AR(p) Model

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Class Web Page (https://nmimoto.github.io/477/index.html) – R resource page (https://nmimoto.github.io/R/index.html)

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1. Causal Representation of AR(1)

Alternative Notation for AR(1)

- We began with writing AR(1) as

$$X_t = \phi X_{t-1} + e_t,$$

- We could also write AR(1) as

$$X_t - \phi X_{t-1} = e_t,$$

  and using the **backward operator**, write

$$\underbrace{(1-\phi B)}_{\Phi(B)}X_t = e_t.$$

- $\Phi(z)$ is called **characteristic polynomial** of AR(1).
- Backward operator makes it go back a day:

$$BX_t = X_{t-1}.$$

Causal Representation for AR(1)

- If AR(1) representation is

$$X_t = \phi X_{t-1} + e_t,$$

then I can write the same thing for yesterday,

$$X_{t-1} = \phi X_{t-2} + e_{t-1}$$

- Now combining the two, I can write

$$\begin{aligned} X_t &= \phi X_{t-1} + e_t \\ &= \phi\left\{\phi X_{t-2} + e_{t-1}\right\} + e_t \\ &= \phi^2 X_{t-2} + \phi e_{t-1} + e_t \end{aligned}$$

- Do it again, this time for X_{t-2} and we get

$$\begin{aligned} X_t &= \phi^2 X_{t-2} + \phi e_{t-1} + e_t \\ &= \phi^2\left(\phi X_{t-3} + e_{t-2}\right) + \phi e_{t-1} + e_t \\ &= \phi^3 X_{t-3} + \phi^2 e_{t-2} + \phi e_{t-1} + e_t \end{aligned}$$

- We can keep doing this, and get

$$X_t = \phi^k X_{t-k} + \phi^{k-1} e_{t-(k-1)} + \cdots + \phi e_{t-1} + e_t$$

- If $|\phi| < 1$, then letting $k \to \infty$ will yield

$$X_t \quad = \quad e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots \quad = \quad \sum_{i=0}^{\infty} \phi^i \, e_{t-i}$$

- So here's **causal representation** for AR(1) process

$$X_t \quad = \quad \sum_{i=0}^{\infty} \phi^i \, e_{t-i} \quad = \quad e_t + \phi \, e_{t-1} + \phi^2 \, e_{t-2} + \cdots$$

- This is called **causal representation** because we can write Y_t as infinite sum of **past** errors (innovations).

Get Mean of AR(1) using Causal Representation

- Now it is easy to see that mean of AR(1)

$$E(X_t) \quad = \quad \sum_{i=0}^{\infty} \phi^i \, E(e_{t-i}) \quad = \quad 0$$

- Recall e_t are White Noise with mean 0 and variance σ.
- Or often, we assume

$$e_t \sim N(0, \sigma^2)$$

Causal Representation and Characteristic Polynomial

- Recall yet another way to write AR(1),

$$(1 - \phi B) \, X_t \quad = \quad e_t.$$

- This means that we can write

$$X_t \quad = \quad \frac{1}{(1 - \phi \, B)} \, e_t.$$

- Compare this with Causal representation

$$X_t \quad = \quad e_t + \phi \, e_{t-1} + \phi^2 \, e_{t-2} + \cdots \quad = \quad 1 + \phi B + \phi^2 B^2 + \cdots \; e_t$$

- So we have the equivalence on the right hand side,

$$\frac{1}{(1 - \phi B)} \quad = \quad 1 + \phi B + \phi^2 B^2 + \phi^3 B^3 + \cdots$$

- This is exactly same as the geometric series,

$$\frac{1}{1-x} \quad = \quad 1 + x + x^2 + x^3 + \cdots$$

- Note that the condition for the geometiric series is

$$|x| < 1 \qquad \text{or} \qquad |\phi| < 1$$

Which is same as the **causal (stationary) condition**.

Another look at Causal Condition

- So if you write AR(1) in causal way,

$$X_t = \sum_{i=0}^{\infty} \phi^i \, e_{t-i},$$

- Then we can calculate variance as

$$\text{Var}(X_t) \quad = \quad \text{Var}\left[\sum_{i=0}^{\infty} \phi^i \, e_{t-i}\right] \quad = \quad \sum_{i=0}^{\infty} \text{Var}\left[\phi^i \, e_{t-i}\right] \quad = \quad \sum_{i=0}^{\infty} \phi^{2i} \text{Var}(e_{t-i}) \quad = \quad \sigma^2 \sum_{i=0}^{\infty} \phi^{2i}$$

This does not converge unless $|\phi| < 1$.

- When it does converge, the variance is $\sigma^2/(1-\phi^2)$.

