Rovira e Virgili University
Master in Computer Science and Mathematics

Modelización mediante ecuaciones diferenciales

Máster en Ingeniería Computacional y Matemáticas

1. Read the case study on Lake Burley Griffin (Section 2.6 of the textbook). The average summer flow rate for the water into and out of the lake is 4×10^6 m^3/month.
   (a) Using this summer flow, how long will it take to reduce the pollution level to 5% of its current level? How long would it take for the lake with the pollution concentration of 10^7 parts/m^3, to drop below the safety threshold? (Assume in both cases that only freshwater enters the lake.)
   (b) Use Maple or MATLAB to replicate the results in the case study, for both constant and seasonal flow and constant and seasonal pollution concentrations entering the lake. Comment on the solutions.

General Data

Water average rate flow $F = 4 \times 10^6$ m^3/month.

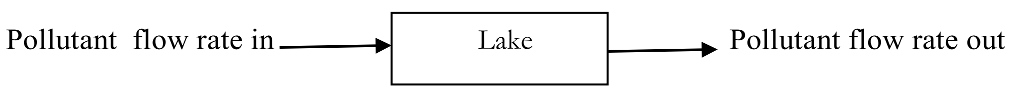
Assumption: Only freshwater enter the lake $C_{in} = 0$

Current level = $C_0 = 10^7$ bacteries/m^3

5% Current level = 0.05×10^7 bacteries/m^3 = 500 000 bacteries/m^3

V=Volume is Costant = 28×10^6 m^3

Safe level = 4×10^6 bacteries/m^3



a. Problems

1.1 How long will it take to reduce the pollution level to 5% of its current level?

Solution:

Current level = C_0 = 10^7 bacteries/m^3

5% Current level = 0.05 X 10^7 bacteries/m^3 = 500 000 bacteries/m^3

freshwater entering the lake Cin = 0

$F = 4 \times 10^6$ m^3/month.

V=Volume is Costant = $28x10^6$ m^3

Let C(t) be the concentration of the pollutant in the lake at time t.

Equation 2.16 from book after deduction

$C(t) = Cin - (Cin - C_0)e^{-F\,t/V}$

$500,000 = 0 - (0 - 10^7)\ e^{-(4x10\uparrow6\ /\ 28x10\uparrow6)t}$

$500,000 = 10^7 e^{-(4x10\uparrow6\ /\ 28x10\uparrow6)t}$

$500,000\ /\ 10^7 = e^{-(4x10\uparrow6\ /\ 28x10\uparrow6)t}$

$0.05 = e^{-(4x10\uparrow6\ /\ 28x10\uparrow6)t}$

Apply if $y = e^x$ then $x = \ln y$

$-\ 0.1428\ t = \ln(0.05)$

$-\ 0.1428\ t = -2.9957$

t = 20.97 months ≈ 21 months is the time for the pollution level to drop to 5% of its current level

1.2 How long would it take for the lake with the pollution concentration of 10^7 parts/m^3, to drop below the safety threshold?

Solution:

Safe level = 4 x 10^6 bacteries/m^3

Current level = C_0 = 10^7 bacteries/m^3

freshwater enter the lake Cin = 0

$F = 4 \times 10^6$ m^3/month.

V=Volume is Costant = $28x10^6$ m^3

Let C(t) be the concentration of the pollutant in the lake at time t.

