CPSC 368 / CPSC 516

Instructions:

- **These problems will not be graded, however, you are encouraged to write solutions.**

Problems:

P.1. The goal of this problem is to bound bit complexities of certain quantities related to linear programs. Let $A \in \mathbb{Q}^{m\times n}$ be a matrix and $b \in \mathbb{Q}^m$ be a vector and let L be the bit complexity of (A, b). (Thus, in particular, $L > m$ and $L > n$.) We assume that $K = \{x \in \mathbb{R}^n : Ax \le b\}$ is a bounded, full-dimensional polytope in $\mathbb{R}^n$.

(a) Prove that there is an integer $M \in \mathbb{Z}$ and a matrix $B \in \mathbb{Z}^{m\times n}$ such that $A = \frac{1}{M}B$ and the bit complexities of M and every entry in B are bounded by L.

(b) Let C be any square and invertible submatrix of A. Consider the matrix norm $\|C\|_2 \coloneqq \max_{x\neq 0} \frac{\|Cx\|_2}{\|x\|_2}$. Prove that there exists a constant d such that $\|C\|_2 \le 2^{O(L\cdot[\log(nL)]^d)}$ and $\|C^{-1}\|_2 \le 2^{O(nL\cdot[\log(nL)]^d)}$.

(c) Prove that every vertex of K has coordinates in $\mathbb{Q}$ with bit complexity $O(nL \cdot [\log(nL)]^d)$ for some constant d.

P.2. Recall that an undirected graph $G = (V, E)$ is said to be bipartite if the vertex set V has two disjoint parts L, R and all edges go between L and R. Consider the case when $n \coloneqq |L| = |R|$ and $m \coloneqq |E|$. A perfect matching in such a graph is a set of n edges such that each vertex has exactly one edge incident to it. Let $\mathcal{M}$ denote the set of all perfect matchings in G. Let $1_M \in \{0,1\}^E$ denote the indicator vector of the perfect matching $M \in \mathcal{M}$. Consider the function

$$f(x) \coloneqq \ln \sum_{M\in\mathcal{M}} e^{\langle x, 1_M\rangle}.$$

(a) Prove that f is convex.

(b) Consider the bipartite perfect matching polytope of G defined as

$$P \coloneqq \mathrm{conv}\{1_M : M \in \mathcal{M}\}.$$

Give a polynomial time separation oracle for this polytope.

(c) Prove that, if there is a polynomial time algorithm to evaluate f given the graph G as input, then one can count the number of perfect matchings in G in polynomial time.

Since the problem of computing the number of perfect matchings in a bipartite graph is #**P**-hard, we have an instance of convex optimization that is #**P**-hard.

P.3. Let $\mathcal{S}$ be a nonempty family of subsets of $\{1, 2, \ldots, n\}$. For a set $S \in \mathcal{S}$, let $1_S \in \mathbb{R}^n$ be the indicator vector of S, i.e., $1_S(i) = 1$ if $i \in S$ and $1_S(i) = 0$ otherwise. Consider a function $f : \mathbb{R}^n \to \mathbb{R}$ given by

$$f(x) \coloneqq \ln \sum_{S\in\mathcal{S}} e^{\langle x, 1_S\rangle}.$$

Prove that the gradient of f is L-Lipschitz continuous for some $L > 0$ that depends polynomially on n with respect to the Euclidean norm.