Quantum Support Vector Machine

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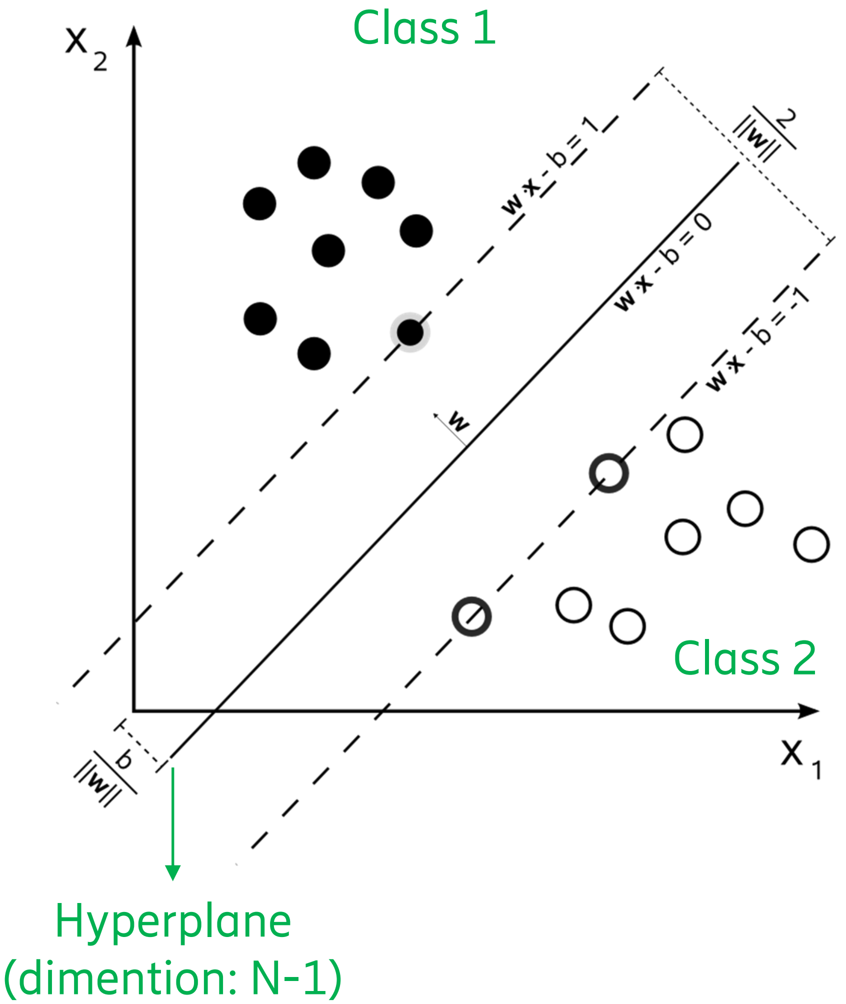
Classical SVM



M training data points: $\{(\vec{x_j}, y_j): \vec{x_j} \in \mathbb{R}^N, y_j = \pm 1\}, j = 1 \dots M$

Goal $\rightarrow$ find the maximum-margin hyperplane $\rightarrow$ maximize $\frac{2}{|\vec{\omega}|}$ $\rightarrow$ minimize $\frac{|\vec{\omega}|^2}{2}$

The constraint:

$$\begin{cases} \vec{w} \cdot \vec{x_j} + b \geq 1 & \text{if } y_j = +1 \ (y_j \ belongs \ to \ class \ 1) \\ \vec{w} \cdot \vec{x_j} + b \leq -1 & \text{if } y_j = -1 \ (y_j \ belongs \ to \ class \ 2) \end{cases} \Longleftrightarrow y_i(\vec{w} \cdot \vec{x_j} + b) \geq 1$$

Computational Complexity: $O(log(\epsilon^{-1})poly(N, M))$

N: Dimension of feature space (input data)

