probability space (Ω, F, P)

Events

sample space

prob measure

$w \in \Omega$

F: set of subsets of Ω

1) $\Omega \in F$

2) $A \in F \Rightarrow A^c \in F$

3) $A, B \in F \Rightarrow A \cup B \in F$ or more generally $A_1, A_2, \ldots \in F$

$$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in F$$

For prob. measure P

- $\forall A \in F$, $P(A) \geq 0$
- $P(\Omega) = 1$
- If $A \cap B = \phi \Rightarrow P(A \cup B) = P(A) + P(B)$

if $A_1, A_2, \ldots$ mutually exclusive

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Continuity of prob: suppose $B_1, B_2, \ldots$ is a sequence of event

- If $B_1 \subset B_2 \subset B_3 \subset \cdots \Rightarrow \lim_{j \to \infty} P(B_j) = P\left(\bigcup_{i=1}^{\infty} B_i\right)$
- if $B_1 \supset B_2 \supset B_3 \cdots \Rightarrow \lim_{j \to \infty} P(B_j) = P\left(\bigcap_{i=1}^{\infty} B_i\right)$

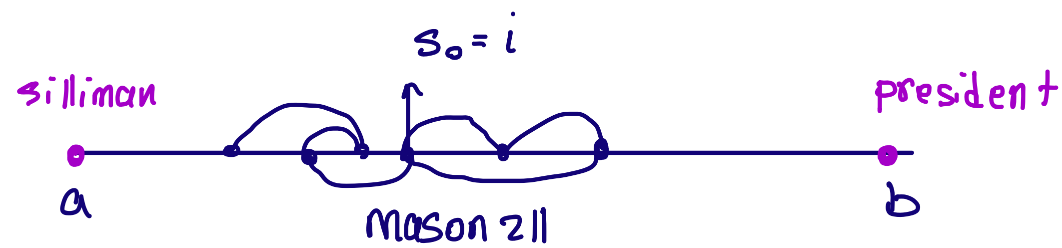
A Random/Stochastic process is an indexed collection of random variables

$$X = (X_t, t \in T)$$

$\xrightarrow{}$ defined on the same prob. space $(\Omega, \mathcal{F}, P)$

- If $T = \mathbb{Z} \Rightarrow X$ is called a discrete process
- If $T = \mathbb{R} \Rightarrow X$ is called a cont. process

