Grover's Search Algorithm: A quantum algorithm to search through the inputs of function $f(x)$ to check whether the function $f(x)$ returns true for that input x.

Quantum Oracle O: On a quantum computer, we can transform the function into a set of quantum gates contributing for a quantum oracle O, and use Grover's search algorithm to find a correct input with $O(\sqrt{2^n})$ iterations.

e.g. If we need the quantum oracle O to pick out string 10, the function that O represents should be:

$$f(x) = \begin{cases} 1, x = 10 \\ 0, x \neq 10 \end{cases}$$

In order to reprent this, we can design the quantum oracle O by:

$$O|x\rangle = (-1)^{f(x)}|x\rangle$$

which will flips the amplitude of the quantum state if x=10, and it can be expressed by the quantum matrix:

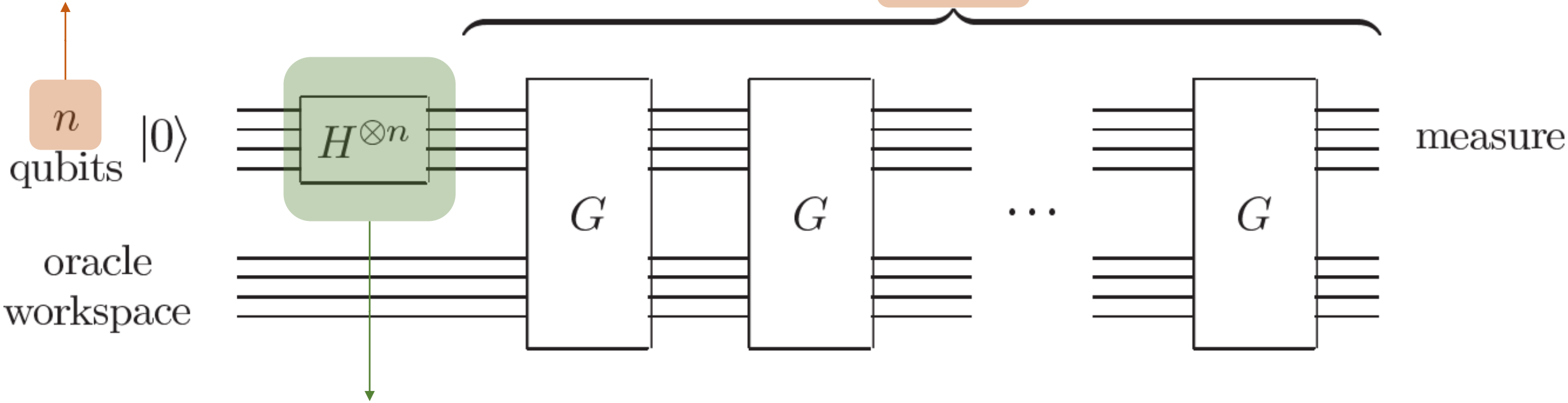
$$O = \begin{array}{c} \\ |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} \begin{array}{c} \begin{array}{cccc} |00\rangle & |01\rangle & |10\rangle & |11\rangle \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Thus, O is a black-box quantum oracle that is already given to the algorithm.

Procedures of Grover's Search Algorithm – Step 1. Create input

$N = 2^n$ is the number of elements that we search from, which means that the index of those elements are from 0 to N-1.

