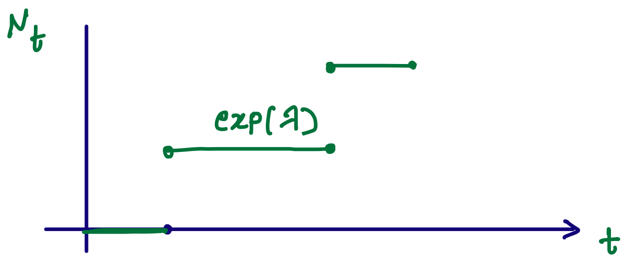
poisson process: N_t is a counting process with rate λ

① $N_t - N_s \perp\!\!\!\perp \{N_r\}_{r\leq s}$ → indep increament

② $N_t - N_s \sim \text{pois}(\lambda(t-s)) \quad \forall\, t \geq s$ → stationary



super position

$X \sim \text{pois}(\lambda)$

$Y \sim \text{Pois}(\gamma)$

$X \perp\!\!\!\perp Y$

1st approach

$$\Pr(X+Y=k) = P(X=0, Y=k) = P(X=0) \times P(Y=k)$$
$$+ \; P(X=1, Y=k-1)$$
$$+ \; \vdots$$

2nd approach

N_t is a poisson process with rate 1.

$N_\nu \sim \text{Pois}(\nu) \quad \sim X$

$N_{\gamma+\nu} - N_\nu \sim \text{Pois}(\gamma) \quad \sim Y$

$X+Y$ has the same distribution as $(N_{\gamma+\nu} - \cancel{N_\nu}) + (\cancel{N_\nu})$

which is $\text{Pois}(\gamma + \mu)$

superposition is the generalization of the above idea

Let us define

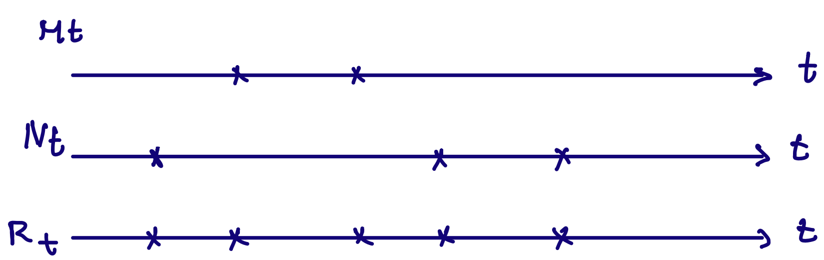
M_t = # undergrads that arrive by time t

N_t = # grads that arrive by time t

R_t = # customers that arrive by time t

$\{M_t\}_{t \geq 0}$ is a poisson process with rate μ

$\{N_t\}_{t \geq 0}$ " " " " " γ



$$R_t - R_s = (M_t - M_s) + (N_t - N_s) \overset{?}{\perp\!\!\!\perp} \{R_u\}_{u \leq s}$$

$$\downarrow$$

$$\{M_u + N_u\}_{u \leq s}$$

$$M_t - M_s \perp\!\!\!\perp \{M_u\}_{u \leq s}$$

$$M_t - M_s \perp\!\!\!\perp \{N_u\}_{u \leq s}$$

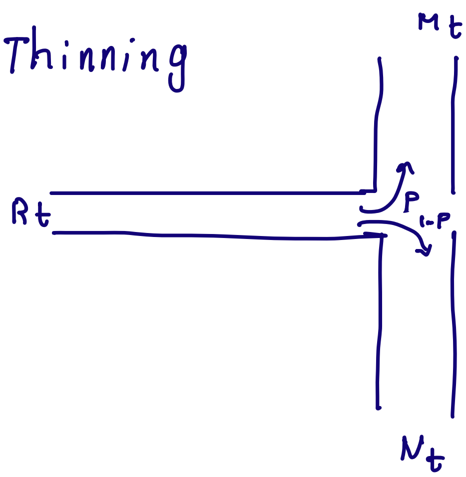
exactly the same reasoning applies to $N_t - N_s$

$$M_t - M_s \sim \text{pois}(\mu(t-s))$$
$$N_t - N_s \sim \text{pois}(r(t-s))$$
$$\Rightarrow R_t - R_s \sim \text{Pois}((\mu+r)(t-s))$$

$\Rightarrow$ R_t is a poisson process with rate $(\mu+r)$

> If $N_t^1, N_t^2, \dots, N_t^k$ are indep poisson processes with rates $\mu_1, \mu_2, \dots, \mu_k$, then $N_t^1 + \dots + N_t^k$ is a poisson process with rate $\mu_1 + \dots + \mu_k$

