reminder

$$P_{ij}(m, m+n+r) = \sum_k P_{ik}(m, m+n)\, P_{kj}(m+n, m+n+r)$$

$$P(X_{m+n+r} = j \mid X_m = i)$$

recurrent: null $= \mu_i = E_i[T_i] = \infty$

recurrent: positive $= \mu_i = \sum n f_{ii}(n) < \infty$

transient $\mu_i = \infty$

- If j is transient $P_{ij}(n) \longrightarrow 0$ as $n \to \infty$ $\forall i$
  " " " null-recurrent " " "

- $d(i) = \gcd\{n : P_{ii}(n) > 0\}$

  if $d(i) = 1 \Rightarrow$ state i is aperiodic

Classifications of Chains.

We say that i communicates with j, if the chain can visit j, starting from i, with strictly positive probability: $P_{ij}(m) > 0$ for some $m \geq 0$

And the notation is $i \to j$

If $i \to j$ and $j \to i \Rightarrow i \leftrightarrow j$

Theorem: if $i \leftrightarrow j$

1. i & j have the same period
2. i is transient iff j is transient
3. i is null-reurrent iff j is null-recurrent

proof 1: $k \in S$, $D_k = \{ m \geq 1, P_{kk}(m) > 0 \}$

so $d(k) = \gcd(D_k)$

Let $i \leftrightarrow j \Rightarrow \exists m, n$ s.t $\alpha := P_{ij}(m) P_{ji}(n) > 0$

$$P_{ii}(m+r+n) \geq P_{ij}(m) P_{jj}(r) P_{ji}(n) \geq \alpha P_{jj}(r)$$

$d(i) \mid m+r+n$ for $r \in \{0\} \cup D_j$

we know $d(i) \mid m+n \Rightarrow d(i) \mid r \quad \forall r \in D_j$

$\Rightarrow d(i) \mid d(j)$

by the same argument $d(j) \mid d(i)$

so, $d(i) = d(j)$

proof(2): if $i \leftrightarrow j \Rightarrow P_{ii}(m+r+n) \geq \alpha P_{jj}(r) \quad \forall r \geq 0$

$$\sum_r P_{jj}(r) < \infty \text{ if } \sum_r P_{ii}(r) < \infty$$

since i is transionl

A set C of states is called

(a) Closed if $p_{ij} = 0$ for all $j \notin C$ & $i \in C$

(b) irreducible if $i \leftrightarrow j$ $\forall\, i,j \in C$

Decomposition theorem : The state space S can be partitioned uniquely as

$$S = T \cup C_1 \cup C_2 \cup \ldots$$

transient

irreducible
closed
recurrent

EX: $S = \{1, 2, \ldots, 6\}$

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 \\ 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

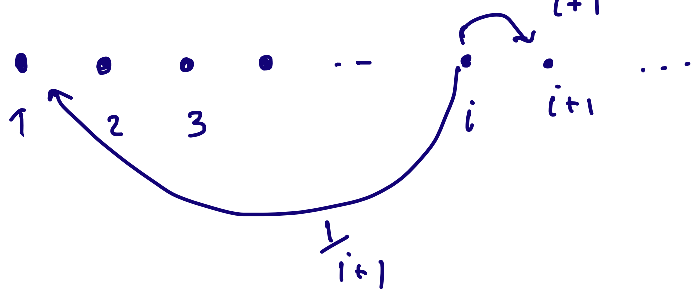
irreducible & closed

$\{3, 4\} \Rightarrow$ transient

Lemma: If S is finite, one state is recurrent & all recurrent states are positive

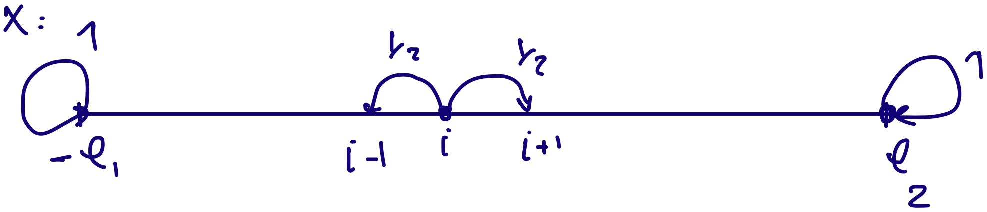
$$1 = \lim_{n\to\infty} \sum_j P_{ij}(n) = \sum \lim_{n\to\infty} P_{ij}(n) = 0$$

EX: All states are null-recurrent $i, i+1$



EX: A symmetric random walk

EX:



All states $-\ell_1 < i < \ell_2$ are transient

$$P_{ii}(n) \to 0 \quad \text{as } n \to \infty$$

let q : probability that the player wins ℓ_2 dollars

$\Rightarrow$ $1-q$: " " " " loses ℓ_1 "

Let $W(n)$ be the gain after playing n round

$$0 = E[W(n)] = \sum_{i=-\ell_1}^{\ell_2} i \underbrace{P_{0,i}(n)}_{P_i(n)}$$

$$0 = \lim E[W(n)] = \ell_2 q - \ell_1(1-q)$$

$$q = \frac{\ell_1}{\ell_1 + \ell_2}$$