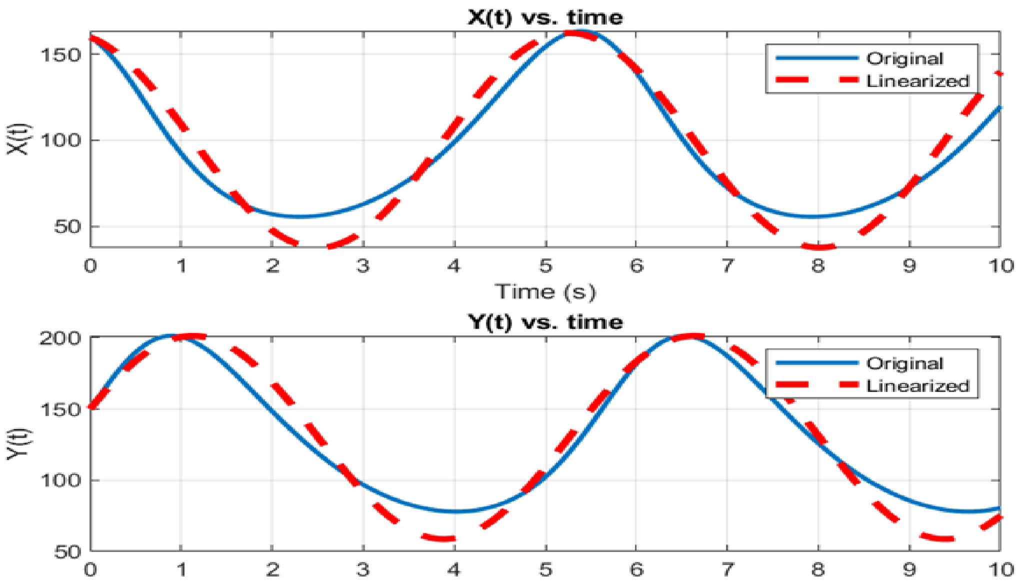
- For $(x_0, y_0) = (101,131)$:



- For $(x_0, y_0) = (110{,}140)$:



- For $(x_0, y_0) = (160{,}150)$:



c. Use the chain rule to find a relationship between the solutions $x = x(t)$ and $y = y(t)$ of the nonlinear system (1).

$$x^t = x(\beta_1 - c_1 y), \qquad y^t = y(-\alpha_2 + c_2 x),$$

$$\frac{dy}{dt} = \frac{dy}{dx} * \frac{dx}{dt} \to \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

$$\frac{dy}{dx} = \frac{y(-a_2 + c_2 x)}{x(\beta_1 - c_1 y)}$$

[3 points]