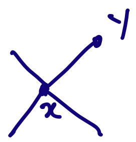
For a finite state space Ω and neighborhood structure $\{N(x) \mid x \in X\}$, let $N = \max_{x \in \Omega} |N(x)|$ and $M \geq N$.

Then,

$$P_{x,y} = \begin{cases} \frac{1}{M} & \text{if } x \neq y \ \& \ y \in N(x) \\ 0 & \text{if } x \neq y \ \& \ y \notin N(x) \\ 1 - \frac{N(x)}{M} & \text{if } x = y \end{cases}$$



$N(x) = 4$ $\quad M = 6 \quad \Rightarrow P_{x,y} = \frac{1}{6}$

insted of $\frac{1}{4}$

If the chain is irrepraducible & apridic $\Rightarrow$ The stationary dist is uniform

Metropolis alg:

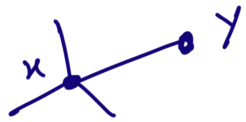
we want a stationary dist $\pi(x) = \frac{b(x)}{B}$ $\left(\pi(x) \propto b(x) \right)$

difficult to find

where for all $x \in \Omega$ we have $b(x) > 0$ & $B = \sum_{x \in \Omega} b(x)$.

Theorem: For a finite space Ω & neighborhood structure $\{N(x) \mid x \in \Omega\}$, Let $N = \max_{x \in \Omega} |N(x)|$ & $M \geq N$.

$$P_{x,y} = \begin{cases} \frac{1}{M} \boxed{\min\left(1, \frac{\pi_y}{\pi_x}\right)} & \text{if } x \neq y \ \& \ y \in N(x) \\ 0 & \text{if } x \neq y \ \& \ x \notin N(x) \\ 1 - \sum_{y \neq x} P_{xy} & \text{if } x = y \end{cases}$$



prob of accepting the move

proof: we prove that the chain is reversible.

let $x \neq y$ & $\pi_x \leq \pi_y \Rightarrow P_{x,y} = \frac{1}{M}$ & $P_{y,x} = \frac{1}{M} \frac{\pi_x}{\pi_y}$

$\Rightarrow \pi_x P_{x,y} = \pi_y P_{y,x}$.

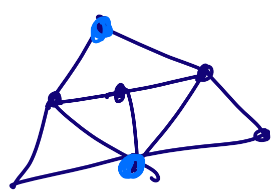
if $\pi_x > \pi_y \Rightarrow$ the same argument $\pi_x P_{x,y} = \pi_y P_{y,x}$

EX: we would like that each indep set I in a graph G has probability propotional to $\lambda^{|I|}$ for some $\lambda > 0$.

we want $\pi_x = \lambda^{|I_x|} / B$ where $B = \sum_x \lambda^{|I_x|}$.

① X_0 is an indep set in G

② To compute X_{i+1}



a) we choose a vertex v uniformly at random from V

b) if $v \in X_i$, set $X_{i+1} = X_i \setminus \{v\}$ w.p $\min(1, \frac{1}{\lambda})$

c) if $v \notin X_i$ and if adding v to X_i still gives an indep set, the $X_{i+1} = X_i \cup \{v\}$ wiph prob $\min(1, \lambda)$

d) otherwise, $X_{i+1} = X_i$

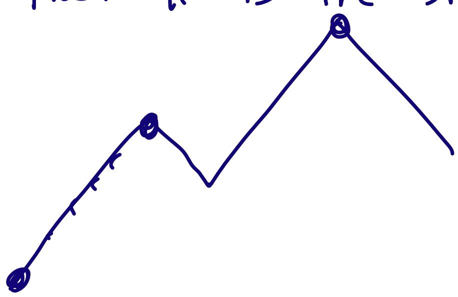
Let's assume that we have a chain with a symmetric transition matrix φ, i.e., $\varphi_{x,y} = \varphi_{y,x}$

so, the stationary dist is uniform.

The Metropolis chain for a probability π & symmetric transiti φ is defined as

$$P_{x,y} = \begin{cases} \varphi_{x,y} \times \overbrace{\left[\min\left(1, \frac{\pi_y}{\pi_x}\right) \right]}^{\text{accept}} & \text{if } x \neq y \\ 1 - \sum_{z:\, z \neq x} \varphi(x,z) \left[\min\left(1, \frac{\pi(z)}{\pi(x)}\right) \right] & \text{if } x = y \end{cases}$$

Then π is the stationary dist.



suppose that Ω is a regular graph so that a random work on Ω has a symmetric transition matrix. fix $\lambda \geqslant 1$

$$\pi_\lambda(x) = \frac{\lambda^{f(x)}}{Z(\lambda)} \qquad \text{where } Z(\lambda) = \sum_{x \in \Omega} \lambda^{f(x)}$$

If $f(y) < f(x)$, then $x \to y$ with prob $\lambda^{f(y)-f(x)}$

Define $\Omega^* = \{ x \in \Omega : f(x) = f^* = \max f(y) \}$

$$\lim_{\lambda \to \infty} \pi_\lambda(x) = \lim_{\lambda \to \infty} \frac{\lambda^{f(x)} / \lambda^{f^*}}{|\Omega^*| + \sum_{x \in \Omega \setminus \Omega^*} \lambda^{f(x)} / \lambda^{f^*}} = \frac{1_{\{x \in \Omega^*\}}}{|\Omega^*|}$$

General case:

For a general irreducible transition matrix ψ and a prob distribution π, we can run the following chain. when at state x, we generate state y from $\psi(x, \cdot)$. we move to y with probability

$$\min\left(\frac{\pi(y)\, \psi(y,x)}{\pi(x)\, \psi(x,y)}, 1 \right)$$

so

$$P(x,y) = \begin{cases} \psi(x,y) \times \min\left(\frac{\pi(y)\, \psi(y,x)}{\pi(x)\, \psi(x,y)}, 1 \right) & \text{if } y \neq x \\ 1 - \sum_{z: z \neq x} \psi(x,z) \times \min\left[\frac{\pi(z)\, \psi(z,x)}{\pi(x)\, \psi(x,z)}, 1 \right] & \text{if } y = x \end{cases}$$

For a uniform dist over nodes of a graph, the acceptance prob

$$\min\left(1, \frac{\deg(x)}{\deg(y)}\right)$$