Equation 2.16 from book after deduction

$C(t) = Cin - (Cin - C_0)e^{-F\,t/V}$

$4 x 10^6 = 0 - (0 - 10^7)\ e^{-(4x10\uparrow6\ /\ 28x10\uparrow6)t}$

$4 x 10^6 = 10^7 e^{-(4x10\uparrow6\ /\ 28x10\uparrow6)t}$

$4 x 10^6\ /\ 10^7 = e^{-(4x10\uparrow6\ /\ 28x10\uparrow6)t}$

$0.4 = e^{-(4x10\uparrow6\ /\ 28x10\uparrow6)t}$

Apply if $y = e^x$ then $x = \ln y$

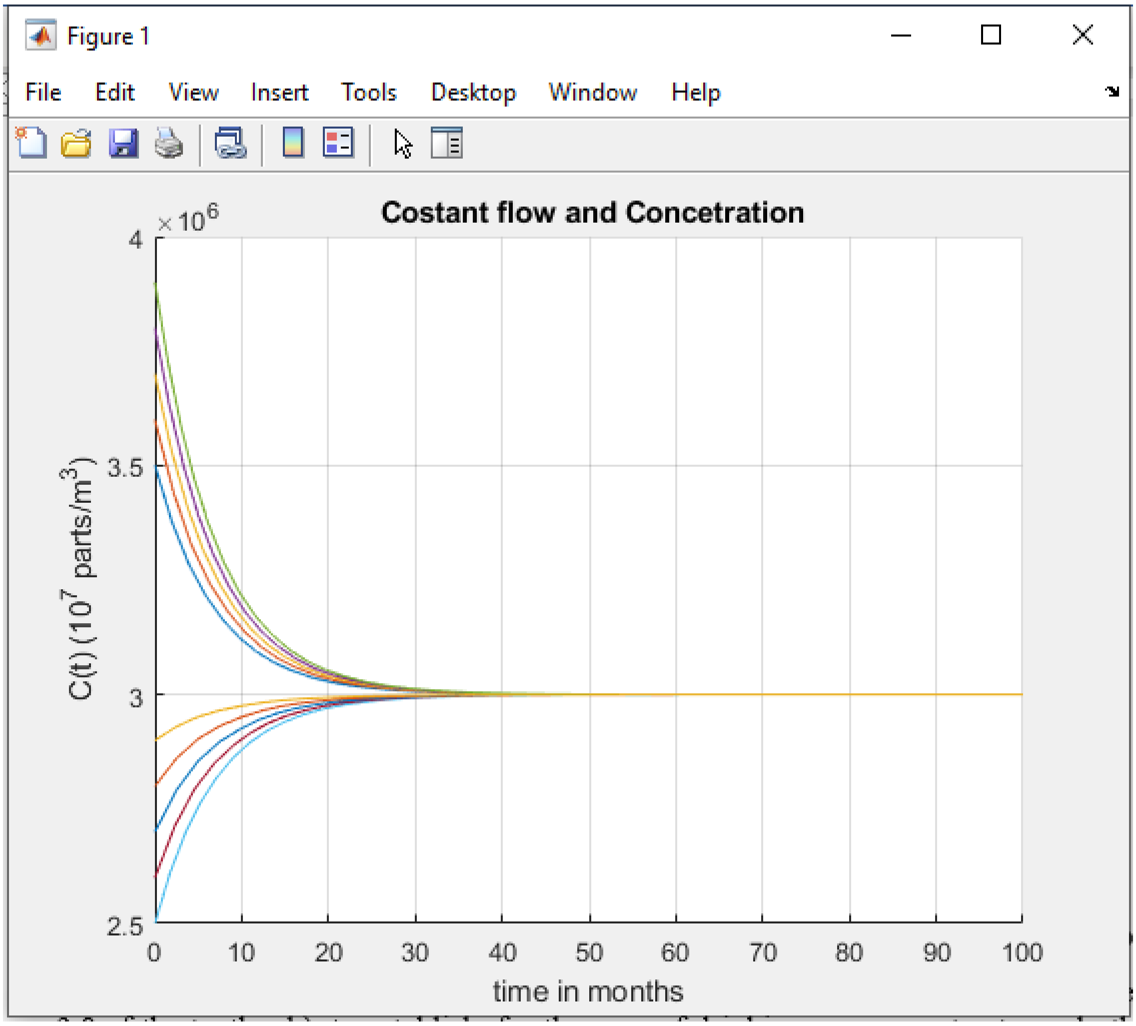
$-\ 0.1428\ t = \ln(0.4)$

$-\ 0.1428\ t = -0.9162$

t = 6.4159 months it will take ≈ 6 months to reach the safety threshold

b. Graphs

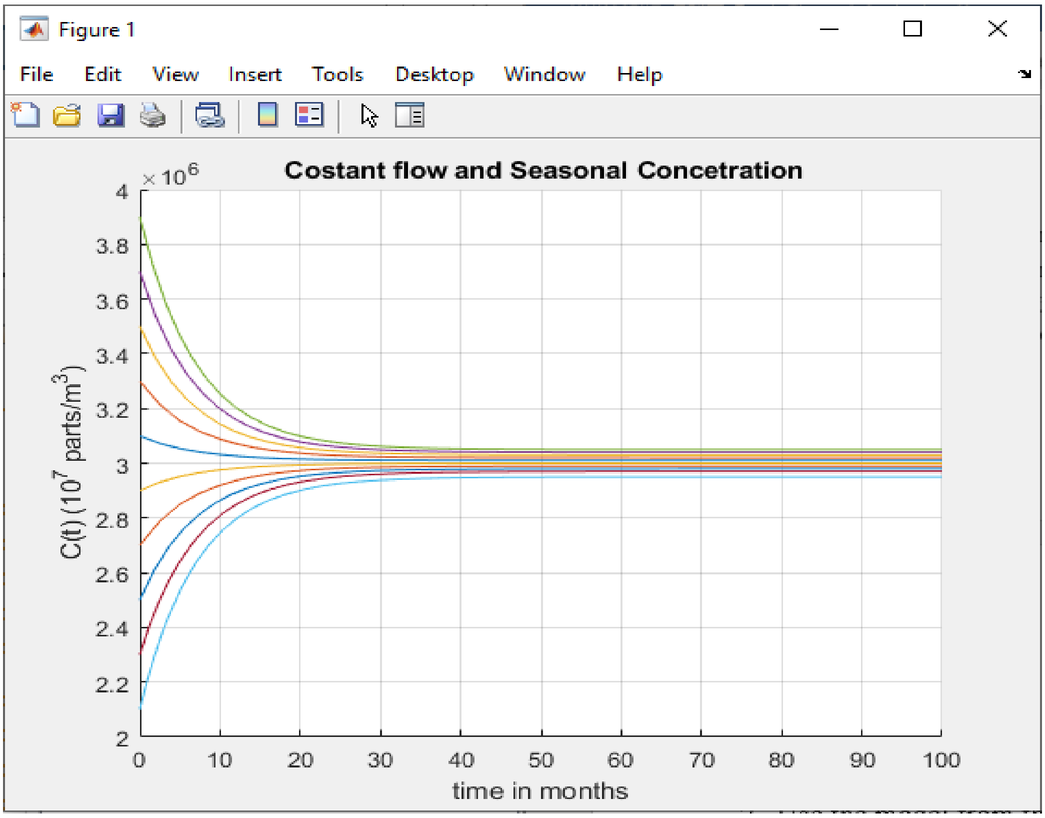
Constant flow and constant concentration in time
c0 range is variable



- c0 range is variable
- For cin < c0 the level of pollution decrease with time to reach cin
- For cin > c0 the level of pollution increase until reach cin.
- In both cases, the level of pollution is approaching cin