Causal Condition of AR(1)

- We can represent AR(1) in causal representation only when $|\phi| < 1$.

$$X_t \quad = \quad e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots$$

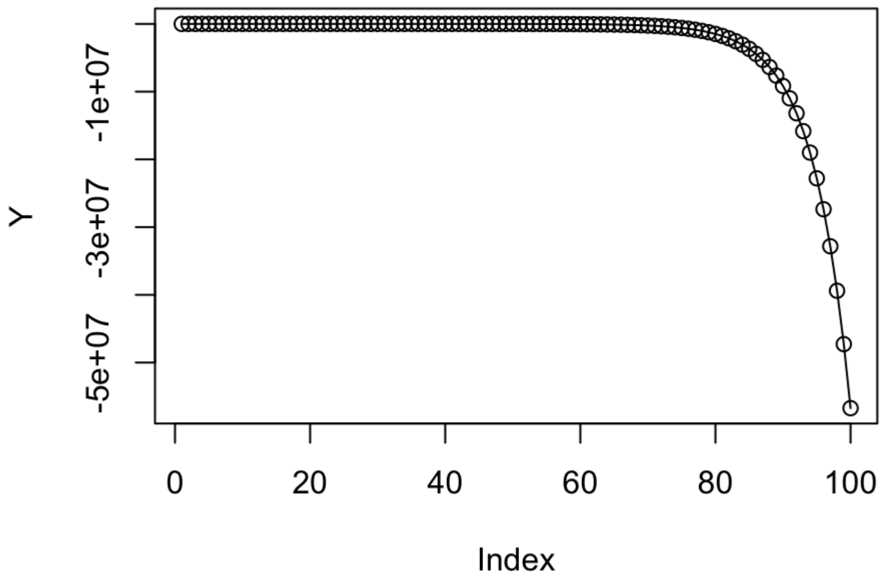
- When AR(1) has $|\phi| > 1$, and if you represent that with past errors, then it is called explosive process, and it's not stationary.
- When AR(1) has $|\phi| > 1$, Y_t can be written as infinite sum of future errors, and it is a unique stationary solution to AR(1) equation.
- When $|\phi| = 1$, then there is no stationary solution.
What is the other name of Y_t when $\phi = 1$?
- We will assume that all AR process we deal with are causal.
- If AR process is not causal, then it can be re-written as causal process with different innovations. (Prob 3.8).

Simulating non-causal AR(1)

Let $\phi = 1.2$. i.e. $\quad \Phi(z) = 1 - 1.2z$

```
Y <- arima.sim(list(ar = c(1.2) ), 100 )      #- gives error because of
 phi=1.2
```

```
#- Hand written simulation
Y <- 0.5
phi <- 1.2
e <- rnorm(100)
for (t in 2:100){
  Y[t] = Y[t-1]*phi + e[t]
}
plot(Y, type="o")
```



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2. AR(p) processes

- Autoregressive process of order p is

$$
\begin{aligned}
X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} - \cdots - \phi_p X_{t-p} &= e_t, \\
\text{where } e_t \sim WN(0, \sigma^2)
\end{aligned}
$$

and $\phi_1, \ldots, \phi_p$ is real valued constant.

- Alternative notation using characteristic polynomial is,

$$
\begin{aligned}
\underbrace{(1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p)}_{\Phi(B)} X_t &= e_t. \\
\Phi(B) \;\; X_t &= e_t.
\end{aligned}
$$

Writing AR(p) in Causal Rep.

- Using the characteristic polynomial,

$$
\begin{aligned}
\Phi(B) X_t &= e_t \\
X_t &= \frac{1}{\Phi(B)} \; e_t
\end{aligned}
$$

- So if we could write AR(p) as causal,

$$
X_t \begin{aligned}
&= \psi_0 + \psi_1 e_t + \psi_2 e_{t-2} + \psi_3 e_{t-3} + \cdots \\
&= (\psi_0 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \cdots) \; e_t
\end{aligned}
$$

- That means we must be able to write polynomial as

$$
\frac{1}{\left(1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p\right)} = \psi_0 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 + \cdots
$$

- When can we do this?

Causal Condition for AR(p)

- From operator theory, we know that if

$$
\text{(complex) root of } \Phi(z) \text{ is outside of the unit circle,}
$$

we can expand the inverse of $\Phi(z)$ as

$$
\frac{1}{\Phi(z)} = \psi_0 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 + \cdots
$$

- This is the condition that allows us to write AR(p) in causal representation.
- We have seen that causal condition for AR(1) was $|\phi| < 1$.

- This was because polynomial

$$\Phi(z) \quad = \quad (1 - \phi z)$$

will have root inside the unit circle if $|\phi| > 1$.

- AR(p) will admit the causal representation if the characteristic polynomial

$$\Phi(z) \quad = \quad 1 + \phi_1 z + \phi_2 z^2 + \phi_3 z^3 + \cdots + \phi_p z^p$$

has all the roots outside of the unit circle. Causal representation will ensure stationarity.

Example: Checking Causality 1

- Check to see of AR(2) model,

$$Y_t \quad = \quad .4Y_{t-1} - .3Y_{t-2} + e_t$$

is causal (stationary).