M: Number of training data points

Image: https://zh.wikipedia.org/wiki/File:Svm_max_sep_hyperplane_with_margin.png

SVM structure

Goal → find the maximum-margin hyperplane → maximize $\frac{2}{|\vec{\omega}|}$ s.t. $y_j(\vec{w}\cdot\vec{x_j}+b)\geq 1$ → minimize $\frac{|\vec{\omega}|^2}{2}$ s.t. $y_j(\vec{w}\cdot\vec{x_j}+b)\geq 1$ → Fulfils KKT condition

↓ introduce the Lagrange multipliers $\vec{\alpha}$

$$\max_{\vec{\alpha}}\min_{\vec{\omega},b} L(\vec{\omega},b,\vec{\alpha})=\frac{|\vec{\omega}|^2}{2}-\sum_{j=1}^{M}\alpha_j y_j(y_j(\vec{\omega}\cdot\vec{x_j}+b)-1),\text{s.t. }\sum_{j=1}^{M}\alpha_j=0\text{ and }\alpha_j y_j\geq 0$$

Partial derivative
$$\begin{cases}\frac{\partial L}{\partial\vec{\omega}}=0\Rightarrow\vec{\omega}=\sum_{j=1}^{M}\alpha_j y_j\vec{x_j}\\ \frac{\partial L}{\partial b}=0\Rightarrow\sum_{j=1}^{M}\alpha_j y_j^2=0\end{cases}$$

↓

Kernel, Inner product ←Problem 1

$$\max_{\vec{\alpha}} L(\vec{\alpha})=\sum_{j=1}^{M}\alpha_j y_j-\frac{1}{2}\sum_{j,k=1}^{M}\alpha_j\vec{x_j}\cdot\vec{x_k}\alpha_k$$

↓

Find optimal $\vec{\alpha}^*$ ←Problem 2

↓

Optimal hyperplane with parameters:
$$\begin{cases}\vec{\omega}=\sum_{j,k=1}^{M}\alpha_j^*\vec{x_j}\\ b=y_i-\vec{\omega}\end{cases}$$

