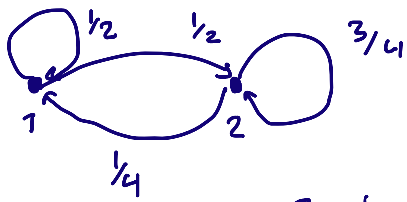


$$f_{11}(n) = \begin{cases} P_{11} = \frac{1}{2} & n = 1 \\ P_{12}\, P_{22}^{\,n-2}\, P_{21} = \frac{1}{2}\left(\frac{3}{4}\right)^{n-2} \frac{1}{4} & n \geq 2 \end{cases}$$

$$\mu_1 = \sum n\, f_{11}(n) = \frac{1}{2} + \sum_{n \geq 2} n \cdot \frac{1}{2}\left(\frac{3}{4}\right)^{n-2} \times \frac{1}{4} = \underline{\underline{3}}$$

$$\mu_2 = \frac{3}{2}$$

Stationary distribution & limit theorem

X_n as $n \to \infty$

The vector π is called the stationary dist of the chain if the entries of π sastify the following properties

a) $\pi_j \geq 0$ & $\sum_j \pi_j = 1$

b) $\pi = \pi \cdot P$ or in other words $\pi_j = \sum_i \pi_i \cdot P_{ij}$

$$\pi = \pi \cdot P$$

$$\pi \cdot P^2 = (\pi \cdot P) \cdot P = \pi \cdot P = \pi$$

$$\pi \cdot P^n = \pi$$

* If X_0 has distribution $\pi \Rightarrow X_n$ has distribution π

Theorem: An irreducible chain has a stationary dist. iff all the states are positive recurrent. In that case, π is unique & is given by $\pi_i = \frac{1}{\mu_i}$ where μ_i is mean recurrent time of state i.

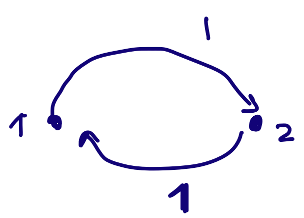
$$P = \begin{pmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{pmatrix} \qquad \pi = \pi \cdot P$$

$$(\pi_1, \pi_2) = (\pi_1, \pi_2) \begin{pmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{pmatrix}$$

$$- \; \pi_1 = \frac{1}{2}\pi_1 + \frac{1}{4}\pi_2$$

$$\pi_2 = \frac{1}{2}\pi_1 + \frac{3}{4}\pi_2 \quad \Rightarrow \quad \pi = (\underbrace{1/3}_{\frac{1}{\mu_1}}, 2/3)$$

$$- \; \pi_1 + \pi_2 = 1$$



$$P_{11}(n) = \begin{cases} 0 & n \text{ is odd} \\ 1 & n \text{ is even} \end{cases}$$

$$\pi = (1/2, 1/2)$$

Even though π exists, $P_{11}(n)$ does not converge a $n \to \infty$

MC limit theorem: For an irreducible apriodic chain, we have

$$P_{ij}(n) \longrightarrow \frac{1}{\mu_j} \quad \text{as} \quad n \to \infty$$

① for both transient and null-recurrent states $\mu_j = \infty$

so, $P_{ij}(n) \to 0$

② $P_{ij}(n)$ does not depend on the starting point $X_0 = i$

③ If the chain is positive recurrent then

$$P_{ij}(n) \longrightarrow \pi_j = \frac{1}{\mu_j}$$

④ If $X = \{X_n\}$ is an irreducible MC with period d, then

$$Y = \{Y_n = X_{nd} : n \geq 0\} \text{ is apiodic}$$

$$P_{jj}(nd) = P_j(Y_n = j) \longrightarrow \frac{d}{\mu_j}$$

Let's try to see what happens when MC is irreducible, finite, & apriodic.

$$\pi = \pi \cdot P$$

Theorem: Let P be the transition matrix of the chain

- $\lambda_1 = 1 \Rightarrow$ corresponding eigenvector π
- $|\lambda_r| < 1$

Let $\lambda_1, \dots, \lambda_N$ ($N = |S|$) be all distinct eigenvalues with the corresponding eigen vectors π_r

$$P = \underbrace{\begin{pmatrix} | & & | \\ \pi_1 & \cdots & \pi_N \\ | & & | \end{pmatrix}}_{B^{-1}} \underbrace{\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix}}_{\Lambda} \underbrace{\begin{pmatrix} - & \pi_1 & - \\ - & \pi_2 & - \\ & \vdots & \\ - & \pi_N & - \end{pmatrix}}_{B}$$

$$P^n = B^{-1} \Lambda^n B = B^{-1} \begin{pmatrix} \lambda_1^n & & \\ & \ddots & \\ & & \lambda_N^n \end{pmatrix} B$$

$$P^n \to B^{-1} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & \cdots & 0 \end{pmatrix} B$$

$$= \begin{pmatrix} - & \pi & - \\ - & \pi & - \\ & \vdots & \\ - & \pi & - \end{pmatrix}$$

$$P_{ij}(n) \to \pi_j \quad \text{as} \quad n \to \infty$$

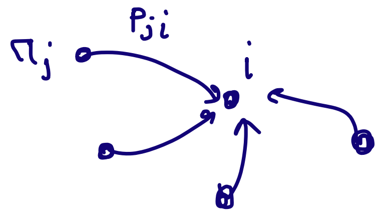
$$P_{ij}(n) = (P^n)_{ij}$$

Let's assume that π is the stationary distribution

For any state i of the chain

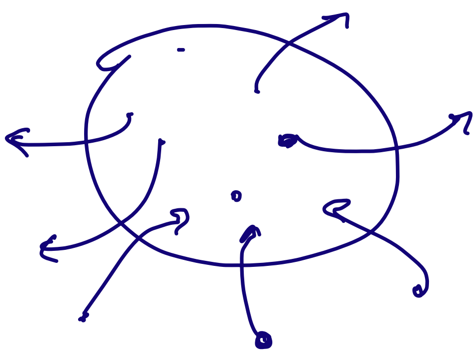
$$\sum_j \pi_j P_{ji} = \pi_i = \pi_i \sum_j P_{ij}$$

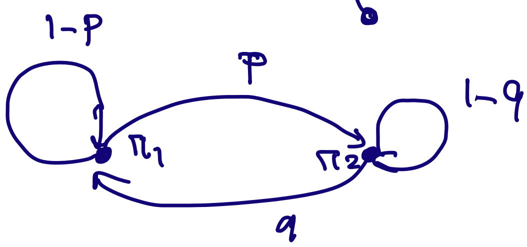
or



$$\sum_{j \neq i} \pi_j P_{j,i} = \sum_{j \neq i} \pi_i P_{ij}$$

Let S be a set of states of a finite, irreducible a priodic MC. In the stationary distribution, the prob that the chain leaves the set S is equal to the probability to enter S





$$\pi_1 \times p = \pi_2 \cdot q \qquad \oplus \qquad \pi_1 + \pi_2 = 1$$

$$\pi_1 = \frac{q}{p+q} \quad \& \quad \pi_2 = \frac{p}{p+q}$$