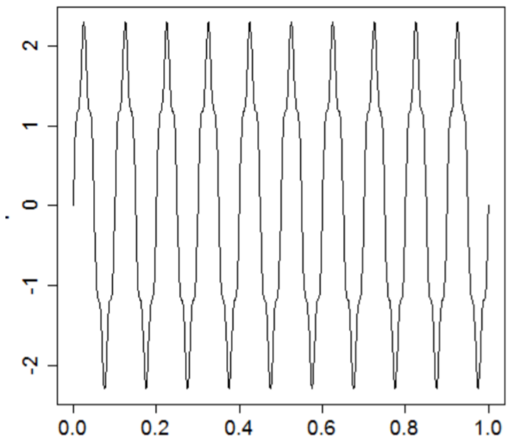
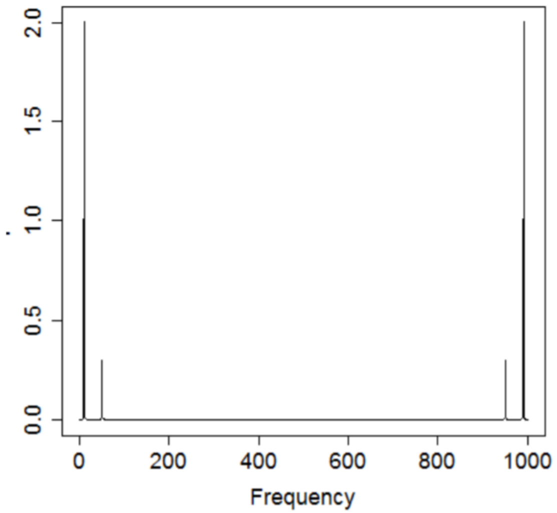
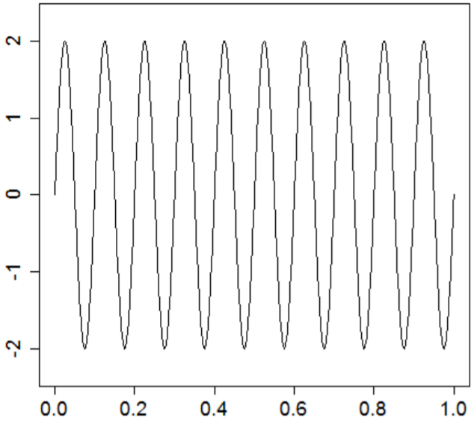
The Quantum Fourier Transform and Its Applications