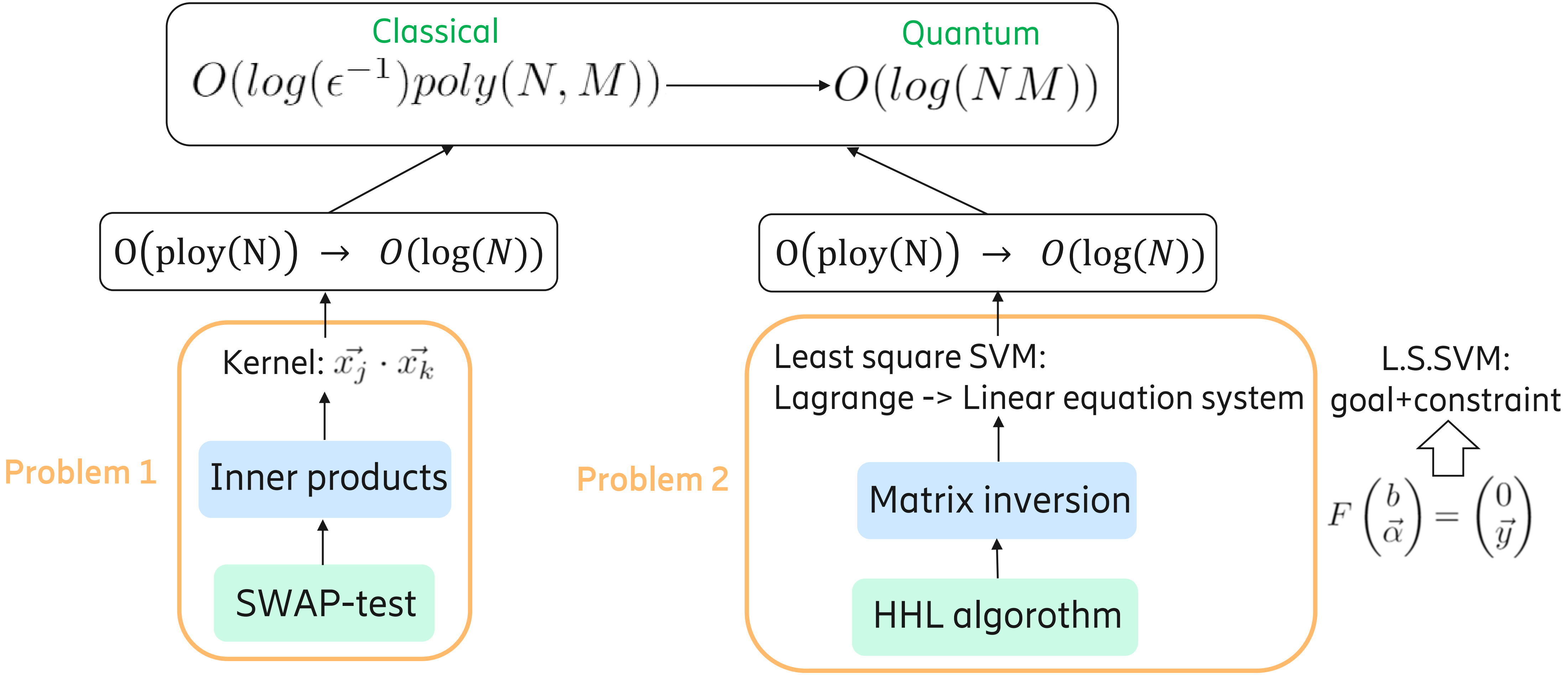
Quantum SVM

Classical → Quantum

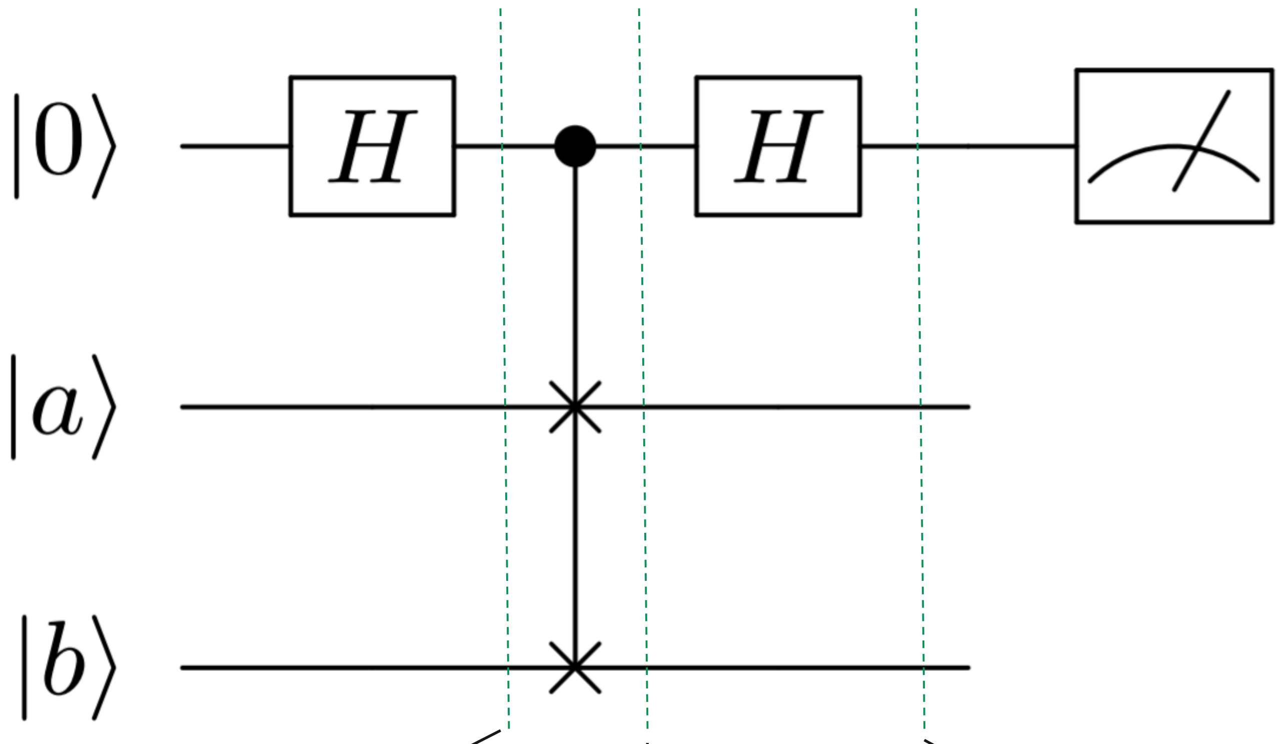
Computational Complexity: $O(log(\epsilon^{-1})poly(N,M)) \longrightarrow O(log(NM))$

