

# S&DS 551 Midterm Cheat Sheet

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## Random Walks

- $S_n : +S_0 + \sum_{j=1}^n X_j$
- $T$  stopping time boundary  $a < b$
- $\mathbb{P}\{T = \infty \mid S_0 = i\} = 0$
- $\mathbb{P}\{S_T = b \mid S_0 = i\} = \frac{i-a}{b-a}$
- $\mathbb{E}[T \mid S_0 = i] = (i-a)(b-i)$
- $T_a := \min\{n : S_n = a\}$
- For  $0 = a < b$ ,  $\{S_T = 0\} \subseteq \{T_0 < \infty\}$
- $\mathbb{P}\{T_0 < \infty \mid S_0 = i\} \geq \mathbb{P}\{S_T = 0 \mid S_0 = i\} = 1 - \frac{i}{b} \rightarrow 1$
- $\mathbb{E}[T_0 \mid S_0 = i] = \lim_{b \rightarrow \infty} \mathbb{E}[T \mid S_0 = i] \rightarrow \infty \cdot \delta_{i,0}$

## Branching Processes

- $X_0 = 1, X_{n+1} = \sum_{i=1}^{X_n} X_{n,i}$  where  $X_{n,i} : \Omega \rightarrow \mathbb{N}$  is iid
- $\mathbb{E}[X_{n+1}] = \mu, \mathbb{E}[X_n] = \mu^{n+1}, \mu = \mathbb{E}[X_{n,i}]$
- $\eta_n := \mathbb{P}\{X_n = 0\}$
- $\eta_{n+1} = \sum_{k \geq 0} p_k \eta_n^k =: g(\eta_n)$
- $\eta = g(\eta)$  fixed point
- $g(1) = 1, g'(1) = \mu, g''(x) \geq 0$

## Poisson Processes

- $Z \sim \text{Po}(\lambda)$
- $\mathbb{P}\{Z = k\} = e^{-\lambda} \lambda^k / k!$
- $\mathbb{E}[Z] = \lambda, \text{Var}[Z] = \lambda$
- *Counting Process*:
  - 1)  $N_0 = 0$ , 2)  $N_t$  increasing in  $t$ , 3) Increase at most 1
- *Poisson Process*
  - 1)  $N_t \sim \text{Po}(\lambda t)$ , 2) independent increments, 3)  $N_t - N_s \stackrel{d}{=} N_{t-s}$
- $T_k$ :  $k$ -th arrival time

- $N_t$ : count at time  $t$
- $\{T_k \leq t\} \equiv \{N_t \geq k\}$
- $T_k \sim \Gamma(k, \lambda)$
- $\mathbb{E}[T_k] = k/\lambda, \text{Var}[T_k] = k/\lambda^2$
- $N_t^{(i)}$  iid Poisson process with params  $\lambda_i$ ,  
 $\sum_i N_t^{(i)}$  Poisson process with param  $\sum_i \lambda_i$
- $R_t$  Poisson process rate  $\lambda$ ,  
subsample  $k$ -th arrival with prob  $p \in \Delta^k$ ,  
 $R_t^{(1)}, \dots, R_t^{(k)}$  ind. Poisson process with params  $p_i \lambda$
- $X \sim \text{Exp}(\mu), Y \sim \text{Exp}(\gamma)$  ind.,  $\mathbb{P}\{X \wedge Y = X\} = \mu/(\mu + \gamma)$
- $N_t$  Poisson process param  $\lambda(t)$  if
  - 1) independent increments, 2)  $N_t - N_s \sim \text{Po}\left(\int_s^t \lambda(u) du\right)$  for  $t \geq s$

