Yale University
S&DS 551, Spring 2023
Homework 2

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Problem 1.

(a)

Our goal is to determine

$$\mathbb{P}\{|Q_{n+1}| = j \mid |Q_n| = i_n, \dots, |Q_1| = i_1\}$$

and show that it only depends on Q_n. We assume that the trajectory has non-zero probability, so that $j \in \{i_n \pm 1\}$ and $i_k \in \{i_{k-1} \pm 1\}$.

Firstly, we notice that if there is some $k \in [n]$ such that $i_k = 0$, we may as well as consider

$$\mathbb{P}\{|Q_{n+1}| = j \mid |Q_n| = i_n, \dots, |Q_{k+1}| = i_{k+1}\}.$$

This is because $|Q_k| = 0 \iff Q_k = 0$. In other words, the trajectory of $|Q_n|$ past k depends only on Q_n, which depends only on $Q_{n-1}, \dots, Q_k$ and Q_k is fixed if we condition on $|Q_k| = 0$. Thus without loss of generality, we may assume that $i_k > 0$ for all $k \in [n]$.

Case I: $j = 0$ In this case, we must have $|Q_n| = 1$ and $Q_n \in \{\pm 1\}$. In either of the two cases, the desired probability is $\frac{1}{2}$.

Case II: $j > 0$ Since we assume that $i_k > 0$ for all $k \in [n]$, there are exactly two trajectories for which $|Q_n| = i_n, \dots, |Q_1| = i_1$. either $Q_1, \dots, Q_n > 0$ or $Q_1, \dots, Q_n < 0$.

Regardless of the two cases, the probability of moving from $Q_n = \pm i_n$ to $Q_{n+1} = \pm j$ is exactly $\frac{1}{2}$ by symmetry.

All in all, $\{|Q_n|\}$ is Markov with transition probabilities

$$\begin{aligned}
P_{0,1} &= 1 \\
P_{i,i+1} &= \frac{1}{2} && i \neq 0 \\
P_{i,i-1} &= \frac{1}{2} && i \neq 0
\end{aligned}$$

(b)

Consider M_{n+1}.

Case I: $M_n = 0$ If $M_n = 0$, then $\max_k Q_k = Q_n$ and either the maximum increases with probability $\frac{1}{2}$, in which case $M_{n+1} = 0$ again, or $Q_{n+1} = Q_n - 1$ with probability $\frac{1}{2}$, in which case $M_{n+1} = 1$.

Case II: $M_n > 0$ If $M_n > 0$, then $Q_n < \max_k Q_k$ so that $Q_{n+1} \leq M_n$. It follows that the maximum remains the same. So $M_{n+1} = M_n \pm 1$, each with probability $\frac{1}{2}$.

All in all, $\{M_n\}$ is indeed a Markov chain with transition probabilities

$$\begin{aligned}
P_{0,0} &= \frac{1}{2} \\
P_{0,1} &= \frac{1}{2} \\
P_{i,i+1} &= \frac{1}{2} && i > 0 \\
P_{i,i-1} &= \frac{1}{2}. && i > 0
\end{aligned}$$

Problem 2.

We claim this is false. Take Q_n to be a simple random walk and $P_n := \max_{0\le k\le n} Q_k - Q_n$. We already know that P_n, Q_n are Markov.

Now,

$$X_n := P_n + Q_n = \max_{0\le k\le n} Q_k$$

cannot be Markov. To see this, we have

$$\mathbb{P}\{X_4 = 2 \mid X_0 = 0, X_1 = 1, X_2 = 1, X_3 = 1\} = \frac{1}{4}$$
$$\mathbb{P}\{X_4 = 2 \mid X_0 = 0, X_1 = 1, X_2 = 2, X_3 = 1\} = 1.$$

The first conditional probability occurs with the Q_n trajectory $0, 1, 0, 1$ or $0, 1, 0, -1$, where only the first trajectory has any chance to arrive at $X_4 = 2$. The second conditional probability occurs with the Q_n trajectory $0, 1, 2, 1$.

Problem 3.

We argue by induction. The base $n = 1$ holds by assumption. Suppose inductively that P^n is stochastic (double stochastic, sub-stochastic) and consider P^{n+1}.

Stochastic We have

$$\begin{aligned}\sum_j P_{ij}^{n+1} &= \sum_j \sum_k P_{ik} \cdot P_{kj}^n \\ &= \sum_k P_{ik} \sum_j P_{kj}^n \\ &= \sum_k P_{ik} \cdot 1 \\ &= 1.\end{aligned}$$

The third inequality holds by the induction hypothesis and the last inequality holds by assumption.

Double Stochastic We have already shown that that the rows sum to 1. It suffices to show that the columns sum to 1. But we can reduce this to the row case. Indeed,

$$\begin{aligned}\sum_i P_{ij}^{n+1} &= \sum_i (P_{ji}^{n+1})^T \\ &= \sum_i (P_{ji}^T)^{n+1} \\ &= 1.\end{aligned}$$

Here we used the fact that P^T is stochastic and so its powers are also stochastic as proven above.