training, testing

HHL based qSVM



Problem 1 – Inner product & SWAP-test



$$P(|0\rangle) = |\langle 0|\psi_3\rangle|^2$$
$$= |\frac{1}{2}\langle 0|0\rangle\,(|a,b\rangle + |b,a\rangle) + \frac{1}{2}\langle 0|1\rangle\,(|a,b\rangle - |b,a\rangle)|^2$$
$$= \frac{1}{4}|(|a,b\rangle + |b,a\rangle)|^2$$
$$= \frac{1}{4}(\langle b|b\rangle\langle a|a\rangle + \langle b|a\rangle\langle a|b\rangle + \langle a|b\rangle\langle b|a\rangle + \langle a|a\rangle\langle b|b\rangle)$$
$$= \frac{1}{2} + \frac{1}{2}|\langle a|b\rangle|^2$$

Inner product

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0,a,b\rangle + |1,a,b\rangle)$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|0,a,b\rangle + |1,b,a\rangle)$$

$$|\psi_3\rangle = \frac{1}{2}|0\rangle\,(|a,b\rangle + |b,a\rangle) + \frac{1}{2}|1\rangle\,(|a,b\rangle - |b,a\rangle)$$

Image: https://arxiv.org/pdf/1804.10068.pdf

Problem 2 – Least Square SVM

Slack variable

Constraint: $y_j(\vec{\omega}\cdot\vec{x_j}+b)\geqslant 1 \xrightarrow{y_j^2=1} \vec{\omega}\cdot\vec{x_j}+b=y_j-y_j e_j$

New Lagrange function: $L(\vec{\omega},b,\vec{e},\vec{\alpha})=\frac{|\vec{\omega}|^2}{2}+\frac{\gamma}{2}\sum_{j=1}^{M}e_j^2-\sum_{j=1}^{M}\alpha_j y_j(\vec{\omega}\cdot\vec{x_j}+b-y_j+y_j e_j)$

Penalty term

⇩

Linear equation system: $F\begin{pmatrix} b \\ \vec{\alpha} \end{pmatrix}\equiv\begin{pmatrix} 0 & -\vec{1}^T \\ \vec{1} & K+\gamma^{-1}I \end{pmatrix}\begin{pmatrix} b \\ \vec{\alpha} \end{pmatrix}=\begin{pmatrix} 0 \\ \vec{y} \end{pmatrix}$

