S&DS 351 / S&DS 551 / MATH 251: Stochastic Processes
Assignment 2: Extra Hints for P2, P3 and P4

Due: 11:59 PM EST, Tuesday, February 14, 2023

Problem 2 We need to notice the following fact: if the following limit exists:

$$t := \lim_{n\to\infty} (a_{n+1} - a_n).$$

Then, it holds that:

$$\lim_{n\to\infty} \frac{a_n}{n} = t.$$

Actually, you can directly use this theorem without proof in your homework.

Problem 3 In this problem, we need to use one of the results in our course material:

$$\mathbb{P}(T_b = n) = \frac{b}{n}\mathbb{P}(S_n = b).$$

Another question is how to make use of $T_b < \infty$. Actually, we need to notice that:

$$\mathbb{E}(N_b | T < \infty) = 1 + \mathbb{E}(N_0)$$

where N_0 is the total number of visitation of the origin (with the trivial visit at time 0 excluded). Please figure out why.

Problem 4 A very tough mathematical question in this problem is that: how to obtain the general formula of a sequence with the following recursive relation:

$$a_{n+1} = \frac{q}{1 - pa_n}.$$

Actually, we can use a simple mathematical trick here. Notice that:

$$a_{n+1} - 1 = \frac{q}{1 - pa_n} - 1 = \frac{p \cdot (a_n - 1)}{1 - pa_n}, \tag{1}$$

and

$$a_{n+1} - \frac{q}{p} = \frac{q}{1 - pa_n} - \frac{q}{p} = \frac{pq \cdot (a_n - q/p)}{p(1 - pa_n)}. \tag{2}$$

Now we divide Equation (1) by Equation (2), and get:

$$\frac{a_{n+1} - 1}{a_{n+1} - q/p} = \frac{p}{q} \cdot \frac{a_n - 1}{a_n - q/p}.$$

Denote $u_n := \frac{a_n - 1}{a_n - q/p}$, then we know that $u_{n+1} = \frac{p}{q} \cdot u_n$, which means $\{u_n\}$ is a geometric series. We can easily write down the general expression of u_n, which leads to the expression of a_n.