Claire Yang 2019/02/14

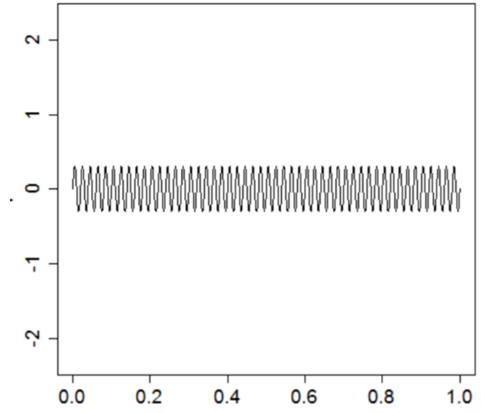
Discrete Fourier Transform







Signal: f=10, A=2



Noise: f=50, A=0.3

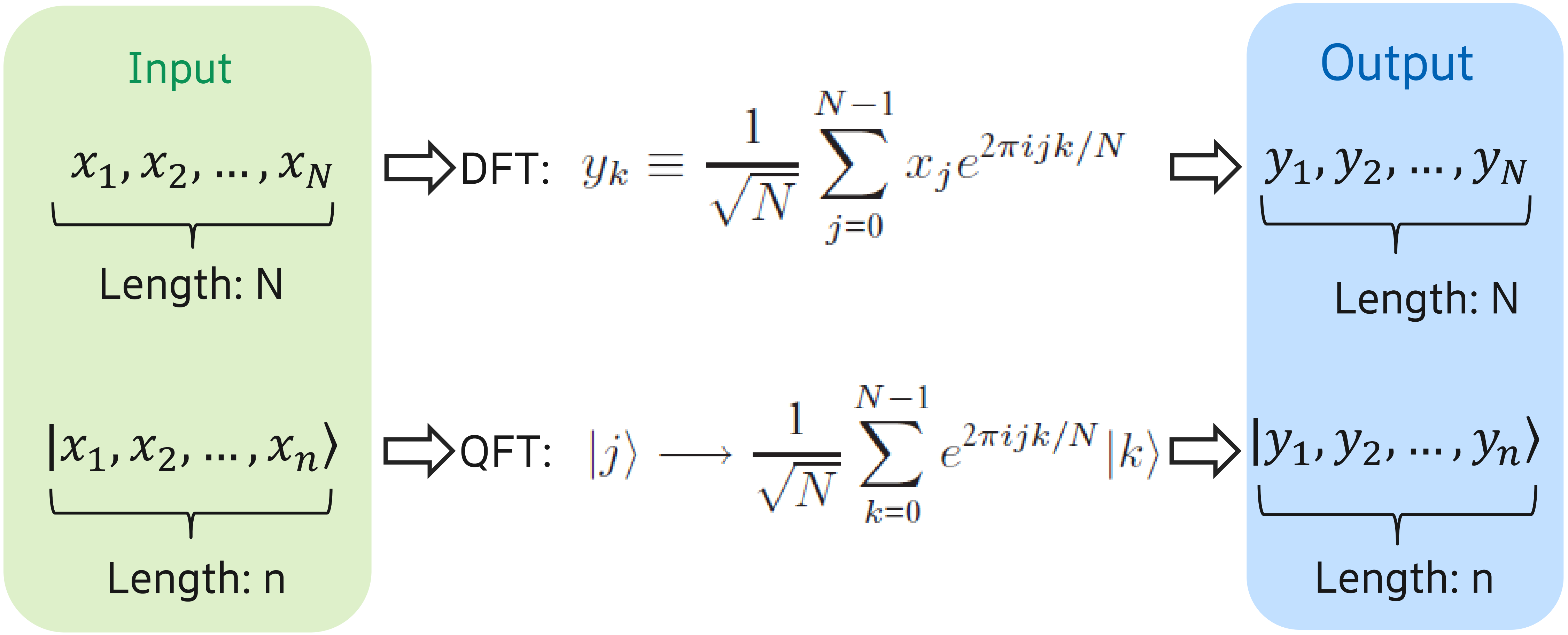
QFT is the quantum analogue of DFT:

— Map signal from **time** domian to **frequency** doain

— Basis for **quantum phase estimation**

Pictures: https://arxiv.org/pdf/1804.10068.pdf

Quantum Fourier Transform



Input

$$\underbrace{x_1, x_2, \dots, x_N}_{\text{Length: N}}$$

DFT: $y_k \equiv \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j e^{2\pi ijk/N}$

Output

$$\underbrace{y_1, y_2, \dots, y_N}_{\text{Length: N}}$$

$$\underbrace{|x_1, x_2, \dots, x_n\rangle}_{\text{Length: n}}$$

QFT: $|j\rangle \longrightarrow \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi ijk/N} |k\rangle$

$$\underbrace{|y_1, y_2, \dots, y_n\rangle}_{\text{Length: n}}$$

$$N = 2^n$$

Quantum Fourier Transform

QFT: $|j\rangle \longrightarrow \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i jk/N} |k\rangle$

$$\underbrace{|x_1, x_2, \dots, x_n\rangle}_{\text{Length: n}} \Rightarrow \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n & \omega_n^2 & \omega_n^3 & \cdots & \omega_n^{N-1} \\ 1 & \omega_n^2 & \omega_n^4 & \omega_n^6 & \cdots & \omega_n^{2(N-1)} \\ 1 & \omega_n^3 & \omega_n^6 & \omega_n^9 & \cdots & \omega_n^{3(N-1)} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega_n^{N-1} & \omega_n^{2(N-1)} & \omega_n^{3(N-1)} & \cdots & \omega_n^{(N-1)(N-1)} \end{bmatrix} \Rightarrow \underbrace{|y_1, y_2, \dots, y_n\rangle}_{\text{Length: n}}$$