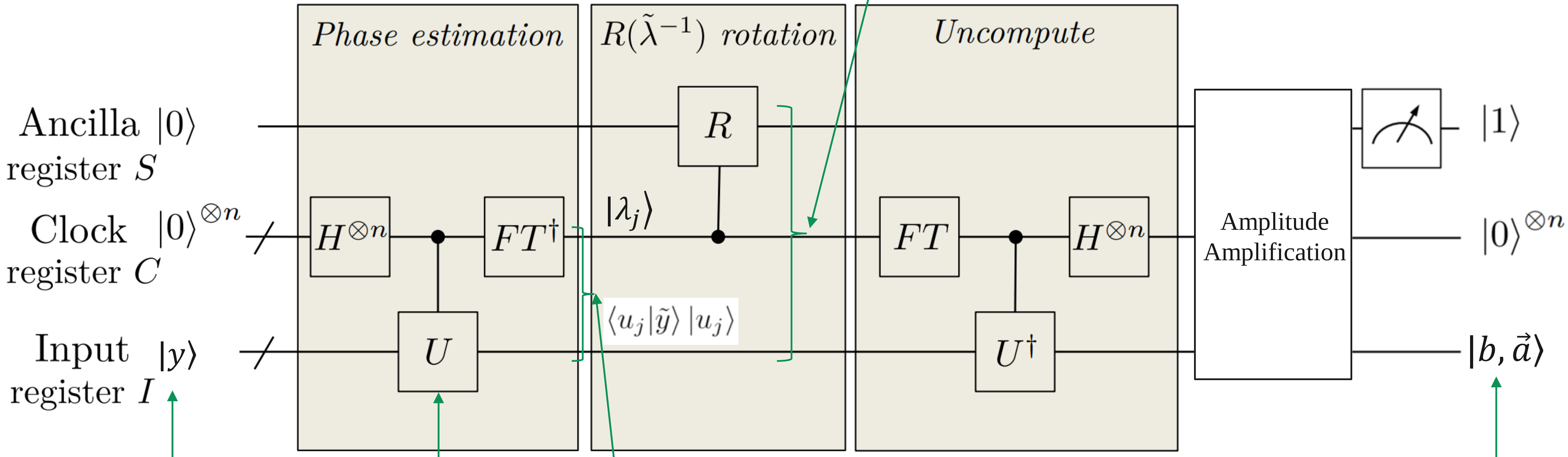
⇩

HHL

Training process

Solve $F\begin{pmatrix} b \\ \vec{\alpha} \end{pmatrix} = \begin{pmatrix} 0 \\ \vec{y} \end{pmatrix}$

$\sum_{j=1}^{M+1}\left(\sqrt{1-\frac{C^2}{\lambda_j^2}}|0\rangle + \frac{C}{\lambda_j}|1\rangle\right)\langle u_j|\tilde{y}\rangle\, |u_j\rangle|\lambda_j\rangle$



$|\tilde{y}\rangle = \sum_{j=1}^{M+1}\langle u_j|\tilde{y}\rangle\, |u_j\rangle$

(1 qubit longer than $|\vec{y}\rangle$)

$F \to \mathrm{U} = e^{-i\hat{F}t}$

$\sum_{j=1}^{M+1}\langle u_j|\tilde{y}\rangle\, |u_j\rangle|\lambda_j\rangle$

$\sum_{j=1}^{M+1}\frac{\langle u_j|\tilde{y}\rangle}{\lambda_j}|u_j\rangle$

Difficult point: how to enact this exponentiation?

Image: https://arxiv.org/pdf/1802.08227.pdf

Difficult point in training process - Enact $e^{-i\hat{F}t}$

$$e^{-i\hat{F}\Delta t}$$

$$\downarrow$$

$$\hat{K} = K/trK$$

$$\downarrow$$

$$e^{-i\hat{K}\Delta t}$$

$$F = \begin{pmatrix} 0 & \vec{1}^T \\ \vec{1} & K + \gamma^{-1} I \end{pmatrix}, \quad \hat{F} = (J + K + \gamma^{-1} I)/trF, \quad J = \begin{pmatrix} 0 & \vec{1}^T \\ \vec{1} & 0 \end{pmatrix}$$

Eliminate γ^{-1}

$$e^{-i\hat{F}\Delta t} = e^{-i\Delta t I/trF} e^{-i\Delta t J/trF} e^{-i\Delta t K/trF} + O(\Delta t^2)$$

$$= e^{-i\Delta t' I/trK} e^{-i\Delta t' J/trK} e^{-i\Delta t' K/trK} + O(\Delta t^2)$$

