S&DS 351 / S&DS 551 / MATH 251: Stochastic Processes
Assignment 3

Due: 11:59 PM EST, Tuesday, February 28, 2023

Problem 1

(a) Let $\{Q_n : n \geqslant 0\}$ be a simple random walk that starts from the origin, and show that $|Q_n|$ defines a Markov chain; then please find out the transition probabilities of this chain.

(b) Let $M_n = \max_{0 \leqslant k \leqslant n} Q_k - Q_n$, then show that M_n also defines a Markov chain, and find out the transition probabilities of this chain.

(a) Since Q_n is a simple random walk, we know that we may express it in the form $Q_n = \sum_{i=1}^{n} X_i$, where $X_i \sim -1 + 2 \times \text{Bern}(p)$ for some p, and all X_i are i.i.d. Consider the quantity,

$$\mathbb{P}(|Q_{n+1}| = q_{n+1} \bigm| |Q_n| = q_n ... |Q_1| = q_1)$$

First, if $q_{n-1} = 0$, it is clear to see that the above is 1 with probability 1. Additionally, if $q_{n-1} \neq q_n \pm 1$, the above probability is obviously zero. Otherwise, let τ be the most recent visit to the origin: $\tau = \max\{j \in \{1...n\} : |Q_j| = 0\}$ (this is a function of our Markov chain). It's clear that $|Q_{n+1}|$ does not depend on $|Q_1|...|Q_{\tau-1}|$, since we know $Q_\tau = 0$, and $Q_\tau ... Q_{n+1}$ is a Markov Chain. Therefore, the above is equal to,

$$\mathbb{P}(|Q_{n+1}| = q_{n+1} \bigm| |Q_n| = q_n ... |Q_\tau| = 0) = \frac{\mathbb{P}(|Q_{n+1}| = q_{n+1}, |Q_n| = q_n \ldots ||Q_\tau| = 0)}{\mathbb{P}(|Q_n| = q_n \ldots ||Q_\tau| = 0)}$$

Note that there are precisely two ways for $|Q_\tau| = 0 \ldots |Q_{n+1}| = q_{n+1}$, since we know we never visit the origin between time τ and n. Indeed, we can condition on $Q_{\tau+1}$:

$$= \frac{\mathbb{P}(|Q_{n+1}| = q_{n+1} \ldots |Q_{\tau+1} = 1)\mathbb{P}(Q_{\tau+1} = 1|Q_\tau = 0) + \mathbb{P}(|Q_{n+1}| = q_{n+1} \ldots |Q_{\tau+1} = -1)\mathbb{P}(Q_{\tau+1} = -1|Q_\tau = 0)}{\mathbb{P}(|Q_n| = q_{n+1} \ldots |Q_{\tau+1} = 1)\mathbb{P}(Q_{\tau+1} = 1|Q_\tau = 0) + \mathbb{P}(|Q_n| = q_{n+1} \ldots |Q_{\tau+1} = -1)\mathbb{P}(Q_{\tau+1} = -1|Q_\tau = 0)}$$

Which simplifies a bit:

$$= \frac{\mathbb{P}(|Q_{n+1}| = q_{n+1} \ldots |Q_{\tau+1} = 1)p + \mathbb{P}(|Q_{n+1}| = q_{n+1} \ldots |Q_{\tau+1} = -1)q}{\mathbb{P}(|Q_n| = q_{n+1} \ldots |Q_{\tau+1} = 1)p + \mathbb{P}(|Q_n| = q_{n+1} \ldots |Q_{\tau+1} = -1)q}$$

Now, let $U = |\{j \in \{\tau + 2...n\} : |Q_j| = |Q_{j-1}| + 1\}|$. When $Q_{\tau+1} = 1$, these correspond to the number of up moves; otherwise, they are down moves. Likewise, let $L = |\{j \in \{\tau + 2...n\} : |Q_j| = |Q_{j-1}| - 1\}|$. Finally, let $\delta = q_{n+1} - q_n$. So that if $Q_{\tau+1} = 1$ and $\delta = 1$, we know $Q_{n+1} = Q_n + 1$. If $\delta = -1$, $Q_{n+1} = Q_n - 1$, and vice versa when $Q_{\tau+1} = -1$. Thus,

$$= \frac{p^\delta q^{1-\delta} p^U q^L \cdot p + q^\delta p^{1-\delta} q^U p^L \cdot q}{p^U q^L \cdot p + q^U p^L \cdot q}$$

$$= p^\delta q^{1-\delta} \frac{p^{U+1} q^L}{p^{U+1} q^L + q^{U+1} p^L} + q^\delta p^{1-\delta} \frac{q^{U+1} p^L}{p^{U+1} q^L + q^{U+1} p^L}$$