N×N matrix

where $\omega_n := e^{\frac{2\pi i}{2^n}}$

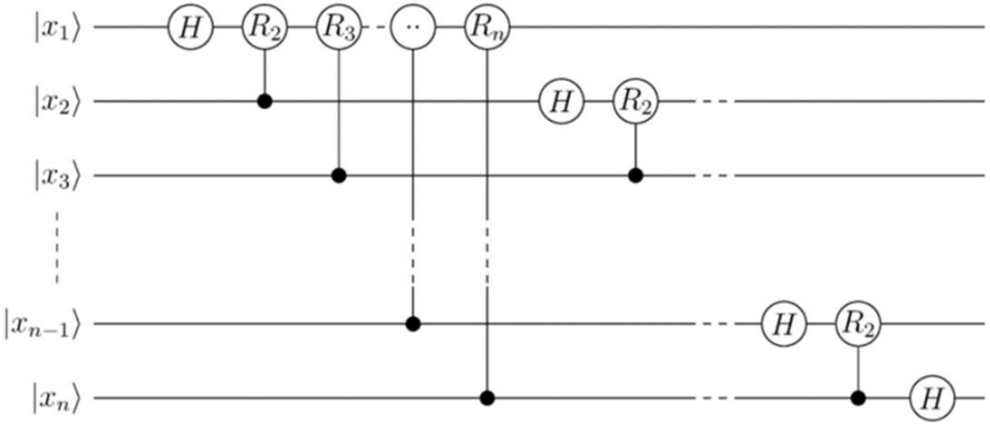
Example (2 qubits):