Constant flow and seasonal concentration in time

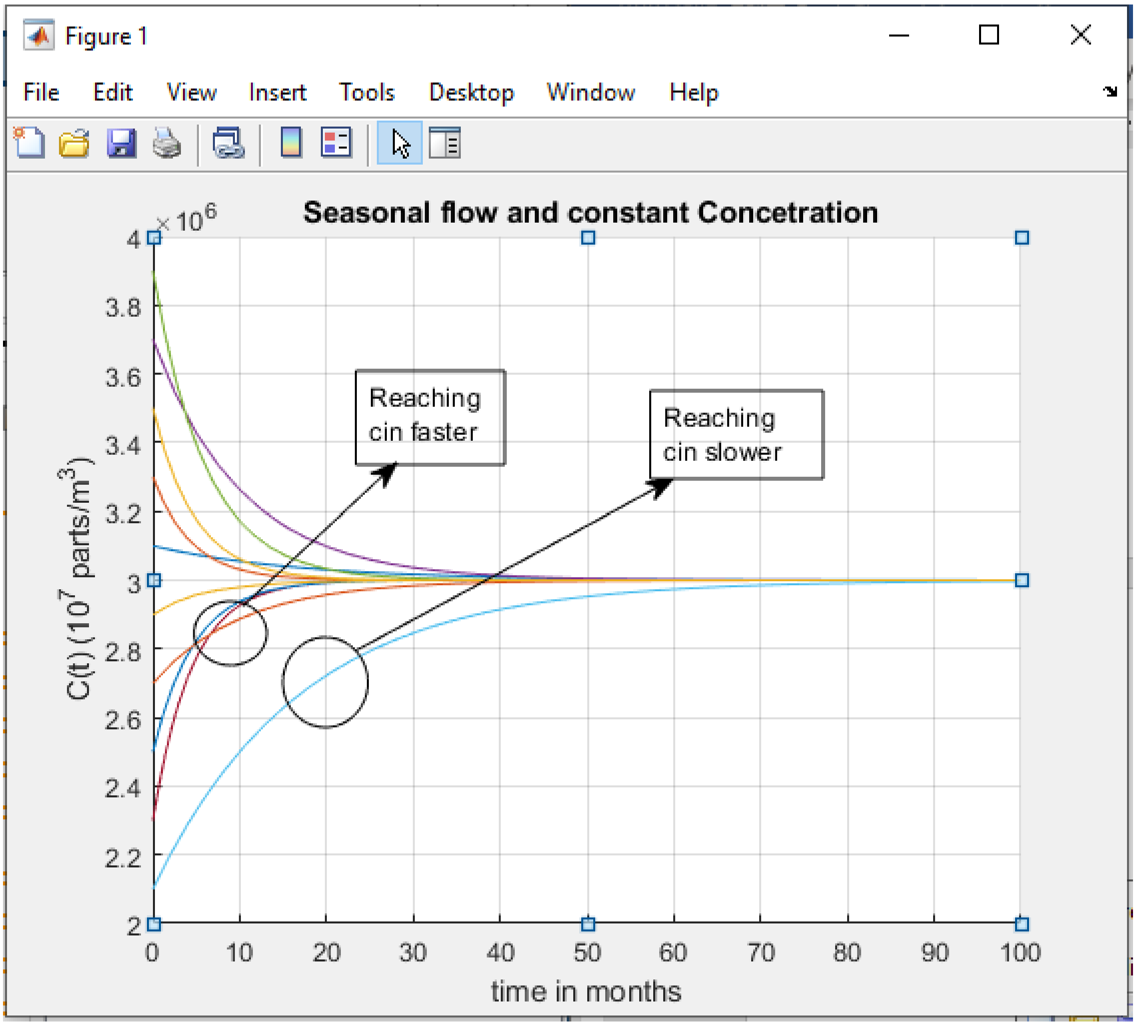
Graph using values around cin $3x10^6$



Graph using $cin(t) = 10^6 (10 + 10 \cos(2\pi t))$ with values around $3x10^6$ for cin

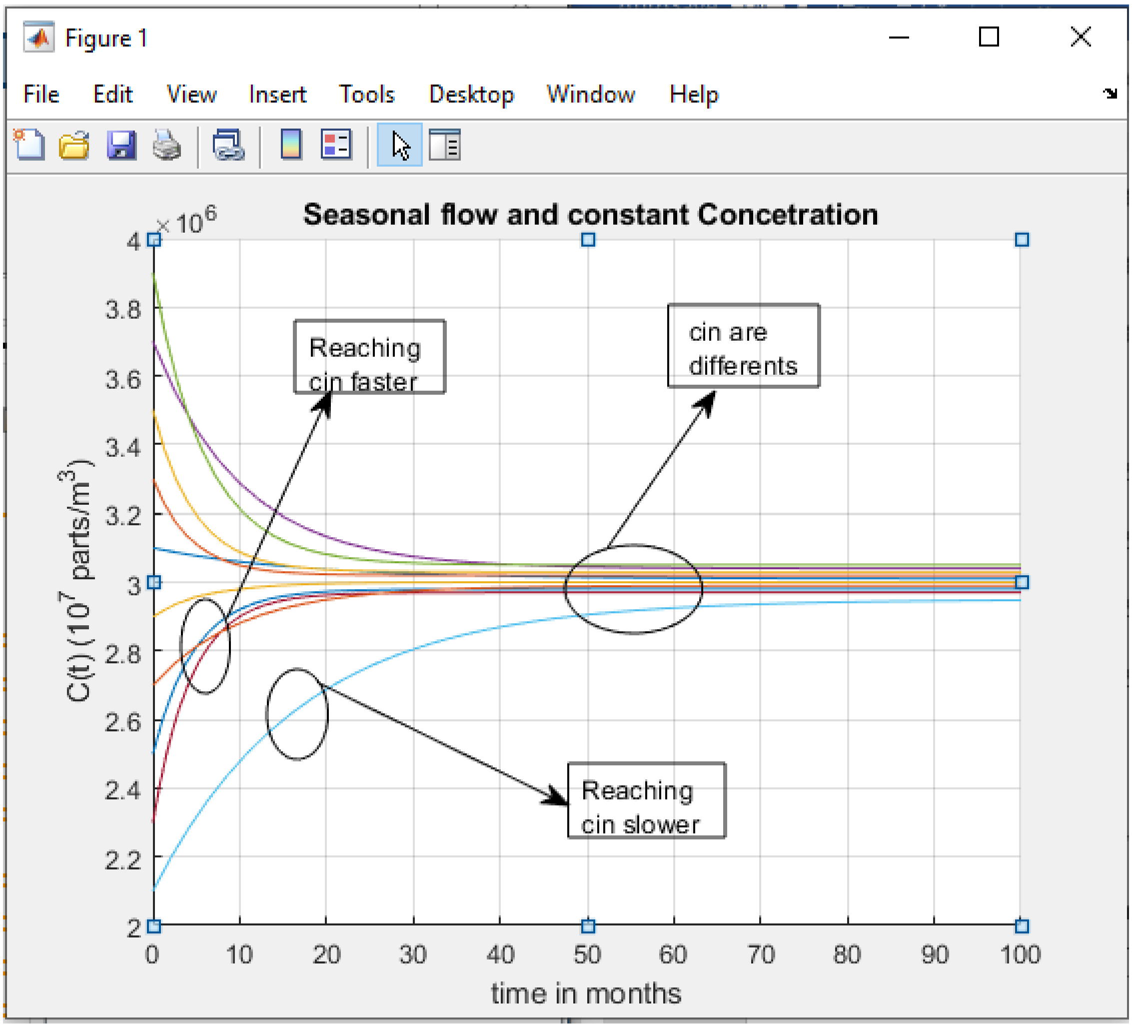
- For cin < c0 the level of pollution decrease with time to reach cin
- For cin > c0 the level of pollution increase until reach cin.
- In both cases, the level of pollution is approaching to cin but with some variation of the concentration over time.
- The approach to cin, in this case, is slower and look like the values get close to cin but no get equal.

