# Yale University
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# Homework 2

**Chang Feng (Felix) Zhou cz397**

## Problem 1.

First, we note that the random walk $S$ can only return to 0 at an even time step.

We wish to count the number of binary strings of length $2n$ that is *balanced* (equal number of 0s and 1s) subject to the condition that any prefix is not balanced. This corresponds to a random walk which returns to 0 at time $2n$ but not before that.

By symmetry, it suffices to double the number of binary strings on $n$ 1s and $n-1$ 0s such that any prefix contains strictly more 1s than 0s (the 0-th character is prepended as 0 to ensure balance). We claim that there are

$$\frac{1}{2n-1}\binom{2n-1}{n-1}$$

such strings.

Before we prove the claim. Note that if the claim holds, then we are done. This is because the desired probability is then given by

$$\frac{\frac{1}{2n-1}\binom{2n-1}{n-1}\cdot 2}{2^{2n}} = \frac{1}{2n-1}\binom{2n}{n}2^{-2n}$$

as required.

To see the claim, we observe that there is a bijection between the number of binary strings on $n$ 1s and $n-1$ 0s such that any prefix contains strictly more 1s than 0s and the number of binary strings on $n$ 1s and $n$ 0s such that any prefix contains at least as many 1s as 0s. Indeed, the bijection is obtained by prepending a 0 to the $(2n-1)$-bit string. But the cardinality of the latter set is precisely given by the well-known $(n-1)$-th Catalan number

$$C_n = \frac{1}{2n-1}\binom{2n-1}{n-1} = \frac{1}{2n-1}\binom{2n-1}{n}.$$

Finally, we have

$$
\begin{aligned}
\mathbb{E}[T^\alpha] &= \sum_{n=1}^{\infty} \mathbb{P}\{T = 2n\} \cdot (2n)^\alpha \\
&= \sum_{n=1}^{\infty} \frac{(2n)^\alpha}{2n-1} \binom{2n}{n} 2^{-2n} \\
&= \sum_{n=1}^{\infty} \frac{(2n)^\alpha}{2n-1} \frac{(2n)!}{n!n!} 2^{-2n} \\
&\approx \sum_{n=1}^{\infty} \frac{1}{2n-1} \frac{(2n)^{2n+\frac{1}{2}+\alpha} e^{-2n} \sqrt{2\pi}}{\left(n^{n+\frac{1}{2}} e^{-n} \sqrt{2\pi}\right)^2} 2^{-2n} \\
&= \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{1}{2n-1} \frac{(2n)^{2n+\frac{1}{2}+\alpha}}{n^{2n+1}} 2^{-2n} \\
&= \frac{2^{\frac{1}{2}+\alpha}}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{1}{2n-1} n^{\alpha-\frac{1}{2}}.
\end{aligned}
$$

For $\alpha < \frac{1}{2}$, this is series is bounded above by a convergent $p$-series

$$\sum_{n\geq 1} \frac{1}{n^p}$$

for some $p > 1$. On the other hand, for $\alpha \geq \frac{1}{2}$, this series is bounded below by the divergent harmonic series

$$\sum_{n\geq 1} \frac{1}{n}.$$

This concludes the proof.

**Problem 2.**

**Problem 3.**

**Problem 4.**

**Problem 5.**