S&DS 351 / S&DS 551 / MATH 251: Stochastic Processes
Assignment 2

Due: 11:59 PM EST, Tuesday, February 14, 2023

General Instructions

Honor Code: These questions should be completed individually. When submitting the assignment, you agree to adhere to the Yale Honor Code. Please read carefully to understand what it entails!

Submission Instructions: You should submit your answers in a PDF file. LaTeX is highly preferred due to the need of formatting equations. Details of solutions are required for all the problems. No late periods are allowed without communicating with Professor Karbasi in advance.

Submitting Answers: Prepare answers to your homework in a single PDF file. Make sure that the answer to each question is on a separate page. The number of the question should be at the top of each page.

Problem 1

Consider a symmetric simple random walk S with $S_0 = 0$. Let $T = \min\{n \geq 1 : S_n = 0\}$ be the time of the first return of the walk to its starting point. Show that

$$\mathbb{P}(T = 2n) = \frac{1}{2n-1}\binom{2n}{n}2^{-2n},$$

and deduce that $\mathbb{E}[T^\alpha] < \infty$ if and only if $\alpha < \frac{1}{2}$. you may need Stirling's formula: $n! \sim n^{n+\frac{1}{2}}e^{-n}\sqrt{2\pi}$.

Problem 2

Let $\{X_n : n \geq 1\}$ be independent, identically distributed random variables taking integer values. Let $S_0 = 0, S_n = \sum_{i=1}^n X_i$. The *range* R_n of $S_0, S_1, \ldots, S_n$ is the number of distinct values taken by the sequence. Show that $\mathbb{P}(R_n = R_{n-1} + 1) = \mathbb{P}(S_1 S_2 \cdots S_n \neq 0)$, and deduce that, as $n \to \infty$,

$$\frac{1}{n}\mathbb{E}[R_n] \to \mathbb{P}(S_k \neq 0 \text{ for all } k \geq 1).$$

Hence show that, for the simple random walk, $n^{-1}\mathbb{E}[R_n] \to |p - q|$ as $n \to \infty$.

Problem 3

Consider a simple random walk starting at 0 in which each step is to the right with probability p $(= 1 - q)$. Let T_b be the number of steps until the walk first reaches b where $b > 0$. Show that $\mathbb{E}[T_b \mid T_b < \infty] = b/|p - q|$.

Problem 4

Consider a branching process whose family sizes have the geometric mass function $f(k) = qp^k, k \geq 0$, where $p + q = 1$, and let Z_n be the size of the nth generation. Let $T = \min\{n : \mathrm{Z}_n = 0\}$ be the extinction time, and suppose that $\mathrm{Z}_0 = 1$. Find $\mathbb{P}(T = n)$. For what values of p is it the case that $\mathbb{E}[T] < \infty$?

Problem 5

Consider a branching process G_t with $G_0 = 1$ and offspring distribution $\mathsf{Poisson}(2)$. We know that the process will either go extinct or diverge to infinity, and the probability that it is any fixed finite value should converge to 0 as $t \to \infty$. In this exercise you will investigate how fast such probabilities converge to 0. In particular, consider the probability $\mathbb{P}(G_t = 1)$, and find the limiting ratio

$$\lim_{t\to\infty} \frac{\mathbb{P}(G_{t+1} = 1)}{\mathbb{P}(G_t = 1)}.$$