Graph using values around cin $3x10^6$



Graph using the equation $F(t) = 10^6 (1 + 6 \sin(2\pi t))$ m3/year for find the values of flow $F(1)= 1.65x10^6$ m^3/year, $F(10)= 6.33x10^6$ m^3/year, $F(20)= 5.87x10^6$ m^3/year, $F(60)= 2.75x10^6$ m^3/year, $F(80)= 4.63x10^6$ m^3/year.

- For cin < c0 the level of pollution decrease with time to reach cin
- For cin > c0 the level of pollution increase until reach cin.
- Depending of the flow variation cin is going to be reached faster or slower as we can see in the graph



Graph using the equation $F(t) = 10^6 (1 + 6 \sin(2\pi t))$ m3/year and the equation $cin(t) = 10^6 (10 + 10 \cos(2\pi t))$ for find the values of flow and concentration

$F(1)= 1.65 \times 10^6$ m^3/year, $F(10)= 6.33 \times 10^6$ m^3/year, $F(20)= 5.87 \times 10^6$ m^3/year, $F(60)= 2.75 \times 10^6$ m^3/year, $F(80)= 4.63 \times 10^6$ m^3/year.

- For cin < c0 the level of pollution decrease with time to reach cin
- For cin > c0 the level of pollution increase until reach cin.
- Depending on the flow variation cin is going to be reached faster or slower as we can see in the graph.
- cin is different accord with the season as we can see they are stabilized in its individual values

Conclusion: a range of initial concentration was used but the general behavior tends to reach the initial concentration values entering the lake this is indication of the importance to control what is entering the lake assuming constant volume.

2.5 points]

2. Use the model from the case study on alcohol consumption (Dull, dizzy or dead, Section 2.8 of the textbook), to establish, for the case of drinking on an empty stomach, the following:

(a) Use Maple or MATLAB to generate graphs to investigate the effects of alcohol on a woman of 60 kg, over a period of time.

Enty stomach $k1 = k2 \approx 6$ n = drink 3 glasses and stop drinking

n initial drinks are consumed and no more alcohol is taken

$C1(0) = c0 = n * cs = 3 * cs = 3 * 0.034 = 0.102$ I = 0 there is not subsecuents drinks

$C2(0) = 0$ bloodstrean initial amount Woman weight = 60 kg

Blood fluid in a woman =0.67 x W = 0.67 * 60 = 40.02 liters

Cs= 14/40.02 then BAL =0.034 Rate remove alcohol from blood = 8g/hrs

Associated reduction in BAL 8/(40.02*10)= 0.019 then k3 = 0.019 BAL

Drinking continuoslly over time

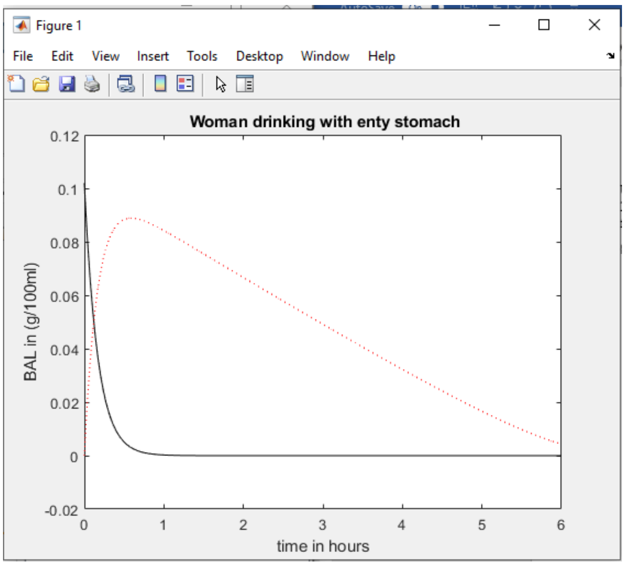
$I = (n/T)cs = (3/6)*0.034 = 0.017$, $c0 = 0$

Attached matlab code

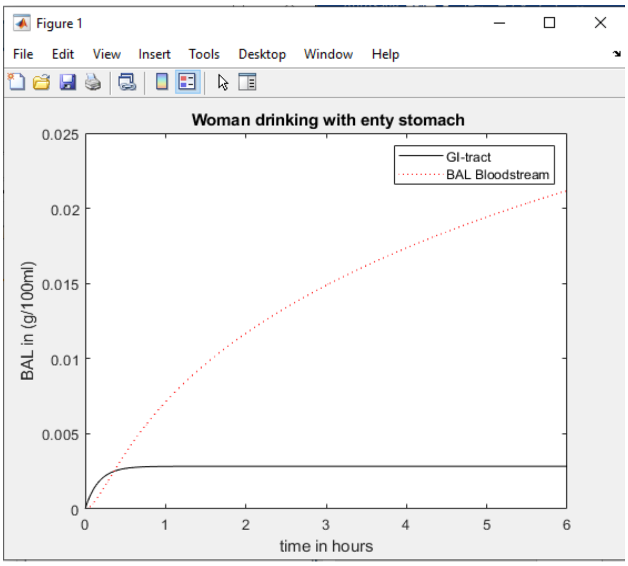
$dC1 / dt = -k1C1, \quad C1(0) = c0,$

$dC2 / dt = k2C1 - (k3C2 / (C2 + M)), \; C2(0) = 0$

Drinking and stop



Drinking continuously



(b) Compare these results with those for a man of the same weight.

Enty stomach $k1 = k2 \approx 6$ n = drink 3 glasses and stop drinking

n initial drinks are consumed and no more alcohol is taken

$C1(0) = c0 = n * cs = 3 * cs = 3 * 0.028 = 0.085$ I = 0 there is not subsecuents drinks

