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GRAPH THEORY

A review of graph theory lays the foundation for the mathematical considerations in this work. Following the discussion of directed graphs as a basis for the networks introduced in this work, network measures . Random graph models are essential to this work and are discussed in Sections **??** and **??**, the latter constituting a discussion of .

1.1 DIRECTED GRAPHS

Here we introduce the various categories of directed graphs. The main reference for this section is the work of Bang-Jensen and Gutin (2008). For the formal definition below we follow nLab (2014).

Definition 1.1 (Directed graphs)**.** A *directed pseudograph* G consists of two finite V, the *set of vertices* of G, and E, the *set of edges* of G, and two maps $s, t : E \to V$, the *source* and *target functions* of G. A *directed multigraph* is a directed pseudograph without loops, that is the map $d = (s, t) : E \to V^2$ already maps maps to $V^2 \setminus \Delta_V$, where $V^2 = V \times V$ denotes the cartesian product and $\Delta_V = \{(x, x) \mid x \in V\} \subseteq V^2$ the diagonal. Similarily, a *directed loop graph* is a directed pseudograph where d is injective. Finally, a *simple directed graph* can be defined as a directed pseudograph where d is both injective and already maps to $V^2 \setminus \Delta_V$.

Thus, in simple directed graphs, neither parallel edges nor loops (edges between the same vertex) are allowed, whereas directed multigraphs and directed loop graphs admit one of them respectively.

Say something about what "directed graph" means here, do all definition until random graphs work for all types of directed graphs?

Given a directed graph G, we denote with $V(G)$ the set of vertices of G and call it the **vertex set** of G. Analogously, the **edge set** $E(G)$ of G denotes the set of edges of G. This means, for a directed graph specified as $G = (V, E, s, t)$, we have $V(G) = V$ and $E(G) = E$.

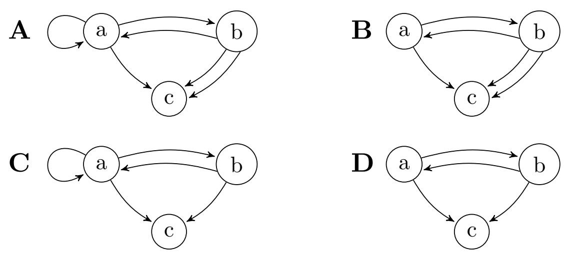

Figure 1.1: Typical examples of the directed graph types A) directed pseudograph **B)** directed multigraph **C)** directed loop graph **D)** simple directed graph.

A **morphism** $\phi : G \to H$, between two directed graphs $G = (V_G, E_G, s_G, t_G)$ and $H = (V_H, E_H, s_H, t_H)$, consists of a pair of maps $\phi_V : V_G \to V_H$ and $\phi_E : E_G \to E_H$, such that

$$s_H \circ \phi_E = \phi_V \circ s_G \quad \text{and} \quad t_H \circ \phi_E = \phi_V \circ t_G,$$

that is such that the following diagram commutes:

$$\begin{tikzcd} E_G \arrow[r, "\phi_E"] \arrow[d, shift right, "s_G"'] \arrow[d, shift left, "t_G"] & E_H \arrow[d, shift right, "s_H"'] \arrow[d, shift left, "t_H"] \\ V_G \arrow[r, "\phi_V"] & V_H \end{tikzcd}$$

A morphism $\varphi : G \to H$, between two directed pseudographs G and H is an **isomorphism**, if the maps $\varphi_V : V_G \to V_H$ and $\varphi_E : E_G \to E_H$ are bijections. Two directed pseudographs are called *isomorphic* if there exists an isomorphism inbetween them.

Remark. The last definition implies that, if there exists an isomorphism $\varphi : G \to H$, an isomorphism $\psi : H \to G$ can be found. This isomorphism is, of course, easily constructed via $\psi_V : V_H \to V_G, v \mapsto \varphi_V^{-1}(v)$, $\psi_E : E_H \to E_G, e \mapsto \varphi_E^{-1}(e)$.

Definition 1.2 (Weighted directed graphs)**.** An *edge-weighted directed graph* is a directed graph G along with a mapping $\omega : E(G) \to \mathbb{R}$, called the *weight function*. Similarly, a *vertex-weighted directed graph* is a directed graph with a mapping $\nu : V(G) \to \mathbb{R}$.

Equivalent definiton for directed loop graphs

Remark. A directed graph G can be equivalently defined as a pair of finite sets V, the *set of vertices* of G, and $E \subseteq V^2$ the *set of edges* of G. For an edge $(x, y) \in E$, we call x the *source* and y the target of the edge (x, y). Source and target functions are then uniquely determined as the projections on the first and second component,

$$s = \mathrm{pr}_1, t = \mathrm{pr}_2 : E(G) \to V.$$

Conversely, the edge set $E(G) \subseteq V^2$ can be determined from the source and target functions as $E := \{(s(e), t(e)) \mid e \in E\}$. The trivial identities $(x, y) = (\mathrm{pr}_1(x, y), \mathrm{pr}_2(x, y))$ and $\mathrm{pr}_1(s(e), t(e)) = s(e)$ with

$\mathrm{pr}_2(s(e), t(e)) = t(e)$ quickly verify the equivalence of the definitions. Given a directed loop graph G, we often assume the graph to be given in this form and write edges as $e = (x, y)$. Note that this concept is more complicated to introduce for directed pseudographs, since parallel edges e and e' should to be differentiated in the egde set of G, establishing the need for $E(G)$ to be a multi- or indexed set, notions we are trying to avoid in this document.

From now on any *directed graph* is assumed to be a directed loop graph. Although most, if not all, concepts work for directed pseudographs just as well, we want to start to heavily use the canonical edge representation, which when talking about pseudograps makes problems as mentioned before.

Remark (More Notation). - Check, do I really need this? For a pair of vertex sets $X, Y \subseteq V(G)$ of a directed graph G we write

$$(X, Y)_G = \{(x, y) \in E(G) | x \in X, y \in Y\}$$

for the set of edges with source in X and target in Y. For vertex sets with a single element $X = x$, we also write $(x, Y)_G$ and mean the edges with source x and target in Y.

Definition 1.3 (In- and out-degree)**.** For a directed graph G the **in-degree** $d_G^-(x)$ of a vertex x is defined as the number of edges of G with target x, that is

$$d_G^-(x) = |(V(G), x)_G| \,.$$

Similarily, the **out-degree** $d_G^+(x)$ of x is defined as

$$d_G^+(x) = |(x, V(G))_G| \,,$$

the number of edges in G with source x.

A basic property of the in- and out-degree in directed graphs is that number of in-degrees of every vertex, as well the sum of every out-degree, equal the total number of edges:

Proposition 1.4. *In every directed graph G, we have*

$$\sum_{x \in V(G)} d^-(x) = \sum_{x \in V(G)} d^+(x) = |E(G)|.$$

Proof. Since $(V(G), x)_G \cap (V(G), y)_G = \emptyset$ for $x \neq y$, we can write

$$\sum_{x \in V(G)} d^-(x) = \left| \bigcup_{x \in V(G)} (V(G), x)_G \right| = |(V(G), V(G))_G| = |E(G)|.$$

Analogously for the out-degree. □

BIBLIOGRAPHY

Bang-Jensen, Jørgen and Gregory Z. Gutin (2008). *Digraphs: Theory, Algorithms and Applications*. 2nd ed. London: Springer. DOI: 10.1007/978-1-84800-998-1.

nLab (2014). *Graph*. URL: http://ncatlab.org/nlab/revision/graph/56.