Sub-stochastic Similarly, we have

$$\begin{aligned}\sum_j P_{ij}^{n+1} &= \sum_j \sum_k P_{ik} \cdot P_{kj}^n \\ &= \sum_k P_{ik} \sum_j P_{kj}^n \\ &\leq \sum_k P_{ik} \cdot 1 \\ &\leq 1.\end{aligned}$$

The first inequality holds by the induction hypothesis and the last inequality by assumption.

By induction, we conclude the proofs.

Problem 4.

(a)

Since the state space is finite, there is at least one positive recurrent state. Since $a, b > 0$, we see that our Markov chain is irreducible and thus positive recurrent. Combining these two facts demonstrate that our Markov chain has a unique stationary distribution by our work in class.

We claim that $[b, a]$ is a left 1-eigenvector. Indeed,

$$\begin{aligned}[b,a]P &= [b(1-a)+ab, ab+a(1-b)] \\ &= [b,a].\end{aligned}$$

It follows that

$$\left[\frac{b}{a+b}, \frac{a}{a+b}\right]$$

is the stationary distribution.

This concludes the proof.

(b)

Suppose that $\pi_n = [p, q]$ where $p, q \geq 0$ and $p + q = 1$. We have

$$\begin{aligned}\pi_{n+1}P &= [p(1-a)+qb, pa+q(1-b)] \\ \pi_{n+1}(1) - \pi(1) &= p(1-a) + qb - \frac{b}{a+b} \\ &= \frac{(a+b)p(1-a) + (a+b)(1-p)b - b}{a+b} \\ &= \frac{(a+b)p - b - (a+b)pa + (a+b)b - (a+b)pb}{a+b} \\ &= \frac{[(a+b)p - b][1-a-b]}{a+b} \\ \pi_n(1) - \pi(1) &= p - \frac{b}{a+b} \\ &= \frac{p(a+b) - b}{a+b} \\ \frac{\pi_{n+1}(1) - \pi(1)}{\pi_n(1) - \pi(1)} &= 1 - a - b\end{aligned}$$

as desired.

Problem 5.

We assume the time unit is hours. Let N_t denote the poisson process with parameter $\lambda = 6$, so N_t is the number of customers arrived by time t. Let T_k denote the arrival time of the k-th customer.

(a)

We wish to compute $\mathbb{E}[N_{12}]$. Since $N_{12} \sim \text{Po}(12\lambda) = \text{Po}(72)$, we have

$$\mathbb{E}[N_{12}] = 72.$$

(b)

We wish to determine

$$\mathbb{P}\{N_{0.5} \leq 3\}.$$

Since $N_{0.5} \sim \text{Po}(0.5\lambda) = \text{Po}(3)$, we have

$$\mathbb{P}\{N_{0.5} \leq 3\} = \sum_{n=0}^{3} \frac{e^{-3}3^n}{n!} \approx 0.64723.$$

(c)

We compute

$$\begin{aligned}
&\mathbb{P}\{N_{1/3} = 2, N_{1/2} - N_{1/6} = 2\} \\
&= \mathbb{P}\{N_{1/6} = 2, N_{2/6} - N_{1/6} = 0, N_{3/6} - N_{2/6} = 2\} \\
&\quad + \mathbb{P}\{N_{1/6} = 1, N_{2/6} - N_{1/6} = 1, N_{3/6} - N_{2/6} = 1\} \\
&\quad + \mathbb{P}\{N_{1/6} = 0, N_{2/6} - N_{1/6} = 2, N_{3/6} - N_{1/6} = 0\} \\
&= \mathbb{P}\{N_{1/6} = 2\}^2\mathbb{P}\{N_{1/6} = 0\} + \mathbb{P}\{N_{1/6} = 1\}^3 + \mathbb{P}\{N_{1/6} = 0\}^2\mathbb{P}\{N_{1/6} = 2\} \\
&= \frac{e^{-2}}{4}e^{-1} + e^{-3} + e^{-2}\frac{e^{-1}}{2} && N_{1/6} \sim \text{Po}(1) \\
&= \frac{7}{4}e^{-1} \\
&\approx 0.64378902205.
\end{aligned}$$

(d)

We wish to determine

$$\mathbb{P}\{T_4 \in [1/3, 1/2]\}.$$

From our work in class,

$$\begin{aligned}
\mathbb{P}\{T_4 \in [1/3, 1/2]\} &= \mathbb{P}\{T_4 \leq 1/2\} - \mathbb{P}\{T_4 \leq 1/3\} \\
&= \mathbb{P}\{N_{1/2} \geq 4\} - \mathbb{P}\{N_{1/3} \geq 4\} \\
&\approx 0.3527681112177687412685 - 0.142876539501452951338 \\
&\approx 0.20989157171.
\end{aligned}$$

(e)

We wish to determine

$$\mathbb{P}\{N_{1/2} - N_{1/3} \geq 1 \mid N_{1/3} = 3\}.$$

But by the independence of increments, this is simply

$$\begin{aligned}
\mathbb{P}\{N_{1/2} - N_{1/3} \geq 1\} &= \mathbb{P}\{N_{1/6} \geq 1\} \\
&= 1 - \mathbb{P}\{N_{1/6} = 0\} \\
&= 1 - \exp(-1/6) \\
&\approx 0.15352.
\end{aligned}$$