1 NETWORK MODEL

Referring to anisotropic characteristics in local cortical circuits of the rat's brain, a network model implementing anisotropic tissue geometry is developed. The introduction of a rewiring algorithm and qualitative anisotropy measure lay the foundation for the analysis of structural aspects of this model in Chapter **??**.

1.1 REWIRING

eliminate anisotropy to find structures caused by it

Distance-dependency as identified in the last section may already account for many of the structural features present in anisotropic networks. A central question of this study is: What structural aspects in the network are truly features of the anisotropy in connectivity? Although a quantitative measure for anisotropy will only be introduced in the next section, already here we are able to qualitatively observe the strong directionality in connectivity - edges originating from one node "point in the same direction", effectively aligning with the orientation of the axonal projection of the source node (cf. **??**). To answer the question above, we need to introduce a method that eliminates this directionality, making networks essentially isotropic in connectivity. Then, structural features present in the original anisotropic networks, but not in their rewired, isotropic counterparts may be attributed to anisotropy.

Rewiring as introduced here, provides the transition from anistropic connectivity to networks isotropic in connectivity, closely resembling purely distance-dependent networks. Applying this process only partially then allows us to analyse structural features as they change with a varying degree of isotropy, asserting the importance of this process to our study. In designing the specific rewiring algorithm we identify two requirements that our implementation should satisfy:

1. elimination of anisotropy in connectivity
2. preservation of distance-dependent connectivity

The second point is especially important to us, as we want to impose isotropy on the network at "minimal cost", that is by changing as little as possible about the other characteristics of the network's connectivity. The following process respects both of the points above:

> For every edge between vertices v and v' with inter-vertex distance x, identify neurons with distance to v in the range of $(x-\varepsilon, x+\varepsilon)$ as potential new targets. Then pick at random one of these vertices (including v') as a new target for the current edge, if such an edge doesn't already exist (Figure 1.1).

In the graph theoretic context we formally define a rewiring as follows:

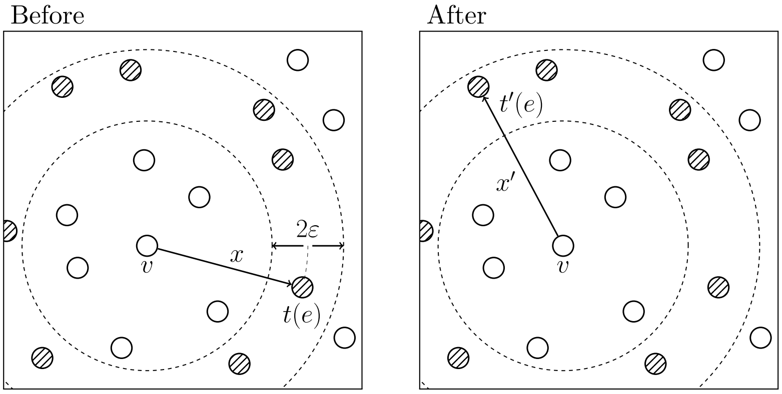

Figure 1.1: Rewiring transforms anisotropic geometric graphs to networks with isotropic connectivity For a given edge e with a distance x from its source vertex v to its target vertex $t(e)$, potential new targets (striped) are found in within a distance $(x-\varepsilon, x+\varepsilon)$ of v. The rewired edge then projects from v to a new target $t'(e)$, randomly chosen from the set of vertices within in this range. Inter-vertex distance between v and $t'(e)$ differs by less than ε from x, ensuring that for small ε the original distance-dependent connectivity is preserved. (Note that all targets within range are eligible for rewiring as no other edges exist. In general this is not the case.)