Number of iterations

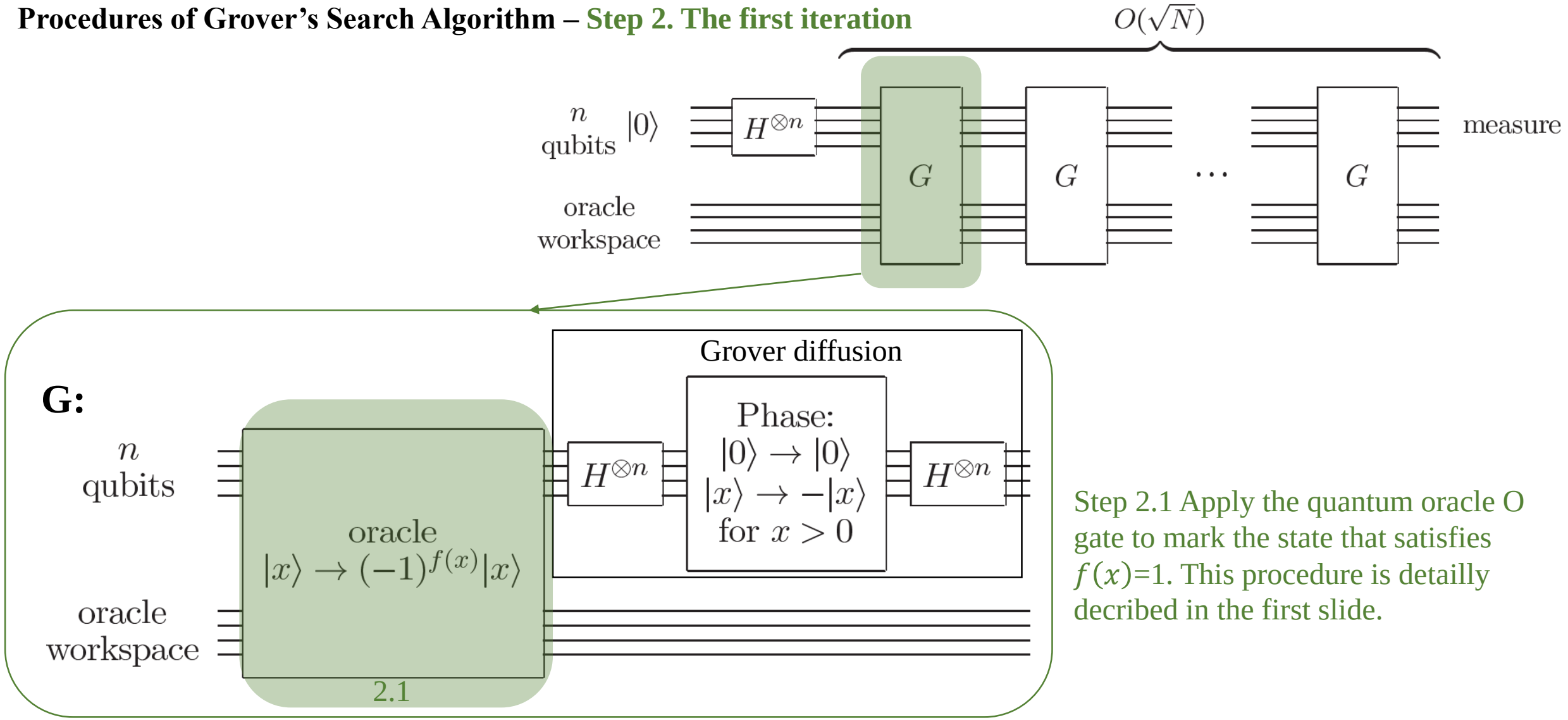


Step 1. Apply the Hadamard transforms, which is to create the possible inputs with equally weighted superposition, $|\psi\rangle = \frac{1}{\sqrt{n}} \sum_{x=0}^{n=1} |x\rangle$.

(e.g. When n=2, the input should be $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$.

Reference & picture: Michael A Nielsen and Isaac Chuang. Quantum computation and quantum information, 2002.

