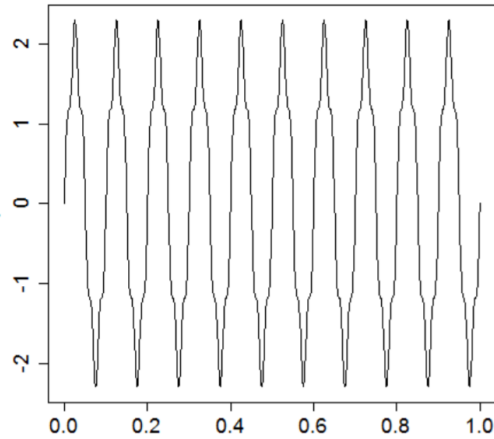
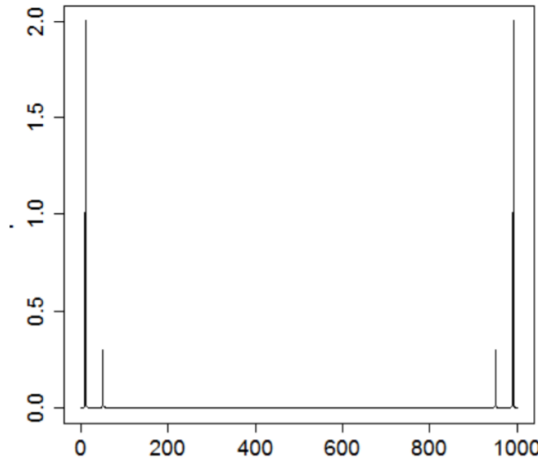
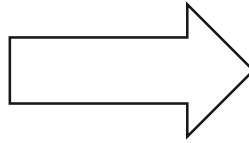


