The Quantum Fourier Transform and Its Applications

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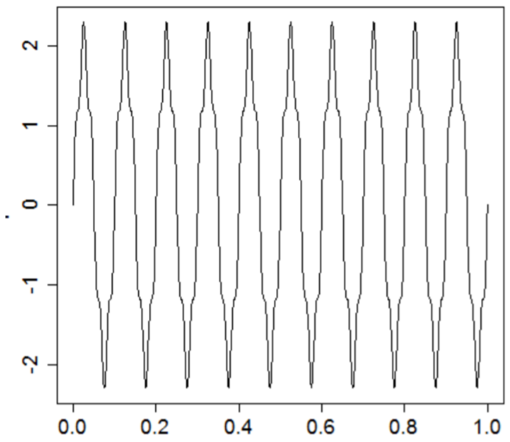
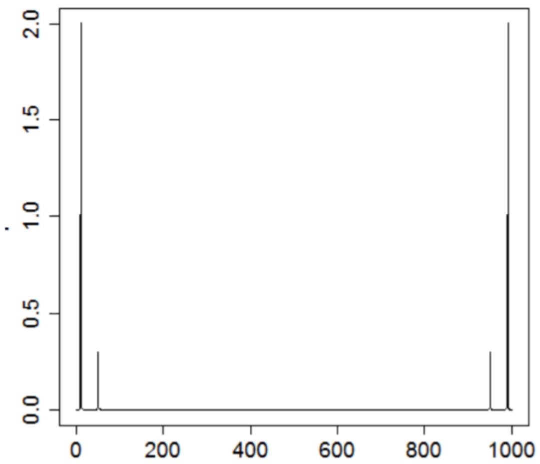
2019-03-19

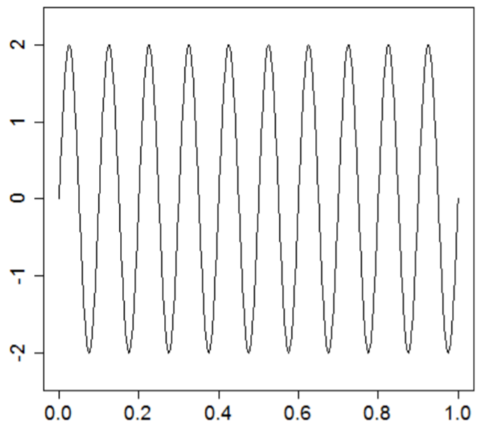
Discrete Fourier Transform



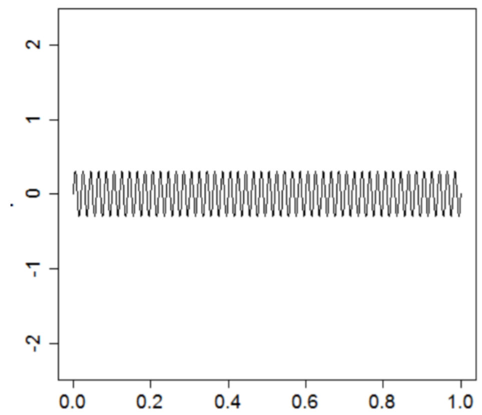
Input signal



Input signal in frequency domain



True signal: f=10, A=2



Noise signal: f=50, A=0.3

QFT is the quantum analogue of DFT:

— Map signal from **time** domain to **frequency** domain on quantum computer with a faster speed

— Basis for **quantum phase estimation**

Pictures: https://arxiv.org/pdf/1804.10068.pdf

Quantum Fourier Transform

Input

$$\underbrace{x_1, x_2, \ldots, x_N}_{\text{Length: N}} \Rightarrow \text{DFT: } y_k \equiv \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j e^{2\pi i jk/N} \Rightarrow \underbrace{y_1, y_2, \ldots, y_N}_{\text{Length: N}}$$

$$\underbrace{|x_1, x_2, \ldots, x_n\rangle}_{\text{Length: n}} \Rightarrow \text{QFT: } |j\rangle \longrightarrow \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i jk/N} |k\rangle \Rightarrow \underbrace{|y_1, y_2, \ldots, y_n\rangle}_{\text{Length: n}}$$

Output

$$N = 2^n$$

Quantum Fourier Transform

QFT: $|j\rangle \longrightarrow \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi ijk/N} |k\rangle$

$$\underbrace{|x_1, x_2, \ldots, x_n\rangle}_{\text{Length: n}} \Rightarrow \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n & \omega_n^2 & \omega_n^3 & \cdots & \omega_n^{N-1} \\ 1 & \omega_n^2 & \omega_n^4 & \omega_n^6 & \cdots & \omega_n^{2(N-1)} \\ 1 & \omega_n^3 & \omega_n^6 & \omega_n^9 & \cdots & \omega_n^{3(N-1)} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega_n^{N-1} & \omega_n^{2(N-1)} & \omega_n^{3(N-1)} & \cdots & \omega_n^{(N-1)(N-1)} \end{bmatrix} \Rightarrow \underbrace{|y_1, y_2, \ldots, y_n\rangle}_{\text{Length: n}}$$