$$F_4 = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

Quantum Fourier Transform

Quantum circuit for QFT

$$\underbrace{|x_1, x_2, \dots, x_n\rangle}_{\text{Length: n}} \Rightarrow$$

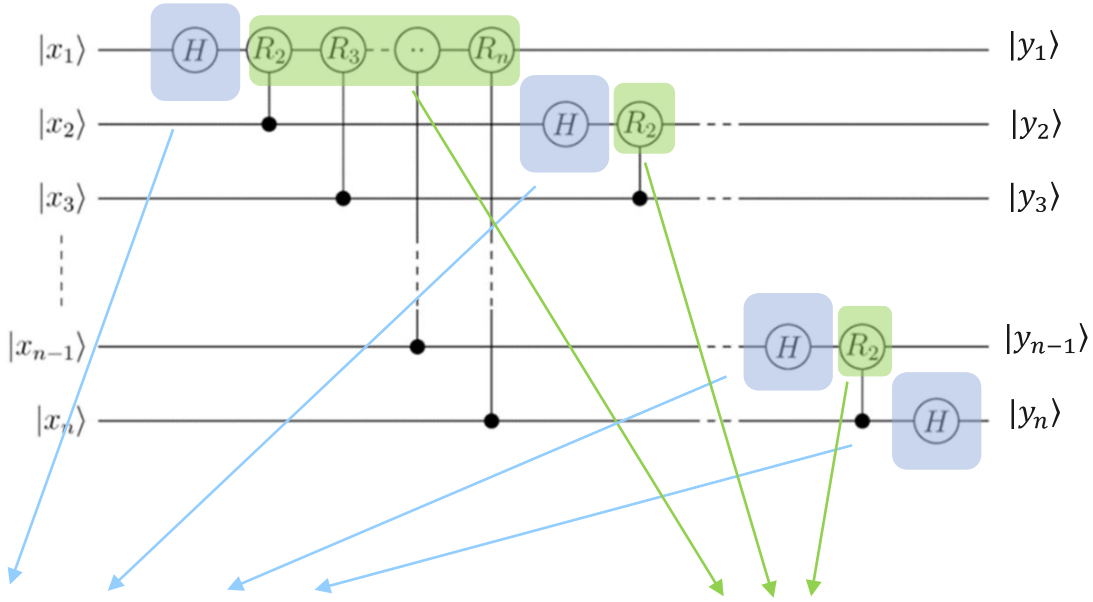


$$\Rightarrow \underbrace{|y_1, y_2, \dots, y_n\rangle}_{\text{Length: n}}$$

Picture: https://www.wikiwand.com/en/Quantum_Fourier_transform

Quantum Fourier Transform

Quantum circuit for QFT



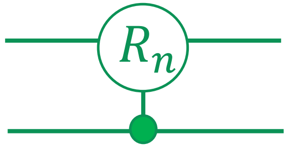
Hadamard gate: create superposition

$$|0\rangle \rightarrow \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

Controlled phase gate

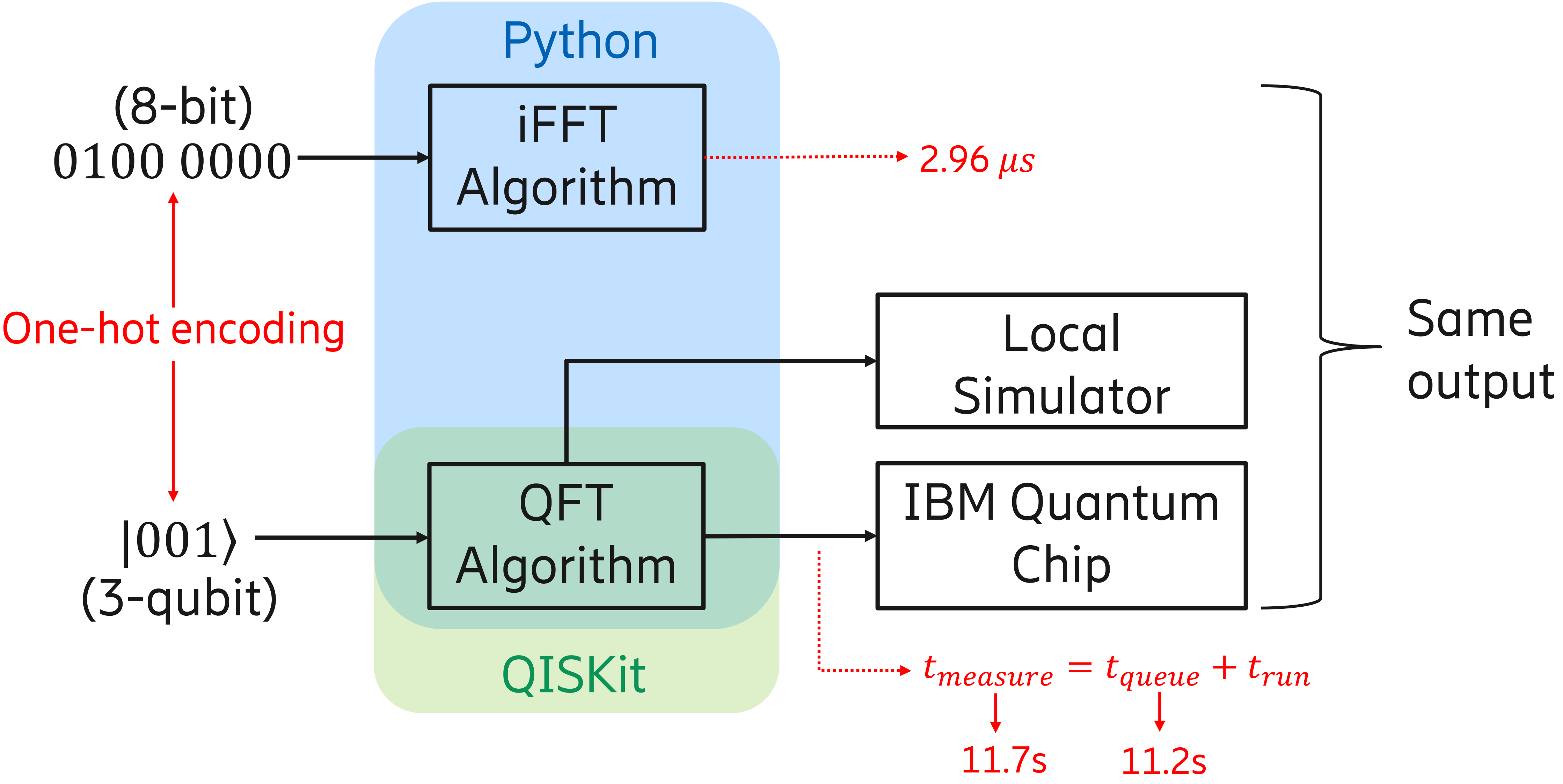
Target qubit