The Quantum Fourier Transform and Its Applications

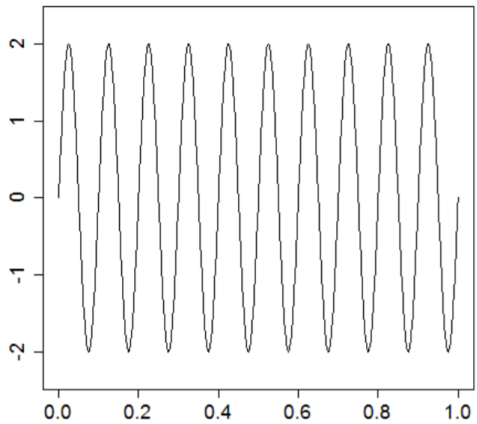
Discrete Fourier Transform



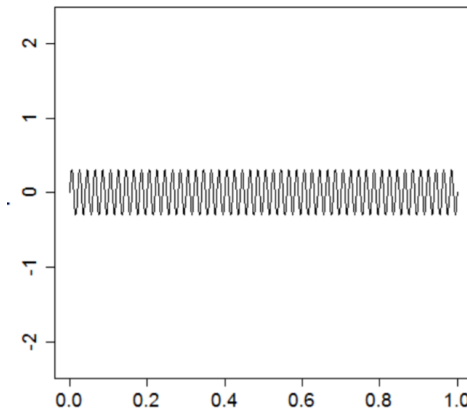
Input signal



Input signal in frequency domain



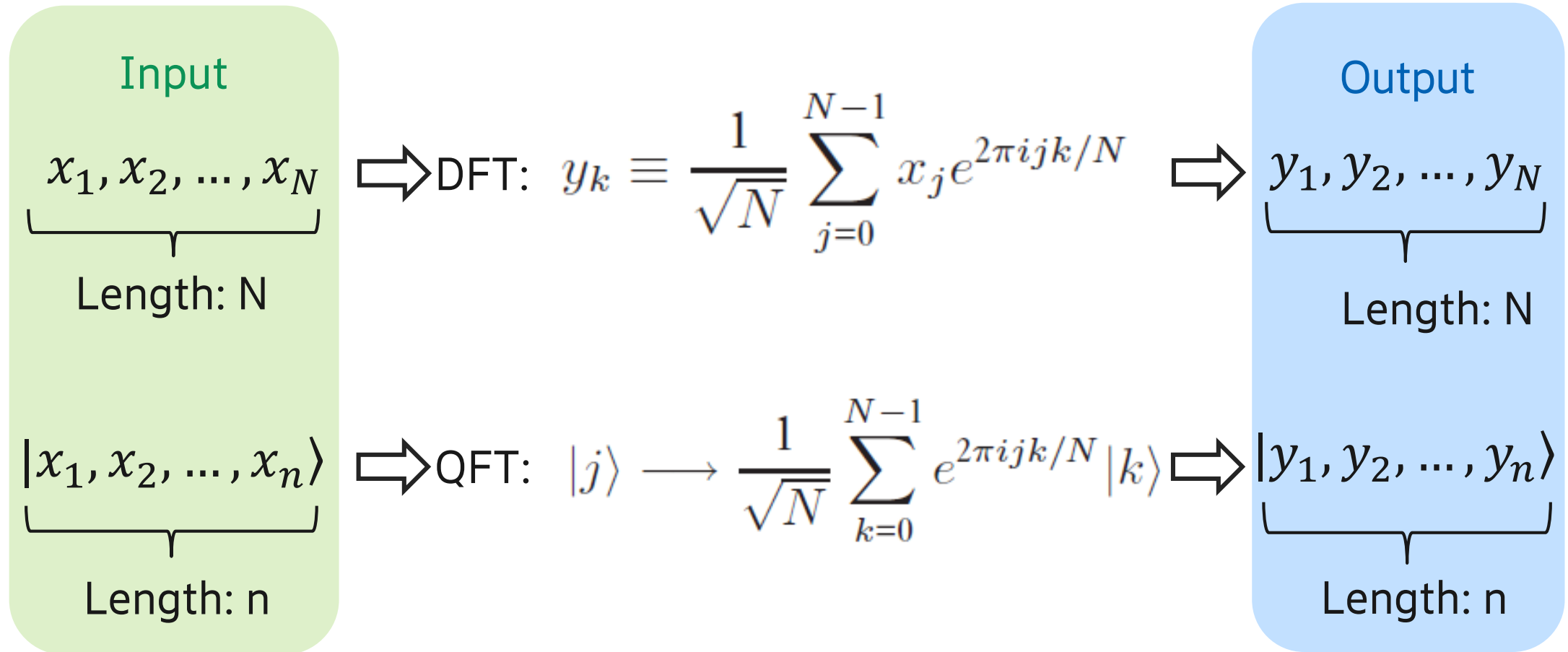
True signal: $f=10$, $A=2$



Noise signal: $f=50$, $A=0.3$

- QFT is the quantum analogue of DFT:
- Map signal from **time** domain to **frequency** domain on quantum computer with a faster speed
 - Basis for **quantum phase estimation**

Quantum Fourier Transform



Quantum Fourier Transform

$$\text{QFT: } |j\rangle \longrightarrow \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i j k / N} |k\rangle$$

$$\underbrace{|x_1, x_2, \dots, x_n\rangle}_{\text{Length: } n} \Rightarrow \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \omega_n^3 & \dots & \omega_n^{N-1} \\ 1 & \omega_n^2 & \omega_n^4 & \omega_n^6 & \dots & \omega_n^{2(N-1)} \\ 1 & \omega_n^3 & \omega_n^6 & \omega_n^9 & \dots & \omega_n^{3(N-1)} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega_n^{N-1} & \omega_n^{2(N-1)} & \omega_n^{3(N-1)} & \dots & \omega_n^{(N-1)(N-1)} \end{bmatrix} \Rightarrow \underbrace{|y_1, y_2, \dots, y_n\rangle}_{\text{Length: } n}$$

N×N matrix

where $\omega_n := e^{\frac{2\pi i}{2^n}}$

Example (2 qubits):

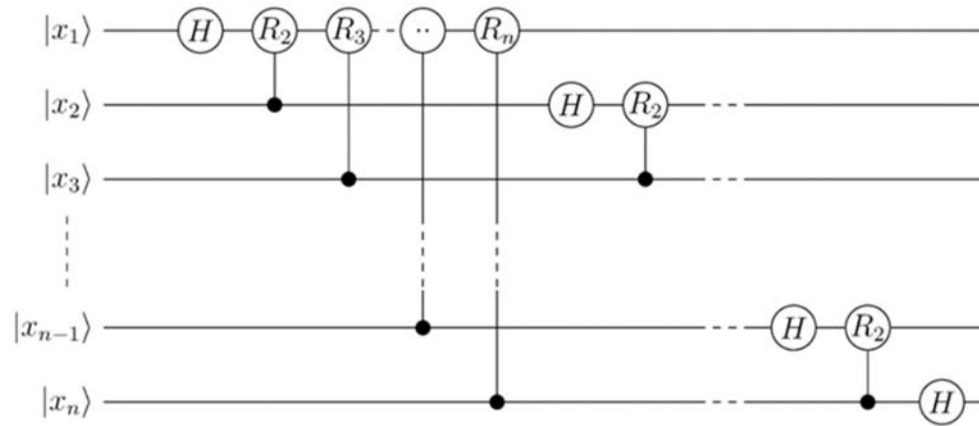
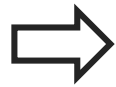
$$F_4 = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

Quantum Fourier Transform

$$\text{QFT: } |j\rangle \longrightarrow \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i j k / N} |k\rangle$$