N×N matrix

where $\omega_n := e^{\frac{2\pi i}{2^n}}$

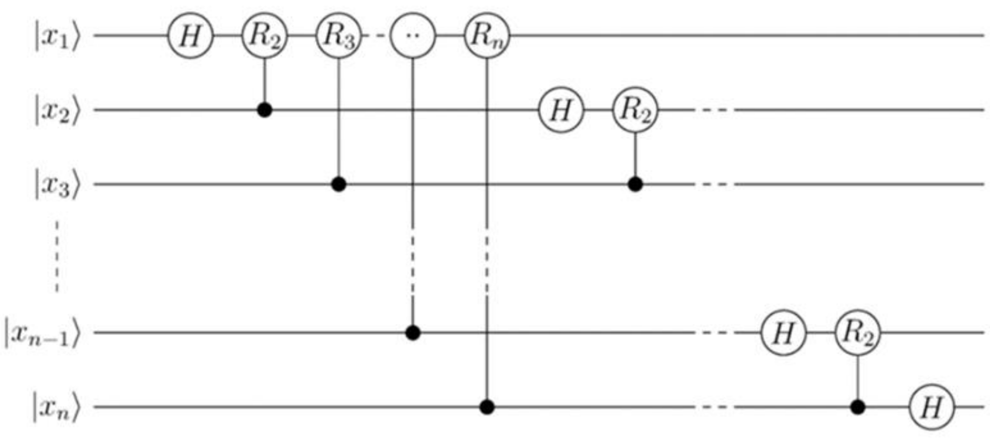
Example (2 qubits):

$$F_4 = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

Quantum Fourier Transform

QFT: $|j\rangle \longrightarrow \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi ijk/N} |k\rangle$

$\underbrace{|x_1, x_2, \dots, x_n\rangle}_{\text{Length: n}} \Rightarrow$



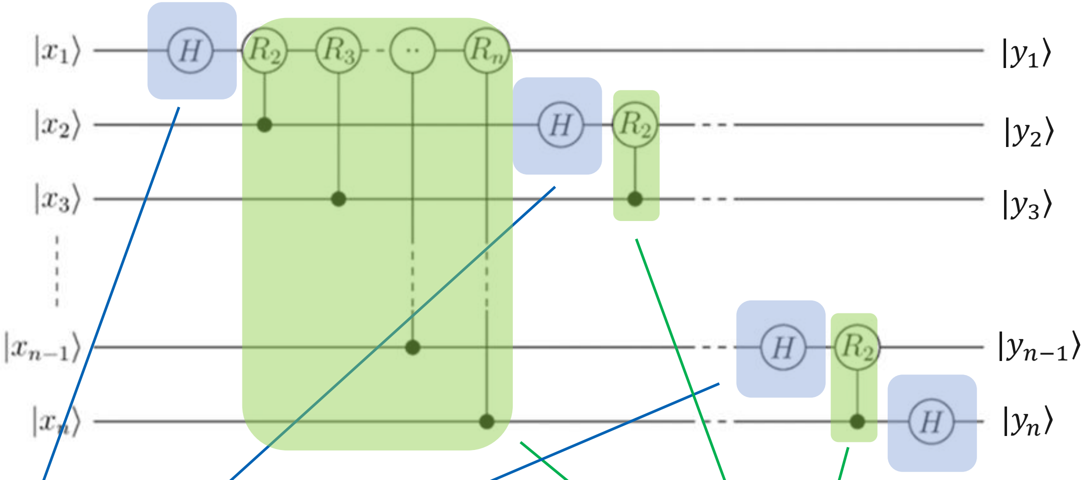
$\Rightarrow \underbrace{|y_1, y_2, \dots, y_n\rangle}_{\text{Length: n}}$

Quantum circuit for QFT

Picture: https://www.wikiwand.com/en/Quantum_Fourier_transform

Quantum Fourier Transform

Quantum circuit for QFT



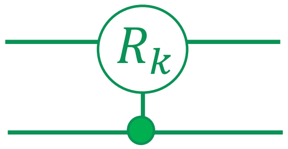
Hadamard gate: create superposition

$$|0\rangle \rightarrow \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$|1\rangle \rightarrow \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Controlled phase gate