$C2(0) = 0$ bloodstrean initial amount Men weight = 60 kg

Blood fluid in a men =0.82 x W = 0.82 * 60 = 49.2 liters

Cs= 14/49.2 then BAL =0.028 Rate remove alcohol from blood = 8g/hrs

Associated reduction in BAL 8/(49.2*10)= 0.016 then k3 = 0.016 BAL

Drinking continuoslly over time

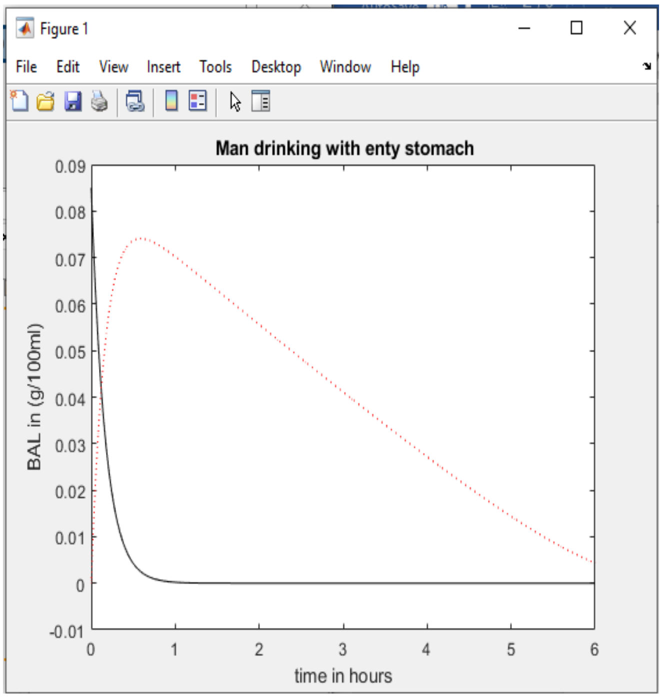
I = (n/T)cs =(3/6)*0.028=0.014, c0 =0

Attached matlab code

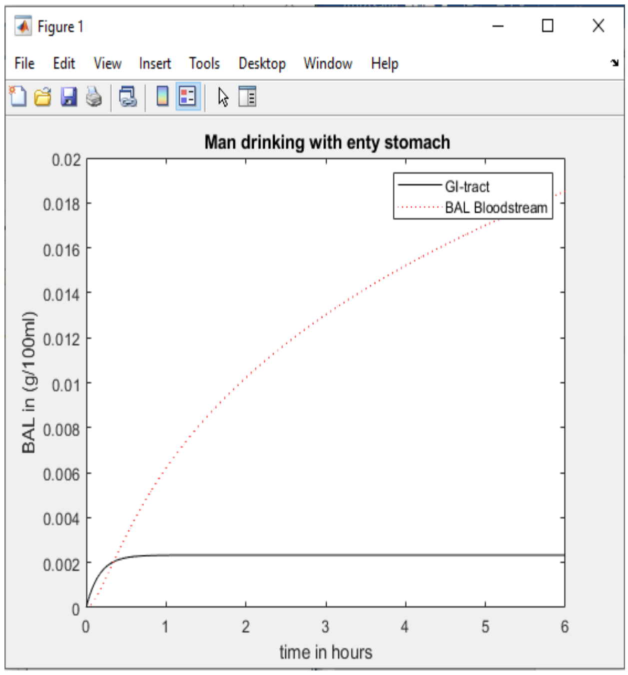
$dC1 / dt = -k1C1, \quad C1(0) = c0,$

$dC2 / dt = k2C1 - (k3C2 / (C2 + M)) , C2(0) = 0$

Drinking and stop

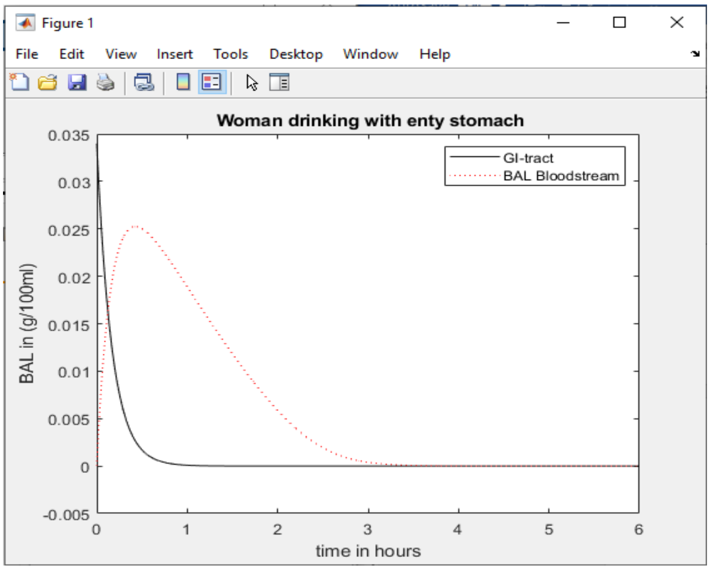


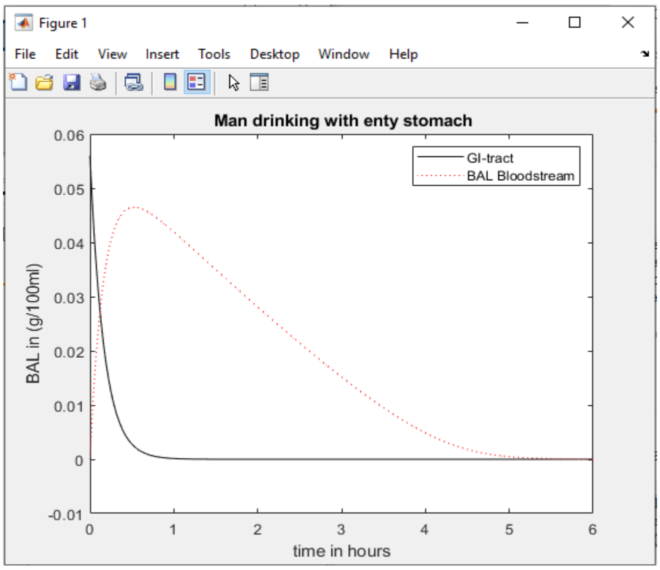
Drinking continuously



With the same amount of drinks the alcohol will reach the bloodstream of a woman with a higher peak than in a man, making her reach the legal BAL faster probably this will increase the risk for her to go over 0.05 BAL and get a DUI(drive under the influence).

(c) Assuming the legal limit to be 0.05 BAL (the Australian limit), establish roughly how much alcohol the man and woman above can consume each hour and remain below this limit.





The man can consume 2 drinks and the women can consume 1 drink in order to not get over BAL 0.05

(d) Repeat (a)-(c) for the case of drinking together with a **meal.**

(d.a) Use Maple or MATLAB to generate graphs to investigate the effects of alcohol on a woman of 60 kg, over a period of time.