Definition 1.1. Let G be an anisotropic geometric graph with $|V(G)| = n$. Then we define a *rewiring* of G to be probability space over G^n_Φ, induced by the following process: For every edge $e \in E(G)$ uniformly at random pick a potential new target $t'(e)$ from the set $M_e = T_e \setminus K_e$, where T_e is the set of all vertices that differ in their distance to $s(e)$ less than ε from the distance of $s(e)$ to $t(e)$,

$$T_e = \{v \in V(G) \setminus s(e) \mid |\mathrm{d}(s(e), v) - \mathrm{d}(s(e), t(e))| < \varepsilon\}$$

and K_e the set vertices that already are connected to $s(e)$ by another rewired edge,

$$K_e = \{v \in V(G) \mid \exists\, e' \in E'(G) : s(e') = s(e), t(e') = v\},$$

where $E'(G)$ is the set of all edges that have been rewired already.

Note that in the way Definition 1.1 is formulated, it is possible for M_e to be empty for some edge e. In this case no new edge is realized and the resulting, rewired network has $|E(G)| - 1$ edges. In practice this happens negligibly seldom, out of approximately on average only 25.68 edges, with a standard deviation of 4.51 and accounting for roughly 0.02% of the rewired edges, are “lost” in this process (`4afc2727`).

We formulated Definition 1.1 in such a way, that distance-dependent connectivity is preserved. Here we verify by the following estimation:

Let $\tilde{C}(x)$ be the distance-dependent connectivity profile of a rewiring R_ε of an anisotropic graph $G_{n,w}$. Denote with $C(x)$ the distance-dependent connection probability of the $G_{n,w}$. We can estimate the

$$\begin{aligned}\mathbf{E}[\tilde{C}(x)] - C(X) &= \int_{x-\varepsilon}^{x+\varepsilon} f(x')C(x')\,dx - C(x) \\ &= \frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} C(x') - C(x)\,dx \\ &= \frac{1}{2\varepsilon}\left\{\int_{x-\varepsilon}^{x} C(x') - C(x)\,dx - \int_{x}^{x+\varepsilon} C(x') - C(x)\,dx\right\} \\ &= \frac{1}{2\varepsilon}\end{aligned}$$

The bandwidth parameter ε this way simulatenously governs how many rewired new targets are available and how well the distance is preserved. Setting $\varepsilon =$??, we find that in the distance dependent we observe as well as estimation Figure 1.2 while at the same time ensuring that enough for rewiring (Figure A.1).

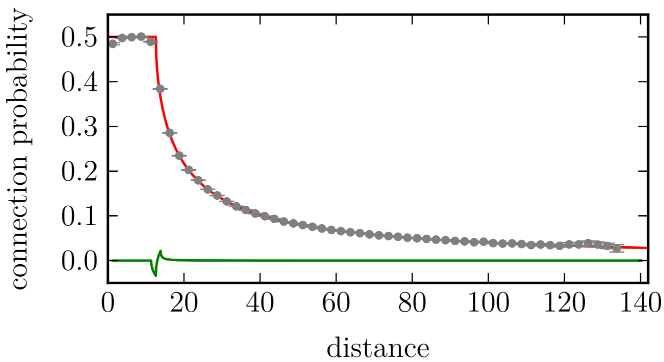

Figure 1.2: Predicted distance-dependent connection probability profile is matched by numerical results Averaging distance-dependent connection probabilities over the 25 sample graphs, we find the expected profile calculated in Theorem **??** is matched perfectly by the numerical results. (`4f4dfcf1`)

As a generalization of Definition 1.1, we define a partial rewiring $R_{\varepsilon,\eta}$, finding new targets only for a fraction η of all edges:

Definition 1.2. Let $\varepsilon > 0$ and $0 \le \eta \le 1$. A *partial rewiring* $R_{\varepsilon,\eta}$ of an anisotropic geometric graph $G_{n,w}$ is then a rewiring R_ε of $G_{n,w}$, in which every edge is rewired with a probability of η, otherwise it remains. To avoid the occurrence of multiple edges, K_e is then extended to include the targets of all edges originating from $s(e)$ that will not be rewired.