Target qubit

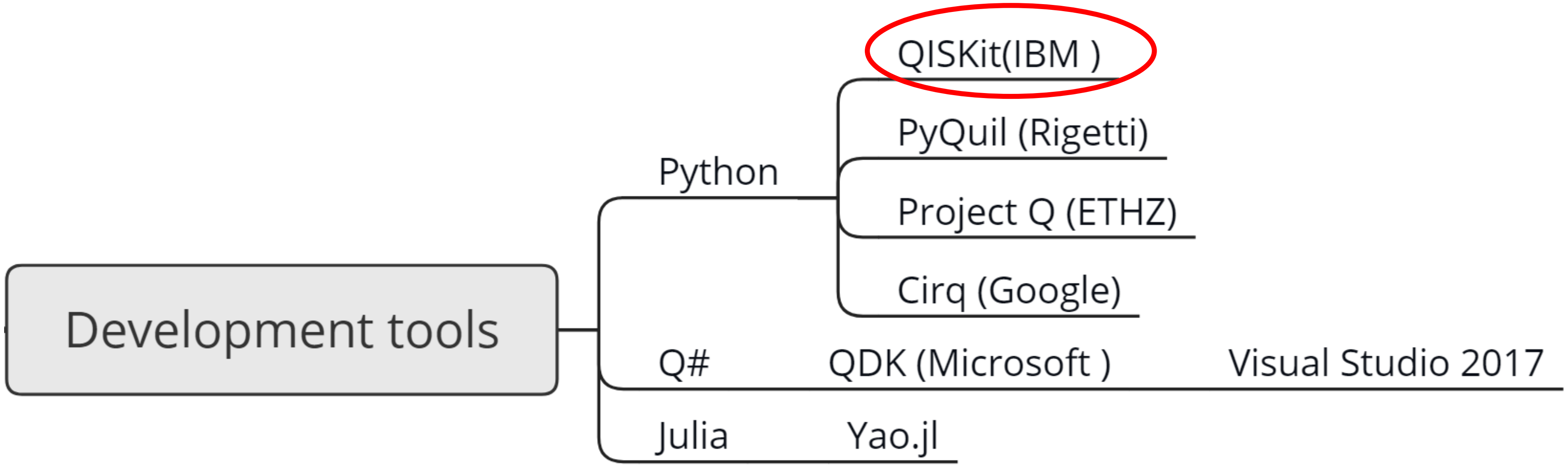


Control qubit

$$R_k \equiv \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi i}{k}} \end{bmatrix}$$

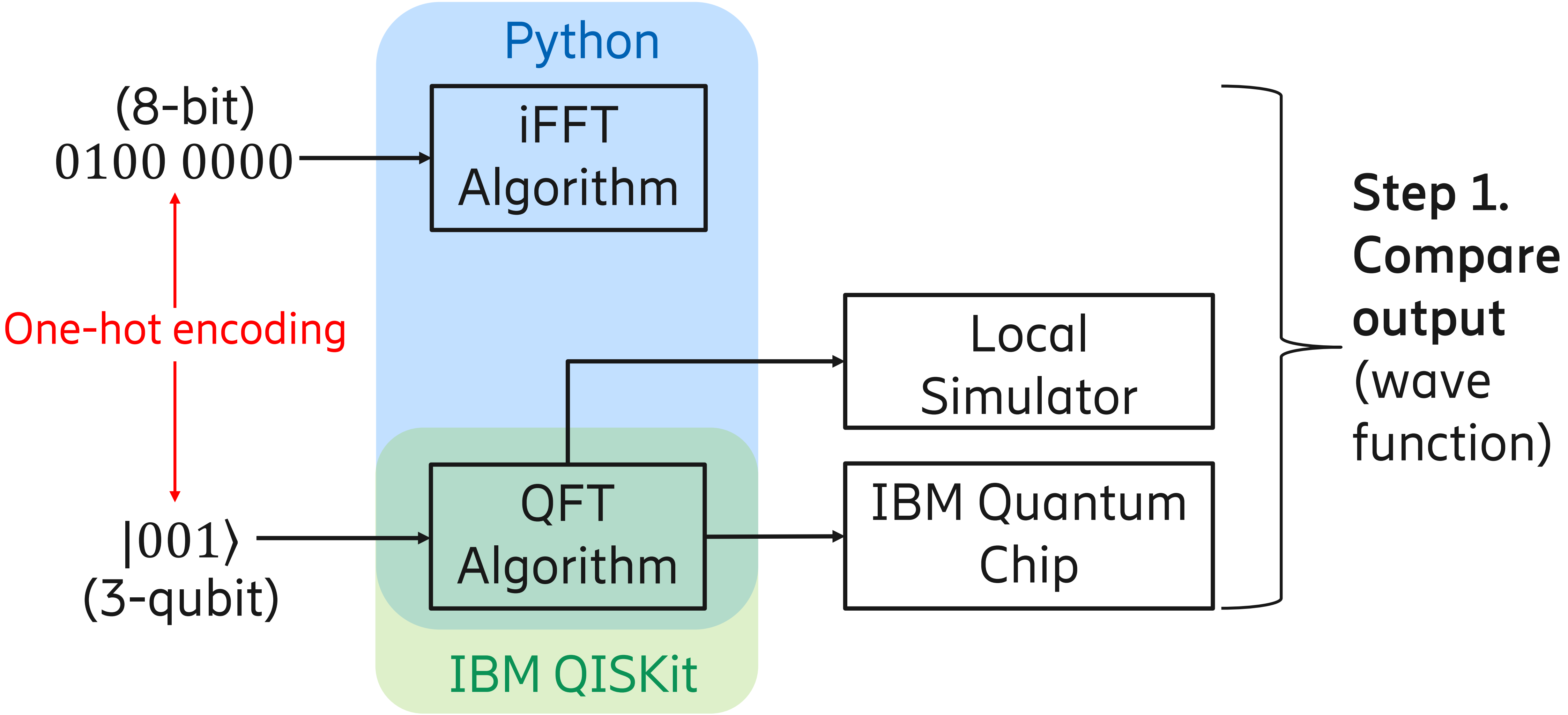
Picture: https://www.wikiwand.com/en/Quantum_Fourier_transform

