CPSC 368 / CPSC 516

Assignment due by Tuesday, 31 January 2023, 9 a.m.

Instructions:

- These problem sets are meant to be worked on alone – no groups.
- Each answer should be supported by a rigorous mathematical proof.
- You are strongly encouraged to type the solution in LaTeX.
- You are strongly encouraged to discover the solutions by yourself as that will help you understand the techniques better and will help you do well in the exam. In case you take help of any kind it has to be clearly acknowledged for each problem.

Problems:

P.1. Which of the following is True?

(a) $n^{1/\sqrt{\log n}} \leq (\log n)^{O(1)}$

(b) $n^{\sqrt{n}} \leq 2^{O(n)}$

(c) $2^n \leq n^{O(\log n)}$

P.2. For each of the following functions, compute the gradient and the Hessian, and write the second-order Taylor approximation.

(a) $f(x) = \sum_{i=1}^{m} (a_i^\top x - b_i)^2$, for $x \in \mathbb{R}^n$ where $a_1, \ldots, a_m \in \mathbb{Q}^n$, and $b_1, \ldots, b_m \in \mathbb{Q}$.

(b) $f(x) = \log\left(\sum_{j=1}^{m} e^{\langle x, v_j \rangle}\right)$ where $v_1, \ldots, v_m \in \mathbb{Q}^n$.

(c) $f(X) = \mathrm{Tr}(AX)$ where A is a real-symmetric $n \times n$ matrix and X runs over real-symmetric matrices.

(d) $f(X) = -\log \det X$, where X runs over positive definite matrices.

P.3. Consider a real $m \times n$ matrix A with $n \leq m$ and a vector $b \in \mathbb{R}^m$. Let $p(x) := \arg\min_{x \in \mathbb{R}^n} \|Ax - b\|_2^2$. Assuming that A is of full rank, derive a formula for $p(x)$ in terms of A and b.

P.4. Given an $n \times n$ real PD matrix A and $u \in \mathbb{R}^n$, show that

$$\|u\|_A := \sqrt{u^\top A u}$$

is a norm. What aspect of being a norm for $\|u\|_A$ breaks down when A is just guaranteed to be PSD (and not PD)? And when A has negative eigenvalues?