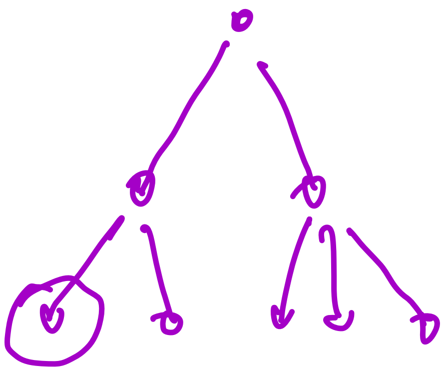
Branching Processes

Chapter 8 - RrH text

Galton - Watson process



$X_0 = 1$

$X_1 = 2$

$X_2 = 5$

X_n = # descendants in n^{th} generation

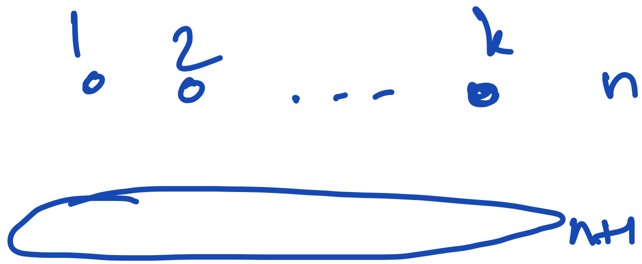
$p_k \rightarrow$ probability that a person has k children

$p_0 + p_1 + \cdots = 1$

$$\boxed{\mathbb{E}(X_n) = ?}$$

μ is the expected no. of children an individual has

$$\mu = \sum_{k=0}^{\infty} k p_k$$



Law of Total Prob.

$$\mathbb{E}(X_{n+1}) = \sum_{k=0}^{\infty} \underbrace{\mathbb{E}(X_{n+1} \mid X_n = k)}_{} \mathbb{P}(X_n = k)$$

$$= \sum_{k=0}^{\infty} k\mu \, \mathbb{P}(X_n = k) = \mu \, \mathbb{E}(X_n)$$

$\mathbb{E}(X_{n+1}) = \mu\, \mathbb{E}(X_n)$ $\qquad$ $X_0 = 1$

$$\boxed{\mathbb{E}(X_n) = \mu^n}$$

$\mu < 1$, $\mathbb{E}(X_n) \to 0$

$\mu > 1$, $\mathbb{E}(X_n) \to \infty$

$\mu = 1$, $\mathbb{E}(X_n) = 1$

Expected Population vs Extinction Probability

Slightly Contrived Example

$\{X_n\}_{n=1}^{\infty}$ $\qquad$ $\left(\mathbb{P}(X_n = 0) = \mathbb{P}(X_n = 2^n) = \frac{1}{2} \right)$

$\mathbb{E}(X_n) \to \infty$ but $\mathbb{P}(X_n = 0) = \frac{1}{2}$

Extinction Probability

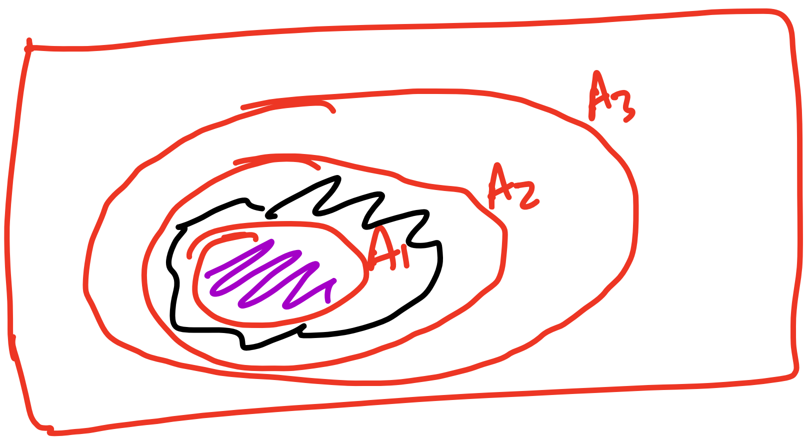
$$\eta := \mathbb{P}\left(\lim_{n\to\infty} X_n = 0\right), \quad \eta_n = \mathbb{P}(X_n = 0)$$

$$\eta_0 = \mathbb{P}(X_0 = 0) = 0$$

$$\left(\{X_n = 0\} \subset \{X_{n+1} = 0\}\right) \text{ so } \eta_1 \le \eta_2 \le \eta_3 \ldots$$

So $\underbrace{\eta = \mathbb{P}\left(\lim_{n\to\infty} X_n = 0\right)} \stackrel{\checkmark}{=} \underbrace{\lim_{n\to\infty} \mathbb{P}\left(\underline{X_n = 0}\right)} = \lim_{n\to\infty} \eta_n$