Example: Drunk on Hillhouse

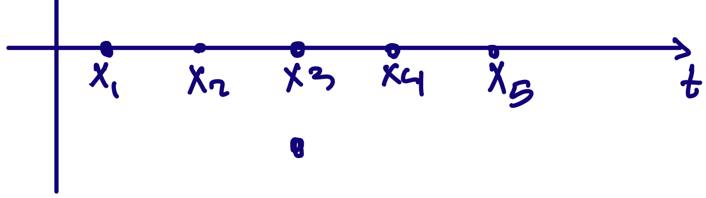


- start from S_0
- In the k-step, you move by an amount X_k

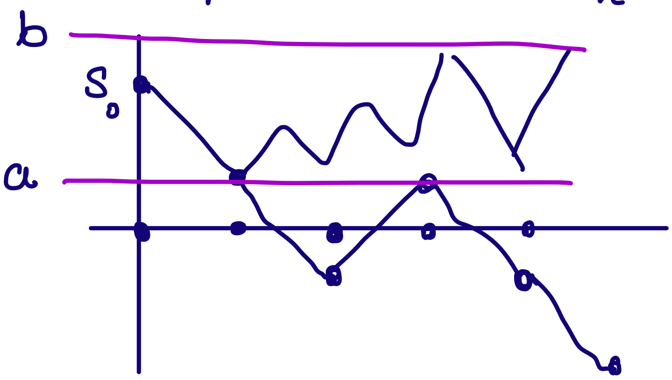
$$S_n = S_0 + X_1 + X_2 + \dots + X_n$$

Assume that $P(X_i = +1) = P(X_i = -1) = \frac{1}{2}$

one process $X = (X_t, t \in \mathbb{N})$

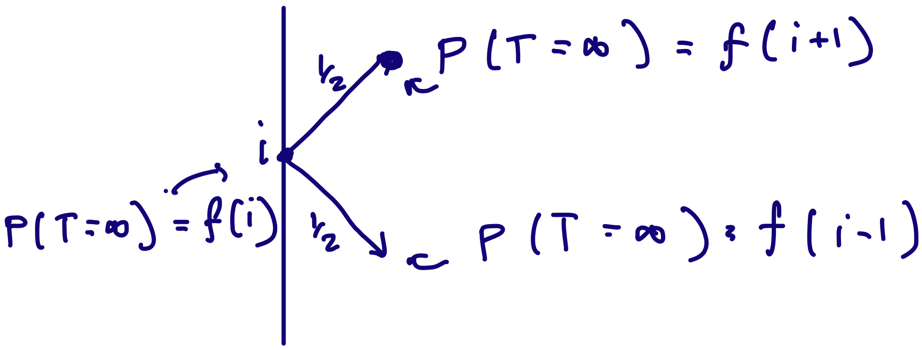


the other process is S_n



Let $\{a, b\}$ be the boundaries

$$f(i) = P(T = \infty \mid S_0 = i) \qquad a \leq i \leq b$$



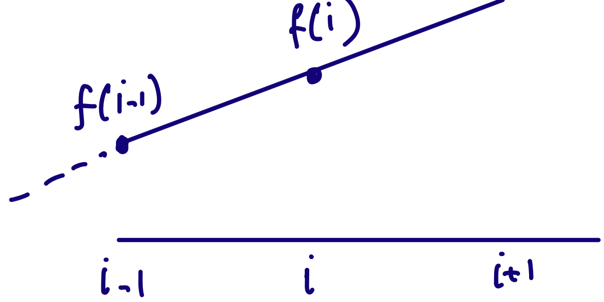
$$\boxed{P(B) = \sum_{A_i} P(B \mid A_i)\, P(A_i)}$$

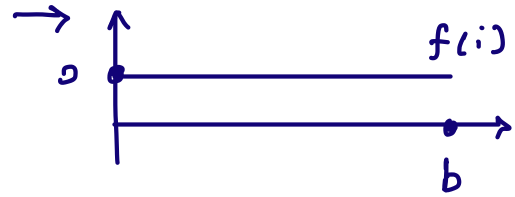
$$f(i) = P(T = \infty \mid S_1 = i+1, S_0 = i)\, P(S_1 = i+1 \mid S_0 = i)$$
$$+ P(T = \infty \mid S_1 = i-1, S_0 = i)\, P(S_1 = i-1 \mid S_0 = i)$$

$$\Rightarrow f(i) = \frac{1}{2} f(i+1) + \frac{1}{2} f(i-1) \qquad (1)$$

$$f(a) = f(b) = 0 \qquad (2)$$

$f(i+1)$.





we know that S_T will be either a or b

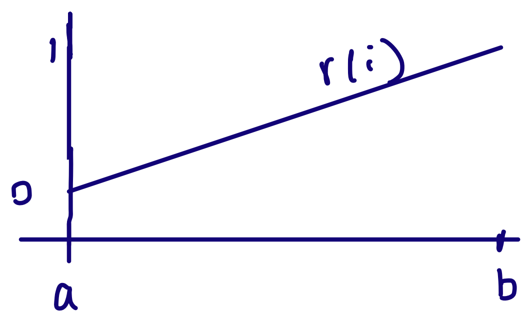
$$r(i) = pr(S_T = b \mid S_0 = i)$$

with the same argument

$$r(i) = \frac{1}{2} r(i+1) + \frac{1}{2} r(i-1) \Rightarrow i \to r(i) \text{ is a straight line}$$

$r(a) = 0$

$r(b) = 1$



$$p(S_T = b \mid S_0 = i) = \frac{i-a}{b-a} \qquad a \leq i \leq b$$