n-step transition prob

$$P_{ij}(0,n) = P_{ij}(m, m+n) = P(X_{m+n} = j \mid X_m = i)$$
$$= P(X_n = j \mid X_0 = i)$$

$$\rightsquigarrow (P^n)_{ij}$$

$$\mu_i^{(n)} = P(X_n = i)$$

which means that the initial distribution $\mu^{(0)}$

$$\mu^{(n)} = \mu^{(0)} P^n$$

Let

$$P_i(A) = P(A \mid X_0 = i)$$

$$E_i(X) = E[X \mid X_0 = i]$$

Def: State i is called recurrent if

$$P_i(X_n = i \text{ for some } n \geq 1) = 1$$

if

$P_i(X_n = i \text{ for some } n \geq 1) < 1$ then

state i is transient

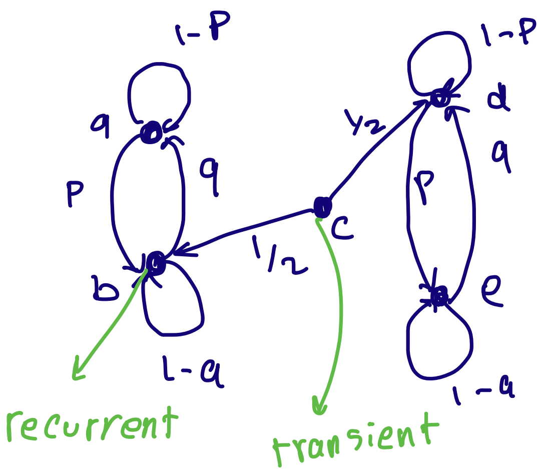
Let

$$f_{ij}(n) = P_i(X_1 \neq j, X_2 \neq j, \dots, X_{n-1} \neq j, X_n = j)$$

So

$$f_{ij} = \sum_{n \geq 1}^{\infty} f_{ij}(n)$$

$\rightsquigarrow$ the prob that the chain visits state j ever if we start from i



Note that a state j is recurrent iff $f_{jj} = 1$

we define the generating functions

$$P_{ij}(s) = \sum_{n=0}^{\infty} s^n p_{ij}(n) \qquad F_{ij}(s) = \sum_{n=0}^{\infty} s^n f_{ij}(n)$$

$$p_{ij}(0) = P(X_0 = j \mid X_0 = i) = \delta_{ij}$$

$$f_{ij}(0) = 0$$

Theorem

a) $P_{ii}(s) = 1 + F_{ii}(s)\, P_{ii}(s)$

b) $P_{ij}(s) = F_{ij}(s)\, P_{jj}(s)$ if $i \neq j$

proof: Let $A_m = \{X_m = j\}$

$B_m = \{X_r \neq j \text{ for } 1 \leq r < m,\ X_m = j\}$

$$P_i(A_m) = \sum_{r \geq 1}^{m} P_i(A_m \cap B_r)$$

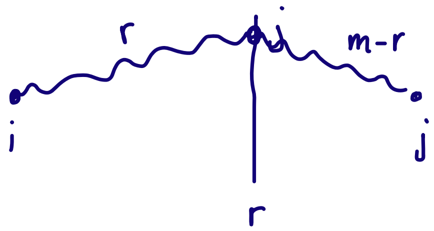
$$= \qquad P(A_m \mid B_r, X_0 = i)\, P_i(B_r)$$

due to Markov

$\|$

$$P(A_m \mid B_r)\, P_i(B_r)$$

$$P_{ij}(m) = \sum_{r \geq 1}^{m} f_{ij}(r)\, P_{jj}(m-r)$$



multiply by s^m and sum over $m \geq 1$ and the property of convolution

$$P_{ij}(s) - \delta_{ij} = F_{ij}(s)\, P_{jj}(s)$$

Lemma 1: State j is recurrent if $\sum_n P_{jj}(n) = \infty$
and if this holds then $\sum_n P_{ij}(n) = \infty$
for all i s.t $f_{ij} > 0$

Let's show that j is recurrent iff $\sum_n P_{jj}(n) = \infty$

$$P_{jj}(s) = \frac{1}{1 - F_{jj}(s)}$$

As $s \uparrow 1$ then $P_{jj}(s) \to \infty$ iff $F_{jj}(1) = f_{jj} = 1$

$$\lim_{s \uparrow 1} P_{jj}(s) = \sum_n P_{jj}(n) \to \infty$$

Lemma 2: if j is transient $\sum_n P_{jj}(n) < \infty$ & if this holds
then $\sum_n P_{ij}(n) < \infty$ for all i.

if j is transient $\lim_{n \to \infty} P_{ij}(n) = 0$

Let $N(i)$: # times we visit state i if we start from i

$$P(N(i) = \infty) = \begin{cases} 1 & \text{if } i \text{ is recurrent} \\ 0 & \text{if } i \text{ is transient} \end{cases}$$

Let

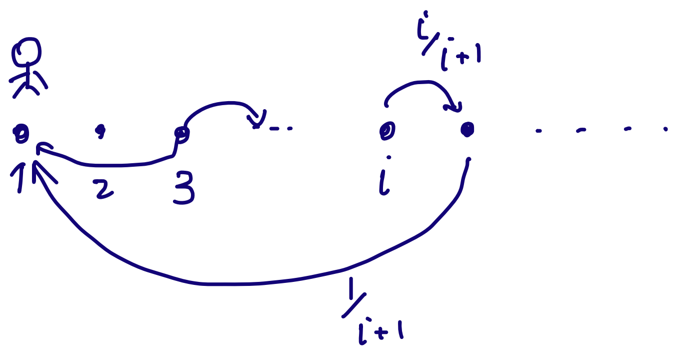
$$T_j = \min\{n \geq 1,\ X_n = j\}$$

& $T_j = \infty$ if this never happens

$P_i(T_i = \infty) > 0$ if i is transient

$$\mu_i = E_i[T_i] = \begin{cases} \sum_n n \cdot f_{ii}(n) & \\ \infty & \text{if } i \text{ is transient} \end{cases}$$

EX:



the prob of not reurning to state 1 after n steps

$$1/2 \times 2/3 \times 3/4 \cdots \times \frac{n}{n+1} = \frac{1}{n+1}$$

what is the average time of getting back to 1

$$\sum_n \frac{1}{n(n+1)} \times n = \sum \frac{1}{n+1} \to \infty$$

Def: for a recurrent state i $\begin{cases} \text{null} & \mu_i = \infty \\ \text{Positive} & \mu_i < \infty \end{cases}$

Theorem: state i is null recurrent iff $P_{ii}(n) \to 0$

The period of a state i, denoted by $d(i)$

is $d(i) = \gcd\{n : P_{ii}(n) > 0\}$

we say i is apriodic if $d(i) = 1$