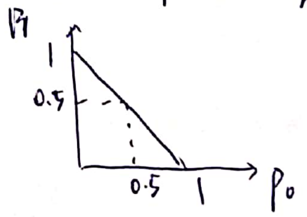
Classical Probability Theory

Probability — coin flipping

$P(X=H)=p_0$

$P(X=T)=p_1$

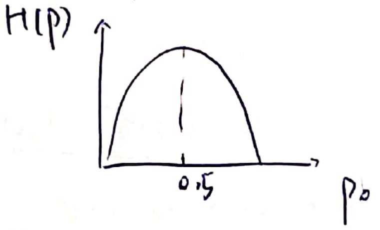
$p_i \geq 0,\ p_i \in \mathbb{R},\ \sum_i p_i = 1 \rightarrow p_1 = 1-p_0$



Entropy ⇒ characterise the unpredictability.

$H(p) = -\sum_i p_i \log_2 p_i$

When all outcome with the same probability, this is the most unpredictable case.



Geometry

$\vec{p} = \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} \quad p_i \geq 0$

↑ stochastic vector

normalize to 1 with $\begin{cases} \text{norm 1}: a_0 + a_1 = 1 \\ \text{norm 2}: \sqrt{|a_0|^2 + |a_1|^2} = 1 \end{cases}$

$\sum_i p_i = \sum_i |p_i| = \|\vec{p}\|_1 = 1$

We ~~...~~ normalise the P vector in 1 norm

[In quantum states, the normalization will be in a different norm]

Transform probability distributions — transform stochastic vectors

$M\vec{p} = \vec{p}\,',\ p_i \geq 0$

↑ stochastic matrix

$\|\vec{p}\,'\|_1 = 1$

In quantum ~~states~~ calculations, the stochastic matrix should be unitary, and ~~they~~ it can transform quantum states to other quantum states.

Quantum States

Quantum state

$|\psi\rangle = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$ $a_i \in \mathbb{C}$ (complex values)

$\| |\psi\rangle \|_2 = 1$ → the normalization of this vector happens in norm 2.

$\sqrt{|a_0|^2 + |a_1^2|} = 1$

Two-level quantum states: qubits.

Superposition:

A quantum state is also called a wave function.

$|\psi\rangle = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = a_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = a_0|0\rangle + a_1|1\rangle$

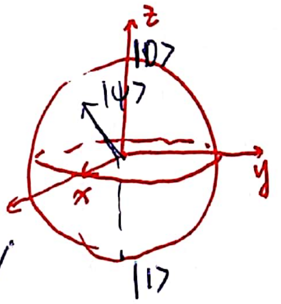
↓ collapse of the wave function

random to deterministic ⇐ out come 0 with prob. $|a_0|^2$

state after the measurement: $|0\rangle$

Bloch Sphere:

Now we have a 2-dimensional complex space, which will take 4-dimensions to visualize, with the restriction ($\sqrt{|a_0|^2+|a_1|^2} = 1$) on the degree of freedom. Thus, we'll ~~have~~ use a 2-dimentional object to visualize ~~quan~~ qubit states.



Classical:

Attention: the orthognality is a little ~~bit~~ different in this ~~s~~phere.

[the $|0\rangle$ and $|1\rangle$ are on the same line in this sphere].

Every single point on this sphere is a qubit state.

Every single probability distribution lies on a straight line.

Interference: (干涉)

Can do on quantum computers, but can't on classical ones.

$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$, $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) = |1\rangle$

Multiple qubits.

Tensor product

$|\psi\rangle = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$

$|\psi'\rangle = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$

$|\psi\rangle \otimes |\psi'\rangle = \begin{bmatrix} a_0 b_0 \\ a_0 b_1 \\ a_1 b_0 \\ a_1 b_1 \end{bmatrix}$

$|0\rangle \otimes |0\rangle = |0\rangle|0\rangle = |00\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$|0\rangle \otimes |1\rangle = |01\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$|1\rangle \otimes |0\rangle = |10\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$|1\rangle \otimes |1\rangle = |11\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

- Convention: rightmost qubit is qubit 0.

Beyond product states

$|\psi\rangle \otimes |\psi'\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$

$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \in \mathbb{C}^2 \otimes \mathbb{C}^2$ it's in the same space as a product vector, but can't be written in a product vector.