Simplifying,

$$= p^\delta q^{1-\delta}\frac{1}{1+q^{U+1-L}p^{L-U-1}} + q^\delta p^{1-\delta}\frac{1}{p^{U+1-L}q^{L-U-1}+1}$$

Finally, note that $U-L+1$ gives the total displacement from Q_τ to Q_n. Thus, $U-L+1=q_n$

$$= p^\delta q^{1-\delta}\frac{1}{1+q^{q_n}p^{-q_n}} + q^\delta p^{1-\delta}\frac{1}{p^{q_n}q^{-q_n}+1} = p^\delta q^{1-\delta}\frac{1}{1+(q/p)^{q_n}} + q^\delta p^{1-\delta}\frac{1}{1+(p/q)^{q_n}}$$

Which only depends on q_n, not the previous history! Therefore, $\{|Q_n|\}_n$ is a Markov Chain. And depending on the value of δ, this means that,

$$\mathbb{P}(|Q_{n+1}| = q_{n+1} \mid |Q_n| = q_n) = \begin{cases} \frac{p}{1+(q/p)^{q_n}} + \frac{q}{1+(p/q)^{q_n}} & q_{n+1} = q_n + 1 \text{ (when } \delta = 1) \\ \frac{q}{1+(q/p)^{q_n}} + \frac{p}{1+(p/q)^{q_n}} & q_{n+1} = q_n - 1 \text{ (when } \delta = -1) \\ 1 & q_n = 0, q_{n+1} \\ 0 & \text{otherwise} \end{cases}$$

(b) Suppose we know that $M_0 = m_0, \ldots, M_{n-1} = m_{n-1}$. This, in particular, tells us that

$$\max_{0\le k\le n-1} Q_k - Q_{n-1} = m_{n-1}.$$

Observe also that,

$$M_n = \max_{0\le k\le n}(Q_k - Q_n) = \max\left\{Q_{n-1} - Q_n, \max_{0\le k\le n-1}(Q_k - Q_n)\right\}$$

Where the latter is equal to,

$$\max_{0\le k\le n-1}(Q_k - Q_n) = \max_{0\le k\le n-1}(Q_k - Q_{n-1} + Q_{n-1} - Q_n) = m_{n-1} + Q_{n-1} - Q_n.$$

Thus,

$$M_n = \max\left\{Q_{n-1} - Q_n, m_{n-1} + Q_{n-1} - Q_n\right\} = \max\left\{-X_n, m_{n-1} - X_n\right\} = \max\left\{0, m_{n-1}\right\} - X_n,$$

which has no dependence on $m_0, \ldots, m_{n-2}$. Therefore,

$$\mathbb{P}(M_n = m_n \mid M_0 = m_0, M_1 = m_1, \ldots, M_{n-1} = m_{n-1}) = \mathbb{P}(M_n = m_n \mid M_{n-1} = m_{n-1}),$$

meaning the walk possesses the Markov property, and the transition probabilities of this Markov chain is

$$\mathbb{P}(M_n = m_n \mid M_{n-1} = m_{n-1}) = \begin{cases} p & \text{if } m_n = \max(0, m_{n-1}) - 1 \\ 1-p & \text{if } m_n = \max(0, m_{n-1}) + 1 \\ 0 & \text{otherwise} \end{cases}$$

Problem 2

> Let P_n and Q_n be two Markov chains on the set of integers. Is their sum $X_n = P_n + Q_n$ necessarily to be a Markov chain?

