Let $Y_t = \phi Y_{t-1} + \varepsilon_t$ $\qquad\qquad \varepsilon_t \sim iid(0, \sigma^2)$

1. (i) Determine the characteristic polynomial:

- Using the backshift / lag operator B, $Y_t = \phi Y_{t-1} + \varepsilon_t$ becomes:

$$Y_t = \phi B Y_t + \varepsilon_t$$
$$Y_t(1 - \phi B) = \varepsilon_t$$

define $\Phi(B) = 1 - \sum_{i=1}^{p} \phi_i B^i = 1 - \phi_1 B - \phi_2 B^2 + \ldots - \phi_i B^i$

where $\underline{\Phi(B) = 1 - \phi B}$ is called characteristic polynomial of an AR(1) process.

$$\Rightarrow Y_t \Phi(B) = \varepsilon_t$$

(ii) Conditions for stationarity

(1.) Const. mean?

$$E(Y_t) = \phi E(Y_{t-1}) + \underbrace{E(\varepsilon_t)}_{=0} = \phi \underbrace{E(Y_{t-1})}$$

$E(Y_0) = 0 \Rightarrow E(Y_t) = 0 \Rightarrow$ const. ✓ $\qquad E(Y_0) = 0$?

(2.) Const. variance?

$$Var(Y_t) = \underbrace{Var(\phi Y_{t-1})}_{Var(ax) = a^2 Var(x)} + \underbrace{Var(\varepsilon_t)}_{\sigma^2}$$

$$Var(Y_t) = \phi^2 Var(Y_{t-1}) + \sigma^2$$
$$(1 - \phi^2) Var(Y_t) = \sigma^2$$
$$Var(Y_t) = \sigma^2 / \underbrace{(1 - \phi^2)} \Rightarrow \text{const.} ✓$$

$$1 - \phi^2 > 0$$
$$1 > \phi^2$$
$$\underline{\underline{|\phi| < 1}} \; !$$

(3.) Const. Covariance

$$Cov(Y_t, Y_{t+h}) = Cov(Y_t, \phi^h Y_t + \cancel{\sum_{i=0}^{h-1} \phi^i \varepsilon_{t+h-i}})$$

not dependent on t because $\underline{\varepsilon_t \sim iid(0, \sigma^2)}$

$$\phi^h Cov(Y_t, Y_t) = \underbrace{\phi^h Var(Y_t)}$$
$$= \phi^h \left(\frac{\sigma^2}{1 - \phi^2}\right) \Rightarrow \text{const.} ✓$$

(iii) Conditions for causality

... can be determined by the roots of the characteristic polynomial

$$\phi(z) = 0$$

$$1 - \phi z = 0$$

$$1 = \phi z$$

$$z = \frac{1}{\phi} \quad |\phi| < 1 \text{ for stationary processes}$$

$$\Rightarrow |z| > 1$$

2. (i) Mean of AR(1) process

$$Y_t = \phi Y_{t-1} + \varepsilon_t$$

$$E(Y_t) = \phi E(Y_{t-1}) + E(\varepsilon_0) = \phi \mu + 0$$

$$\mu = \phi \mu + 0$$

$$\mu(1 - \phi) = 0$$

$$\underline{\underline{E(Y_t) = \mu = 0}} \checkmark$$

(ii) Variance of AR(1) process

$$[\mathrm{Var}(ax) = a^2 \mathrm{Var}(x)]$$

$$\mathrm{Var}(Y_t) = \mathrm{Var}(\phi_1 Y_{t-1}) + \mathrm{Var}(\varepsilon_t)$$

$$= \phi_1^2 \mathrm{Var}(Y_{t-1}) + \sigma^2$$

$\mathrm{Var}(Y_{t-1}) = \mathrm{Var}(Y_t)$ for a stationary AR(1) process

$$\mathrm{Var}(Y_t) = \phi^2 \mathrm{Var}(Y_t) + \sigma^2$$

$$(1 - \phi^2)\mathrm{Var}(Y_t) = \sigma^2$$

$$\mathrm{Var}(Y_t) = \sigma^2 / (1 - \phi^2)$$

$$\mathrm{Var}(Y_t) > 0 \Rightarrow 1 - \phi^2 > 0$$

$$1 > \phi^2$$

$$\underline{|\phi| < 1}$$

(iii) ACF of AR(1)

lag = 1 ;(t+1)

$$Cov(y_t, y_{t+1}) = \gamma = E(y_t\, y_{t+1}) = E(y_t(\phi y_t + \varepsilon_{t+1}))$$

$$= E(\phi y_t^2 + y_t \varepsilon_{t+1}) = \phi Var(y_t)$$

$$Corr.(y_t, y_{t+1}) = \rho = \frac{Cov(y_t, y_{t+1})}{Var(y_t)} = \phi$$

lag = h

$$y_t = \phi y_{t-1} + \varepsilon_t$$

$$y_{t-h}\, y_t = \phi y_{t-h}\, y_{t-1} + y_{t-h}\, \varepsilon_t$$

$$E(\underbrace{y_{t-h} y_t}) = E(\phi y_{t-h} y_{t-1}) + E(y_{t-h}\, \varepsilon_t)$$

$$\gamma_h = \phi\, \gamma_{h-1}$$

$$\gamma_h = \phi^h \gamma_0$$

$$\Rightarrow \rho_h = \frac{\gamma_h}{Var(y_t)} = \frac{\phi^h Var(y_t)}{Var(y_t)} = \phi^h$$

3. Moments of the AR(1) process

$$f(y_t \mid y_{t-1}, \ldots, y_{t-h}) = f(y_t \mid y_{t-1}) \rightarrow \text{only depends on last prev.}$$

$$E(y_t \mid y_{t-1}) = \phi y_{t-1}$$

$$Var(y_t \mid y_{t-1}) = \sigma^2$$

$Y_t = \phi Y_{t-1} + \varepsilon_t \qquad Y_{t-2} = \phi Y_{t-3} + \varepsilon_t$

$Cov(Y_t, Y_{t-2}) = Cov(\phi Y_{t-1} + \varepsilon_t, \phi Y_{t-3} + \varepsilon_t)$