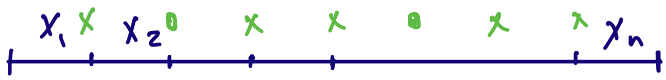
Poisson process:



n bins

A customer arrives with probability p

X_k is Bernouli random variable with parameter p

we assume that X_i's are independent

$$S_n = \sum_{k=1}^{n} X_k \Rightarrow E[S_n] = np = \mu \Rightarrow p = \frac{\mu}{n}$$

Let's see what happens in the cont. limit

Let N be the # cutomers arrived in one hour

$$P(N=k) = \lim_{\substack{n\to\infty \\ p=\mu/n}} P(S_n = k) = \underbrace{\lim_{\substack{n\to\infty \\ p=\mu/n}} \binom{n}{k}\left(\frac{\mu}{n}\right)^k\left(1-\frac{\mu}{n}\right)^{n-k}}_{*}$$

$$* = \underbrace{\frac{n!}{(n-k)!\, n^k}}_{\to 1} \frac{\mu^k}{k!} \underbrace{\left(1-\frac{\mu}{n}\right)^n}_{\to e^{-\mu}} \underbrace{\left(1-\frac{\mu}{n}\right)^{-k}}_{\to 1}$$

$$\lim_{\substack{n\to\infty \\ p=\mu/n}} * = \frac{e^{-\mu}\mu^k}{k!}$$

A random variable Z with distribution

$P(Z=k) = \frac{e^{-\mu}\mu^k}{k!}$ is called a poisson r.v.

As a sanity check

$$\sum_{k=0}^{\infty} pr(N=k) = e^{-\mu} \sum \frac{\mu^k}{k!} = e^{-\mu} \cdot e^{\mu} = 1$$

$$E[N] = \sum_{k=0}^{\infty} k \; pr(N=k) = e^{-\mu} \sum \frac{\mu^k}{(k-1)!}$$

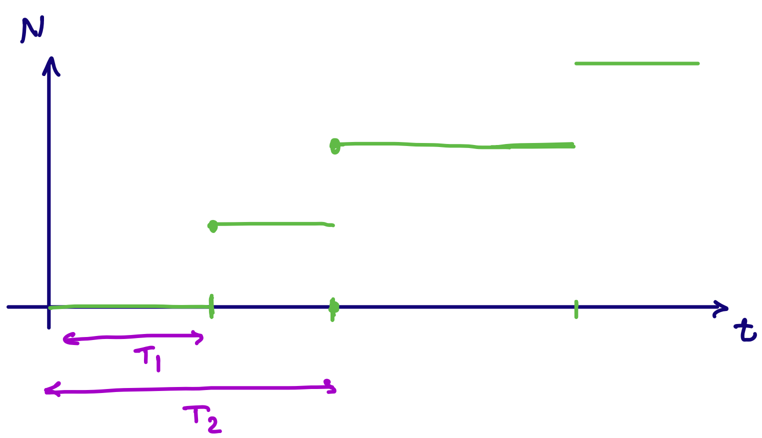
$$= e^{-\mu} \cdot \mu \sum_{k=1}^{\infty} \frac{\mu^{k-1}}{(k-1)!}$$

$$= e^{-\mu} \cdot \mu \cdot e^{\mu}$$

$$= \mu$$

T_k : The time that the k-th customer arrives

N_t : The # cutomers arrived by time t



N_1 : # customers arrived in the first hour

$N_{0.5}$: # " " " " " ½ hour

$N_t \sim Pois(\mu t)$ $\quad t \geq 0$

$$\{T_k \leq t\} \equiv \{N_t \geq k\}$$

$$pr(T_5 \leq t) = P(N_t \geq 5) = 1 - pr(N_t < 5)$$

$$= 1 - \sum_{n=0}^{4} P(N_t = n)$$

$$= 1 - \sum_{n=0}^{4} \frac{e^{-\mu t} (\mu t)^n}{n!}$$

More generally

$$P(T_k \leq t) = 1 - \sum_{n=0}^{k-1} \frac{e^{-\mu t} (\mu t)^n}{n!}$$

$$\text{Gamma}(\mu, k)$$

In particular

$$P(T_1 \leq t) = 1 - e^{-\mu t}$$

$\rightarrow$ exponential dist

$$P(T_1 > t) = e^{-\mu t}$$

If T is an exponential r.v with rate μ

$$P(T > t+s \mid T > s) = P(T > t)$$

A random process $\{N_t\}$ is called a counting process

1) $N_0 = 0$

2) N_t is increasing in t

3) N_t increases by 1 every time that it changes

Poisson process

a) $N_t \sim \text{pois}(\lambda t)$

b) $N_t - N_s \perp\!\!\!\perp \{N_r\}_{r \le s}$ $\quad t \geq s$

c) $N_t - N_s$ has the same dist as N_{t-s} $\quad t \geq s$

$$Pr(N_s = i, N_t = j) \qquad t > s$$

$$\parallel$$

$$P(N_s = i, N_t - N_s = j - i) = Pr(N_s = i) \times P(N_t - N_s = j - i)$$

$$= \frac{e^{-\lambda s} (\lambda s)^i}{i!} \times \frac{e^{-\lambda(t-s)} (\lambda(t-s))^{j-i}}{(j-i)!}$$

$$P(T_1 > t) = e^{-\lambda t}$$

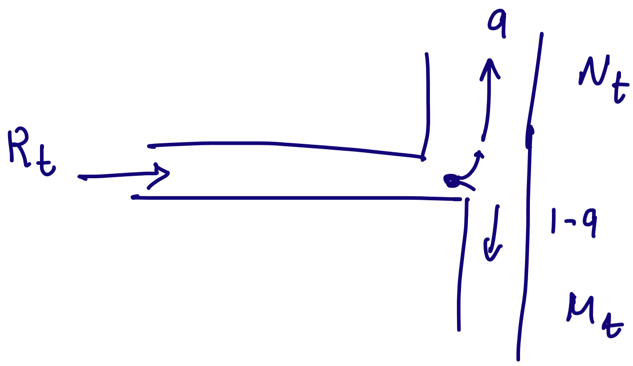
$$P(T_2 - T_1 > s \mid T_1 = t) = e^{-\lambda s}$$

$$T_1, T_2 - T_1, T_3 - T_2, \dots$$

$$E[T_k] = E[T_1] + E[T_2 - T_1] + \dots + E[T_k - T_{k-1}]$$

$$= \frac{k}{\lambda}$$

$$Var(T_k) = k / \lambda^2$$



$X \sim \mathrm{Pois}(\mu)$

$Y \sim \mathrm{Pois}(\nu)$

$X \perp Y$, $X + Y$?

$\downarrow$

$\mathrm{Pois}(\mu + \nu)$