CPSC 368 / CPSC 516

Instructions:

- **These problems will not be graded, however, you are encouraged to write solutions.**

Problems:

P.1. In this problem we apply the Multiplicative Weight Update (MWU) framework to approximately find equilibria in two player zero-sum games. Consider the following theorem about the MWU method. (Here, $\Delta_n := \{p : \sum_{i=1}^n p_i = 1 \text{ and } p \geq 0\}$.)

MWU Theorem. *Let $g^t \in \mathbb{R}^n$ be vectors with $\|g^t\|_\infty \leq 1$ for every $t = 0, \ldots, T-1$, and let $0 < \delta \leq \frac{1}{2}$. Then, starting with $p^0 := \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$, the MWU algorithm produces a sequence of probability distributions $p^1, \ldots, p^{T-1} \in \Delta_n$ such that*

$$\sum_{t=0}^{T-1} \langle p^t, g^t \rangle - \inf_{p \in \Delta_n} \sum_{t=0}^{T-1} \langle p, g^t \rangle \leq \frac{\ln n}{\delta} + \delta T.$$

Let $A \in \mathbb{R}^{n \times m}$ be a matrix with $A(i,j) \in [0,1]$ for all $i \in [n]$ and $j \in [m]$. We consider a game between two players: the row player and a column player. The game consists of one round in which the row player picks one row $i \in \{1, 2, \ldots, n\}$ and the column player picks one column $j \in \{1, 2, \ldots, m\}$. The goal of the row player is to minimize the value $A(i,j)$ which they pay to the column player after such a round, the goal of the column player is the opposite (to maximize the value $A(i,j)$).

The min-max theorem asserts that:

$$\max_{q \in \Delta_m} \min_{i \in \{1,\ldots,n\}} \mathbb{E}_{J \leftarrow q} A(i, J) = \min_{p \in \Delta_n} \max_{j \in \{1,\ldots,m\}} \mathbb{E}_{I \leftarrow p} A(I, j). \tag{1}$$

Here $\mathbb{E}_{I \leftarrow p} A(I,j)$ is the expected loss of the row player when using the randomized strategy $p \in \Delta_n$ against a fixed strategy $j \in \{1, 2, \ldots, m\}$ of the column player. Similarly, define $\mathbb{E}_{J \leftarrow q} A(i, J)$. Formally,

$$\mathbb{E}_{I \leftarrow p} A(I,j) := \sum_{i=1}^n p_i A(i,j) \quad \text{and} \quad \mathbb{E}_{J \leftarrow q} A(i,J) := \sum_{j=1}^m q_j A(i,j)$$

Let opt be the common value of the two quantities in (1) corresponding to two optimal strategies $p^\star \in \Delta_n$ and $q^\star \in \Delta_m$ respectively. Our goal is to use the MWU framework to construct, for any $\varepsilon > 0$, a pair of strategies $p \in \Delta_n, q \in \Delta_m$ such that:

$$\max_j \mathbb{E}_{I \leftarrow p} A(I,j) \leq \text{opt} + \varepsilon \qquad \text{and} \qquad \min_i \mathbb{E}_{J \leftarrow q} A(i,J) \geq \text{opt} - \varepsilon.$$

(a) Prove the following "easier" direction of Equation (1):

$$\max_{q \in \Delta_m} \min_{i \in \{1,\ldots,n\}} \mathbb{E}_{J \leftarrow q} A(i, J) \leq \min_{p \in \Delta_n} \max_{j \in \{1,\ldots,m\}} \mathbb{E}_{I \leftarrow p} A(I, j).$$

(b) Give an algorithm, which given $p \in \Delta_n$ constructs a $j \in \{1, 2, \ldots, m\}$ which maximizes $\mathbb{E}_{I \leftarrow p} A(I,j)$. What is the running time of your algorithm? Show that for such a choice of j we have $\mathbb{E}_{I \leftarrow p} A(I,j) \geq \text{opt}$.

We will follow the MWU scheme with $p^0, \ldots, p^{T-1} \in \Delta_n$ and the vector g^t at step t being $g^t := Aq^t$, where $q^t := e_j$ with j chosen as to maximize $\mathbb{E}_{I \leftarrow p^t} A(I,j)$. (Recall that e_j is the vector with 1 at coordinate j and 0 otherwise.)

(c) Prove that $\|g^t\|_\infty \leq 1$ and $\langle p^\star, g^t \rangle \leq \text{opt}$ for every $t = 0, 1, \ldots, T-1$.

(d) Use the MWU theorem mentioned above to show that for T large enough:

$$\mathrm{opt} \leq \frac{1}{T} \sum_{t=0}^{T-1} \langle p^t, g^t \rangle \leq \mathrm{opt} + \varepsilon.$$

What is the smallest value of T which suffices for this to hold? Conclude that for some $1 \leq t < T$ it holds that $\max_j \mathbb{E}_{I \leftarrow p^t} A(I, j) \leq \mathrm{opt} + \varepsilon$.

(e) Let $q \coloneqq \frac{1}{T} \sum_{t=0}^{T-1} q^t$. Prove that for T as in part (d): $\min_i \mathbb{E}_{J \leftarrow q} A(i, J) \geq \mathrm{opt} - \varepsilon$.

(f) What is the total running time of the whole procedure to find an ε-approximate pair of strategies p and q we set out to find at the beginning of this problem?

P.2. Consider a general linear feasibility problem which asks for a point x satisfying a system of inequalities

$$\langle a_i, x \rangle \geq b_i$$

for $i = 1, 2, \ldots, m$, where $a_1, a_2, \ldots, a_m \in \mathbb{R}^n$ and $b_1, b_2, \ldots, b_m \in \mathbb{R}$. The goal of this problem is to give an algorithm that, given an error parameter $\varepsilon > 0$, outputs a point x such that

$$\langle a_i, x \rangle \geq b_i - \varepsilon \tag{2}$$

for all i whenever there is a solution to the above system of inequalities. We also assume the existence of an *oracle* that, given vector $p \in \Delta_m$, solves the following relaxed problem: does there exist an x such that

$$\sum_{i=1}^{m} \sum_{j=1}^{n} p_i a_{ij} x_j \geq \sum_{i=1}^{m} p_i b_i. \tag{3}$$

Assume that when the oracle returns a feasible solution for a p, the solution x that it returns is not arbitrary but has the following property:

$$\max_i |\langle a_i, x \rangle - b_i| \leq 1.$$

Prove the following theorem:

Theorem 0.1. *There is an algorithm that, if there exists an x such that $\langle a_i, x \rangle \geq b_i$ for all i, outputs an $\bar{x}$ that satisfies (2). The algorithm makes at most $O\left(\frac{\ln m}{\varepsilon^2}\right)$ calls to the oracle for the problem mentioned in (3).*