$|\psi\rangle \otimes |\psi'\rangle = \begin{bmatrix} a_0 b_0 \\ a_0 b_1 \\ a_1 b_0 \\ a_1 b_1 \end{bmatrix} = a_0 b_0 |00\rangle + a_0 b_1 |01\rangle + a_1 b_0 |10\rangle + a_1 b_1 |11\rangle$

Associate $|\Phi^+\rangle = |\psi\rangle \otimes |\psi'\rangle$

$\therefore \begin{cases} a_0 b_0 = \frac{1}{\sqrt{2}} \\ a_0 b_1 = 0 \\ a_1 b_0 = 0 \\ a_1 b_1 = \frac{1}{\sqrt{2}} \end{cases}$ $\rightarrow$ a_0 or b_1 has to be 0,

but if $a_0 = 0$, $a_0 b_0 \neq \frac{1}{\sqrt{2}}$,

if $b_1 = 0$, $a_1 b_1 \neq \frac{1}{\sqrt{2}}$

Thus $|\Phi^+\rangle$ can't be writen as product state.

Such states are called entangled states.

Measurement

Bra-ket notation:

Ket: $|\psi\rangle = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$ → $a_0 = x_0 + i y_0$

$\bar{a}_0 = x_0 - i y_0$

Bra: $\langle\psi| = |\psi\rangle^{\dagger} = [\bar{a}_0 \ \ \bar{a}_1]$

Conjugate transpose

Dot Product

$\langle\psi|\psi\rangle = |a_0|^2 + |a_1|^2 = 1$ $= \| |\psi\rangle \|_2^2$ ⇒ bra-ket

$\langle 0|1\rangle = [1 \ \ 0]\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$ ⇒ scala value

ket-bra

$|0\rangle\langle 0| = |0\rangle\langle 0| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}[1 \ \ 0] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ ⇒ matrix

Measurements

Intuition: measurement is very similar to random variable.

measurement outcome ~ value a random variable

Born Rule

Outcome 0 with prob. $|a_0|^2$

State afterwards: $|0\rangle$ "collapse of the wavefunction"

The measurement outcome is actually a projection.

e.g. $|0\rangle\underbrace{\boxed{\langle 0|\psi\rangle}}_{\text{scala}} = \cancel{|0\rangle} \ |0\rangle \cdot a_0$

$\langle\psi|0\rangle\langle 0|\psi\rangle = \| a_0 |0\rangle \|_2^2 = |a_0|^2$

State afterwards : $\dfrac{|0\rangle\langle 0|\psi\rangle}{\sqrt{\langle\psi|0\rangle\langle 0|\psi\rangle}}$ if we observe the output i.

MIXED STATES

Mixed States

$|\psi\rangle$: pure quantum state

$\rho = |\psi\rangle\langle\psi|$ density matrix

Operations on kets can be rewritten as operations on density matrices.

e.g. Born rule : $\mathrm{Tr}[|0\rangle\langle 0|\rho]$ → the probability of seeing 0.

trace, which is the sum of diagnal elements.

Why density matrix?

$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, $p_i \geq 0$, $\sum p_i = 1$ → for mixed states

→ classical ignorance: this is sth. that we don't know about the underlying quantum system.

classical probability distributions over pure states

e.g. ① $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ → $\rho = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$ → density matrix for an equal superposition

② $\rho' = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$ → desity matrix for the equally mixed state of $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$

→ diagnal

off-diagnal also called coherence

In ② ~~exp~~ example, the off-diagnal elements are gone. they're also called coherence, and their presence indicates that the states is quantum. The smaller these values are, the closer the quantum state is to a classical probability distribution.

not just a mixed state, but also a maximally mixed state.

A maximally mixed state is the equivalent of a uniform distribution the entropy is maximum, and we have no predictive power of what's going to happen next.

Measuring multi-qubit systems

$|\phi\rangle^+ = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ → entangled

if we measure the ~~0th~~ 0th qubit (the right most one) and get the outcome 0:

$(\mathbb{1} \otimes |0\rangle\langle 0|)|\phi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle$

$\mathbb{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Then if we measure the other qubit, we must get 0 deterministicly.

Partial trace: ~~matrix~~ marginal probability

$$S = |\phi^+\rangle\langle\phi^+| = \begin{bmatrix} 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 \end{bmatrix}$$

$$Tr_1\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix} = \begin{bmatrix} a+f & c+h \\ i+n & k+p \end{bmatrix}$$

$$\therefore Tr_1[S] = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$ $\Rightarrow$ this is the same result with the maximally mixed state that ~~we int~~ introduced in previous page.

This means that if we ~~may~~ marginalize out one of the qubits in this system, then we end up with a uniform distribution.

$\downarrow$ after measuring 1 qubit, the other is deterministic

the entropy

we have no perdictive power over what's going to happen in that remaining quantum system.