Enty stomach k1 > k2, k1≈ 6, k2= k1/2 ≈ 3 n = drink 3 glasses and stop drinking

n initial drinks are consumed and no more alcohol is taken

C1(0) = c0 = n * cs = 3 * cs =3 * 0.034 = 0.102 I = 0 there is not subsecuents drinks

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Drinking continuoslly over time

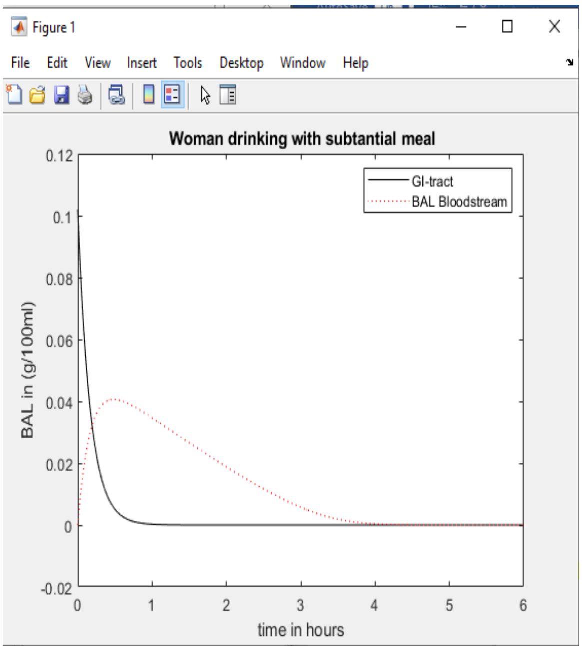
I = (n/T)cs =(3/6)*0.034=0.017, c0 =0

Attached matlab code

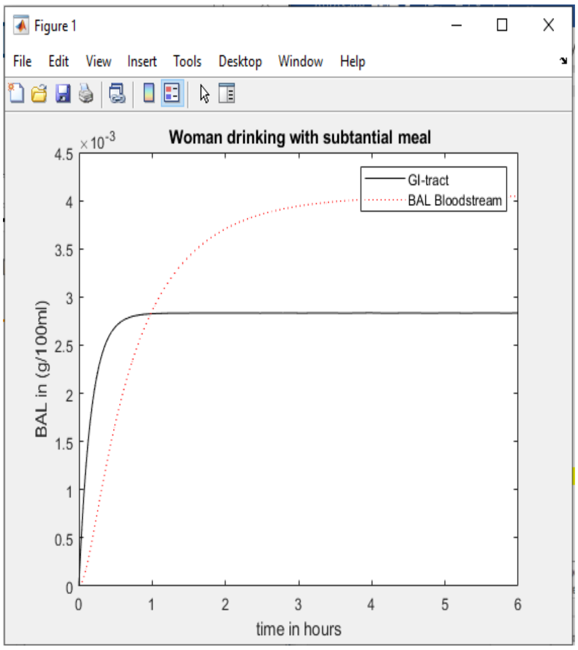
$dC1 / dt = -k1C1, \quad C1(0) = c0,$

$dC2 / dt = k2C1 - (k3C2 / (C2 + M)) , C2(0) = 0$

Dinking and Stop



Drinking continuously



(d.b) Compare these results with those for a man of the same weight.

Enty stomach k1 > k2, k1≈ 6, k2= k1/2 ≈ 3 n = drink 3 glasses and stop drinking

n initial drinks are consumed and no more alcohol is taken

C1(0) = c0 = n * cs = 3 * cs =3 * 0.028 = 0.085 I = 0 there is not subsecuents drinks

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Associated reduction in BAL 8/(49.2*10)= 0.016 then k3 = 0.016 BAL