Thinning



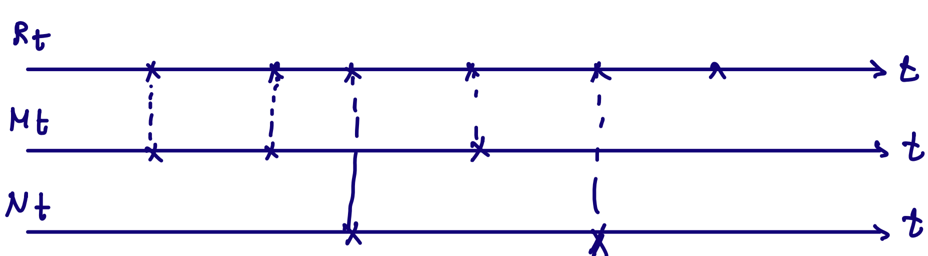
R_t = # customers arrived by time t

M_t = # customers going Up

N_t = # customers going Down

X_k = which door customer k took

$$P\{X_k = U\} = p \quad \& \quad P(X_k = D) = (1-p)$$



$$M_t = \sum_{k=1}^{R_t} \mathbb{1}\{X_k = U\}$$

$\sum^{R_t}$

$$N_t = \sum_{k\geq 1} 1\!\!1\{X_k = D\}$$

$$P\{M_t = k\} = \sum_{r\geq 0}^{\infty} P(M_t = k \mid R_t = r)\, P(R_t = r)$$

$$= \sum_{r\geq 0}^{\infty} P\left\{ \sum_{i\geq 1}^{R_t} 1\!\!1\{X_i = U\} = k \mid R_t = r \right\} \times P(R_t = r)$$

$$= \sum_{r\geq 0}^{\infty} P\left\{ \sum_{i\geq 1}^{r} 1\!\!1\{X_i = u\} = k \mid R_t = r \right\} P(R_t = r)$$

$$\sum_{r\geq 0}^{\infty} P\left\{ \sum_{i\geq 1}^{r} 1\!\!1\{X_i = U\} = k \right\} P(R_t = r)$$

$$P\{M_t = k\} = \sum_{r\geq k}^{\infty} \binom{r}{k} p^k (1-p)^{r-k} \frac{e^{-\mu t} (\mu t)^r}{r!}$$

$$= \sum_{r\geq k}^{\infty} \frac{\cancel{r!}}{k!\,(r-k)!} p^k (1-p)^{r-k} e^{-\mu t} \frac{(\mu t)^r}{\cancel{r!}}$$

$$= \underbrace{\frac{e^{-\mu t} (p\mu t)^k}{k!}}_{\downarrow} \underbrace{\sum_{r\geq k}^{\infty} \frac{((1-p)\mu t)^{r-k}}{(r-k)!}}_{e^{(1-p)\mu t}}$$

$$= \qquad \times \; e^{(1-p)\mu t}$$

$$= \frac{e^{-\mu p t} (\mu p t)^k}{k!}$$

It is easy to show that M_t has indep increament

M_t is a poisson process with rate $p \cdot \mu$

N_t " " " " $(1-p)\mu$

Note that

$M_t + N_t = R_t$

$$P(M_t = m, N_t = n) = P(M_t = m, R_t = m+n)$$

$$= P\left(\sum_{i=1}^{R_t} 1_{\{X_i = U\}} = m, R_t = m+n\right)$$

$$= P\left(\sum_{i=1}^{m+n} 1_{\{X_i = U\}} = m, R_t = m+n\right)$$

$$= P\left(\sum_{i=1}^{m+n} 1_{\{X_i = U\}} = m\right) P(R_t = m+n)$$

$$= \binom{m+n}{m} p^m (1-p)^n \, e^{-\mu t} \frac{(\mu t)^{m+n}}{(m+n)!}$$

$$\vdots$$

$$= \frac{e^{-\mu p t} (p\mu t)^m}{m!} \times \frac{e^{-(1-p)\mu t} ((1-p)\mu t)^n}{n!}$$

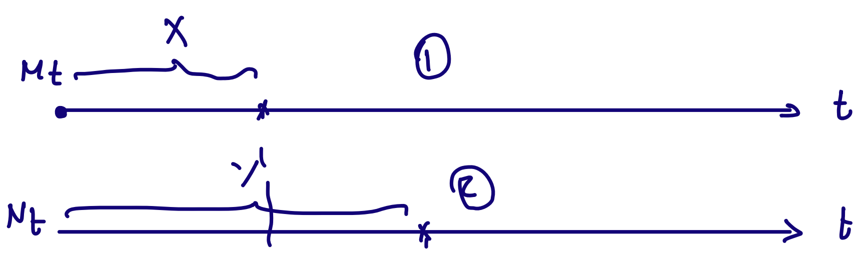
$$= P(M_t = m) \times P(N_t = n)$$

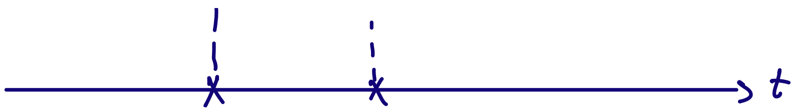
Ex: Let $X \sim \exp(\mu)$ & $Y \sim \exp(r)$

$X \wedge Y$

Let M_t be a poisson process with rate μ

" N_t is " " " " " r





$\mu + \gamma$

$R_t = M_t + N_t$

for thinning of $R_t \longrightarrow M_t$ we need to set the probability to $\frac{\mu}{\mu+\gamma}$

$R_t \longrightarrow N_t$ subsampling $\frac{\gamma}{\mu+\gamma}$

$$P(X \leq Y) = P(\underline{X \wedge Y} = X) = P(\text{first label is } 1)$$

$$= \frac{\mu}{\mu+\gamma}$$

$X \wedge Y$ is the first arrival of $R_t \sim \exp(\mu+\gamma)$