$|x_1, x_2, \dots, x_n\rangle$

Length: n



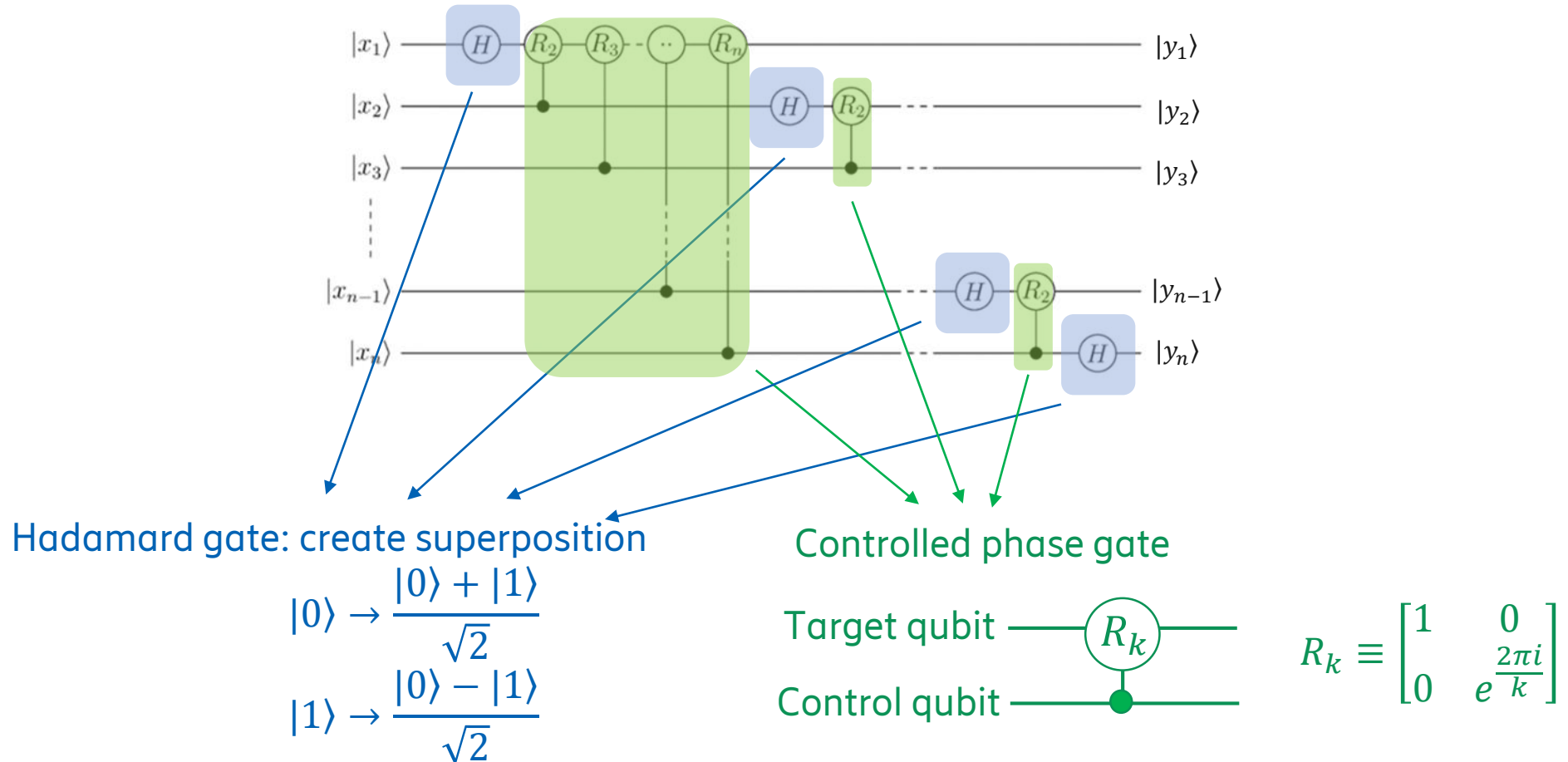
Quantum circuit for QFT

$|y_1, y_2, \dots, y_n\rangle$

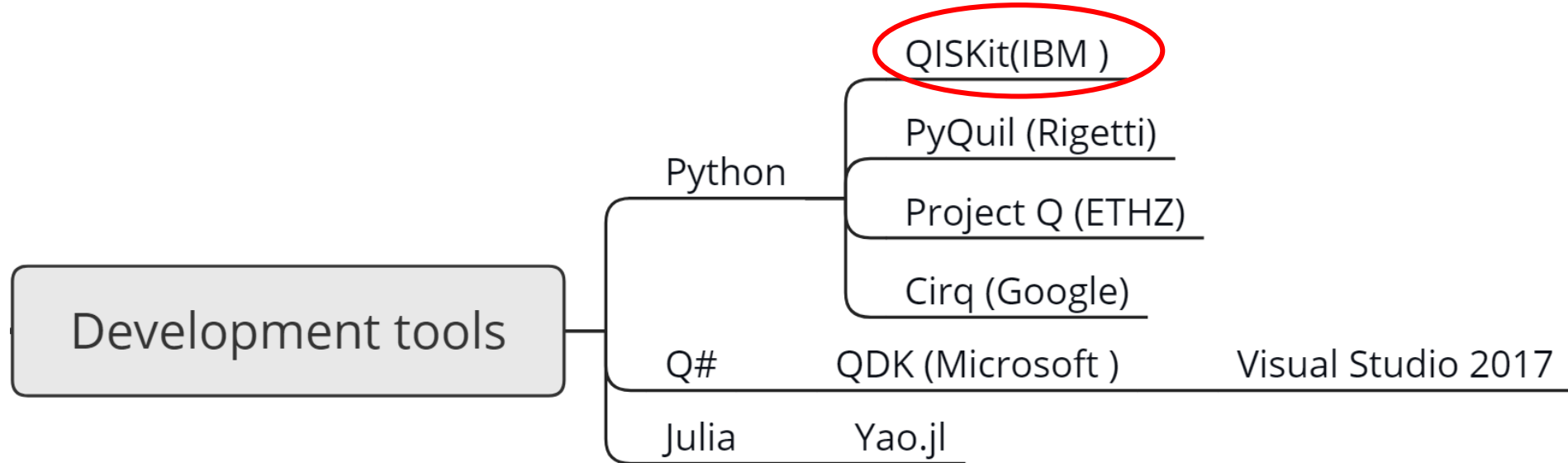
Length: n

Quantum Fourier Transform

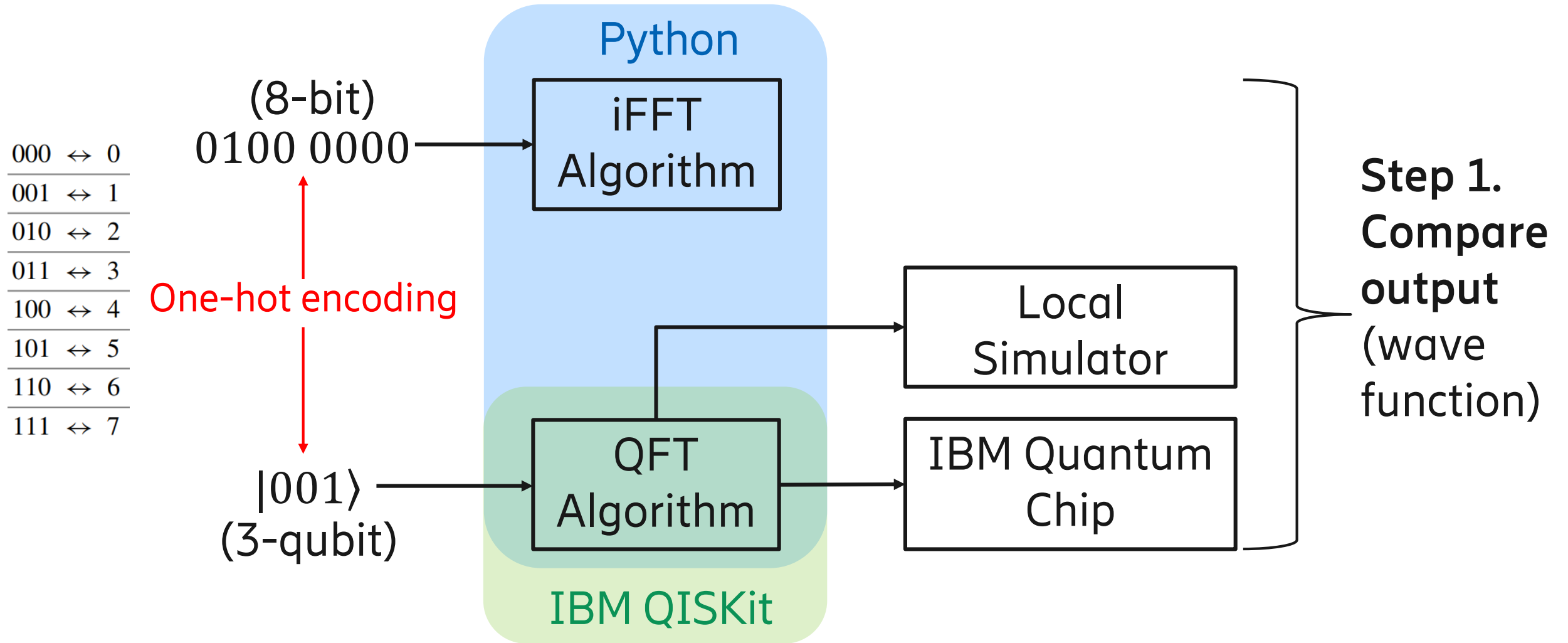
Quantum circuit for QFT



Development tools

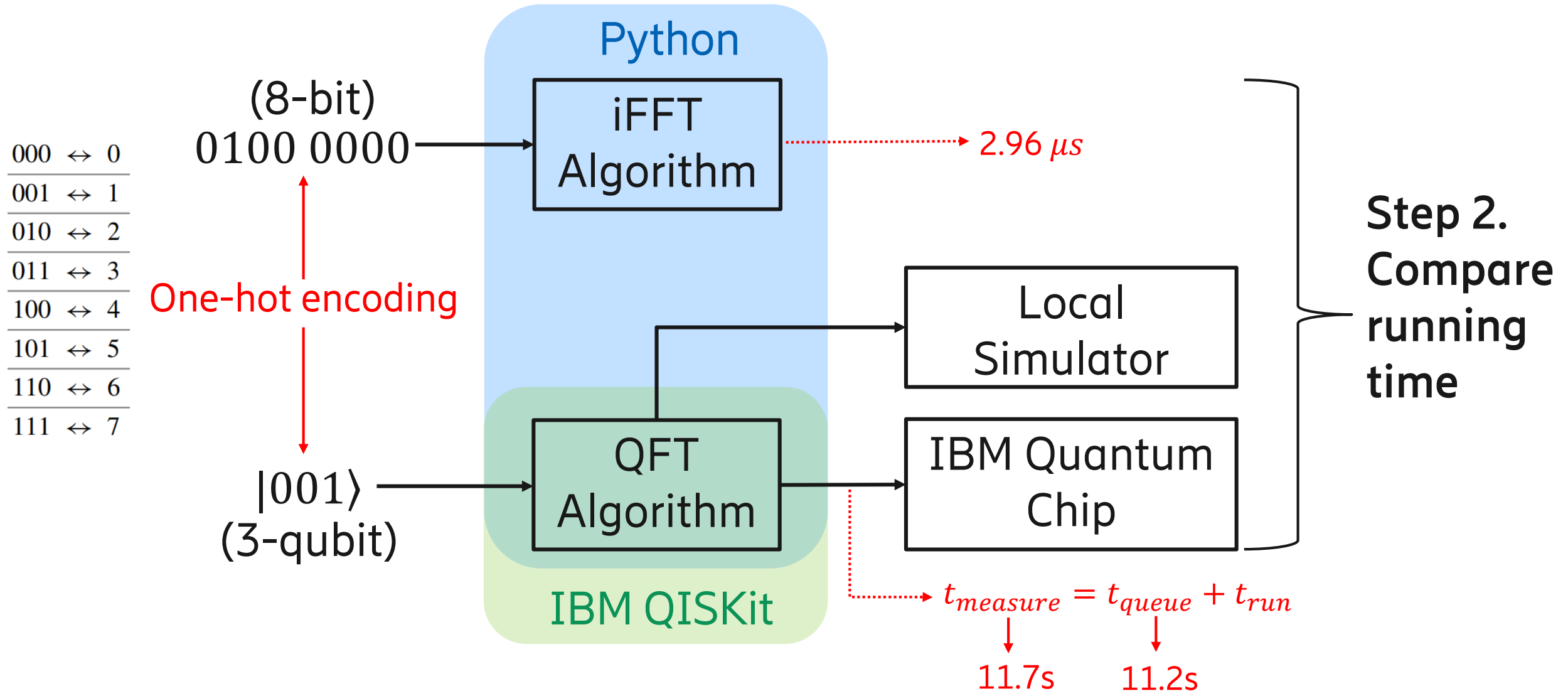


Implementation and comparison with iFFT