Rescale time by a factor $\frac{trK}{trF}$

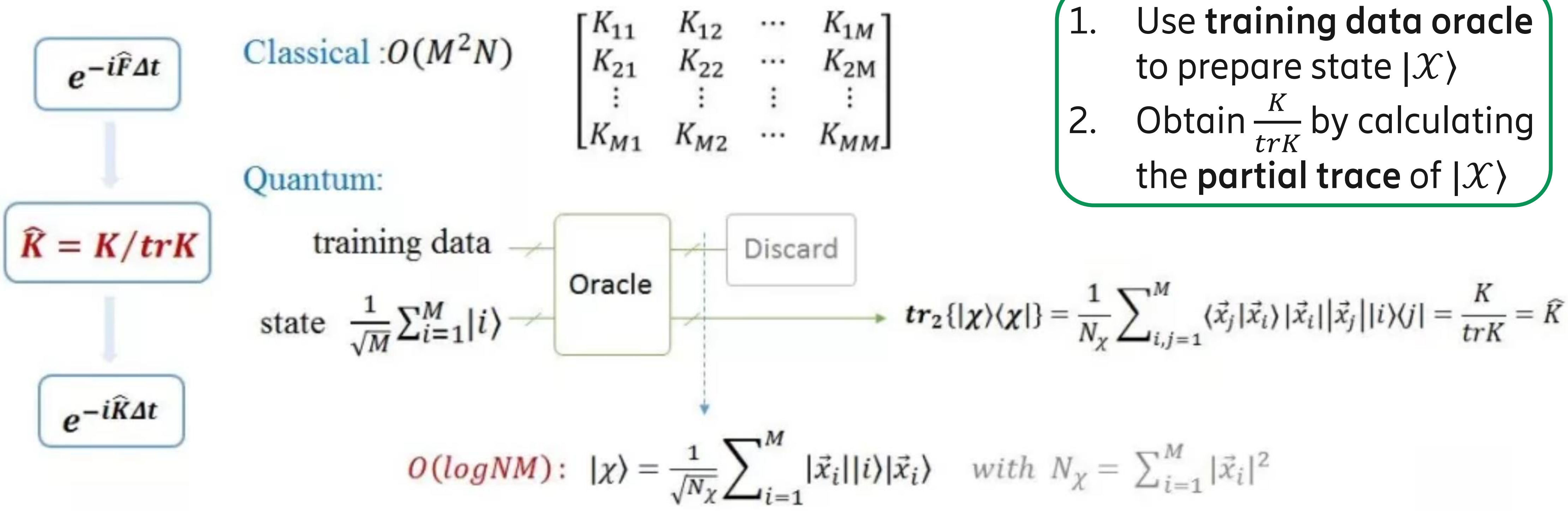
$\gamma^{-1} I$: *is trivial*

J: *eigenvalues*: $\lambda_{\pm}^{star} = \pm\sqrt{M}$, *eigenstates*: $|\lambda_{\pm}^{star}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \pm \frac{1}{\sqrt{M}} \sum_{k=1}^{M} |k\rangle\right)$

K/trK : density operator

Image:https://mp.weixin.qq.com/s?__biz=MzUxMDQzNzEzNA==&mid=2247483963&idx=1&sn=7ca3a577f3f02ed8e8b3aeff407cb5bb&chksm=f903b05ace74394c33c2cdae2d7989cf841c4fbab6095cbb224bfd3892de1f9c08c8a38d1a2b&scene=21#wechat_redirect

Difficult point in training process – Enact $\frac{K}{trK}$



1. Use **training data oracle** to prepare state $|\mathcal{X}\rangle$
2. Obtain $\frac{K}{trK}$ by calculating the **partial trace** of $|\mathcal{X}\rangle$

Image:https://mp.weixin.qq.com/s?__biz=MzUxMDQzNzEzNA==&mid=2247483963&idx=1&sn=7ca3a577f3f02ed8e8b3aeff407cb5bb&chksm=f903b05ace74394c33c2cdae2d7989cf841c4fbab6095cbb224bfd3892de1f9c08c8a38d1a2b&scene=21#wechat_redirect

