S&DS 551 Midterm Cheat Sheet

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Random Walks

- $S_n : +S_0 + \sum_{j=1}^n X_j$
- T stopping time boundary $a < b$
- $\mathbb{P}\{T = \infty \mid S_0 = i\} = 0$
- $\mathbb{P}\{S_T = b \mid S_0 = i\} = \frac{i-a}{b-a}$
- $\mathbb{E}[T \mid S_0 = i] = (i-a)(b-i)$
- $T_a := \min\{n : S_n = a\}$
- For $0 = a < b$, $\{S_T = 0\} \subseteq \{T_0 < \infty\}$
- $\mathbb{P}\{T_0 < \infty \mid S_0 = i\} \geq \mathbb{P}\{S_T = 0 \mid S_0 = i\} = 1 - \frac{i}{b} \to 1$
- $\mathbb{E}[T_0 \mid S_0 = i] = \lim_{b\to\infty} \mathbb{E}[T \mid S_0 = i] \to \infty \cdot \delta_{i,0}$

Branching Processes

- $X_0 = 1, X_{n+1} = \sum_{i=1}^{X_n} X_{n,i}$ where $X_{n,i} : \Omega \to \mathbb{N}$ is iid
- $\mathbb{E}[X_{n+1}] = \mu, \mathbb{E}[X_n] = \mu^{n+1}, \mu = \mathbb{E}[X_{n,i}]$
- $\eta_n := \mathbb{P}\{X_n = 0\}$
- $\eta_{n+1} = \sum_{k\geq 0} p_k \eta_n^k =: g(\eta_n)$
- $\eta = g(\eta)$ fixed point
- $g(1) = 1, g'(1) = \mu, g''(x) \geq 0$

Poisson Processes

- $Z \sim \text{Po}(\lambda)$
- $\mathbb{P}\{Z = k\} = e^{-\lambda}\lambda^k/k!$
- $\mathbb{E}[Z] = \lambda, \text{Var}[Z] = \lambda$
- *Counting Process*:
1) $N_0 = 0$, 2) N_t increasing in t, 3) Increase at most 1
- *Poisson Process*
1) $N_t \sim \text{Po}(\lambda t)$, 2) independent increments, 3) $N_t - N_s \overset{d}{=} N_{t-s}$
- T_k: k-th arrival time
- N_t: count at time t
- $\{T_k \leq t\} \equiv \{N_t \geq k\}$
- $T_k \sim \Gamma(k, \lambda)$
- $\mathbb{E}[T_k] = k/\lambda, \text{Var}[T_k] = k/\lambda^2$
- $N_t^{(i)}$ iid Poisson process with params λ_i,
$\sum_i N_t^{(i)}$ Poisson process with param $\sum_i \lambda_i$
- R_t Poisson process rate λ,
subsample k-th arrival with prob $p \in \Delta^k$,
$R_t^{(1)}, \dots, R_t^{(k)}$ ind. Poisson process with params $p_i\lambda$
- $X \sim \text{Exp}(\mu), Y \sim \text{Exp}(\gamma)$ ind., $\mathbb{P}\{X \wedge Y = X\} = \mu/(\mu+\gamma)$
- N_t Poisson process param $\lambda(t)$ if
1) independent increments, 2) $N_t - N_s \sim \text{Po}\left(\int_s^t \lambda(u)du\right)$ for $t \geq s$