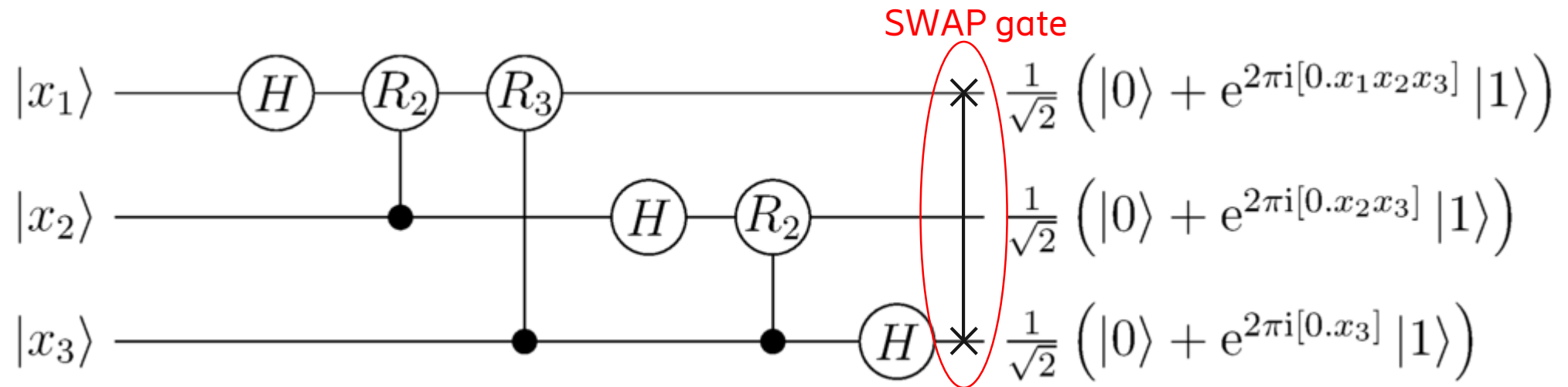
mismatching: QFT = iFFT, iQFT = FFT

Implementation and comparison with iFFT



Estimation of the QFT running time

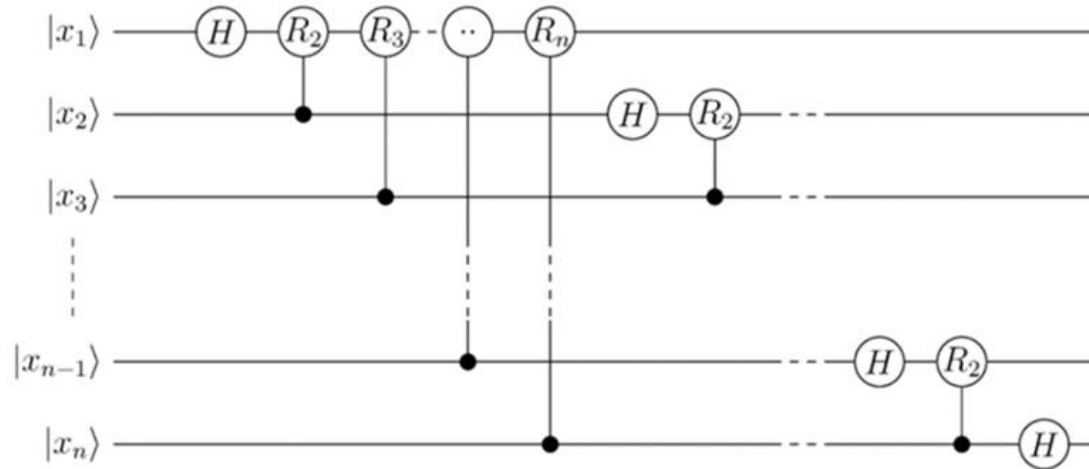
- Add up the delays on the longest path of the compiled circuit
- Single qubit gate: 80 ns, two qubits gate: $170 < t < 348$ ns



- e.g. For 3-qubit circuit, the estimated running time is among: (920, 1632)ns
 3 single qubit gates + 4 two qubits gate (SWAP gate)
- Computational cost for n qubits (or for 2^n bits): QFT $\Theta(n^2)$, FFT $\Theta(n \cdot 2^n)$

Practical application

- The algorithm itself is easy to be generalized from 3-qubit system to n-qubit system.



- But it is hard to create a complex enough input qubit string, e.g. Sine wave.

