# CPSC 368 / CPSC 516

## Assignment due by Tuesday, 14 February 2023, 9 a.m.

**Instructions:**

- These problem sets are meant to be worked on alone – no groups.
- Each answer should be supported by a rigorous mathematical proof.
- You are strongly encouraged to type the solution in LaTeX.
- You are strongly encouraged to discover the solutions by yourself as that will help you understand the techniques better and will help you do well in the exam. In case you take help of any kind it has to be clearly acknowledged for each problem.

**Problems:**

P.1. Consider the generalized negative entropy function $f(x) = \sum_{i=1}^{n} x_i \log x_i - x_i$ over $\mathbb{R}^n_{>0}$.

(a) Write the gradient and Hessian of $f$.

(b) Prove $f$ is strictly convex.

(c) Prove that $f$ is not strongly convex with respect to the $\ell_2$-norm.

(d) Write the Bregman divergence $D_f$. Is $D_f(x, y) = D_f(y, x)$ for all $x, y \in \mathbb{R}^n_{>0}$?

(e) Prove that $f$ is 1-strongly convex with respect to $\ell_1$-norm when restricted to points in the subdomain $\{x \in \mathbb{R}^n_{>0} : \sum_{i=1}^{n} x_i = 1\}$.

P.2. Consider the following subset of $\mathbb{R}^n$

$$P \coloneqq \{x \in \mathbb{R}^n \colon\ |\langle a_i, x\rangle| \leq 1 \quad \text{for } i = 1, 2, \ldots, m\},$$

where $a_1, a_2, \ldots, a_m \in \mathbb{R}^n$ are vectors.

Let $d$ denote the dimension of the linear subspace of $\mathbb{R}^n$ spanned by $a_1, a_2, \ldots, a_m$. We define a function

$$F(x) \coloneqq -\sum_{i=1}^{m} \log(1 - (\langle a_i, x\rangle)^2)$$

for all $x \in \mathbb{R}^n$ where the above formula makes sense and set $F(x) = +\infty$ otherwise.

(a) Prove that the set $P$ is bounded[1] if and only if $d = n$.

(b) Compute the gradient $g(x)$ and the Hessian $H(x)$ of $F$.

(c) Prove that $F$ is a convex function. What is the domain of $F$ (the set of points where $F$ is finite)?

(d) What is the global minimum of $F$? (Assume $d = n$.)

(e) For any $x$ in the domain of $F$ define $\mathcal{E}_x \coloneqq \{h \in \mathbb{R}^n \colon h^\top H(x)h \leq 1\}$. Prove that $\mathcal{E}_x$ is a convex set and that $\mathcal{E}_x \subseteq P$.

[1] A set $K \subseteq \mathbb{R}^n$ is called bounded if there exists an $r > 0$ such that $\|x\| \leq r$ for every $x \in K$.