## Markov Chains

- *Time-homogeneous Markov chain*,  
 $\mathbb{P}\{X_{n+1} = j \mid \forall 0 \leq k \leq n, X_k = i_k\}$   
 $= \mathbb{P}\{X_{n+1} = j \mid X_n = i_n\}$
- thm:  $P_{ij}(s) = \delta_{ij} + F_{ij}(s)P_{ij}(s)$
- *recurrent state*  $i$  if  $\mathbb{P}_i\{\exists n \geq 1, X_n = i\} = 1$
- lem: If  $\sum_n P_{jj}(n) = \infty$ ,
  - a)  $j$  is recurrent, b)  $\sum_n P_{ij}(n) = \infty$  for all  $i$   $f_{ij} > 0$
- *transient state*  $i$  if  $\mathbb{P}_i\{\exists n \geq 1, X_n = i\} < 1$  - lem: If  $\sum_n P_{jj}(n) < \infty$ ,
  - a)  $j$  is transient, b)  $\sum_n P_{ij}(n) < \infty$  for all  $i$
- If  $j$  transient,  $\lim_n P_{ij}(n) = 0$  for all  $i$
- recurrent  $j$  *null-recurrent* if  $\mathbb{E}_i \min\{n \geq 1 : X_n = j\} = \infty$ ,  
else *positive-recurrent*
- $j$  null-recurrent  $\iff P_{ii}(n) \rightarrow 0$
- *period* state  $j$ :  $d(j) = \gcd\{n : P_{jj}(n) > 0\}$
- $j$  aperiodic if  $d(j) = 1$
- $i$  communicates with  $j$ ,  $i \rightarrow j$  if  $\exists m, P_{ij}(m) > 0$
- thm: If  $i \leftrightarrow j$ , 1)  $d(i) = d(j)$ , 2)  $i$  transient  $\iff j$  3)  $i$  null-recurrent  $\iff j$
- $C \subseteq S$  is *closed* if cannot transition out,

- *irreducible* if  $i \leftrightarrow j \forall i, j \in C$  - lem: If  $|S| < \infty$ , a)  $\geq 1$  recurrent state, b) all recurrent states are positive
- thm: Irreducible MC has stationary distribution  $\iff$  all states positive recurrent, then  $\pi_i = 1/\mu_i$  uniquely (mean recurrent time)
- thm: Irreducible, aperiodic MC satisfies  $P_{ij}(n) \rightarrow 1/\mu_j$
- thm: Irreducible, finite, aperiodic MC has  $\lambda_1 = 1$  and  $|\lambda_r| < 1$  for all  $r > 1$
- $\forall T \subseteq S, \sum_{i \in T, j \notin T} \pi_i P_{ij} = \sum_{i \in T, j \notin T} \pi_j P_{ji}$
- Irreducible, aperiodic, positive-recurrent MC is *ergodic*
- thm: RW on  $G$  aperiodic  $\iff G$  not bipartite
- thm: RW on conneted non-bipartite graph  $\rightarrow \pi_v = d(v)/(2|E|)$
- $X_n$  Irreducible, positive-recurrent MC, stationary  $\pi, X_0 \sim \pi$ , *reversed chain*  $Y_n := X_{N-m}$
- $Y_n$  is MC with transition  $Q_{ij} = \pi_j P_{ji} / \pi_i$
- $X_n$  *reversible* if  $\pi_j P_{ji} = \pi_i P_{ij}$
- $X_n$  reversible wrt  $\pi$  then  $\pi$  stationary distribution

## MCMC

test

## Useful Facts

- $\sum_{i=1}^n i = n(n+1)/2$
- $\sum_{i=1}^n i^2 = (n+1)(2n+1)/6$
- For  $N : \Omega \rightarrow \mathbb{N}$ ,
- $\mathbb{E}[N] = \sum_{n \geq 1} \mathbb{P}\{N \geq n\}$
- $\mathbb{E}[N^2] = \sum_{n \geq 1} \mathbb{P}\{N \geq n\} [n^2 - (n-1)^2]$
- Catalan number  $C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{(n+1)!n!} = \prod_{k=2}^n \frac{n+k}{k}$
- *Convolution* of seq  $a_n, b_n$  is  $c_n := \sum_{i=0}^n a_i b_{n-i}$   
generating func  $G_c(s) = G_a(s) \cdot G_b(s)$
- Generating func of  $X : \Omega \rightarrow \mathbb{N}$ ,  
 $G_X(s) := \mathbb{E}[s^X] = \sum_{i \geq 0} \mathbb{P}\{X = i\} s^i$
- Sum of MC not necessarily a MC