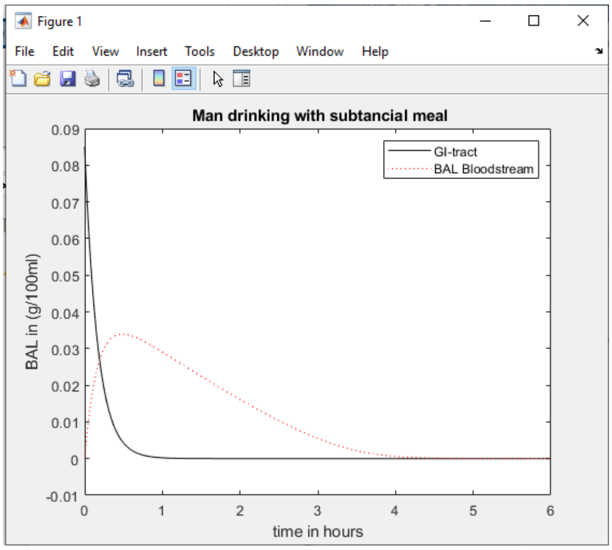
Drinking continuoslly over time

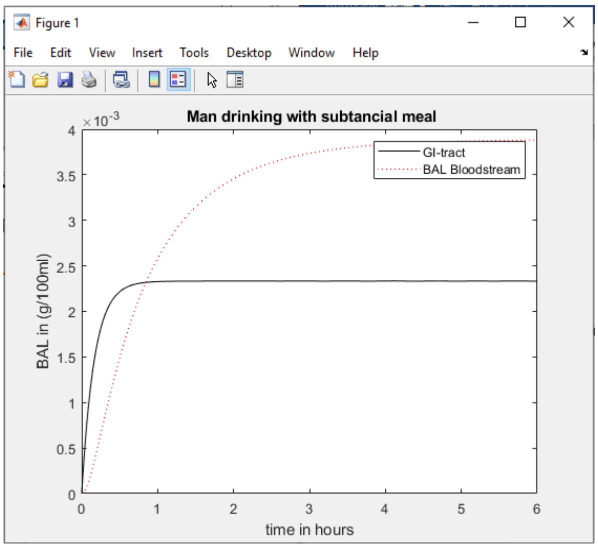
I = (n/T)cs =(3/6)*0.028=0.014, c0 =0

Attached matlab code

dC1 / dt = −k1C1, C1(0) = c0,

dC2 / dt = k2C1 – (k3C2 / (C2 + M)) , C2(0) = 0

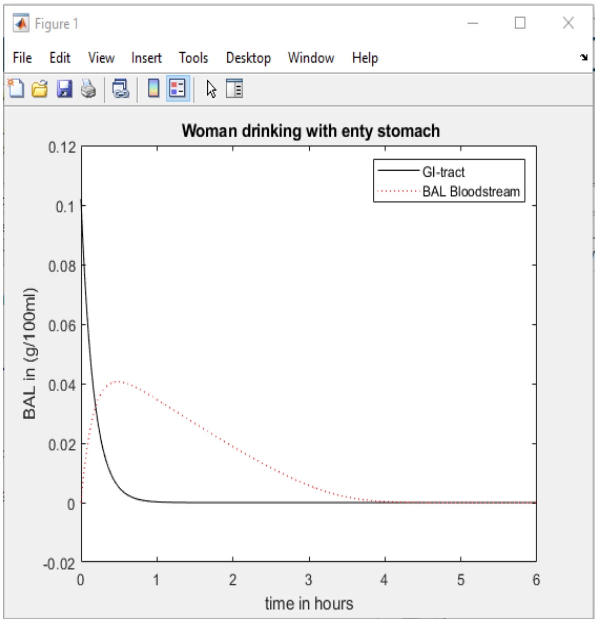


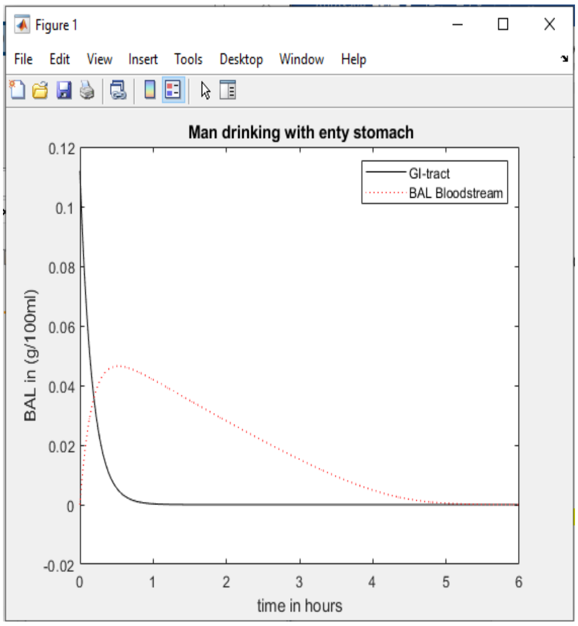


food will help a lot to reduce the speed at which the alcohol is passed to the bloodstream but the path still the same if a woman is compared with a man.

With the same amount of drinks the alcohol will reach the bloodstream of a woman with a higher peak than in a man, making her reach the legal BAL faster probably this will increase the risk for her to go over 0.05 BAL and get a DUI(drive under the influence).

(d.c) Assuming the legal limit to be 0.05 BAL (the Australian limit), establish roughly how much alcohol the man and woman above can consume each hour and remain below this limit.





The man can consume 4 drinks of Alcohol and the women can consume 3 drink of alcohol in order to no get over BAL 0.05

[2.5 points]

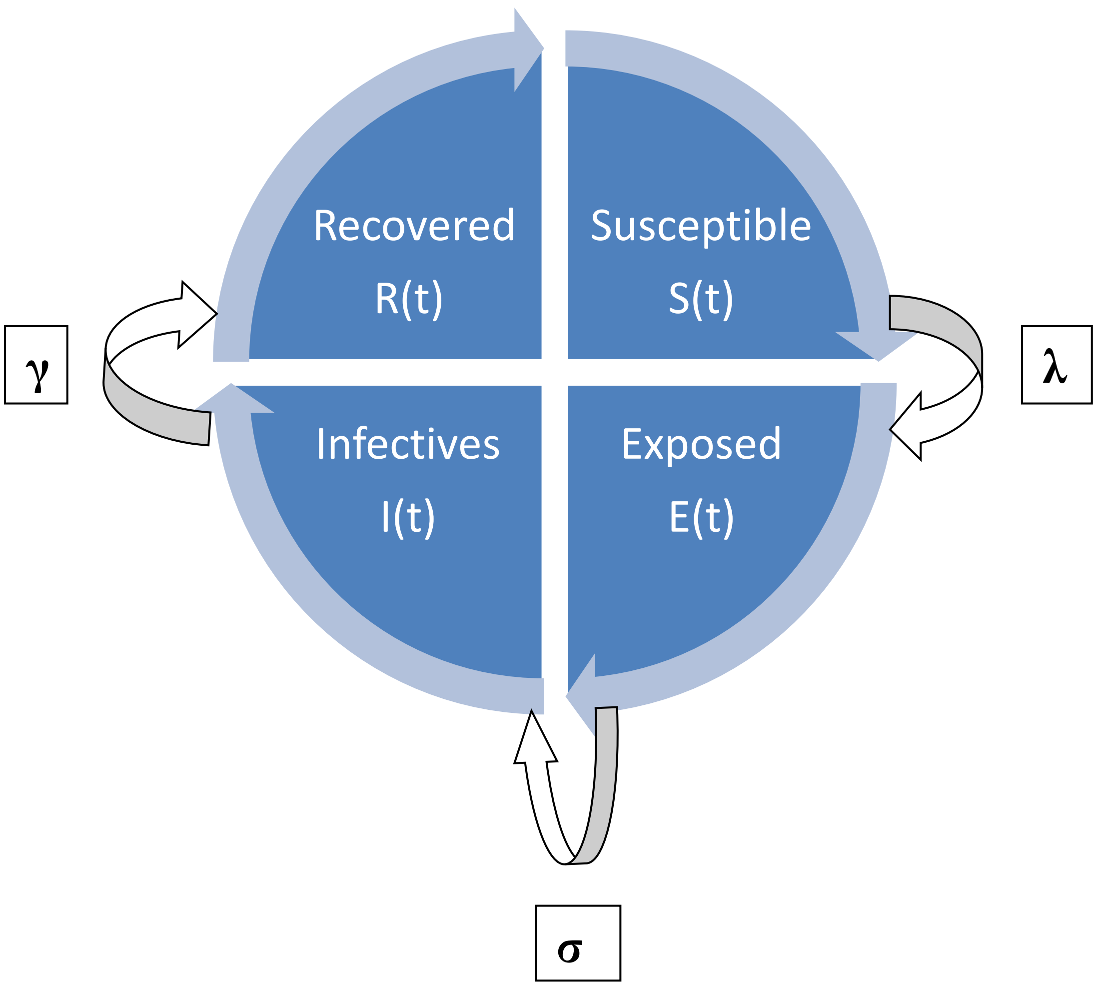
3. Many diseases have a *latent period*, which is when there is a period of time between infection and when an infected individual becomes infectious. One example is measles, where the latent period is approximately 5 days.