Difficult point in training process - Enact $e^{-i\hat{K}t}$

$e^{-i\hat{F}\Delta t}$

$\hat{K} = K/trK$

$e^{-i\hat{K}\Delta t}$

Given by the qPCA articale

To simulate non-sparse symmetric or Hermitian matrics:

Density matrix exponentiation: n copies of $\hat{K}$

For some quantum state ρ, $e^{-i\hat{K}\Delta t}\rho e^{i\hat{K}\Delta t} \equiv e^{-iL_{\hat{K}}\Delta t}(\rho)$, $L_{\hat{K}} = [\hat{K},\rho]$ $\;(L_{\hat{K}} = [\hat{K},\cdot])$

$$e^{-iL_{\hat{K}}\Delta t}(\rho) \approx tr_1\{e^{-iS\Delta t}\hat{K} \otimes \rho e^{iS\Delta t}\} = \rho - i\Delta t[\hat{K},\rho] + O(\Delta t^2)$$

$$S = \sum_{m,n=1}^{M} |m\rangle\langle n| \otimes |n\rangle\langle m|$$

M^2 by M^2 matrix, the SWAP matrix

Image:https://mp.weixin.qq.com/s?__biz=MzUxMDQzNzEzNA==&mid=2247483963&idx=1&sn=7ca3a577f3f02ed8e8b3aeff407cb5bb&chksm=f903b05ace74394c33c2cdae2d7989cf841c4fbab6095cbb224bfd3892de1f9c08c8a38d1a2b&scene=21#wechat_redirect

Classification process

Input: $|b,\vec{\alpha}\rangle = \frac{1}{\sqrt{C}}\left(b|0\rangle + \sum_{k=1}^{M} \alpha_k |k\rangle\right)$, $|\vec{x}\rangle$

Output: $y \in \{-1, +1\}$

Algorithm:

1. By calling the training data oracle, construct $|\tilde{u}\rangle$ and the query state $|\tilde{x}\rangle$:

$$|\tilde{u}\rangle = \frac{1}{\sqrt{N_{\tilde{u}}}}\left(b|0\rangle|0\rangle + \sum_{k=1}^{M} \alpha_k |\vec{x}_k||k\rangle|\vec{x}_k\rangle\right),\ N_{\tilde{u}} = b^2 + \sum_{k=1}^{M} \alpha_k^2 |\vec{x}_k|^2$$

$$|\tilde{x}\rangle = \frac{1}{\sqrt{N_{\tilde{x}}}}\left(|0\rangle|0\rangle + \sum_{k=1}^{M} |\vec{x}||k\rangle|\vec{x}\rangle\right),\ N_{\tilde{x}} = M|\vec{x}|^2 + 1$$

2. Perform a swap test.

Using an ancilla, construct : $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle|\tilde{u}\rangle + |1\rangle|\tilde{x}\rangle)$, measure the ancilla in $|\phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

Success $P = |\langle\psi|\phi\rangle|^2 = \frac{1}{2}(1 - \langle\tilde{u}|\tilde{x}\rangle)$, $\quad \langle\tilde{u}|\tilde{x}\rangle = 1/\sqrt{N_{\tilde{x}} N_{\tilde{u}}}\left(b + \sum_{k=1}^{M} \alpha_k |\vec{x}_k||\vec{x}|\langle\vec{x}_k|\vec{x}\rangle\right)$

3. If $P < \frac{1}{2}$, we classify $|\vec{x}\rangle$ as +1; otherwise, -1.

Image:https://mp.weixin.qq.com/s?__biz=MzUxMDQzNzEzNA==&mid=2247483963&idx=1&sn=7ca3a577f3f02ed8e8b3aeff407cb5bb&chksm=f903b05ace74394c33c2cdae2d7989cf841c4fbab6095cbb224bfd3892de1f9c08c8a38d1a2b&scene=21#wechat_redirect

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