Procedures of Grover's Search Algorithm – Step 2. The first iteration

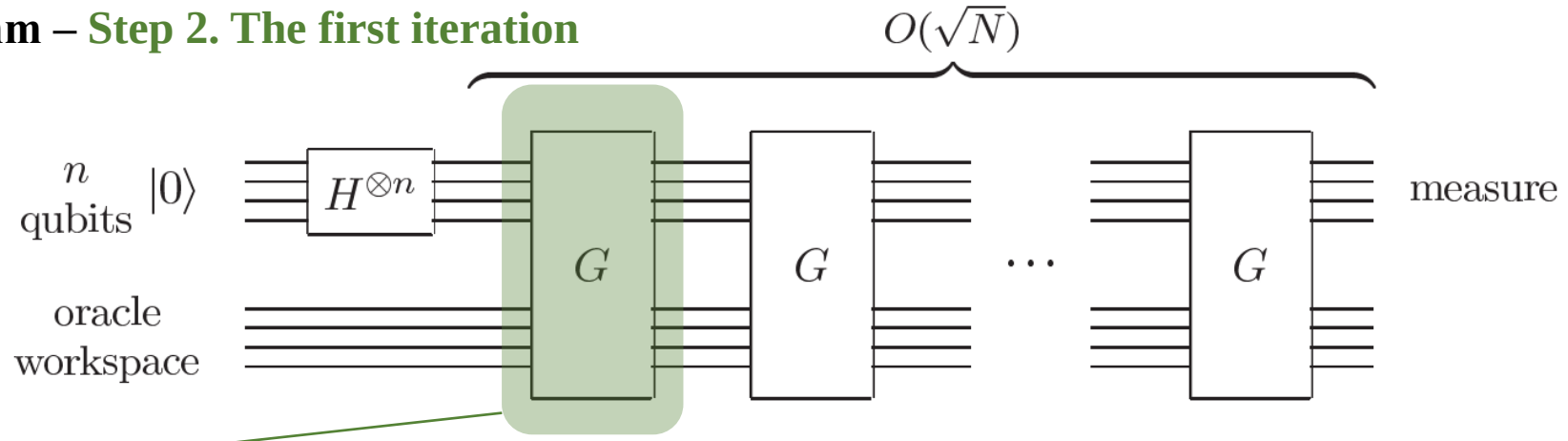


Step 2.1 Apply the quantum oracle O gate to mark the state that satisfies $f(x)$=1. This procedure is detailly decribed in the first slide.

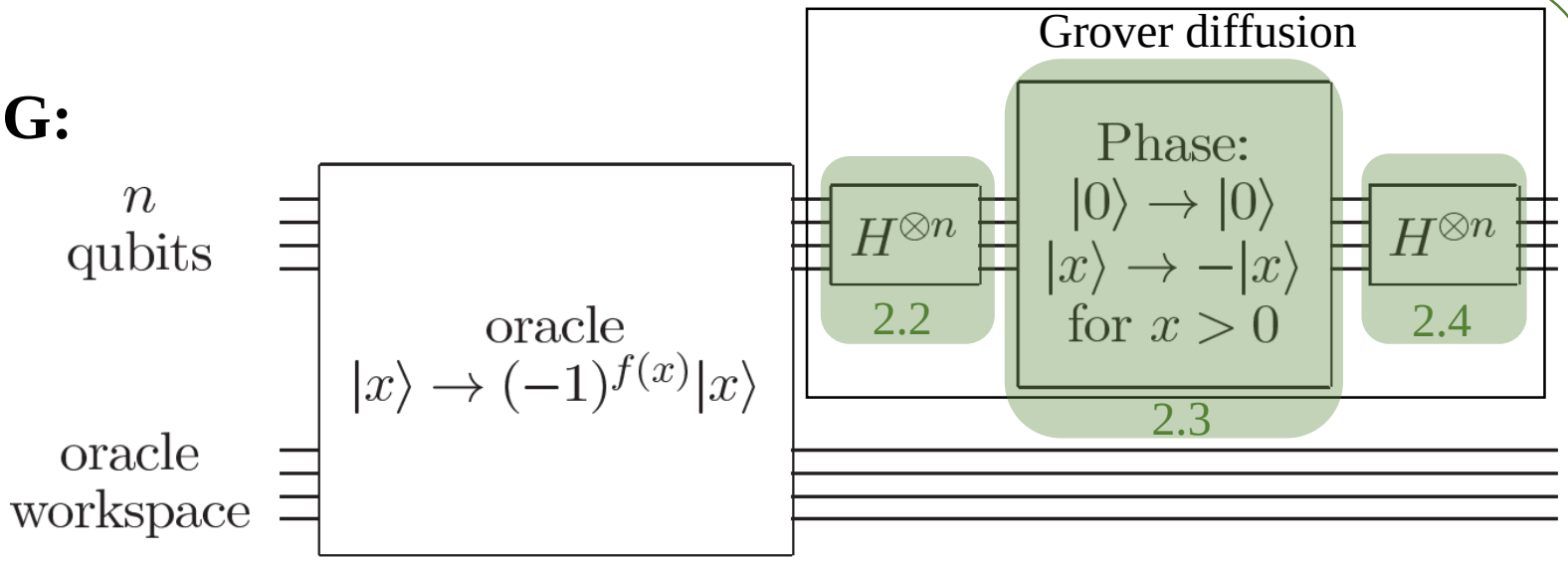
Reference & picture: Michael A Nielsen and Isaac Chuang. Quantum computation and quantum information, 2002.