Example: 01000000 \rightarrow $|001\rangle$

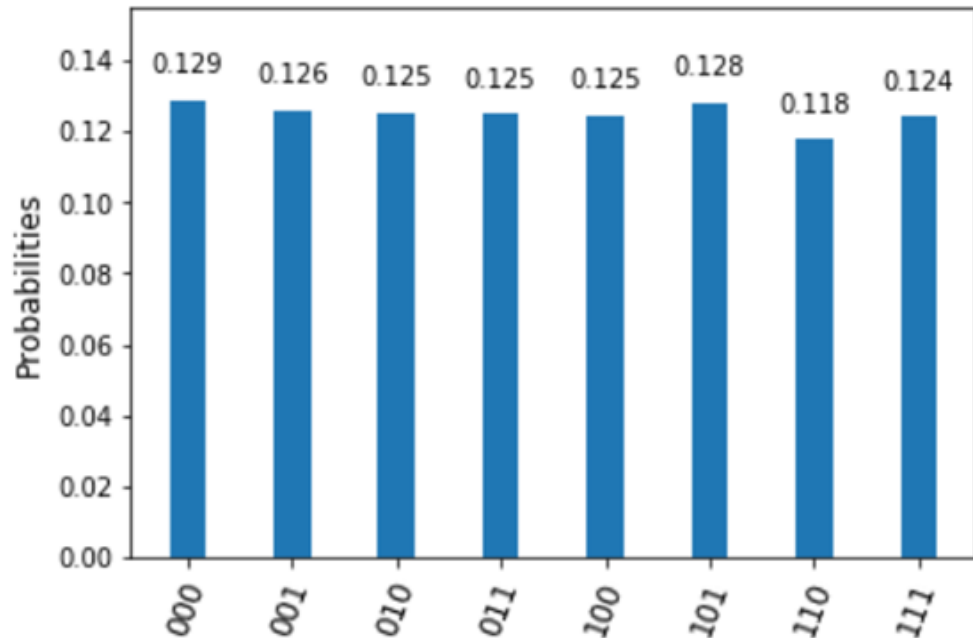
$$12345678 \rightarrow \frac{1}{\sqrt{M}} (|000\rangle + 2|001\rangle + 3|010\rangle + 4|011\rangle + 5|100\rangle + 6|101\rangle + 7|110\rangle + 8|111\rangle)$$

Where $M = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2$ is the normalization factor.

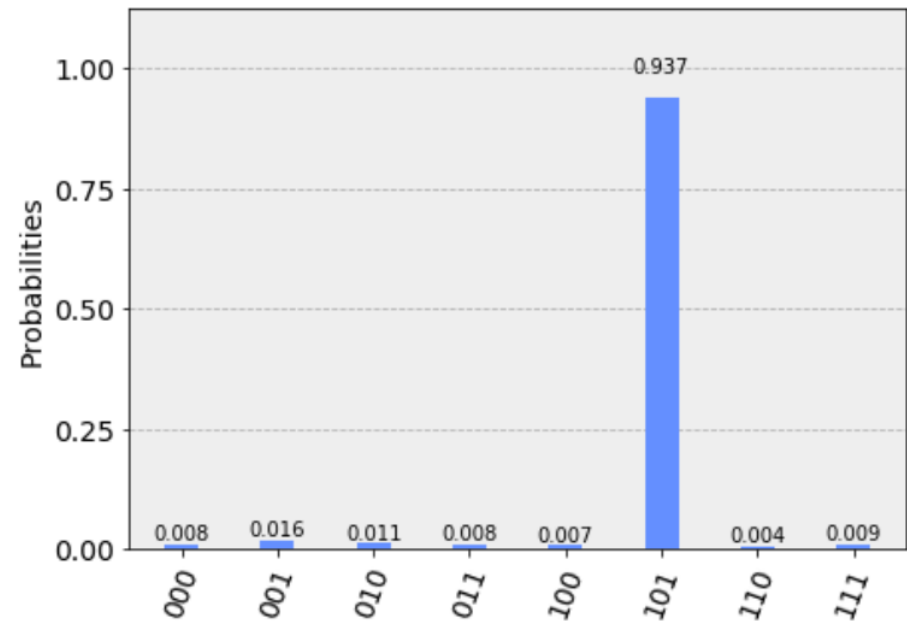
Practical application

$$12345678 \rightarrow \frac{1}{\sqrt{M}} (|000\rangle + 2|001\rangle + 3|010\rangle + 4|011\rangle + 5|100\rangle + 6|101\rangle + 7|110\rangle + 8|111\rangle)$$

Where $M = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2$ is the normalization factor.



