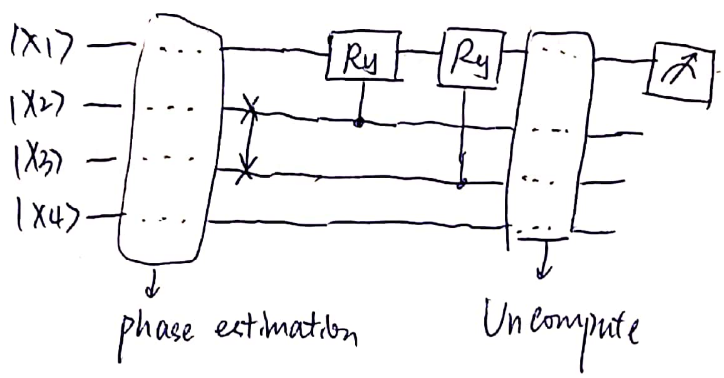
Controlled Rotation for HHL algorithm

① SWAP $|x_2\rangle$ and $|x_3\rangle$

Before this step, the state of $|x_2x_3x_4\rangle$ is $\beta_1|01\rangle|u_1\rangle+\beta_2|10\rangle|u_2\rangle$



But after the SWAP of $|x_2\rangle$ and $|x_3\rangle$, the state of $|x_2x_3x_4\rangle$ becomes:

$\beta_1|10\rangle|u_1\rangle+\beta_2|01\rangle|u_2\rangle$

This is because, previously $|x_2x_3\rangle$ stands for λ_1 (=1, binary 01) and λ_2 (=2, binary 10). But what we need is the λ_1^{-1} and λ_2^{-1}, we can use this SWAP process to get λ_j^{-1}:

SWAP
$10 \longleftrightarrow 01$

$\because$ $\lambda_1=1 \rightarrow \frac{1}{\lambda_1}=1 \rightarrow \frac{2}{\lambda_1}=2\ (10)$

$\lambda_2=2 \rightarrow \frac{1}{\lambda_2}=\frac{1}{2} \rightarrow \frac{2}{\lambda_2}=1\ (01)$

$\therefore$ The relationship between λ_j and λ_j^{-1} is the SWAP operation

However this step add an additional "2" to the system.

This number gives the λ_j^{-1} an equal scale, so won't have any effect on the final result.

Other examples for this:

e.g.1. For a 4×4 matrix with 4 eigenvalues

$\lambda_1=1 \quad \lambda_2=2 \quad \lambda_3=4 \quad \lambda_4=8$

$\frac{1}{\lambda_1}=1 \quad \frac{1}{\lambda_2}=\frac{1}{2} \quad \frac{1}{\lambda_3}=\frac{1}{4} \quad \frac{1}{\lambda_4}=\frac{1}{8}$

$\underbrace{8\cdot\frac{1}{\lambda_1}=8}_{1000} \quad \underbrace{8\cdot\frac{1}{\lambda_2}=4}_{0100} \quad \underbrace{8\cdot\frac{1}{\lambda_3}=2}_{0010} \quad \underbrace{8\cdot\frac{1}{\lambda_4}=1}_{0001}$

This "8" is a similar scale number.

e.g.3
$\lambda_1=1 \rightarrow \frac{1}{\lambda_1}=1 \rightarrow \frac{3}{\lambda_1}=3\ (11)$
$\lambda_2=3 \rightarrow \frac{1}{\lambda_2}=\frac{1}{3} \rightarrow \frac{3}{\lambda_2}=1\ (01)$
$11 \xleftrightarrow{X(|x_2\rangle)} 01$ add NOT gate on $|x_2\rangle$

n is the multiple of 2.

e.g.2. The binary "a" and "$\frac{n}{a}$" have the same form, except for the decimal point.
like 0001 and 00.01

② Apply gate Ry to

Our goal is to realize: $R|0\rangle|\lambda_j^{-1}\rangle=\lambda_j|1\rangle|\lambda_j^{-1}\rangle+\sqrt{1-\lambda_j^2}\,|0\rangle|\lambda_j^{-1}\rangle$

R_y is the approximate realization of R, which act as:

$$R_y|0\rangle = \sin(\lambda_j^{-1})|1\rangle + \cos(\lambda_j^{-1})|0\rangle$$

Then if we use $|\lambda_j^{-1}\rangle$ as the control bit to act R_y on $|0\rangle$ (the ancilla qubit):

$$R_y|0\rangle|\lambda_j^{-1}\rangle = \sin(\lambda_j^{-1})|1\rangle|\lambda_j^{-1}\rangle + \cos(\lambda_j^{-1})|0\rangle|\lambda_j^{-1}\rangle$$

R_y is a Rotation gate around the y-axis of the Bloch sphere.