Development tools



Implementation and comparison with iFFT

000	$\leftrightarrow$	0
001	$\leftrightarrow$	1
010	$\leftrightarrow$	2
011	$\leftrightarrow$	3
100	$\leftrightarrow$	4
101	$\leftrightarrow$	5
110	$\leftrightarrow$	6
111	$\leftrightarrow$	7



mismatching: QFT = iFFT, iQFT = FFT

Implementation and comparison with iFFT

000	↔	0
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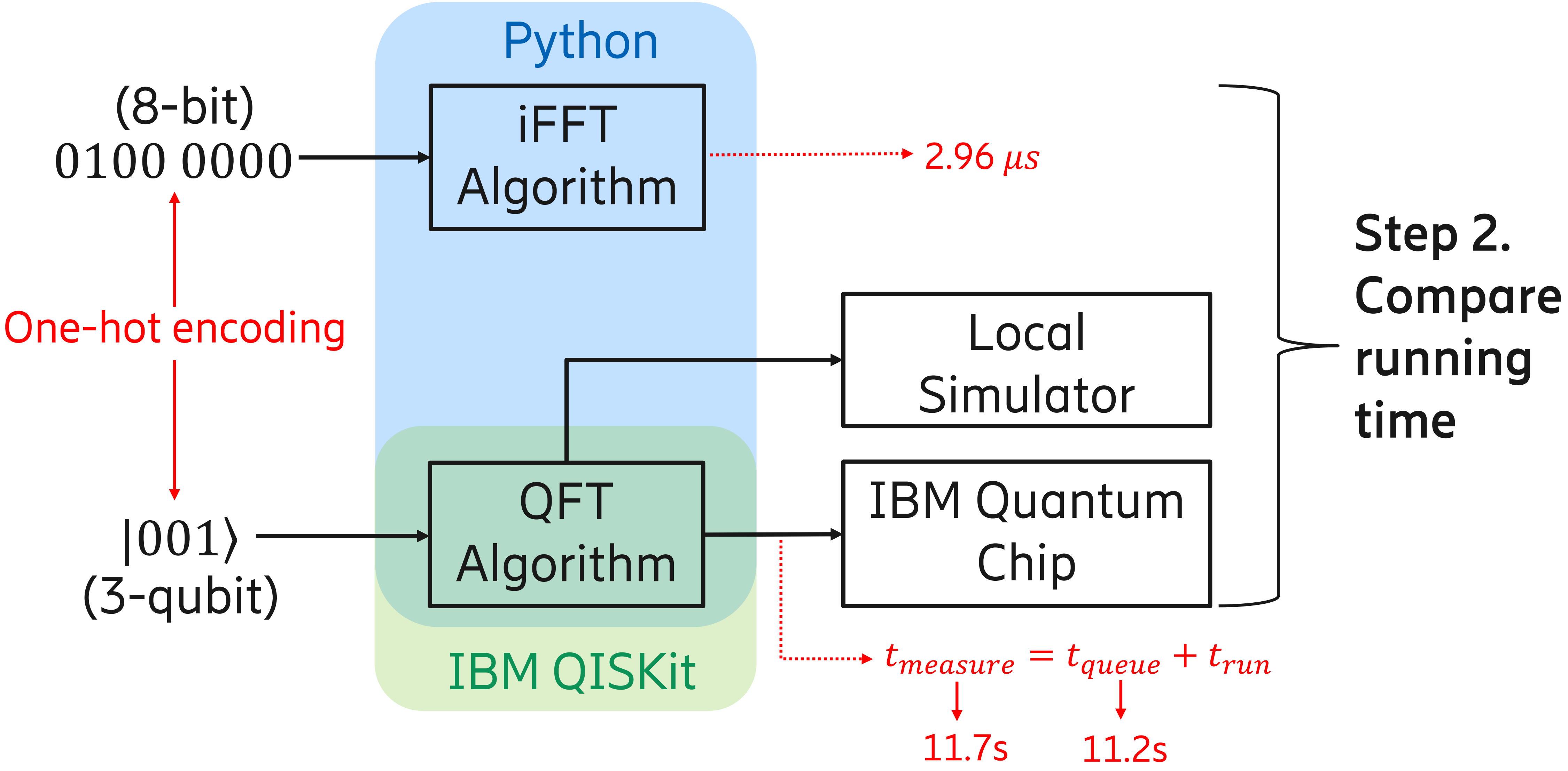
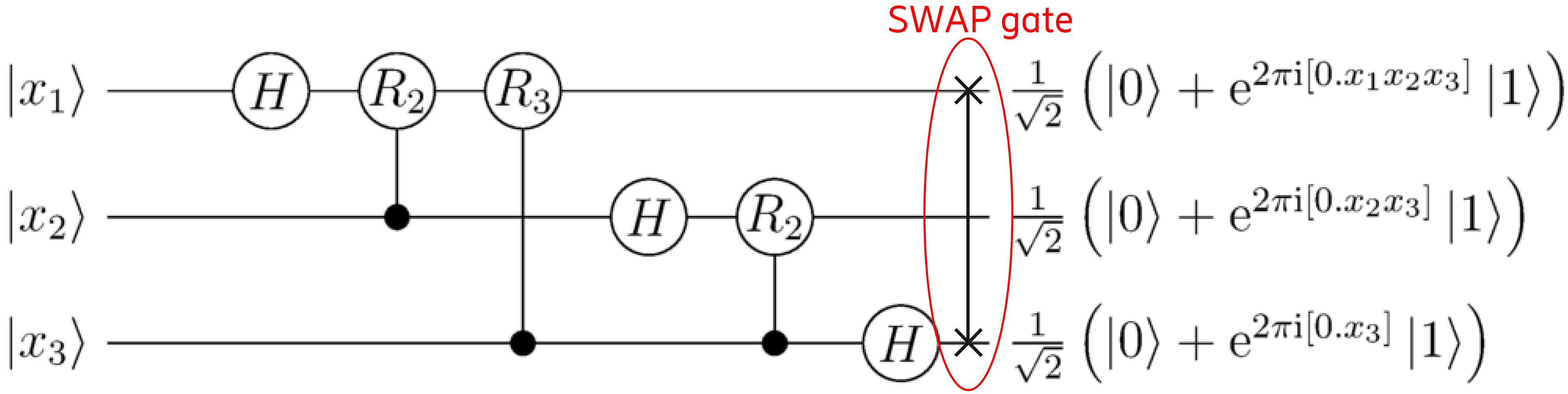