Use Hadamard gates



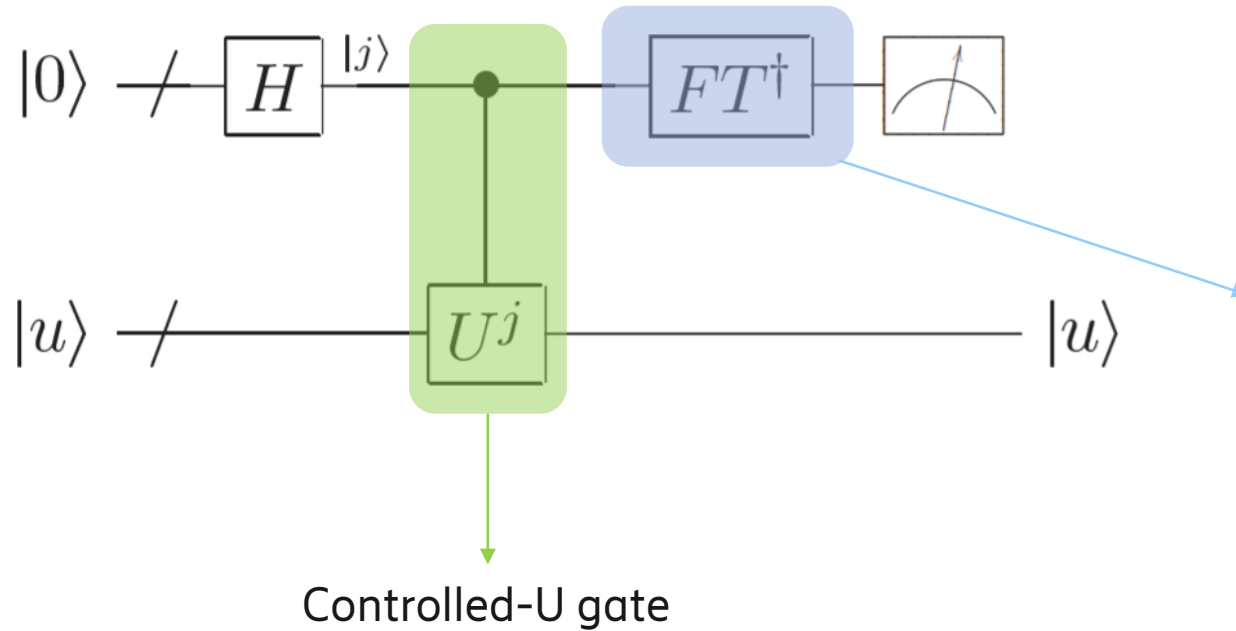
Grover's search

Generating an input state with probability distribution of interest can be as hard as implementing the quantum computing algorithm.

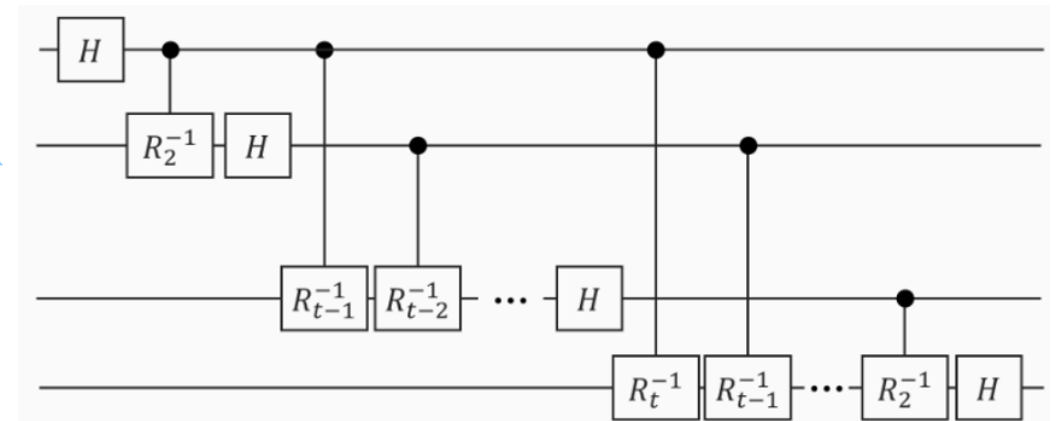
Picture: https://github.com/Qiskit/qiskit-tutorials/blob/master/community/algorithms/grover_algorithm.ipynb

Application in other algorithms

Quantum phase estimation: to calculate the eigenvalue of U



Inverse quantum Fourier transform:

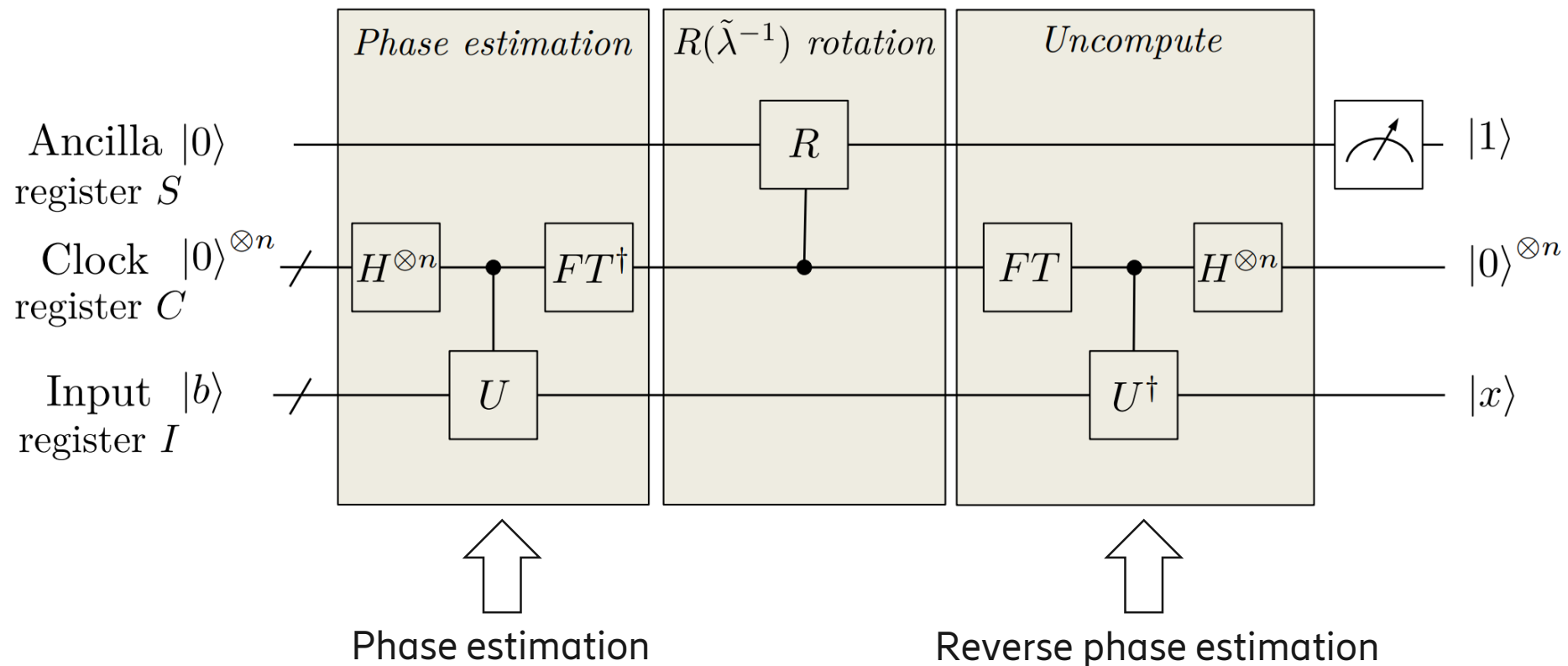


$$|j\rangle \rightarrow \frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} e^{-2\pi i j k / 2^n} |k\rangle$$