No. Consider two trivial Markov chain generated by an i.i.d. process. For instance, let $Y_1, Y_2, \ldots \sim \text{Bern}(1/2)$. Then let $P_n = Y_n$ and $Q_n = Y_{n+2}$. P_n and Q_n are Markov chains, individually. However, we will see their sum is not, because we exploit time. And so,

- $P_n + Q_n = Y_n + Y_{n+2}$
- $P_{n-1} + Q_{n-1} = Y_{n+1} + Y_{n-1}$
- $P_{n-2} + Q_{n-2} = Y_n + Y_{n-2}$

Clearly, $Y_n + Y_{n+2}$ is independent with $Y_{n+1} + Y_{n-1}$, since all the Y_i's are i.i.d. On the other hand, $Y_n + Y_{n+2}$ and $Y_n + Y_{n-2}$ are obviously not independent. For example,

$$\begin{aligned} \mathbb{P}(P_n + Q_n \geq 1 | P_{n-1} + Q_{n-1} = 2, \quad P_{n-2} + Q_{n-2} = 2) &= 1, \text{ yet} \\ \mathbb{P}(P_n + Q_n \geq 1 | P_{n-1} + Q_{n-1} = 2 \qquad\qquad\qquad\qquad\quad) &= 3/4 \end{aligned}$$

So $P_n + Q_n$ is not a Markov chain.

Problem 3

For a stochastic matrix $P \in \mathbb{R}^{n \times n}$, it is called doubly stochastic if for $\forall j$, it holds that $\sum_i P_{ij} = 1$. It is called sub-stochastic if for $\forall j$, it holds that $\sum_i P_{ij} \leqslant 1$. Prove that, if P is stochastic (or respectively, double stochastic, sub-stochastic), then for all positive integer n, P^n is stochastic (or respectively, double stochastic, sub-stochastic).

If P is stochastic, then it has $\mathbf{1}$ as a right eigenvector of eigenvalue 1: $P\mathbf{1} = \mathbf{1}$. Therefore, P^n also has $\mathbf{1}$ as a right eigenvector, so it is stochastic.

Observe that doubly stochastic matrices have $\mathbf{1}$ as a left eigenvector as well; so $\mathbf{1}^\top P = \mathbf{1}^\top$. Likewise, if a matrix is doubly stochastic, then P^n also has $\mathbf{1}$ as a left eigenvector, so it is doubly stochastic. It remains to handle the sub-stochastic case, which is slightly more delicate.

I claim that if P and Q are substochastic, then so is PQ. Indeed, for all $j \in [n]$, by the formula for matrix multiplication,

$$\sum_i (PQ)_{ij} = \sum_i \sum_k P_{ik} Q_{kj} = \sum_k \left[Q_{kj} \left(\sum_i P_{ik} \right) \right] \overset{P \text{ is sub-stochastic}}{\leq} \sum_k [Q_{kj} \cdot 1] = \sum_k Q_{kj} \overset{Q \text{ is sub-stochastic}}{\leq} 1$$

It therefore easily follows from induction that P^n is sub-stochastic by setting $Q = P^{n-1}$.

Problem 4

Consider a general 2-state Markov chain with probability transition matrix $P = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$ where P is not the identity matrix ($ab \neq 0$). We denote the two states as 1 and 2.

(a) Find the stationary distribution π of P and show that it is unique.

(b) Denote the distribution of X_n as π_n, show that the ratio $\dfrac{\pi_{n+1}(1) - \pi(1)}{\pi_n(1) - \pi(1)}$ is constant in n and find a simple expression for this ratio in terms of a and b. This shows that π_n converges geometrically fast to π and identifies the rate of convergence.

(a) Since we assume that $ab \neq 0$, it follows that the Markov chain is irreducible and has a unique stationary distribution. However, we can also determine the stationary distribution π by solving the left eigenvector equation $\pi P = \pi$. Indeed, this equation implies the following row-wise equalities:

$$\begin{array}{ccccc} \pi(1) & = & \pi(1)(1-a) & + & \pi(2)\,b \\ \pi(2) & = & \pi(1)\,a & + & \pi(2)(1-b) \end{array}$$

Rearranged, this says $\pi(1)a = \pi(2)b$. Therefore, $\pi(2) = \pi(1)(a/b)$. If we also enforce $\pi(1) + \pi(2) = 1$, we find

$$1 = \pi(1) + \pi(2) = \pi(1) + \pi(1)\left(\frac{a}{b}\right) = \pi(1)\left(\frac{a+b}{b}\right) \therefore \pi(1) = \frac{b}{a+b}$$

Likewise, $\pi(2) = \dfrac{a}{a+b}$.

