1 GRAPH THEORY

A review of graph theory lays the foundation for the mathematical considerations in this work. Following the discussion of directed graphs as a basis for the networks introduced in this work, common network measures are reviewed. Random graph models are integral to this work and are discussed in Sections **??** and **??**, the latter introducing geometric random graphs and performing geometric, probabilistic computations essential for the analytical discussions in later chapters.

1.1 DIRECTED GRAPHS

Here we introduce the various categories of directed graphs. The main reference for this section is Bang-Jensen and Gutin (2008), for the formal definition below however, we follow nLab (2014).

Definition 1.1 (Directed graphs)**.** A *directed pseudograph* G consists of two finite V, the *set of vertices* of G, and E, the *set of edges* of G, and two maps $s, t : E \to V$, the *source* and *target functions* of G. A *directed multigraph* is a directed pseudograph without loops, that is the map $d = (s, t) : E \to V^2$ already maps maps to $V^2 \setminus \Delta_V$, where $V^2 = V \times V$ denotes the cartesian product and $\Delta_V = \{(x, x) \mid x \in V\} \subseteq V^2$ the diagonal. Similarily, a *directed loop graph* is a directed pseudograph where d is injective. Finally, a *simple directed graph* can be defined as a directed pseudograph where d is both injective and already maps to $V^2 \setminus \Delta_V$.

Thus, in simple directed graphs, neither parallel edges nor loops (edges between the same vertex) are allowed, whereas directed multigraphs and directed loop graphs admit one of them respectively. We refer to any of the four graph types simply as a directed graph and only specify the type when needed.

Given a directed graph G, we denote with $V(G)$ the set of vertices of G and call it the **vertex set** of G. Analogously, the **edge set** $E(G)$ of G denotes the set of edges of G. This means, for a directed graph specified as $G = (V, E, s, t)$, we have $V(G) = V$ and $E(G) = E$.

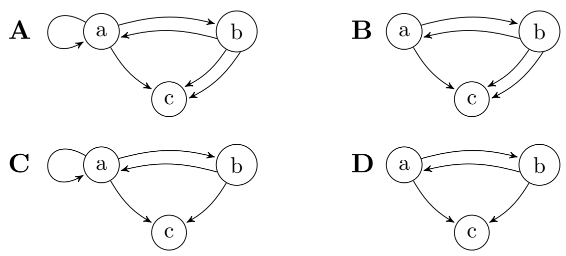

Figure 1.1: Typical examples of the directed graph types A) directed pseudograph **B)** directed multigraph **C)** directed loop graph **D)** simple directed graph.

A **morphism** $\phi : G \to H$, between two directed graphs $G = (V_G, E_G, s_G, t_G)$ and $H = (V_H, E_H, s_H, t_H)$, consists of a pair of maps $\phi_V : V_G \to V_H$ and $\phi_E : E_G \to E_H$, such that

$$s_H \circ \phi_E = \phi_V \circ s_G \quad \text{and} \quad t_H \circ \phi_E = \phi_V \circ t_G,$$

that is such that the following diagram commutes:

$$\begin{tikzcd} E_G \arrow[r, "\phi_E"] \arrow[d, shift right, "s_G"'] \arrow[d, shift left, "t_G"] & E_H \arrow[d, shift right, "s_H"'] \arrow[d, shift left, "t_H"] \\ V_G \arrow[r, "\phi_V"] & V_H \end{tikzcd}$$

A morphism $\varphi : G \to H$, between two directed pseudographs G and H is an **isomorphism**, if the maps $\varphi_V : V_G \to V_H$ and $\varphi_E : E_G \to E_H$ are bijections. Two directed pseudographs are called *isomorphic* if there exists an isomorphism inbetween them.

Remark. The last definition implies that, if there exists an isomorphism $\varphi : G \to H$, an isomorphism $\psi : H \to G$ can be found. This isomorphism is, of course, easily constructed via $\psi_V : V_H \to V_G, v \mapsto \varphi_V^{-1}(v)$, $\psi_E : E_H \to E_G, e \mapsto \varphi_E^{-1}(e)$.

Definition 1.2 (Weighted directed graphs)**.** An *edge-weighted directed graph* is a directed graph G along with a mapping $\omega : E(G) \to \mathbb{R}$, called the *weight function*. Similarly, a *vertex-weighted directed graph* is a directed graph with a mapping $\nu : V(G) \to \mathbb{R}$.