Estimation of the QFT running time

- Add up the delays on the longest path of the compiled circuit
- Single qubit gate: 80 ns, two qubits gate: 170<t<348 ns



- e.g. For 3-qubit circuit, the estimated running time is among: (920, 1632)ns

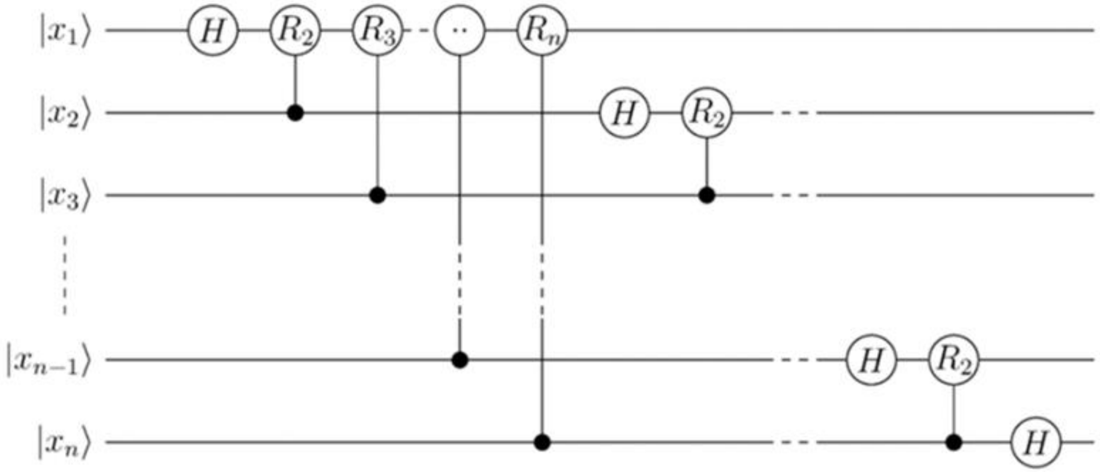
  3 single qubit gates + 4 two qubits gate (SWAP gate)

- Computational cost for n qubits (or for 2^n bits): QFT $\Theta(n^2)$, FFT $\Theta(n \cdot 2^n)$

Picture: https://www.wikiwand.com/en/Quantum_Fourier_transform

Practical application

— The algorithm itself is easy to be generalized from 3-qubit system to n-qubit system.



— But it is hard to create a complex enough input qubit string, e.g. Sine wave.

Example: $01000000 \rightarrow |001\rangle$

$$12345678 \rightarrow \frac{1}{\sqrt{M}}(|000\rangle + 2|001\rangle + 3|010\rangle + 4|011\rangle + 5|100\rangle + 6|101\rangle + 7|110\rangle + 8|111\rangle)$$

Where $M = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2$ is the normalization factor.

Picture: https://www.wikiwand.com/en/Quantum_Fourier_transform

Practical application

$$12345678 \rightarrow \frac{1}{\sqrt{M}}(|000\rangle + 2|001\rangle + 3|010\rangle + 4|011\rangle + 5|100\rangle + 6|101\rangle + 7|110\rangle + 8|111\rangle)$$