(b) Observe that $\pi_{n+1}(1) = \pi_n(1)(1-a) + \pi_n(2)b$ by the law of total probability. But of course, we know $\pi_n(2) = 1 - \pi_n(1)$. So $\pi_{n+1}(1) = \pi_n(1)(1-a-b) + b$. Therefore,

$$\begin{aligned} \frac{\pi_{n+1}(1) - \pi(1)}{\pi_n(1) - \pi(1)} &= \frac{\pi_n(1)\Big(1-(a+b)\Big) + b - \frac{b}{a+b}}{\pi_n(1) - \frac{b}{a+b}} \\ &= \frac{\pi_n(1)\Big(1-(a+b)\Big) - \frac{b}{a+b}\Big(1-(a+b)\Big)}{\pi_n(1) - \frac{b}{a+b}} \\ &= \frac{\Big(\pi_n(1) - \frac{b}{a+b}\Big)\Big(1-(a+b)\Big)}{\pi_n(1) - \frac{b}{a+b}} = 1-(a+b) \end{aligned}$$

Problem 5

> The number of customers coming to a store (per hour) satisfies the Poisson process with $\lambda = 6$. Given the fact that the store opens at 8am.
>
> (1) The store opens 12 hours a day. What's the expected total number of customers in a day?
>
> (2) What's the probability that there are no more than 3 customers coming before 8:30am?
>
> (3) What's the probability that there are exactly 2 customers coming before 8:20am and there are exactly 2 customers coming between 8:10-8:30am?
>
> (4) What's the probability that the fourth customer comes between 8:20-8:30am?
>
> (5) Given that there are exactly 3 customers coming before 8:20am, what's the probability that the fourth customer comes before 8:30am?

(1) Given that $t = 12$, we know the Poisson process has $\mathbb{E}[N(t)] = t\lambda = 12 \cdot 6 = 72$, where $N(t)$ is the number of customers coming in t hours.

(2) Now letting $t = 1/2$, we know that $\mathbb{P}(N(t) \leq 3) = F_{t\lambda}(3)$, where $F_{t\lambda}$ is the CDF of the Poisson distribution with rate $t\lambda = 3$. We know that

$$\mathbb{P}\left(\{\leq 3 \text{ customers before 8:30}\}\right) = \mathbb{P}\left(N\left(\frac{1}{2}\right) \leq 3\right)$$

$$= \sum_{k=0}^{3} \mathbb{P}\left(N\left(\frac{1}{2}\right) = k\right) = \sum_{k=0}^{3} e^{-\lambda/2}\frac{(\lambda/2)^k}{k!} = e^{-3}\left(\frac{3^0}{1} + \frac{3^1}{1} + \frac{3^2}{2} + \frac{3^3}{6}\right) = 13e^{-3} \approx 0.6472$$

(3) We can break this up into the following possibilities, depending on how many customers come in between 8:10 and 8:20

- (0 Customers) Two customers come before 8:10 and two customers come between 8:20 and 8:30, and none come in between 8:10 and 8:20)
- (1 Customer) One customer comes in before 8:10, another between 8:10 and 8:20, and another between 8:20 and 8:30
- (2 Customers) Two customers come between 8:10 and 8:20, and none come in any other time between 8-8:10 or 8:20-8:30.

Recall that $\mathbb{P}(N(t) = k) f_{t\lambda}(k) = e^{-t\lambda} \dfrac{(t\lambda)^k}{k!}$. Then, the probability of each case happen is

- (0 Customers)

$$\begin{aligned} p_0 &= \mathbb{P}(\{2 \text{ customers before } 8{:}10\}) \times \mathbb{P}(\{0 \text{ customers btw } 8{:}10 \text{ - } 8{:}20\}) \times \mathbb{P}(\{2 \text{ customers btw } 8{:}20 \text{ - } 8{:}30\}) \\ &= \mathbb{P}\left(N\left(\frac{1}{6}\right) = 2\right) \times \mathbb{P}\left(N\left(\frac{1}{6}\right) = 0\right) \times \mathbb{P}\left(N\left(\frac{1}{6}\right) = 2\right) \\ &= \left[e^{-\lambda/6}\left(\frac{(\lambda/6)^2}{2!}\right)\right]^2 \times e^{-\lambda/6}\left(\frac{(\lambda/6)^0}{0!}\right) = \left[e^{-1}\left(\frac{1^2}{2!}\right)\right]^2 \times e^{-1}\left(\frac{1^0}{0!}\right) = \frac{1}{4}e^{-3}. \end{aligned}$$

