Recap:

Stationary dist $\pi P^n = \pi$ for any $n \geq 1$

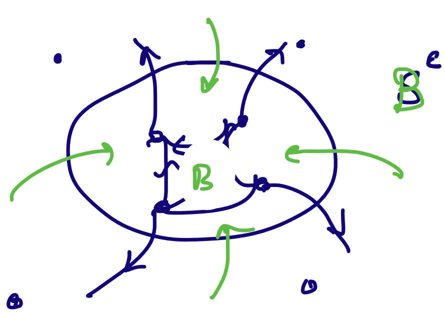
- We showed that for an irreducible MC, it has a stationary dist iff all the states are positive recurrent & if that is the case then $\pi_i = \frac{1}{\mu_i}$ where μ_i is the mean recurrent time of i.

- For an irreducible apiodic chain we have

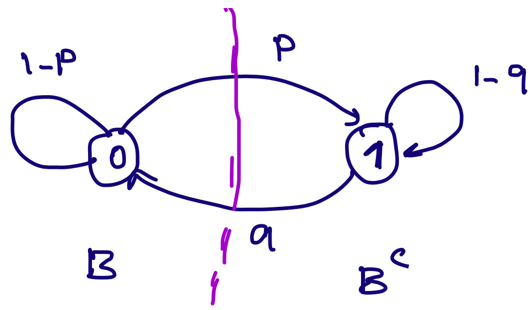
$$P_{ij}(n) \longrightarrow \frac{1}{\mu_j} \quad \text{as} \quad n \to \infty$$

An ergodic chain is a chain that is
1. irreducible
2. apriodic
3. positive recurrent

For a finite & ergodic chain, the probability that the chain leaves the set B is = to the prob that it enters B



Ex:



$$\begin{cases} \pi_0 \times p = \pi_1 \times q \\ \pi_0 + \pi_1 = 1 \end{cases}$$

$$\Rightarrow \pi_0 = \frac{q}{p+q}, \quad \pi_1 = \frac{p}{p+q}$$

$$\pi_0 = \pi_0(1-p) + \pi_1 q$$

$$\pi_1 = \pi_0 p + (1-q)\pi_1$$

$$\pi_0 + \pi_1 = 1$$

$$\Rightarrow \pi_0 = \frac{q}{p+q}, \quad \pi_1 = \frac{p}{p+q}$$

Let's consider a bounded queue where at each step exactly one of the two things happen.

- if the queue has fewer than n customers, with prob λ, a new customer joins the queue
- if the queue is not empty, with prob μ the head of the line is served and leaves the queue
- with the remaining prob, nothing happens

Mathematically

$$P_{i,i+1} = \lambda \quad \text{if} \quad i < n$$

$$P_{i,i-1} = \mu \quad \text{if} \quad i > 0$$

$$P_{ii} = \begin{cases} 1-\lambda & i = 0 \\ 1-\lambda-\mu & 1 \le i \le n-1 \end{cases}$$

$$\begin{cases} 1-\lambda-\mu & 1 \le i \le n-1 \\ 1-\mu & i=n \end{cases}$$

At the stationary dist

$$\pi_0 = (1-\lambda)\pi_0 + \mu \pi_1$$

$$\pi_i = \lambda \pi_{i-1} + (1-\lambda+\mu)\pi_i + \mu \pi_{i+1} \quad 1 \le i \le n-1$$

$$\pi_n = (1-\mu)\pi_n + \lambda \pi_{n-1}$$

so

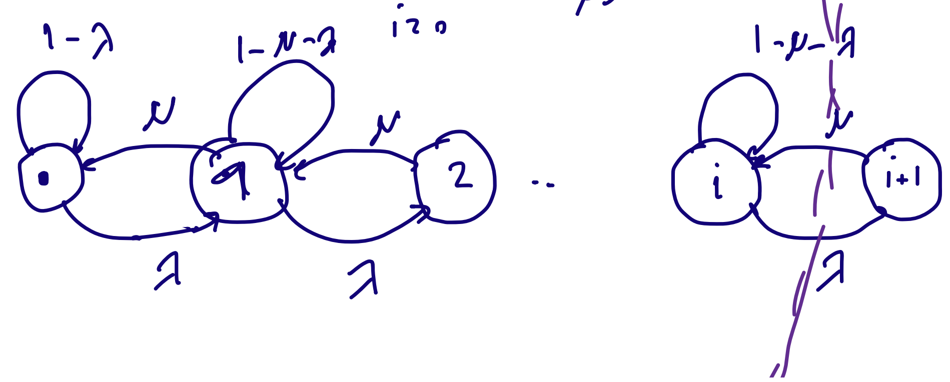
$$\pi_i = \pi_0 \left(\frac{\lambda}{\mu}\right)^i$$

$$\sum_{i=0}^{n} \pi_i = 1$$

this means that $\sum_{i=0}^{n} \pi_0 \left(\frac{\lambda}{\mu}\right)^i = 1$

$$\Rightarrow \quad \pi_0 = \frac{1}{\sum_{i=0}^{n} \left(\frac{\lambda}{\mu}\right)^i}$$

$$\pi_i = \frac{(\lambda/\mu)^i}{\sum_{i=0}^{n} (\lambda/\mu)^i}$$



$$\pi_i \lambda = \mu \pi_{i+1}$$

$$\Rightarrow \pi_i = \pi_0 \left(\frac{\lambda}{\mu}\right)^i$$

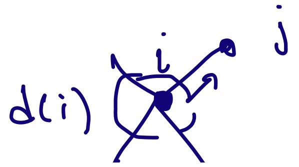
In case there is no upper limit n on the # customers the MC is no longer finite. MC has a stationary dist iff the following set of linear equations has solution with $\pi_i > 0$ for all i.

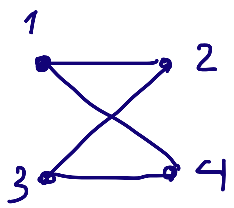
$$\pi_0 = (1-\lambda)\pi_0 + \mu \pi_1$$

$$\pi_i = \lambda \pi_{i-1} + (1-\lambda-\mu)\pi_i + \mu \pi_{i+1} \qquad i \geq 1$$

$$\Rightarrow \pi_i = \frac{(\lambda/\mu)^i}{\sum_{i \geq 0}^{\infty} (\lambda/\mu)^i} = \left(1 - \frac{\lambda}{\mu}\right)(\lambda/\mu)^i$$

Let $G(V,E)$ be a finite undirected & connected graph. The random walk on G works as follow. If a particle is at vertex i, and if i had $d(i)$ outgoing edges, the probability that i goes to vertex j when $(i,j) \in E$ will be $\frac{1}{d(i)}$





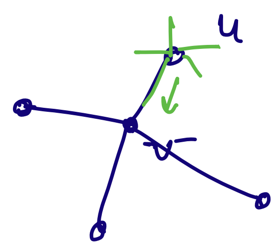
Theorem: A radom warlk on G is apriodic iff G is not bipartite

Theorem: A radom walk on G converges to the stationary dist π where

$$\pi_v = \frac{d(v)}{2|E|}$$

$$\sum_{v \in V} d(v) = 2|E| \Rightarrow \sum \pi_v = 1$$

$$\pi_v = \sum_{u \in N(v)} \frac{1}{\cancel{d(u)}} \times \frac{\cancel{d(u)}}{2|E|}$$



$$= \frac{d(v)}{2|E|}$$