Equivalent definiton for directed loop graphs

Remark. A directed graph G can be equivalently defined as a pair of finite sets V, the *set of vertices* of G, and $E \subseteq V^2$ the *set of edges* of G. For an edge $(x, y) \in E$, we call x the *source* and y the target of the edge (x, y). Source and target functions are then uniquely determined as the projections on the first and second component,

$$s = \mathrm{pr}_1, t = \mathrm{pr}_2 : E(G) \to V.$$

Conversely, the edge set $E(G) \subseteq V^2$ can be determined from the source and target functions as $E := \{(s(e), t(e)) \mid e \in E\}$. The trivial identities $(x, y) = (\mathrm{pr}_1(x, y), \mathrm{pr}_2(x, y))$ and $\mathrm{pr}_1(s(e), t(e)) = s(e)$ with

$\mathrm{pr}_2(s(e), t(e)) = t(e)$ quickly verify the equivalence of the definitions. Given a directed loop graph G, we often assume the graph to be given in this form and write edges as $e = (x, y)$. Note that this concept is more complicated to introduce for directed pseudographs, since parallel edges e and Xe' should to be differentiated in the egde set of G, establishing the need for $E(G)$ to be a multi- or indexed set, notions we are trying to avoid in this document.

From now on any directed graph is assumed to be a directed loop graph. Although most, if not all, concepts work for directed pseudographs just as well, we want to start to heavily use the canonical edge representation, which when talking about pseudograps makes problems as mentioned before.

For a pair of vertex sets $X, Y \subseteq V(G)$ of a directed graph G we write

$$(X, Y)_G = \{(x, y) \in E(G) | x \in X, y \in Y\}$$

for the set of edges with source in X and target in Y. Specifically we write $T(x) = (x, V(G))_G$ for the set of *targets* for edges originating from the vertex x and $S(x) := (V(G), x)_G$ for the set of *sources* for edges projecting to x.

Notation for target and source sets

Definition 1.3 (In- and out-degree)**.** For a directed graph G the **in-degree** $d^-_G(x)$ of a vertex x is defined as the number of edges of G with target x, that is

$$d^-_G(x) = |S(x)|.$$

Similarily, the **out-degree** $d^+_G(x)$ of x is defined as

$$d^+_G(x) = |T(x)|,$$

the number of edges in G with source x.

A basic property of the in- and out-degree in directed graphs is that number of in-degrees of every vertex, as well the sum of every out-degree, equal the total number of edges:

Proposition 1.4. *In every directed graph G, we have*

$$\sum_{x \in V(G)} d^-(x) = \sum_{x \in V(G)} d^+(x) = |E(G)|.$$

Proof. Since $(V(G), x)_G \cap (V(G), y)_G = \emptyset$ for $x \neq y$, we can write

$$\sum_{x \in V(G)} d^-(x) = \left| \bigcup_{x \in V(G)} (V(G), x)_G \right| = |(V(G), V(G))_G| = |E(G)|.$$

Analogously for the out-degree. $\square$

Let G be a (simple) directed graph. A **walk** W in G is an alternating sequence $(x_1, e_1, x_2, e_2, x_3, \ldots, x_{n-1}, e_{n-1}, x_n)$ of of vertices x_i and edges e_i from G, such that

$$s(e_i) = x_i \quad \text{and} \quad t(e_i) = x_{i+1}, \text{ for } i = 1, .., n-1,$$

that is, such that the vertices are connected by the edges inbetween them. We denote the set of vertices $(x_1, \ldots, x_n)$ of W as $V(W)$ and the sequence of edges $(e_1, \ldots, e_{n-1})$ as $E(W)$.

The vertices x_1 and x_n are called the *end vertices* of W and we also say that W is an (x, y)-walk. The **length** of W is defined as the length of the sequence of edges; a walk consisting of only one vertex has length zero.

Definition 1.5 (Distance)**.** The **distance** of two vertices x,y in a directed graph G, is defined as the minimum length of an (x, y)-walk, if any such walk exists, otherwise $\text{dist}(x, y) = \infty$. In short,

$$\text{dist}(x, y) = \inf\{|E(W)| \mid W \text{ is } (x, y)\text{−walk}\}.$$

Proposition 1.6. *The distance function* $\text{dist} : V(G) \times V(G) \to \mathbb{N}$ *of a directed graph G satisfies the triangle equality,*

$$\text{dist}(x, z) \leq \text{dist}(x, y) + \text{dist}(y, z), \text{ for } x, y, z \in V(G).$$

Proof. Let x, y, z be vertices in G. If either no (x, y)-walk or (y, z)-walk exists, the inequality holds by definition. Other wise, let W be an (x, y)-walk of minimal length and let U be a (y, z)-walk of minimal length. Certainly, by concatenating W and U we obtain an (x, z)-walk of length $|E(W)| + |E(U)| = \text{dist}(x, y) + \text{dist}(y, z)$, proofing that

$$\text{dist}(x, z) \leq \text{dist}(x, y) + \text{dist}(y, z).$$

□

Definition 1.7 (Average path length)**.** The *average path length* of a directed graph G with $V(G) = n$ is defined as

$$l = \frac{1}{n(n-1)} \sum_{x \neq y \in V(G)} \text{dist}(x, y).$$

In practice, vertex pairs with $\text{dist}(x, y) = \infty$, that is pairs that are not connected by a walk, are disregarded in the computation and the average path length is determined in the connected components of the graph, ensuring that l is finite.