Markov Chains

- *Time-homogeneous Markov chain*,
$\mathbb{P}\{X_{n+1} = j \mid \forall 0 \leq k \leq n, X_k = i_k\}$
$= \mathbb{P}\{X_{n+1} = j \mid X_n = i_n\}$
- thm: $P_{ij}(s) = \delta_{ij} + F_{ij}(s)P_{ij}(s)$
- *recurrent state* i if $\mathbb{P}_i\{\exists n \geq 1, X_n = i\} = 1$
- lem: If $\sum_n P_{jj}(n) = \infty$,
a) j is recurrent, b) $\sum_n P_{ij}(n) = \infty$ for all i $f_{ij} > 0$
- *transient state* i if $\mathbb{P}_i\{\exists n \geq 1, X_n = i\} < 1$
- lem: If $\sum_n P_{jj}(n) < \infty$,
a) j is transient, b) $\sum_n P_{ij}(n) < \infty$ for all i
- If j transient, $\lim_n P_{ij}(n) = 0$ for all i
- recurrent j *null-recurrent* if $\mathbb{E}_i \min\{n \geq 1 : X_n = i\} = \infty$, else *positive-recurrent*
- j null-recurrent $\iff$ $P_{ii}(n) \to 0$
- *period* state j: $d(j) = \gcd\{n : P_{jj}(n) > 0\}$
j *aperiodic* if $d(j) = 1$
- i communicates with j, $i \to j$ if $\exists m, P_{ij}(m) > 0$
- thm: If $i \leftrightarrow j$,
1) $d(i) = d(j)$, 2) i transient $\iff$ j 3) i null-recurrent $\iff$ j
- $C \subseteq S$ is *closed* if cannot transition out,
irreducible if $i \leftrightarrow j$ $\forall i, j \in C$
- lem: If $|S| < \infty$,
a) ≥ 1 recurrent state, b) all recurrent states are positive
- thm: Irreducible MC has stationary distribution $\iff$ all states positive recurrent, then $\pi_i = 1/\mu_i$ uniquely (mean recurrent time)
- thm: Irreducible, aperiodic MC satisfies $P_{ij}(n) \to 1/\mu_j$
- thm: Irrreducible, finite, aperiodic MC has $\lambda_1 = 1$ and $|\lambda_r| < 1$ for all $r > 1$
- $\forall T \subseteq S, \sum_{i\in T, j\notin T} \pi_i P_{ij} = \sum_{i \in T, j\notin T} \pi_j P_{ji}$
- *ergodic:* Irreducible, aperiodic, positive-recurrent MC
- thm: RW on G aperiodic $\iff$ G not bipartite
- thm: RW on conneted non-bipartite graph $\to$ $\pi_v = d(v)/(2|E|)$
- X_n Irreducible, positive-recurrent MC, stationary π, $X_0 \sim \pi$, *reversed chain* $Y_n := X_{N-m}$
- Y_n is MC with transition $Q_{ij} = \pi_j P_{ji}/\pi_i$
- X_n *reversible* if $\pi_j P_{ji} = \pi_i P_{ij}$
- X_n reversible wrt π then π stationary distribution

MCMC

Wish to "tweak" MC so that stationary distribution is the desired π.

- *Metropolis algorithm:* Given MC with symmetric transition $\Psi_{x,y}$,

$$P_{x,y} = \begin{cases} \Psi_{x,y}\min(1, \pi_y/\pi_x), & y \neq x \\ 1 - \sum_{z\neq x} P_{x,z}, & y = x \end{cases}$$

- *Metropolis-Hastings filter:* Given MC with transition $\Psi_{x,y}$,

$$P_{x,y} = \begin{cases} \Psi_{x,y}\min(1, \pi_y/\pi_x \cdot \Psi_{y,x}/\Psi_{x,y}), & y \neq x \\ 1 - \sum_{z\neq x} P_{x,z}, & y = x \end{cases}$$

- *Gibbs sampler:* State S, $G = (V, E)$, configurations $\Omega \subseteq S^V$. Define $\Omega(x, v) := \{y \in \Omega : \forall w \neq v, y(w) = x(w)\}$.
1) Choose $v \in V$ at random,
2) $P_{x,y}^v = \begin{cases} \pi(y)/\pi(\Omega(x,v)), & y \in \Omega(x,v) \\ 0, & y \notin \Omega(x,v) \end{cases}$

Useful Facts

- $\sum_{i=1}^n i = n(n+1)/2$
- $\sum_{i=1}^n i^2 = (n+1)(2n+1)/6$
- For $N : \Omega \to \mathbb{N}$,
1) $\mathbb{E}[N] = \sum_{n\geq 1} \mathbb{P}\{N \geq n\}$
2) $\mathbb{E}[N^2] = \sum_{n\geq 1} \mathbb{P}\{N \geq n\}\left[n^2 - (n-1)^2\right]$
- Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!} = \prod_{k=2}^n \frac{n+k}{k}$
- *Convolution* of seq a_n, b_n is $c_n := \sum_{i=0}^n a_i b_{n-i}$
generating func $G_c(s) = G_a(s) \cdot G_b(s)$
- Generating func of $X : \Omega \to \mathbb{N}$,
$G_X(s) := \mathbb{E}[s^X] = \sum_{i\geq 0} \mathbb{P}\{X = i\} s^i$
- Sum of MC not necessarily a MC