Control qubit



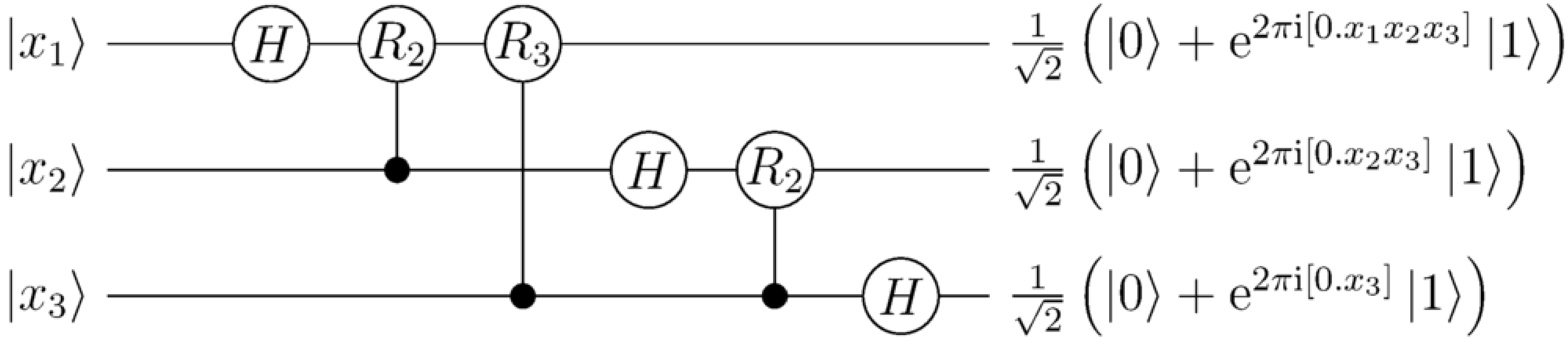
Picture: https://www.wikiwand.com/en/Quantum_Fourier_transform

Implementation and comparison with iFFT



Estimation of the QFT running time

- Add up the delays on the longest path of the compiled circuit
- Single qubit gates: 10 ns, two qubit gate: 100 ns



$|x_1\rangle$
$|x_2\rangle$
$|x_3\rangle$
H
R_2
R_3
H
R_2
H
$\frac{1}{\sqrt{2}}\left(|0\rangle + \mathrm{e}^{2\pi\mathrm{i}[0.x_1x_2x_3]}|1\rangle\right)$
$\frac{1}{\sqrt{2}}\left(|0\rangle + \mathrm{e}^{2\pi\mathrm{i}[0.x_2x_3]}|1\rangle\right)$
$\frac{1}{\sqrt{2}}\left(|0\rangle + \mathrm{e}^{2\pi\mathrm{i}[0.x_3]}|1\rangle\right)$

