1

NETWORK MODEL

Referring to anisotropic characteristics in local cortical circuits of the rat's brain, a network model implementing anisotropic tissue geometry is developed. The introduction of a rewiring algorithm and qualitative anisotropy measure lay the foundation for the analysis of structural aspects of this model in Chapter **??**.

1.1 DISTANCE DEPENDENT CONNECTIVITY

random graph models in Section ??

In Gilbert's random graph model $G(n, p)$, probability of connection p is independently chosen and a fixed value for all vertex pairs. The anisotropic geometric graph model introduced in Section **??** is itself a random graph model - node positions as well as preferred directions of connection are uniformly at random distributed. In contrast to Gilbert's model however, neither is the probability of connection between a given vertex pair independent of the realization of other edges in the graph, nor is it a fixed value - probabilities strongly depend on internode distance in the anisotropic geometric graph model introduced.

Analyzing dependencies in the anisotropic model, specifically by identifying prevalent patterns of connectivity and relating these modes of non-randomness to biological findings, is the main focus of Chapter **??**. However, such structural correlations may not necessarily be an inherent feature of the network's anisotropy - distance dependent connectivity alone, as imposed by the model's specific geometry, may be the cause for emerging dependencies. It is therefore a crucial initial task to map the anisotropic model's distance dependent connection probability. Inferring connection probability as a function of internode distance and comparing it with computational results, in this section we explore distance connectivity of the anisotropic network model, securing a vital component in the analysis of structural features.

Theorem 1.1. *Let (G, Φ, a) represent an arbitrary realization of the anisotropic random graph model $G(n, w)$. Define $C : [0, \sqrt{2}] \to [0, 1]$ as the distance-dependent connection probability profile of (G, Φ), that is such that $C(x)$ is the probability that for a vertex pair $(v, v') \in V(G)^2 \setminus \Delta_{V(G)}$ in distance $x = \|\Phi(v) - \Phi(v')\|$ the vertex v projects to vertex v'. Then*

$$C(x) = \begin{cases} \frac{1}{2} & \text{for } x \leq w/2 \\ \frac{1}{\pi} \arcsin\left(\frac{w}{2x}\right) & \text{for } x > w/2. \end{cases}$$

Proof. Let v, v' be a pair of vertices in $V(G)^2 \setminus \Delta_{V(G)}$ in Euclidean distance x of each other. The vector difference $\Phi(v') - \Phi(v)$ may then be written as $xe^{i\theta}$, with $0 \leq \theta < 2\pi$. We have

$$R_{-\alpha(v)} xe^{i\theta} = xe^{i(\theta - \alpha(v))}.$$

Only for suitable combination of θ and $\alpha(v)$ an edge from v to v' exists. Assuming $\alpha(v)$ fixed, we calculate the probability of connection depending on the random choice of θ. We can assume $\alpha(v) = 0$, otherwise the same argument holds for $\theta' = \theta - \alpha(v)$.

From **??** we obtain the necessary and sufficient conditions

$$x \cos\theta \geq 0 \quad \text{and} \quad |x \sin\theta| \leq \frac{w}{2}.$$

Choosing uniformly at random θ in the range of $[0, 2\pi)$, the first condition is satisfied with a probability of $\frac{1}{2}$. Consider for the second condition $\theta \in [0, \pi)$. We have

$$\sin\theta \leq \frac{w}{2x},$$

and for $x \leq \frac{w}{2}$ the inequality holds for all θ by definition of $\sin\theta$. In the case of $x > \frac{w}{2}$, we note that for the first condition to hold θ must already be in $[0, \frac{\pi}{2})$ and can thus write the second condition θ as

$$\theta \leq \arcsin\frac{w}{2x},$$

yielding $C(x)$ by combining the considerations above and using the symmetry of sine for θ in the third and fourth quadrant. $\square$

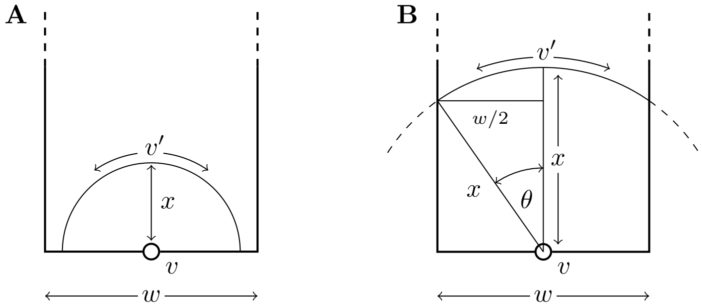

Figure 1.1: Illustrating the proof of Theorem 1.1 Distance-dependent connectivity profile $C(x)$ in an anisotropic geometric graph calculated from geometric dependencies. **A)** In the case of $x \leq w/2$, target v' may be located anywhere on the shown semicircle and therefore receives input from v with probability $1/2$. **B)** For $x > w/2$, suitable positions for target v' are dependent on x. The geometric relation $\sin\theta = w/2x$ leads to the distance-dependent connectivity profile as described in Theorem 1.1.

We can verify this result by extracting the distance-dependent connection probabilities in the sample graphs created in Section **??**. Combining data of all 25 graphs, we find that connection probabilities perfectly match the theoretical prediction (Figure 1.2). Additionally we're able to extend the reference sample graphs by distance-dependent networks (Definition **??**). Using Theorem 1.1 in conjunction with the sample graph parameter set ($N = 1000$, $s = 100$[1], $w = 25.2$) we easily ob-

distance-dependent sample graphs as reference

1 The generalization of Theorem 1.1 to allow for arbitrary side-length s is trivial and omitted here

tain the expected distance-dependent connectivity profile for the created sample graphs and, using this profile, generate purely distance-dependent networks[2]. Being highly interested in structural features not explained by distance-dependent connectivity, the numerical analysis in this work will heavily rely on these networks to identify aspects that are inherent to the anisotropy in connectivity.

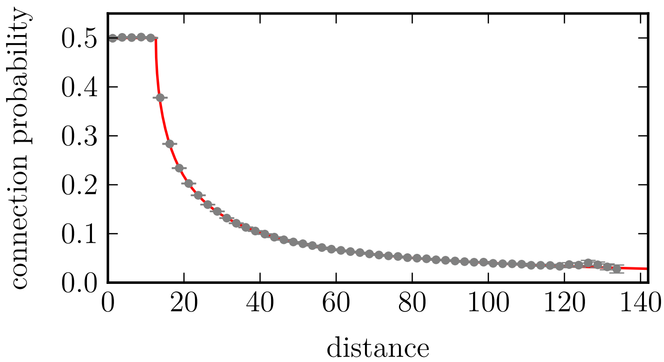

Figure 1.2: Predicted distance-dependent connection probability profile is matched by numerical results Averaging distance-dependent connection probabilities over the 25 sample graphs, we find the expected profile calculated in Theorem 1.1 is matched perfectly by the numerical results. (`dbffa88e`)

1.2 SUMMARY AND DISCUSSION

2 label: `N1000-dist_depend-flat_graph-00-24.xml.gz`