S&DS 351 / S&DS 551 / MATH 251: Stochastic Processes
Assignment 3

Due: 11:59 PM EST, Tuesday, February 28, 2023

General Instructions

Honor Code: These questions should be completed individually. When submitting the assignment, you agree to adhere to the Yale Honor Code. Please read carefully to understand what it entails!

Submission Instructions: You should submit your answers in a PDF file. LaTeX is highly preferred due to the need of formatting equations. Details of solutions are required for all the problems. No late periods are allowed without communicating with Professor Karbasi in advance.

Submitting Answers: Prepare answers to your homework in a single PDF file. Make sure that the answer to each question is on a separate page. The number of the question should be at the top of each page.

Problem 1

(a) Let $\{Q_n : \ n \geqslant 0\}$ be a simple random walk that starts from the origin, and show that $|Q_n|$ defines a Markov chain; then please find out the transition probabilities of this chain.
(b) Let $M_n = \max_{0 \leqslant k \leqslant n} Q_k - Q_n$, then show that M_n also defines a Markov chain, and find out the transition probabilities of this chain.

Problem 2

Let P_n and Q_n be two Markov chains on the set of integers. Is their sum $X_n = P_n + Q_n$ necessarily to be a Markov chain?

Problem 3

For a stochastic matrix $P \in \mathbb{R}^{n\times n}$, it is called doubly stochastic if for $\forall j$, it holds that $\sum_i P_{ij} = 1$. It is called sub-stochastic if for $\forall j$, it holds that $\sum_i P_{ij} \leqslant 1$. Prove that, if P is stochastic (or respectively, double stochastic, sub-stochastic), then for all positive integer n, P^n is stochastic (or respectively, double stochastic, sub-stochastic).e

Problem 4

Consider a general 2-state Markov chain with probability transition matrix $P = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$ where P is not the identity matrix ($ab \neq 0$). We denote the two states as 1 and 2.

(a) Find the stationary distribution π of P and show that it is unique.
(b) Denote the distribution of X_n as π_n, show that the ratio $\frac{\pi_{n+1}(1)-\pi(1)}{\pi_n(1)-\pi(1)}$ is constant in n and five a simple expression for this ratio in terms of a and b. This shows that π_n converges geometrically fast to π and identifies the rate of convergence.

Problem 5

The number of customers coming to a store satisfies the Poisson process with $\lambda = 6$. Given the fact that the store opens at 8am.
(1) The store opens 12 hours a day. What's the expected total number of customers in a day?
(2) What's the probability that there are no more than 3 customers coming before 8:30am?
(3) What's the probability that there are exactly 2 customers coming before 8:20am and there are exactly 2 customers coming between 8:10-8:30 am?
(4) What's the probability that the fourth customer comes between 8:20-8:30am?
(5) Given that there are exactly 3 customers coming before 8:20am, what's the probability that the fourth customer comes before 8:30am?