   Extend the basic SIR epidemic model to one with an additional population class $E(t)$, corresponding to individuals who have been **exposed** to the disease, so they are no longer susceptibles, but are not yet infectious. You may assume in this **SEIR** model that the per-capita rate at which an individual in the exposed class becomes an infective is constant. You may also suppose that those who recover from the disease have lifelong immunity.

Basic SIR model extended to the SEIR Model

For many important infections, there is a significant incubation period during which the individual has been infected but is not yet infectious yet. During this period the individual is in compartment E (for exposed).”Wikipedia compartmental model”.The SEIR differs from the SIR model in the addition of a latency period. Individuals who are exposed (E) have had contact with an infected person, but are not themselves infectious.” Arizona State SEIR Model ”

The SEIR model



Assumptions:

- We assume that the populations of susceptibles and contagious infectives are large so that random differences between individuals can be neglected
- We ignore births and deaths in this model and assume the disease is spread by contact
- Latent period for the disease, setting it equal to five (5) days
- We assume all those who recover from the disease have lifelong immunity.
- We also assume that, at any time, the population is homogeneously mixed, that is, we assume that the contagious infectives and susceptibles are always randomly distributed over the area in which the population lives.

Formulating The differential equation

Word equation :

- S is the fraction of susceptible individuals (those able to contract the disease),
- E is the fraction of exposed individuals (those who have been infected but are not yet infectious),
- I is the fraction of infective individuals (those capable of transmitting the disease),
- R is the fraction of recovered individuals (those who have become immune).

- The only way the number of susceptible can change is the loss of those who become exposes.

Rate of change in number of susceptibles = - {rate susceptibles exposed}

- Susceptibles becoming exposed and exposed becoming infectives

Rate of change in number of exposed = {rate susceptibles exposed} – { rate exposed become infectives}

- Exposed becoming infectives and infectives becoming recovered

Rate of change in number of infectives = {rate expose become infectives} – {rate infectives become recovered}

Rate of change in the number of recovery = {rate infectives become recovered}

Rate subseptible become exposed = $\lambda(t)E(t)$
Rate expose become become infetives = $\sigma E(t)$
Rate infectives recovery = $\gamma I(t)$

$dS/dt = -\lambda(t)S(t)$, $dE/dt = \lambda(t)S(t) - \sigma E(t)$, $dI/dt = \sigma E(t) - \gamma I(t)$, $dR/dt = \gamma I(t)$

c = rate individual made contacts = contacts per time
p = probablility that contact result in an infection
$N(t)$ = total population size = $S(t) + I(t) + E(t) + R(t)$
$N'(t) = S'(t) + I'(t) + E'(t) + R'(t) = 0$ population size is constant
$\lambda(t)$ = force of infection it is no constant depend on the current number of infectives $I(t)$, probability p, rate c, and population

$\lambda(t) = c^* \, p * (I(t) / N(t))$ if $\beta f = c^*p$
$\lambda(t) = \beta f \; * (I(t) / N(t))$
$\beta = \beta f / N$ = transmission coheficient
γ = recovery rate
σ = percapita rate individual become infectives is constant

Governing equation with population size constant

$$dS/dt = -(\beta f *S*I) / N = \beta*S*I$$

$$dE/dt = \lambda(t)S(t) - \sigma E(t) = (\beta*S*I) - (\sigma*E)$$

$$dI/dt = \sigma E(t) - \gamma I(t) = (\sigma*E) - (\gamma*I)$$

$$dR/dt = \gamma I(t) = \gamma*I$$

[2.5 points]

4. A 100-liter tank originally contains 50 liters of freshwater. Beginning at time t = 0, water containing 50% of pollutants flows into the tank at a rate 2 liters per minute and the well-stirred mixture leaves at a rate of 1 liter per minute.

Data

Tank volume = 100 liters

V1 = 50 liters freshwater

Water with Pollutants (t=0) = 50% pollutants

In Rate 2 liter/min

Out Mixture leave rate =1 liter/min

(a) Find the volume of mixture in the tank as a function of time.

V1 = 50 liter freshwater

Vrate_in = 2 liter / minutes

Vrate_out = 1 liter / minute

Net change is a gain

V = V1 + (Vrate_in - Vrate_out) = 50 + (2-1)t = 50 + t

(b) Formulate a differential equation for the volume of pollutants in the tank.

Rate of change of S(t) = {rate at which S(t) enter the tank} – {rate at which S(t) exits the tank}

Rate at which S(t) enter the tank = (flow rate of liquid entering) x (concentration of substance in liquid entering)

Rate at which S(t) exits the tank = (flow rate of liquid exiting) x (concentration of substance in liquid exiting)

Concentration = S/V = Amount of pollutant in the tank at any time, t / Volume of water in the tank at any time, t

Volume of pollutant = concentration of pollutant

amount of pollutant in the tank at t = 0, S (0) = 0

$dS/dt = (2 * 0.5) - (1 * S/(50 + t)) = 1 – S/(50+t)$

then $S'(t) + (S(t) / (50 + t)) = 1$

$S'+P(t)S = S(t)$

$P(t) = 1/(50+t), S(t) = 1$

(c) Hence show that the concentration of pollutants at the time the tank overflows is approximately 48%.

$S'(t) + (S(t) / (50 + t)) = 1$

$S' + P(t)S = S(t)$

$P(t) = 1/(50+t)$, $S(t) = 1$

$u(t) = e^{\int P(x)dt} = e^{\int dt/(50+t)} = 50+t$

General solutión

$S*u(t) = \int S(t)u(t)dt + c$

$S(t+50) = \int 1 * (t+50)\, dt + c$

$S(0) = 0$ then $c = 0$

$S(t) = (t^2/2 + 50t) / (t+50)$

Tank overflow when $50 + t = 100$ then $t = 50$ minutes

$C(t) = S(t)/V(t)$

$V(t) = 50 + t = 100$

Way 1

$S(t) / 100 = ((t^2/2 + 50t) / (t+50)) / 100 =$ **0.375 =37.5%**

Way 2

$C(t) = S(t) / (50 + t) = ((t^2/2 + 50t) / (t+50)) / (50 + t)$

After work by hand I got

$C(t) = t(t + 100) / 2(t + 50)^2$

$C(50) = (50*(50+100)) / (2*(50 + 50)^2) = 7500/20000 = 0.375 = 37.5\%$