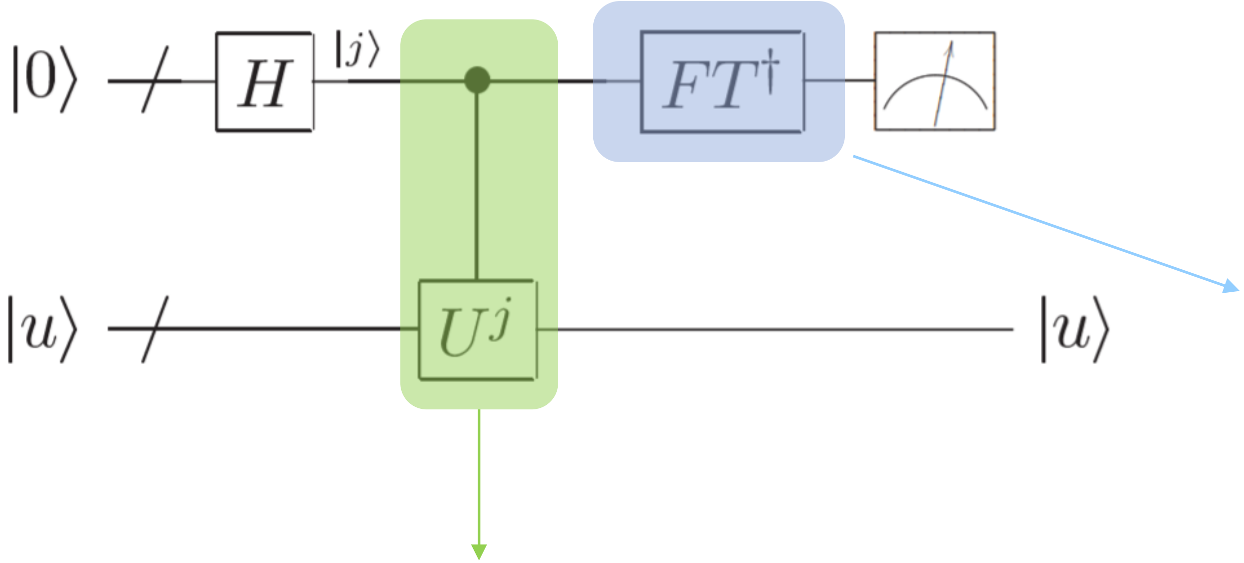
- For 3-qubit circuit, the estimated running time is:

  6 single qubit gates + 1 two qubits gate (SWAP gate) = 160ns

- Computational cost for n qubits (or for 2^n bits): QFT $\Theta(n^2)$, FFT $\Theta(n \cdot 2^n)$

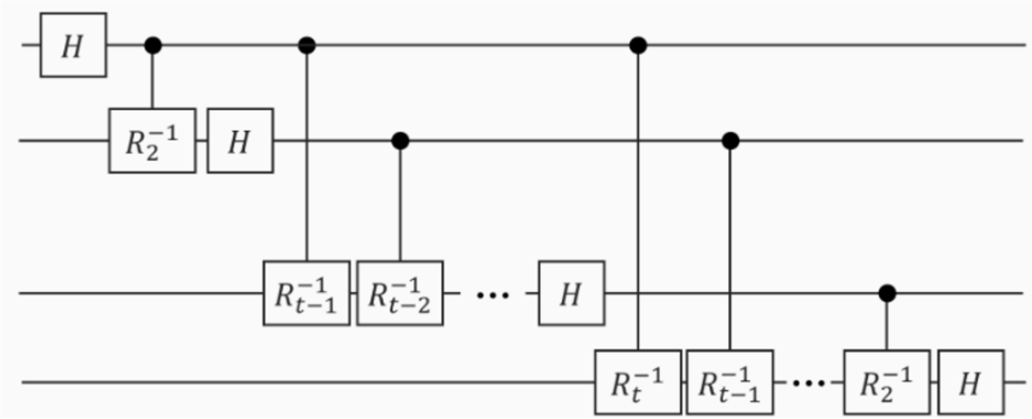
Application: Quantum Phase Estimation

Quantum phase estimation: to calculate the eigenvalue of U



Controlled-U gate

Inverse quantum Fourier transform:

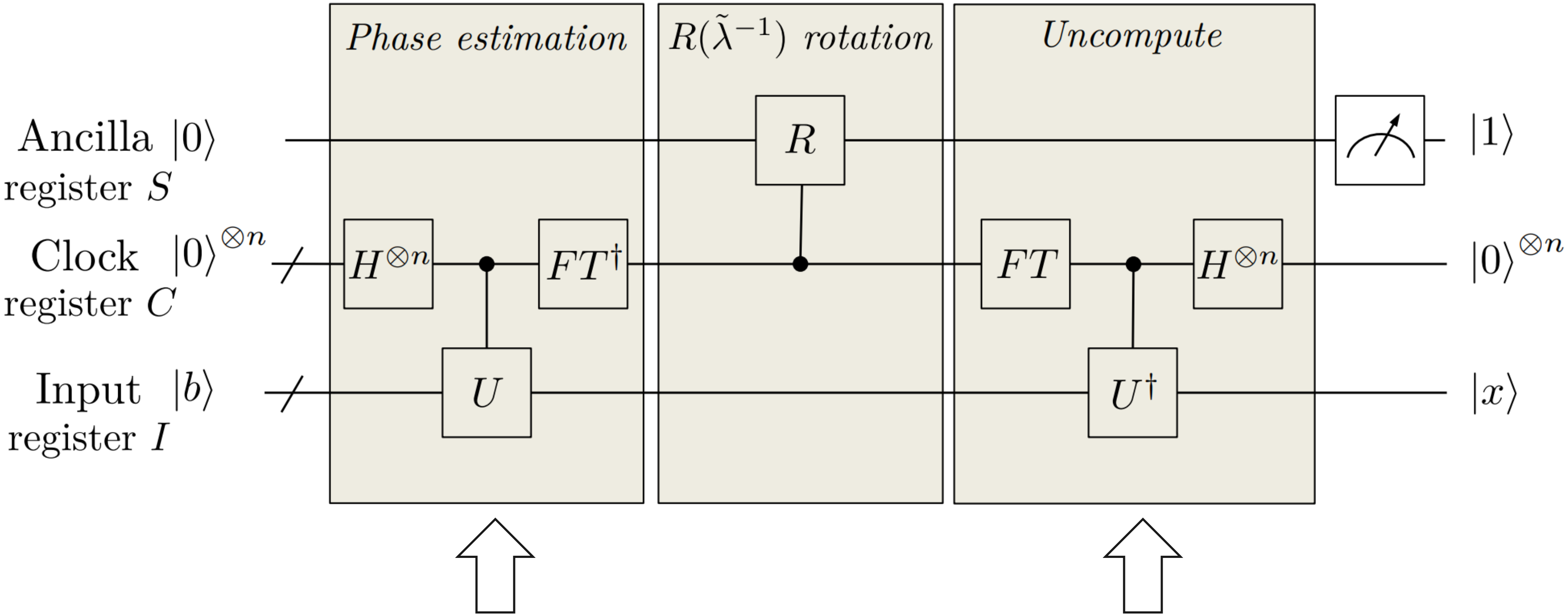


$$|j\rangle \longrightarrow \frac{1}{2^{\frac{n}{2}}}\sum_{k=0}^{2^n-1} e^{-2\pi i jk/2^n}|k\rangle$$

$$R_k{}^{-1} \equiv \begin{bmatrix} 1 & 0 \\ 0 & e^{-\frac{2\pi i}{k}} \end{bmatrix}$$

Picture: https://www.wikiwand.com/en/Quantum_Fourier_transform

Application: HHL Algorithm- Quantum algorithm for linear systems of equations



Quantum phase estimation

Reverse quantum phase estimation

Usage: quantum machine learning algorithms including bayesian inference, qSVM, qPCA...

Picture: https://arxiv.org/pdf/1802.08227.pdf

Materials

- Textbook: Quantum Computation and Quantum Information
- Online courses:
  - edX & TU Delft: Quantum Cryptography
  - edX & University of Toronto: Quantum Machine Learning