$$R_k^{-1} \equiv \begin{bmatrix} 1 & 0 \\ 0 & e^{-\frac{2\pi i}{k}} \end{bmatrix}$$

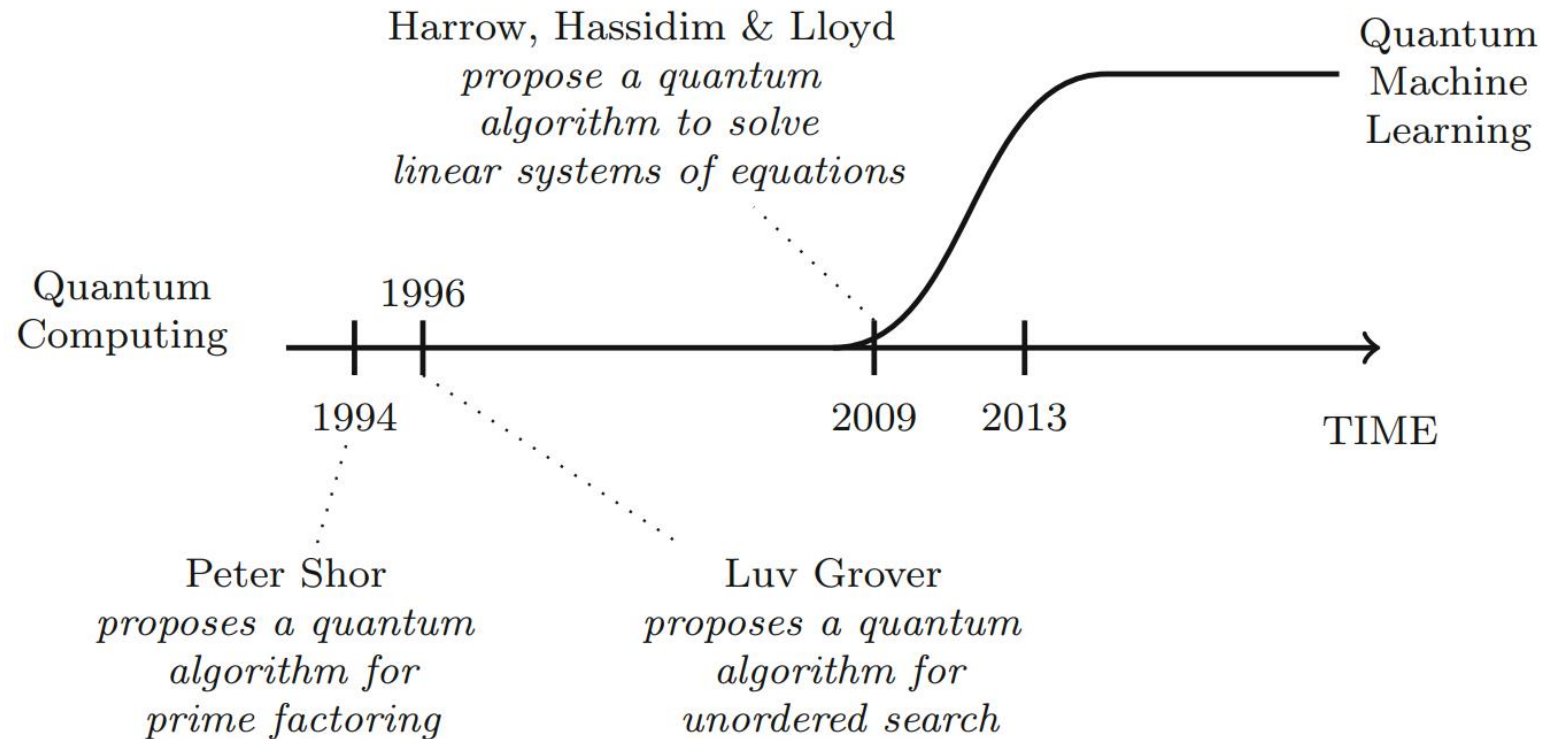
Application in other algorithms

HHL Algorithm - Quantum algorithm for linear systems of equations
Solve $\vec{A}x = \vec{b}$ using a quantum computer



Application in other algorithms

HHL Algorithm- Milestone for QML



Usage: quantum machine learning algorithms including bayesian inference, qSVM, qPCA...

Not only the quantum Fourier transform itself is important, but it is also the fundamental unit of many other quantum algorithms.

Materials

- Books:

 - Quantum Computation and Quantum Information, 2002

 - Quantum Machine Learning: What Quantum Computing Means to Data Mining, 2014

 - Supervised Learning with Quantum Computers, 2018

- Online courses:

 - edX & University of Toronto: Quantum Machine Learning

 - edX & TU Delft: Quantum Cryptography

 - edX TUDelft: The building blocks of a quantum computer I

 - edX TUDelft: The building blocks of a quantum computer II