Quick Pf



$A_1 \subset A_2 \subset A_3 \cdots$

$\bigcup_n A_n = \lim_{n\to\infty} A_n$

$B_1 = A_1, \quad B_2 = A_2 \setminus A_1, \quad B_3 = A_3 \setminus A_2, \ldots \quad B_n = A_n \setminus A_{n-1}$

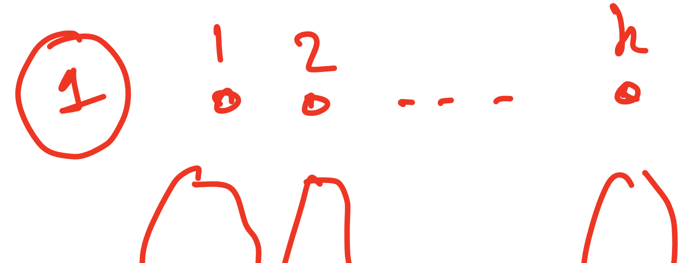
$\mathbb{P}\left(\lim_n A_n\right) = \mathbb{P}\left(\bigcup_n A_n\right) = \mathbb{P}\left(\bigcup_n B_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(B_n)$

$$= \underbrace{\lim_{n\to\infty} \mathbb{P}(A_n)}$$

$$\eta = \lim_{n\to\infty} \eta_n \qquad \eta_n = \mathbb{P}(X_n = 0)$$

$$\eta_{n+1} = \mathbb{P}(X_{n+1}=0) = \sum_{k=0}^{\infty} \underbrace{\mathbb{P}(X_{n+1}=0 \mid X_1 = k)}\,\mathbb{P}(X_1 = k)$$

$$\{X_{n+1} = 0 \mid X_1 = k\}$$



(within n generations, the line of descendants from k ppl must all go extinct)

$$\mathbb{P}(X_{n+1}=0 \mid X_1 = k) = \prod_{i=1}^{k} P(X_n = 0) = \eta_n^{k}$$

$$\eta_{n+1} = \sum_{k=0}^{\infty} \eta_n^{k} p_k$$

$$\eta_3 = g(g(g(0)))$$

Let $g(x) = \sum_{k=0}^{\infty} x^k p_k$

$$\left[\eta_0 = 0, \quad \eta_{n+1} = g(\eta_n) \quad n \geq 0\right]$$

$$\eta = \lim_{n\to\infty} \eta_n = \lim_{n\to\infty} g(\eta_{n-1}) = g\left(\lim_{n\to\infty} \eta_{n-1}\right) = g(\eta)$$

$$\eta = g(\eta)$$

$$x = g(x) \qquad x \in [0,1]$$

$f_1(x) = x$ ✓

$f_2(x) = g(x)$ ✓

$$g(x) = \sum^{\infty} x^k p_k$$

$k=0$

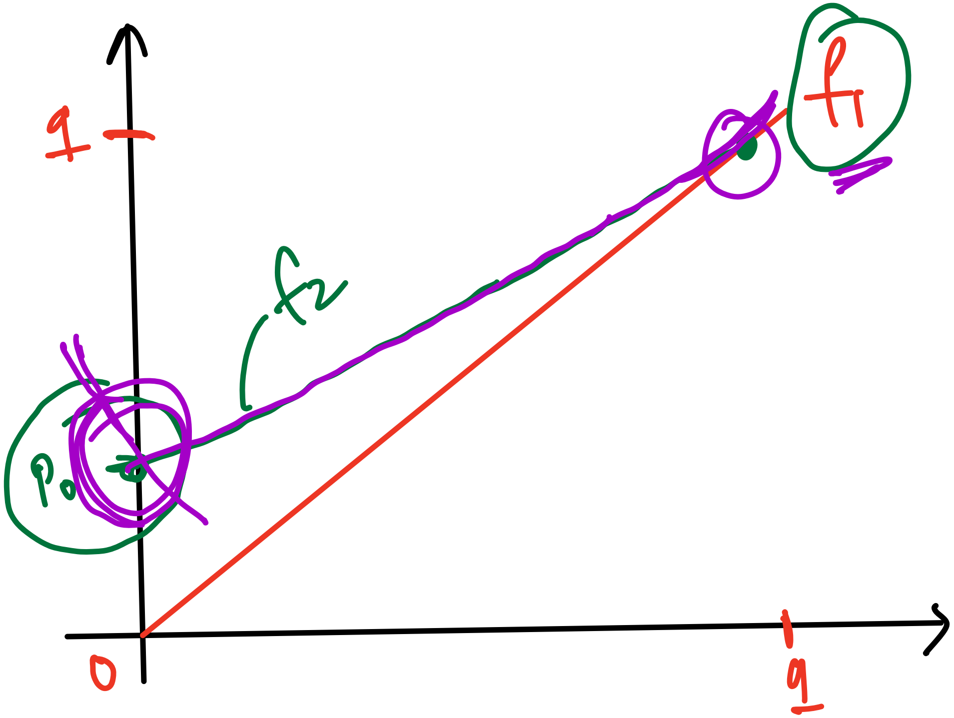
$$g(1) = \sum_{k=0}^{\infty} p_k = 1 \quad , \quad g(0) = p_0$$