1.2 SUMMARY AND DISCUSSION

Part I

APPENDIX

A

APPENIDX

A.1 MATHEMATICA

In[1]:=
```
f[d_] = Piecewise[{{1 / (s * (d) ^ (1 / 2)) - 1 / (s^2), 0 < d < s^2}, {0, d > s^2}}]
```

Out[1]=
$$\begin{cases} -\frac{1}{s^2} + \frac{1}{\sqrt{d}\, s} & 0 < d < s^2 \\ 0 & \text{True} \end{cases}$$

In[2]:=
```
g[x_] := Convolve[f[d], f[d], d, x, Assumptions → {d ∈ Reals, x ∈ Reals}]
Simplify[g[x], {s > 0, x ∈ Reals}]
```

Out[3]=
$$\begin{cases} \frac{\pi s^2 - 4 s \sqrt{x} + x}{s^4} & x > 0 \,\&\&\, s^2 \geq x \\ -\frac{2 s^2 + x + \frac{4 s^3}{\sqrt{-s^2+x}} - \frac{4 s x}{\sqrt{-s^2+x}} - 2 s^2 \mathrm{ArcTan}\left[\frac{s}{\sqrt{-s^2+x}}\right] + i s^2 \mathrm{Log}\left[s - i \sqrt{-s^2+x}\right] - i s^2 \mathrm{Log}\left[s + i \sqrt{-s^2+x}\right]}{s^4} & s^2 < x \,\&\&\, 2 s^2 > x \\ 0 & \text{True} \end{cases}$$

In[4]:=
```
h[x_] := g[x^2] * 2 * x
```

In[5]:=
```
Simplify[h[x], {s > 0, x ∈ Reals, x > 0}]
```

Out[5]=
$$2 x \left(\begin{cases} \frac{\pi s^2 - 4 s x + x^2}{s^4} & s \geq x \\ -\frac{2 s^2 + x^2 + \frac{4 s^3}{\sqrt{-s^2+x^2}} - \frac{4 s x^2}{\sqrt{-s^2+x^2}} - 2 s^2 \mathrm{ArcTan}\left[\frac{s}{\sqrt{-s^2+x^2}}\right] + 2 s^2 \mathrm{ArcTan}\left[\frac{\sqrt{-s^2+x^2}}{s}\right]}{s^4} & s < x \,\&\&\, \sqrt{2}\, s > x \\ 0 & \text{True} \end{cases} \right)$$

In[6]:=
```
(*For s == 1, h becomes*)
```

In[7]:=
```
Simplify[h[x], {s == 1, x ∈ Reals, x > 0}]
```

Out[7]=
$$2 x \left(\begin{cases} \pi + (-4 + x)\, x & x \leq 1 \\ -2 - x^2 + 4 \sqrt{-1 + x^2} - 2\, \mathrm{ArcCot}\left[\frac{1}{\sqrt{-1+x^2}}\right] + 2\, \mathrm{ArcTan}\left[\frac{1}{\sqrt{-1+x^2}}\right] & 1 < x < \sqrt{2} \\ 0 & \text{True} \end{cases} \right)$$