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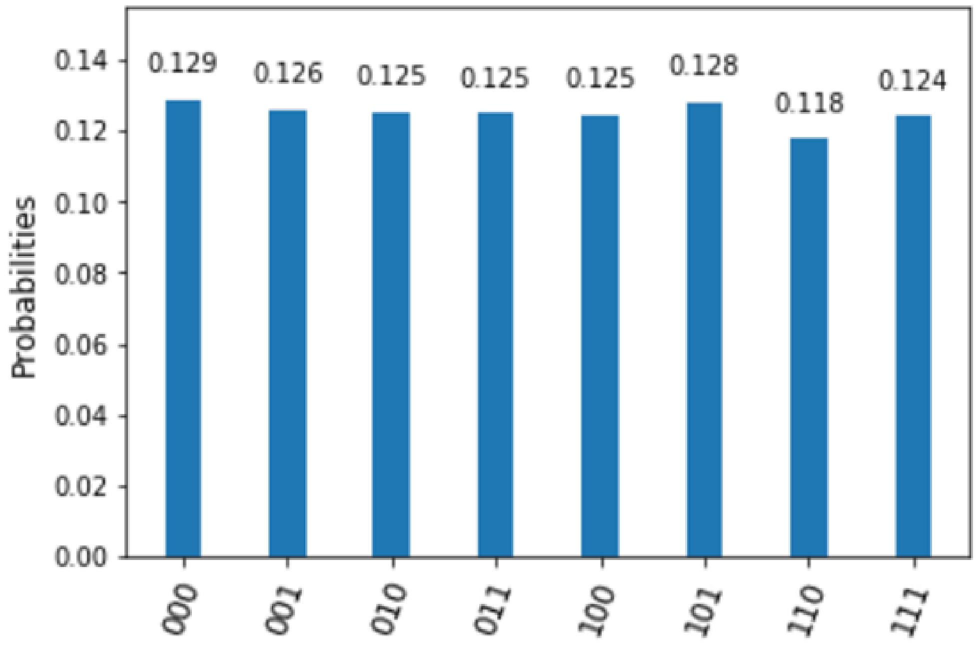
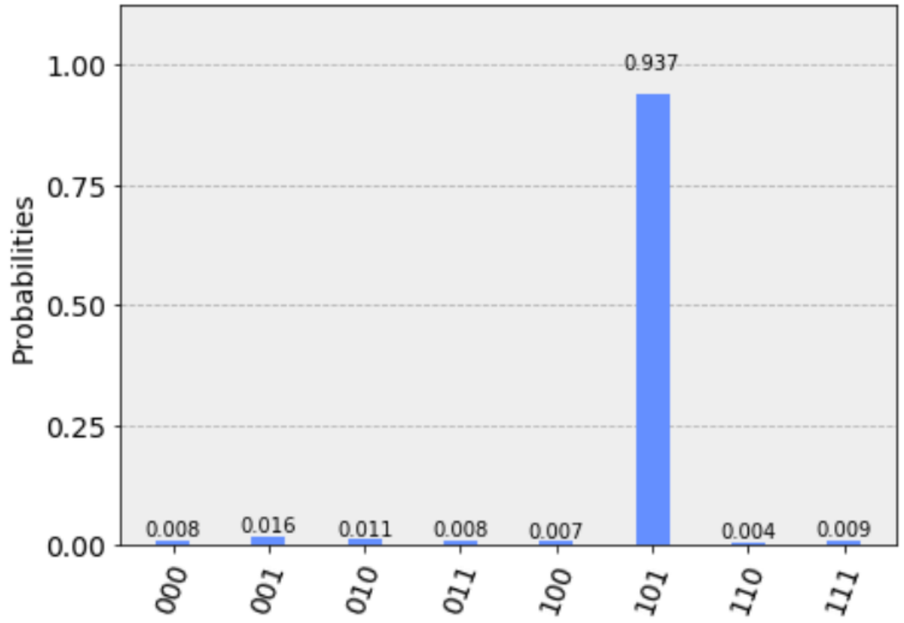


Use Hadamard gates



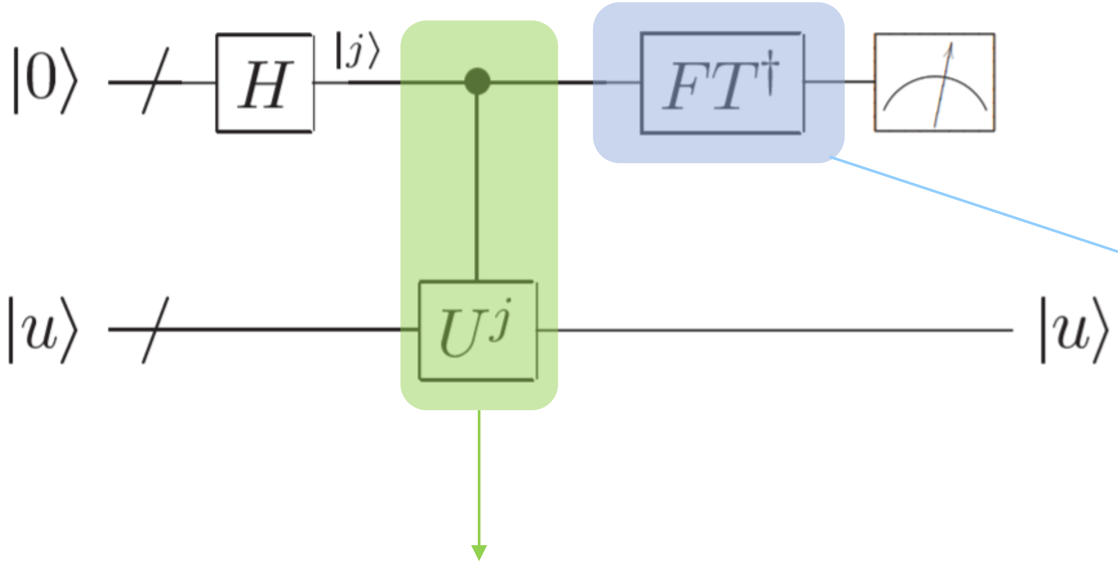
Grover's search

Generating an input state of interest can be as hard as implementing the quantum computing algorithm.