Procedures of Grover's Search Algorithm – Step 2. The first iteration

Step 2.2 Apply the Hadamard transforms to the n qubits.
Step 2.3 Inverse $|x\rangle$ than if $|x\rangle \neq |0\rangle$.
Step 2.4 Again apply the Hadamard transforms to the n qubits.



G:

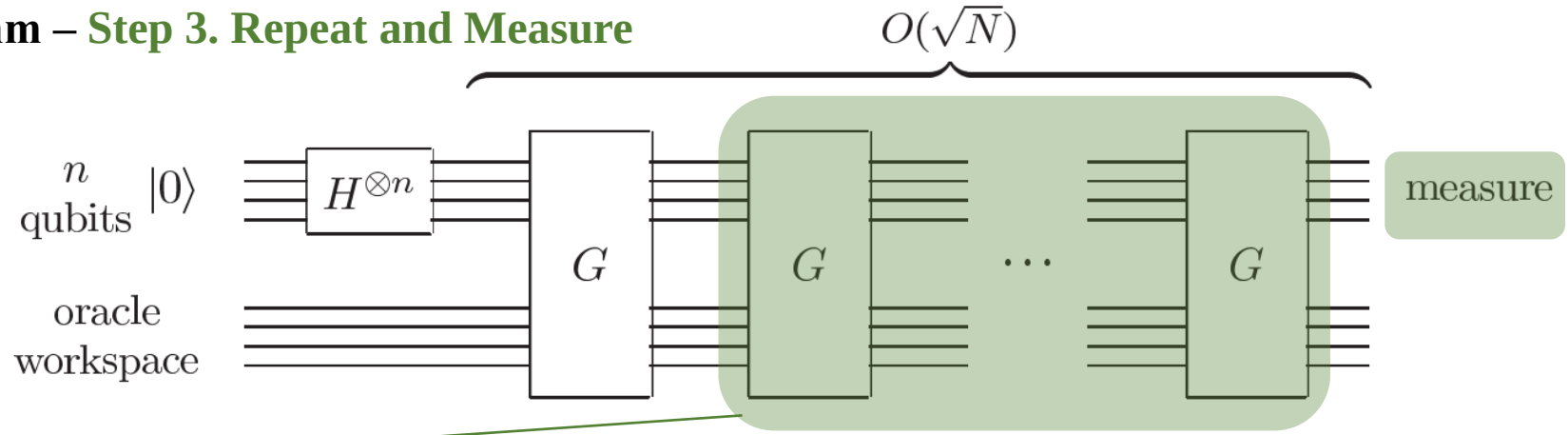


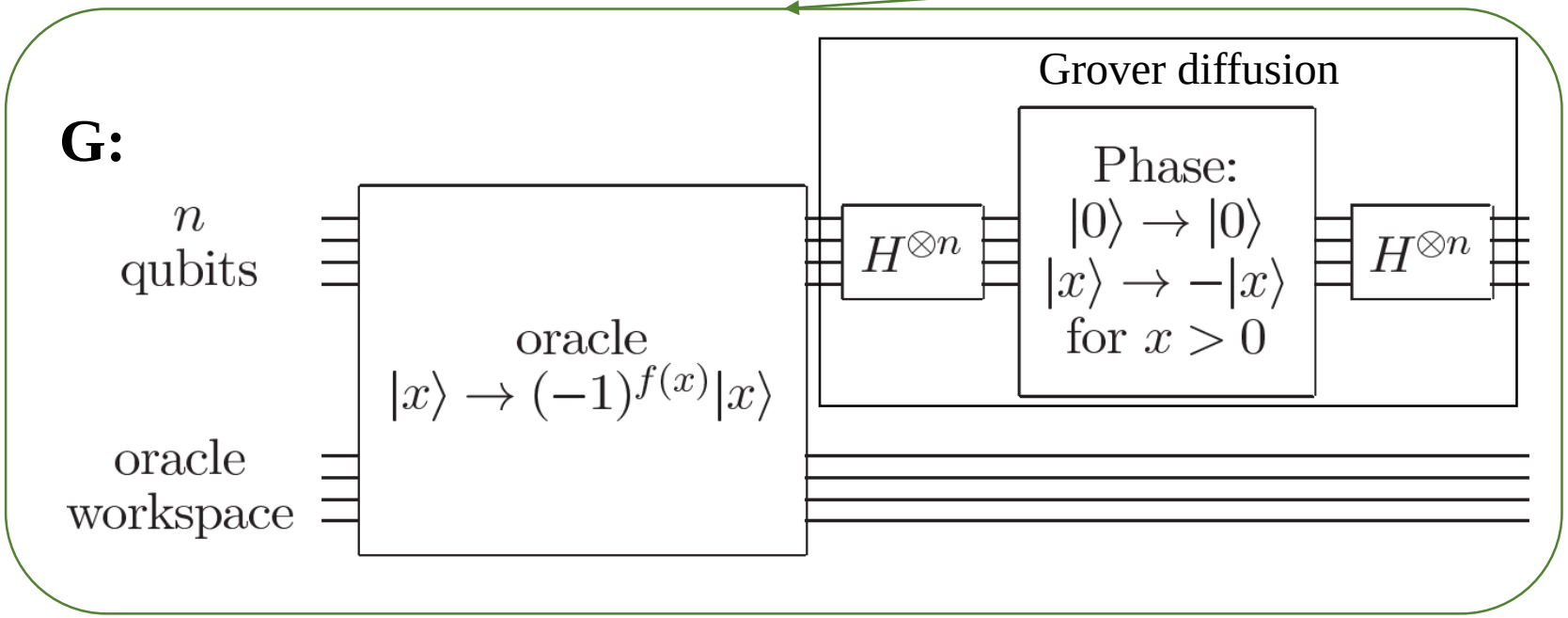
Thus steps 2.2 to 2.4 can be expressed by:
$H^{\otimes n}(2|0\rangle\langle 0| - I)\, H^{\otimes n} = 2|\psi\rangle\langle\psi| - I$.
Thus the G operator is:
$G = (2|\psi\rangle\langle\psi| - I)O$.

These three steps are called Grover diffusion, which is to amplify the probability amplitude of the target element.

Reference & picture: Michael A Nielsen and Isaac Chuang. Quantum computation and quantum information, 2002.

