LECTURE #9: FIXME

1 A Model

Recall the Kac-Rice formula for counting critical points of "nice" functions (e.g. Gaussian processes).

$$\mathbb{E}\,\mathsf{Crit}(f, A) = \int_A \mathbb{E}\left[\mathsf{det}\,\boldsymbol{\nabla}^2 f(x) \mid \boldsymbol{\nabla} f(x) = 0\right] p_{\boldsymbol{\nabla} f(x)}(0) dx.$$

We now introduce the model which we analyze (c.f. "Elastic Manifold" in physics).

$$f(x) = \frac{\alpha}{2}\|x\|^2 + g(x)$$

where g is the some centered, stationary Gaussian process, which we explain below.

Recall from probability theory that a stochastic process $G_x = g(x)$ is a *Gaussian process (GP)* if every finite-dimensional distribution of G_x satisfies

$$(G_{x_1}, \dots, G_{x_m}) \sim \mathcal{N}(\mu_{x_1,\dots,x_m}, \Sigma_{x_1,\dots,x_m}).$$

For instance, Brownian motion is a Gaussian process.

We will specifically concern ourselves with centered Gaussian processes with a very specific covariance function.

$$\begin{aligned}(G_{x_1}, \dots, G_{x_m}) &\sim \mathcal{N}(0, \Sigma) \\ \Sigma_{ij} &= K(x_i, x_j) \\ &= K(x_i - x_j).\end{aligned}$$

Here $K(\cdot)$ is a *kernel* function (e.g. rbf kernel). This choice of covariance ensures that out Gaussian process is *stationary*, i.e.

$$(g(x))_{x\in\mathbb{R}^N} \stackrel{d}{=} (g(x - x_0))_{x\in\mathbb{R}^N}.$$

The particular Gaussian process we examine is given by

$$g(x) = \sum_{i_1,\dots,i_d\in[N], s_1,\dots,s_d\in\{0,1\}} \prod_{a=1}^{d} \left[\mathsf{cos}(x_{i_a})\mathbb{1}\{s_i = 0\} + \mathsf{sin}(x_{i_a})\mathbb{1}\{s_i = 1\}\right] W_{i_1,\dots,i_d,s_1,\dots,s_d}.$$

Here $W_{i_1,\dots,i_d,s_1,\dots,s_d} \sim \mathcal{N}(0, 1)$ i.i.d.

2 Applying the Kac-Rice Formula

In order to apply the Kac-Rice formula, we will need up to compute second-order information about $f(x)$. As a preliminary, let us first consider $g(x)$. We have

$$\begin{aligned}
K(x,y) &= \mathbb{E}g(x)g(y) \\
&= \sum_{i_1,\ldots,i_d,s_1,\ldots,s_d} \prod_{a=1}^{d} \left[\mathsf{cos}(x_{i_a})\,\mathsf{cos}(y_{i_a})\mathbb{1}\{s_a = 0\} + \mathsf{sin}(x_{i_a})\,\mathsf{sin}(y_{i_a})\mathbb{1}\{s_a = 1\}\right] \\
&= \left[\sum_{i=1}^{N} \mathsf{cos}(x_i)\,\mathsf{cos}(y_i) + \mathsf{sin}(x_i)\,\mathsf{sin}(y_i)\right]^d \\
&= \left[\sum_{i=1}^{N} \mathsf{cos}(x_i - y_i)\right]^d \\
&=: S(x,y)^d
\end{aligned}$$

In particular, for $x = y$, we have $K(x,x) = N^d$,

Note that this computation shows that $g(x)$ is indeed stationary. The second equality comes from staring at the terms in the expanded summation and realizing that if any indices are not precisely the same, then the expectation of the term is 0.

In order to apply the Kac-Rice formula, we need to understand the joint distribution of $(f(x), \boldsymbol{\nabla} f(x), \boldsymbol{\nabla}^2 f(x)) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N$. It is known that the derivative of a centered Gaussian process with differentiable kernel is another Gaussian process whose kernel is just the derivative of original kernel. We have

$$\begin{aligned}
f(x) &= \frac{\alpha}{2}\|x\|^2 + g(x) \\
\boldsymbol{\nabla} f(x) &= \alpha x + \boldsymbol{\nabla} g(x) \\
\boldsymbol{\nabla}^2 f(x) &= \alpha I_N + \boldsymbol{\nabla}^2 g(x).
\end{aligned}$$

This is a "massive Gaussian vector" say with parameters (μ_x, Σ_x). By inspection,

$$\begin{aligned}
\mu_x &= \left(\frac{\alpha}{2}\|x\|^2, \alpha x, \alpha I_N\right) \\
\Sigma_x &= \mathsf{Cov}\left(g(x), \boldsymbol{\nabla} g(x), \boldsymbol{\nabla}^2 g(x)\right).
\end{aligned}$$

It remains to compute these covariances depicted in Figure 1.

