Generating functions (GF

$a = \{a_i\}$

$$G_a(s) = \sum_{i \geq 0}^{\infty} a_i s^i \qquad \text{for } s \in \mathbb{R} \text{ s.t the sum converges}$$

Let's have two sequences $\{a_i\}$ & $\{b_i\}$

$$C_n = a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0$$

$$G_c(s) = \sum_{n \geq 0}^{\infty} c_n s^n = \sum_{n=0}^{\infty} \left( \sum_{i=0}^{n} a_i b_{n-i} \right) s^n$$

$$= \sum_{i \geq 0}^{n} a_i s^i \sum_{n \geq i}^{\infty} b_{n-i} s^{n-i}$$

$$= G_a(s) \times G_b(s)$$

Let $X$ be a discrete rv taking only non-negative values $\{0, 1, \dots\}$ with probability $f(i) = P(X = i)$

GF of $X$

$$G_X(s) = E[s^X] = \sum_i P(X=i) s^i = \sum_{|s| \leq 1} s^i f(i)$$

Note that $G_X(1) = 1$

$$E[X] = G_X'(1)$$

If $X \perp Y \Rightarrow G_{X+Y}(s) = G_X(s) \times G_Y(s)$

- $P(X=c)=1 \Rightarrow G_X(s) = s^c$
- $P(X=1)=p$ & $P(X=0)=1-p \Rightarrow G_X(s)=(1-p)+p\cdot s$
- If $X$ is a poisson r.v

$$G(s) = \sum s^k \frac{\lambda^k}{k!} e^{-\lambda} = e^{\lambda(s-1)}$$

## Markov Chain

A random process $\{X_n\}$ where each $X_n$ takes value from a set $S$ is called MC if

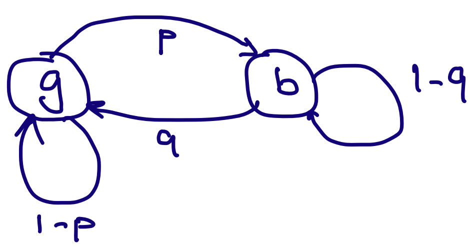
$$P(X_{n+1}=x_{n+1} \mid X_n = x_n, \ldots, X_0 = x_0) = P(X_{n+1}=x_{n+1} \mid X_n = x_n)$$

we only consider time-homogeneous if

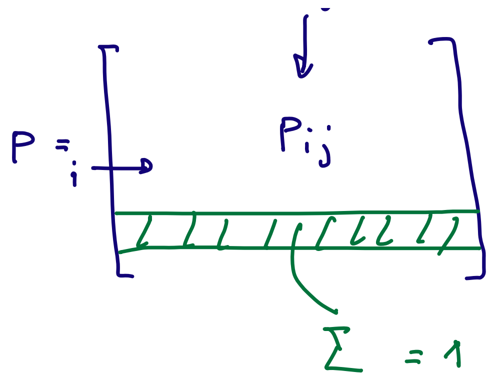
$$P(X_{n+1}=j \mid X_n = i) = P_{ij}$$

sickness : $P_{gb} = p \qquad P_{gg} = 1-p$

$P_{bg} = q \qquad P_{bb} = 1-q$




j




Any $P$ must satisfy two prop

- $0 \leq P_{ij} \leq 1$
- $\sum_{j \in S} P_{ij} = 1$

$$P_{X_0}\{\cdot\} = P(\cdot \mid X_0 = x_0)$$

$$P_{X_0}\{X_3 = x_3, X_2 = x_2, X_1 = x_1)$$

$$= P_{X_0}(X_3 = x_3 \mid X_2 = x_2, X_1 = x_1) \times P_{X_0}(X_2 = x_2 \mid X_1 = x_1) \times P_{X_0}\{X_1 = x_1)$$

$$P(X_3 = x_3 \mid X_2 = x_2) \times P(X_2 = x_2 \mid X_1 = x_1) \times P(X_1 = x_1 \mid X_0 = x_0)$$

$$= P_{x_2 x_3} \times P_{x_1 x_2} \times P_{x_0, x_1}$$

$$P_{X_0}\{X_1 = x_1, \dots, X_n = x_n) = P_{x_0 x_1} \times P_{x_1 x_2}, \dots \times P_{x_{n-1}, x_n}$$

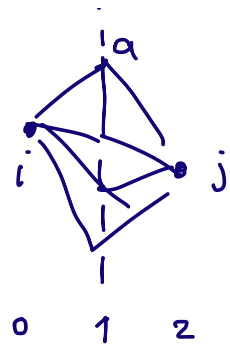
Ex: $P_g(X_1 = g, X_2 = g, X_3 = b, X_4 = g) = P_{gg} P_{gg} P_{gb} P_{bg}$

$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} \qquad = (1-p)^2 p q$$

recall $\quad P(X_1 = j \mid X_0 = i) = P_{ij}$

what is $\quad P(X_2 = j \mid X_0 = i)$ ?

$$= \sum_{a\in S} P(X_2=j, X_1=a \mid X_0=i)$$




$$= \sum_{a\in S} P(X_2=j \mid X_1=a, X_0=i) \times P(X_1=a \mid X_0=i)$$

$$= \sum_{a\in S} P(X_2=j \mid X_1=a) \cdot P(X_1=a \mid X_0=i)$$

$$= \sum_{a\in S} P_{ia} P_{aj}$$

$$= (P^2)_{i,j}$$

$$P(X_k=j \mid X_0=i) = (P^k)_{ij}$$

EX: Let $\{X_n\}$ be a MC. Let's assume that every time we visit a state $j$ we get a reward $f(j)$. So, the rework we get at time $k$ is $f(X_k)$. Let

$$r(i) = E_i\left(\sum_{k=0}^{n} f(X_k)\right) = E\left[\sum f(X_k) \mid X_0=i\right]$$

accumulated reward up to tim $n$ if MC starts from $i$.

$$E_i[Z] = E[Z \mid X_0 = i]$$

$$r(i) = \sum_{k \geq 0}^{n} E_i(f(X_k)) \doteq E[f(X_k) \mid X_0 = i)$$

$$= \sum_{k \geq 0}^{n} \sum_{j \in S} f(j) \quad P(X_k = j \mid X_0 = i)$$

$$= \sum_{k \geq 0}^{n} \sum_{j \in S} (P^k)_{ij} \, f(j)$$

Let $|S| = d$

$$r = \begin{pmatrix} r(i_1) \\ r(i_2) \\ \vdots \\ r(i_d) \end{pmatrix}, \quad f = \begin{pmatrix} f(i_1) \\ \vdots \\ f(i_d) \end{pmatrix}, \quad P = \Big[ \qquad \Big]_{d \times d}$$

$$r = \sum_{k \geq 0}^{n} P^k f = [I + P + \cdots + P^n] \cdot f$$

Let $\mu_i^{(n)} = P(X_n = i)$

$$\mu^{(k)} = \mu^{(0)} \cdot P^k$$