Procedures of Grover's Search Algorithm – Step 3. Repeat and Measure

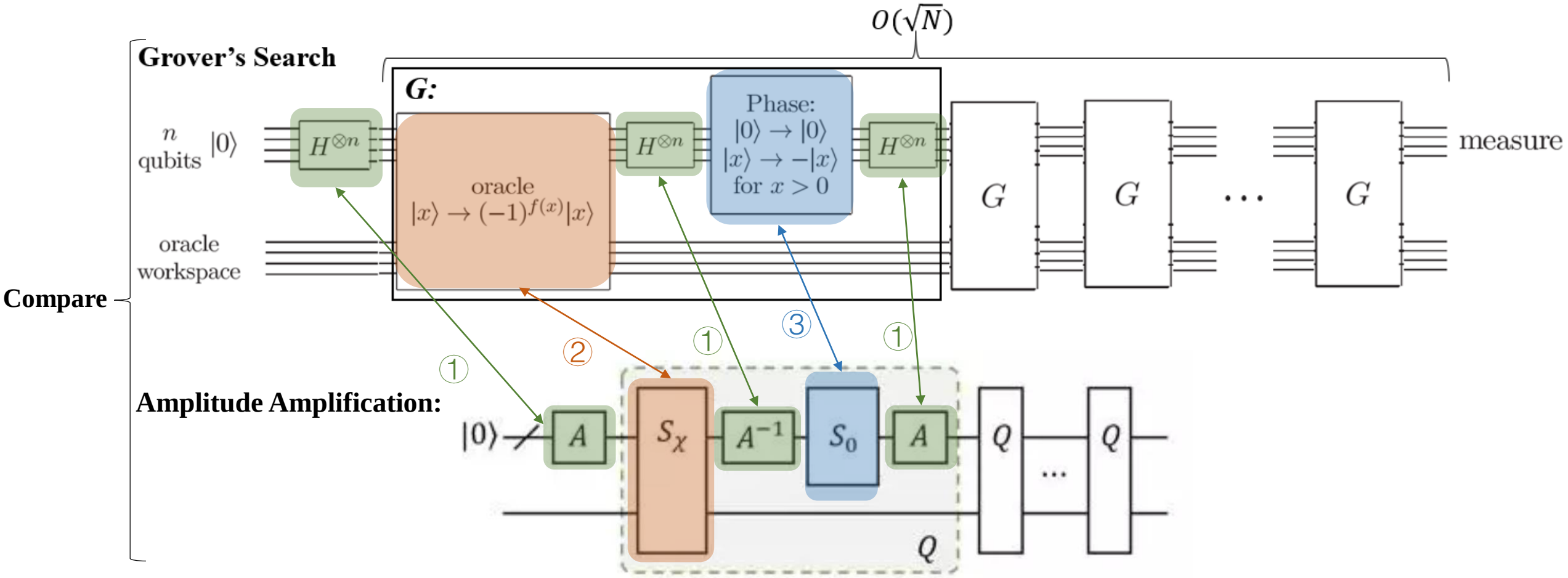




Step 3. Repeat the operation G for around $\frac{\pi}{4}\sqrt{N}$ times. Then measure the n qubits, to get the x with high probability.

Reference & picture: Michael A Nielsen and Isaac Chuang. Quantum computation and quantum information, 2002.

Amplitude Amplification: An application of Grover's search, and it can be used to Improve the probability of success of many classical and quantum algorithms, e.g. HHL algorithm and many quantum ML algorithms based on HHL.



Reference & picture:

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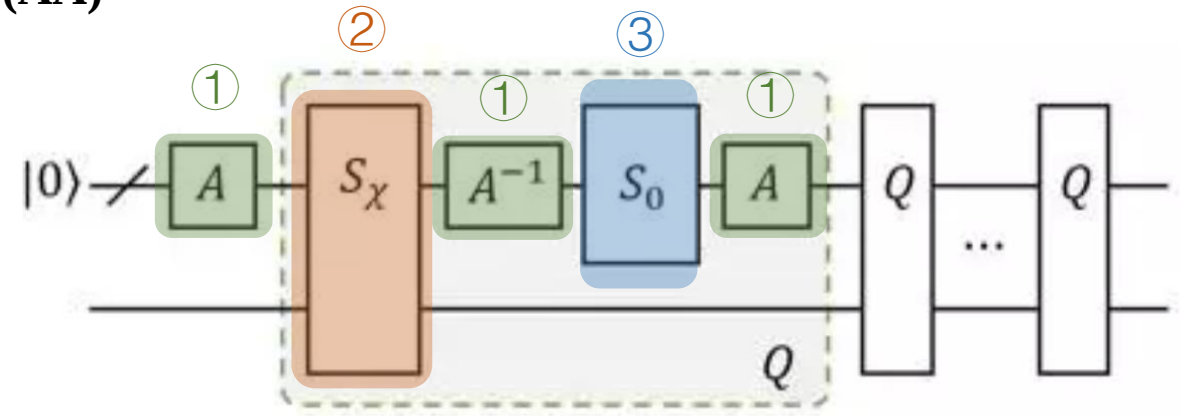
Amplitude Amplification (AA)



① The AA algorithm changes the Hadamard gate in the Grover's search algorithm into gate A, which can map $|0\rangle$ into a superposition with arbitrary weights.

$$A|0\rangle = \sum_{x \in X} a_x |x\rangle$$

② The Oracle O in Grover's search is a special case of $S_\chi(\varphi)$ in AA. when $\varphi = \pi$, it will becomes the oracle O.

$$S_\chi(\varphi): |x\rangle \rightarrow \begin{cases} e^{i\varphi}|x\rangle \; if \; \chi(x) = 1 \\ |x\rangle \quad if \; \chi(x) = 0 \end{cases}$$

③ The Grover diffusion in Grover's search is a special case of $S_0(\phi)$ in AA. when $\phi = \pi$, it will becomes the Grover diffusion.

$$S_o(\phi): |x\rangle \rightarrow \begin{cases} e^{i\phi}|x\rangle \; if \; x = 1 \\ |x\rangle \quad if \; x \neq 0 \end{cases}$$

Reference & picture:

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