$$g'(x) = \sum_{k=1}^{\infty} k x^{k-1} p_k \quad , \quad g'(1) = \mu$$

$\mu = 1$

$$g''(x) = \sum_{k=2}^{\infty} k(k-1) x^{k-2} p_k \geq 0$$

g is cvx



$f_1'(x) = 1$

Say $\mu \leq 1$

$\mu = 1$

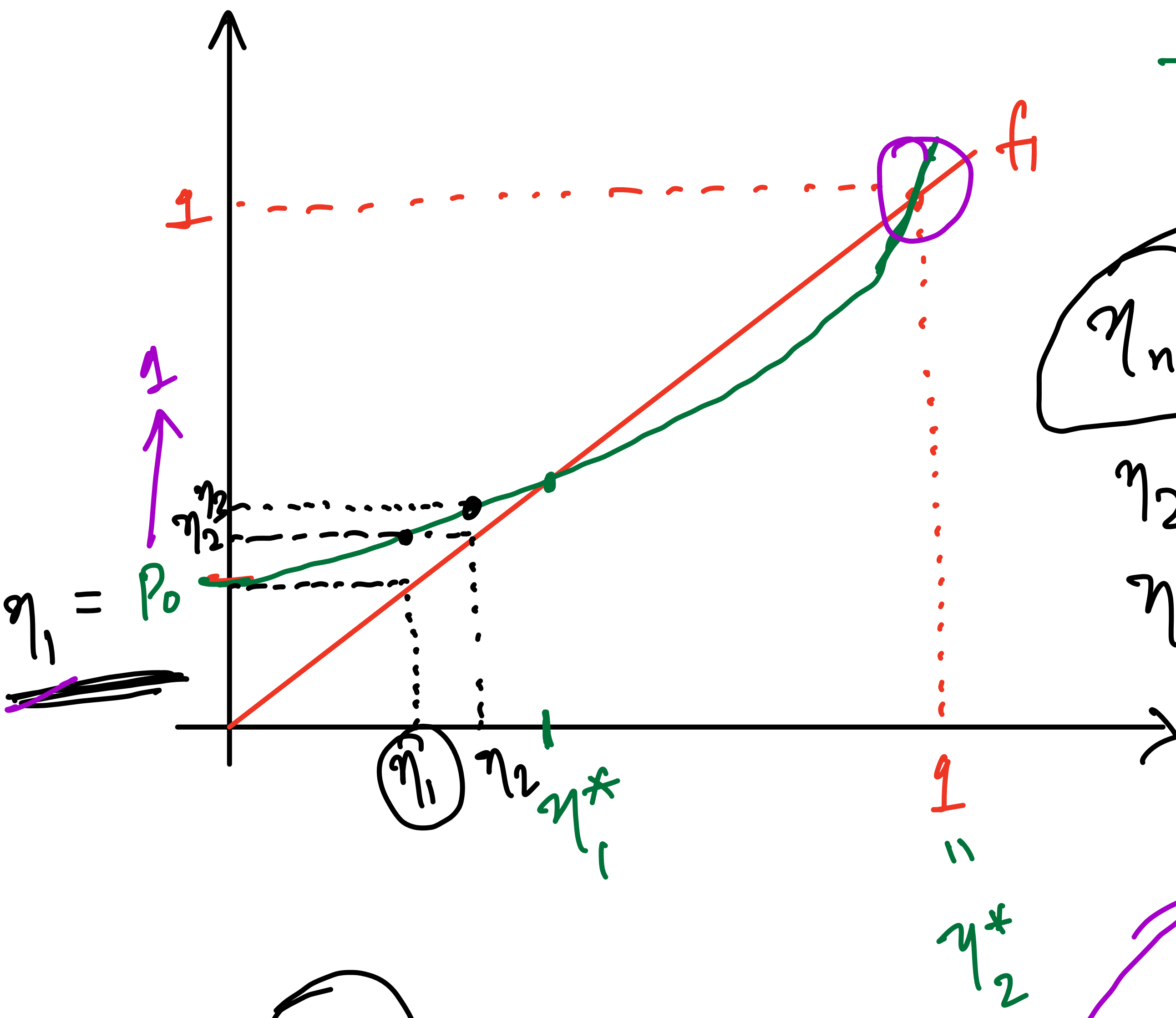
$f_2(x) = f_1(x)$

$x = 1$

$\mu \leq 1$

$\eta = 1$

$\mu > 1$



$\eta_{n+1} = g(\eta_n)$

$\eta_2 = g(\eta_1)$

$\eta_3 = g(\eta_2)$

η_1^*

$\|\eta_m - \eta_{m+1}\| \le \varepsilon$

$\eta_1 = P_0$

$\eta_{k+1} = g(\eta_k)$

when $\mu > 1$, the smaller solution to $x = g(x)$ is the extinction prob.

✓ Extinction Probability in varion regimes