Picture: https://github.com/Qiskit/qiskit-tutorials/blob/master/community/algorithms/grover_algorithm.ipynb

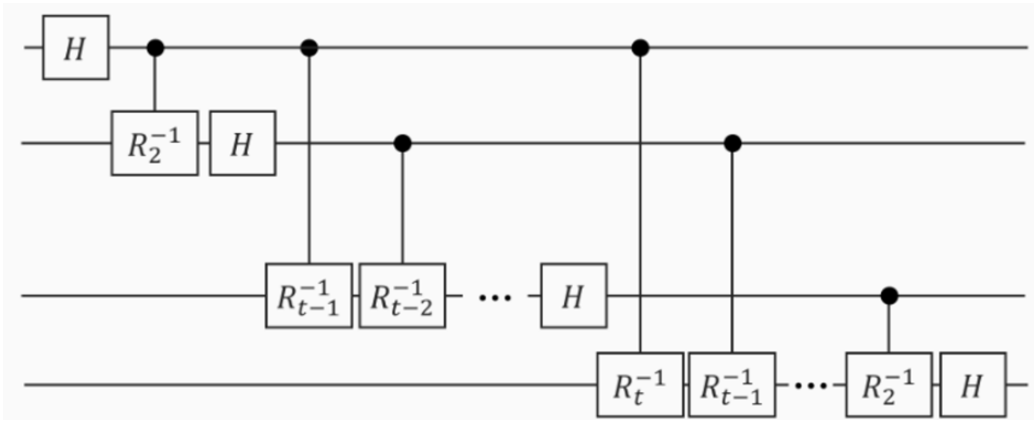
Application in other algorithms

Quantum phase estimation: to calculate the eigenvalue of U



Controlled-U gate

Inverse quantum Fourier transform:



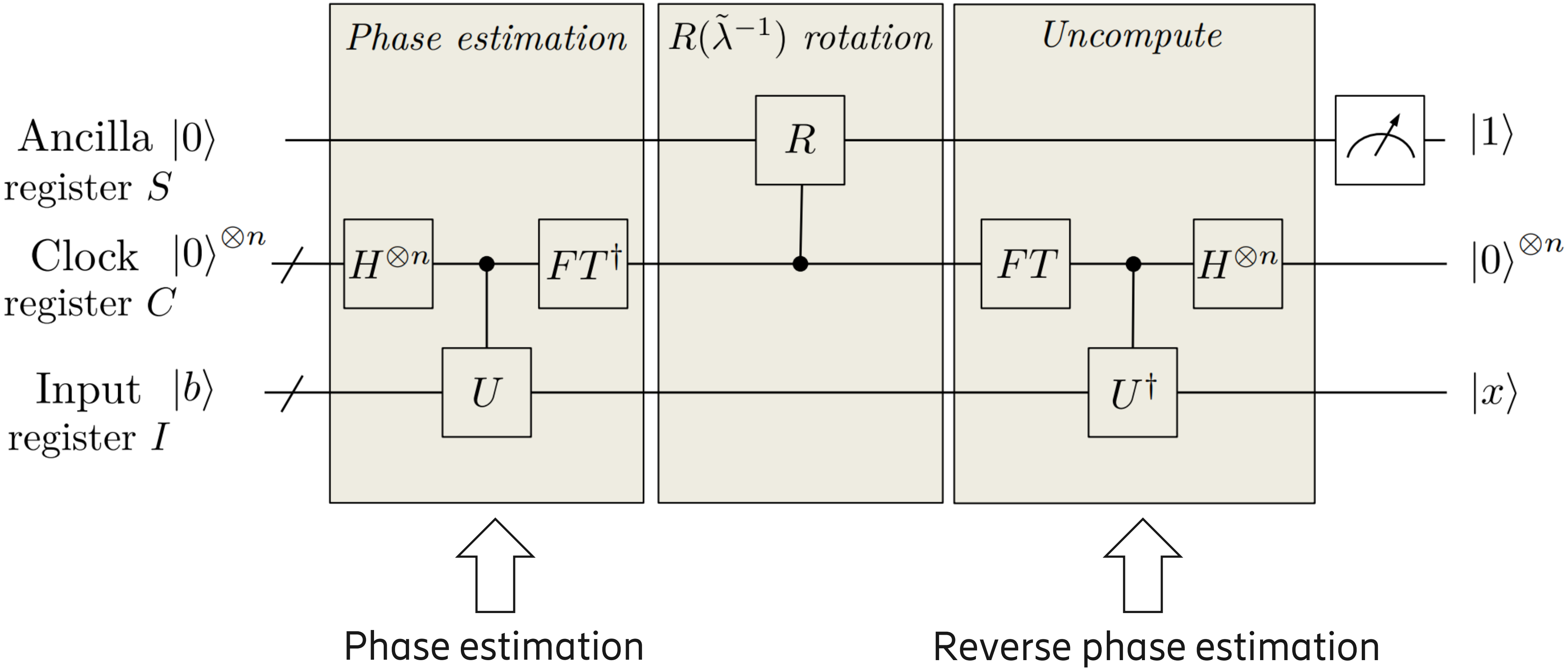
$$|j\rangle \longrightarrow \frac{1}{2^{\frac{n}{2}}} \sum_{k=0}^{2^n-1} e^{-2\pi i jk/2^n} |k\rangle$$

$$R_k{}^{-1} \equiv \begin{bmatrix} 1 & 0 \\ 0 & e^{-\frac{2\pi i}{k}} \end{bmatrix}$$