- (1 Customer)

$$\begin{aligned} p_1 &= \mathbb{P}(\{1 \text{ customer before } 8{:}10\}) \times \mathbb{P}(\{1 \text{ customer btw } 8{:}10 \text{ - } 8{:}20\}) \times \mathbb{P}(\{1 \text{ customer btw } 8{:}20 \text{ - } 8{:}30\}) \\ &= \mathbb{P}\left(N\left(\frac{1}{6}\right) = 1\right) \times \mathbb{P}\left(N\left(\frac{1}{6}\right) = 1\right) \times \mathbb{P}\left(N\left(\frac{1}{6}\right) = 1\right) \\ &= \left[e^{-\lambda/6}\left(\frac{(\lambda/6)^1}{1!}\right)\right]^3 = \left[e^{-1}\left(\frac{(1)^1}{1!}\right)\right]^3 = e^{-3}. \end{aligned}$$

- (2 Customers)

$$\begin{aligned} p_2 &= \mathbb{P}(\{0 \text{ customers before } 8{:}10 \text{ and btw } 8{:}20 \text{ - } 8{:}30\}) \times \mathbb{P}(\{2 \text{ customers btw } 8{:}10 \text{ - } 8{:}20\}) \\ &= \mathbb{P}\left(N\left(\frac{1}{3}\right) = 0\right) \times \mathbb{P}\left(N\left(\frac{1}{6}\right) = 2\right) \\ &= e^{-\lambda/3}\left(\frac{(\lambda/3)^0}{0!}\right) \times e^{-\lambda/6}\left(\frac{(\lambda/6)^2}{2!}\right) = e^{-2}\left(\frac{2^0}{0!}\right) \times e^{-1}\left(\frac{1^2}{2!}\right) = \frac{1}{2}e^{-3}. \end{aligned}$$

So the probability that there are exactly 2 customers coming before 8:20am and there are exactly 2 customers coming between 8:10-8:30am is

$$p_0 + p_1 + p_2 = \frac{1}{4}e^{-3} + e^{-3} + \frac{1}{2}e^{-3} = \frac{7}{4}e^{-3} \approx 0.0871.$$

(4) Observe that the fourth customer coming between 8:20 and 8:30 is equivalent to at least four customers

arriving before 8:30, but also, at most 3 arrived before 8:20. Thus,

$$
\begin{aligned}
&\mathbb{P}(\{\text{fourth customer btw 8:20 - 8:30}\}) \\
&= \mathbb{P}(\{\geq 4 \text{ customers before 8:30}\}) - \mathbb{P}(\{\geq 4 \text{ customers before 8:20}\}) \\
&= (1 - \mathbb{P}(\{< 4 \text{ customers before 8:30}\})) - 1 + (\mathbb{P}(\{< 4 \text{ customers before 8:20}\})) \\
&= \mathbb{P}(\{\leq 3 \text{ customers before 8:20}\}) - \mathbb{P}(\{\leq 3 \text{ customers before 8:30}\}) \\
&= \mathbb{P}\left(N\left(\frac{1}{3}\right) \leq 3\right) - \mathbb{P}\left(N\left(\frac{1}{2}\right) \leq 3\right) \\
&= \sum_{k=0}^{3} \mathbb{P}\left(N\left(\frac{1}{3}\right) = k\right) - \sum_{k=0}^{3} \mathbb{P}\left(N\left(\frac{1}{2}\right) = k\right) \\
&= \sum_{k=0}^{3} e^{-\lambda/3} \frac{(\lambda/3)^k}{k!} - \sum_{k=0}^{3} e^{-\lambda/2} \frac{(\lambda/2)^k}{k!} \\
&= \sum_{k=0}^{3} e^{-2} \frac{2^k}{k!} - \sum_{k=0}^{3} e^{-3} \frac{3^k}{k!} \\
&= e^{-2}\left(\frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!}\right) + e^{-3}\left(\frac{3^0}{0!} + \frac{3^1}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!}\right) \\
&= \frac{19}{3} e^{-2} - 13e^{-3} \approx 0.2099
\end{aligned}
$$

(5) Since a Poisson process is a memoryless stochastic point process, so this is the same as the probability that there is at least one customer coming between 8:20 and 8:30.

$$
\mathbb{P}\left(N\left(\frac{1}{6}\right) \geq 1\right) = 1 - \mathbb{P}\left(N\left(\frac{1}{6}\right) < 1\right) = 1 - \mathbb{P}\left(N\left(\frac{1}{6}\right) = 0\right) = 1 - e^{-1} \approx 0.6321
$$