The concept of a small-world property in graphs was introduced by Watts and Strogatz (1998). Networks associated with the property are characterized by a small average path length, while however most nodes are organized in "cliques", connecting to nodes that are themselves neighbors. A measure capturing this property is the (local) clustering coefficient:

Definition 1.8 (Clustering coefficient)**.** The *clustering coefficient* of a vertex x in a directed graph G is defined as ratio of realized and possible edges between the neighbors of x. If N_x is the neighborhood of all vertices reciprocally connected to x,

$$N_x = \{v \mid v \in T(x) \wedge v \in S(x)\},$$

then the clustering coefficient is given by

$$\mathrm{clust}(x) = \frac{|(N_x, N_x)_G|}{|N_x|(|N_x| - 1)}.$$

Small-worldness is then described by a low average path length and high clustering coefficient, usually considered as the mean of all vertices.

2

STRUCTURAL ASPECTS

Subjecting the anisotropic network model to a critical examination of its structural features, we identify prevalent patterns of connectivity and relate theoretical and computational results to findings from experiments in the rat's visual cortex.

Small-world networks, as described in Section 1.2, are characterized by a small average path length and comparably high clustering coefficient. In brain networks, small-world properties are frequently discussed as the stemming from few long-rage projections. While most often reported on the macroscale (Sporns and Zwi 2004), small-worldness is also found in local cortical networks (Perin et al. 2011).

Here we are interested in exploring the question whether network anisotropy has effect on the small-worldness of geometric networks. While anisotropic networks display a relatively high clustering coefficient, $c = \pm$ at network size $N = 1000$ (compared with $p = 0.116$ in random networks) and a comparable path length $l_{\text{aniso}} =$ and $l_{\text{random}} = 1.8820 \pm 0.0001$) in comparison with random graphs, we find that

However, using distance-dependent networks as a reference, we find that successively eliminating anisotropy through rewiring contributes positively to the small-world property; with rising isotropy in the network, the characteristic path length declines, while the network clustering coefficient increases resulting together in rewired networks to display a higher degree of small-worldness (Figure 2.1).

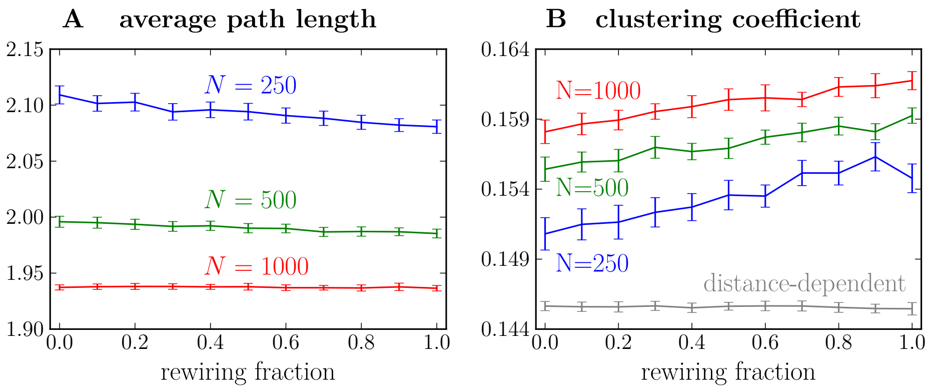

Figure 2.1: Anisotropy does not contribute to small-worldness In increasingly rewired networks, trends show a decreasing average path length and rising clustering coefficient and thus possibly a higher degree of small-worldness in the rewired, isotropically connected networks. **A)** Average path lengths for network sizes $N = 250$, 500 and 1000, where vertex pairs with no existing are discarded. Individual value pairs are obtained by averaging over a trial size of 20, 15 and 5 respectively; errorbars are SEM. **B)** Network configuration as in A), additionally showing clustering coefficients for distance-dependent networks. (`064f9b10`)

In distance dependent networks the average path length is in general smaller (**??**), matching those of a random network.

The fact that rewired anisotropic networks show an overall higher clustering coefficient as purely distance-dependent networks is here not fully explained, but presumably relates to the difference in the out-degree distribution (**??**),