2.1 Covariance Computations

Now, taking an expectation is simply an integral. In the case of "nice" functions, we know from elementary calculus that we are able to exchange the order of the integral and differential. Staring long enough yields the following identity.

$$\mathbb{E}\left[\frac{\partial^a g}{\partial x_{i_1} \ldots \partial x_{i_a}}(x) \frac{\partial^b g}{\partial y_{j_1} \ldots \partial y_{j_b}}(y)\right] = \frac{\partial^{a+b} K}{\partial x_{i_1} \ldots \partial x_{i_a} \partial y_{j_1} \ldots \partial y_{j_b}}(x,y).$$

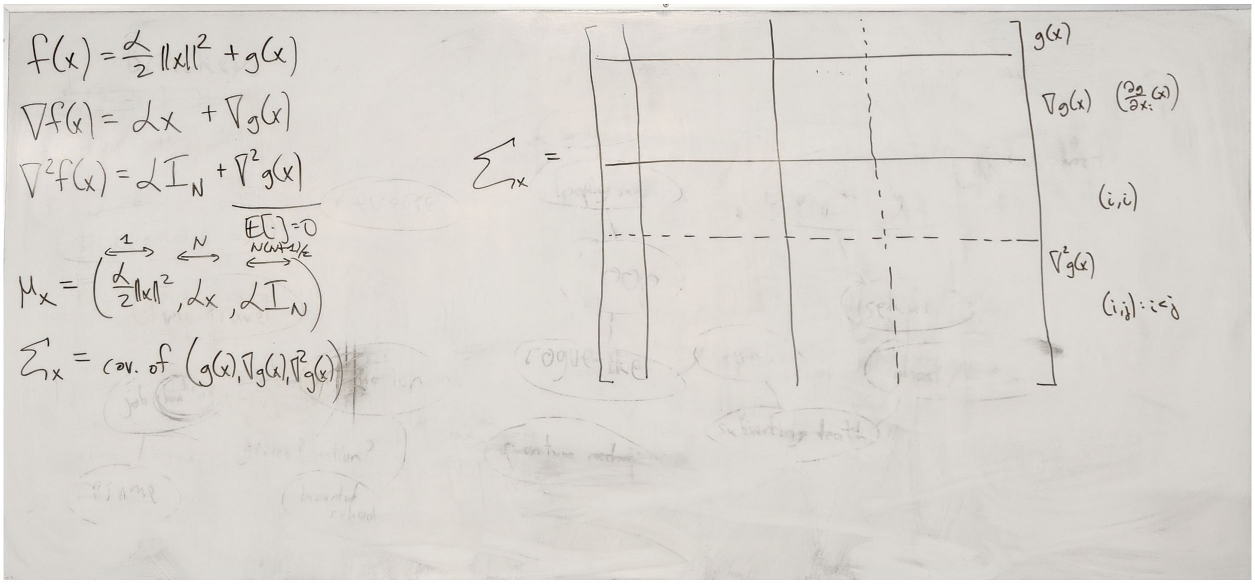

Figure 1: A big covariance matrix.

3 An Explicit Random Matrix Model

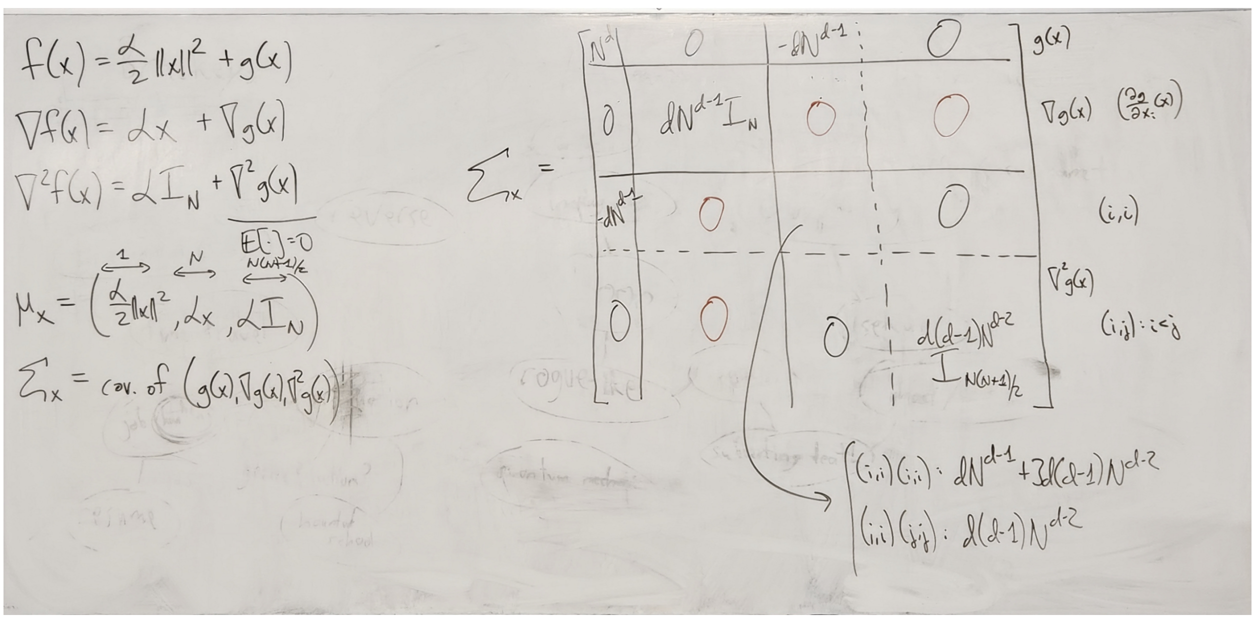

Figure 2: A filled out covariance matrix.