In[8]:=
```
(*Expected Value*)
s := 1.
Integrate[x * h[x], {x, 0, Sqrt[2]}]
```

Out[9]= 0.521405

Mathematica A.1: Computation of probability density function for distance between to random points in square of side length s as supplement to proof of Theorem ??. Note that form of final result `Out[7]` differs from solution given in ??. While proof of equivalence could not be achieved analytically, expressions given are numerically equivalent, see Mathematica A.2.

```
In[1]:= f[d_] = Piecewise[{{1 / (s * (d) ^ (1 / 2)) - 1 / (s^2), 0 < d < s^2}, {0, d > s^2}}]

Out[1]= { -1/s^2 + 1/(Sqrt[d] s)    0 < d < s^2
        { 0                         True

In[2]:= g[x_] := Convolve[f[d], f[d], d, x, Assumptions → {d ∈ Reals, x ∈ Reals}]
        Simplify[g[x], {s > 0, x ∈ Reals}]

Out[3]= { (π s^2 - 4 s Sqrt[x] + x) / s^4      x > 0 && s^2 ≥ x
        { -(2 s^2 + x + (4 s^3)/Sqrt[-s^2 + x] - (4 s x)/Sqrt[-s^2 + x] - 2 s^2 ArcTan[s/Sqrt[-s^2 + x]] + i s^2 Log[s - i Sqrt[-s^2 + x]] - i s^2 Log[s + i Sqrt[-s^2 + x]]) / s^4      s^2 < x && 2 s^2 > x
        { 0                                      True

In[4]:= h[x_] := g[x^2] * 2 * x

In[5]:= Simplify[h[x], {s > 0, x ∈ Reals, x > 0}]

Out[5]= 2 x ( { (π s^2 - 4 s x + x^2) / s^4      s ≥ x
              { -(2 s^2 + x^2 + (4 s^3)/Sqrt[-s^2 + x^2] - (4 s x^2)/Sqrt[-s^2 + x^2] - 2 s^2 ArcTan[s/Sqrt[-s^2 + x^2]] + 2 s^2 ArcTan[Sqrt[-s^2 + x^2]/s]) / s^4      s < x && Sqrt[2] s > x
              { 0                                True )

In[6]:= (*For s == 1, h becomes*)

In[7]:= Simplify[h[x], {s == 1, x ∈ Reals, x > 0}]

Out[7]= 2 x ( { π + (-4 + x) x        x ≤ 1
              { -2 - x^2 + 4 Sqrt[-1 + x^2] - 2 ArcCot[1/Sqrt[-1 + x^2]] + 2 ArcTan[1/Sqrt[-1 + x^2]]        1 < x < Sqrt[2]
              { 0                     True )

In[8]:= (*Expected Value*)
        s := 1.
        Integrate[x * h[x], {x, 0, Sqrt[2]}]

Out[9]= 0.521405
```

Mathematica A.2: Computation of probability density function for distance between to random points in square of side length s as supplement to proof of Theorem **??**. Note that form of final result `Out[7]` differs from solut

A.2 SUPPLEMENTARY FIGURES

Chapter 1

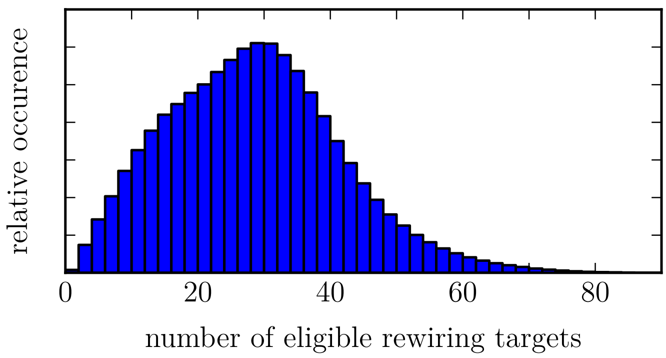

Figure A.1: (c7ee86d7)

??

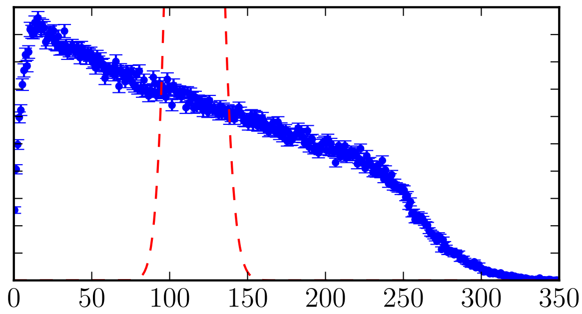

Figure A.2: (c7ee86d7)