Picture: https://www.qtumist.com/info/PE-H5/index.html

Application in other algorithms

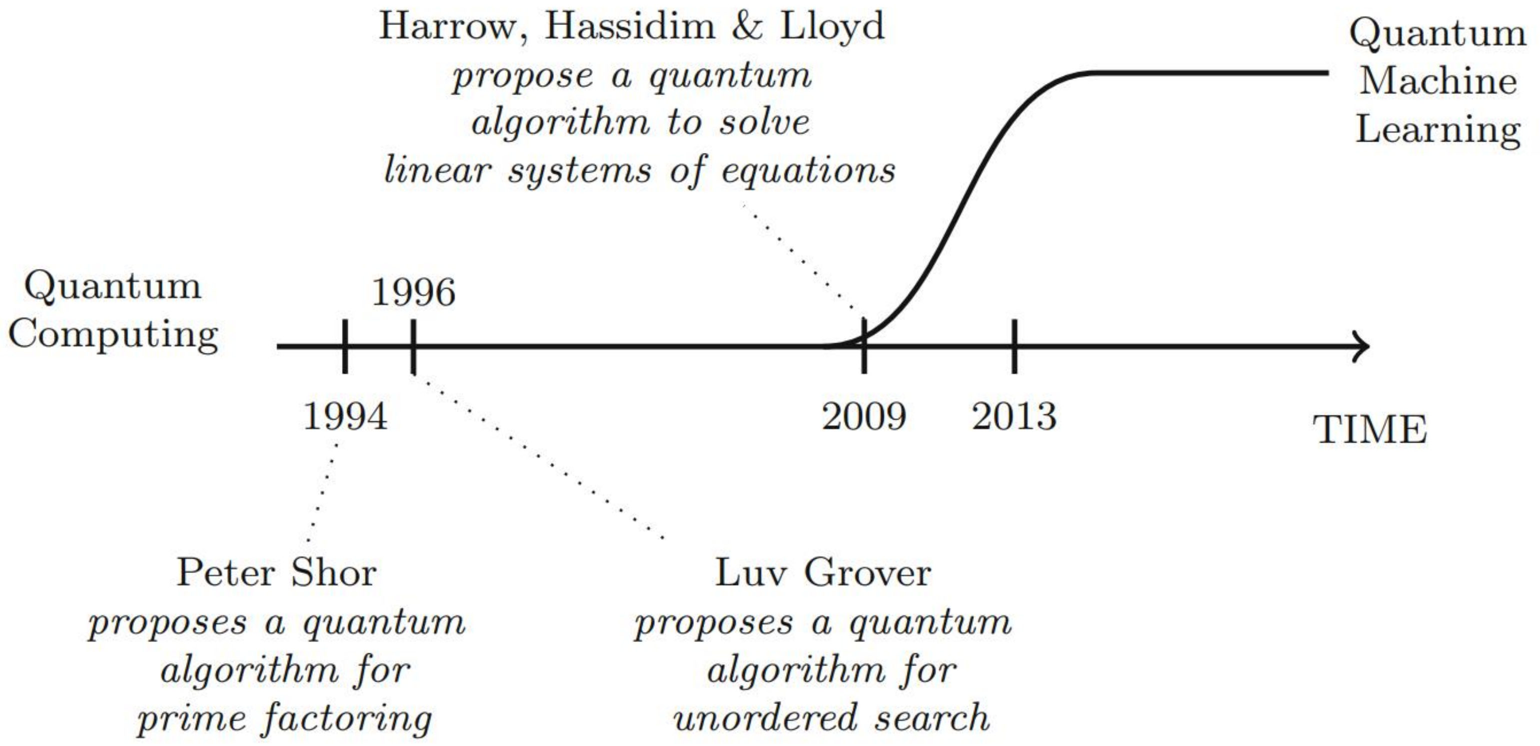
HHL Algorithm - Quantum algorithm for linear systems of equations
Solve $\vec{A}x = \vec{b}$ using a quantum computer



Picture: https://arxiv.org/pdf/1802.08227.pdf

Application in other algorithms

HHL Algorithm- Milestone for QML



Usage: quantum machine learning algorithms including bayesian inference, qSVM, qPCA…

Not only the quantum Fourier transform itself is important, but it is also the fundamental unit of many other quantum algorithms.

Picture: M. Schuld and F. Petruccione. Supervised learning with quantum computers. Springer, 2018.

Materials

- Books:

  Quantum Computation and Quantum Information, 2002

  Quantum Machine Learning: What Quantum Computing Means to Data Mining, 2014

  Supervised Learning with Quantum Computers, 2018

- Online courses:

  edX & University of Toronto: Quantum Machine Learning

  edX & TU Delft: Quantum Cryptography

  edX TUDelft: The building blocks of a quantum computer I

  edX TUDelft: The building blocks of a quantum computer II