- We have to look at the root of

$$\Phi(z) \quad = \quad 1 - .4z + .3z^2$$

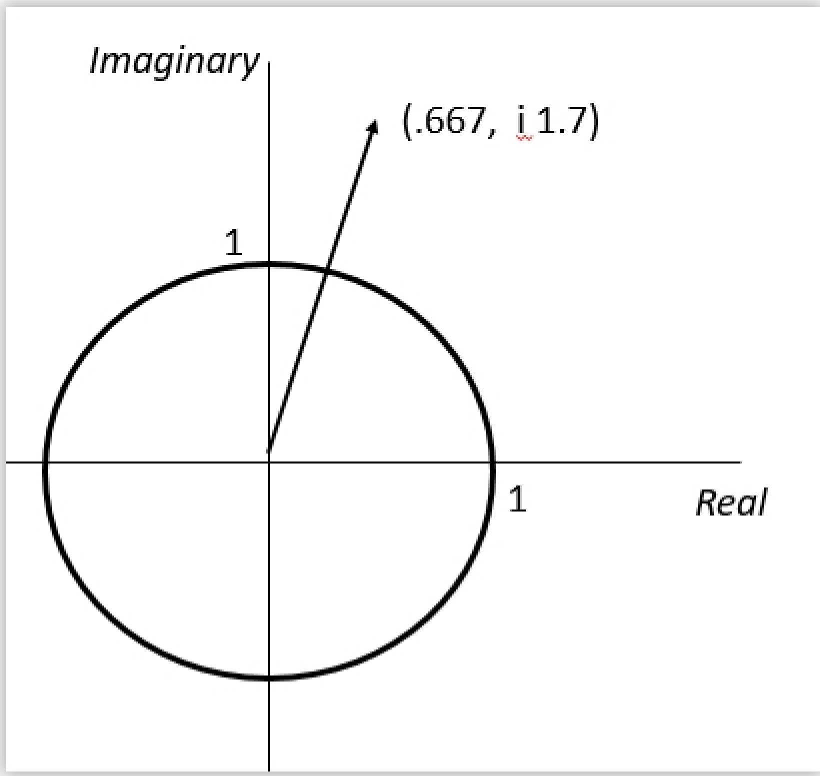
which is

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{.4 \pm \sqrt{(.4)^2 - 4(.3)(1)}}{2(.3)} = .667 \pm i1.7$$

- Their distance form the origin is

$$\sqrt{(.667)^2 + (i1.7)^2} = 1.826$$

So the roots are outside the complex unit circle. Thus this AR(2) is causal.



Example: Checking Causality 2

- Check to see of AR(2) model,

$$Y_t = -.7Y_{t-1} - .6Y_{t-2} + e_t$$

  is causal or not.

- We have to look at the root of

$$\Phi(z) = 1 + .7z + .6z^2,$$

  which has roots $-.583 1.15 i $.

```
z <- polyroot(c(1,.7,.6))    # find root of 1+.7z+.6z^2
z                            #- see the root
```

```
## [1] -0.583333+1.15169i -0.583333-1.15169i
```

```
Mod(z)                       #- Distance from origin
```

```
## [1] 1.290994 1.290994
```

- They are at at distance $\sqrt{(-.583)^2 + (1.15i)^2} = 1.29$ from origin. Therefore, this AR(2) is causal.

Example: Checking Causality 3

- Given

$$X_t = .7X_{t-1} + .6X_{t-2} + e_t,$$

  Check the causality.

- we look at the root of

$$\Phi(z) = 1 - .7z - .6z^2,$$

```
z <- polyroot(c(1,-.7,-.6))
z
```

```
## [1]  0.8333333+0i -2.0000000+0i
```

```
Mod(z)
```

```
## [1] 0.8333333 2.0000000
```

- The polynomial has roots .833, .2. Therefore, this AR(2) is not causal.

Summary

- AR(p) is defined as

$$\begin{aligned} X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} - \cdots - \phi_p X_{t-p} &= e_t, \\ (1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p) X_t &= e_t. \\ \Phi(B)\ X_t &= e_t. \end{aligned}$$

  where $\phi_1, \ldots, \phi_p$ is real valued constant, and $e_t \sim WN(0, \sigma^2)$.

- AR(p) can be written in causal representation,

$$X_t = \sum_{i=0}^{\infty} \phi^i \, e_{t-i},$$

  when its characteristic polynomial $\Phi(z)$ has all the roots **outside** of the unit circle on the imaginary plane.

- When the AR(p) can be written as causal process, then it is stationary.
- You can use **polyroot()** function in R to calculate the roots of polynomial, and **Mod()* to calculate their distance from the origin.

```
z <- polyroot(c(1,.7,.6))    # find root of 1+.7z+.6z^2
z                            # see the root
Mod(z)                       # Distance from origin
```