S&DS 351 / S&DS 551 / MATH 251: Stochastic Processes
Assignment 1

Due: 11:59pm ET Tuesday January 31 2023

General Instructions

Honor Code: These questions should be completed individually. When submitting the assignment, you agree to adhere to the Yale Honor Code. Please read carefully to understand what it entails!

Submission Instructions: You should submit your answers in a PDF file. LaTeX is highly preferred due to the need of formatting equations. Details of solutions are required for all the problems. No late periods are allowed without communicating with Professor Karbasi in advance.

Submitting Answers: Prepare answers to your homework in a single PDF file. Make sure that the answer to each question is on a separate page. The number of the question should be at the top of each page.

Problem 1: An urn contains n balls numbered $1, 2, \ldots, n$. We remove k balls at random (without replacement) and add up the numbers of all remaining balls. Find the mean and variance of the total sum.

Problem 2: Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables for which $\mathbb{E}(X_1^{-1})$ exists. Show that, if $m \leqslant n$, then $\mathbb{E}(S_m/S_n) = m/n$, where $S_m = X_1 + X_2 + \ldots + X_m$.

Problem 3: Let $\{X_r : r \geqslant 1\}$ be independent and uniformly distributed on the interval $[0, 1]$. Let $0 < x < 1$ and define

$$N = \min\{n \geqslant 1 : \ X_1 + X_2 + \ldots + X_n > x\}.$$

Show that $\mathbb{P}(N > n) = x^n/n!$, and hence find the mean and variance of N.

Problem 4: Let X, Y and Z be independent and uniformly distributed on the interval $[0, 1]$. Find the joint density function of XY and Z^2, and show that $\mathbb{E}(XY < Z^2) = \frac{5}{9}$.

Problem 5: From a set of 52 poker cards (without 2 jokers), we keep taking cards randomly one by one with replacement, until all the cards taken by us can cover all 4 shades.
(1) Compute the probability that we have picked exactly n cards.
(2) Verify that, after taking summation over $n = 1, 2, \ldots$, the sum of the probabilities above equals to 1.